# Part III, Chapter 10

# Mesh orientation

Orienting the edges and the faces of a mesh is crucial when working with finite elements whose degrees of freedom invoke normal or tangential components of vector fields. This notion is important also when working with high-order scalar-valued finite elements to enumerate consistently all the degrees of freedom in each mesh cell sharing the edge or the face in question. In this chapter we focus on matching meshes (see Definition 8.11), and we assume that the meshes are affine. We first explain how to orient meshes. Then we introduce the important notion of generation-compatible orientation. Finally we study whether simplicial, quadrangular and hexahedral meshes can be equipped with a generation-compatible orientation.

## 10.1 How to orient a mesh

Let us consider a three-dimensional matching mesh. The geometric entities to be oriented are the mesh edges $E \in \mathcal{E}_h$ and the mesh faces $F \in \mathcal{F}_h$ (one can also orient the vertices and the cells of the mesh, but for simplicity we will not introduce these notions here). The edges of the mesh are oriented by specifying how to circulate along them. This is done by fixing one unit vector tangent to each edge. The faces of the mesh are oriented by specifying how to cross them. This is done by fixing one unit normal vector on each face. Orienting the mesh thus means that we fix once and for all the following collections of unit vectors:

$$\{\boldsymbol{\tau}_E\}_{E\in\mathcal{E}_h}, \qquad \{\boldsymbol{n}_F\}_{F\in\mathcal{F}_h}. \tag{10.1}$$

Since the mesh is affine, the mesh edges are straight and the mesh faces are planar. Hence one single tangent vector is enough to orient each edge and one normal vector is enough to orient each face.

Let us now consider a two-dimensional mesh. Then the mesh edges and the mesh faces are identical one-dimensional manifolds in $\mathbb{R}^2$, but they are oriented differently. The orientation of the mesh edges is done as in the three-dimensional case by fixing once and for all a unit tangent vector along the edge, whereas the mesh faces are oriented by rotating the unit tangent vectors anti-clockwise, i.e., for every edge $E$ oriented by the vector $\boldsymbol{\tau}_E$, we set

$$\boldsymbol{n}_E := \boldsymbol{R}_{\frac{\pi}{2}}(\boldsymbol{\tau}_E), \tag{10.2}$$

where the matrix of $\boldsymbol{R}_{\frac{\pi}{2}}$ relative to the canonical basis of $\mathbb{R}^2$ is $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

It is useful to define the following subsets: For any mesh edge $E \in \mathcal{E}_h$ and for every mesh face $F \in \mathcal{F}_h$, let

$$\mathcal{T}_E := \{K \in \mathcal{T}_h \mid E \subset K\}, \qquad \mathcal{T}_F := \{K \in \mathcal{T}_h \mid F \subset K\}, \tag{10.3}$$

be the collection of the mesh cells sharing $E$ and $F$, respectively. The cardinality of the subset $\mathcal{T}_E$ cannot be ascertained a priori, whereas we have $\mathcal{T}_F = \{K_l, K_r\}$ for every interface $F := \partial K_l \cap \partial K_r \in \mathcal{F}_h^\circ$ and $\mathcal{T}_F = \{K_l\}$ for every boundary face $F := \partial K_l \cap \partial D \in \mathcal{F}_h^\partial$; see Definition 8.10.

**Remark 10.1 (Face orientation in 3D).** The faces of cells in three-dimensional meshes have connected boundaries. Hence, instead of assigning a normal vector to each face, one can also orient the faces by specifying how to circulate along their boundary. The two ways of orienting faces are equivalent once an orientation for the ambient space $\mathbb{R}^3$ has been fixed (by using the right-hand convention for example). The boundary-based orientation is more intrinsic since it does not require to embed the faces into $\mathbb{R}^3$. In this book we adopt the normal-based orientation introduced in (10.1) since it is more convenient to use with finite elements. □

**Remark 10.2 (Incidence matrices).** Consider a three-dimensional mesh where the vertices, edges, faces, and cells have been enumerated from 1 to $N_\mathrm{v}$, $N_\mathrm{e}$, $N_\mathrm{f}$, and $N_\mathrm{c}$, respectively. Assume that the mesh has been oriented. Incidence matrices can then be defined as follows. The matrix $\mathcal{M}^{\mathrm{ev}} \in \mathbb{R}^{N_\mathrm{e}\times N_\mathrm{v}}$ is s.t. $\mathcal{M}^{\mathrm{ev}}_{ml} := 1$ if $\boldsymbol{z}_l$ is a vertex of $E_m$ and $\boldsymbol{\tau}_{E_m}$ points toward $\boldsymbol{z}_l$, $\mathcal{M}^{\mathrm{ev}}_{ml} := -1$ if $\boldsymbol{\tau}_{E_m}$ points in the opposite direction, and $\mathcal{M}^{\mathrm{ev}}_{ml} := 0$ if $\boldsymbol{z}_l$ is not a vertex of $E_m$. The matrix $\mathcal{M}^{\mathrm{fe}} \in \mathbb{R}^{N_\mathrm{f}\times N_\mathrm{e}}$ is s.t. $\mathcal{M}^{\mathrm{fe}}_{ml} := 1$ if $E_l$ is an edge of $F_m$ and the orientation of $E_l$ prescribed by $\boldsymbol{\tau}_{E_l}$ and that induced by $\boldsymbol{n}_{F_m}$ on $E_l \subset \partial F_m$ using the right-hand convention are the same, $\mathcal{M}^{\mathrm{fe}}_{ml} := -1$ if these orientation are opposite, and $\mathcal{M}^{\mathrm{fe}}_{ml} := 0$ if $E_l$ is not an edge of $F_m$. The matrix $\mathcal{M}^{\mathrm{cf}} \in \mathbb{R}^{N_\mathrm{c}\times N_\mathrm{f}}$ is s.t. $\mathcal{M}^{\mathrm{cf}}_{ml} := 1$ if $F_l$ is a face of $K_m$ and $\boldsymbol{n}_F$ points toward the outside of $K_m$, $\mathcal{M}^{\mathrm{cf}}_{ml} := -1$ if $\boldsymbol{n}_F$ points inward, and $\mathcal{M}^{\mathrm{cf}}_{ml} := 0$ if $F_l$ is not a face of $K_m$. The incidence matrices $\mathcal{M}^{\mathrm{ev}}$, $\mathcal{M}^{\mathrm{fe}}$, and $\mathcal{M}^{\mathrm{cf}}$ can be viewed as discrete counterparts of the gradient, curl, and divergence operators, respectively. In particular we have $\mathcal{M}^{\mathrm{fe}}\mathcal{M}^{\mathrm{ev}} = 0_{\mathbb{R}^{N_\mathrm{f}\times N_\mathrm{v}}}$ and $\mathcal{M}^{\mathrm{cf}}\mathcal{M}^{\mathrm{fe}} = 0_{\mathbb{R}^{N_\mathrm{c}\times N_\mathrm{e}}}$. We refer the reader to Bossavit [36], Bochev and Hyman [27], Bonelle and Ern [32], Gerritsma [102] and references therein for further insight into this topic. □

## 10.2 Generation-compatible orientation

Let $\mathcal{T}_h$ be an oriented mesh and let $K \in \mathcal{T}_h$ be a mesh cell. Recall that the cell $K$ is generated using a geometric mapping $\boldsymbol{T}_K : \widehat{K} \to K$. One of the key results from the previous chapter, Lemma 9.13, deals with the preservation of the moments of the normal and tangential components of fields defined on $K$. Let $\widehat{F}$ be a face of $\widehat{K}$ and let $\widehat{E}$ be an edge of $\widehat{K}$. Let $F := \boldsymbol{T}_K(\widehat{F})$ and $E := \boldsymbol{T}_K(\widehat{E})$ be the corresponding face and edge of $K$. Let $\widehat{\boldsymbol{n}}_{\widehat{F}}$ be a unit vector normal to $\widehat{F}$ and let $\widehat{\boldsymbol{\tau}}_{\widehat{E}}$ be a unit vector tangent to $\widehat{E}$. Recall from (9.14) that $\boldsymbol{\Phi}_K^{\mathrm{d}}(\widehat{\boldsymbol{n}}_{\widehat{F}})(\boldsymbol{x}) := \epsilon_K \|(\mathbb{J}_K^{-\mathsf{T}}\widehat{\boldsymbol{n}}_{\widehat{F}})(\widehat{\boldsymbol{x}})\|_{\ell^2}^{-1}(\mathbb{J}_K^{-\mathsf{T}}\widehat{\boldsymbol{n}}_{\widehat{F}})(\widehat{\boldsymbol{x}})$ is a unit vector normal to $F$ and that $\boldsymbol{\Phi}_K^{\mathrm{c}}(\widehat{\boldsymbol{\tau}}_{\widehat{E}})(\boldsymbol{x}) := \|(\mathbb{J}_K\widehat{\boldsymbol{\tau}}_{\widehat{E}})(\widehat{\boldsymbol{x}})\|_{\ell^2}^{-1}(\mathbb{J}_K\widehat{\boldsymbol{\tau}}_{\widehat{E}})(\widehat{\boldsymbol{x}})$ is a unit vector tangent to $E$, where $\mathbb{J}_K$ is the Jacobian matrix of $\boldsymbol{T}_K$, $\epsilon_K := \frac{\det(\mathbb{J}_K)}{|\det(\mathbb{J}_K)|} = \pm 1$, and $\boldsymbol{x} := \boldsymbol{T}_K(\widehat{\boldsymbol{x}})$. With the Piola transformations $\psi_K^{\mathrm{g}}$, $\boldsymbol{\psi}_K^{\mathrm{c}}$, and $\boldsymbol{\psi}_K^{\mathrm{d}}$ defined in Definition 9.8, Lemma 9.13 states that the following holds for all $\boldsymbol{v} \in \boldsymbol{C}^0(K)$ and all $q \in C^0(K)$:

$$\int_F (\boldsymbol{v}\cdot\boldsymbol{\Phi}_K^{\mathrm{d}}(\widehat{\boldsymbol{n}}_{\widehat{F}}))(\boldsymbol{x})q(\boldsymbol{x})\,\mathrm{d}s = \int_{\widehat{F}} (\boldsymbol{\psi}_K^{\mathrm{d}}(\boldsymbol{v})\cdot\widehat{\boldsymbol{n}}_{\widehat{F}})(\widehat{\boldsymbol{x}})\psi_K^{\mathrm{g}}(q)(\widehat{\boldsymbol{x}})\,\mathrm{d}\widehat{s}, \tag{10.4a}$$

$$\int_E (\boldsymbol{v}\cdot\boldsymbol{\Phi}_K^{\mathrm{c}}(\widehat{\boldsymbol{\tau}}_{\widehat{E}}))(\boldsymbol{x})q(\boldsymbol{x})\,\mathrm{d}l = \int_{\widehat{E}} (\boldsymbol{\psi}_K^{\mathrm{c}}(\boldsymbol{v})\cdot\widehat{\boldsymbol{\tau}}_{\widehat{E}})(\widehat{\boldsymbol{x}})\psi_K^{\mathrm{g}}(q)(\widehat{\boldsymbol{x}})\,\mathrm{d}\widehat{l}. \tag{10.4b}$$

Since we are going to define face and edge dofs for vector-valued finite elements by using the right-hand side in (10.4), we want to make sure that the results do not depend on the mapping $\boldsymbol{T}_K : \widehat{K} \to K$. For instance, let $F \in \mathcal{F}_h$ be an interface, i.e., $F := \partial K_l \cap \partial K_r$ so that $\mathcal{T}_F = \{K_l, K_r\}$. One way to ascertain that the right-hand side of (10.4a) gives the same results on both sides of $F$ simply consists of requiring that

$$\boldsymbol{n}_F = \boldsymbol{\Phi}_K^{\mathrm{d}}(\widehat{\boldsymbol{n}}_{\widehat{F}}), \quad \forall K \in \mathcal{T}_F, \text{ with } \widehat{F} := \boldsymbol{T}_K^{-1}(F), \tag{10.5}$$

that is, letting $\widehat{F}_l := \boldsymbol{T}_{K_l}^{-1}(F)$ and $\widehat{F}_r := \boldsymbol{T}_{K_r}^{-1}(F)$, we would like that $\boldsymbol{n}_F = \boldsymbol{\Phi}_{K_l}^{\mathrm{d}}(\widehat{\boldsymbol{n}}_{\widehat{F}_l}) = \boldsymbol{\Phi}_{K_r}^{\mathrm{d}}(\widehat{\boldsymbol{n}}_{\widehat{F}_r})$. This idea is illustrated in Figure 10.1.

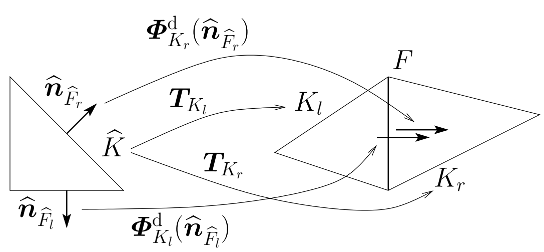



**Fig. 10.1** Orientation transfer for face normals.

Similarly, given a mesh edge $E \in \mathcal{E}_h$ oriented by the fixed unit tangent vector $\boldsymbol{\tau}_E$, we want to ascertain that for every mesh cell $K$ of which $E$ is an edge, i.e., for all $K \in \mathcal{T}_E$ (see (10.3)), we have $\boldsymbol{\tau}_E = \boldsymbol{\Phi}^{\mathrm{c}}_K(\widehat{\boldsymbol{\tau}}_{\widehat{E}})$ where $\widehat{E} := \boldsymbol{T}_K^{-1}(E)$. This leads to the following important notion.

**Definition 10.3 (Generation-compatible orientation).** *Let $\mathcal{T}_h$ be an oriented mesh specified by the collections of unit tangent vectors $\{\boldsymbol{\tau}_E\}_{E\in\mathcal{E}_h}$ and unit normal vectors $\{\boldsymbol{n}_F\}_{F\in\mathcal{F}_h}$ as in* (10.1). *We say that this orientation is generation-compatible if there is an orientation of the reference cell $\widehat{K}$ specified by the unit tangent vectors $\{\widehat{\boldsymbol{\tau}}_{\widehat{E}}\}_{\widehat{E}\in\mathcal{E}_{\widehat{K}}}$ and the unit normal vectors $\{\widehat{\boldsymbol{n}}_{\widehat{F}}\}_{\widehat{F}\in\mathcal{F}_{\widehat{K}}}$ and a collection of geometric mappings $\{\boldsymbol{T}_K\}_{K\in\mathcal{T}_h}$ such that for all $E \in \mathcal{E}_h$ and all $F \in \mathcal{F}_h$,*

$$\boldsymbol{\tau}_E = \boldsymbol{\Phi}^{\mathrm{c}}_K(\widehat{\boldsymbol{\tau}}_{\widehat{E}}), \qquad \forall K \in \mathcal{T}_E,\ \widehat{E} := \boldsymbol{T}_K^{-1}(E), \tag{10.6a}$$

$$\boldsymbol{n}_F = \boldsymbol{\Phi}^{\mathrm{d}}_K(\widehat{\boldsymbol{n}}_{\widehat{F}}), \qquad \forall K \in \mathcal{T}_F,\ \widehat{F} := \boldsymbol{T}_K^{-1}(F). \tag{10.6b}$$

The key consequence of the notion of generation-compatible mesh is the following result which says that the moments of the normal and tangential components of vector fields are preserved by the transformations $\psi^{\mathrm{g}}_K, \boldsymbol{\psi}^{\mathrm{c}}_K, \boldsymbol{\psi}^{\mathrm{d}}_K$.

**Lemma 10.4 (Preservation of moments of normal and tangential components).** *Assume that the orientation of $\mathcal{T}_h$ is generation compatible and use the above notation. The following holds true for all $\boldsymbol{v} \in \boldsymbol{C}^0(K)$ and all $q \in C^0(K)$:*

$$\int_F (\boldsymbol{v}\cdot\boldsymbol{n}_F)(\boldsymbol{x})q(\boldsymbol{x})\,\mathrm{d}s = \int_{\widehat{F}} (\boldsymbol{\psi}^{\mathrm{d}}_K(\boldsymbol{v})\cdot\widehat{\boldsymbol{n}}_{\widehat{F}})(\widehat{\boldsymbol{x}})\psi^{\mathrm{g}}_K(q)(\widehat{\boldsymbol{x}})\,\mathrm{d}\widehat{s}, \tag{10.7a}$$

$$\int_E (\boldsymbol{v}\cdot\boldsymbol{\tau}_E)(\boldsymbol{x})q(\boldsymbol{x})\,\mathrm{d}l = \int_{\widehat{E}} (\boldsymbol{\psi}^{\mathrm{c}}_K(\boldsymbol{v})\cdot\widehat{\boldsymbol{\tau}}_{\widehat{E}})(\widehat{\boldsymbol{x}})\psi^{\mathrm{g}}_K(q)(\widehat{\boldsymbol{x}})\,\mathrm{d}\widehat{l}. \tag{10.7b}$$

*Proof.* Apply Lemma 9.13. □

Whether it is possible to orient a mesh in a generation-compatible way is not guaranteed for general meshes. However we will see in the following sections that this is indeed possible for simplicial meshes in any dimension, for quadrangular meshes, and for hexahedral meshes (possibly up to an additional subdivision of the cells). The key idea to achieve this is the increasing vertex-index enumeration technique introduced in the next section.

**Remark 10.5 (Faces in 2D).** Recall that the mesh edges and faces are identical one-dimensional manifolds in $\mathbb{R}^2$, and that we have adopted the convention that once the edges are oriented, the faces are oriented by rotating the unit tangent vectors anti-clockwise; see (10.2). It is proved in Exercise 10.1 that $\boldsymbol{R}_{\frac{\pi}{2}}(\boldsymbol{\Phi}^{\mathrm{c}}_K(\boldsymbol{z})) = \boldsymbol{\Phi}^{\mathrm{d}}_K(\boldsymbol{R}_{\frac{\pi}{2}}(\boldsymbol{z}))$ for all $\boldsymbol{z} \in \mathbb{R}^2$. Hence if (10.6a) holds true, then (10.7b) holds true as well, because in this case

$\boldsymbol{n}_E := \boldsymbol{R}_{\frac{\pi}{2}}(\boldsymbol{\tau}_E) = \boldsymbol{R}_{\frac{\pi}{2}}(\boldsymbol{\Phi}^{\mathrm{c}}_K(\widehat{\boldsymbol{\tau}}_{\widehat{E}})) = \boldsymbol{\Phi}^{\mathrm{d}}_K(\boldsymbol{R}_{\frac{\pi}{2}}(\widehat{\boldsymbol{\tau}}_{\widehat{E}})) =: \boldsymbol{\Phi}^{\mathrm{d}}_K(\widehat{\boldsymbol{n}}_{\widehat{E}})$. In conclusion one only needs to prove (10.6a) in dimension two. □

## 10.3 Increasing vertex-index enumeration

The increasing vertex-index enumeration technique described in this section is the key tool to orient meshes in a generation-compatible way. The technique is illustrated for various types of meshes in §10.4 and §10.5.

Let us enumerate the edges and the faces of $\widehat{K}$ from 1 to $n_{\mathrm{ce}}$ and from 1 to $n_{\mathrm{cf}}$, respectively. Orienting the reference cell $\widehat{K}$ consists of prescribing the following unit vectors:

$$\{\widehat{\boldsymbol{\tau}}_{\widehat{E}_n}\}_{n\in\{1:n_{\mathrm{ce}}\}}, \qquad \{\widehat{\boldsymbol{n}}_{\widehat{F}_n}\}_{n\in\{1:n_{\mathrm{cf}}\}}.$$

Recalling the connectivity arrays `j_ce` and `j_cf` defined in (8.12), any mesh edge $E_l$ for all $l \in \{1:N_{\mathrm{e}}\}$ satisfies $E_l = \boldsymbol{T}_{K_m}(\widehat{E}_n)$ with $(m,n) \in \{1:N_{\mathrm{c}}\} \times \{1:n_{\mathrm{ce}}\}$ s.t. $\mathtt{j\_ce}(m,n) = l$. Similarly any mesh face $F_l$ for all $l \in \{1:N_{\mathrm{f}}\}$ satisfies $F_l = \boldsymbol{T}_{K_m}(\widehat{F}_n)$ with $(m,n) \in \{1:N_{\mathrm{c}}\} \times \{1:n_{\mathrm{cf}}\}$ s.t. $\mathtt{j\_cf}(m,n) = l$.

**Definition 10.6 (Increasing vertex-index enumeration).** *A mesh $\mathcal{T}_h$ is said to be oriented according to the* increasing vertex-index convention *if:*

(i) *Every edge $E_n$ with vertices $\boldsymbol{z}_p, \boldsymbol{z}_q$, $p < q$, is oriented by the vector $\boldsymbol{\tau}_{E_n} := \|\boldsymbol{t}_{p,q}\|^{-1}_{\ell^2}\boldsymbol{t}_{p,q}$ with $\boldsymbol{t}_{p,q} := \boldsymbol{z}_q - \boldsymbol{z}_p$;*
(ii) *Every face $F_n$ in dimension two is oriented by the vector $\boldsymbol{R}_{\frac{\pi}{2}}(\boldsymbol{\tau}_{F_n})$ (here $F_n$ is viewed as an edge, and $\boldsymbol{R}_{\frac{\pi}{2}}$ is the rotation of angle $\frac{\pi}{2}$ in $\mathbb{R}^2$ defined in* (10.2)*), and every face $F_n$ in dimension three is oriented by the vector $\boldsymbol{n}_{F_n} := \|\boldsymbol{t}_{p,q}\times\boldsymbol{t}_{p,r}\|^{-1}_{\ell^2}(\boldsymbol{t}_{p,q}\times\boldsymbol{t}_{p,r})$, where $p < q < r$ are the three global indices of the vertices of $F_n$.*

The increasing vertex-index enumeration is illustrated in Figure 10.2 for the unit simplex and the unit cuboid in dimension two and dimension three.

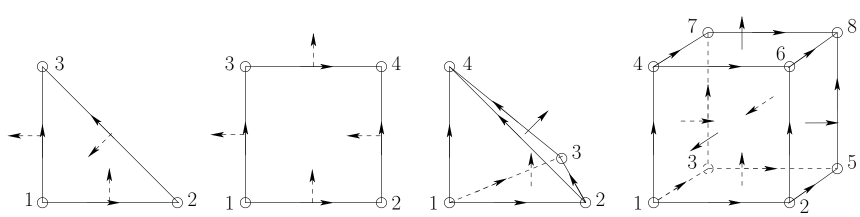



**Fig. 10.2** Enumeration of the vertices and orientation of the edges and faces in the reference simplex and the reference cuboid in dimensions two and three.

| | |
|---|---|
| 2D triangle | $\widehat{\boldsymbol{z}}_1 = (0,0)$, $\widehat{\boldsymbol{z}}_2 = (1,0)$, $\widehat{\boldsymbol{z}}_3 = (0,1)$ |
| 3D tetrahedron | $\widehat{\boldsymbol{z}}_1 = (0,0,0)$, $\widehat{\boldsymbol{z}}_2 = (1,0,0)$, $\widehat{\boldsymbol{z}}_3 = (0,1,0)$, $\widehat{\boldsymbol{z}}_4 = (0,0,1)$ |
| 2D square | $\widehat{\boldsymbol{z}}_1 = (0,0)$, $\widehat{\boldsymbol{z}}_2 = (0,1)$, $\widehat{\boldsymbol{z}}_3 = (1,0)$, $\widehat{\boldsymbol{z}}_4 = (1,1)$ |
| 3D cube | $\widehat{\boldsymbol{z}}_1 = (0,0,0)$, $\widehat{\boldsymbol{z}}_2 = (1,0,0)$, $\widehat{\boldsymbol{z}}_3 = (0,1,0)$, $\widehat{\boldsymbol{z}}_4 = (0,0,1)$<br>$\widehat{\boldsymbol{z}}_5 = (1,1,0)$, $\widehat{\boldsymbol{z}}_6 = (1,0,1)$, $\widehat{\boldsymbol{z}}_7 = (0,1,1)$, $\widehat{\boldsymbol{z}}_8 = (1,1,1)$ |

**Table 10.1** Enumeration of the vertices in the reference simplex and in the reference cuboid in dimensions two and three.

Unless specified otherwise we enumerate the vertices of the reference element $\widehat{K}$ by using the convention described in Table 10.1. Moreover $\widehat{K}$ is oriented by using the convention of the increasing vertex-index enumeration as in Figure 10.2.

## 10.4 Simplicial meshes

Recall that the reference simplex $\widehat{K}$ is oriented by using the increasing vertex-index technique. Let us show that it is possible to find a generation-compatible orientation for every three-dimensional affine mesh $\mathcal{T}_h$ composed of simplices (the construction proposed thereafter is actually independent of the space dimension). The key idea is to orient $\mathcal{T}_h$ by using the increasing vertex-index enumeration. More precisely, let $\{\boldsymbol{z}_n\}_{n\in\{1:N_{\rm v}\}}$ be the mesh vertices. For every edge $E_l$ with end vertices $\boldsymbol{z}_p, \boldsymbol{z}_q$, where $p<q$, we orient $E_l$ by introducing $\boldsymbol{t}_{p,q} := \boldsymbol{z}_q - \boldsymbol{z}_p$ and by setting

$$\boldsymbol{\tau}_{E_l} := \|\boldsymbol{t}_{p,q}\|_{\ell^2}^{-1}\boldsymbol{t}_{p,q}. \tag{10.8}$$

For every face $F_l$ defined by its three vertices, say $\boldsymbol{z}_p, \boldsymbol{z}_q, \boldsymbol{z}_r$ with $p<q<r$, we orient $F_l$ by introducing $\boldsymbol{t}_{p,q} := \boldsymbol{z}_q - \boldsymbol{z}_p$, $\boldsymbol{t}_{p,r} := \boldsymbol{z}_r - \boldsymbol{z}_p$, and by setting

$$\boldsymbol{n}_{F_l} := \|\boldsymbol{t}_{p,q}\times\boldsymbol{t}_{p,r}\|_{\ell^2}^{-1}(\boldsymbol{t}_{p,q}\times\boldsymbol{t}_{p,r}). \tag{10.9}$$

Let us now construct the geometric mapping $\boldsymbol{T}_K$ for all $K \in \mathcal{T}_h$. Let $\boldsymbol{z}_p, \boldsymbol{z}_q, \boldsymbol{z}_r, \boldsymbol{z}_s$ be the four vertices of $K$ ordered by increasing vertex-index, i.e., $p<q<r<s$. We define $\boldsymbol{T}_K$ by setting

$$\boldsymbol{T}_K(\widehat{\boldsymbol{z}}_1) := \boldsymbol{z}_p, \quad \boldsymbol{T}_K(\widehat{\boldsymbol{z}}_2) := \boldsymbol{z}_q, \quad \boldsymbol{T}_K(\widehat{\boldsymbol{z}}_3) := \boldsymbol{z}_r, \quad \boldsymbol{T}_K(\widehat{\boldsymbol{z}}_4) := \boldsymbol{z}_s. \tag{10.10}$$

Hence the global index of the mesh vertex $\boldsymbol{T}_K(\widehat{\boldsymbol{z}}_n)$ increases with $n$. Using the connectivity array `j_cv` defined by (8.12), we have $\texttt{j\_cv}(m,1) = p$, $\texttt{j\_cv}(m,2) = q$, $\texttt{j\_cv}(m,3) = r$, and $\texttt{j\_cv}(m,4) = s$, where $m$ is the global enumeration index of the mesh cell $K$. Notice that (10.10) is sufficient to define $\boldsymbol{T}_K$ entirely since we assumed that the mesh is affine. We emphasize that in the present construction the mapping $\boldsymbol{T}_K$ is invertible, but its Jacobian determinant can be positive or negative.

**Example 10.7 (Orienting a tetrahedron).** Consider a tetrahedron whose vertices have global indices 35, 42, 67, and 89, as shown in Figure 10.3. The orientation of the (five visible) edges is materialized by dark arrows. The unit normal vector $\boldsymbol{n}_F$ defined by the increasing-vertex enumeration points toward the outside of tetrahedron for the face defined by the indices $\{35, 42, 67\}$, and it points inward for the face defined by the indices $\{42, 67, 89\}$, etc. $\square$

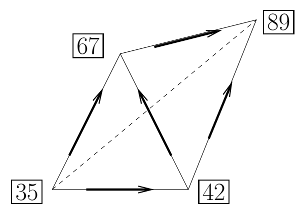



**Fig. 10.3** Illustration of Example 10.7.

**Theorem 10.8 (Simplicial mesh orientation).** *Let $\mathcal{T}_h$ be a simplicial mesh. Let $\widehat{K}$ be oriented by using the increasing vertex-index enumeration. For all $K \in \mathcal{T}_h$, let $\boldsymbol{T}_K$ be defined by the increasing vertex-index convention* (10.10). *Then the orientation of $\mathcal{T}_h$ based on the increasing vertex-index enumeration is generation-compatible.*

*Proof.* (1) Let us prove (10.6a). Let $E_l$ be an edge with vertices $\boldsymbol{z}_p, \boldsymbol{z}_q$, $p < q$. Let $(m, n)$ be s.t. $E_l = \boldsymbol{T}_{K_m}(\widehat{E}_n)$, i.e., $\texttt{j\_ce}(m, n) = l$. Let $\widehat{\boldsymbol{z}}_i, \widehat{\boldsymbol{z}}_j$ with $i < j$ be the vertices of the edge $\widehat{E}_n$ of $\widehat{K}$. The increasing vertex-index convention (10.10) for the geometric mappings implies that $\boldsymbol{T}_{K_m}(\widehat{\boldsymbol{z}}_i) = \boldsymbol{z}_p$ and $\boldsymbol{T}_{K_m}(\widehat{\boldsymbol{z}}_j) = \boldsymbol{z}_q$. Moreover the orientation for $\widehat{K}$ implies that $\widehat{\boldsymbol{\tau}}_{\widehat{E}_n} = \|\widehat{\boldsymbol{t}}_{i,j}\|_{\ell^2}^{-1}\widehat{\boldsymbol{t}}_{i,j}$ with $\widehat{\boldsymbol{t}}_{i,j} := \widehat{\boldsymbol{z}}_j - \widehat{\boldsymbol{z}}_i$, so that $\boldsymbol{\Phi}^{\mathrm{c}}_{K_m}(\widehat{\boldsymbol{\tau}}_{\widehat{E}_n}) = \|\mathbb{J}_{K_m}\widehat{\boldsymbol{\tau}}_{\widehat{E}_n}\|_{\ell^2}^{-1}\mathbb{J}_{K_m}\widehat{\boldsymbol{\tau}}_{\widehat{E}_n} = \|\mathbb{J}_{K_m}\widehat{\boldsymbol{t}}_{i,j}\|_{\ell^2}^{-1}\mathbb{J}_{K_m}\widehat{\boldsymbol{t}}_{i,j}$. Since $\boldsymbol{T}_{K_m}$ is affine, we have

$$\mathbb{J}_{K_m}\widehat{\boldsymbol{t}}_{i,j} = \boldsymbol{T}_{K_m}(\widehat{\boldsymbol{z}}_j) - \boldsymbol{T}_{K_m}(\widehat{\boldsymbol{z}}_i) = \boldsymbol{z}_q - \boldsymbol{z}_p = \boldsymbol{t}_{p,q},$$

and we conclude that $\boldsymbol{\Phi}^{\mathrm{c}}_{K_m}(\widehat{\boldsymbol{\tau}}_{\widehat{E}_n}) = \|\boldsymbol{t}_{p,q}\|_{\ell^2}^{-1}\boldsymbol{t}_{p,q} = \boldsymbol{\tau}_{E_l}$.
(2) Let us prove (10.6b) in dimension three. Let $F_l$ be a face with vertices $\boldsymbol{z}_p, \boldsymbol{z}_q, \boldsymbol{z}_r$, $p < q < r$. Let $(m, n)$ be s.t. $F_l = \boldsymbol{T}_{K_m}(\widehat{F}_n)$, i.e., $\texttt{j\_cf}(m, n) = l$. Let $\widehat{\boldsymbol{z}}_i, \widehat{\boldsymbol{z}}_j, \widehat{\boldsymbol{z}}_k$ with $i < j < k$ be the vertices of the face $\widehat{F}_n$ of $\widehat{K}$. Reasoning as above, we have $\mathbb{J}_{K_m}\widehat{\boldsymbol{t}}_{i,j} = \boldsymbol{t}_{p,q}$ and $\mathbb{J}_{K_m}\widehat{\boldsymbol{t}}_{i,k} = \boldsymbol{t}_{p,r}$. Using the identity $\mathbb{A}^{-\mathsf{T}}(\boldsymbol{x}\times\boldsymbol{y}) = \det(\mathbb{A})^{-1}(\mathbb{A}\boldsymbol{x}\times\mathbb{A}\boldsymbol{y})$ for every $3\times 3$ invertible matrix $\mathbb{A}$ and all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3$ (see Exercise 9.6), we have

$$\mathbb{J}_{K_m}^{-\mathsf{T}}(\widehat{\boldsymbol{t}}_{i,j}\times\widehat{\boldsymbol{t}}_{i,k}) = \det(\mathbb{J}_{K_m})^{-1}(\boldsymbol{t}_{p,q}\times\boldsymbol{t}_{p,r}).$$

Moreover, since $\widehat{\boldsymbol{n}}_{\widehat{F}_n}$ and $\widehat{\boldsymbol{t}}_{i,j}\times\widehat{\boldsymbol{t}}_{i,k}$ are collinear and point in the same direction, the definition (9.14a) implies that

$$\boldsymbol{\Phi}^{\mathrm{d}}_{K_m}(\widehat{\boldsymbol{n}}_{\widehat{F}_n}) = \epsilon_{K_m}\|\mathbb{J}_{K_m}^{-\mathsf{T}}(\widehat{\boldsymbol{t}}_{i,j}\times\widehat{\boldsymbol{t}}_{i,k})\|_{\ell^2}^{-1}\mathbb{J}_{K_m}^{-\mathsf{T}}(\widehat{\boldsymbol{t}}_{i,j}\times\widehat{\boldsymbol{t}}_{i,k}).$$

Since $\|\mathbb{J}_{K_m}^{-\mathsf{T}}(\widehat{\boldsymbol{t}}_{i,j}\times\widehat{\boldsymbol{t}}_{i,k})\|_{\ell^2} = |\det(\mathbb{J}_{K_m})|^{-1}\|\boldsymbol{t}_{p,q}\times\boldsymbol{t}_{p,r}\|_{\ell^2}$, we conclude that

$$\begin{aligned}\boldsymbol{\Phi}^{\mathrm{d}}_{K_m}(\widehat{\boldsymbol{n}}_{\widehat{F}_n}) &= \epsilon_{K_m}|\det(\mathbb{J}_{K_m})|\|\boldsymbol{t}_{p,q}\times\boldsymbol{t}_{p,r}\|_{\ell^2}^{-1}\det(\mathbb{J}_{K_m})^{-1}(\boldsymbol{t}_{p,q}\times\boldsymbol{t}_{p,r})\\ &= \|\boldsymbol{t}_{p,q}\times\boldsymbol{t}_{p,r}\|_{\ell^2}^{-1}(\boldsymbol{t}_{p,q}\times\boldsymbol{t}_{p,r}) = \boldsymbol{n}_{F_l}.\end{aligned}$$ □

(3) Finally, owing to Remark 10.5, the argument in Step (1) implies that (10.6b) holds true in dimension two. □

**Remark 10.9 (Positive Jacobian determinant).** If one insists on building geometric mappings such that $\det(\mathbb{J}_K) > 0$, the above orientation of the edges and the faces of the mesh is still generation-compatible if one uses two reference tetrahedra; see Ainsworth and Coyle [6]. □

## 10.5 Quadrangular and hexahedral meshes

We state without proof a result by Agelek et al. [4] on quadrangular and hexahedral meshes.

**Theorem 10.10 (Quad/Hex mesh orientation).** (i) *Let the reference square or cube be oriented using the increasing vertex-index enumeration technique.* (ii) *Let $\mathcal{T}_h$ be a quadrangular mesh. It is possible to orient the mesh to make it generation-compatible.* (iii) *Let now $\mathcal{T}_h$ be a hexahedral mesh and let $\mathcal{T}_{\frac{h}{2}}$ be obtained from $\mathcal{T}_h$ by cutting each hexahedron into eight smaller hexahedra. It is possible to orient $\mathcal{T}_{\frac{h}{2}}$ to make it generation-compatible.*

Let us provide some further insight into this result. Let us start with the faces since orientating the faces is simple and independent of the space dimension. Consider the undirected graph whose vertices are the mesh faces and the edges are the mesh cells. We say that two mesh faces $F_1$, $F_2$ are connected through $K$ iff $F_1$, $F_2$ are faces of $K$ that are $\boldsymbol{T}_K$-parallel (i.e., images by $\boldsymbol{T}_K$ of faces of $\widehat{K}$ that are parallel). Since each face is connected to either one (boundary face) or two cells (interface), all the connected components of the graph thus constructed are either closed loops or chains whose extremities are boundary faces. In either case the connected components of the graph realize a partition of the faces of $\mathcal{T}_h$. We then assign the same orientation to all the faces in the same connected component of the graph.

Let us now orient the edges. For quadrangular meshes, the edges are oriented by rotating clockwise the unit normal vector; see the second panel in

Figure 10.2 and the left panel of Figure 10.4 where the dashed lines connect the edges/faces that are in the same equivalence class. For hexahedral meshes, we further need to devise a specific orientation of the edges. Let $\mathcal{E}_h$ be the collection of the mesh edges. We say that two edges of a cell $K$ are $\boldsymbol{T}_K$-parallel if they are images by $\boldsymbol{T}_K$ of edges in $\widehat{K}$ that are parallel. We then define a binary relation $\mathcal{R}$ on $\mathcal{E}_h$. Let $E, E' \in \mathcal{E}_h$ be two mesh edges. We say that $E\mathcal{R}E'$ if either $E$ and $E'$ belong to the same cell $K$ and are $\boldsymbol{T}_K$-parallel or there is a collection of cells $K_1, \ldots, K_L$, all different, and a collection of edges $E =: E_1, \ldots, E_{L+1} := E'$ such that $E_l$ and $E_{l+1}$ are both edges of $K_l$, $l \in \{1{:}L\}$, and $E_l$, $E_{l+1}$ are $\boldsymbol{T}_{K_l}$-parallel. This defines an equivalence relation over the edges which in turn generates a partition of $\mathcal{E}_h$. Unfortunately it is not always possible to give the same orientation to all the edges belonging to the same equivalence class since in dimension three edges in the same equivalence class may actually be sitting on a Möbius strip. An example of non-orientable mesh (in the sense defined above) composed of hexahedra is shown in the right panel of Figure 10.4. Theorem 10.10 then says that after subdivision, this mesh becomes orientable in a generation-compatible way, and more generally, every mesh composed of hexahedra is orientable after one subdivision.

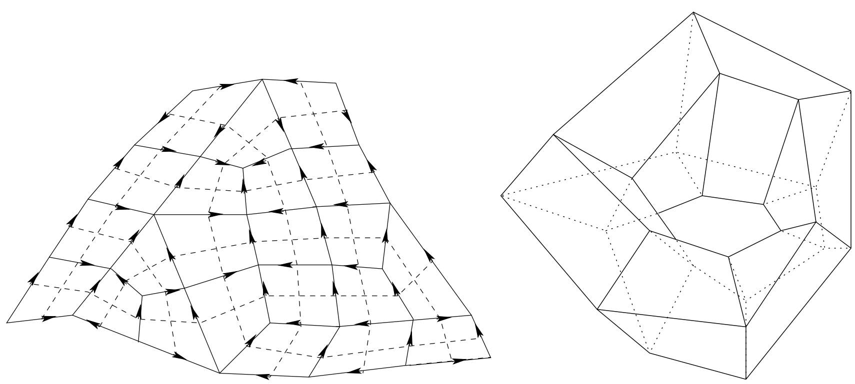


**Fig. 10.4** Orientation of the edges in a mesh composed of quadrangles (left). Non-orientable three-dimensional mesh composed of hexahedra (right).

Assuming that the mesh edges have been oriented as discussed above, it is now possible to build the geometric mappings $\boldsymbol{T}_K$ such that the above mesh orientation is generation-compatible. The idea is that for each mesh cell $K$, there is only one vertex such that all the edges sharing it are oriented away from it. This vertex is called *origin* of the cell. Then we choose $\boldsymbol{T}_K$ such that $\boldsymbol{T}_K$ maps $\widehat{\boldsymbol{z}}_1$ to the origin of $K$ (recall that $\widehat{\boldsymbol{z}}_1$ is the only vertex of $\widehat{K}$ such that all the edges sharing it are oriented away from it; see Figure 10.2). This choice implies that the image by $\boldsymbol{T}_K$ of $\widehat{\boldsymbol{z}}_4$ (if $d = 2$) and of $\widehat{\boldsymbol{z}}_8$ (if $d = 3$) is the vertex of $K$ opposite to the origin. Finally the image by $\boldsymbol{T}_K$ of the remaining two (if $d = 2$) or six (if $d = 3$) vertices can be chosen arbitrarily. One criterion to limit the choices can be to fix a sign for $\det(\mathbb{J}_K)$. In dimension two, one choice gives a positive sign and the other gives a negative sign, whereas in dimension three, three choices give a positive sign and three choices give a negative sign.

## Exercises

**Exercise 10.1 (Faces in 2D).** Let $\boldsymbol{R}_{\frac{\pi}{2}}$ be the rotation of angle $\frac{\pi}{2}$ in $\mathbb{R}^2$. (i) Let $\mathbb{A}$ be an inversible $2\times 2$ matrix. Prove that $\mathbb{A}^{-\mathsf{T}}\boldsymbol{R}_{\frac{\pi}{2}} = \frac{1}{\det(\mathbb{A})}\boldsymbol{R}_{\frac{\pi}{2}}\mathbb{A}$. (ii) Prove that $\boldsymbol{\Phi}_K^{\mathrm{d}}(\boldsymbol{R}_{\frac{\pi}{2}}(\boldsymbol{z})) = \boldsymbol{R}_{\frac{\pi}{2}}(\boldsymbol{\Phi}_K^{\mathrm{c}}(\boldsymbol{z}))$ for all $\boldsymbol{z} \in \mathbb{R}^2$.

**Exercise 10.2 (Connectivity arrays `j_cv`, `j_ce`).** Consider the mesh shown in Figure 10.5, where the face enumeration is identified with large circles and the cell enumeration is identified with squares. (i) Write the connectivity ar-

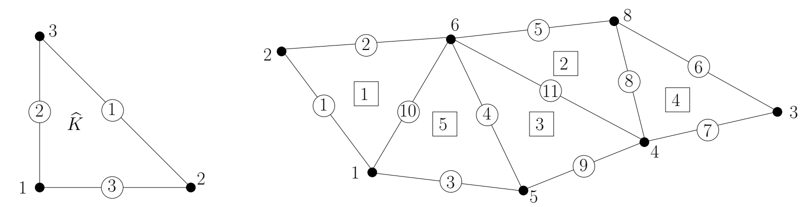



**Fig. 10.5** Illustration for Exercise 10.2.

rays `j_cv` and `j_ce` based on increasing vertex-index enumeration. (ii) Give the sign of the determinant of the Jacobian matrix of $\boldsymbol{T}_K$ for each triangle.

**Exercise 10.3 (Connectivity array `j_geo`).** Consider the mesh shown in Figure 10.6 and based on the $\mathbb{P}_{2,2}$ geometric Lagrange element. (i) Write

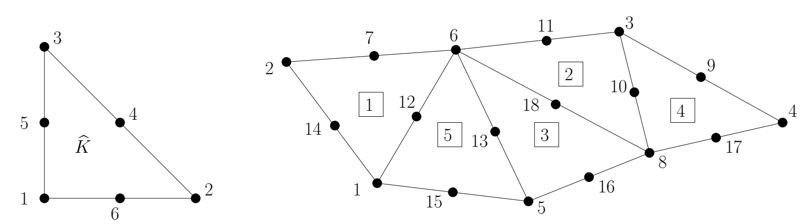



**Fig. 10.6** Illustration for Exercise 10.3.

the connectivity array `j_geo` based on increasing vertex-index enumeration.

(ii) Give the sign of the determinant of the Jacobian matrix of $\boldsymbol{T}_K$ for each triangle.

**Exercise 10.4 (Orientation of quadrangular mesh).** (i) Using the enumeration and the orientation conventions proposed in the chapter, orient the mesh shown in Figure 10.7, where the cell enumeration is identified with shaded rectangles. (ii) Give the connectivity array `j_geo` so that the mesh

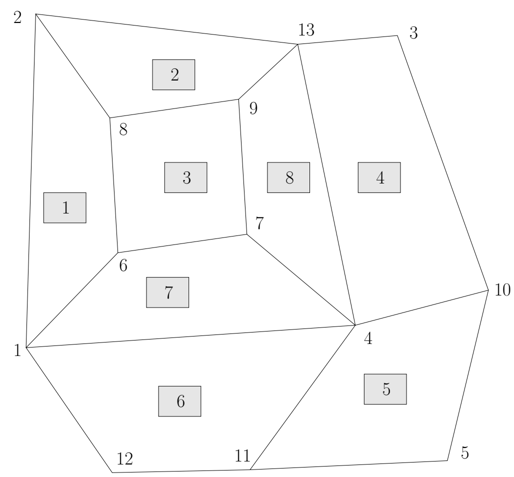



**Fig. 10.7** Illustration for Exercise 10.4.

orientation is generation-compatible and the determinant of the Jacobian matrix of $\boldsymbol{T}_K$ is positive for even quadrangles and negative for odd quadrangles.

**Exercise 10.5 (Mesh extrusion).** (i) Let $K$ be a triangular prism. Denote by $\boldsymbol{e}_3$ the unit vector in the vertical direction. Let $\boldsymbol{z}_1, \boldsymbol{z}_2, \boldsymbol{z}_3$ be the three vertices of the bottom triangular face of $K$, and let $\boldsymbol{z}_4, \boldsymbol{z}_5, \boldsymbol{z}_6$ be the three vertices of its top triangular face, so that the segments $[\boldsymbol{z}_p, \boldsymbol{z}_{p+3}]$ are parallel to $\boldsymbol{e}_3$ for every $p \in \{1, 2, 3\}$. Propose a way to cut $K$ into three tetrahedra. (ii) Let $\mathcal{T}_h$ be a two-dimensional oriented mesh composed of triangles. Let $\mathcal{T}_h'$ be a copy of $\mathcal{T}_h$ obtained by translating $\mathcal{T}_h$ in the third direction $\boldsymbol{e}_3$, say $\mathcal{T}_h' := \mathcal{T}_h + \boldsymbol{e}_3$. Propose a way to cut all the prisms thus formed to make a matching mesh composed of tetrahedra.

## Solution to exercises

**Exercise 10.1 (Faces in 2D).** (i) Let us set $\mathbb{A} := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. We have

$$\mathbb{A}^{-\mathsf{T}}\boldsymbol{R}_{\frac{\pi}{2}} = \frac{1}{\det(\mathbb{A})}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \frac{1}{\det(\mathbb{A})}\begin{pmatrix} -c & -d \\ a & b \end{pmatrix}.$$

We also have

$$\boldsymbol{R}_{\frac{\pi}{2}}\mathbb{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix}.$$

This proves the claim.
(ii) Using the above result and the definition of $\boldsymbol{\Phi}_K^{\mathrm{d}}$ and $\boldsymbol{\Phi}_K^{\mathrm{c}}$, we obtain

$$\begin{aligned}\boldsymbol{\Phi}_K^{\mathrm{d}}(\boldsymbol{R}_{\frac{\pi}{2}}(\boldsymbol{z})) &= \epsilon_K \frac{\mathbb{J}_K^{-\mathsf{T}}\boldsymbol{R}_{\frac{\pi}{2}}(\boldsymbol{z})}{\|\mathbb{J}_K^{-\mathsf{T}}\boldsymbol{R}_{\frac{\pi}{2}}(\boldsymbol{z})\|_{\ell^2}} = \epsilon_K \frac{|\det(\mathbb{J}_K)|}{\det(\mathbb{J}_K)} \frac{\boldsymbol{R}_{\frac{\pi}{2}}(\mathbb{J}_K\boldsymbol{z})}{\|\boldsymbol{R}_{\frac{\pi}{2}}(\mathbb{J}_K\boldsymbol{z})\|_{\ell^2}} \\ &= \boldsymbol{R}_{\frac{\pi}{2}}\left(\frac{\mathbb{J}_K\boldsymbol{z}}{\|\mathbb{J}_K\boldsymbol{z}\|_{\ell^2}}\right) = \boldsymbol{R}_{\frac{\pi}{2}}(\boldsymbol{\Phi}_K^{\mathrm{c}}(\boldsymbol{z})).\end{aligned}$$

**Exercise 10.2 (Connectivity arrays `j_cv`, `j_ce`).** (i) The connectivity arrays are

$$\mathtt{j\_cv} = \begin{pmatrix} 1 & 2 & 6 \\ 4 & 6 & 8 \\ 4 & 5 & 6 \\ 3 & 4 & 8 \\ 1 & 5 & 6 \end{pmatrix} \qquad \mathtt{j\_ce} = \begin{pmatrix} 2 & 10 & 1 \\ 5 & 8 & 11 \\ 4 & 11 & 9 \\ 8 & 6 & 7 \\ 4 & 10 & 3 \end{pmatrix}$$

(ii) The signs of the determinants are as follows:

$$\begin{bmatrix} \text{index of } K\text{:} & 1 & 2 & 3 & 4 & 5 \\ \text{sign}(\det \mathbb{J}_K)\text{:} & - & - & - & - & + \end{bmatrix}$$

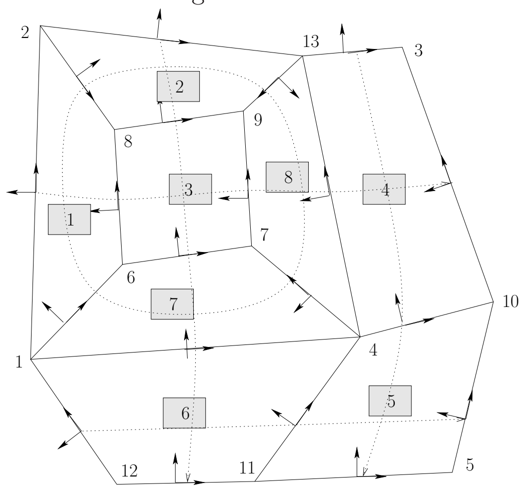
**Exercise 10.3 (Connectivity array `j_geo`).** (i) The following array `j_geo` is based on increasing vertex-index enumeration:

$$\mathtt{j\_geo} = \begin{pmatrix} 1 & 2 & 6 & 7 & 12 & 14 \\ 3 & 6 & 8 & 18 & 10 & 11 \\ 5 & 6 & 8 & 18 & 16 & 13 \\ 3 & 4 & 8 & 17 & 10 & 9 \\ 1 & 5 & 6 & 13 & 12 & 15 \end{pmatrix}.$$

(ii) The signs of the determinants are as follows:

$$\begin{bmatrix} \text{index of } K\text{:} & 1 & 2 & 3 & 4 & 5 \\ \text{sign}(\det \mathbb{J}_K)\text{:} & - & + & - & - & + \end{bmatrix}$$

**Exercise 10.4 (Orientation of quadrangular mesh).** (i) Here is a generation-compatible orientation of the edges and faces




The edges belonging to the same connected component of the edge/cell graph are linked by a dotted curve.
(ii) If one wishes that the determinant of the Jacobian matrix is positive for even quadrangles and negative for odd quadrangles, the geometric connectivity is as follows:

$$\texttt{j\_geo} = \begin{bmatrix} 1 & 2 & 6 & 8 \\ 2 & 8 & 13 & 9 \\ 6 & 8 & 7 & 9 \\ 4 & 10 & 13 & 3 \\ 11 & 4 & 5 & 10 \\ 12 & 11 & 1 & 4 \\ 1 & 6 & 4 & 7 \\ 4 & 13 & 7 & 9 \end{bmatrix}$$

Note that for each cell $K_m$, $m \in \{1{:}6\}$, $\texttt{j\_geo}(m, 1)$ gives the index of the vertex that is the origin of $K_m$ (such that the two edges sharing it are oriented away from it).

**Exercise 10.5 (Mesh extrusion).** (i) We first orient the edges of the bottom face using the increasing vertex-index enumeration. Then one needs to find a strategy to cut the three vertical faces. The key idea is to use the orientation of the edges of the bottom face. The cutting of the face whose vertices are $(\boldsymbol{z}_1, \boldsymbol{z}_2, \boldsymbol{z}_4, \boldsymbol{z}_5)$ is done by connecting $\boldsymbol{z}_1$ with $\boldsymbol{z}_5$, i.e., the cut starts from $\boldsymbol{z}_1$ and is done along the vector $(\boldsymbol{z}_2 - \boldsymbol{z}_1) + \boldsymbol{e}_3$. The cutting of the face whose vertices are $(\boldsymbol{z}_1, \boldsymbol{z}_3, \boldsymbol{z}_4, \boldsymbol{z}_6)$ is done by connecting $\boldsymbol{z}_1$ with $\boldsymbol{z}_6$, i.e., the cut starts from $\boldsymbol{z}_1$ and is done along the vector $(\boldsymbol{z}_3 - \boldsymbol{z}_1) + \boldsymbol{e}_3$. The cutting of the face whose vertices are $(\boldsymbol{z}_2, \boldsymbol{z}_3, \boldsymbol{z}_5, \boldsymbol{z}_6)$ is done by connecting $\boldsymbol{z}_2$ with $\boldsymbol{z}_6$, i.e., the cut starts from $\boldsymbol{z}_2$ and is done along the vector $(\boldsymbol{z}_3 - \boldsymbol{z}_2) + \boldsymbol{e}_3$. The proposed cutting produces three tetrahedra, with vertices $(\boldsymbol{z}_1, \boldsymbol{z}_4, \boldsymbol{z}_5, \boldsymbol{z}_6)$,

$(\boldsymbol{z}_1; \boldsymbol{z}_2, \boldsymbol{z}_5, \boldsymbol{z}_6)$, and $(\boldsymbol{z}_1, \boldsymbol{z}_2, \boldsymbol{z}_3, \boldsymbol{z}_6)$.
(ii) The key idea is to use the orientation of the edges of $\mathcal{T}_h$ to do the cutting of the vertical faces of the prisms produced by translating $\mathcal{T}_h$ in the $\boldsymbol{e}_3$ direction. Let $E$ be an edge of $\mathcal{T}_h$ with vertices $\boldsymbol{z}_p$, $\boldsymbol{z}_q$ and orientation vector $\boldsymbol{\tau}_E$, and assume that $\boldsymbol{z}_q - \boldsymbol{z}_p$ and $\boldsymbol{\tau}_E$ have the same orientation (notice that if $p < q$, then $\boldsymbol{z}_q - \boldsymbol{z}_p$ and $\boldsymbol{\tau}_E$ have the same orientation if the the increasing vertex-index enumeration technique is used). Let $\boldsymbol{z}_r := \boldsymbol{z}_p + \boldsymbol{e}_3$ and $\boldsymbol{z}_s := \boldsymbol{z}_q + \boldsymbol{e}_3$. Then we cut the vertical face whose vertices are $(\boldsymbol{z}_p, \boldsymbol{z}_q, \boldsymbol{z}_r, \boldsymbol{z}_s)$ by connecting $\boldsymbol{z}_p$ with $\boldsymbol{z}_s$, i.e., the cut starts from $\boldsymbol{z}_p$ and is done along the vector $\boldsymbol{\tau}_E + \boldsymbol{e}_3$. Notice that for the two prisms sharing the same rectangular face, the proposed strategy provides for a unique way to cut the face in question. As a result the mesh of tetrahedra thus formed is a matching mesh.