papers

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Contents

1 Locally compact groups acting on trees, the type I conjecture and non-amenable von Neumann algebras (S.Raum and C.Houdayer)

Cyril Houdayer and Sven Raum[HR16]

Conjecture 1.0.1. *Let G be a group acting on a tree$T$. G is type I if and only if $G \curvearrowright T$ is boundary transitive.*

Rem 1.0.1. *$G \curvearrowright T$ is boundary transitive if and only if $G \curvearrowright S_n(v)$ for all $v \in V(T)$ is transitive.*

Rem 1.0.2 (A)**.** *$G \curvearrowright S_2(v)$ for all $v \in V(T)$ is transitive if and only if for any v_1, v_2, w_1, $w_2 \in V(T)$, there exists $g \in G$ s.t. $gv_i = w_i$.*

Thm 1.1. *If G do not satisfy (A), $L(G)$ is not amenable.*

This theorem is based on the Bass-Serre theory. The following lemma is the key lemma.

Lem 1.1. *Let G_i be a locallycompact group and A be a common compact subgroup. Assume $K\backslash G_1/K \geq 3$ and $K\backslash G_2/K \geq 2$. Then, $L(G_1 *_A G_2)$ is non-amenable.*

Thm 1.2 (Dykema)**.** *Let M_i be a hyperfinite von Neumann algebra. Assume $\dim M_i \geq 2$ and $\dim M_1 + \dim M_2 \geq 5$. Then, $M_1 * M_2$ is non-amenable.*

2 Some prime factorization results for type II_1 factprs(Ozawa and Popa)

3 Boudnaries of reduced free group C*-algebras (Narutaka Ozawa)

[Oza06] Let Γ be a free group with rank n $(2 \leq n < infty)$. A measure μ on $\partial\Gamma$ is called quasi-invariant if for any measurable subset $A \subset \partial\Gamma$ and any $s \in \Gamma$, one has $\mu(sA) = 0$ if and only if $\mu(A) = 0$. A measure μ on $\partial\Gamma$ is called doubly ergodic if the diagonal action of Γ on $(\partial\Gamma^2, \mu^{\otimes 2})$ is ergodic. Let a measure μ on Γ be quasi-invariant and doubly-erogodic.

Thm 3.1. *Under the above condition, If*

$$\theta : C(\partial\Gamma) \rtimes_r \Gamma) \to L^\infty(\partial\Gamma, \mu) \rtimes \Gamma$$

is a completely positive map with $\theta|_{C_r^*(\Gamma)} = \mathrm{id}_{C_r^*(\Gamma)}$, then $\theta = \mathrm{id}$.

Cor 3.1.1. *$C(\partial\Gamma) \rtimes_r \Gamma)$ sits between $C_r^*(\Gamma)$ and its injective envelop $I(C_r^*(\Gamma))$.*

Prop 3.1. *If*

$$\varphi : C(\partial\Gamma) \to L^\infty(\partial\Gamma, \mu)$$

is a unital positive Γ-equivariant map, then $\varphi = \mathrm{id}$

This proposition is key.

proof of proposition. Any Γ-equivariant Borel map from $\partial\Gamma$ to $\mathcal{M}(\partial\Gamma)$ has image in $\mathcal{M}_{\leq 2}(\partial\Gamma)$ $\mu^{\otimes 2}$-a.e.

Fix a dense Γ-invariant subalgebra $\mathcal{C}$ which is algebraically generated by a countable set. There exist a Γ-equivariant Borel map

$$\varphi_* : \partial\Gamma \ni \xi \mapsto \varphi_*^\xi \in \mathcal{M}(\partial\Gamma)$$

s.t. for μ-a.e. $\xi \in \partial\Gamma$,

$$\forall f \in \mathcal{C} \int f(\eta) d\varphi_*^\xi(\eta) = \varphi(f)(\xi).$$

We consider the Γ-equivariant Borel map

$$\partial\Gamma^2 \ni (\xi, \eta) \mapsto \varphi_*^\xi + \delta_\xi + \delta_\eta.$$

By the first argument, $\varphi_*^\xi = t\delta_\xi + (1-t)\delta_\eta$. By μ-invarinat, $\varphi_*^\xi = \delta_\xi$ a.e. So, $\varphi = \mathrm{id}$. □

proof of theorem. There exist a faithful normal conditinal expectation $E : L^\infty(\partial\Gamma, \mu) \rtimes \Gamma L^\infty(\partial\Gamma, \mu)$. We suffices to show that $E \circ \theta|_{C(\partial\Gamma)}$ is a unital positive Γ-equivariant map. We note taht $E(s.x) = s.E(x)$. □

4 Simplicity of the C*-algebra associated with the free group on two generators

[Pow75]

Thm 4.1. *For any $x \in C_r^*(\mathbb{F}_2)_{sa}$ and $\varepsilon > 0$, there exist an integer n, elements $g_i \in \mathbb{F}_2$ and $t_i > 0$ for $i = 1, \ldots, n$ with $\sum t_i = 1$ s.t.*

$$\left\| \tau(x)I - \sum_{i=1}^n t_i \lambda_{g_i} x \lambda_{g_i^{-1}} \right\| < \varepsilon$$

Thm 4.2. *$C_r^*(\mathbb{F}_2)$ is simple.*

At first, we prove theorem 4.2 from theorem 4.1

Proof. $C_r^*(\mathbb{F}_2)$ has a faithful tracial state. □

Lem 4.1. *Suppose $x = \sum_i \{\alpha_i \lambda_{g_i} + \overline{\alpha_i} \lambda_{g_i^{-1}}\} \in \mathbb{C}[\mathbb{F}_2]$, where $g_i \neq e$. Then, there exists $h_i \in \mathbb{F}_2$ s.t.*

$$\left\| \frac{1}{20} \sum_{i=1}^{20} \lambda_{h_i} x \lambda_{h_i^{-1}} \right\| < \frac{1}{20}\|x\|.$$

Proof. There exist a integer k s.t. for any i, $b^k g_i b^{-k}$ begins and ends a non-zero power of b. For $r = 1, \cdots 20$, $h_r = a^r b^k$. Let K_r be a closed subspace of $L^2(\mathbb{F}_2)$ s.t. $f(g) = 0$ unless g begins a^r. $\lambda_{h_r g_i h_r^{-1}}$ maps $K_r^\perp$ to K_r. So, $\lambda_{h_r g_i h_r^{-1}} x \lambda_{h_r g_i h_r^{-1}}$ maps $K_r^{/perp}$ to K_r. Let P_r be a orthogonal projection to K_r. Note $P_r P_s = \delta_{rs} P_r$, so they are orthoginal. Let $B = \frac{1}{20}\sum_{i=1}^{20} \lambda_{h_i} x \lambda_{h_i^{-1}}$. Suppose $f \in H$ and $\|f\| = 1$.

$$\sum_{r=1}^{20} \|P_r f\| \leq \|f\| = 1.$$

There exist a p s.t. $\|P_p f\| \leq \frac{1}{20}$.

$$\begin{aligned} |\langle f, Bf\rangle| &\leq \frac{1}{20}\sum_{r=1}^{20} |\langle f, \lambda_{h_i} x \lambda_{h_i^{-1}} f\rangle| \\ &\leq \frac{19}{20}\|x\| + \frac{1}{20}|\langle f, \lambda_{h_i} x \lambda_{h_i^{-1}} f\rangle|. \end{aligned}$$

Since $\lambda_{h_p g_i h_p^{-1}}$ maps $K_p^\perp$ to K_p,

$$\begin{aligned} |\langle f, \lambda_{h_p} x \lambda_{h_p^{-1}} f\rangle| &\leq ||x||\{\|P_p f\|^2 + 2\|P_p f\|\|(1-P_p)f\|\} \\ &\leq \frac{1}{20} + 2\frac{1}{\sqrt{20}} \cdot 1 < \frac{1}{2}. \end{aligned}$$

So, we have

$$|\langle f, Bf\rangle| \leq \frac{19}{20}\|x\| + \frac{1}{40}\|x\| = \frac{39}{40}\|x\|.$$

Since $\|B\| = \sup_{\|f\|=1}\langle f, Bf\rangle$, this completes the proof of the lemma. □

proof of theorem 4.1. This theorem follows from the previous lemma and finite dimensional approximation. □

This proof apply to the uniqueness of a tracial state.

5 A Galois Correspondence for compact Groups of Automorphisms of von Neumann Algebras with a Generalization to Kac Algebra (M.Izumi, R.Longo, S.Popa)

[ILP98]

Nakamura and Takeda [NT60], and Suzuki showed such(closed subgroup version) galois correspondence in the ccase of II_1-factor and G a finite group whose action on M is minimal, i.e. $M^{G'} \cap M = \mathbb{C}$. Kishimoto [Kis77] showed a galois ocrrespondence, between normal closed subgroups of a compact (minimal) group G and globally G-invariant intermediate von Neumann algebras, following methods in the analysis of chemical potential in Quantum Statistical Mechanics. Generalizations of this result concerning dual actions of aloccaly compact group G were dealt by Takesaki [AKTH77], in case of G abelian, and by Nakagami in more generality.

Another kind of Galis correspondence was provided by H.Choda [Cho78]. In concerns in particular the crossed product of a factor by an outer action of a discrete group and characterizes the intermediate von Neumann subakgebras that are crossed product by a discrete subgrups. An important assumption here is the existence of a normal conditional expectation onto the intermediate subalgebras.

Thm 5.1. *Let G be a discrete group and α ab outer action of G on a factor N. Then, the map*

$$H \mapsto N \rtimes_\alpha H$$

gives one-to-one correspondence between the lattice of all subgroups of G and that of all subgroups of G and that of all intermedidate sufactors of $N \subset N \rtimes_\alpha G$.

Thm 5.2. *Let G be compact group and α a minimal action of G on a factor M. The, the map*

$$H \mapsto M^H$$

gives one-to-one correspondence between the lattice of all closed subgroups of G and that of all intermediate subfactors of $M \supset M^G$.

6 Inclusions of simple C*-algebras

[Izu02]

Thm 6.1 ([Izu02], Corollary 6.6)**.** *Let α be an outer action of a finite group G on a simple σ-unital C*-algebra A. Then,*

1. *There exists a one-to-one correspondence between the set of subgroups of G and the set of intermedeiate C*-algebras between $A \rtimes_\alpha G \supset A$. Thecorrespondence is given as follows: For a subgroup H, the corresponding intermediate C*-algebra is $A \rtimes_\alpha G$.*

2. *There exists a one-to-one correspondence between the set of subgroups of g and the set of intermediate C*-subalgebras between $A \supset A^\alpha$. The correspondence is given as follows: For a subgroup H, the corresponding intermediate C*-algebra is the fixed point subalgebra of A under the restriction of α to H.*

7 On tensor poducts of von Neumann algebras

[GK96]

Thm 7.1 ([GK96], Thoerem A)**.** *If $\mathcal{M}$ is a factor, $\mathcal{I}$ is a von Neumann algebra, and $\mathcal{B}$ is a von Neumann subalgebra of $\mathcal{M}\overline{\otimes}\mathcal{I}$ that contains $\mathcal{M}\overline{\otimes}\mathbb{C}I$, then $\mathcal{B} = \mathcal{M}\overline{\otimes}\mathcal{J}$, where $\mathcal{J}$ is a von Neumann subalgebra of $\mathcal{J}$.*

8 Splitting for subalgebras of tensor products

[Zac01]

Thm 8.1 ([Zac01], Corollary 3.4)**.** *Let A be a unital C^*-algebra. Then A is simple and has property (S) iff for any unital C -algebra B and C s.t. $A \otimes 1 \subset C \subset A \otimes B$ we have $C = A \otimes B_0$ for some subalgebra $B_0 \subset B$.*

9 A criterion for splitting C*-algebras in tensor products

[Zsi00]

Thm 9.1 ([Zsi00], Theorem)**.** *Let A, D, C be unital C^*-algebras, A simple and nuclear, such that*

$$A \otimes 1_D \subset D \subset A \otimes_{\min} D.$$

Then

$$C = A \otimes_{\min} B$$

for some C^*-subalgebra $B \subset D$.

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