Notes on discrete groups and Furstenberg boundaries

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Let G be a (countable) discrete group.

1 Introduction

We will argue C*-simplicity. So, we recall group C*-algebras.

Def 1.1. *Let λ be a left regular representation of G, i.e.*

$$G \ni g \mapsto \lambda_g \, [= \delta_h \mapsto \delta_{gh}] \in B(l^2(G)).$$

We define the reduced group C-algebra $C_r^*(G)$ by the closure of $\overline{\mathrm{span}}\{\lambda_g\}_{g \in G}$.*

For any descrete group G, the reduced C-algebra $C_r^*(G)$ has a canonical tracial state $\tau_0 := \langle \dot{\delta}_e, \delta_e \rangle$.*

Def 1.2. *A group G is called C*-simple, if $C_r^*(G)$ is simple, i.e. $C_r^*(G)$ has no closed two-sided ideal.*

A group G has the unique trace property if $C_r^*(G)$ has a unique trace.

Powers [Pow75] proved that free groups is C*-simple.

Thm 1.1 ([Pow75])**.** *Let $\mathbb{F}_2$ be a free group of rank 2. For any $a \in C_r^*(G)$ and any $\varepsilon > 0$, there are $g_i, \ldots, g_n \in G$ and a partition of unity $\sum_{i=1}^n c_i = 1$ s.t.*

$$\|\tau_0(a) - \sum_{i=1}^n c_i \lambda_{g_i} a \lambda_{g_i}^* \| < \varepsilon$$

In particular, $C_r^*(\mathbb{F}_2)$ is simple and has a unique trace. In other words, $\mathbb{F}_2$ is C-simple and has the unique trace property.*

Variants of Powers' proof became the main method for establishing these property. Many many results.

Kalantar and Kennedy [KK17] made the new method for proving C*-simplicity.

Thm 1.2 ([KK17])**.** *Let G be a discrete group and $\partial_F G$ be the Furstenberg boundary of G. The followings are equivalent.*

1. *G is C*-simple;*
2. *$C(\partial_F G) \rtimes_r G$ is simple;*
3. *$C(B) \rtimes_r G$ is simple for some G-boundary B;*

4. G acts on $\partial_F G$ topologically freely;

5. G acts on B topologically freely for some G-boundary B:

Rem 1.2.1 ([BKKO17])**.** *G acts on $\partial_F G$ freely if and only if G acts on B topologically freely for some G-boundary B.*

So, in order to prove C*-simplicity, it suffices to find the boundary on which G acts topologically freely. For example, for amalgamated free products or HNN-extension, we probably consider the ideal boundary of its Bass-Serre tree. So, many existing results are proved more simplily.

2 G-boundary

Def 2.1. *Let G be a group and X be a locally compact group on which G acts. The action of G is called minimal if X has no non-trivial G-invariant closed subspace, that is for any $x \in X$, $\overline{G \cdot x} = X$. The action of G is called strongly proximal if for any $\mu \in \mathcal{P}(X)$, $\overline{G \cdot \mu}$ has a dirac mass, that is for any $\mu \in \mathcal{P}(X)$, there exist $x \in X$ and a net $g_i \in G$ s.t. for any $f \in C_0(X)$,*

$$g_i.\mu(f) = \mu(g_i^{-1}.f) \to f(x).$$

X is called G-boundary if X is compact and the action of G is minimal and strongly proximal.

Ex 1 ($\mathbb{F}_2$)**.** *We consider a Caley graph of $\mathbb{F}_2$. Let a, b be generators of $\mathbb{F}_2$. Let $V(T)$ be a set of all words of $\mathbb{F}_2$ and $E(T) = \{(v, w) \in V(T) \times V(T) |$ there exists $x \in \{a, a^{-1}, b, b^{-1}\}$ s.t. $v = wx\}$.*

参考文献

[BKKO17] Emmanuel Breuillard, Mehrdad Kalantar, Matthew Kennedy, and Narutaka Ozawa. C*-simplicity and the unique trace property for discrete groups. *Publications mathématiques de l'IHÉS*, 126(1):35–71, 2017.

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