

① Let X denote the # of throws req'd to get a 6
 $\mathcal{X} = \{1, 2, 3, \dots\}$

$$P(X=x) = \left(\frac{5}{6}\right)^{x-1} \frac{1}{6} \quad x \in \mathcal{X}$$

$$= 0 \quad \text{o/w}$$

$$E(X) = \sum_1^{\infty} x \left(\frac{5}{6}\right)^{x-1} \frac{1}{6} = \frac{1}{6} \sum_1^{\infty} x \left(\frac{5}{6}\right)^{x-1} = \left(1 + 2\left(\frac{5}{6}\right) + 3\left(\frac{5}{6}\right)^2 + \dots\right) \frac{1}{6}$$

$$= \frac{1}{6} \left[\left(1 + \frac{5}{6} + \left(\frac{5}{6}\right)^2 + \dots\right) + \frac{5}{6} \left(1 + \frac{5}{6} + \left(\frac{5}{6}\right)^2 + \dots\right) + \left(\frac{5}{6}\right)^2 \left(1 + \frac{5}{6} + \left(\frac{5}{6}\right)^2 + \dots\right) + \dots \right]$$

$$= \frac{1}{6} \left[\frac{1}{1 - \frac{5}{6}} + \frac{5}{6} \frac{1}{1 - \frac{5}{6}} + \left(\frac{5}{6}\right)^2 \frac{1}{1 - \frac{5}{6}} + \dots \right]$$

$$= \frac{1}{6} \left[6 \left(1 + \frac{5}{6} + \left(\frac{5}{6}\right)^2 + \dots\right) \right] = 6$$

②

X : length of run of heads or tails starting with trial 1
 $\mathcal{X} = \{1, 2, \dots\}$

$$P(X=x) = \underbrace{(1-p)^x p}_{\text{run of } x \text{ T's}} + \underbrace{p^x (1-p)}_{\text{run of } x \text{ H's}}$$

$$E(X) = \sum_1^{\infty} x \left((1-p)^x p + p^x (1-p) \right)$$

$$= p(1-p) \left(\sum_1^{\infty} x (1-p)^{x-1} + \sum_1^{\infty} x p^{x-1} \right)$$

(2)

$$= p(1-p) \left(\frac{1}{p^2} + \frac{1}{(1-p)^2} \right) \leftarrow \text{as in (1)}$$

$$= \frac{1-2p+2p^2}{p(1-p)}$$

(3) (a) $E(|X|) = \sum_1^{\infty} |x| \frac{1}{x(x+1)} = \sum_1^{\infty} \frac{1}{x+1}$ not convergent
 $\Rightarrow E(X)$ does not exist

(b) $E|X| = \int_{|x|>1} |x| \frac{1}{2x^2} dx = \infty$
 $\Rightarrow E(X)$ does not exist

(c) $E|X| = \int_{-\infty}^{\infty} \frac{|x|}{\pi} \frac{1}{1+x^2} dx = \frac{2}{\pi} \int_0^{\infty} \frac{x}{1+x^2} dx$
 $= \frac{1}{\pi} \left(\log(1+x^2) \right) \Big|_0^{\infty} = \infty$
 $\Rightarrow E(X)$ does not exist

(4) Trivial calculations
 e.g. (a) $E(X) = a \int_0^1 x^a dx = \frac{a}{a+1}$; $E(X^2) = \frac{a}{a+2}$
 $V(X) = E(X^2) - (E(X))^2 = \frac{a}{a+2} - \left(\frac{a}{a+1} \right)^2$
 $= - \dots$

(5) $E(X) = \frac{c}{a} \int_{\mu}^{\infty} x \left(\frac{x-\mu}{a} \right)^{c-1} e^{-\left(\frac{x-\mu}{a} \right)^c} dx$

$$y = \left(\frac{x-\mu}{a} \right)^c \quad dy = \frac{c}{a} \left(\frac{x-\mu}{a} \right)^{c-1} dx$$

$$\Rightarrow E(X) = \int_0^{\infty} \left(a y^{1/c} + \mu \right) e^{-y} dy = a \Gamma\left(\frac{1}{c}+1\right) + \mu$$

$$E(X^2) = \frac{c}{a} \int_{-\infty}^{\infty} x^2 \left(\frac{x-\mu}{a} \right)^{c-1} e^{-\left(\frac{x-\mu}{a} \right)^c} dx$$

(3)

$$\text{as in } E(X) = \int_0^{\infty} (ay^{1/c} + \mu) e^{-y} dy$$

$$= a^2 \sqrt{\frac{2}{c}+1} + 2a\mu \sqrt{\frac{1}{c}+1} + \mu^2$$

$$V(X) = E(X^2) - (E(X))^2$$

$$= \left(a^2 \sqrt{\frac{2}{c}+1} + 2a\mu \sqrt{\frac{1}{c}+1} + \mu^2 \right) - \left(a \sqrt{\frac{1}{c}+1} + \mu \right)^2$$

$$= \dots$$

(6)

$$\int_0^m 3x^2 dx = \int_m^1 3x^2 dx = \frac{1}{2}$$

$$\Rightarrow m = \dots$$

(7)

$$\int_0^{\infty} (1-F(x)) dx = \int_0^{\infty} \int_x^{\infty} f_X(y) dy dx$$

$$= \int_0^{\infty} \underbrace{\int_0^y f_X(x) dx}_{0 < x < y < \infty} dy = \int_0^{\infty} y f_X(y) dy = E(X)$$

(8)

Let Z denote the r. v. \Rightarrow

Z : score in a shot $\mathcal{X}_Z = \{0, 2, 3, 4\}$

$$P(Z=0) = P(X > \sqrt{3}) = \frac{2}{\pi} \int_{\sqrt{3}}^{\infty} \frac{1}{1+x^2} dx = \frac{2}{\pi} \tan^{-1} x \Big|_{\sqrt{3}}^{\infty} = \frac{1}{3}$$

$$P(Z=2) = P(1 < X < \sqrt{3}) = \frac{2}{\pi} \int_1^{\sqrt{3}} \frac{1}{1+x^2} dx = \frac{1}{6}$$

④

$$P(z=3) = P\left(\frac{1}{\sqrt{3}} < x < 1\right) = \frac{2}{\pi} \int_{\frac{1}{\sqrt{3}}}^1 \frac{1}{1+x^2} dx = \frac{1}{6}$$

$$P(z=4) = P\left(0 < x < \frac{1}{\sqrt{3}}\right) = \frac{2}{\pi} \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{1+x^2} dx = \frac{1}{3}$$

Expt → score,

$$\begin{aligned} E(z) &= 0 \times \frac{1}{3} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{3} \\ &= \frac{1}{3} + \frac{1}{2} + \frac{4}{3} = \dots \end{aligned}$$

⑨ (a) $M_X(t) = E(e^{tx}) = \sum_{x=0}^n e^{tx} \binom{n}{x} p^x (1-p)^{n-x}$

$$= \sum_{x=0}^n \binom{n}{x} (pet)^x (1-p)^{n-x}$$

$$= (1-p + pet)^n$$

$$\left. \frac{d}{dt} M_X(t) \right|_{t=0} = \left. n(1-p + pet)^{n-1} pet \right|_{t=0} \leftarrow q = 1-p$$

$$= np = E(x)$$

$$\frac{d^2}{dt^2} M_X(t) = n(n-1)(1-p + pet)^{n-2} (pet)^2 + n(1-p + pet)^{n-1} pet$$

$$\left. \frac{d^2 M_X(t)}{dt^2} \right|_{t=0} = n(n-1)p^2 + np = \mu_2' = E(x^2)$$

$$\begin{aligned} V(x) &= E x^2 - (E x)^2 = n(n-1)p^2 + np - n^2 p^2 \\ &= np(1-p) = npq \end{aligned}$$

Sly (b)

(5)

(9)

$$X \sim G(\alpha, \beta)$$

(c)

$$\begin{aligned} M_X(t) = E(e^{tx}) &= \frac{1}{\Gamma(\alpha) \beta^\alpha} \int_0^\infty e^{tx} x^{\alpha-1} e^{-x/\beta} dx \\ &= \frac{1}{\Gamma(\alpha) \beta^\alpha} \int_0^\infty x^{\alpha-1} e^{-x(\frac{1}{\beta} - t)} dx \\ &\quad \text{(region of existence } t < \frac{1}{\beta}) \\ &= \frac{\Gamma(\alpha)}{\Gamma(\alpha) \beta^\alpha} \cdot \frac{1}{(\frac{1}{\beta} - t)^\alpha} = \left(\frac{1}{1 - \beta t} \right)^\alpha \end{aligned}$$

$$\begin{aligned} E(X) &= \left. \frac{d}{dt} M_X(t) \right|_{t=0} = -\alpha (1 - \beta t)^{-\alpha-1} (-\beta) \Big|_{t=0} \\ &= \alpha \beta \end{aligned}$$

$$\begin{aligned} E(X^2) &= \left. \frac{d^2}{dt^2} M_X(t) \right|_{t=0} \\ &= \alpha \beta \left(-(\alpha+1) (1 - \beta t)^{-\alpha-2} (-\beta) \right) \Big|_{t=0} \\ &= \alpha(\alpha+1) \beta^2 \end{aligned}$$

$$V(X) = E(X^2) - (E(X))^2 = \alpha^2 \beta^2 + \alpha \beta^2 - \alpha^2 \beta^2 = \alpha \beta^2$$

shy (e).

(10)

$$M_X(t) = e^{-5t} \frac{1}{2} + e^{4t} \frac{1}{6} + e^{5t} \frac{1}{8} + \frac{5}{24} e^{25t}$$

a 4 pt distⁿ

p.m.f.

$X = x$	-5	4	5	25
$P(X = x)$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{8}$	$\frac{5}{24}$

(6)

d.f. $F_X(x) = \begin{cases} 0 & x < -5 \\ \frac{1}{2} & -5 \leq x < 4 \\ \frac{1}{2} + \frac{1}{6} & 4 \leq x < 5 \\ \frac{1}{2} + \frac{1}{6} + \frac{1}{8} & 5 \leq x < 25 \\ \frac{1}{2} + \frac{1}{6} + \frac{1}{8} + \frac{5}{24} = 1 & x \geq 25 \end{cases}$

(11) X is (+)ve valued r.v., by Markov's inequality

$$P(X \geq 2\mu) = P(X \geq 2\mu) \leq \frac{E(X)}{2\mu} = \frac{1}{2}$$

(12) $P(-2 < X < 8) = P\left(\frac{-2-3}{2} < \frac{X-E(X)}{\sqrt{V(X)}} < \frac{8-3}{2}\right)$

$$E(X) = 3$$

$$V(X) = 4$$

$$= P\left(-\frac{5}{2} < \frac{X-E(X)}{\sqrt{V(X)}} < \frac{5}{2}\right)$$

$$= P\left(|X-\mu| \leq \frac{5}{2} \sqrt{V(X)}\right)$$

$$= 1 - P\left(|X-\mu| \geq \frac{5}{2} \sqrt{V(X)}\right)$$

$$\geq 1 - \frac{V(X)}{\frac{25}{4} \cdot V(X)} \quad (\text{Chebyshev's inequality})$$

$$= 1 - \frac{4}{25} = \frac{21}{25}$$

(13) $E(X) = -\frac{1}{8} + \frac{1}{8} = 0 = \mu$; $V(X) = EX^2 = \frac{1}{8} + \frac{1}{8} = \frac{1}{4} = \sigma^2$

By Chebyshev's inequality

$$\forall \epsilon > 0, \quad P(|X-\mu| \geq \epsilon) \leq \frac{\sigma^2}{\epsilon^2}$$

$$\text{i.e. } P(|X| \geq \epsilon) \leq \frac{1}{4\epsilon^2}$$

Also,
$$P(|X| \geq \epsilon) = \begin{cases} \frac{1}{8} + \frac{1}{8} = \frac{1}{4} & 0 < \epsilon \leq 1 \\ 0 & \epsilon > 1 \end{cases}$$

i.e. $P(|X| \geq \epsilon) = \frac{1}{4} \quad \forall \epsilon \geq 0 < \epsilon \leq 1$

\Rightarrow for $\epsilon = 1$, bound from Chebyshev's inequality is attained exactly and hence cannot be improved

(14) Let X be the r.v. denoting the # of components functioning
 $X \sim B(n, p)$

$$P(\text{system works effectively}) = P(X \geq \lceil n/2 \rceil + 1)$$

(a) $P(5 \text{ comp. system works}) = p_5$

i.e. $p_5 = P_5(X \geq 3) = \binom{5}{3} p^3 (1-p)^2 + \binom{5}{4} p^4 (1-p) + p^5$

& $p_3 = \binom{3}{2} p^2 (1-p) + p^3 = P(3 \text{ comp system works})$

$$p_5 > p_3$$

$$\text{If } \binom{5}{3} p^3 (1-p)^2 + \binom{5}{4} p^4 (1-p) + p^5 > \binom{3}{2} p^2 (1-p) + p^3$$

simplify to get the condition as $p > \frac{1}{2}$.

(b) $P_{2k+1}(X \geq k+1) = p_{2k+1} = P_{2k-1}(X \geq k+1)$

$$+ P_{2k-1}(X=k) P_2(X \geq 1)$$

$$+ P_{2k-1}(X=k-1) P_2(X=2)$$

$$\text{i.e. } p_{2k+1} = p_{2k-1} (X \geq k+1) + p_{2k-1} (X=k) (1-(1-p)^2) + p_{2k-1} (X=k-1) p^2$$

$$\begin{aligned} \text{Further } p_{2k-1} &= p_{2k-1} (X \geq k) \\ &= p_{2k-1} (X=k) + p_{2k-1} (X \geq k+1) \end{aligned}$$

$$\begin{aligned} \text{Since } p_{2k+1} &= p_{2k-1} (X \geq k+1) + p_{2k-1} (X=k) \\ &\quad \xleftarrow{\hspace{1.5cm}} - p_{2k-1} (X=k) (1-p)^2 \\ &\quad + p_{2k-1} (X=k-1) p^2 \end{aligned}$$

$$\Rightarrow p_{2k+1} = p_{2k-1} - p_{2k-1} (X=k) (1-p)^2 + p_{2k-1} (X=k-1) p^2$$

$$\Rightarrow p_{2k+1} > p_{2k-1} \quad \text{if}$$

$$p_{2k-1} (X=k) (-(1-p)^2) + p_{2k-1} (X=k-1) p^2 > 0$$

$$\text{i.e. } \binom{2k-1}{k} p^k (1-p)^{k-1} (-1-p^2+2p) + \binom{2k-1}{k-1} p^{k-1} (1-p)^k p^2 > 0$$

$$\text{i.e. } p^k (1-p)^{k-1} (-1-p^2+2p+p-p^2) > 0$$

$$\text{i.e. } -1-2p^2+3p > 0$$

$$\text{i.e. } (2p-1)(1-p) > 0$$

$$\text{i.e. } p > \underline{\underline{\frac{1}{2}}} \quad \text{required condition.}$$

(15) X : # of interviews attempt to get 5 interviews.

$$P(X=x) = \binom{x-1}{4} \left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)^{x-5} \times \frac{2}{3} \quad ; x=5, 6, \dots$$

reqd prob $P(X \leq 8) = P(X=5) + P(X=6) + P(X=7) + P(X=8)$

$$= \binom{4}{4} \left(\frac{2}{3}\right)^5 + \binom{5}{4} \left(\frac{2}{3}\right)^5 \left(\frac{1}{3}\right)^1 + \dots + \dots$$

$$= \dots$$

(16) $P(\text{selecting Box 1}) = P(\text{selecting Box 2}) = \frac{1}{2}$

suppose Box 2 is found empty ~~at (N+1)th~~, then Box 2 has been chosen $(N+1)^{\text{th}}$ times, at this time Box 1 contains K matches if it has been chosen $N-K$ times.

\Rightarrow $\left. \begin{array}{l} \text{choosing Box 2} \equiv \text{success.} \\ \text{choosing Box 1} \equiv \text{failure} \end{array} \right\} \text{Bernoulli trial.}$
 $p = \frac{1}{2}$

\Rightarrow Box 2 found empty with K matches left in Box 1
 $\equiv N-K$ failures preceding $(N+1)^{\text{th}}$ success

$$\begin{aligned} \text{prob} &= \binom{N+(N-K)}{N} \left(\frac{1}{2}\right)^N \left(\frac{1}{2}\right)^{N-K} \cdot \frac{1}{2} \\ &= \binom{2N-K}{N} \left(\frac{1}{2}\right)^{2N-K+1} \end{aligned}$$

By Box 1 found empty with K matches in Box 2

$$\text{prob} = \binom{2N-K}{N} \left(\frac{1}{2}\right)^{2N-K+1}$$

$$\Rightarrow \text{reqd prob} = \binom{2N-K}{N} \left(\frac{1}{2}\right)^{2N-K}.$$