My Studies

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Contents

1 character

1.1 C*-algebra

simple, completely positive map, K-group, masa, Cartan subalgebra, representation, norm on tensor(faithful), embedding, hereditary subalgebra left ideal,

1.2 von Neumann algebra

trace, factor, GNS, spectral decomposition, M_p, weight, modular automorphism, spectrum, solid, amenable, rigid,

1.3 example

$M_n(\mathbb{C})$ $L(G)$ $C_r(G), C(G)$ $\mathcal{R}$ $M \rtimes G$ *ultraproduct GNS* AF-algebra Toeplitz algebra Powers factor

2 Takesaki

2.1 Egorov and Lusin

Thm 2.1 (Egorov's lemma)**.** *Let be $(\Omega,\ \mathcal{F},\ \mu)$ a finite measure space. Let be $\{f_n\}_{n\in\mathbb{N}}$ and f measurable functions. Suppose $\{f_n\}_{n\in\mathbb{N}} \to f$ a.e. Then, for each $\epsilon > 0$, There exists $\mathcal{F} \subset A$ s.t. $\mu(\Omega - A) < \epsilon$ and $\{f_n\}_{n\in\mathbb{N}}$ convergent f uniformly on A.*

Thm 2.2 (Lusin's theorem)**.** *Same assumption. Then, for each $\epsilon > 0$, There exists $\mathcal{F} \subset A$ s.t. $\mu(\Omega - A) < \epsilon$ and f is continuous on$A$.*

Thm 2.3 (noncommutative Egorov's theorem)**.** *Let (M, H) be von Neumann algebra and $A \subset M$ be a bounded subset. Suppose any $a \in \overline{A}^s$, any $\varphi \in M_{*,+}$, any $e \in P(M)$, any $\epsilon > 0$. Then, there exists $e_0 \in P(M)$ s.t. $e_0 \leq e$ and a sequence $\{a_n\}_{n\in\mathbb{N}} \subset A$ s.t. $\lim_{n\to\infty} ||(a - a_n)e_0|| = 0$, $\varphi(e - e_0) < \epsilon$.*

Thm 2.4 (noncommutative Lusin's theorem)**.** *Let A be a C*-algebra. Suppose $M = \overline{A}^s$. Suppose $0 \neq \varphi \in M_{*,+}$, $0 \neq e \in P(M)$, ϵ, $\delta > 0$.*

- *For each $a \in M$,there exists $e \geq e_0 \in P(M)$ and $a_0 \in A$ s.t. $ae_0 = a_0e_0$ and $||a_0|| \leq (1 + \delta)||ae_0||$.*
- *For each $a \in M_{sa}$, there exsists $e \geq e_0 \in P(M)$ and $a_0 \in A_{sa}$ s.t. $ae_0 = a_0e_0$, $||a_0|| \leq \min\{2(1 + \delta)||ae_0||, ||a|| + \delta\}$.*
- *Let $1 \in A$. For each $a \in \mathcal{U}(M)$, there exists $e \geq e_0 \in P(M)$ and $a_0 \in \mathcal{U}(A)$ s.t. $ae_0 = a_0e_0$, $||a_0 - 1|| \leq ||a - 1|| + \delta$.*

If $\varphi = \mathrm{Tr}$, the following theorem is followed.

Thm 2.5 (Transitivity theorem)**.** *Let A be a irredusible C*-algebra on H. For eqch finite projection $e \in P(M)$, $Ae = B(H)e$.*

Ω	M
$\mathcal{F}$	P(M)
μ	φ ($\in M_{*,+}$)
measurable function	M
$f : \Omega \to \mathbb{C}$	$a : M \to M$

2.2 universal representation

Thm 2.6. *Let A be a C*-algebra and (π,H) be a representation of A.*

$$\begin{tikzcd} A \arrow[r, "\pi"] \arrow[d, "\iota"] & \pi(A)'' \\ A^{**} \arrow[ur, "\tilde{\pi}"] & \end{tikzcd}$$

2.3 Polar decomposition

Thm 2.7. *closed left invariant subspace* $\longleftrightarrow$ σ*-w closed right ideal*
$M_* \supset V = J^0 = M_*e \longleftrightarrow M \supset J = V^0 = (1-e)M$

Ex 1. $l^2(\mathbb{Z})$

Thm 2.8. *Let* X *and* Y *be Banach sp.s. Let* B_X *be a unit open ball. If* $T(B_X) = B_Y$*,then injective.*

Thm 2.9. *Let* $A < B$ *be C*-algebras. Let* $E : B \to A$ *be a surjective map, which statisfies* $E(a) = a\,(a \in A)$, $||E(x)|| \leq ||x||$. *Then, it is conditional expectation.*

- $E(x^*x) \geq 0\,(x \in B)$
- $E(axb) = aE(x)b\,(a, b \in A,\, x \in B)$
- $E(x)^*E(x) \leq E(x^*x)$

Thm 2.10. ω *is normal* $\Leftrightarrow$ *for all* $\{p_n\} \subset P(M)$, $\omega(\sum p_n) = \sum \omega(p_n)$.

Thm 2.11. $\varphi \in M_*$. *There exsit unique* $v \in M$ *and* $\omega \in M_{*,+}$ *s.t.* $\varphi = v\omega$, $v^*v = s(\omega) = s_r(\varphi)$, $vv^* = s_l(\varphi)$.

Def 2.1. *Let* (M_i, τ_i) *be tracial von Neumann algebras.*
(M, τ) *is called free product von Neumann algebra, if it satisfies the following properties.*
$\varphi_i : M_i \to M$ *are hom and* $\tau_i = \tau \circ \varphi_i$, $\varphi_i(M_i)$ *are free with respect to /tau, and generates* M.
von Neumann subalgebrasM_i *of von Neumann algebra*M *is free with respect to faithful normal state, if for each* $x_i \in M_{k_i}$*with* $k_1 \neq \cdots \neq k_n$, $\tau(x_i) = 0$, $\tau(x_1 \cdots x_n) = 0$.

Thm 2.12. *Let* (M_i, τ_i) *be tracial von Neumann algebras. Let* (M, τ) *be its free product von Neumann algebra. Let* Q *be deffuse von Neumann subalgebra of* M_1. *Then,* $Q' \cap M \subset M_1$. *Specially,* $Z(M) \subset Z(M_1)$*,therefore if* M_1 *is a* II_1*factor , so is* M.

Proof. A masa of tracial deffuse von Neumann algebra is deffuse, since conditinal expectation.
separable abelian diffuse von Neumann algebra is isomorphic to $L^\infty([0,1], \mu)$(μ:Leb), and faithful state is given by integration, so $u_n = exp(2\pi int)$ is $\tau(u_n) = 0$, and $w - \lim u_n = 0$(Riemann-Lebesgue).

It suffices to show $x \in \ker(\tau_1)$ is 0.
Since E_{M_1} is faithful, it suffices to show $E_{M_1}(x * x) = 0$.
For each $x, y \in M$with$E_{M_1}(x) = 0$ and $E_{M_2}(y) = 0$, it suffices to show $\lim ||E_{M_1}(xu_ny)|| = 0$.
Since $ker(\tau_1)$ideal, $E_{M_1}(xu_ny) = \tau_1(bu_nd)x_1acy_1$. Because b, d are $ker(\tau_1) \cup \{1\}$, QED. $\square$

Thm 2.13. *Let* (M, τ) *be a tracial von Neumann algbera,* $e_B \in P(L^2(M))$ *be a projection onto* $L^2(B)$. *Suppose* $B = \langle B, e_B \rangle$.

- $B = M \cap \{e_B\}'$;

2.4 amplification

Thm 2.14. *Let* $\{M_1, H_1\}$*and* $\{M_2, H_2\}$ *be vN algebras.*
If π *is a normal hom of* M_1 *onto* M_2*, then there exists a Hilbert sp.* K, $e' \in \mathcal{P}(M_1' \otimes B(K))$*and isometry* U

of $e'(H_1 \otimes K)$ *onto* H_2 *s.t.* $\pi(x) = U(x \otimes 1_K)_{e'}U^*(x \in M_1)$.

$$\begin{tikzcd} e'(H_1 \otimes K) \arrow[r, "(x\otimes 1_K)_{e'}"] \arrow[d, "U"] & e'(H_1 \otimes K) \arrow[d, "U"] \\ H_2 \arrow[r, "\pi(x)"] & H_2 \end{tikzcd}$$

Proof. Let $\{\xi_i\} \subset H_2$ be a maximal family s.t. $H_{2,i}[\pi(M_1)\xi_i]$ are mutually orthogonal. Then, $H_2 = \oplus_i H_{2,i}$.
Let K_i be a separable infinite dimensional Hilbert sp, K be a $\otimes K_i$and π_1 be a amplification of M_1. Since ${}^t\pi(\omega_{\xi_i}) \in M_*^+$, there exists $\zeta_i \in H \otimes K_i$ s.t. ${}^t\pi_1(\omega_{\zeta_i}) = {}^t\pi(\omega_{\xi_i})$.
Let e' be a projection of $H \otimes K$ onto $\otimes_i[\pi_1(M_1)\zeta_i]$, which bvelongs to $\pi_1(M_1)'$.
Let isometry U_i be a extention of $[\pi_1(M_1)\zeta_i]$ onto H_i and $U = \oplus_i U_i$. □

2.5 semifinite

Prop 2.1. *Let M be a von Neumann algebra with faithful normal semifinite trace. If $N < M$ is a von Neumann subalgebra and $\tau|_N$ is semifinite, there exists faithful normal conditional expectation E of M onto N s.t. $\tau = \tau \circ E$.*

3 fundamental theory of C*-algebras(not only ozbr)

3.1 completely positive

Def 3.1. *$\varphi : A \to B$ is completely positive(c.p.), if for all n, $\varphi \otimes \mathrm{id}_n : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is positive.*
contractive c.p.(c.c.p.)

Thm 3.1. *Let A be unital C*-algebras and $\varphi : A \to B(H)$ be a unital *-hom. Then, there exisit a Hilbert space $\hat{H}$, $\pi : A \to B(\hat{H})$ and $V : H \to \hat{H}$ s.t. $\varphi = V^*\pi(a)V$ $(a \in A)$:*

$$\begin{tikzcd} H \arrow[r, "\varphi(a)"] \arrow[d, "V"] & H \arrow[d, "V"] \\ \hat{H} \arrow[r, "\pi(a)"] & \hat{H} \end{tikzcd}$$

Proof. We define inner product on $\hat{H}$ by $\langle \sum a_i \otimes \xi_i, \sum b_i \otimes \zeta_i \rangle := \sum_{i,j} \langle \varphi(b_j^* a_i)\xi_i, \zeta_j \rangle$. We denote a completion of $A \odot H/N$ by $\hat{H}$, where $N = \{a \in A \odot H | \varphi(a^*a) = 0\}$.
We define V by a extension of $\xi \to 1_A \otimes \xi$. We define π by a extension of $a_i \otimes \xi_i \to aa_i \otimes \xi_i$.
We remark $V^*(a \otimes xi) = \varphi(a)\xi$. □

Prop 3.1. *Let π be a non-zero homomorphism. Then, π is a $*$-homomorphism iff $||\pi|| = 1$.*

Proof. At first, we show that $x \in B(H)$ is unitarty iff $||x|| = ||x^{-1}|| = 1$. Only if:it follows x is isometric.
We suffices to show for $u \in \mathcal{U}(H)$, $\pi(u)$ is unitary. □

Thm 3.2. *Let A be a unital C*-algebra and $E \subset A$ be an operator subsystem. Then, every c.c.p. map $\varphi : E \to B(H)$ extends to a c.c.p. map $\overline{\varphi} : A \to B(H)$.*

Prop 3.2. *Let A be a unital C*-algebra. A map $\varphi : A \to M_m(\mathbb{C})$ is c.p. if and only if $\hat{\varphi}$ is positive on $M_n(\mathbb{C})$, where $\hat{\varphi}((a_{ij})) = \sum \varphi(a_{ij})$.*
$\mathrm{CP}(A, M_n(\mathbb{C}))\varphi \mapsto \hat{\varphi} \in M_n(\mathbb{C})_+^*$ *is a bijective correspondence.*

Def 3.2 (nuclear)**.** *Let A, B be a C*-algebras. Let $\theta : A \to B$ be a map.*
φ is called nuclear if there exist c.c.p. maps $\varphi_n : A \to M_{k(n)}$ and $\psi_n : M_{k(n)} \to B$ s.t. $\psi_n \circ \varphi_n \to \theta$ in the point-norm topology.

Thm 3.3. *Let A be a C*-algebra. A is nuclear if and only if A has a property(T):$|| \cdot ||_{max} = || \cdot ||_{min}$*

Proof. φ □

3.2 Kadison-Schwartz inequality

Thm 3.4 (Kadison-Scwartz inequality [Kad52])**.** *Let A be a* C**-algebra. Let φ be a positive linear map from A to $B(H)$ s.t. $\|\varphi\| \leq 1$.. Then, for each $a \in A_h$,*

$$\varphi(a^2) \geq \varphi(a)^2.$$

Proof. We may assume A is unital. We may assume $A = C(\Omega)$, where Ω is a compact Hausdorff space. By the GNS construction, there exists a injective $*$-homomorphism from A to $B(K)$. $C(\Omega)$ is abelian, the above map is u.c.i and φ is u.c.p. By the injectivity, there exists a u.c.p map $B(K) \to B(H)$ extending φ, we continue to denote by φ. We suffices to show that for $\alpha_i \in \mathbb{R}$ and characteristic functions E_i of disjoint borel subsets of X,

$$\varphi((\sum \alpha_i E_i)^2) \geq (\varphi(\sum \alpha_i E_i))^2.$$

By disjointness,

$$\varphi(\sum \alpha_i^2 E_i) \geq (\varphi(\sum \alpha_i E_i))^2.$$

So, we suffices to show $\varphi(E_i) \geq \varphi(E_i)^2$. Since $\|E_i\| = 1$ and $\|\varphi\| \leq 1$, $\varphi(E_i) \leq 1$. Also, φ is positive, $\varphi(E_i)$ is self-adjoint. So, $\varphi(E_i) \geq \varphi(E_i)^2$. $\square$

3.3 operator system

Thm 3.5 ([CE77])**.** *Let R be an injective envelope system and completely isometoric map $R \to B(H)$. Then, there exists a unital complete order isomorphism of R onto an essentially unique unital* C**-algebra. The latter is conditionally complete, i.e. any increasing net in R_h ehich is bounded above hs a least upper bound in R_h.*

Espesially, the C**-algebra may be faithfully represented as a AW*-algebra.*

Furthermorem, if R is a σ-weakly closed, then the C**-algebra may be faithfully represented as a von Neumann algebra.*

3.4 implimitivity theorem

Thm 3.6. *Let Γ be a discrete group and Λ be a subgroup of Γ. Then,*

$$c_0(\Gamma/\Lambda) \rtimes_r \Gamma \cong \mathbb{K}(l^2(\Gamma/\Lambda)) \otimes C_r^*(\Gamma)$$

https://math.dartmouth.edu/ dana/cpcsa/draft-31Jan06.pdf

3.5 Gelfand Duality

Prop 3.3 ([Suz18])**.** *Let X, Y be locally compaact spaces. Let $\varphi : C_0(X) \to C_0(Y)$ be a isometric *-homomorphism s.t. $\varphi(C_0(X))C_0(Y) \subset C_0(Y)$ is norm-dense. Then, $\varphi* : Y \to X$ is a proper quotient map.*

Proof. Proper: Let K be a compact subset of X. By compactness, finite relative compact open subsets, whose closure relative comapact open subset, covers K. So, we may assume there exist $f \in C_c(X)$ s.t. $K = supp(f)$.

$$\begin{aligned} supp(\varphi(f)) &= \{y \in Y : 0 \neq y(\varphi(f)) = (y \circ \varphi)(f) = \varphi^*(y)(f)\} \\ &= (\varphi^*)^{-1}(supp(f)). \end{aligned}$$

Quatient: surjectivity is OK. Since $\varphi(C_0(X))C_0(Y) \subset C_0(Y)$ is norm-dense, We can think one-point compactification and $\varphi(\infty_Y) = \infty_X$. Then,

$$\begin{array}{ccc} C_0(X) & \xrightarrow{\varphi} & C_0(Y) \\ \downarrow & & \downarrow \\ C(\tilde{X}) & \xrightarrow{\tilde{\varphi}} & C(\tilde{Y}) \end{array}$$

and $\tilde{\varphi}^*|_Y = \varphi^*$. Since $\tilde{\varphi}^*$ is a cntinuous surjective map from a compact space to compact space, it is an open map. So, a quotient topology on Y w.r.t. φ^* correspondens to an original topology on Y. $\square$

Rem 3.6.1. *Let X be a locally compact space. Then, $y_i \to \infty \Leftrightarrow$ for any $f \in C_0(Y)$, $f(y_i)/rightarrow0$.*

4 crossed product

Def 4.1. *Let (X, μ) be a measure space, G be a countable discrete group. We define $\mathcal{A} = \mathrm{L}^\infty(X, \mu)$ and $H = L^2(X, \mu)$. Let α be a action on $\mathcal{A}$, or $\alpha : G \to Aut(\mathcal{A})$ is a homomorphism.*

- *α is free $\Leftrightarrow$ α_g is free, in other words $xa = a\alpha_g(a) \Rightarrow a = 0$;*
- *α is ergodic $\Leftrightarrow$ there is no invariant projection in $\mathcal{A}$ under $\alpha(G)$ other than 0 or 1.*

Let $K = H \otimes l^2(G)$. We define $\pi : \mathcal{A} \to B(K)$ by $\pi(a)\xi(g) = \alpha_g^{-1}(a)\xi(g)$. The von Neumann algebra generated by $\pi(\mathcal{A})$ and $\{\lambda_g\}$ is called crossedproduct of $\mathcal{A}$ by G w.r.t. α, and we denote $\mathcal{A} \rtimes_\alpha G or \mathcal{A} \rtimes G$.

Thm 4.1.
- *$\pi(\mathcal{A})$ is masa in $\mathcal{A} \rtimes G \Leftrightarrow \alpha$ is free.*
- *$\mathcal{A} \rtimes G$ is a factor $\Leftrightarrow \alpha$ is free and ergodic.*

Prop 4.1. *$x = \sum \pi(a_g)\lambda_g$, $y = sum\pi(b_g)\lambda_g$.*

- *$\lambda_g \pi(f) \lambda_{g^{-1}} = \pi(\alpha_g(f))$;*
- *$xy = \sum_{g' \in G} \sum_{h \in G} \pi(a_{g'h^{-1}} \alpha_{g'h^{-1}}(b_n))\lambda_{g'}$;*
- *$x^* = \sum \pi(\alpha_g(\overline{a_{g^{-1}}}))\lambda_g$:*

Prop 4.2. *There exists a faithful normal conditional expextation E of M onto $\mathcal{A}$ s.t. $E(\sum_{fin.} \pi(a_g)\lambda_g) = a_e$.*

When X is a finite measure space and , we define state τ by $\langle \cdot (1 \otimes \delta_e), 1 \otimes \delta_e \rangle$. It satisfies $\mu \circ E = \tau$.

5 Stable Rank

Def 5.1. *Let A be a unital C*-algebra. For each $n \in \mathbb{N}$, define*

$$\mathrm{Lg}_n(A) := \{(a_1, a_2, \dots, a_n) \in A^n | Aa_1 + Aa_2 + \cdots + Aa_n = A\}.$$

Then the stable rank of A is defined to be the value

$$\min\{n \in \mathbb{N} | \mathrm{Lg}_n(A) \, is \, norm \, dense \, in \, A\}.$$

Prop 5.1 ([Rie83], Proposition 3.1)**.** *A unital C*-algebra has stable rank one if and only if the set of invertible elements of A is norm dense in A.*

Def 5.2. *Let A be a C*-algebra. Denoted by $\mathrm{ZD(A)}$ the set of two-sided zero divisor in A, i.e., the set of elements x in A for which $ax = xb = 0$ for some non-zero elements a and b in A.*

Prop 5.2 ([Rør91], proposition 3.2)**.** *For each unital C*-algebra A, $\overline{\mathrm{ZD}}(A)$ consists precisely of all elements x in A that are not one-sided invertible.*

Proof. Let X be a set of all elements x in A that are not one-sided invertible. At first, we show that $\mathrm{ZD}(A) \subset X$. Let $x \in \mathrm{ZD}(A)$. Since $\mathcal{R}(x) \neq H$ and $\ker(x) \neq H$, $x \in X$. We show that X is closed. Suppose $a, x \in A$ and $x_n \in X$ s.t. $ax = 1$ and $x_n \to x$. There exists n s.t. $\|ax_n - 1\| < 1$. So, $ax_n \in \mathrm{GL}(A)$, this is contradiction. So, we suffices to show that $\overline{\mathrm{ZD}}(A) \supset X$. Let $x \in X$ and $\varepsilon > 0$. Then, neither $|x|$ nor $|x^*|$ is invertible (since if $|x|$ is invertible, $x^*x = |x|^2$ is invertible). Suppose $f_\varepsilon, g : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous funsctions s.t.

$$f(x) := \begin{cases} 0 & (x \in [0, \varepsilon] \\ x - \varepsilon & (x \in [\varepsilon, \infty) \end{cases}, \; g : \begin{cases} g(0) = 1 \\ \mathrm{supp}(g) \subset [0, \varepsilon]. \end{cases}$$

Let $x = v|x|$ by polar decomposition. Let $a := g(|x^*|)$, $b := g(|x|)$ and $x_\varepsilon := vf_\varepsilon(|x|)$. Then, $ax_\varepsilon = 0 = x_\varepsilon b$. Since $0 \in \sigma(|x)$ and $0 \in \sigma(|x^*|)$, $a, b \neq 0$. So, $x_\varepsilon \in \mathrm{ZD}(A))$. Since $\|x - x_\varepsilon\| \leq \varepsilon$, $x \in \overline{\mathrm{ZD}}(A)$. $\square$

6 modular theory

6.1 Left Hilbert algebra

Def 6.1. *Let A be a complex involution$\sharp$ algbera.*
Let H be a completion of A.
A is called by left Hilbert algebra, if it satisfies the following properties.

- *For each $\xi \in A$, $A \ni \eta \to \xi\eta$ is continuous;*
- *For any $\xi, \eta, \zeta \in A$, $\langle \xi\eta, \zeta \rangle = \langle \eta, \xi\zeta \rangle$;*
- *$A^2 \subset A$ is dense;*
- *$A \ni \xi \to \xi^\sharp$:preclosed:*

Def 6.2. *S^A is defined by the closure of $A \ni \xi \to \xi^\sharp$.*
We often write S instead of S^A.

Def 6.3. $\mathcal{L}(A) := \overline{span\{L_\xi | \xi \in A\}}^w$.

Prop 6.1.
- *For any $\eta \in \mathcal{D}_{S^*}$, $A \ni \xi \to \xi\eta$ is preclosed and we call it R_η;*
- *Then, $(R_\eta)^*\xi = L_\xi S^*\eta (\xi \in A)$;*

Cor 6.0.1. *If $\eta \in A'$, then $S^*\eta \in A'$ and $S^*(S^*\eta) = \eta$, $R_{S^*\eta} = (R_\eta)^*$.*

Cor 6.0.2. *If $\eta_1, \eta_2 \in A'$ and $x' \in \mathcal{L}(A)'$, then $R_{\eta_1}x'\eta_2 \in A'$ and $S^*R_{\eta_1}x'(\eta_2) = R_{S^*\eta_2}{x'}^*S^*\eta_1$, $R_{R_{\eta_1}x'(\eta_2)} = R_{\eta_1}x'R_{\eta_2}$.*

Cor 6.0.3. *If $\eta \in \mathcal{D}_{S^*}$, then there exists $\{\eta_n\}, \{\zeta_n\} \subset A'$, s.t. $R_{\eta_n}\zeta_n \to \eta$ and $S^*R_{\eta_n}\zeta_n \to S^*\eta$.*

Def 6.4. $A' := \{\eta \in \mathcal{D}_{S^*} | R_\eta isbounded\}$

Prop 6.2. *Suppose $\eta, \zeta \in H$ and $x' \in B(H)$. The following are equivalent.*

1. *$\eta \in A'$, $S^*\eta = \zeta$, $R_\eta = x'$;*
2. *for any $\xi \in A$, $L_\xi\eta = x'\xi$, $L_\xi\zeta = {x'}^*\xi$:*

By the above proposition, we get a right Hilbert moduleA',which is endowed the operations

- $\eta_1\eta_2 = R_{\eta_2}\eta_1$;
- $\eta^\flat = S^*\eta$:

Thm 6.1. *Let A_1 be a dense subalgebra of right Hilbert algebraA_2. Then, A_1 is a right Hilbert subalgebra of A_2. Moreover, the followins are equvalent.*

- *${A_1}' = {A_2}'$;*
- *${A_1}'' = {A_2}''$;*
- *$S^{A_1} = S^{A_2}$:*

6.2 weight

Prop 6.3. *Let M be a vN algebra with cyclic separating vector ξ_0.*
For each $\xi \in H$, We define $\mathcal{D}_{L^o_\xi}$, $L^o_\xi(x'\xi_0) = x'\xi$. Then,

$$\begin{aligned}\mathfrak{P} := &\{\xi \in H | L^o_\xi : \textit{positive}\} \\ = &\{\xi \in H | {\omega'}_{\xi,\xi_0} \geq 0\} \\ = &\{A\xi_0 | A : \textit{positive selfadjoint affiliated to} M, \xi_0 \in \mathcal{D}_A\}\end{aligned}$$

Proof. Friedrichs extention. □

Lem 6.1. *Let M be a vN algebra with cyclic separating vector ξ_0.*
Then, for any $\phi \in M_*^+$, there exists an unique $\xi \in \mathfrak{P}_S$ s.t. $\phi = \omega_\xi$. Specially, there exists positive selfadjoint operator A affiliated to M s.t. $\phi = \omega_{A\xi_0}$.

Proof. There exists $\zeta \in H$ s.t. $\phi = \omega_\xi$.
$\omega_{\zeta,\xi_0} = v'|\omega_{\zeta,\xi_0}|$.
$\xi = v'^*\zeta$. □

Thm 6.2. *Let $A \subset H$ be a left Hilbert algebra. Then,*

- *$JA'' = A'$, $JA' = A''$;*
- *$S^*J\xi = JS\xi(\xi \in A'')$;*
- *$SJ\eta = JS^*\eta(\eta \in A')$;*
- *$R_{J\xi} = JL_\xi J$, $L_{J\eta} = JR_\eta J$:*

Def 6.5. *We call $\varphi : M^+ \to [0, \infty]$ weight, when it is $\mathbb{R}_{\geq}$-linear.*

- *fatithful if $\varphi(a) = 0 \Leftrightarrow a = 0$;*
- *semifinite if $\mathfrak{M}_\varphi^+ = \mathfrak{N}_\varphi = \{x \in M^+ | \varphi(x) < \infty\}$;*
- *normal if sum of M_*^+:*

Prop 6.4. *Let φ be a weight on M. The followings are equivalent.*

- *commute with σ-w sum;*
- *commute with increasing net;*
- *lower σ-w semicontinuous;*
- *sup of M_*^+;*
- *sum of M_*^+:*

Proof. ($2 \Rightarrow 5$): Connes' inverse theorem. □

Thm 6.3. *Let $A \subset B$ be a left Hilbert algebra.*
We define weight φ_A on M^+ by

$$\varphi_A(a) = \begin{cases} ||\xi||^2 & (\exists \xi \in A'' s.t. a^{\frac{1}{2}} = L_\xi) \\ \infty & \textit{otherwise} \end{cases}$$

Then, φ is a semifinite faithful normal weight.

By the following lemma, increading and additive follows.

Lem 6.2. *Let a, $b \in M^+$ s.t. $a \geq b$.*
We define $v : [b^{\frac{1}{2}}H] \oplus [b^{\frac{1}{2}}H]^{\perp} \to H$ by $b^{\frac{1}{2}}\eta + \theta \mapsto a^{\frac{1}{2}}\eta$.
Then, v belongs to M and $b^{\frac{1}{2}}v^* a^{\frac{1}{2}} = a$.

Proof. (additive) Let $a^{\frac{1}{2}} = L_\xi$ and $b^{\frac{1}{2}} = L_\xi$. Then, $v^*v + w^*w$ is a projection. $(a+b)^{\frac{1}{2}} = L_{v^*\xi + w^*\zeta}$. □

Lem 6.3. *Let $A \subset H$ be a left Hilbert algebra. Then, there exisit σ-finite projections $\{e_i\} \subset \mathcal{L}(A)^+$ s.t.*

- *$\sum e_i = 1$;*
- *For each i, there exist increasing sequence $\{a_{i,n}\} \subset \mathfrak{M}^+$ converging e_i;*
- *For each n and each $r \in \mathbb{Q}$, there exists $m(n,r)$ s.t. $\sigma_r(a_{i,n}) \leq a_{i,m(n,r)}$:*

Proof. Numbering $\mathbb{Q}$ and iikanji net. □

Suppose $a_{1,i}^{\frac{1}{2}} = L_{\xi_1}$ and $(a_{n,i}^{\frac{1}{2}} - a_{1,n-1}^{\frac{1}{2}})^{\frac{1}{2}} = L_{\xi_n}$.
We define $\varphi' = \sum_i \sum_n \omega_{\xi_{i,n}}$. $J\varphi' J$ is faithful semifinite normal weight and satisfies the following properties.

- $\varphi \leq \varphi_A$;
- $\varphi(a) = \varphi_A(a) a \in \mathfrak{M}_A^+$;
- $\varphi(\sigma_t(a)) = \varphi(a) a \in \mathcal{L}(A)^+$:

In fact, It is same as φ_A.

Def 6.6. *We define Tomita algebra by the following way.*

$$\mathfrak{T} := \left\{ \xi \in \cap_{\alpha \in \mathbb{C}} \mathcal{D}_{\triangle^\alpha} \;\middle|\; \begin{array}{l} for\, each,\ \alpha \in \mathbb{C}, \triangle^\alpha \xi \in \mathcal{A}' \cap \mathcal{A}'' (In\, fact,\ we\, only\, need\, it.), \\ \mathcal{D}_{\triangle^\alpha L_\xi \triangle^{-\alpha}} = \mathcal{D}_{\triangle^{-\alpha}}, \triangle^\alpha L_\xi \triangle^{-\alpha} \subset L_{\triangle^\alpha \xi}, \\ \mathcal{D}_{\triangle^\alpha R_\xi \triangle^{-\alpha}} = \mathcal{D}_{\triangle^{-\alpha}} and\ \triangle^\alpha R_\xi \triangle^{-\alpha} \subset R_{\triangle^\alpha \xi} \end{array} \right\}$$

Thm 6.4. *Let $\mathfrak{A} \subset H$ be a left Hilbert algebra. Then, $\mathfrak{T}$ is a left Hilbert subalgebra of $\mathfrak{A}''$. and $\mathfrak{T}' = \mathfrak{A}'$, $\mathfrak{T}'' = \mathfrak{A}''$.*
Moreover,

6.3 KMS-condition and noncommutative Radon-Nikodym derivative

Def 6.7. *Let φ be a weight on M and $\{\pi_t\}$ be a one-parameter group of *-automophism of M.*

- *varphi satisfies the Kubo-Martin-Schwinger condition (KMS-condition) for $x, y \in \mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^*$ w.r.t $\{\pi_t\}$, if there exists an 1-c.a. function $F_{x,y}$ s.t. $F(it) = \varphi(x\pi_t(y))$ and $F(it+1) = \varphi(\pi_t(y)x)$.*
- *Moreover, $\{\phi, \pi_t\}$ satisfies the modular condition, if it satisfies KMS-condition for any two elements in $\mathcal{N}_\varphi \cap \mathcal{N}_\varphi^*$ and $\{\pi_t\}$ leaves invariant the weght φ.*

Thm 6.5.

- *$\{\varphi, \sigma_t^\varphi\}$ satisfies the modular condition.*
- *Conversely, if $\{\varphi, \pi_t\}$ satisfies the modular condition $(\mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^* \to (\mathfrak{M}_\varphi)^2)$, $\sigma_t^\varphi = \pi_t$.*

Thm 6.6. *Let A be C^*-algebra, φ be a faithful state on A and σ_t^φ be a one-parameter automorphismgroup on A. If σ_t^φ satisfies KMS-condition with rispect to φ, we can extend φ and σ_t^φ to $\tilde{\varphi}$ and $\tilde{\sigma_t^\varphi}$ on $\pi_\varphi(A)''$and $\tilde{\varphi}$ is a faithful normal state.*

Proof. Use KMS and cyclicvector. □

Def 6.8.

- *$M_\infty^\varphi := \{x \in M | it \mapsto \sigma_t^\varphi(x)$ has an entire analytic extension$\}$;*
- *$M_0^\varphi := \{x \in M | \sigma_t^\varphi(x) = x \forall t\}$:*

Thm 6.7 (A.Connes)**.** *Let M be a von Neumann algebra. Let φ and ψ be a faithful semifintie normal weight. Then, there exsits one-parameter group $\{u_t\} \subset \mathcal{U}(M)$ s.t.*

- $u_{t+s} = u_t \sigma_t^{\varphi}(u_s)$;
- $u_t^* = \sigma_t^{\varphi}(u_{-t})$;
- $\sigma_t^{\psi}(x) = u_t \sigma_t^{\varphi}(x) u_t^*$:

Proof. We define faithful normal semifinite weght θ on $M_2(\mathbb{C})$ by $\theta((a_{i,j})) = \varphi(a_{1,1}) + \psi(a_{2,2})$. By $u = e_{11} - e_{22} \in M_0^{\theta}$, e_{11} and e_{22} belong to M_0^{θ}.
Using KMS-condition, $\sigma_t^{\theta}(\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} \sigma_t^{\varphi}(x) & 0 \\ 0 & 0 \end{pmatrix}$. Similarily, σ_t^{ψ}.
$\sigma_t^{\theta}(e_{21}) = \begin{pmatrix} 0 & 0 \\ u_t & 0 \end{pmatrix}$. □

Rem 6.7.1. $\sigma_t^{\theta(\varphi,\varphi)} = \sigma_t^{\varphi} \otimes \mathrm{id}_2$

Thm 6.8. *u_t in above Theorem is uniquely determined by the above property and the following condition: for all $x \in \mathfrak{N}_{\psi} \cap \mathfrak{N}_{\varphi}^*$ and $y \in \mathfrak{N}_{\varphi} \cap \mathfrak{N}_{\psi}^*$, there exists 1-c.a. function F s.t. $F(it) = \varphi(x u_t \sigma_t^{\varphi}(y))$ and $F(1+it) = \psi(\sigma_t^{\psi}(y) u_t^* x)$.*

In fact, the above two theorem is true($u_t u_t^* = s(\psi) = u_0$, $u_t^* u_t = \sigma_t^{\varphi}(s(\psi))$, $x \to xs(\psi)$ $y \to s(\psi)y$ in the additional condition), when ψ is normal semifinite weght.

Proof.

$$(x_{ij}) \in \mathfrak{N}_{\theta}(\cap \mathfrak{N}_{\theta}^*) \Leftrightarrow \begin{cases} x_{11} \in \mathfrak{N}_{\varphi}(\cap \mathfrak{N}_{\varphi}^*) \\ x_{22} \in \mathfrak{N}_{\psi}(\cap \mathfrak{N}_{\psi}^*) \\ x_{12} \in \mathfrak{N}_{\psi}(\cap \mathfrak{N}_{\varphi}^*) \\ x_{21} \in \mathfrak{N}_{\varphi}(\cap \mathfrak{N}_{\psi}^*) \end{cases}$$

□

We denote u_t by $[D\psi : D\varphi]$.

Cor 6.8.1. *Let $\varphi \in W_{nsf}(M)$ and ψ_1, $\psi_2 \in W_{ns}(M)$. Then,*
$[D\psi_1 : D\varphi] = [D\psi_2 : D\varphi]$ $Leftrightarrow$ $\psi_1 = \psi_2$.

Proof. We only prove if part.
$s(\psi_1) = [D\psi_1 : D\varphi]_0 = [D\psi_2 : D\varphi]_0 = s(\psi_2)$.
Since $[D\psi_2 : D\psi_1]_t [D\psi_1 : D\varphi]_t = [D\psi_2 : D\varphi]$, $[D\psi_2 : D\psi_1]_t = 1$.
We may prove that if φ, $\psi \in W_{nsf}(M)$ and $[D\varphi : D\psi] = 1$, $\varphi = \psi$.
Since $\sigma_t^{\theta(\varphi,\psi)} = \sigma_t^{\theta(\varphi,\varphi)}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in M_2(\mathbb{C})^{\theta(\varphi,\psi)}$,

$$\begin{aligned} \varphi(x) &= \theta(\varphi,\psi)(\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}) \\ &= \theta(\varphi,\psi)(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}) \\ &= \psi(x). \end{aligned}$$

□

Thm 6.9. *Let $\varphi \in W_{nsf}(M)$ and $\psi \in W_{ns}(M)$. The followings are equivalent.*

1. $\psi \circ \sigma_t^{\varphi} = \psi$;
2. $u_t := [D\psi : D\varphi]_t \in M^{\psi}$;

3. $[D\psi : D\varphi]_t \in M^\varphi$*;*

4. $\{[D\psi : D\varphi]\}_{t\in\mathbb{R}}$ *is a s-continuous group of unitary elements in* $\mathcal{U}(M_{s(\psi)})$*;*

5. there exists a positive self-adjoint A *affiliated to* M^φ *s.t.* $\psi = \varphi_A$*:*

Furthermore, if ψ *is faithful, then also the following statement is equivalent to those above:*

- $\varphi \circ \sigma_t^\psi = \varphi$.

Proof. $(1 \Rightarrow 2)$: $\psi(u_t x u_t^*) = \psi(u_t \sigma_t^\psi(\sigma_{-t}^\varphi(x))) = \psi(x)$.
$\psi(u_t^* x u_t) = \psi(u_t \sigma_t^\varphi(\sigma_{-t}^\psi(x))) = \psi(\sigma_t^\varphi(u_{-t}\sigma_t^\varphi(x)u_{-t}^*)) = \psi(u_{-t}\sigma_t^\varphi(x)u_{-t}^*) = \psi(\sigma_t^\psi(x)) = \psi(x)$.
$(2 \Leftrightarrow 3 \Rightarrow)$: $\sigma_t^\varphi(u_t) = u_t^*\sigma_t^\psi(u_t)u_t = \sigma_t^\psi(u_t) = u_t$
$(3 \Rightarrow 4)$: $u_t^* u_t = \sigma_t^\varphi(s(\psi))$.
$(4 \Rightarrow 2)$: Since $u_s^* u_s$ is σ_t^φ-invariant, $u_{s+t} = u_s \sigma_t^\varphi(u_t)$, so $u_t = \sigma_s^\varphi(U_t)$.
$(3 \Leftrightarrow 4 \Rightarrow 5)$: $[D\varphi_A : D\varphi]_t = A^{it} = [D\psi : D\varphi]_t$. By the above corollary, $\varphi_A = \psi$.
$(5 \Rightarrow 1)$: Since A^{it} affiliated to M^{φ_A}, $\psi \circ \sigma_t^\varphi(x) = \varphi_A \circ \sigma_t^\varphi(x) = \lim \varphi((Ae_n)^{\frac{1}{2}}\sigma_t^\varphi(x)(Ae_n)^{\frac{1}{2}}) = \lim \varphi(\sigma_t^\varphi((Ae_n)^{\frac{1}{2}}x(Ae_n)^{\frac{1}{2}})) = \lim \varphi((Ae_n)^{\frac{1}{2}}x(Ae_n)^{\frac{1}{2}}) = \varphi_A(x)$. $\square$

Def 6.9. *If the above conditions are satisfied, we say that* ψ *commutes with* φ*.*

6.4 Pedersen-Takesaki construction

Def 6.10. *Let* φ *be a normal semifinite weight and* a *be a positive element in* M^φ*. We define* $\varphi_a := \varphi(a^{\frac{1}{2}} \cdot a^{\frac{1}{2}})$*.*

Rem 6.9.1.

- $\mathfrak{N}_\varphi \subset \mathfrak{N}_{\varphi_a}$, $\mathfrak{M}_\varphi \subset \mathfrak{M}_{\varphi_a}$*;*
- φ_a *is a normal semifinite weight;*
- *If* φ *is fatithful and* a *is invertible,* φ_a *is fatihful and* $\mathfrak{N}_\varphi = \mathfrak{N}_{\varphi_a}$, $\mathfrak{M}_\varphi = \mathfrak{M}_{\varphi_a}$*:*

Thm 6.10. *Let* a *be invertible and* φ *be faithful. Then,* $H_\varphi = H_{\varphi_a}$, $S_{\varphi_a} = S_\varphi$, $\pi_\varphi = \pi_{\varphi_a}$ *and* $\sigma_t^{\varphi_a}(x) = a^{it}\sigma_t^\varphi(x)a^{-it}$*.*

Proof. $\langle x, y\rangle_{\varphi_a} = \langle x, J_\varphi \pi_\varphi(a) J_\varphi y\rangle_\phi$. Since, $||a^{-1}||^{-1} \leq J_\varphi \pi_\varphi(a) J_\varphi ||a||$, $H_\varphi = H_{\varphi_a}$. The assertions without the last one are clear,
Adjoint of S_{φ_a} w.r.t. $\langle\cdot,\cdot\rangle_{\varphi_a}$ $S_{\varphi_a}^*$ is $J_\varphi\pi_\varphi(a)^{-1}J_\varphi S_\varphi^* J_\varphi \pi_\varphi(a) J_\varphi$. So, $\triangle_{\varphi_a} = J_\varphi \pi_\varphi(a)^{-1} J_\varphi \pi_\varphi(a) \triangle_\varphi$. Since $J_\varphi\pi_\varphi(a)^{-1}J_\varphi$, $\pi_\varphi(a)$ and $\triangle_\varphi$ commute with each other,

$$\begin{aligned}
\pi_{\varphi_a}(\sigma_t^{\varphi_a}(x)) &= \triangle_{\varphi_a}^{it}\pi_{\varphi_a}(x)\triangle_{\varphi_a}^{-it} \\
&= J_\varphi\pi_\varphi(a)^{it}J_\varphi\pi_\varphi(a)^{it}\,\triangle_\varphi^{it}\,\pi_{\varphi_a}(x)\,\triangle_\varphi^{-it}\,\pi_\varphi(a)^{-it}J_\varphi\pi_\varphi(a)^{-it}J_\varphi \\
&= J_\varphi\pi_\varphi(a)^{it}J_\varphi\pi_\varphi(a^{it}\sigma_t^\varphi(x)ia^{-it})J_\varphi\pi_\varphi(a)^{-it}J_\varphi \\
&= \pi_\varphi(a^{it}\sigma_t^\varphi(x)ia^{-it}).
\end{aligned}$$

$$\begin{tikzcd}
M \arrow[r, "\sigma_t^{\varphi_a}"] \arrow[d, "\pi_{\varphi_a}"] & M \arrow[d, "\pi_{\varphi_a}"] \\
\pi_{\varphi_a}(M) \arrow[r, "\triangle_{\varphi_a}^{it}\cdot\triangle_{\varphi_a}^{-it}"] & \pi_{\varphi_a}(M).
\end{tikzcd}$$

$\square$

Prop 6.5. *Let* φ *be a semifinite normal weight and* A*,* B *be a positive self-adjoint affiliated to* M^φ*, then* $\varphi_A + \varphi_B = \varphi_{A+B}$*.*

Proof. $u := w\text{-}\lim_{\varepsilon\to 0} A^{\frac{1}{2}}(A+B+\varepsilon)^{-\frac{1}{2}} \in M^\varphi$. $\square$

Def 6.11. *Let* A_k*,* A*,* B *be a self-adjoint positive op's.*

- $A \leq B \Leftrightarrow (1+B)^{-1} \leq (1+A)^{-1}$;
- $A_k \uparrow A \Leftrightarrow (1+A_k)^{-1} \downarrow (1+A)^{-1}$.

Rem 6.10.1. • *;;;;*

Thm 6.11. *Let A and increasing net $\{A_i\}_{i\in I}$ be a positive self-adjoint op's on H s.t. $A_i \leq A$. Then, there exists positive self-adjoint op. B s.t. $A_i \uparrow B$.*

This follows from the following proposition.

Prop 6.6. *Let $\{A_i\}$ be an increasing net of positive self-adjoint operators. There exists a positive selfadjoint op. A s.t. $A_i \uparrow A$ if and only if $D = \{\xi \in H | \lim_i ||A_i^{\frac{1}{2}}\xi|| < \infty\}$ is dense in H. In this case, $D = D_{A^{\frac{1}{2}}}$.*

Thm 6.12. *Let φ be a normal semifinite weight and A be a positive s.a. op. affiliated to M^φ. We define $\varphi_A(x) := \lim_n \varphi((Ae_n)^{\frac{1}{2}} x (Ae_n)^{-\frac{1}{2}}))$. Then, $\varphi_A \in W_{ns}(M)$ and $\sigma_t^{\varphi_A} = A^{it}\sigma_t^\varphi(x)A^{-it}$ $(x \in M_{s(A)})$.*

Cor 6.12.1. $[D\varphi_A : D\varphi]_t = A^{it}$.

Proof. Let $B = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$.
Then, $\begin{pmatrix} 0 & 0 \\ [D\varphi_A : D\varphi]_t & 0 \end{pmatrix} = \sigma_t^{\theta(\varphi,\varphi_A)}(\begin{pmatrix} 0 & 0 \\ s(A) & 0 \end{pmatrix}) = \sigma_t^{\theta(\varphi,\varphi)_B}(\begin{pmatrix} 0 & 0 \\ s(A) & 0 \end{pmatrix}) = B^{it}\sigma_t^{\theta(\varphi,\varphi)}(\begin{pmatrix} 0 & 0 \\ s(A) & 0 \end{pmatrix})B^{-it}$. □

Prop 6.7. *Let φ be a semifinite weight and $v \in M$ be a partial isometry s.t. $vv^* \in M^\varphi$. Then, $\varphi_v := \varphi(v \cdot v^*) \in W_n(M)$.*

Proof. Let $e_i \in \mathfrak{M}_\varphi$ be $e_i \nearrow 1$. We define $v^*v =: p$ and $vv^* =: q$. Since $qe_iq \in \mathfrak{M}_\varphi$, $1 - p + v^*e_iv \in \mathfrak{M}_{\varphi_a}$. □

Thm 6.13. *Let φ be a normal semifinite faithful weight $v \in M$ be a partial isometry s.t. $vv^* \in M^\varphi$. Then, $\sigma_t^{\varphi_v}(x) = v^*\sigma_t^\varphi(vxv^*)v$ $(x \in M_{v^*v})$.*

Proof. modular condition. □

Thm 6.14. *Let φ be a normal semifinite faithful weight $v \in M$ be a partial isometry s.t. $vv^* \in M^\varphi$. Then, $[D\varphi_v : D\varphi] = v^*\sigma_t^\varphi(v)$ and $\sigma_t^\varphi(v) = v[D\varphi_v : D\varphi]$.*

Proof. Let $u = \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix}$.
Then,

$$\begin{aligned}
\begin{pmatrix} 0 & 0 \\ [D\varphi_u : D\varphi]_t & 0 \end{pmatrix} &= \sigma_t^{\theta(\varphi,\varphi_v)}(\begin{pmatrix} 0 & 0 \\ s(\varphi_v) & 0 \end{pmatrix}) \\
&= \sigma_t^{\theta(\varphi,\varphi)_u}(\begin{pmatrix} 0 & 0 \\ v^*v & 0 \end{pmatrix}) \\
&= \begin{pmatrix} 1 & 0 \\ 0 & v^* \end{pmatrix} \sigma_t^{\theta(\varphi,\varphi)}(\begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix} \begin{pmatrix} 0 & 0 \\ v^*v & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & v^* \end{pmatrix}) \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ v^*\sigma_t^\varphi(v) & 0 \end{pmatrix}
\end{aligned}$$

□

6.5 Connes' inverse problem

Def 6.12. *Let G be a locally compact group and $\sigma : G \to \mathrm{Aut}(M)$.*
σ-cocycle $Leftrightarrow$ $w : G \to M$, s^*-conti s.t.

- $w(gh) = w(g)\sigma_g(w(h))$;
- $w(g^{-1}) = \sigma_g^{-1}(w(g)^*)$:

We denote all σ-cocycle by $Z_\sigma(G; M)$.

Rem 6.14.1.
- $w(g)w(g)^* = w(e)$, $w(g)^*w(g) = \sigma_g(w(e))$.
- $[D\psi : D\varphi]$ *is a σ^φ-cocycle.*

Thm 6.15. *For all $\varphi \in W_{nsf}(M)$ and for all $w \in Z_\sigma(G; M)$, there only exists $\psi \in W_{ns}(M)$ s.t. $[D\psi : D\varphi]$.*

Proof. $L^2(\mathbb{R}) \cong l^2(\mathbb{Z})$.

$\Phi(x) := \sum \varphi(x_{ii})$ is a normal semifinite faithful weight and $\sigma_t^\Phi \overline{\otimes} \iota$.

Let $u_t \in B(L)$ be a left regular representation. By Stone's theorem, there exisits A aff to $(M \overline{\otimes} B(l^2(\mathbb{Z})))^\Phi$ s.t.$1 \otimes u_t = A^{it}$. $\Phi' := \Phi_A$. Then, $\sigma_t^{\Phi'} = \sigma_t^\varphi \overline{\otimes} \mathrm{Ad}u_t$.

We define $W \in M \overline{\otimes} B(L^2(\mathbb{R}))$ by $W\zeta(t) := w(t)\zeta(t)$. Then, $\sigma_t^{\Phi'}(W*)(s) = \sigma_t^\varphi(w(s-t)^*)$, so $W\sigma_t^{\Phi'}(W*)(s) = w(t)$, so $W\sigma_t^{\Phi'}(W*) = w(t) \otimes 1$. Since $\sigma_t^{\Phi'}(W * W)$, $\Psi := \Phi'(W^* \cdot W) \in W_{ns}(M)$ and $\sigma_t^\Psi = \mathrm{Ad}(w(t)) \circ \sigma_t^\varphi \overline{\otimes} \mathrm{Ad}(u_t)$.

Similarily, we define $\Psi' := \Psi_{A^{-1}}$. Then, $\sigma_t^{\Psi'} = \mathrm{Ad}(w(t)) \circ \sigma_t^\varphi \overline{\otimes} \iota$.

Let $p \in P(B(l^2(\mathbb{Z})))$ be a minimal projection. Since $M \cong (1 \otimes p)(M \overline{\otimes} B(l^2(\mathbb{Z})))$, we define $\psi' := \Psi'_{(1 \otimes p)}$. Then, $\sigma_t^{\psi'}(x) = \sigma_t^{\psi'}(x \otimes p) = \mathrm{Ad}(w(t)) \circ \sigma_t^\varphi(x)$.

Let $w'(t) := [D\psi' : D\varphi]_t$ and $a(t) := w'(t)^* w(t)$. Since $\mathrm{Ad}(w(t)) \circ \sigma_t^\varphi = \sigma_t^{\varphi'} = \mathrm{Ad}(w(t)) \circ \sigma_t^\varphi$, $a(t)$ belongs to $(U)(Z(M))$ and is s-continuous. So, there exists A affiliated to $M^{\psi'}$. $\psi := \psi'_A$. Then,

$$
\begin{aligned}
[D\psi : D\phi]_t &= [D\psi : D\psi']_t [D\psi' : D\varphi]_t \\
&= a(t)w'(t) = w'(t)a(t) = w(t)
\end{aligned}
$$

□

6.6 continuous decomposition

Rem 6.15.1. *If e, $f \in M^\varphi$, $\varphi(pxp) + \varphi((f-p)x(f-p)) = \varphi(fxf)$.*

6.7 Existence of conditional expectation

Thm 6.16 ([Tak72])**.** *Let M be a von Neumann algebra and N be a subalgera of M. Let φ be a nsff weight on M and $\varphi|_N$ be a semifinite weight on N. Then, the following conditions are equivalent.*

(1) *N is invariant for σ_t^φ;*

(2) *there exists a normal conditinal expectation ε from M onto N s.t. $\dot\varphi(x) = \dot\varphi \circ \varepsilon(x)$ $x \in \mathfrak{M}$:*

Rem 6.16.1. *By modular condtion and (2), $\sigma_t^\varphi = \sigma_t^{\varphi \circ \epsilon}$ on N.*
$\triangle^{it}\eta_\varphi(x) = \eta_\varphi(\sigma_t^\varphi(x))$.

Proof. We assume (1). Let $A = \mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^*$ and H be a completion of A. Let $B = \mathfrak{N}_{\varphi|_N} \cap \mathfrak{N}_{\varphi|_N}$ and K be a completion of B. Let E be a orthogonal projection from H onto K. Then, $B = A \cap N$.
Since $\triangle_A^{it}\eta_\varphi(x) = \eta_\varphi(\sigma_t^\varphi(x)) = \eta_\varphi(\sigma_t^{\varphi|_N}(x)) = \triangle_B^{it}\eta_\varphi(x)$ $(x \in \mathfrak{N} \cap N)$, $\triangle_B^{it}\xi = \triangle_A^{it}\xi$ $(\xi \in K)$. Since E commutes with $\triangle_A$, $\triangle_B = \triangle_A|_K$ and $J_B = J_A|_K$.
$A \cap K = B$ and $A' \cap K = B'$. Indeed, $A' \cap K \subset B' \Rightarrow A \cap K = J(A' \cap K) \subset JB' = B$. Let $\rho : \mathcal{L}(B) \to \mathcal{A}$ $x \mapsto \pi_M \circ \pi_N^{-1}(x)$ and $\rho' : \mathcal{L}(B)' \to \mathcal{A}'$ $x \mapsto J\pi_M \circ \pi_N^{-1}(JxJ)J$. Then, $\rho(L_\xi^B) = L_\xi^A$ $(\xi \in B)$ and $\rho'(L_\xi^{B'}) = L_\xi^{A'}$ $(\xi \in B')$.
Therefore, $B_0 = A_0 \cap K$, where A_0 and B_0 are Tomita algebras.

Also, $B = EA$, $B' = EA'$, $B_0 = EA_0$, $E(\xi\eta) = \xi E(\eta)$ $(\xi \in B, \eta \in A)$ and $E(\xi\eta) = E(\xi)\eta$ $(\xi \in A', \eta \in B')$. Since $L^B_{E\xi} = EL^A_\xi E$ (ξinA), $\mathcal{L}(B) = E\mathcal{L}(A)E|_K$, we can define $\varepsilon(x) := \pi_N^{-1}(E\pi_M(x)E)$
For $x \in \mathfrak{M}^+$, $\varphi(x) = \sup\{\langle \pi_M(x)\eta, \eta\rangle | \eta \in B', ||L^{B'}_\eta|| \leq 1\}$. ($geq$ part follows from semifinite). By semifiniteness, ε is faithful.

We assume (2). Then, $E\eta(x) = \eta \circ \varepsilon(x)$ $(x \in \mathfrak{N})$. So, $EA = B$. Since $E\xi^\sharp = (E\xi)^\sharp$ $(\xi \in A)$, $ES\xi = SE\xi$ $(\xi \in \mathcal{D}_S)$.
By $S = (1-2E)S(1-2E)$ and $S^* = (1-2E)S^*(1-2E)$, $\triangle = (1-2E)\triangle(1-2E)$. So, E and $\triangle$ commute. So, $\triangle^{it}B = B$. By the previous remark, so we get this result.

□

7 KMS-state

7.1 definition of KMS-state

Def 7.1. *Let $\mathcal{A}$ be a C^*-algebra. Let α be a representation of $\mathbb{R}$ as a automorphism on $\mathcal{A}$ s.t. $t \mapsto \alpha_t(A)$ continuous. For $0 \leq \beta < \infty$, we say that an invariant state φ of $\mathcal{A}$ is a β-KMS state if $\varphi(\alpha_t(A)B) = \varphi(B\alpha_{t+i\beta}(A))$ for $A \in \mathcal{A}^a$ and $B \in \mathcal{A}$, where $\mathcal{A}^a$ is a set of all analitic elements i.e. $t \mapsto \alpha_t(a)$ has a analytic extension on $\mathbb{C}$.*
When $\beta = 0$, φ is called chaotic state. When $\beta = 1$, φ is called $\{\alpha_t\}$-KMS state.

Rem 7.0.1. *It definition is equivalent to KMS condition of modular theory.*

Prop 7.1. *Let $\mathcal{A} = M_n(\mathbb{C})$, $A > 0$ and $\alpha_t(x) = A^{it}xA^{-it}$. Then, φ is a $\{\alpha_t\}$-KMS state if and only if $\varphi(x) = \frac{\mathrm{Tr(A^{-1}x)}}{\mathrm{Tr(A^{-1})}}$.*

Proof. We may asuume $\mathrm{Tr(A^{-1})} = 1$.
"only if" part follows from that $z \mapsto \mathrm{Tr(A^{-1}xA^zyA^{-z})}$ is analytic.
Let ψ be an another $\{\alpha_t\}$-KMS state. Then, there exists a positive matrix B s.t. $\psi = \mathrm{Tr(B\cdot)}$. By KMS condition, for x, $y \in M_n(\mathbb{C})$,

$$\mathrm{Tr(BxA^{i(t+i)}y^{-i(t+i)})} = \mathrm{Tr(BA^{it}yA^{-it}x)},$$
$$\mathrm{Tr(A^{-1}yAB(A^{-it}xA^{it}))} = \mathrm{Tr(ByA^{it}xA^{it})}.$$

Since $\mathrm{Tr((A^{-1}yAB - By)A^{-it}xA^{it})} = 0$, $A^{-1}yAB = By$. So, $AB \in M_n(\mathbb{C})'$. □

7.2 Cuntz algebra $\mathcal{O}_n$

[OP78] For $t \in \mathbb{R}$, let ρ_t be an automorphism of $\mathcal{O}_n$ s.t. $\rho_t(S_i) = e^{it}S_i$.

Thm 7.1. *The C^*-dynamical system $(\mathcal{O}_n, \mathbb{T}, \rho)$ has exactly one KMS state. Furthermore, the onlyaddmissible β-value is $\log n$ if $n < \infty$, amd ∞ if $n = \infty$.*

Proof. We only prove $n < \infty$. otherwise is yokuwakaran. Let $\varphi_n = \tau_n \circ E_0$, where τ_n is a unique tracial state on $\mathcal{F}_n$. Since $P_n \subset \mathcal{O}_n^a$ and P_n is dense in $\mathcal{O}_n$, by Phragmen-Lindelöf, we suffices to show the case in P_n. By unique decomposition, we may assume $A = S^{k*}a$, $B = bS^k$, where a, $b \in \mathcal{F}_n$. $\varphi_n(\rho_t(A)B) = n^k e^{-ikt}\varphi_n(ab)$, $\varphi_n(B\rho_{t+i\beta}(A)) = n^k e^{-i(t+i\beta)k}\varphi_n(ab)$.

□

8 Group von Neumann algebra

Let G be a locally von Neumann algebra and $\triangle = \triangle_G$ be a modular function. Let $\mu = \mu_G$ be a left Haar measure on G.

Lem 8.1. *Let H be a open subgroup of G. Then, $\mu|_H$ is also a left Haar measure.*

Proof. Suppose $K \subset G$ is a compact subset s.t. $\mu(K) > 0$. By $G = \sqcup g_i H$ and compactness of K, $K \subset \cup_{k=1}^{n} g_k H$. $\mu(K) \leq \sum \mu(K) = 0$. Contradiction. $\square$

Def 8.1. *We define a group von Neumann algebra $L(G)$ by the weak closure of $\lambda(G)$, where λ is a left regular representation.*
When λ is a right regular representation, we denote $R(G)$.

$C_c(G)$ is a left Hilbert algebra [21.3]. Then, modular operator $\triangle_{C_c(G)}$ of $C_c(G)$ and modular function $\triangle_G$ are correspondence. So, $L(G)$ has a J-map.

Def 8.2 ([Tak13])**.** *The weight on $L(G)$ associated with the full Hilbert algebra is called the Plancherel weight and denoted by ψ_G.*

8.1 Fourie algebras

Def 8.3 ([Tak13])**.** *$A(G) := \{\xi * \eta^\vee | \xi, \eta \in L^2(G)\}$ is called the Fourie algebra, where $\xi^\vee(g) = \xi(g^{-1})$.*
Identifying $A(G)$ with $L(G)_*$ under the correspondence $\overline{\eta} * \xi^\vee \leftrightarrow \omega_{\xi,\eta}$, $A(G)$ is a commutative Banach algebra.

We remark that existence of J-map implies $M_* = \{\omega_{\xi,\eta} | \xi, \eta \in H\}$.

Thm 8.1. *$A(G)$ is a dense *-subalgebra of $C_0(G)$.*

Lem 8.2. *For ξ, $\eta \in L^2(G)$, $\xi * \eta \in C_0(G)$, and $\langle \lambda(g)\xi, \eta \rangle = (\overline{\eta} * \xi^\vee)(g)$.*

Proof. In the case of $C_c(G)$, OK. $|\langle \lambda(g)\xi, \eta \rangle - \langle \lambda(g)\xi_n, \eta_n \rangle|$ convergences to 0 uniformly. $\square$

proof of theorem. We only prove multiplicative. We define $W : L^2(G \times G) \to L^2(G \times G)$ by $(W\xi)(g,h) = \xi(g, gh)$. Then, $W \in L^\infty(G)\overline{\otimes}L(G)$ and $W^*(\lambda(g) \otimes 1)W = \lambda(g) \otimes \lambda(g)$.
We deine $\pi : L(G) \to B(L^2(G \times G))$ by $x \mapsto W^*(x \otimes 1)W$. By the previous remark, $\omega_{\xi,\eta} =^t \pi(\omega_{\xi_1,\eta_1} \otimes \omega_{\xi_2,\eta_2})$, for $\xi_1, \xi_2, \eta_1\eta_2 \in L^2(G)$. $\overline{\eta_1} * \xi_1^\vee(g)\overline{\eta_2} * \xi_2^\vee(g) = \overline{\eta} * \xi^\vee(g)$. $\square$

8.2 Hecke algebras

Let G be a locally compact totally disconnected group (i.e. locally profinite group) and μ be a Haar measure on G.

Def 8.4. *For compact open subset K, we define the avraging projection p_K associated to K by*

$$p_K := \frac{1}{\mu(K)} \int_K \lambda_G(k) d\mu(k).$$

Let K be a compact open subgroup of G.

Def 8.5. *A Hecke algebra $C_c(G, K)$ associated to a Hecke pair $(G, K)$is defined by $\chi_K * C_c(G) * \chi_K \subset Cc(G)$ or $p_K C_c p_K \subset L(G)$.*
A Hecke von Neumann algebra $L(G, K)$ associated to a Hecke pair (G, K) by $p_K L(G) p_K$.

Rem 8.1.1. *Since p_K is left K invariant, $C_c(G, K) = \chi_K * C_c(G) * \chi_K = \{f \in C_c(G) | f(kgk') = f(g)$ for all $k, k' \in K\}$.*

Rem 8.1.2. *Since the above remark,* $\mathrm{dim C_c(G, K) = |K\backslash G/K|}$.

Rem 8.1.3. *$p_K \lambda_G(g) p_K = \chi_{KgK}$ in $L(G)$.*

Proof. Since K is a compact open subgroup,

$$KgK = \bigcup_{k \in K} kgK = \bigcup_{fin} kgK = \bigsqcup_{i=1}^{N} k_i gK$$

by proposition17.1. Then, $\mu(KgK) = N\mu(K)$. There exist compact open sebsets K_i of K s.t. $K_igK = k_igK$, since $K \ni k \mapsto kgK \in KgK$ is continuous. Then, $K_i = K \cap k_igKg^{-1}$ and $\mu(K_i) = \frac{1}{N}\mu(K)$, since $k_ik_j^{-1} : K_j \to K_i$ is bijective.

$$\begin{aligned}
p_K\lambda_G(g)p_K &= \int_K\int_K \lambda(kgk')d\mu(k')d\mu(k)\frac{1}{\mu(K)^2} \\
&= \sum_{i=1}^N \int_{K_i}\int_K \lambda(kgk')d\mu(k')d\mu(k)\frac{1}{\mu(K)^2} \\
&= \sum_{i=1}^N \int_{K\cap(k_ig)^{-1}Kg}\int_K \lambda(k_ighk')d\mu(k')d\mu(h)\frac{1}{\mu(K)^2} \\
&= \sum_{i=1}^N \lambda(k_ig)\int_{K\cap(k_ig)^{-1}Kg}\lambda(h)\int_K \lambda(k)d\mu(k')d\mu(h)\frac{1}{\mu(K)^2} \\
&= \sum_{i=1}^N \lambda(k_ig)\int_{K\cap(k_ig)^{-1}Kg}(\lambda(h)p_K)(=p_K)d\mu(h)\frac{1}{\mu(K)} \\
&= \sum_{i=1}^N \lambda(k_ig)p_K\frac{1}{N}.
\end{aligned}$$

$$\begin{aligned}
\chi_{KgK} &= \frac{1}{\mu(KgK)}\int_{KgK}\lambda(h)d\mu(h) \\
&= \frac{1}{N\mu(K)}\sum_{i=1}^N\int_{k_igK}\lambda(h)d\mu(h) \\
&= \frac{1}{N\mu(K)}\sum_{i=1}^N\lambda(k_ig)\int_K\lambda(h)d\mu(h) \\
&= \frac{1}{N}\sum_{i=1}^N\lambda(k_ig)p_K.
\end{aligned}$$

□

9 Amalgamated free product von Neumann algevras

[Ued99]

Thm 9.1. *For each n.s.f.f. weight φ on N, we have*

$$\sigma_t^{\varphi\circ E} = *_{N \atop s\in S}\sigma_t^{\varphi\circ E_s} \text{ for } (t\in\mathbb{R}).$$

10 ultraproduct

Let $\omega \in \beta\mathbb{N}\backslash\mathbb{N}$ be a free ultrafilter.

Def 10.1. $\mathcal{J}_\omega(M) := \{(x_\nu) \in l^\infty(M) | \lim_{\nu\to\omega} x_\nu = 0\}$;
$\mathcal{M}^\omega(M) := \{x \in l^\infty(M) | x\mathcal{J}_\omega(M) + \mathcal{J}_\omega(M)x \subset \mathcal{J}_\omega(M)\}$;
(ultraproduct von Neumann algebra)$M^\omega(M) := \mathcal{M}^\omega(M)/\mathcal{J}_\omega(M)$:

11 filter

Def 11.1. *A $\mathcal{F}$ is called by a free ultrafilter if it satisfies the followings.*

- $\mathcal{F} \neq \emptyset$ *and* $\mathcal{F} \neq \mathcal{P}(X)$*;*
- $A \in F, A \subset B \Rightarrow B \in \mathcal{F}$*;*
- $A, B \in \mathcal{F}, \exists C \in \mathcal{F} s.t. C \subset A \cap C$*;*
- *(ultra)* $A \subset \mathcal{P}(X), either A or X \backslash A$*;*
- *(free)* $\cap \mathcal{F} = \emptyset$*:*

Prop 11.1. $\beta\mathbb{N} \backslash \mathbb{N} \ni \omega \leftrightarrow \mathcal{F} : free ultrafilter on \mathbb{N}$.
$\mathcal{F} := \{S \cap \mathbb{N} | S \subset \beta\mathbb{N}, \omega \in S^i\}$.
$\{\omega\} := \cap\{\overline{S} | S \in \mathcal{F}\}$*.tabunatterukedoayashii.*

Rem 11.0.1. $\lim_{n\to\omega}(x_n)_{n\in\mathbb{N}} = \alpha \Leftrightarrow \forall\varepsilon > 0, \exists U \in \mathcal{F}_\omega, \forall V \subset U, |x_n - \alpha| < \varepsilon(n \in V)$.

Rem 11.0.2. *If* $x_n = (-1)^n$, $\lim_{n\to\omega}(x_n)_{n\in\mathbb{N}}$ *exists* 1 *or* -1*, but* $\lim_{n\to\infty}(x_n)_{n\in\mathbb{N}}$ *dose not exist.*

12 Groupoid C*-algebras

12.1 groupoid

Def 12.1. *An G is a groupoid if G is a set, has a subset $G^{(0)}$ of G (called by a unit space), maps $s, r : G \to G^{(0)}$ and*

$$G^{(2)} \to G, \ (x, y) \mapsto xy,$$

where $G^{(2)} := \{(x, y) | s(x) = r(y)\}$, and satisfies the following properties:

1. *for each $x \in G^{(0)}$, $x = s(x) = r(x)$;*
2. *for each $(x, y) \in G^{(2)}$, $s(xy) = s(y)$ and $r(xy) = r(x)$;*
3. *for each $(x, y) \in G^{(2)}$ and $(y.z) \in G^{(2)}$, $(xy)z = x(yz)$;*
4. *for each $x \in G$, $r(x)x = x = xs(x)$;*
5. *for each $x \in G$, there exists the unique x^{-1} s.t. $xx^{-1} = r(x)$ and $x^{-1}x = s(x)$:*

Rem 12.0.1. *Let Γ be a group and X be a Γ-space. Then, $X \rtimes \Gamma$ is a groupoid under the multiplication $(x, g)(y, h) = (y, gh)$, $s(x, g) = x$ and $r(x, g) = g.x$.*

We assume G is a locally compact *étale* groupoid and $G^{(0)}$ is compact open.

Def 12.2 ([Suz17])**.** *Let C be a compact subset of G and $\varepsilon > 0$. Let K be a compact subgroupoid of G containing the unit space $G^{(0)}$. We say thet K is (C, ε)-invariant if the following inequality holds for all $s \in G^{(0)}$.*

$$\frac{\sharp(CKs \backslash Ks)}{\sharp(Ks)} < \varepsilon.$$

Def 12.3 ([Suz17])**.** *We say that a groupoid G is almost fintie if it satisfies teh following conditions.*

1. *The union of all compact open G-sets covers G.*
2. *For any compact subset $C \subset G$ and $\varepsilon > 0$, there is a (C, ε)-invariant elementary subgroupoid K of G.*

Prop 12.1. *Let $G = X \rtimes \Gamma$, where X is a compact space and Γ is a discrete group. If G is alamost finite, then Γ is amenable.*

Proof. We show that Γ has Følner condition. Let E be a compact subset of Γ and $\varepsilon > 0$. Let $p : X \times \Gamma \to \Gamma$ be a projection onto Γ. By almost fintieness of G, there exists a $(X \times (E \cup E^{-1}), \varepsilon)$-invariant elementary subgroupoid K of G. So, for all $s \in G^{(0)}$.

$$\frac{\sharp((X \times (E \cup E^{-1}))Ks \backslash Ks)}{\sharp(Ks)} < \varepsilon.$$

Let $F := p(Ks)$. Then, $\sharp(Ks) = \sharp(F)$. Let $s \in G^{(0)}$ and $t \in E$. Then, we remark $\sharp((X \times \{t\})Ks) = \sharp(tF)$ and $\sharp(Ks \backslash ((X \times \{t\})Ks)) = \sharp(((X \times \{t^{-1}\})Ks) \backslash Ks)$.

$$\frac{\sharp((X \times \{t\})Ks \backslash Ks)}{\sharp(Ks)} \leq \frac{\sharp((X \times E)Ks \backslash Ks)}{\sharp(Ks)} < \varepsilon$$

and

$$\frac{\sharp(Ks \backslash ((X \times \{t\})Ks))}{\sharp(Ks)} = \frac{\sharp((X \times \{t^{-1}\})Ks \backslash Ks)}{\sharp(Ks)} < \varepsilon$$

□

12.2 groupoid C*-algebras

We construct the C*-algebra from a groupoid. We assume G is a locally compact *étale* groupoid and $G^{(0)}$ is compact open.

We consider $C_c(G)$. For $x \in G^{(0)}$, we define

$$G_x := \{y \in G | s(y) = x\} \, and \, G^x := \{y \in G | r(y) = x\}.$$

For $f, g \in C_c(G)$, we define

$$f * g(x) := \sum_{yz=x} f(y)g(z) = \sum_{\beta \in G_x} f(x\beta^{-1})g(\beta).$$

Prop 12.2. *Let Γ be a discrete group and X be a compact Γ-space. Then,*

$$C_r^*(G) = C(X) \rtimes \Gamma.$$

Proof.

$$\begin{aligned} C(X) \rtimes \Gamma &\to C_r^*(G), \\ f &\mapsto \varphi(f), \\ s &\mapsto 1 \otimes \delta_s, \end{aligned}$$

where

$$\varphi(f) := \begin{cases} f(x) & (x \in G^{(0)} = X \times \{e\}) \\ 0 & (x \notin G^{(0)}) \end{cases}$$

□

Rem 12.0.2. *When X is a locally compact space including a non-compact case, the map*

$$fu_s \mapsto \varphi(f)(1 \otimes \delta_e)$$

gives an isomorphism from $C_0(X) \rtimes \Gamma$ onto $C_r^*(X \rtimes \Gamma)$.

12.3 orbit equivalence relation groupoid

Let X be a locally compact space and Γ be a contable discrete group.

Def 12.4. *We define a orbit equivalence relation groupoid associated to $\Gamma \curvearrowright X$, dneoted by $\mathcal{R}_{\Gamma \curvearrowright X}$ or $\mathcal{R}$.*

$$\mathcal{R} := \{(\gamma\Gamma_x, x) | x \in X,\, \gamma \in \Gamma\},$$
$$\mathcal{R}^0 = \{(\Gamma_x, x) | x \in X\},$$

and for each $(\gamma\Gamma_x, x) \in \mathcal{R}$, $(\gamma\Gamma_x, x)^{-1} = (\gamma^{-1}\Gamma_{\gamma x}, \gamma x)$

Rem 12.0.3. *This groupoid is principal.*

As below, we consider a orbit equivalnce relation groupoid $\mathcal{R}_{\Gamma \curvearrowright X}$.

Lem 12.1. *Let C, C_1, C_2 be compact subsets of X and γ, $\tau \in \Gamma$. Then,*

$$\chi^*_{C\times\{\gamma\}} = \chi_{\gamma C\times\{\gamma^{-1}\}}$$
$$\chi_{C_1\times\{\gamma\}} * \chi_{C_2\times\{\tau\}} = .$$

Espesially, when X is compact, $\chi_{X\times\{\gamma\}}$ is a unitary and $\chi_{X\times\{\gamma\}} * \chi_{X\times\{\tau\}} = \chi_{X\times\{\gamma\tau\}}$.

Proof. Let $\xi, \eta \in C_c(G)$. Let $V = \chi_{C\times\{\gamma\}}$ and $W = \chi_{\gamma C\times\{\gamma^{-1}\}}$. For $x \in X$,

$$\begin{aligned}\langle V\xi, \eta\rangle(x) &= \sum_{\beta\in\mathcal{R}_x} \overline{(V\xi)}(\beta)\eta(\beta)\\ &= \sum_{\beta\in\mathcal{R}_x}\sum_{\alpha\in\mathcal{R}_{s(\beta)=x}} \chi_{C\times\{\gamma\}}(\beta\alpha^{-1})\overline{\xi}(\alpha)\eta(\beta)\end{aligned}$$

Let $\beta = (x, \sigma\Gamma_x)$ and $\alpha = (x, \theta\Gamma_x)$. Then, $\beta\alpha^{-1} = (\theta x, \sigma\theta^{-1}\Gamma_{\theta x})$. Therefore,

$$\begin{aligned}\beta\alpha^{-1} \in C\times\{\gamma\} &\Leftrightarrow \theta x \in C \textit{ and } \sigma\theta^{-1}\Gamma_{\theta x} = \gamma\Gamma_{\theta x}\\ &\Leftrightarrow \theta x \in C \textit{ and } \sigma\Gamma_x = \gamma\theta\Gamma_x.\end{aligned}$$

So,

$$\langle V\xi, \eta\rangle(x) = \sum_{\sigma\in\Gamma/\Gamma_x} \overline{\xi}(x, \gamma^{-1}\sigma\Gamma_x)\eta(x, \sigma\Gamma_x)\chi_C(\gamma^{-1}\sigma x).$$

Similarily,

$$\langle \xi, W\eta\rangle(x) = \sum_{\beta\in\mathcal{R}_x}\sum_{\alpha\in\mathcal{R}_x} \overline{\xi}(\beta)\chi_{\gamma C\times\{\gamma^{-1}\}}(\beta\alpha^{-1})\eta(\alpha)$$

Then,

$$\begin{aligned}\beta\alpha^{-1} \in \gamma C\times\{\gamma^{-1}\} &\Leftrightarrow \theta x \in \gamma C \textit{ and } \sigma\theta^{-1}\Gamma_{\theta x} = \gamma^{-1}\Gamma_{\theta x}\\ &\Leftrightarrow \gamma^{-1}\theta x \in C \textit{ and } \sigma\Gamma_x = \gamma^{-1}\theta\Gamma_x.\end{aligned}$$

So,

$$\begin{aligned}\langle \xi, W\eta\rangle(x) &= \sum_{\sigma\in\Gamma/\Gamma_x} \overline{\xi}(x, \sigma\Gamma_x)\eta(x, \gamma\sigma\Gamma_x)\chi_C(\sigma x)\\ &= \sum_{\theta=\gamma\sigma\in\Gamma/\Gamma_x} \overline{\xi}(x, \gamma^{-1}\theta\Gamma_x)\eta(x, \theta\Gamma_x)\chi_C(\gamma^{-1}\theta x).\end{aligned}$$

So, $\langle V\xi, \eta\rangle = \langle \xi, W\eta\rangle$ and therefore $V^* = W$. $\square$

13 Cuntz-Pimsner algebra

13.1 Construction of Toeplitz-Pimsner algebra and Cuntz-Pimsner algebra

Def 13.1. *Let A be a C^*-algebra. An A-B C^*-correspondence is a (right) Hilbert B-module H with a faithful $*$-representation $\pi_H : A \to B(H)$, where $B(H)$ is a all of adjointable B-linear bounded maps.*
In paticular, we call C^*-correspondence over A if $A = B$.

Let $\mathcal{F}(H) := \oplus_{n\geq 0} H^{\otimes_A n}$, where $H^{\otimes_A 0} = A$. Let $\pi_{\mathcal{F}(H)}(a) = T_a \oplus (\oplus_{n\geq 0}\pi_H(a) \otimes id^{\otimes n})$, where T_a is a left multiplication. Then, $\mathcal{F}(H)$ is a C*-correspondence over A.
For $\xi \in H$, we define $T_\xi \in B(\mathcal{F}(H))$ by $T_\xi(\hat{a}) = \xi a \in H$ and $T_\xi(\xi_1 \otimes \cdots \otimes \xi_n) = \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n$.

Def 13.2. $\mathcal{T}(H) = C^*(A \cup \{T_\xi | \xi \in H\})$ *is called a Toeplitz-Pimsner algebra.*

Let $I_H = A \cap K(H)$, where $K(H) = C^*(\{\theta_{\xi,\eta} | \xi, \eta \in H\})$ $(\theta_{\xi,\eta}(\zeta) = \xi\langle \eta, \zeta\rangle)$, which is an ideal of A. Let $K(\mathcal{F}(H)I_H)$ is a ideal of $B(\mathcal{F}(H)I_H)$ and contained in $\mathcal{T}(H)$. Let $Q_I : B(\mathcal{F}(H)I_H) \to B(\mathcal{F}(H)I_H)/K(\mathcal{F}(H)I_H)$ be a quatient map.

Def 13.3. $\mathcal{O}(H) := Q_I(\mathcal{T}(H))$ *is called a Cuntz-Pimsner algebra.*

14 K-theory

15 Boundary

We assume G is discrete group.

15.1 injective envelope

Def 15.1. *Let $\mathcal{S}$ be a operator system, i.e. a unital self-adjoint subspace of a unital* C**-algebra.*

We say G-operator system if there is a homomorphism from G into the group of order isomorphism on $\mathcal{S}$ that sends the identity element of G to the unit of $\mathcal{S}$.

A G-operator system $\mathcal{U}$ is G-injective if for every unital c.i. G-equivariant map $\iota : \mathcal{S} \to \mathcal{T}$ and every unital c.p. G-equvariant map $\psi : \mathcal{S} \to \mathcal{U}$, there exists unital c.p. G-equivariant map $\hat{\varphi} : \mathcal{T} \to \mathcal{U}$ s.t. $\hat{\varphi} \circ \iota = \psi$

$$\begin{tikzcd}
\mathcal{S} \arrow[rr, "\iota(c.i.)"] \arrow[dr, "\psi(c.p.)"'] & \circlearrowleft & \mathcal{T} \arrow[dl, dashed, "\hat{\varphi}(c.p.)"] \\
& \mathcal{U} &
\end{tikzcd} .$$

A G-extension of $\mathcal{S}$ is a pair $(\mathcal{T}, \iota)$ consisting of a G-operator space $\mathcal{T}$ and c.i. G-equivariant $\iota : \mathcal{S} \to \mathcal{T}$.

A G-extenstion $(\mathcal{U}, \iota)$ is G-injective.

It is G-essensial if for every unital c.p. G-equivarinat map $\varphi : \mathcal{U} \to \mathcal{T}$ s.t. $\varphi \circ \iota$ is c.i. on $\mathcal{S}$ is necessarily c.i.

$$\begin{tikzcd}
\mathcal{S} \arrow[r, "\iota(c.i.)"] & \mathcal{U} \arrow[d, "\varphi(c.p.\Rightarrow c.i.)"] \\
& \mathcal{T}.
\end{tikzcd}$$

It is G-rigid if for every unital c.p. G-equivarinat map $\varphi : \mathcal{U} \to \mathcal{U}$ s.t. $\varphi \circ \iota = \iota$ on $\mathcal{S}$, φ is necessarily the identity map on $\mathcal{U}$.

$$\begin{tikzcd}
\mathcal{U} \arrow[rr, "\varphi(c.p.)\Rightarrow \mathrm{id}"] & \circlearrowleft & \mathcal{U} \\
& \mathcal{U} \arrow[ul, "\iota"] \arrow[ur, "\iota"'] &
\end{tikzcd} .$$

It is G-injective envelope of $\mathcal{S}$ if it is G-injective and G-essential.

Rem 15.0.1. *[Ham79] Every G-injective envelope of $\mathcal{S}$ is G-rigid.*

Rem 15.0.2. *Unital completely isometric map is unital completely positive.*

Thm 15.1 (Hamana)**.** *Let G be a discrete group, and $\mathcal{S}$ be a G-oprator system. Then, $\mathcal{S}$ has a G-injective envelope $(I_G(\mathcal{S}), \kappa)$. This injective envelope is uniwue, un the sense that for every G-injective envelope $(\mathcal{U}, \iota)$ of $\mathcal{S}$, there exists a uc.i. G-equivarent map $\varphi : I_G(\mathcal{S}) \to \mathcal{U}$ s.t. $\varphi \circ \kappa = \iota$.*

For an injective C*-algebra $\mathcal{A}$, $I_G(\mathcal{A})$ is injective C*-algebra w.r.t the Choi-Effros product.

15.2 Furstenberg boundary

Def 15.2. *Let G be a discrete group. The Hamana boundary $\partial_H G$ of G is the compact space s.t. $I_G(\mathbb{C}) = C(\partial_H G)$. By contrvariance, the G-action on $C(\partial_H G)$ induces a G-action on $\partial_H G$ which we will refer to as the G-action on $\partial_H G$).*

Thm 15.2. *$\partial_H G$ is a point if and only if G is amenable.*

Def 15.3. *Let G be a group and X be a compact G-space.*

The G-action on X is minimal if for every x in X, the G-orbit Gx is dense in X.

The G-action on X is strongly proximal if for every probability measure $\nu \in \mathcal{P}(X)$, the weak closure of the G-orbit $G\nu$ contains a point mass δ_x for some $x \in X$.*

X is a G-boundary if the G-action on X is both minimal and strongly proximal, i.e. for every probability measure $\nu \in \mathcal{P}(X)$, $\overline{G\nu}^{w*} \supset X$.

Rem 15.2.1. *The Hamana boundary is a G-boundary.*

Proof. ne ☐

Rem 15.2.2. *∂F_2 is a F_2-boundary, but $\partial\mathbb{Z}$ is not a $\mathbb{Z}$-boundary.*

Lem 15.1. *Let G be a group, let M be a minimal compact G-space and let B be a compact G-boundary. There is at most one unital positive G-equivariant map from $C(B)$ to $C(M)$, and if such a map exists, then it is a unital injective $*$-homomorphism.*

Thm 15.3. *Let G be a discrete group. Then, $\partial_F G \cong \partial_H G$.*

Prop 15.1. *The map taking a probability measure $\nu \in \mathcal{P}(\partial_F G)$ to $P_\nu(C(\partial_F G))$ is a bijection between $\mathcal{P}(\partial_F G)$ and the collection of unital isometoric G-equivarinat copies of $C(\partial_F G)$ in $l^\infty(G)$. The image $P_\nu(C(\partial_F G))$ is a C*-subalgebra if and only if ν is a point mass.*

Thm 15.4. *Let G be a discrete group. Then G is exact if and only if the G-action on $\partial_F G$ is amenable.*

15.3 C*-simplicity

Thm 15.5. *Let G be a discrete group. There is a canonical nuclear C*-algebra $N(C_r^*(G)) = C(\partial_F G) \rtimes_r G$ s.t.*

$$C_r^*(G) \subset N(C_r^*(G) \subset I(C_r^*(G)),$$

where $I(C_r^*(G))$ denotes the injective envelope of $C_r^*(G)$. The algebra $N(C_r^*(G))$ is simple if $C_r^*(G)$ is simple, and prime if and only if $C_r^*(G)$ is prime.

Def 15.4. *Let G be adiscrete group, and let X be a compact G-space.*

The G-action on X is topologically free if for every $s \in G \backslash \{e\}$, the set

$$X \backslash X^s = \{x \in X | sx \neq x\}$$

is dense in X.

Thm 15.6. *Let G be a discrete group. Then the followings are equivalent:*

1. *The group G is* C**-simple.*
2. $C(\partial_F G) \rtimes_r G$ *is simple.*
3. $C(B) \rtimes_r G$ *is simple for some G-boundary B.*
4. *The G-action on $\partial_F G$ is topologically free.*
5. *The G-action on some G-boundary is topologically free.*

16 Trees

Def 16.1. *A graph Γ consists of a set $Vert\Gamma$, a set $Y = Edge\Gamma$ and two maps $Y \to X \times X$, $y \mapsto (o(y), t(y))$ and $Y \to Y$, $y \mapsto \overline{y}$ (inversion) satisfying $y = \overline{\overline{y}}$, $y \neq \overline{y}$ and $o(y) = t(\overline{y})$.*

Def 16.2. *Let T be a tree i.e. a connected graph with no loop.*

- *For vertices $x, y \in V(T)$, we define a distance $d(x, y)$ by a length of a geodesic path (i.e. a path with no loop) from x to y.*
- *A map $\varphi : T \to T$ is called automorphism if it is isometric bijective and we denote $Aut(T)$ by all automorphisms.*
- *For $x, y \in V(T)$, $d(x, y) =$ the length of the geodesic from x to y. We define the topology on T by this distance.*
- *Two geodesic paths $(x(n))_n$, $(y(n))_n$ are equivalent if there is $m_0, n_0 \in \mathbb{N}$ s.t. $x(n + n_0) = y(n + m_0)$.*
- *The ideal boudary ∂T of T is defined as the set of all equivalent classes of infinite geodesic paths.*
- *For $x \in \overline{T} := T \cup \partial T$ and a finite set $F \subset T$, we define $U_{x,F} := \{y \in \overline{T} | [x, y] \cap F = \emptyset\}$. $\{U_{x,F}\}$ is a basis of the topology on $\overline{T}$.*

Rem 16.0.1.

- *$\overline{T}$ is compact.*
- *$Aut(T)$ is totally desconnected.*

Def 16.3. *Let $G \curvearrowright X$ with no inversion($\Rightarrow G \cdot y \neq \overline{G \cdot y}$). Let $V(G\backslash T) = G\backslash V(X)$, $E(G\backslash X) = G\backslash E(X)$, $V(G\backslash E(X)) \to V(G\backslash X) \oplus V(G\backslash X)$ $G \cdot y \mapsto (G \cdot o(y), G \cdot t(y)))$, and $E(G\backslash X) \to E(G\backslash X)$ $G \cdot y \mapsto \overline{G \cdot y}$. Then, $G\backslash X = (V(G\backslash X), E(G\backslash X))$ is a graph.*

As below, we assume $G \curvearrowright X$ with no inversion.

16.1 amalgamated products and a seqment

Thm 16.1. *Let X be a graph, which is $G\backslash X \cong$ a segment $T = (\{P, Q\}, \{y, \overline{y}\}) < X$. Then, the followings are equivalent.*

- *X is a tree;*
- *A homomorphism $\varphi : G_P *_{G_y} G_Q \to G$ induced by $G_P(G_Q) \hookrightarrow G_y$ is an isomorphism.*

Rem 16.1.1. *There is no element $g \in G$ s.t. $gP = Q$. Indeed, if not, $V(G\backslash X)$ is a point.*

This theorem follows from the two following lemmas.

Lem 16.1. *X is connected iff φ is surjective.*

Proof. Let X' be a connected component of X containig T. Let $G' := \{g \in G \mid gX' = X'\}$. Let G'' be a subgroup generated by G_P and G_Q. We prove $G' = G''$.
For $g \in G_P \cup G_Q$, y and gy have a common vertex. So, $gX' = X'$. Thus, $G'' \subset G'$.
Since $G''T$ and $(G - G'')T$ are disjoint, $X' \subset G''T$. For $g \in G'$, $gT \subset gX' \subset G''T$, so there exists a $h \in G''$ s.t. either $gP = hP$ or $gQ = hQ$ by the above remark. we may assume $gP = hP$. Then, $h^{-1}gP = P$, so $h^{-1}g \in G_P$. Hence, $g = hh^{-1}g \in G''$. So, $G' \subset G''$. Therefore, $G' = G''$.
X is connected $\Leftrightarrow X = X' \Leftrightarrow G = G' \Leftrightarrow G = G''$. $\square$

Lem 16.2. *X has no loop iff φ is injective.*

Proof. Assume X has a loop $c := (c_1, \ldots, c_n)$. We may assume $c_1 = y$. Let $P_k := t(c_k)$ $(k \geq 1)$. There exist a $g_k \in G$ and a $y_i \in \{y, \overline{y}\}$ s.t. $c_k = g_k y_k$ By the above remark, $y_{k+1} = \overline{y_k}$. Hence, $h_k := g_{k+1}^{-1} g_k \in G_{t(y_k)}$. If $h_k \in G_y$, it is contradiction to $y_{k+1} = \overline{y_k}$. So, $h_k \notin G_y$. $P = t(c) = t(g_n y_n) = g_1 h_1^{-1} \cdots h_n^{-1} P$. Since $g_1 = 1$, $h_1^{-1} \cdots h_n^{-1} \in G_P$. There exists a $k \in G_P$ s.t. $kh_1 \cdots h_n = 1$. This is contradiction to $kh_1 \cdots h_n \neq 1$, by the definition 17.2 of amalgamated free product. $\square$

16.2 Amalgamated products and trees

Def 16.4. *A graph of groups is consisting of a connected nonempty graph Y and groups G_P $(P \in V(Y))$ (resp.G_y $(y \in E(Y))$), and denoted by (G, Y) s.t. $G_y \to G_{t(y)}$, which is denoted by $a \mapsto a^y$, and $G_y = G_{\overline{y}}$.
For a graph of groups (G, Y), we denote $\lim(G, Y)$ by G_T.*

Thm 16.2. *Let (G, Y) be a graph of groups. There exists a graph X on which G_T acts s.t. T is a fundamental domain of X with respect to G_T and $(G_T)_P = G_P (P \in V(T))$(resp. $(G_T)_y = G_y (y \in E(Y))$).
Moreover, X is a tree.*

Thm 16.3. *Let G be a group acting on a graph X. Let T be a tree whose fundamental domain with respect to G is a tree T. Let $\varphi : G_T \to G$ be a homomorphism induced by $G_P \to G$, which is surjective, since X is connnected. Let $\psi : \tilde{X} \to X$ be a homomorphism which is uniquely determined by $T \to T$ and φ. The followings are equivalent.*

- *X is a tree.*
- *ψ is an isomorphism.*
- *φ is an isomorphism.*

16.3 Fundamental groups of a graph of groups

Def 16.5. *Let (G, Y) be a graph of groups.
We denote $F(G, Y)$ by the quotient of $*_{P \in V(T)} * F$ by the normal subgroup generated by $y\overline{y}$ and $ya^y y^{-1} (a^{\overline{y}})^{-1} (y \in E(Y), a \in G_y)$, where F is a free group generated by $E(Y)$.
For $P_0 \in V(Y)$,*

$$\pi_1(G, Y, P_0) := \left\{ |c, \mu| := r_0 y_1 r_1 \cdots y_n r_n \;\middle|\; \begin{array}{c} c = (y_1, y_2, \ldots, y_n) \text{ is path in } Y \\ \text{s.t. } o(y_1) = t(y_n), r_i \in G_{t(y_i)} \end{array} \right\}.$$

For a maximal tree $T < Y$, we denote $\pi_1(G, Y, T)$ by the quotient $F(G, Y)$ by the normal subgroup generated by $y = 1$ $(y \in E(T))$. Then,we denote g_y by the image in $\pi_1(G, Y, T)$ of $y \in E(Y)$.

Rem 16.3.1. *$F(G, Y)$ has the relation $ya^y y^{-1} = a^{\overline{y}}$ $(y \in E(Y), a \in G_y)$. $\pi_1(G, Y, T)$ has the above relation and the relation $g_y = 1$ $y \in E(T)$.*

Rem 16.3.2. *$\pi_1(G, Y, P_0)$ and $\pi_1(G, Y, T)$ are isomorphic, so denoted by $\pi_1(G, Y)$. This follows from the following proposition.*

Lem 16.3. *$p : F(G, Y) \to \pi_1(G, Y, T)$ induces an isomorphism $\overline{p} : \pi_1(G, Y, P_0) \to \pi_1(G, Y, T)$.*

Proof. For $P \in V(T)$, we define $c_P = (y_1, \ldots, y_n)$ by the path from P_0 to P with no backtracking and $\gamma_P := y_1 \cdots y_n$. For $x \in G_P$ and $y \in E(Y)$, let $x' := \gamma_P x {\gamma_P}^{-1} \in \pi_1(G, Y, P_0)$ and $y' := \gamma_{o(y)} y {\gamma_{t(y)}}^{-1} \in \pi_1(G, Y, P_0)$. Then, for $y \in E(T)$, either $c_{t(y)} = (c_{o(y)}, y)$ or $c_{o(y)} = (c_{t(y)}, \overline{y})$ is satisfied, so $y' = 1$. Also, $\overline{y}' = \gamma_{o(\overline{y})} \overline{y} {\gamma_{t(\overline{y})}}^{-1} = \gamma_{t(y)} \overline{y} {\gamma_{o(y)}}^{-1}$, so $y'\overline{y}' = \overline{y}'y' = 1$. For $y \in E(Y)$ and $a \in G_y$,

$$\begin{aligned} y'(a^y)'{y'}^{-1} &= \gamma_{o(y)} y {\gamma_{t(y)}}^{-1} \gamma_{t(y)} a^y {\gamma_{t(y)}}^{-1} \gamma_{t(y)} y^{-1} {\gamma_{o(y)}}^{-1} \\ &= \gamma_{o(y)} a^{\overline{y}} {\gamma_{o(y)}}^{-1} \\ &= (a^{\overline{y}})'. \end{aligned}$$

So, there exists a unique homomorphism $f : \pi_1(G, Y, T) \to \pi_1(G, Y, P_0)$ s.t. $f(\overline{x}) = x'$ and $f(\overline{y}) = y'$.
Since $p \circ f = \mathrm{id}$, $\overline{p}$ is injective. By the construction, $\overline{p}$ is surjective. $\square$

Ex 2. *If $G_y = \{e\}$, $\pi_1(G, Y, T) = *_{P \in V(Y)} G_P * F$.*

Ex 3. *If Y is a seqment, $\pi_1(G, Y, T) = G_P *_{G_y} G_Q$.*

Ex 4. *If Y is a loop, $\pi_1(G, Y, T) =$HNN-extension. It is generated by G_P and g_y, and satisfies the relation $g_y a^y g_y^{-1} = a^{\overline{y}}$*

Def 16.6. *Let (G, Y) be a graph of groups and $X < Y$ be a (connected) subgraph.*
We define $(G, Y/X)$ as below.
$V(Y/X) := V(Y) \sqcup \{V(X)\}$, $E(Y/X) := E(Y) \backslash E(X)$ and

$$o(y) := \begin{cases} o(y) & (o(y) \notin V(X)) \\ V(X) & (o(y) \in V(X)) \end{cases}, o(y) := \begin{cases} t(y) & (t(y) \notin V(X)) \\ V(X) & (t(y) \in V(X)). \end{cases}$$

$G_{\{X\}} := \pi_1(G, X)$.

Rem 16.3.3. *$\pi_1(G, Y) = \pi_1(G, Y/X)$.*

Rem 16.3.4 (cf. Rem.17.1.1)**.** *Let (G, Y) be a graph of groups. Let $T < Y$ be a maximal tree in Y.*
Then, $Y' := Y/T$ is an order-$\mathrm{rank}(\pi_1(\mathrm{Y}))$ bouquet graph and $\pi_1(Y) \cong \pi_1(Y')$.
Let $\tilde{T}$ be a universal covering of Y' and its covering map is denoted by π.
We define $(G, \tilde{T})$ by $G_{\tilde{P}} := G_{\pi(\tilde{P})}$ $(\tilde{P} \in V(\tilde{T}))$ (resp. $G_{\tilde{y}} := G_{\pi(\tilde{y})}$ $(\tilde{y} \in E(\tilde{T}))$.
Then, $\pi_1(Y')$ acts on $\tilde{T}$ by the deck transformation $Deck(\tilde{T}/Y') \cong \pi_1(Y')$, so $\pi_1(Y')$ acts on $\pi_1(G, \tilde{T})$.
We define $\Theta : \pi_1(G, Y') \to \pi_1(G, \tilde{T}) \rtimes \pi_1(Y')$ by

$$\begin{aligned} \pi_1(G, Y', P_0) &\to \pi_1(G, \tilde{T}, \tilde{T}) \rtimes \pi_1(Y'), \\ r_0 y_1 r_1 \cdots y_n r_n &\mapsto (r_0 \cdots r_n, y_1 \cdots y_n), \end{aligned}$$

where $Q_0 \in V(\tilde{T})$ and $r_0 \in G_{Q_0}, r_1 \in G_{y_1(Q_0)}, r_2 \in G_{y_1 y_2(Q_0)}, \cdots, r_n \in G_{y_1 \cdots y_n(Q_0)}$. Θ is isomorphism.
We define a homomorphism $\pi_1(G, \tilde{T}, \tilde{T}) \rtimes \pi_1(Y') \to \pi_1(G, Y', P_0)$ by

$$(r_0 \cdots r_n, y) \mapsto r_0 \pi(\gamma_1) r_1 \cdots \pi(\gamma_n) r_n \pi(\zeta),$$

where for $r_i \in G_{P_i}$ $(P_0 = Q)$, γ_i is a geodedic from P_i to P_{i+1} in T and ζ is a geodedic from P_n to Q in T.
In conclusion, $\pi_1(G, Y) = \pi_1(G, Y') \cong \pi_1(G, \tilde{T}) \rtimes \pi_1(Y)$.

Def 16.7. *Let (G, Y) be a graph of groups. For the path $c := (y_1, \ldots, y_n)$, and $\mu = (r_0, \ldots, r_n)$ $(a_i \in G_{t(y_i)}$, $r_0 \in G_{o(y_1)})$, we define*

$$|c, \mu| := r_0 y_1 r_2 \cdots r_n y_n$$

One says that (c, μ) is reduced if it satisfies the following condition: If $n = 0$ one has $r_0 \neq 0$; if $n \leq 1$ one has $r_i \notin G_{y_i}^{y_i} := \mathrm{Im}(\mathrm{G_{y_i}} \to \mathrm{G_{t(y_i)}})$ for each index i s.t. $y_{i+1} = \overline{y_i}$.

Thm 16.4. *If (c, μ) is a reduced word, the associated element $|c, \mu|$ of $F(G, Y)$ is $\neq 1$.*

The following corollary follows from $\pi_1(G, Y, P_0) \cong \pi_1(G, Y, T)$.

Cor 16.4.1. *Let $T < Y$ be a maximal tree and let (c, μ) be a reduced word whose type c is a closed path. Then, $\overline{|c, \mu|} \neq 1$ in $\pi_1(G, Y, T)$.*

16.4 Universal covering relative to a graph of groups

Let (G,Y) be a graph of groups. Let $T < Y$ be a maximal tree. Let $A < Y$ be an orientation (i.e. $Y = A \sqcup \overline{A}$). For $y \in E(Y)$,

$$|y| := \begin{cases} y & (y \in A) \\ \overline{y} & (y \notin A) \end{cases}, \ e(y) := \begin{cases} 0 & (y \in A) \\ 1 & (y \notin A) \end{cases}.$$

We construct the following objects.

- graph $\tilde{X} = \tilde{X}(G,Y,T)$;
- $\pi := \pi_1(G,Y,T)$ acts on $\tilde{X}$;
- $p : \tilde{X} \to Y$ induces an isomorphism $\pi \backslash \tilde{X} \cong Y$;
- sectios $V(Y) \to V(\tilde{X})$ and $E(Y) \to E(\tilde{X})$ of p, which is denoted by $P \mapsto \tilde{P}$ and $y \mapsto \tilde{y}$;
- $\pi_{\tilde{P}} = G_P$, $\pi_{\tilde{y}} = G^{\overline{|y|}}_{|y|}$:

Let $V(\tilde{X}) := \sqcup_{P \in V(Y)} \pi/\pi_P$ and $E(\tilde{X}) := \sqcup_{y \in E(Y)} \pi/\pi_y$, where $\pi_P := G_P$ $(P \in V(Y))$ and $G_y := G^{\overline{|y|}}_{|y|}$ $(y \in E(Y))$.

We denote the image of 1 in π/π_P (resp. π/π_y) by $\tilde{P}$ (resp. $\tilde{y}$).
For $g \in \pi$ and $y \in E(Y)$,

$$\begin{aligned} \overline{g\tilde{y}} &:= g\tilde{\overline{y}}, \\ o(g\tilde{y}) &:= g g_y^{-e(y)} \tilde{o(y)}, \\ t(g\tilde{y}) &:= g g_y^{1-e(y)} \tilde{t(y)}. \end{aligned}$$

Then, $\pi_y = \pi_{\overline{y}}$. Also, for $h \in \pi_{\tilde{y}}$, $h g_y^{-e(y)} \tilde{o(y)} = g_y^{-e(y)} \tilde{o(y)}$.

Thm 16.5. *The above graph $\tilde{X}$ is a tree.*

Proof. Connectedness is ganbaru.
We show that $\tilde{X}$ have no closed path with no backtracking.

□

16.5 ping-pong lemma

In this section, we consider a tree as a connected set in $\mathbb{R}^2$.

Def 16.8. *Let γ be a isometric bijection of a tree T. We define $l(\gamma) := \inf_{x \in T} d(x, \alpha x)$*

- *If γ fix a point in T, then γ is called elliptic;*
- *If γ does not fix any point in T, then γ is called hyperbolic:*

Rem 16.5.1. *For γ, $\mathrm{Fix}(\gamma)$ is a tree.*

Prop 16.1. *Let γ be a hyperbolic element. Then, there exist the unique γ-invariant line. We denote this line by $\mathrm{Axis}(\gamma)$.*

Rem 16.5.2. *Suppoese there exist $x \in T$ s.t. $d(x, \gamma^2 x) = 2d(x, \gamma x)$. γx is in $[x, \gamma^2 x]$, since γ is hyperbolic. $[x, \gamma x, \gamma^2 x]$ generate a γ-invariant line.*

Proof. Let $y \in T$. If $x \in [\gamma x, \gamma^2 x]$, then it contradicts hyperbolicity. If $\gamma x \in [x, \gamma^2 x]$, then it is what we want. So, we may asuume x, γx, $\gamma^2 x$ are common points. we remark $\gamma x \neq \gamma^2 x$.
Let α (resp. β) be geodesic from x to γx (resp. $\gamma^{-1}x$). Let n be a maximal number which satisfies $\alpha(k) = \beta(k)$ $(1 \leq k \leq n)$ and let $a = \alpha(n) = \beta(n)$, which is the crux of a triangle with vertecies x, γx, $\gamma^{-1}x$. Let $b = \gamma a$ and $c = \gamma^{-1}a$, which is not a by hyperbolicity. Then, b in $[x, \gamma x]$, since a in $[x, \gamma^{-1}x]$. If $d(b, x) < d(x, a)$, then x in $[x, \gamma x]$, so a, γa, $\gamma^{-1}a$ is on the same line. So, we may assume $d(b, x) > d(x, a)$. Similarily, we may asuume c in $[x, \gamma^{-1}]$ and $d(c, x) < d(x, a)$. So, $d(\gamma^{-1}a, \gamma a) = 2d(\gamma^{-1}a, a)$. □

Rem 16.5.3. *For isometry g,* $g\mathrm{Axis}(\gamma) = \mathrm{Axis}(g\gamma g^{-1})$

Prop 16.2. *Let γ, $\delta \in \mathrm{Aut}(T)$.*

1. *If γ, δ are elliptic and $\mathrm{Fix}(\gamma) \cap \mathrm{Fix}(\delta) = \emptyset$, then $\gamma\delta$ is hyperbolic with $l(\gamma\delta) = 2d(\mathrm{Fix}(\gamma), \mathrm{Fix}(\gamma))$.*
2. *If γ, δ are hyperbolic and $\mathrm{Axis}(\gamma) \cap \mathrm{Fix}(\delta) = \emptyset$, then $\gamma\delta$ is hyperbolic with $l(\gamma\delta)l(\gamma)+l(\delta)+2d(\mathrm{Axis}(\gamma), \mathrm{Axis}(\delta))$ and $\mathrm{Axis}(\gamma\delta)$ intersects $\mathrm{Axis}(\gamma)$ and $\mathrm{Axis}(\delta)$.*

Proof. There exist $x \in \mathrm{Fix}(\gamma)$ and $y \in \mathrm{Fix}(\delta)$ s.t. $d(x, y) = d(\mathrm{Fix}(\gamma), \mathrm{Fix}(\delta))$. Let α be a geodesic from x to y. Then, $\alpha \cup \overline{\delta\alpha}$ is a geodesic from δx to y. Also, $\delta\alpha$ is a geodesic path form $\delta\mathrm{Fix}(\gamma)$ to $\mathrm{Fix}(\delta)$. Indeed, if $\delta\alpha \cap \alpha = [y, z]$ is not a vertex, that is $y \neq z$, this is contradiction to $\mathrm{Fix}(\gamma) \cap \mathrm{Fix}(\delta) = \emptyset$. So, $d(x, (\gamma\delta)^2 x) = 2d(x, \gamma\delta x)$.

There exist $x \in \mathrm{Axis}(\gamma)$ and $y \in \mathrm{Axis}(\delta)$ s.t. $d(x, y) = d(\mathrm{Axis}(\gamma), \mathrm{Axis}(\delta))$.

$$\begin{array}{cccccc} \mathrm{Axis}(\gamma) \to & \mathrm{Axis}(\delta) \to & \mathrm{Axis}(\delta) \to & \gamma\mathrm{Axis}(\delta) \to & \gamma\mathrm{Axis}(\delta) \to & \delta\gamma\mathrm{Axis}(\delta) = \delta\mathrm{Axis}(\gamma) \\ x \to & y \to & \gamma y \to & \gamma x \to & \delta\gamma x \to \delta\gamma y & \end{array}$$

So, $d(x, (\delta\gamma)^2 x) = 2d(x, \delta\gamma x)$. □

Lem 16.4. *Let e, $e' \in E(T)$ and an isometry γ s.t. $\gamma e = e'$. Let $x = o(e)$ and $y = t(e)$. If $d(x, \gamma x) = d(y, \gamma y)$, then γ is hyperbolic.*

Proof. ? □

Lem 16.5 (ping-pong lemma)**.** *Let γ, δ be hyperbolic elements whose axes has a intersection. If this intersection is compact, there exist $n \in \mathbb{N}$ s.t. γ^n and δ generate a free group of rank 2.*

Proof. Let $K := \mathrm{Axis}(\gamma) \cap \mathrm{Axis}(\delta)$. $n = |K|$. □

16.6 Amenabilty and hyperbolic element

Thm 16.6 ([Neb88])**.** *Let T be a locally finite tree and $G < \mathrm{Aut}(T)$ be a closed subgroup. Then, G is amenable if and only if one of the followinf statements holds*

- *G fixes a vertex;*
- *G stabilize an edge;*
- *G fix a point in ∂T;*
- *G stabilize a pair of points in ∂T:*

Prop 16.3. *Let $G < \mathrm{Aut}(T)$ be a closed non-amenable subgroup. There exists a hyperbolic element in G.*

Lem 16.6. *If for any $g, h \in G$, $\mathrm{Fix}(g) \cap \mathrm{Fix}(h) \neq \emptyset$, then $\cap_{fin.}\mathrm{Fix}(g) \neq \emptyset$.*

Proof. We remark $\mathrm{Fix}(g)$ is a tree. If $\mathrm{Fix}(g) \cap \mathrm{Fix}(h) \cap \mathrm{Fix}(k) = \emptyset$, there exists a cycle. Contradiction. □

proof of proposition. We assume G has no hyperbolic element. If there exist elliptic elements $g, h \in G$ s.t. $\mathrm{Fix}(g) \cap \mathrm{Fix}(h) = \emptyset$, there exist a hyperbolic element. So, for any $g, h \in G$, $\mathrm{Fix}(g) \cap \mathrm{Fix}(h) \neq \emptyset$. Since $\overline{T}$ is a complete metric space,

$$\cap_{g \in G}\mathrm{Fix}_{\overline{T}}(g) \neq \emptyset.$$

It is contradiction to non-amenability of G. □

17 Group

Def 17.1 (amalgamated product)**.** *Let Γ_i be a group and Λ be a group with homomorphisms $\varphi_i : \Lambda$ into each Γ_i. We denote by N a normal subgroup generated by $\varphi_1(\gamma)\varphi_2(\gamma)^{-1}$. $\Gamma_1 * \Gamma_2/N$ is called amalgamated product.*

Def 17.2 (amalgamated free product)**.** *Let Γ_i be a group and Λ be a common subgroup (i.e. Λ come with an injective homomorphism $\varphi_i : \Lambda$ into each Γ_i). Then, the amalgamated free product $\Gamma = \Gamma_1 *_\Lambda \Gamma_2$ is the group satusfyubg the following properties:*

- *Γ contains Γ_1 and Γ_2 as subgroups and Γ is generated by Γ_1 and Γ_2;*
- *$\Gamma_1 \cap \Gamma_2 = \Lambda$ in Γ;*
- *$s_1 \cdots s_n a \neq e$ whenever $n \geq 1$, $a \in \Lambda$ and $s_k \in \Gamma_{i_k} \backslash \Lambda$ with $i_k \neq i_{k+1}$ for $1 \leq k < n$;*
- *if we choose systems S_i of represetatives of Γ_i/Λ and let $S_i^0 = S_i \backslash \{e\}$ (we always assume that the representative of the coset Λ is e), then any element s in Γ can be uniquely written as $s = s_1 \cdots s_n a$, where $a \in \Lambda$ and $s_k \in S_{i_k}^0$ such that $i_k \neq i_{k+1}$ for $1 \leq k < n$:*

Rem 17.0.1. $\Gamma_1 *_\Lambda \Gamma_2 = \Gamma_1 * \Gamma_2/($*the smallest normal subgroup containig* $\varphi_1(\lambda)^{-1}\varphi_2(\lambda))$.

Rem 17.0.2 (universality)**.** *If*

$$\begin{tikzcd}
N \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] \arrow[dr, phantom, "\circlearrowleft"] & \Gamma_1 \arrow[d, "f_1"] \\
\Gamma_2 \arrow[r, "f_2"] & \Gamma,
\end{tikzcd}$$

then,

$$\begin{tikzcd}
N \arrow[rr, "\varphi_1"] \arrow[dd, "\varphi_2"'] \arrow[dr, phantom, "\circlearrowleft"] & & \Gamma_1 \arrow[dl, "\iota_1"] \arrow[dd, "f_1"] \\
& \Gamma_1 *_N \Gamma_2 \arrow[dr, "\exists f"] \arrow[d, phantom, "\circlearrowleft"] \arrow[r, phantom, "\circlearrowleft"] & \\
\Gamma_2 \arrow[ur, "\iota_2"] \arrow[rr, "f_2"] & & \Gamma.
\end{tikzcd}$$

Prop 17.1. *Let G be a group and H be a subgroup.*
Then, for $g_1, g_2 \in G$, $g_1H = g_2H \Leftrightarrow g_1H \cap g_2H \Leftrightarrow g_1^{-1}g_2 \in H$.

Thm 17.1. *Let G be a group and A be a subgroup of G. Let $\theta : A \to G$ be a injective homomorphism. Then, there exist a group G' which is generated by G and s s.t. $\theta(a) = sas^{-1}$ $(a \in A)$.*

Proof.

$$\begin{tikzcd}
\cdots G & & G & & G \cdots \\
\cdots & A \arrow[ul, "\theta"'] \arrow[ur, hook] & & A \arrow[ul, "\theta"'] \arrow[ur, hook] & \cdots .
\end{tikzcd}$$

Amlgamation is as above. Let $G_n = G$ $(n \in \mathbb{Z})$. Let $u : G_n \to G_{n+1}$ be a canonical isomorphism. Let $H_n := *_A{}_{k=-n}^{n} G_k$ $(n \in \mathbb{N})$. We define $\varphi_n : H_n \to H_{n+1}$ by $g \mapsto 1 * 1 * u(g)$.

$$\begin{tikzcd}
H_1 \arrow[r] \arrow[dr, "\varphi_1"] & H_2 \arrow[r] \arrow[dr, "\varphi_2"] & H_3 \arrow[r] \arrow[dr, "\varphi_2"] & \cdots & \\
H_1 \arrow[r] & H_2 \arrow[r] & H_3 \arrow[r] & H_4 \arrow[r] & \cdots .
\end{tikzcd}$$

Since the above diagram is commutative, there exist a $\sigma : *_A G \to *_A G$ s.t. $\sigma(a) = \theta(a)$ $(a \in A)$. Since $\mathbb{Z}$ acts on $*_A G$ by $n \mapsto \sigma^n$, $G' := *_A G \rtimes \mathbb{Z}$. Actually,

$$(e,1)(a,0)(e,1)^{-1} = (e,1)(a,-1) = (\sigma(a),0) = (\theta(a),0).$$

□

Def 17.3. *The above G' is called the HNN extension of G relative to θ and denoted by $G*_\theta$.*

Rem 17.1.1. *In the sense of the above construction, $G*_\theta = *_A G \rtimes \mathbb{Z} \cong G * \mathbb{Z}/(sas^{-1} = \theta(a))$, where s is a generator of $\mathbb{Z}$.*

Proof. A homomorphism $G * \mathbb{Z}/(sas^{-1} = \theta(a)) \to *_A G \rtimes \mathbb{Z}$ is definef by the universality.
A homomorphism $\varphi : *_A G \rtimes \mathbb{Z} \to G * \mathbb{Z}/(sas^{-1} = \theta(a))$ is defined by

$$(g_1 g_2 \cdots g_n, r) \mapsto k_1 g_1 (k_2 - k_1) g_2 \cdots (k_n - k_{n-1}) g_n (r - k_n),$$

where $g_i \in G_{k_i}$.

$$\begin{aligned}
\varphi(g_1 g_2 \cdots g_n, r)\varphi(h_1 h_2 \cdots h_m, s) &= k_1 g_1 (k_2 - k_1) g_2 \cdots (k_n - k_{n-1}) g_n (r - k_n + l_1) h_1 \\
&\quad \cdot (l_2 - l_1) h_2 \cdots (h_m - h_{m-1}) h_m (s - l_m). \\
\varphi((g_1 g_2 \cdots g_n, r)(h_1 h_2 \cdots h_m, s)) &= \varphi(g_1 g_2 \cdots g_n r(h_1 h_2 \cdots h_m), r+s) \ (r(h_i) \in G_{l_i + r}) \\
&= k_1 g_1 (k_2 - k_1) g_2 \cdots (k_n - k_{n-1}) g_n ((l_1 + r) - k_n) h_1 \\
&\quad \cdot ((l_2 + r) - (l_1 + r)) h_2 \cdots h_{m-1} (s + r - (l_m + r)) \\
&= k_1 g_1 (k_2 - k_1) g_2 \cdots (k_n - k_{n-1}) g_n (r - k_n + l_1) h_1 \\
&\quad \cdot (l_2 - l_1) h_2 \cdots (h_m - h_{m-1}) h_m (s - l_m).
\end{aligned}$$

By the above calculation, the definition is well-defined. □

18 Locally compact groups

18.1 topological gropus

Thm 18.1. *Let G be a connected topological group. Let U be a neighborhoff of e s.t. $U = U^{-1}$. Then, for any $g \in G$, there exists $g_1, \ldots, g_k$ s.t. $g = g_1 \ldots g_k$.*

Thm 18.2. *Let G be a compact group acting on topological space X. Then, a quotient map $\pi : X \to G \backslash X$ is proper (i.e. $\pi^{-1}(cpt.) = cpt.$).*

Proof. Let $F \subset X \backslash G$ be a compact subset. Let $\{U_i\}_{i \in I}$ be a open covering of F.
For $x \in \pi^{-1}(F)$, there exist a $i_x \in I$ s.t. $\pi(x) \in U_{i_x}$. For $g \in G$, there exist a open neighborhood $V_{x,g} \subset G$ of g and a open neighborhood $W_{x,g}$ of x s.t. $V_{x,g} \cdot W_{x,g} \subset U_{i_x}$.
By compactness of G, $G = \cup_{fin.} V_{x,g_n}$. Let $W_x := \cap W_{x,g_n}$. Then, $G \cdot W_x \subset U_{i_x}$.
Since $F \subset \cup_{x \in \pi^{-1}(F)} \pi(GW_x)$ and $G \cdot W_x$ is open, $F \subset \cup_{fin.} \pi(G \cdot W_x)$.
So, $\pi^{-1}(F) \subset \cup_{fin.} G \cdot W_x \subset \cup_{fin.} U_{i_x}$. □

Prop 18.1. *Let G be a topological group and H be a topological subgroup of G with finite index. G is topologically finitely generated if ond only if so is H.*

Proof. We assume H is topologically finitely generated. Let F be a fintie generator of H. Then, $G = \sqcup_{fin.} g_i H$. So, $F \cup \{g_i\}$ is a finite generator of G.

We assume G is topologically finitely generated. Let Y be a finite generator of G. Let $[\cdot] : G \to G$ $g \mapsto [g]$ be a left H-invariant map. We may assume $e = [e]$. We have, for any $g \in G$

$$g[g]^{-1} \in H,\ [[g]h] = [gh],\ [g] = [[g]].$$

$T = \{[g]y[[g]y]^{-1} | g \in G,\ y \in Y\}$ is finite and generator of H. Finiteness is follows from finite index. Indeed, suppose $h = g_1 g_2 \cdots g_r \in H$ ($g_i \in Y$). For simplicity, $r = 3$.

$$h = g_1[g_1]^{-1} \cdot [g_1]g_2[[g_1]g_2]^{-1} \cdot [[g_1]g_2]g_3[[[g_1]g_2]g_3]^{-1}.$$

Each element belongs to T. □

18.2 locally compact group

Let G be a locally comapct group. Let $\mu(=\mu_G)$ be a left Haar measure on G. Let $\triangle(=\triangle_G)$ be a modular function on G.

Cor 18.2.1. *H be a compact subgroup. A quotient map $\pi : G \to G/H$ is proper.*

Prop 18.2. *Then, G is tottaly disconnected iff G has a fundamental system of neighborhoods for a unit e consistinng of compact open subgroups.*

18.3 Haar measure

Prop 18.3. *Let K be a compact open subgroup of G. Then, $\ker(\triangle_G)$ is a clopen normal subgrop containig K.*

Def 18.1. *We denote $\ker(\triangle_G)$ by G_0.*

Prop 18.4. *Let G be a totally disconnected. Let K be a compact normal subgroup of G. Let $\mu = \mu_K$. Then, μ is invariant under the conjugation action of G.*

Proof. We suffices to show that for any open subgroup $H < K$ and for any $g \in G$, $\mu(H) = \mu(gHg^{-1})$, since a borel set is generated by open subsets and a topological group has fundamental system of neighborhoods of e.
Suppose $\mu(H) < \mu(gHg^{-1})$. $K = \sqcup_{fin.} k_i H$, since H is a open subgroup.

$$\mu(K) = \sum_i \mu(H) < \sum_i \mu(gHg^{-1}) = \sum_i \mu((gk_ig^{-1})gHg^{-1}) = \mu(gKg^{-1}) = \mu(K),$$

which is contradiction. □

Rem 18.2.1. *This proposition is satisfied in the case of general locally compact groups.*

Proof. For $g \in G$, $\mu_g := \mu(g \cdot g^{-1})$ is a left Haar measure. By the uniqueness of a Haar measure, $\mu = \mu_g$. □

Prop 18.5. *Let G_i be a locally compact group and K be a common open subgroup. Then, $G_1 *_K G_2$ is a locally comapct group with respect to the topology generated by K. Moreover, if G_i is unimodular, $G_1 *_K G_2$ is unimodular.*

Proof. Let $\mu := \mu_{G_1 *_K G_2}$. We suffices to show that $\mu(gEg^{-1}) = \mu(E)$ for all $g \in G_i$ and $E \subset K$, because $gE = \sqcup_{g_j} g_j E_i$ ($g \in G_i,\ E_i \subset K$). It follows from unimodularity of G_i. □

18.4 Amenability

Def 18.2. *G is amenable if it has a left invarinat mean, that is a left invariant state on $\mathrm{L}^\infty(\mathrm{G}, \mu)$.*

Prop 18.6. *Let G be a group acting on (X, ν). Assume X has a G-invarinat mean, the action is measure preserving and the satabilizer subgroup G_x is amenable for a.e. $x \in X$. Then, G is amenable.*

Proof. Let μ_x be a left invarinat mean on G_x. Let μ_X be a G-invarinat mean on X. Let $X = \sqcup_i G \cdot x_i$.
For $x \in X$, there exist a $g_x \in G$ s.t. $x = g_x x_i$. For, $f \in \mathrm{L}^\infty(\mathrm{G})$ and $x \in X$, $\theta(f)(x) := \mu_{x_i}((g_x^{-1}.f)|_{G_{x_i}})$. For $g \in G_{x_i}$, $\mu_{x_i}(((g_x g)^{-1}.f)|_{G_{x_i}}) = \mu_{x_i}(g^{-1}.(g_x^{-1}.f)|_{G_{x_i}}) = \mu_{x_i}((g_x^{-1}.f)|_{G_{x_i}})$, so it is well-defined. We define μ by $\mu(f) := \mu_X(\theta(f))$.
For $g,\ h \in G$, $\theta(h.f)(gx_i) = \mu_{x_i}(g_x^{-1}.(h.f)|_{G_{x_i}}) = \mu_{x_i}((h^{-1}g_x)^{-1}f)|_{G_{x_i}}) = (h.\theta(f))(x)$. So, μ is a left invariant mean. □

Cor 18.2.2. *Let G be a locally compact group which is an extension of N by H. Assume N and H are amenable group. Then, G is amenable.*

Ex 5 (amenale)**.** • *abelian groups;*

- *compact groups;*
- $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} = \mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$*, since extension;*

Ex 6 (non-amenable)**.** • $\mathbb{F}_2$*;*

- $\mathbb{Z}/n\mathbb{Z} * \mathbb{Z}/m\mathbb{Z}$ $(n \geq 2, m \geq 3)$*, like* $\mathbb{F}_2$*;*

Thm 18.3 ([BO08])**.** *Let G be a countable discrete group. Then, the followings are wquivalent.*

1. *G is amenabale;*
2. *G satisfies the Følner condition (i.e., for any finite subset $E \subset G$ and $\varepsilon > 0$, there exist a finite subset $F \subset G$ s.t.*

$$\max_{s \in E} \frac{|sF \triangle F|}{|F|} < \varepsilon$$

   );
3. *the trivial representation τ_0 is weakly contained in the regular representation λ (i.e., there exist unit vectors $\xi_i \in l^2(G)$ s.t. $\|\lambda_s(\xi_i) - \xi_i\| \to 0$ for all $s \in G$);*
4. *there exists a net (φ_i) of finitely supported positive definite functions on G s.t. $\varphi_i \to 1$ pointwise;*
5. *$C^*(G) = C^*_r(G)$;*
6. *$C^*_r(G)$ has a character (i.e., one-dimensional representation);*
7. *for any finite subset $E \subset G$, we have*

$$\|\frac{1}{|E|}\sum_{s \in E} \lambda_s\| = 1;$$

8. *$C^*_r(G)$ is nuclear;*
9. *$L(G)$ is semidiscrete.*

19 Representations

19.1 Induced Representations

We assume G is discrete group. Let $H < G$ be a subgroup of G. In the case of locally compact groups, if H is open or compact open, probably OK. Let σ be a unitary representation of H. Let $q : G \to G/H$ be a quotient map.

$$\mathcal{F}_0 := \{f \in C_c(G, H_\sigma) | q(supp(f)) : compact,\ f(y\xi) = \sigma(\xi^{-1})f(y)\ (y \in G,\ \xi \in H)\}.$$

For $f,\, g \in \mathcal{F}_0$,

$$\langle f, g\rangle := \int_{G/H} \langle f(x), g(x)\rangle_\sigma d\mu(xH).$$

We define $\mathcal{F}$ by the completion of $\mathcal{F}_0$ w.r.t. this inner product. G acts on $\mathcal{F}$ by left transformation, denoted by $\mathrm{ind}_{\mathrm{H}}^{\mathrm{G}}(\sigma)$

Rem 19.0.1. *If σ is a trivial representation, $\mathrm{ind}_{\mathrm{H}}^{\mathrm{G}}(\sigma)$ is the natural representation of G on $l^2(G/H)$ by left transformation.*

Prop 19.1. $\mathrm{ind}_{\mathrm{H}}^{\mathrm{G}}(\lambda_{\mathrm{H}}) \cong \lambda_{\mathrm{G}}$.

Proof.

$$\mathcal{F} := \{f \in l^2(G, l^2(H_\sigma)) | ||f|| < \infty (\textit{not } l^2),\ f(y\xi)(s) = f(y)(\xi s)\ (y \in G, \xi, s \in H)\}.$$

We define $\Phi : l^2(G) \to \mathcal{F}$ by $\Phi(f)(y)(\xi) := f(y\xi)$ for $f \in l^2(G)$, $y \in G$, $\xi \in H$). linear is OK.

Surjectivity: For $f \in \mathcal{F}$, $g(y) := f(y, e)$.

Inner product preserving: For $f, g \in l^2(G)$,

$$\begin{aligned}
\langle \Phi(f), \Phi(g) \rangle &= \int_{G/H} \langle \Phi(f)(x), \Phi(g)(x) \rangle_\sigma d\mu_{G/H}(xH) \\
&= \int_{G/H} \int_H \Phi(f)(x, s)\overline{\Phi(g)(x, s)} d\mu_H(s) d\mu_{G/H}(xH) \\
&= \int_{G/H} \int_H f(xs)\overline{g(xs)} d\mu_H(s) d\mu_{G/H}(xH) \\
&= \int_G f(x)\overline{g(x)} d\mu_G(x) = \langle f, g \rangle
\end{aligned}$$

We suffices to check the following diagram is commutative.

$$\begin{tikzcd}
l^2(G) \arrow[r, "\lambda_G"] \arrow[d, "\Phi"] & l^2(G) \arrow[d, "\Phi"] \\
\mathcal{F} \arrow[r, "\mathrm{ind}_{\mathrm{H}}^{\mathrm{G}}(\lambda_{\mathrm{H}})"] & \mathcal{F}.
\end{tikzcd}$$

Let $\sigma = \mathrm{ind}_{\mathrm{H}}^{\mathrm{G}}(\lambda_{\mathrm{H}})$. For $f \in l^2(G)$, $g, y \in G$, $\xi \in H$,

$$\begin{gathered}
(\sigma(g) \circ \Phi)(f)(y, \xi) = \sigma(g)(\Phi(f))(y, \xi) = \Phi(f)(g^{-1}y, \xi) = f(g^{-1}y\xi), \\
(\Phi \circ \lambda_G(g))(f)(y, \xi) = \Phi(\lambda_G(g)(f))(y, \xi) = (\lambda_G(g)(f))(y\xi) = f(g^{-1}y\xi).
\end{gathered}$$

□

20 Modules

Def 20.1. *Let N, M be von Neumann algebras. A Hilbert space H is called N-M bimodule if it is a left N module and a right M module satisfying $n(xm) = (nx)m$.*

21 example

21.1 memo

- $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{F}_2$ is not weakly amenable but has the Haagerup property and AP(broz373)
-

21.2 $M_n(\mathbb{C})$

Prop 21.1. $M_n(\mathbb{C}) = L(\mathbb{Z}/n\mathbb{Z} \rtimes \mathbb{Z}/n\mathbb{Z})$.

Proof. It follows from Takesaki, since $\mathrm{L}^\infty(\mathbb{Z}/\mathrm{n}\mathbb{Z})$ is atomic and the action is free and ergodic. □

Thm 21.1. *Let φ be a normalized trace and ψ be a functional on $M_n(\mathbb{C})$. Then, there exists positive matrix A s.t. $\psi = \varphi(A\cdot) = \langle \cdot A^{\frac{1}{2}}, A^{\frac{1}{2}} \rangle_{HS}$.*

Proof. $\varphi(e_{ij}) = a_{ij}$. □

Thm 21.2. $\triangle_\varphi = L_A R_{A^{-1}}$. $\triangle_\varphi^{it} = L_{A^{it}} R_{A^{-it}}$.
$Jx = \triangle_\varphi^{\frac{1}{2}} x^* \triangle_\varphi^{-\frac{1}{2}}$.

Prop 21.2. *Let $\mathcal{A} = M_n(\mathbb{C})$. $\alpha_t(x) = A^{it} x A^{-it}$. Then, φ is a $\{\alpha_t\}$-KMS state if and only if $\varphi(x) = \frac{\mathrm{Tr}(A^{-1}x)}{\mathrm{Tr}(A^{-1})}$.*

Proof. at KMS state. □

Thm 21.3. *Let $y \in M_n(\mathbb{C})$ be a $Jy = y$. There exists $c \in M_n(\mathbb{C})_{sa}$ s.t. $y = A^{\frac{1}{4}} c A^{-\frac{1}{4}}$. Furthermore, $\mathfrak{P} = A^{\frac{1}{4}} \mathfrak{P}_S A^{-\frac{1}{4}} = \{A^{\frac{1}{4}} x A^{-\frac{1}{4}} | x \text{ is a } self\text{-}adjoint\ positive\}$.*

Thm 21.4. *If A's eigenvalues are different, $M_n(\mathbb{C})_0^\varphi = diagonal\,operators$.*

Thm 21.5. *Let $A = M_{2^n}(\mathbb{C})$, $A = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$ and $\varphi_n := \otimes \frac{1}{1+\lambda} \mathrm{Tr}\left(\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \cdot \right)$ $(0 < \lambda < 1)$. Then, $M_0^{\varphi_n} = \oplus_{k=0}^n M_{{}_nC_k}(\mathbb{C})$.*

Prop 21.3. *Let $k_1 + k_2 + \cdots + k_m = n$, $p_j M_n(\mathbb{C}) p_j = M_{k_j}$. $E : M_n(\mathbb{C}) \to M_{k_1} \oplus \cdots \oplus M_{k_m}$ $E(a) := \sum p_{k_j} a p_{k_j}$ is a conditional expectation.*

21.3 $(C_c(G), \mu) \curvearrowright L^2(G, \mu)$

This is a left Hilbert algebra.

Ex 7. *$G = \mathbb{R}_+^* \rtimes \mathbb{R}$ is not unimodular.*

$\frac{1}{x^2} dx dy$ is a left invariant Haar measure.
$\triangle : G \to \mathbb{R}^*$ $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \mapsto |x|$ *is a modular function.*

Let $\triangle$ be a modular function.
The following lemma followed by Lebaesgue dominated theorem.

Lem 21.1. *Suppose $f \in C(G)$ and $h, f \in L^2(G, \mu)$. The following are equivalent.*

- *$h = gf$;*
- *there exists $f_n \in C_c(G)$ s.t. $f_n \to f$ and $gf_n \to gf$.*

Thm 21.6.
- *$\mathcal{D}_S := \{\xi \in L^2(G, \mu) | \triangle^{\frac{1}{2}} \xi(s) \in L^2(G, \mu)\}$ and $S\xi(s) := \overline{\xi(s^{-1})} \triangle (s)^{-1}\}$;*
- *$\mathcal{D}_{S^*} := \{\xi \in L^2(G, \mu) | \triangle^{\frac{1}{2}} \xi(s) \in L^2(G, \mu)\}$ and $S^*\xi(s) := \overline{\xi(s^{-1})}\}$;*
- *Its modular operator is a modular map;*
- *$J : \triangle^{\frac{1}{2}}(s)\xi(s) \mapsto \overline{\xi(s^{-1})} \triangle (s)^{-1}$:*

Lem 21.2. *If $f \in L^1(G)$, then $f*$ is a bounded linear operator on $B(L^2(G))$.*

Since a modular map does not have zero and continuous, so for any compact set K of G, there exists $\varepsilon > 0$ s.t. $|f(x)| \geq 0$ $(x \in K)$. Therefore,

Thm 21.7. $C_c(G) \subset \mathfrak{T}$.

$e^{-\triangle^2} \xi \in \mathfrak{T}$?

21.4 Araki-Woods factor

$\mathcal{A}_n := \otimes_{j=1}^n M_{k_j} (= M_{k_1 \cdots K_n})$.
$\mathcal{A} := \overline{\cup_n \mathcal{A}_n}$.
Let ϕ_n be a state on M. There exists $T_n \in M_{k_n}(\mathbb{C})$ s.t. $\phi_n = tr(T_n \cdot)$. We can define state $\phi^{(n)}$ on M by $\phi^{(n)} := tr((\otimes_{j=1}^n T_j \cdot))$. By $tr(T_n) = 1$, $\phi^{(n+1)}|_{\mathcal{A}_n} = \phi^{(n)}$.

Thm 21.8. *$\phi = \lim \phi^{(n)}$ is a factorial state.*

Proof. Suppose $Q \in Z(\pi_\phi(\mathcal{A})'')$. By Kaplansky density theorem, there exist $\{\mathcal{A}_n\} \subset \mathcal{A}_\infty = \cup_n \mathcal{A}_n$s.t. $||\pi_\phi(A_k)|| \leq ||Q||$and $\pi_\phi(A_k) \to^s Q$. $B_k = \int_{\mathcal{U}(\mathcal{A}_n)} U A_k U^* dU$. Then, $\pi_\phi(B_k) \to^s Q$. For each $C_1, C_2 \in \mathcal{A}_n$, $\langle \pi_\phi(C_1)x_\phi, Q\pi_\phi(C_2)x_\phi \rangle = \lim \phi(C_1^* C_2 B_k) = \langle x_\phi, Qx_\phi \rangle \langle \pi_\phi(C_1)x_\phi, \pi_\phi(C_2)x_\phi \rangle$, because $B_k \in {\mathcal{A}_n}' \cap \mathcal{A}_m = \mathbb{C}I \otimes (\otimes_{j=n+1}^m M_{k_j}(\mathbb{C}))$. □

21.5 Powers factor

When $k_j = 2$ and ϕ_n is a tensor state of $\frac{1}{1+\lambda}\mathrm{Tr}\left(\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \cdot\right)$ $(0 < \lambda < 1)$, we call this Araki-Woods factor Powers factor.
Let $A = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$.

21.6 crossed product

$x = \sum \pi(a_g)\lambda_g = \pi(a)\lambda_g$, $y = \sum \pi(b_h)\lambda_h = \pi(b)\lambda_h$.

When X is a finite measure space and, we define state τ by $\langle \cdot(1 \otimes \delta_e), 1 \otimes \delta_e \rangle$. It satisfies $\mu \circ E = \tau$.

Thm 21.9. $\triangle(\hat{x}) = \sum \pi(\widehat{\frac{d\mu}{d\mu \circ \alpha_{g^{-1}}} a_g})\lambda_g$.

Proof. $E(yx^*) = \sum \overline{a_g} b_g$. $E((\triangle x)^* y) = \sum \alpha_{g^{-1}}(a'_g b_g)$. Since $\langle Sx, Sy \rangle = \langle y, \triangle x \rangle$, $\sum \mu(\overline{a_g} b_g) = \sum \mu \circ \alpha_{g^{-1}}(a'_g b_g) = \sum \mu(\frac{d\mu \circ \alpha_{g^{-1}}}{d\mu} \overline{a_g} b_g)$. □

Let ρ be a right regular representation, $\xi_0 = 1 \otimes \delta_e$ and $x = \pi(\alpha_g(a_g))\lambda_g)$. Then, $x\xi_0 = a_g \otimes \delta_g$. Then,

$$\begin{aligned}
\triangle x\xi_0 &= \frac{d\mu \circ \alpha_g}{d\mu} a \otimes \delta_g; \\
\triangle^{it} x\xi_0 &= \left(\frac{d\mu \circ \alpha_g}{d\mu}\right)^{it} a \otimes \delta_g; \\
Jx\xi_0 &= \alpha_g \left(\left(\frac{d\mu \circ \alpha_g}{d\mu}\right)^{-\frac{1}{2}} \overline{a}\right) \otimes \delta_{g^{-1}} \\
&= \left(\frac{d\mu \circ \alpha_g}{d\mu}\right)^{\frac{1}{2}} \alpha_g(\overline{a}) \otimes \delta_{g^{-1}} =: u_g(\overline{a}) \otimes \delta_{g^{-1}}; \\
\sigma_t^\varphi(\pi(f)) = \pi(f),\ \sigma_t^\varphi(\lambda_h) &= \pi\left(\left(\frac{d\mu \circ \alpha_g}{d\mu}\right)^{it}\right)\lambda_h; \\
J\pi(f)J = \overline{f} \otimes 1,\ J\lambda_h J &= u_h \otimes \rho_h :
\end{aligned}$$

$\mu \circ \alpha_g =: g^{-1}\mu$.
canonical implementation?

Thm 21.10. $M_0^\varphi \supset L^\infty(X)$.
Especially, if for all $g \in G$, α_g is probability measure presearving, $M_0^\varphi = M$.

22 Cuntz algebra $\mathcal{O}_n$

[OP78]

Def 22.1. *For $2 \leq n \in \mathbb{N}$, Cuntz algebras $\mathcal{O}_n$ is the universal C*-algebra generated by isometries S_1, S_2, ..., S_n s.t. $\sum S_i S_i^* = 1$.*
Moreover, for $n = \infty$, Cuntz algebra $\mathcal{O}_\infty$ is the universal C-algebra generated by isometries S_1, S_2, ..., S_n s.t. $\sum S_i S_i^* \leq 1$.*

The universal C*-algebra is the disjoint union of all elements which satisfies same property.

Prop 22.1 (Universality)**.** *Let S_i be generators of $\mathcal{O}_n$ and T_i be isometries which satisfies $\sum T_i T_i^* = 1$. There exists a *-homomorphism $\phi : \mathcal{O}_n \to \mathfrak{O} := C^*(\{T_i\})$ s.t. $\phi(S_i) = T_i$.*

Proof. Let p be a noncommutative $2n$-variable polynomial. $A := p(T_1, \ldots, T_n, T_1^*, \ldots, T_n^*)$. By GNS construction, there exists a representation π s.t. $||\pi(A)|| = ||A||$. $||A|| = ||\pi(A)|| \leq ||p(S_1, \ldots, S_n, S_1^*, \ldots, S_n^*)||$. □

Rem 22.0.1. *For $t \in \mathbb{R}$, there exists $\rho_t : \mathcal{O}_n \to \mathcal{O}_n$ s.t. $\rho_t(S_i) = e^{it} S_i$.*

Thm 22.1. $K_0(\mathcal{O}_n) = \mathbb{Z}/(n-1)\mathbb{Z}$, $K_1(\mathcal{O}_n) = 0$.

Prop 22.2. *$M_{n^\infty} \cong \mathcal{F}_n \subset \mathcal{O}_n$. There exists a conditinal expectation E_n:$\mathcal{O}_n \to M_{n^\infty}$, which in obtained by an integration.*

Def 22.2. *Let $\alpha = (\alpha_1, \alpha_2, \ldots, alpha_k) \in \{1, 2, \ldots, n\}^k$. We define S^α by $S_{\alpha_1} S_{\alpha_2} \cdots S_{\alpha_k}$. $l(\alpha) := k$.*
We define $\mathcal{F}_n$ by the C-subalgebra generated by $S_\alpha S_\beta^*$'s, where $l(\alpha) = l(\beta)$.*
*We denote by $\mathcal{P}_n$ a *-algebra generated by S_i's.*

Proof. We define φ by a isomorphism $\mathcal{F}_n \to M_{n^\infty}$ s.t. $\varphi(S_\alpha S_\beta{}^*) = e_{\alpha_1\beta_1} \otimes e_{\alpha_2\beta_2} \otimes \cdots \otimes e_{\alpha_k\beta_k} \otimes 1 \otimes \cdots$ for α, β.
For $A \in \mathcal{P}_n$, there exist unique $A_k \in \mathcal{F}_n \cap \mathcal{P}_n$ s.t. $A = \sum_k S_1^{k*} A_{-k} + A_0 + \sum_k A_k S_1^k$. Let $E_0 : \mathcal{O}_n \to \mathcal{F}_n$ be a isomorphism s.t $E_0(A) = A_0$ $(A \in \mathcal{P}_n)$. Then, for $A \in \mathcal{O}_n$,

$$E_0(A) = \int_0^{2\pi} \rho_t(A) dt.$$

□

22.1 Haar measure

Let F be a p-adic field. Let μ be a left Haar measure on GL(F). $\triangle(t_1, \ldots, t_n) := |t_1|^{-n+1} |t_2|^{-n+3} \cdots |t_n|^{n-q}$.

23 boyaki

Riemann-Lebesgue ext to finite measure space?
abelian compact group?
Connes-Takesaki module

24 Subjects

- flow
- rigidity
- core

25 Problem

- Riemann-Lebesgue extension?
- Let u_t be a left regular rep. By Stone's theorem, there exists a positive self-adjpint op. A on $\mathrm{L}^2(\mathbb{R})$. What is A?
- Tomita algebra example
- various spectrum of modular op.
- Naimark's problem:C*algebra with only irreducible representation up to unitary equivalence, this is compact op. It is solved when including sep. C*-algebra and typeI C*-algebra. 1708.04368
- pure extention problem
- modular theory for abelian von Neumann algebra
- Connes' bicentralizer property:
- locally compact case $C_r^*(G) = C^*(G)$.
- locally compact case amenable radical

26 need

- e_ξ;
- semifinite factor $\otimes$;

27 comment bibliography

[OP78] There exists the exactly one KMS-state for an action $\rho_t(S_i) = e^{it}S_i$ of Cuntz algebra $\mathcal{O}_n$.
LaTeX $\flat$ $\spadesuit\heartsuit\clubsuit\diamondsuit$

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