Homework Set-I for PHY-305A: Physics of the Universe

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1. Suppose a particle of mass m is present in the gravitational field of a large mass $M \gg m$. The object with mass M is supposed to be at rest and the smaller object is moving.

   (a) Show that the angular momentum of the smaller mass $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, where $\mathbf{p} = m\mathbf{v}$ and $\mathbf{v}$ is the volocity of the smaller object, must be constant. From this show that the motion of the object must be planer and in plane polar coordinates

   $$mr^2\dot{\theta} = \ell\,, \tag{1}$$

   where ℓ is a constant. Here the dot specifies a time derivative.

   (b) Argue (you may not derive) that the radial equation of motion of the object in the gravitational field of the big mass is given by

   $$m\ddot{r} - mr\dot{\theta}^2 = f(r)\,, \tag{2}$$

   where

   $$f(r) = -\frac{GM}{r^2}\,,$$

   where G is the universal gravitational constant. Can you specify the origin of all the force terms appearing in Eq. (2)?

   (c) In plane polar coordinates the velocity vector is given as $\mathbf{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$. From this you can easily find out the kinetic energy $\frac{1}{2}m\mathbf{v}^2$ of the object in motion. If the potential energy of the object in the gravitational field of the big mass is

   $$V(r) = -\frac{GM}{r}\,, \tag{3}$$

   then show that the total energy of the object in motion is given as

   $$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r)\,. \tag{4}$$

   As the potential is independent of time, the total energy E is assumed to be constant.

   (d) From Eq. (1) we see that $\ell dt = mr^2 d\theta$. Using this one can write the relation between time derivatives and angular derivatives as

   $$\frac{d}{dt} = \frac{\ell}{mr^2}\frac{d}{d\theta}\,.$$

   One can use the above relation and show that

   $$\frac{d^2}{dt^2} = \frac{\ell}{mr^2}\frac{d}{d\theta}\left(\frac{\ell}{mr^2}\frac{d}{d\theta}\right)\,. \tag{5}$$

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Use this equation in conjunction with Eq. (1) to rewrite Eq. (2) as

$$\frac{\ell}{r^2}\frac{d}{d\theta}\left(\frac{\ell}{mr^2}\frac{d}{d\theta}\right) - \frac{\ell^2}{mr^3} = f(r)\,. \tag{6}$$

Using the relation

$$\frac{1}{r^2}\frac{dr}{d\theta} = -\frac{d(1/r)}{d\theta}\,,$$

and defining

$$u \equiv \frac{1}{r}\,, \tag{7}$$

rewrite Eq. (6) as

$$\frac{\ell^2 u^2}{m}\left(\frac{d^2u}{d\theta^2} + u\right) = -f(1/u)\,.$$

We know $f(r) = -dV(r)/dr$ for the Newtonian potential. Using the relation

$$\frac{d}{du} = \frac{dr}{du}\frac{d}{dr} = -\frac{1}{u^2}\frac{d}{dr}\,,$$

rewrite the last equation as

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{\ell^2}\frac{dV(1/u)}{du}\,. \tag{8}$$

This is the orbit equation of the small object in the gravitational field of the large mass.

2. Show that the orbit equation can also be written as

$$\frac{d^2u}{d\theta^2} + u = \frac{mGM}{\ell^2}\,. \tag{9}$$

   (a) Changing variable as

   $$y = u - \frac{mGM}{\ell^2}\,,$$

   show that the last orbit equation transforms to

   $$\frac{d^2y}{d\theta^2} + y = 0\,. \tag{10}$$

   (b) Solving the above equation, where the solution can be taken as $y = B\cos(\theta - \theta')$ for constants B and θ', show that the relationship between r and θ is given by

   $$\frac{1}{r} = \frac{mGM}{\ell^2}[1 + e\cos(\theta - \theta')]\,, \tag{11}$$

   where e is defined as

   $$e \equiv \frac{B\ell^2}{mGM}\,. \tag{12}$$

   What kind of geometrical shapes of the orbits is prdicted by Eq. (11)?

(c) From Eq. (4) we can write

$$\dot{r} = \sqrt{\frac{2}{m}\left(E - V(r) - \frac{\ell^2}{2mr^2}\right)}\,,$$

which can also be written as

$$dt = \frac{dr}{\sqrt{\frac{2}{m}\left(E - V(r) - \frac{\ell^2}{2mr^2}\right)}}\,.$$

Using the relation between dt and $d\theta$ as given in question 1(d), show that the above equation can be converted to an equation containing integrals as:

$$\int d\theta = \int \frac{\ell dr}{mr^2\sqrt{\frac{2}{m}\left(E - V(r) - \frac{\ell^2}{2mr^2}\right)}}\,,$$

where θ' is a constant of integration. Then show that in terms of u the above integral can be written as

$$\theta = \theta' - \int \frac{du}{\sqrt{\frac{2mE}{\ell^2} + \frac{2mGMu}{\ell^2} - u^2}}\,, \tag{13}$$

where we have used the Newtonian potential expression in terms of u.

(d) The indefinite integral appearing in Eq. (13) can be done using the result

$$\int \frac{dx}{\sqrt{\alpha + \beta x + \gamma x^2}} = \frac{1}{\sqrt{-\gamma}} \arccos\left[-\frac{(\beta + 2\gamma x)}{\sqrt{q}}\right]\,, \tag{14}$$

where

$$q = \beta^2 - 4\alpha\gamma\,.$$

Using the above integral and Eq. (13) show that in the present case,

$$\theta = \theta' - \arccos\left[\frac{\frac{\ell^2 u}{mGM} - 1}{\sqrt{1 + \frac{2E\ell^2}{mG^2M^2}}}\right]\,. \tag{15}$$

(e) Write the last equation in terms of $r = 1/u$ to get

$$\frac{1}{r} = \frac{mGM}{\ell^2}\left[1 + \sqrt{1 + \frac{2E\ell^2}{mG^2M^2}}\cos(\theta - \theta')\right]\,. \tag{16}$$

Compare Eq. (11) and the above equation verify that

$$e \equiv \sqrt{1 + \frac{2E\ell^2}{mG^2M^2}}\,. \tag{17}$$

The above expression gives the eccentricity of the orbits. What kind of geometric shapes of the orbits do you expect when $E > 0$, $E = 0$, $E < 0$ and $E = -\frac{mG^2M^2}{2\ell^2}$?

3. It is known that the total energy of a small planet moving in the presence of gravitational force produced by a big star is always negative.

   (a) With the above information and the result of the last question show that the orbit of the planet will be an ellipse, where the star lies at one of the foci.

   (b) Suppose the planet is moving in such an elliptic orbit. Find out the area swept out by this planet in a time interval t to $t + dt$. Here the area swept out by the planet in its orbit has to be calculated by approximating the area with the area of a triangle whose one vertex is at the focus of the ellipse (where the star is present) and the other two vertices are at the positions of the planet at times t and $t + dt$ on the orbit. The triangle approximation becomes exact as dt is very small. Calculate the infinitesimal area dA swept out by the planet in time dt and using Eq. (1), show that

$$\frac{dA}{dt} = \frac{\ell}{2m}\,, \tag{18}$$

   which shows that the rate of change of the area swept out by the planet reamins a constant.

   (c) Suppose the planet moves in an elliptuic orbit with semiminor axis length b and semimajor axis length a. In that case it is known that the total area of the orbit is $A = \pi ab$. Show that for an ellipse

$$b = a\sqrt{1 - e^2}\,, \tag{19}$$

   where e is the eccentricity of the ellipse. It is known that the planet is at one point closest to the star and at another point it is farthest from the star. Both these points lie on the major axis of the elliptic orbit. Show that at these points the radial velocity of the planet vanishes and the distance of these points from the star can be found from the roots of of the quadratic equation (in r)

$$E - \frac{\ell^2}{2mr^2} + \frac{GM}{r} = 0\,.$$

   If r_1 and r_2 are the two roots of the above equation then $2a = r_1 + r_2$. From this information show that

$$a = -\frac{GM}{2E}\,.$$

   Using this equation and Eq. (17) show that

$$\frac{\ell^2}{mGM} = a(1 - e^2)\,. \tag{20}$$

   Using the above equation and Eq. (19) show that

$$b = \sqrt{\frac{a\ell^2}{mGM}}\,.$$

   The time period of the planetary orbit, T, is obtained by dividing the total area of the orbit, A, by the constant rate at which this area is swept by the planet. From this fact and the above information show that

$$T = 2\pi a^{3/2}\sqrt{\frac{m}{GM}}\,. \tag{21}$$