The Variational Formulation

(Poisson Equation)

Lecture 09

September 24, 2013

Variational Formulation

Here we discuss how PDEs can be reformulated in terms of variational problems. For this recall Linear Algebra.
Consider the equation:

$$Ax = b, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$ are given matrix and vector and $x \in \mathbb{R}^n$ is the unknown
Assume also that A is SPD, i.e. A is symmetric and $\exists\ \lambda > 0$, s.t. $\forall y \in \mathbb{R}^n$: $Ay \cdot y \geq \lambda |y|^2$.

Variational Formulation (cont.)

Consider a function

$$Q(x) = \frac{1}{2}(Ax \cdot x) - (b \cdot x), \quad x \in \mathbb{R}^n. \tag{2}$$

Note that

$$Q(x) \geq \lambda |x|^2 - |b||x|,$$

hence, $Q(x) \to +\infty$ as $|x| \to +\infty$ (i.e. $Q(x)$ is weakly coercive)
Thus, $Q(x)$ is bounded from below and attains its minimum at some point $x^* = (x_1^*, \ldots, x_n^*) \in \mathbb{R}^n$. Then the minimizer x^* solves

$$\sum_{j=1}^{n} A_{ij} x_j^* = b_i, \quad i = 1, \ldots, n.$$

Therefore, solving equation (1) is equivalent to finding the minimizer of the function (2).

Variational Formulation (cont.)

This idea can be generalized to many PDEs, and in particular, to the Laplace equation. Define the energy functional:

$$I[v] = \int_\Omega \left(\frac{1}{2} |\nabla v|^2 - vf \right) \, dx,$$

and the class of admissible functions:

$$V = \left\{ v \in C^2(\bar{\Omega}) : \; v = g \text{ on } \partial\Omega \right\}.$$

Theorem

A function $u \in C^2(\bar{\Omega})$ solves the BVP

$$\begin{cases} -\triangle u &= f \quad \text{in } \; \Omega \\ \quad u &= g \quad \text{on } \; \partial\Omega \end{cases}$$

if and only if $\quad u \in V \quad$ *and* $\quad I[u] = \min_{v \in V} I[v].$

References

- Evans pp. 41–43