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Explicit interpolation formulas for the Bell triangle

Masayuki Okabe

Optopian Inc., 2-8-21 *Sakuradai, Hasuda, Saitama 349-01, Japan*

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Abstract

Explicit interpolation formulas are presented mainly using area coordinates for the Bell triangle with 18 degrees of freedom so that C^1 continuity is clearly observed. The Bell triangle is derived by imposing the P_4-admissible constraints on the Argyris triangle, by which the 4-unisolvency is proved. Variable-node expressions combining the Argyris and Bell triangles are further developed.

1. Introduction

Over a triangle K of class C^1, the most famous is the 21-d.o.f. element [1–5] which Ciarlet called the Argyris triangle [6]. It is known that the Argyris triangle is 5-unisolvent, having the unique interpolation basis with degree 5 of approximation. Explicit interpolation formulas for the Argyris triangle have been presented mainly in terms of area coordinates [7, 8].

If we impose the constraints on the Argyris triangle so that the normal slope of the relevant trial function is to be the polynomial of degree 3 on each triangular edge, then we have the 18-d.o.f. element [1, 2, 9, 10] called the Bell triangle [6]. In this paper, we first demonstrate that those constraints are valid for any polynomials below degree 4, which are termed the P_4-admissible constraints. Then the 4-unisolvency of the Bell triangle can be proved.

In practical finite element applications of the Bell triangle, matrix manipulations are inevitable since its interpolation basis is still unknown explicitly [11]. We thus aim to develop the explicit interpolation formulas for the Bell triangle by imposing the P_4-admissible constraints on the Argyris triangle. Then C^1 continuity of the Bell triangle can clearly be observed. Introducing the existence parameters [12–14], we further develop the variable-node expressions combining the Argyris and Bell triangles.

2. Trial function space with the interpolation basis

Consider an arbitrarily finite element K composed of N nodes. For node i, we assign a differential operator D_i. In Hermite interpolation, the trial function ϕ can then be written as

$$\phi = \sum_{i=1}^{N} \phi_i H_i^N , \tag{1}$$

where the discretized variable ϕ_i is given as the $D_i\phi$ value at node i,

$$\phi_i = D_i\phi(i) , \quad i = 1, \ldots, N . \tag{2}$$

Each shape function H_i^N should satisfy the so-called interpolation condition such that

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$$D_j H_i^N(j) = \delta_{ij}\,, \quad i, j = 1, \ldots, N\,. \tag{3}$$

Notice that nodes i and j can be placed at the same point so long as D_i and D_j are different ($i \neq j$).

The trial function space $F^N(K)$ is defined as

$$F^N(K) = \left\{ f\colon E^N(f) = f - \sum_{i=1}^{N} D_i f(i) H_i^N = 0 \right\}, \tag{4}$$

where f denotes an arbitrary function which is sufficiently smooth at least within the finite element K. The basis $\{H_i\}_{i=1}^N$ spans the $F^N(K)$ space which is especially called the interpolation basis.

3. Inclusive relations

We next consider another finite element K eliminating nodes k_q ($q = 1, \ldots, K$) but without changing the original element geometry. Let the interpolation basis be expressed as $\{H_i^{N-K}\}$. Assuming D_i ($i \neq k_q$, $q = 1, \ldots, K$) is invariant, the trial function can then be written as

$$\phi = \sum_{\substack{i \neq k_q \\ 1 \leq q \leq K}} \phi_i H_i^{N-K}\,, \tag{5}$$

and the corresponding trial function space $F^{N-K}(K)$ is given by

$$F^{N-K}(K) = \left\{ f\colon E^{N-K}(f) = f - \sum_{\substack{i \neq k_q \\ 1 \leq q \leq K}} D_i f(i) H_i^{N-K} = 0 \right\}. \tag{6}$$

We suppose that the elimination of node k_q is based on the constraint by

$$C_q(\phi) = \phi_{k_q} - \sum_{\substack{i \neq k_t \\ 1 \leq t \leq K}} \phi_i a_i^q = 0\,, \quad q = 1, \ldots, K\,, \tag{7}$$

where a_i^q designates an appropriate constant.

Evidently the $F^N(K)$ space of (4) includes the $F^{N-K}(K)$ space of (6):

$$F^N(K) \supset F^{N-K}(K)\,. \tag{8}$$

LEMMA 3.1. The constraints C_q of (7) are related to E^N and E^{N-K} in (4) and (6) by

$$E^{N-K}(\phi) = E^N(\phi) + \sum_{q=1}^{K} H_{k_q}^N C_q(\phi)\,. \tag{9}$$

THEOREM 3.2 (In-family relevancy). The interpolation bases $\{H_i^N\}$ and $\{H_i^{N-K}\}$ are related by

$$H_i^{N-K} = H_i^N + \sum_{q=1}^{K} a_i^q H_{k_q}^N\,, \quad i \neq k_q\,. \tag{10}$$

Here the constant a_i^q is given by

$$a_i^q = D_{k_q} H_i^{N-K}(k_q)\,, \quad i \neq k_q\,,\ q = 1, \ldots, K\,. \tag{11}$$

PROOF. After imposition of the constraint C_q of (7), the trial function ϕ of (1) should be identified as that of (5), which immediately gives (10).

At node k_q, (5) can further be written as

$$\phi_{k_q} = \sum_{\substack{i \neq k_t \\ 1 \leq t \leq K}} \phi_i D_{k_q} H_i^{N-K}(k_q)\,, \quad q = 1, \ldots, K\,. \tag{12}$$

The identity of (7) and (12) thus gives (11). □

REMARK 3.3. Noting the interpolation condition (3) related to $\{H_i^N\}$, the in-family relevancy of (10) and (11) assures that the interpolation condition associated with $\{H_i^{N-K}\}$ holds such that

$$D_j H_i^{N-K}(j) = \delta_{ij}\,, \quad i, j \neq k_q\,, \quad q = 1, \ldots, K\,. \tag{13}$$

For further details of the in-family relevancy, see [13].

Let $P_m(K)$ be the polynomial space of degree m over K. If and only if the following relations hold:

$$C_q(f) = D_{k_q} f(k_q) - \sum_{\substack{i \neq k_t \\ 1 \leq t \leq K}} D_i f(i) a_i^q = 0 \quad \forall f \in P_m(K)\,,$$

then we call C_q *the* P_m*-admissible constraint.*

THEOREM 3.4. Suppose that the $F^N(K)$ *space of* (4) *has the degree* m *of approximation*

$$F^N(K) \supset P_m(K)\,. \tag{14}$$

If all the constraints C_q *of* (7) *are* P_m*-admissible* ($q = 1, \ldots, K$), *then the* $F^{N-K}(K)$ *space of* (6) *also has the degree* m *of approximation,*

$$F^{N-K}(K) \supset P_m(K)\,. \tag{15}$$

PROOF. Let $f \in P_m(K)$. Then assumption (14) can be written as $E^N(f) = 0$. Lemma 3.1 thus guarantees that $E^{N-K}(f) = 0$. $\square$

4. Details of the Agryris triangle

The finite element we consider is a triangle K (Fig. 1) with vertices 1, 2 and 3 numbered counterclockwise in the Cartesian system (x, y), twice the area of which is given by

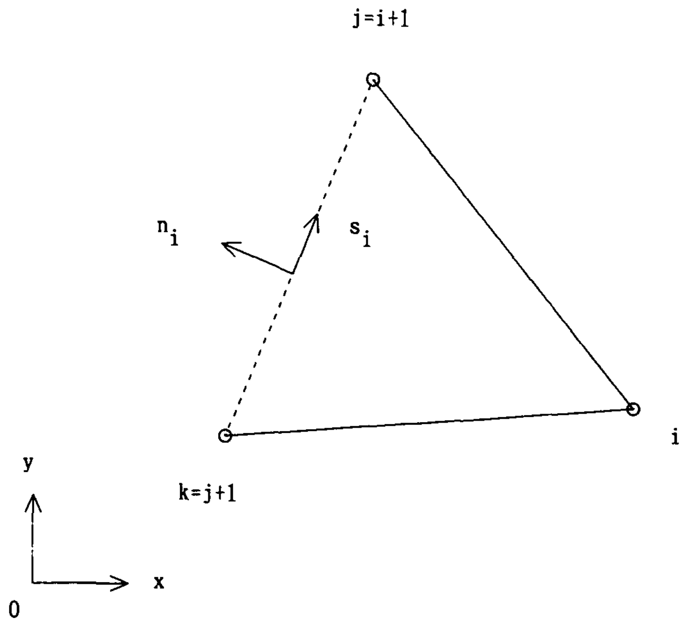

Fig. 1. Triangle K in the Cartesian system (x, y) and the surface Cartesian system (s_i, n_i) on the edge of $\omega_i = 0$.

$$A = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} \tag{16}$$

Here x_i and y_i denote the coordinates of vertex i, and $|\cdot|$ designates the determinant.

We use suffix i with $j = i + 1$ and $k = j + 1$ with modulo 3 ($i = 1, \ldots, 3$). Then the area coordinate ω_i related to vertex i is written as

$$\omega_i = \frac{1}{A} \begin{vmatrix} 1 & x & y \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{vmatrix}, \quad i = 1, \ldots, 3. \tag{17}$$

We denote the unit outward normal by n_i on the edge of $\omega_i = 0$, and $D_{n_i} = \partial/\partial n_i$ designates the differentiation in the n_i direction. The tangential direction s_i along the edge of $\omega_i = 0$ is selected so that s_i and n_i compose a Cartesian system (Fig. 1). The length of the edge of $\omega_i = 0$ is denoted by L_i:

$$L_i = \{(x_j - x_k)^2 + (y_j - y_k)^2\}^{1/2}, \quad i = 1, \ldots, 3. \tag{18}$$

We further prepare the constants e_{ij} and e_{ik} by

$$e_{il} = D_{n_l}\omega_i / D_{n_l}\omega_l, \quad l = j, k, \ i = 1, \ldots, 3. \tag{19}$$

Let $\phi(x, y)$ be the trial function over the triangle K, and let $D^{m,n} = \partial^{m+n}/\partial^m x\, \partial^n y$ with non-negative integers m and n. In the Argyris triangle, six discretized variables are then assigned for vertex i by

$$\phi_i^{m,n} = D^{m,n}\phi(i), \quad 0 \leqq m + n \leqq 2, \ i = 1, \ldots, 3. \tag{20}$$

Another variable is assigned for the midpoint of the edge of $\omega_i = 0$ such that

$$\phi_{jk}^{n_i} = D_{n_i}\phi(jk), \quad i = 1, \ldots, 3. \tag{21}$$

Here $D_{n_i}\phi(jk)$ denotes the $D_{n_i}\phi$ value at $((x_j + x_k)/2, (y_j + y_k)/2)$.

Thus, the trial function of the Argyris triangle is written as

$$E^{21}(\phi) = \phi - \sum_{i=1}^{3} \left\{ \sum_{m=0}^{2} \sum_{n=0}^{2-m} \phi_i^{m,n} H_i^{m,n} + \phi_{jk}^{n_i} H_{jk}^{n_i} \right\} = 0, \tag{22}$$

where $H_i^{m,n}$ and $H_{jk}^{n_i}$ compose the interpolation basis. For configuration of the Argyris triangle, see Fig. 2.

The trial function space $F^{21}(K)$ is defined by

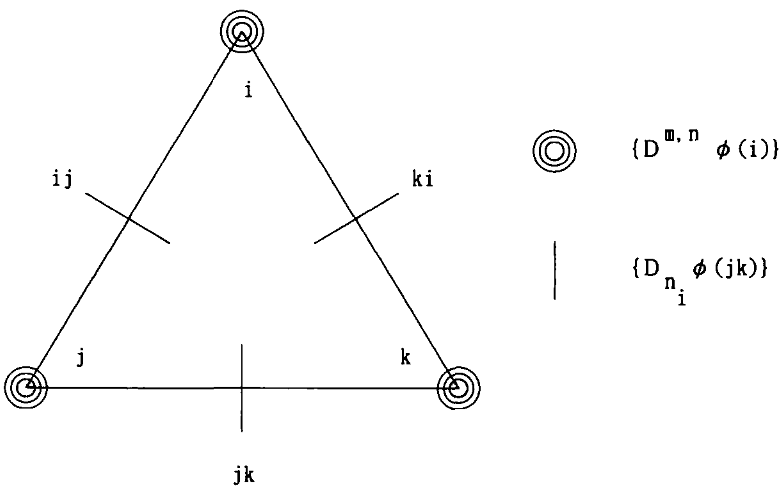

Fig. 2. Configuration of the Argyris triangle.

$$F^{21}(K) = P_5(K) = \{f \in P_5(K)\colon E^{21}(f) = 0\} . \tag{23}$$

THEOREM 4.1. The interpolation basis of the Argyris triangle is given explicitly by

$$H_i^{0,0} = \omega_i^3(10 - 15\omega_i + 6\omega_i^2) - 30(e_{ij}\omega_k + e_{ik}\omega_j)\omega_i^2\omega_j\omega_k ,$$

$$H_i^{1,0} = \omega_i^3(4 - 3\omega_i)(x - x_i) + \{7(x_i - x_k)e_{ij} - 5(x_j - x_k)\}\omega_i^2\omega_j\omega_k^2 + \{7(x_i - x_j)e_{ik} - 5(x_k - x_j)\}\omega_i^2\omega_j^2\omega_k ,$$

$$H_i^{0,1} = \omega_i^3(4 - 3\omega_i)(y - y_i) + \{7(y_i - y_k)e_{ij} - 5(y_j - y_k)\}\omega_i^2\omega_j\omega_k^2 + \{7(y_i - y_j)e_{ik} - 5(y_k - y_j)\}\omega_i^2\omega_j^2\omega_k$$

$$H_i^{2,0} = \omega_i^3(x - x_i)^2/2 - (x_i - x_k)\{(x_i - x_k)e_{ij} - 2(x_j - x_k)\}\omega_i^2\omega_j\omega_k^2/2$$
$$- (x_i - x_j)\{(x_i - x_j)e_{ik} - 2(x_k - x_j)\}\omega_i^2\omega_j^2\omega_k/2 ,$$

$$H_i^{1,1} = \omega_i^3(x - x_i)(y - y_i) - [(y_i - y_k)\{(x_i - x_k)e_{ij} - 2(x_j - x_k)\}$$
$$+ (x_i - x_k)\{(y_i - y_k)e_{ij} - 2(y_j - y_k)\}]\omega_i^2\omega_j\omega_k^2/2$$
$$- [(y_i - y_j)\{(x_i - x_j)e_{ik} - 2(x_k - x_j)\} + (x_i - x_j)\{(y_i - y_j)e_{ik} - 2(y_k - y_j)\}]\omega_i^2\omega_j^2\omega_k/2 ,$$

$$H_i^{0,2} = \omega_i^3(y - y_i)^2/2 - (y_i - y_k)\{(y_i - y_k)e_{ij} - 2(y_j - y_k)\}\omega_i^2\omega_j\omega_k^2/2$$
$$- (y_i - y_j)\{(y_i - y_j)e_{ik} - 2(y_k - y_j)\}\omega_i^2\omega_j^2\omega_k/2 ,$$

$$H_{jk}^{n_i} = 16\omega_i\omega_j^2\omega_k^2/D_{n_i}\omega_i , \quad i = 1, \ldots, 3 . \tag{24}$$

THEOREM 4.2. The first derivatives of shape functions of the Argyris triangle in the n_i direction are given on the edge of $\omega_i = 0$ by

$$D_{n_i}H_j^{0,0} = 0 ,$$
$$D_{n_i}H_j^{1,0} = \tilde\omega_j^2(1 - 2\tilde\omega_j)(5 - 4\tilde\omega_j)(y_j - y_k)/L_i ,$$
$$D_{n_i}H_j^{0,1} = -\tilde\omega_j^2(1 - 2\tilde\omega_j)(5 - 4\tilde\omega_j)(x_j - x_k)/L_i ,$$
$$D_{n_i}H_j^{2,0} = -\tilde\omega_j^2(1 - \tilde\omega_j)(1 - 2\tilde\omega_j)(x_j - x_k)(y_j - y_k)/L_i ,$$
$$D_{n_i}H_j^{1,1} = \tilde\omega_j^2(1 - \tilde\omega_j)(1 - 2\tilde\omega_j)\{(x_j - x_k)^2 - (y_j - y_k)^2\}/L_i ,$$
$$D_{n_i}H_j^{0,2} = \tilde\omega_j^2(1 - \tilde\omega_j)(1 - 2\tilde\omega_j)(x_j - x_k)(y_j - y_k)/L_i ,$$

$$D_{n_i}H_k^{0,0} = 0 ,$$
$$D_{n_i}H_k^{1,0} = \tilde\omega_k^2(1 - 2\tilde\omega_k)(5 - 4\tilde\omega_k)(y_j - y_k)/L_i ,$$
$$D_{n_i}H_k^{0,1} = -\tilde\omega_k^2(1 - 2\tilde\omega_k)(5 - 4\tilde\omega_k)(x_j - x_k)/L_i ,$$
$$D_{n_i}H_k^{2,0} = \tilde\omega_k^2(1 - \tilde\omega_k)(1 - 2\tilde\omega_k)(x_j - x_k)(y_j - y_k)/L_i ,$$
$$D_{n_i}H_k^{1,1} = -\tilde\omega_k^2(1 - \tilde\omega_k)(1 - 2\tilde\omega_k)\{(x_j - x_k)^2 - (y_j - y_k)^2\}/L_i ,$$
$$D_{n_i}H_k^{0,2} = -\tilde\omega_k^2(1 - \tilde\omega_k)(1 - 2\tilde\omega_k)(x_j - x_k)(y_j - y_k)/L_i ,$$

$$D_{n_i}H_{jk}^{n_i} = 16\tilde\omega_j^2\tilde\omega_k^2 , \qquad i = 1, \ldots, 3 , \tag{25}$$

$$D_{n_i}H_i^{m,n} = 0 , \qquad 0 \leqq m + n \leqq 2 ,\ i = 1, \ldots, 3 , \tag{26}$$

$$D_{n_i}H_{ki}^{n_j} = D_{n_i}H_{ij}^{n_k} = 0 , \quad i = 1, \ldots, 3 . \tag{27}$$

Here $\tilde\omega_j$ and $\tilde\omega_k$ denote the length coordinates on the edge of $\omega_i = 0$.

REMARK 4.3. Theorems 4.1 and 4.2 ensure that the Argyris triangle is of class C^1. For derivation of these explicit interpolation formulas, see [8].

5. Bell triangle due to P_4-admissible constraints

We aim to eliminate $\phi_{jk}^{n_i}$ so that the normal slope on the edge of $\omega_i = 0$ should be reduced to the polynomial of degree 3. Noting Theorem 4.2, the constraint C_i imposed on the Argyris triangle can then be written explicitly as

$$\begin{aligned} C_i(\phi) = \phi_{jk}^{n_i} &+ [4(y_j - y_k)(\phi_j^{1,0} + \phi_k^{1,0}) - 4(x_j - x_k)(\phi_j^{0,1} + \phi_k^{0,1}) \\ &- (x_j - x_k)(y_j - y_k)(\phi_j^{2,0} - \phi_k^{2,0} - \phi_j^{0,2} + \phi_k^{0,2}) \\ &+ \{(x_j - x_k)^2 - (y_j - y_k)^2\}(\phi_j^{1,1} - \phi_k^{1,1})]/8L_i = 0\,, \quad i = 1, \ldots, 3\,. \end{aligned} \tag{28}$$

THEOREM 5.1. The constraints C_i of (28) are P_4-admissible ($i = 1, \ldots, 3$) so that $C_i(f) = 0$ $\forall f \in P_4(K)$.

Imposition of constraints C_i ($i = 1, \ldots, 3$) on the Argyris triangle gives a 18-d.o.f. element which is called the Bell triangle [6], the trial function space of which is denoted by $F^{18}(K)$. For configuration of the Bell triangle, see Fig. 3.

THEOREM 5.2. The Bell triangle has degree 4 of approximation

$$F^{18}(K) \supset P_4(K)\,. \tag{29}$$

PROOF. By definition, $F^{21}(K) \supset P_4(K)$. Then Theorems 5.1 and 3.4 complete the proof. □

6. Explicit interpolation formulas for the Bell triangle

We construct here the trial function ϕ of the Bell triangle such that

$$E^{18}(\phi) = \phi - \sum_{i=1}^{3} \sum_{m=0}^{2} \sum_{n=0}^{2-m} \phi_i^{m,n} H_i^{m,n} = 0\,. \tag{30}$$

THEOREM 6.1. The interpolation basis of the Bell triangle is given explicitly by

$$H_i^{0,0} = \omega_i^3(10 - 15\omega_i + 6\omega_i^2) - 30(e_{ij}\omega_k + e_{ik}\omega_j)\omega_i^2\omega_j\omega_k\,,$$

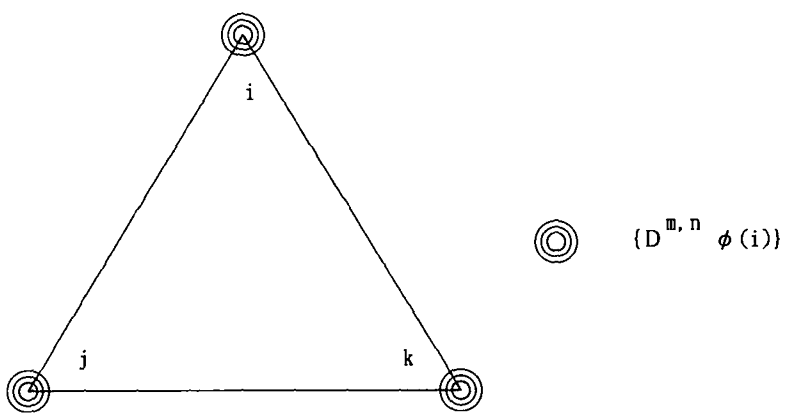

Fig. 3. Configuration of the Bell triangle.

$$H_i^{1,0} = \omega_i^3(4-3\omega_i)(x-x_i) + \{15(x_i-x_k)e_{ij} + 3(x_j-x_k)\}\omega_i^2\omega_j\omega_k^2$$
$$+ \{15(x_i-x_j)e_{ik} + 3(x_k-x_j)\}\omega_i^2\omega_j^2\omega_k\,,$$

$$H_i^{0,1} = \omega_i^3(4-3\omega_i)(y-y_i) + \{15(y_i-y_k)e_{ij} + 3(y_j-y_k)\}\omega_i^2\omega_j\omega_k^2$$
$$+ \{15(y_i-y_j)e_{ik} + 3(y_k-y_j)\}\omega_i^2\omega_j^2\omega_k\,,$$

$$H_i^{2,0} = \omega_i^3(x-x_i)^2/2 - (x_i-x_k)\{5(x_i-x_k)e_{ij} + 2(x_j-x_k)\}\omega_i^2\omega_j\omega_k^2/2$$
$$- (x_i-x_j)\{5(x_i-x_j)e_{ik} + 2(x_k-x_j)\}\omega_i^2\omega_j^2\omega_k/2\,,$$

$$H_i^{1,1} = \omega_i^3(x-x_i)(y-y_i) - [(y_i-y_k)\{5(x_i-x_k)e_{ij} + 2(x_j-x_k)\}$$
$$+ (x_i-x_k)\{5(y_i-y_k)e_{ij} + 2(y_j-y_k)\}]\omega_i^2\omega_j\omega_k^2/2$$
$$- [(y_i-y_j)\{5(x_i-x_j)e_{ik} + 2(x_k-x_j)\} + (x_i-x_j)\{5(y_i-y_j)e_{ik} + 2(y_k-y_j)\}]\omega_i^2\omega_j^2\omega_k/2\,,$$

$$H_i^{0,2} = \omega_i^3(y-y_i)^2/2 - (y_i-y_k)\{5(y_i-y_k)e_{ij} + 2(y_j-y_k)\}\omega_i^2\omega_j\omega_k^2/2$$
$$- (y_i-y_j)\{5(y_i-y_j)e_{ik} + 2(y_k-y_j)\}\omega_i^2\omega_j^2\omega_k/2\,, \quad i=1,\ldots,3\,. \tag{31}$$

THEOREM 6.2. The first derivatives of shape functions of the Bell triangle in the n_i direction are given on the edge of $\omega_i = 0$ by

$$D_{n_i}H_j^{0,0} = 0\,,$$
$$D_{n_i}H_j^{1,0} = -\,\tilde{\omega}_j^2(3-2\tilde{\omega}_j)(y_j-y_k)/L_i\,,$$
$$D_{n_i}H_j^{0,1} = \tilde{\omega}_j^2(3-2\tilde{\omega}_j)(x_j-x_k)/L_i\,,$$
$$D_{n_i}H_j^{2,0} = \tilde{\omega}_j^2(1-\tilde{\omega}_j)(x_j-x_k)(y_j-y_k)/L_i\,,$$
$$D_{n_i}H_j^{1,1} = -\,\tilde{\omega}_j^2(1-\tilde{\omega}_j)\{(x_j-x_k)^2 - (y_j-y_k)^2\}/L_i\,,$$
$$D_{n_i}H_j^{0,2} = -\,\tilde{\omega}_j^2(1-\tilde{\omega}_j)(x_j-x_k)(y_j-y_k)/L_i\,,$$

$$D_{n_i}H_k^{0,0} = 0\,,$$
$$D_{n_i}H_k^{1,0} = -\,\tilde{\omega}_k^2(3-2\tilde{\omega}_k)(y_j-y_k)/L_i\,,$$
$$D_{n_i}H_k^{0,1} = \tilde{\omega}_k^2(3-2\tilde{\omega}_k)(x_j-x_k)/L_i\,,$$
$$D_{n_i}H_k^{2,0} = -\,\tilde{\omega}_k^2(1-\tilde{\omega}_k)(x_j-x_k)(y_j-y_k)/L_i\,,$$
$$D_{n_i}H_k^{1,1} = \tilde{\omega}_k^2(1-\tilde{\omega}_k)\{(x_j-x_k)^2 - (y_j-y_k)^2\}/L_i\,,$$
$$D_{n_i}H_k^{0,2} = \tilde{\omega}_k^2(1-\tilde{\omega}_k)(x_j-x_k)(y_j-y_k)/L_i\,, \quad i=1,\ldots 3\,, \tag{32}$$

$$D_{n_i}H_i^{m,n} = 0\,, \quad 0 \leqq m+n \leqq 2\,, \quad i=1,\ldots,3\,. \tag{33}$$

REMARK 6.3. For one-dimensional interpolations corresponding to Theorem 6.2, see Appendix A.

THEOREM 6.4. The tangential derivatives of shape functions of the Bell triangle on the edge of $\omega_i = 0$ are given by

$$
\begin{aligned}
&D_{s_i}H_j^{0,0} = 30\tilde{\omega}_j^2(1-\tilde{\omega}_j)^2/L_i\,,\\
&D_{s_i}H_j^{1,0} = -\tilde{\omega}_j^2(2-3\tilde{\omega}_j)(6-5\tilde{\omega}_j)(x_j-x_k)/L_i\,,\\
&D_{s_i}H_j^{0,1} = -\tilde{\omega}_j^2(2-3\tilde{\omega}_j)(6-5\tilde{\omega}_j)(y_j-y_k)/L_i\,,\\
&D_{s_i}H_j^{2,0} = \tilde{\omega}_j^2(1-\tilde{\omega}_j)(3-5\tilde{\omega}_j)(x_j-x_k)^2/2L_i\,,\\
&D_{s_i}H_j^{1,1} = \tilde{\omega}_j^2(1-\tilde{\omega}_j)(3-5\tilde{\omega}_j)(x_j-x_k)(y_j-y_k)/L_i\,,\\
&D_{s_i}H_j^{0,2} = \tilde{\omega}_j^2(1-\tilde{\omega}_j)(3-5\tilde{\omega}_j)(y_j-y_k)^2/2L_i\,,\\
&D_{s_i}H_k^{0,0} = -30\tilde{\omega}_k^2(1-\tilde{\omega}_k)^2/L_i\,,\\
&D_{s_i}H_k^{1,0} = -\tilde{\omega}_k^2(2-3\tilde{\omega}_k)(6-5\tilde{\omega}_k)(x_j-x_k)/L_i\,,\\
&D_{s_i}H_k^{0,1} = -\tilde{\omega}_k^2(2-3\tilde{\omega}_k)(6-5\tilde{\omega}_k)(y_j-y_k)/L_i\,,\\
&D_{s_i}H_k^{2,0} = -\tilde{\omega}_k^2(1-\tilde{\omega}_k)(3-5\tilde{\omega}_k)(x_j-x_k)^2/2L_i\,,\\
&D_{s_i}H_k^{1,1} = -\tilde{\omega}_k^2(1-\tilde{\omega}_k)(3-5\tilde{\omega}_k)(x_j-x_k)(y_j-y_k)/L_i\,,\\
&D_{s_i}H_k^{0,2} = -\tilde{\omega}_k^2(1-\tilde{\omega}_k)(3-5\tilde{\omega}_k)(y_j-y_k)^2/2L_i\,,\quad i=1,\ldots,3\,,
\end{aligned}
\tag{34}
$$

$$D_{s_i}H_i^{m,n} = 0\,,\quad 0 \leqq m+n \leqq 2\,,\quad i=1,\ldots,3\,. \tag{35}$$

REMARK 6.5. It is now clear in Theorems 6.1, 6.2 and 6.4 that the Bell triangle is of class C^1. For derivation of these theorems, the relations in Appendix B should be used.

REMARK 6.6. Since

$$D_{s_i}H_{jk}^{n_i} = D_{s_i}H_{ki}^{n_j} = D_{s_i}H_{ij}^{n_k} = 0\,,\quad i=1,\ldots,3\,, \tag{36}$$

the tangential derivative of the trial function of the Argyris triangle can be identified with that of the Bell triangle due to Theorem 6.3. Here $H_{jk}^{n_i}$ is given by (24).

REMARK 6.7. Theorem 5.2 can also be proved by examining

$$E^{18}(f) = 0 \quad \forall f \in P_4(K)\,. \tag{37}$$

It is rather complicated to prove (37) in comparison to the proof of Theorem 5.1.

7. Variable-node expressions combining the Argyris and Bell triangles

Let κ_{n_i} be the existence parameter [12–14] by

$$\kappa_{n_i} = \begin{cases} 1\,, & \text{if } \phi_{jk}^{n_i} \text{ exists}\,,\\ 0\,, & \text{if } \phi_{jk}^{n_i} \text{ disappears}\,,\end{cases} \qquad i=1,\ldots,3\,. \tag{38}$$

Then as realized in Lagrange interpolations [12–15], the Argyris and Bell triangles can be expressed uniformly.

THEOREM 7.1. Variable-node expressions combining the Argyris and Bell triangles are given by

$$
\begin{aligned}
H_i^{0,0} &= \omega_i^3(10-15\omega_i+6\omega_i^2) - 30(e_{ij}\omega_k + e_{ik}\omega_j)\omega_i^2\omega_j\omega_k\,,\\
H_i^{1,0} &= \omega_i^3(4-3\omega_i)(x-x_i) + \{(15-8\kappa_{n_j})(x_i-x_k)e_{ij} + (3-8\kappa_{n_j})(x_j-x_k)\}\omega_i^2\omega_j\omega_k^2\\
&\quad + \{(15-8\kappa_{n_k})(x_i-x_j)e_{ik} + (3-8\kappa_{n_k})(x_k-x_j)\}\omega_i^2\omega_j^2\omega_k\,,
\end{aligned}
$$

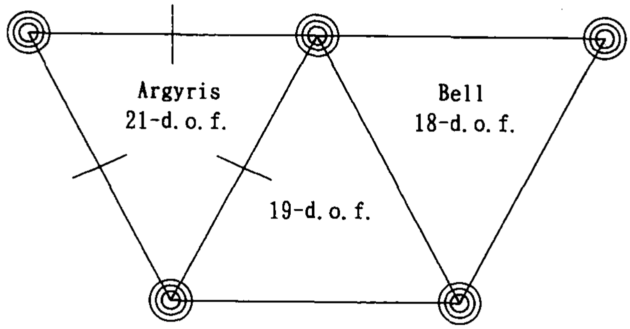

Fig. 4. Connection of the Argyris triangle with the Bell triangle.

$$
\begin{aligned}
H_i^{0,1} &= \omega_i^3(4-3\omega_i)(y-y_i) + \{(15-8\kappa_{n_j})(y_i-y_k)e_{ij} + (3-8\kappa_{n_j})(y_j-y_k)\}\omega_i^2\omega_j\omega_k^2 \\
&\quad + \{(15-8\kappa_{n_k})(y_i-y_j)e_{ik} + (3-8\kappa_{n_k})(y_k-y_j)\}\omega_i^2\omega_j^2\omega_k\,, \\
H_i^{2,0} &= \omega_i^3(x-x_i)^2/2 - (x_i-x_k)\{(5-4\kappa_{n_j})(x_i-x_k)e_{ij} + (2-4\kappa_{n_j})(x_j-x_k)\}\omega_i^2\omega_j\omega_k^2/2 \\
&\quad - (x_i-x_j)\{(5-4\kappa_{n_k})(x_i-x_j)e_{ik} + (2-4\kappa_{n_k})(x_k-x_j)\}\omega_i^2\omega_j^2\omega_k/2\,, \\
H_i^{1,1} &= \omega_i^3(x-x_i)(y-y_i) - [(y_i-y_k)\{(5-4\kappa_{n_j})(x_i-x_k)e_{ij} + (2-4\kappa_{n_j})(x_j-x_k)\} \\
&\qquad\qquad + (x_i-x_k)\{(5-4\kappa_{n_j})(y_i-y_k)e_{ij} + (2-4\kappa_{n_j})(y_j-y_k)\}]\omega_i^2\omega_j\omega_k^2/2 \\
&\quad - [(y_i-y_j)\{(5-4\kappa_{n_k})(x_i-x_k)e_{ik} + (2-4\kappa_{n_k})(x_k-x_j)\} \\
&\qquad + (x_i-x_j)\{(5-4\kappa_{n_k})(y_i-y_j)e_{ik} + (2-4\kappa_{n_k})(y_k-y_j)\}]\omega_i^2\omega_j^2\omega_k/2\,, \\
H_i^{0,2} &= \omega_i^3(y-y_i)^2/2 - (y_i-y_k)\{(5-4\kappa_{n_j})(y_i-y_k)e_{ij} + (2-4\kappa_{n_j})(y_j-y_k)\}\omega_i^2\omega_j\omega_k^2/2 \\
&\quad - (y_i-y_j)\{(5-4\kappa_{n_k})(y_i-y_j)e_{ik} + (2-4\kappa_{n_k})(y_k-y_j)\}\omega_i^2\omega_j^2\omega_k/2\,, \\
H_{jk}^{n_i} &= 16\kappa_{n_i}\omega_i\omega_j^2\omega_k^2/D_{n_i}\omega_i\,, \quad i=1,\ldots,3\,.
\end{aligned} \tag{39}
$$

REMARK 7.2. The interpolation bases in variable-node expressions of (39) cover 18- to 21-d.o.f. triangles. The Argyris triangle can thus be connected C^1 conformably with the Bell triangle through the transitive triangle with 19 or 20 degrees of freedom as shown schematically in Fig. 4.

REMARK 7.3. In Theorems 4.1, 6.1 and 7.1, coordinates x and y can be given in terms of area coordinates by using

$$
\begin{aligned}
x - x_i &= (x_j - x_i)\omega_j + (x_k - x_i)\omega_k\,, \\
y - y_i &= (y_j - y_i)\omega_j + (y_k - y_i)\omega_k\,, \quad i=1,\ldots,3\,.
\end{aligned} \tag{40}
$$

8. Concluding remarks

Mainly in terms of area coordinates, we have presented the interpolation basis of the Bell triangle full-explicitly, by which C^1 continuity can clearly be observed. In practical procedures, we can now utilize the finite element matrices in closed form.

The Bell triangle is derived by imposing some constraints on the Argyris triangle in order to make the normal slope on each triangular edge one order lower. It is demonstrated that those constraints are P_4-admissible and consequently the Bell triangle is 4-unisolvent.

There exist the transitive elements with 19 and 20 degrees of freedom which can be connected C^1

conformably with the Argyris triangle and/or the Bell triangle. Introducing existence parameters for midpoint variables on edges, we then give the interpolation bases covering these four independent triangles in variable-node expressions.

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Appendix A: Some one-dimensional interpolation polynomials

Consider the bar element K of Fig. 5 in the x system composed of vertices j and k at x_j and x_k, respectively. We define the length coordinates $\tilde{\omega}_j$ and $\tilde{\omega}_k$ by

$$\tilde{\omega}_j = (x_k - x)/L\ , \qquad \tilde{\omega}_k = (x - x_j)/L\ , \tag{A.1}$$

where

$$L = x_k - x_j\ . \tag{A.2}$$

Evidently we have

$$\tilde{\omega}_l(x_m) = \delta_{lm}\ , \quad l, m = j, k\ . \tag{A.3}$$

Over K we introduce the trial function $\phi(x)$ expressed as

$$E(\phi) = \phi(x) - \sum_{n=0}^{l} \{\phi_j^n H_j^n(x) + \phi_k^n H_k^n(x)\} = 0\ . \tag{A.4}$$

Here

$$\phi_l^n = D^n \phi(x_l)\ , \quad l = j, k, \tag{A.5}$$

where $D = \mathrm{d}/\mathrm{d}x$.

Let the trial function space P_K be given by

$$P_K = \{f \in P_3(K) \colon E(f) = 0\}\ . \tag{A.6}$$

Then the shape functions in (A.4) can be written explicitly as

$$\begin{aligned} H_j^0 &= \tilde{\omega}_j^2(3 - 2\tilde{\omega}_j)\ , \qquad H_j^l = \tilde{\omega}_j^2(1 - \tilde{\omega}_j)L\ ,\\ H_k^0 &= \tilde{\omega}_k^2(3 - 2\tilde{\omega}_k)\ , \qquad H_k^l = -\tilde{\omega}_k^2(1 - \tilde{\omega}_k)L\ . \end{aligned} \tag{A.7}$$

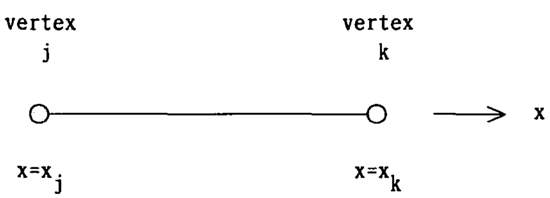

Fig. 5. Bar element K with two points.

Appendix B: Differential relations

For the triangle K, the following differential relations hold;

$$
\begin{aligned}
&D_{n_i}x = -D_{s_i}y = -(y_j - y_k)/L_i\,,\\
&D_{n_i}y = D_{s_i}x = (x_j - x_k)/L_i\,,\\
&D_{n_i}\omega_i = -L_i/A\,,\\
&D_{s_i}\omega_i = 0\,,\\
&D_{s_i}\omega_j = -D_{s_i}\omega_k = 1/L_i\,, \quad i = 1, \ldots, 3\,.
\end{aligned}
\tag{B.1}
$$

Furthermore, we have

$$
\begin{aligned}
&\{(x_i - x_k) + (x_j - x_k)e_{ji}\}L_i^2 = (y_j - y_k)A\,,\\
&\{(y_i - y_k) + (y_j - y_k)e_{ji}\}L_i^2 = -(x_j - x_k)A\,,\\
&\{(x_i - x_j) + (x_k - x_j)e_{ki}\}L_i^2 = (y_j - y_k)A\,,\\
&\{(y_i - y_j) + (y_k - y_j)e_{ki}\}L_i^2 = -(x_j - x_k)A\,, \quad i = 1, \ldots, 3\,.
\end{aligned}
\tag{B.2}
$$

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