DESCRIPTION OF FINITE ELEMENT AND INTEGRATION METHODS

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a Generic Finite Element library in C++

Documentation, part 3

DESCRIPTION OF FINITE ELEMENT AND INTEGRATION METHODS

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Introduction

This documentation describes the different finite element methods and cubature formulas available in GETFEM++.

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Contents

1 Finite element methods

All finite element methods defined in GETFEM++ are interfaced in the file `getfem_fem.h`. A descriptor on a finite element method is available thanks to the function

```
getfem::pfem pf = getfem::fem_descriptor("name of method");
```

where `"name of method"` is a string to be choosen among the existing methods.

1.1 Finite element methods description

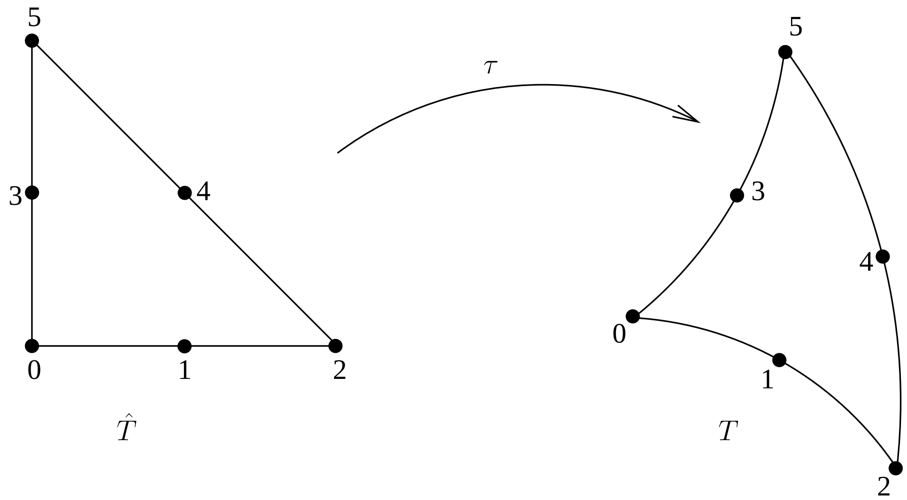


Figure 1: *Example of geometric transformation for a triangle.*

A finite element method is defined on a reference element $\hat{T} \subset \mathbb{R}^P$ by a set of n_d nodes a^i and corresponding base functions

$$\hat{\varphi}^i : \hat{T} \subset \mathbb{R}^P \longrightarrow \mathbb{R}^Q.$$

Each base function corresponds to a degree of freedom (d.o.f). Most finite element methods are scalar, which means that $Q = 1$, but GETFEM++ support also intrinsic vectorial elements. The map between the reference element and the real element is called the geometric transformation and is denoted by

$$\tau : \hat{T} \longrightarrow T,$$

and is generally polynomial (see [2] or [3]). The base functions $\hat{\varphi}^i$ defined on the reference element define a set of base function on the real element defined by

$$\tilde{\varphi}^i(x) = \hat{\varphi}^i(\hat{x}) = \hat{\varphi}^i(\tau^{-1}(x)),$$

If the element is said to be equivalent throught the geometric transformation τ (or τ-equivalent) then base functions on the real element are just defined by

$$\varphi^i(x) = \tilde{\varphi}^i(x).$$

This is generally the case for Lagrange element, but not for Hermite elements (when some dof represent the gradient of the unkown). When the element is not equivalent throught the geometric transformation then GETFEM++ allows to define a square matrix $\tilde{M}$ depending on the real element (i.e. on the geometric transformation) such that base functions on the real element are defined by

$$\varphi^i(x) = \sum_{j=0}^{n_d-1} \tilde{M}_{ij} \tilde{\varphi}^j(x).$$

We denote by

$$[\hat{\varphi}(\hat{x})] = \begin{pmatrix} \hat{\varphi}^0(\hat{x}) \\ \hat{\varphi}^1(\hat{x}) \\ \dots \\ \hat{\varphi}^{n_d-1}(\hat{x}) \end{pmatrix},$$

the $n_d \times Q$ matrix, such that when a function is defined by

$$f(x) = \sum_{i=0}^{n_d-1} \alpha_i \varphi^i(x),$$

one has

$$f(\tau(\hat{x})) = \alpha^T \tilde{M}[\hat{\varphi}(\hat{x})],$$

where α is the vector of components α_i.

1.1.1 Different types of d.o.f.

To each base function of a finite element method corresponds a degree of freedom (d.o.f) which is a linear form on this function. The following table gives the most significant types of d.o.f.

type	expression	commentary
Lagrange type	$\phi(a_i)$	Value of ϕ on the node a_i. The most simple d.o.f. Allows the Lagrange interpolation.
Hierarchical Lagrange type	$\phi(a_i) - \ldots$	Difference between the value of ϕ on the node a_i and the value of some other base functions. This is generally the bubble functions type of d.o.f .
mean type	$\frac{1}{\lvert T \rvert} \int_T \phi(x) dx$	Value of the mean value of ϕ on the element. Exists also for the restriction on a face.
derivative type	$\frac{\partial}{\partial x_i}\phi(a_i)$ or $\frac{\partial}{\partial n}\phi(a_i)$	Value of a derivative of ϕ on the node a_i. This kind of d.o.f. makes the element not to be τ-equivalent. $\frac{\partial}{\partial n}\phi(a_i)$ denotes the normal derivative with respect to a face.
second derivative type	$\frac{\partial^2}{\partial x_i \partial x_j}\phi(a_i)$	Value of a second derivative of ϕ on the node a_i. This kind of d.o.f. makes also the element not to be τ-equivalent.

1.1.2 Graphical codification of d.o.f.

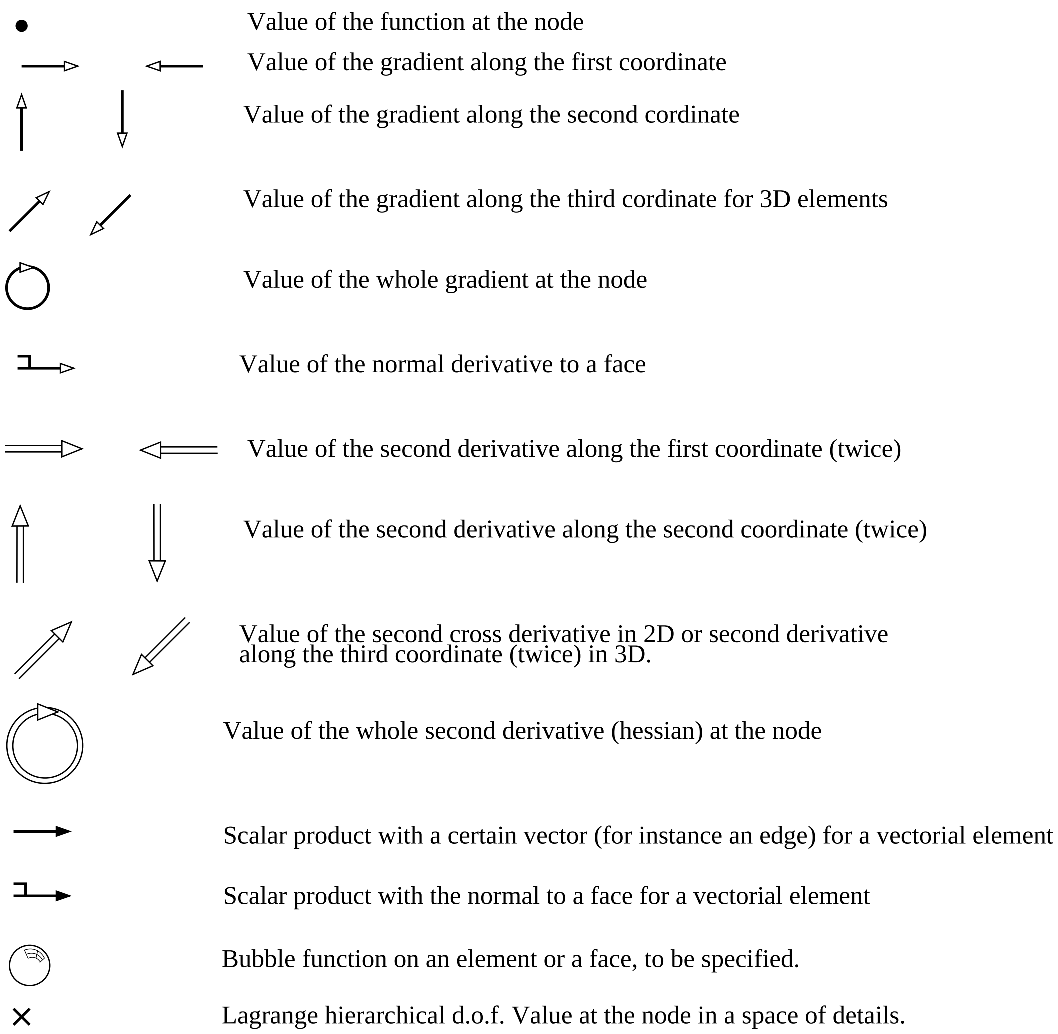


Figure 2: *Symbols representing degree of freedom types*

1.2 Classical “P_K” Lagrange elements on simplices

It is possible to define a classical “P_K” Lagrange element of arbitrary dimension and arbitrary degree. This element has only degrees of freedom which corresponds to the value of the function on a node. The grid of node is the so-called Lagrange grid. Figures 3, 4 and 5 show examples of dimension 1, 2 and 3.

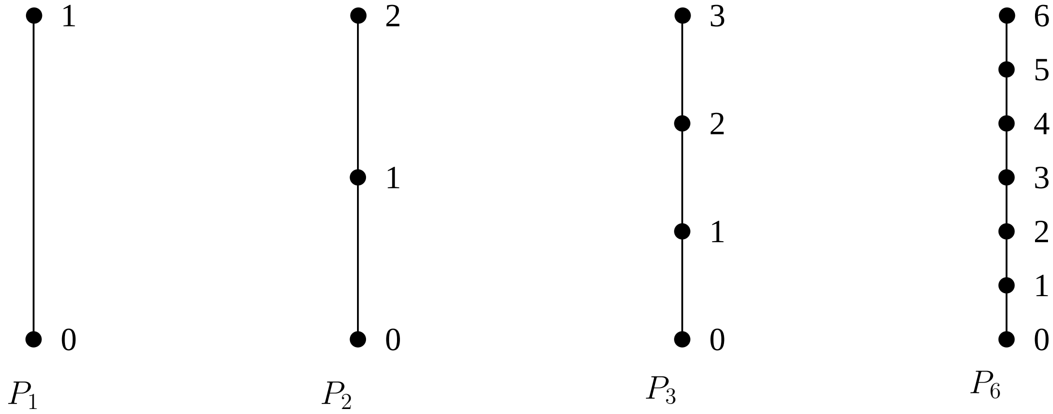


Figure 3: *Examples of classical P_K Lagrange elements on a segment.*

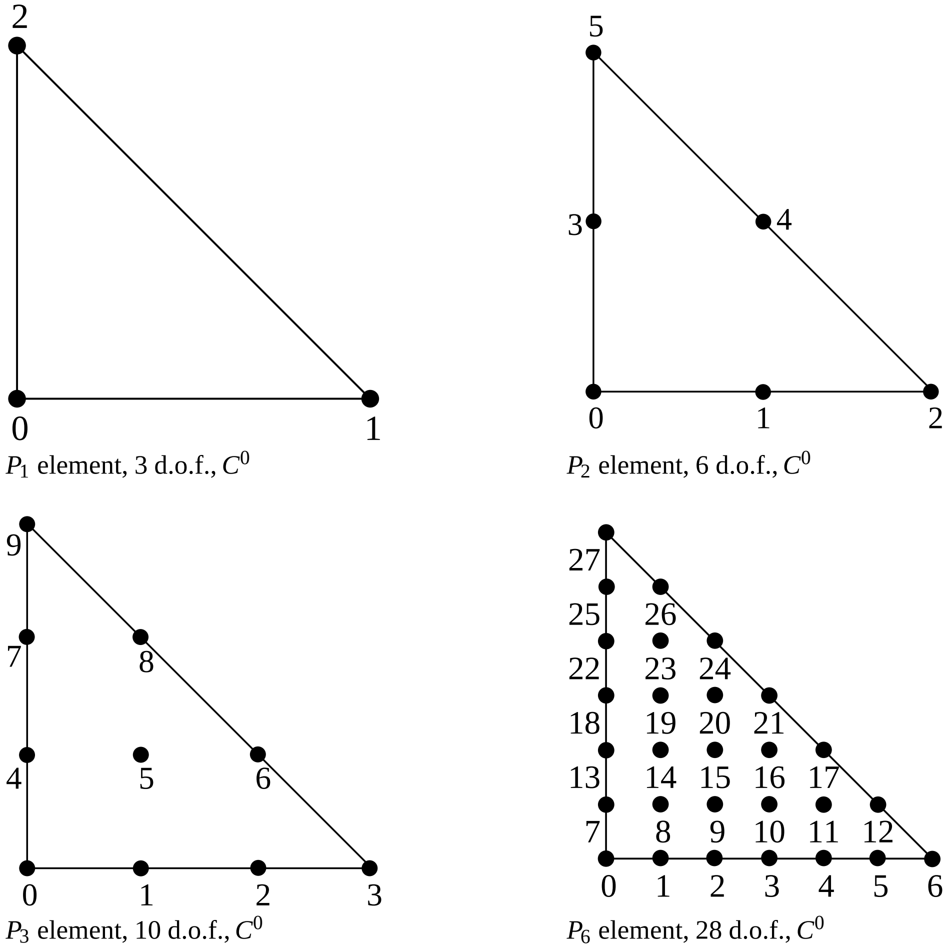
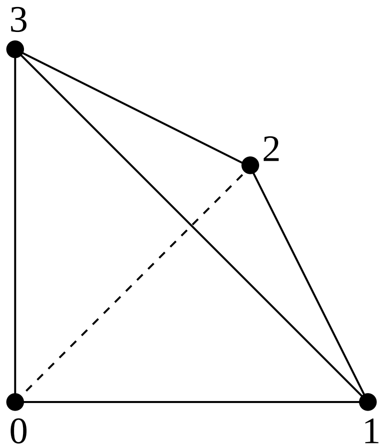
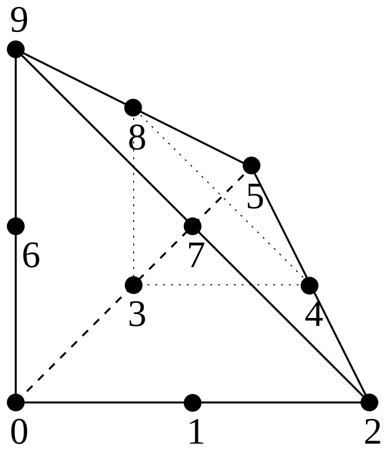


Figure 4: *Examples of classical P_K Lagrange elements on a triangle.*

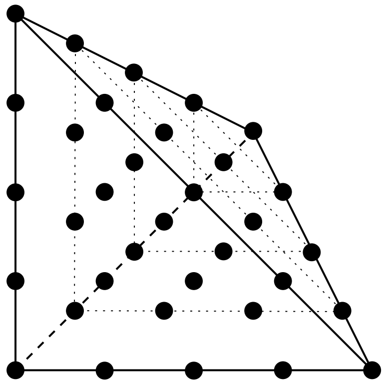
The number of degree of freedom for a classical “P_K” Lagrange element of dimension P and degree K is $\frac{(P+K)!}{P!K!}$. For instance, in dimension 2 ($P=2$), this value is $\frac{(P+1)(P+2)}{2}$, in dimension 3 ($P=3$), this value is $\frac{(P+1)(P+2)(P+3)}{6}$ ...



P_1 element, 4 d.o.f., C^0,



P_2 element, 10 d.o.f., C^0



P_4 element, 35 d.o.f., C^0

Figure 5: *Examples of classical P_K Lagrange elements on a tetrahedron.*

The particular way selected in GETFEM++ to numerate the nodes are also shown in figures 3, 4 and 5. Using another numeration, let

$$i_0, i_1, \ldots i_P,$$

be somme indices such that

$$0 \leq i_0, i_1, \ldots i_P \leq K, \ \text{ and } \sum_{n=0}^{P} i_n = K.$$

Then, the coordinate of a node can be computed as

$$a_{i_0,i_1,\ldots i_P} = \sum_{n=0}^{P} \frac{i_n}{K} S_n, \quad \text{for } K \neq 0,$$

where $S_0, S_1, \ldots S_N$ are the vertices of the simplex (for $K = 0$ the particular choice $a_{0,0,\ldots 0} = \sum_{n=0}^{P} \frac{1}{P+1} S_n$ has been chosen). Then each base function, corresponding of each node $a_{i_0,i_1,\ldots i_P}$ is defined by

$$\phi_{i_0,i_1,\ldots i_P} = \prod_{n=0}^{P} \prod_{j=0}^{i_n-1} \left(\frac{K\lambda_n - j}{j+1} \right).$$

where λ_n are the barycentric coordinates, i.e. the polynomials of degree 1 whose value is 1 on the vertex S_n and whose value is 0 on other vertices. On the reference element, one has

$$\lambda_n = x_n, \ \ 0 \leq n < P,$$

$$\lambda_P = 1 - x_0 - x_1 - \ldots - x_{P-1}.$$

When between two elements of the same degrees (even with different dimensions), the d.o.f. of a common face are linked, the element is of class C^0. This means that the global polynomial is continuous. If you try to link elements of different degrees, you will get some trouble with the unlinked d.o.f. This is not automatically supported by GETFEM++, so you will have to support it (add constraints on these d.o.f.).

For some applications (computation of a gradient for instance) one does not want the d.o.f. of a common face to be linked. This is why there are two versions of the classical "P_K" Lagrange element.

Classical "P_K" Lagrange element "FEM_PK(P, K)"						
Degree	dimension	d.o.f. number	class	vectorial	τ-equivalent	Polynomial
K, $0 \leq K \leq 255$	P, $1 \leq P \leq 255$	$\dfrac{(K+P)!}{K!P!}$	C^0	No ($Q = 1$)	Yes ($\tilde{M} = Id$)	Yes

Discontinuous "P_K" Lagrange element "FEM_PK_DISCONTINUOUS(P, K)"						
Degree	dimension	d.o.f. number	class	vectorial	τ-equivalent	Polynomial
K, $0 \leq K \leq 255$	P, $1 \leq P \leq 255$	$\dfrac{(K+P)!}{K!P!}$	discon-tinuous	No ($Q = 1$)	Yes ($\tilde{M} = Id$)	Yes

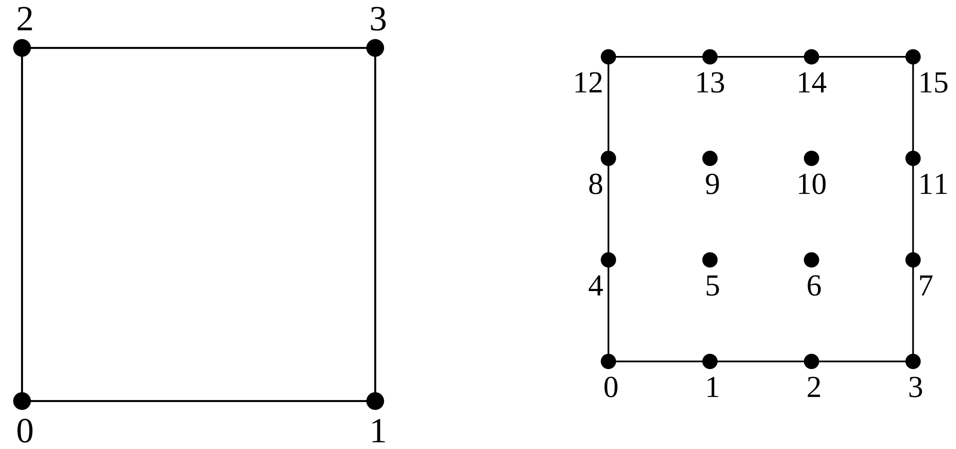
Even thought Lagrange elements are defined for arbitrary degrees, to choose a hight degree can be problematic for a large number of applications due to the "noisy" caracteristic of the lagrange basis. Those element are recommended for the basic interpolation but for p.d.e. applications elements with hierarchical basis are preferable (see the corresponding section).

1.3 Classical Lagrange elements on other geometries

Classical Lagrange elements on parallelepipeds or prisms are obtained as tensorial product of Lagrange elements on simplices. When two element are defined, one on a dimension P_1 and the other in dimension P_2, one obtains the base functions of the tensorial product (on the reference element) as

$$\hat{\varphi}_{ij}(x,y) = \hat{\varphi}^1_i(x)\hat{\varphi}^2_j(y), \quad x \in \mathbb{R}^{P_1}, y \in \mathbb{R}^{P_2},$$

where $\hat{\varphi}^1_i$ and $\hat{\varphi}^2_i$ are respectively the base functions of the first and second element.

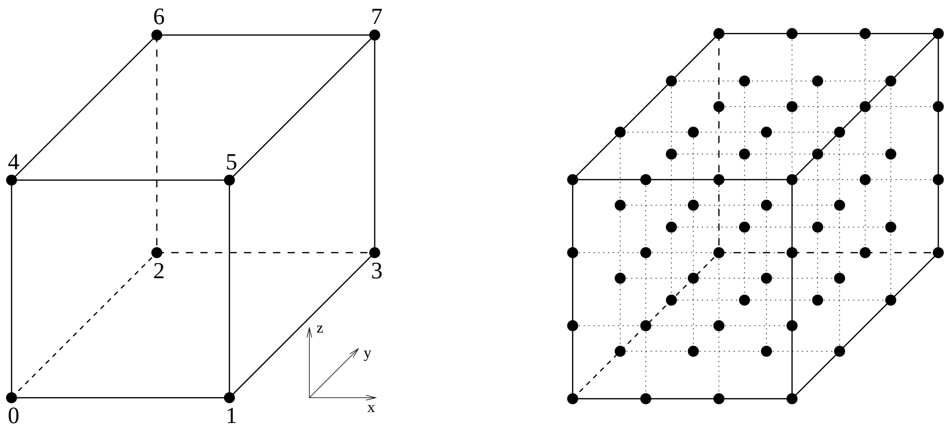


Q_1 element, 4 d.o.f., C^0 $\quad$ Q_3 element, 16 d.o.f., C^0

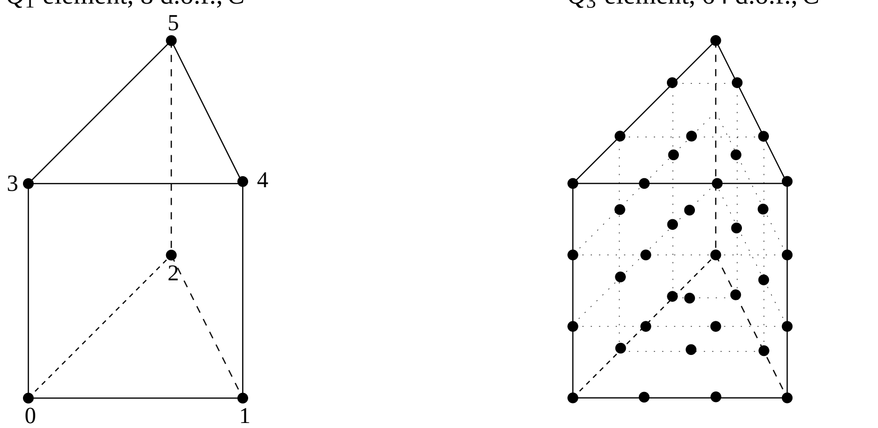
Figure 6: *Examples of classical Lagrange elements in dimension 2*

The Q_K element on a parallelepiped of dimension P is obtained as the tensorial product of P classical P_K element on the segment. Examples in dimension 2 are shown in figure 6 and in dimension 3 in figure 7.

A prism in dimension $P > 1$ is the direct product of a simplex of dimension $P-1$ with a segment. The $P_K \otimes P_K$ element on this prism is the tensorial product of the classical P_K element on a simplex of dimension $P-1$ with the classical P_K element on a segment. For $P = 2$ this coincide with a parallelepiped. Examples in dimension 3 are shown in figure 7. This is also possible not to have the same degree on each dimension. An example is shown on figure 8.



Q_1 element, 8 d.o.f., C^0 $\quad$ Q_3 element, 64 d.o.f., C^0



$P_1 \otimes P_1$ element, 6 d.o.f., C^0 $\quad$ $P_3 \otimes P_3$ element, 40 d.o.f., C^0

Figure 7: *Examples of classical Lagrange elements in dimension 3*

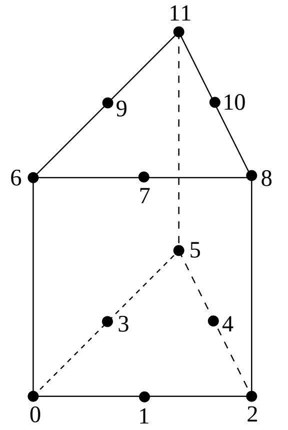
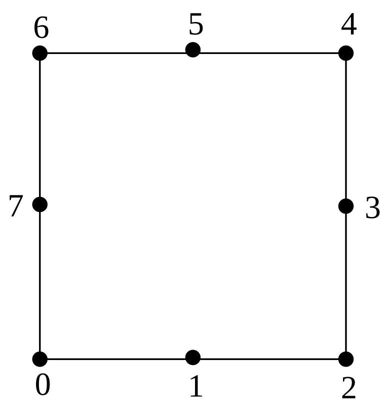


Figure 8: $P_2 \otimes P_1$ *Lagrange element on a prism, 12 d.o.f.,* C^0

Q_K **Lagrange element on parallelepipeds** "FEM_QK(P, K)"						
Degree	dimension	d.o.f. number	class	vectorial	τ-equivalent	Polynomial
KP, $0 \leq K \leq 255$	P, $2 \leq P \leq 255$	$(K+1)^P$	C^0	No $(Q=1)$	Yes ($\tilde{M} = Id$)	Yes

$P_K \otimes P_K$ **Lagrange element on prisms** "FEM_PK_PRISM(P, K)"						
Degree	dimension	d.o.f. number	class	vectorial	τ-equivalent	Polynomial
$2K$, $0 \leq K \leq 255$	P, $2 \leq P \leq 255$	$(K+1) \times \dfrac{(K+P-1)!}{K!(P-1)!}$	C^0	No $(Q=1)$	Yes ($\tilde{M} = Id$)	Yes

$P_{K_1} \otimes P_{K_2}$ **Lagrange element on prisms** "FEM_PRODUCT(FEM_PK(P-1, K_1), FEM_PK(1, K_2))"						
Degree	dimension	d.o.f. number	class	vectorial	τ-equivalent	Polynomial
$K_1 + K_2$, $0 \leq K_1, K_2 \leq 255$	P, $2 \leq P \leq 255$	$(K_2+1) \times \dfrac{(K_1+P-1)!}{K_1!(P-1)!}$	C^0	No $(Q=1)$	Yes ($\tilde{M} = Id$)	Yes



Incomplete Q_2 element, 8 d.o.f., C^0

Incomplete Q_2 Lagrange element on quadrilateral (Quad 8 serendipity element) "FEM_INCOMPLETE_Q2"						
Degree	dimension	d.o.f. number	class	vectorial	τ-equivalent	Polynomial
3	2	8	C^0	No $(Q=1)$	Yes ($\tilde{M} = Id$)	Yes

1.4 Elements with hierarchical basis

The idea behind hierarchical basis is the desciption of the solution at different level : a rought level, a more refined level ... In the same discretisation some degrees of freedom represent the rought description, some other the more rafined and so on. This correspond to imbricated spaces of discretisation. The hierarchical basis contains a basis of each of these spaces (this is not the case in classical Lagrange elements when the mesh is refined).

Among the advantages, the condition number of rigidity matrices can be greatly improved, it allows local raffinement and a resolution with a multigrid approach.

1.4.1 Hiercarchical elements with respect to the degree

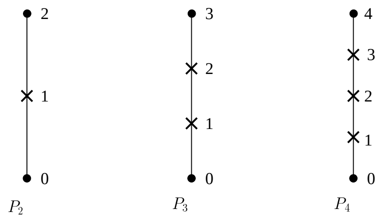


Figure 9: P_K *Hierarchical element on a segment,* C^0

P_K **Classical Lagrange element on simplices but with a hierarchical basis with respect to the degree** `"FEM_PK_HIERARCHICAL(P,K)"`						
Degree	dimension	d.o.f. number	class	vectorial	τ-equivalent	Polynomial
K, $0 \leq K \leq 255$	P, $1 \leq P \leq 255$	$\dfrac{(K+P)!}{K!P!}$	C^0	No ($Q=1$)	Yes ($\tilde{M} = Id$)	Yes

Q_K **Classical Lagrange element on parallelepipeds but with a hierarchical basis with respect to the degree** `"FEM_QK_HIERARCHICAL(P,K)"`						
Degree	dimension	d.o.f. number	class	vectorial	τ-equivalent	Polynomial
K, $0 \leq K \leq 255$	P, $2 \leq P \leq 255$	$(K+1)^P$	C^0	No ($Q=1$)	Yes ($\tilde{M} = Id$)	Yes

P_K **Classical Lagrange element on prisms but with a hierarchical basis with respect to the degree** `"FEM_PK_PRISM_HIERARCHICAL(P,K)"`						
Degree	dimension	d.o.f. number	class	vectorial	τ-equivalent	Polynomial
K, $0 \leq K \leq 255$	P, $2 \leq P \leq 255$	$(K+1) \times \dfrac{(K+P-1)!}{K!(P-1)!}$	C^0	No ($Q=1$)	Yes ($\tilde{M} = Id$)	Yes

some particular choices : P_4 will be build with the basis of the P_1, the additional basis of the P_2 then the additionnal basis of the P_4.

P_6 will be build with the basis of the P_1, the additional basis of the P_2 then the additionnal basis of the P_6 (not with the basis of the P_1, the additional basis of the P_3 then the additionnal basis of the P_6, this is possible to build the latter with `"FEM_GEN_HIERARCHICAL(a,b))`

1.4.2 Composite elements

The principal interest of the composite elements is to build hierarchical elements. But this tool can also be used to build piecewise polynomial elements.

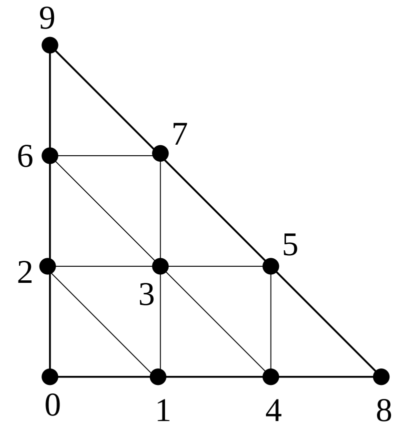


Figure 10: *composite element* "FEM_STRUCTURED_COMPOSITE(FEM_PK(2,1), 3)"

composition of a finite element method on a element with S subdivisions "FEM_STRUCTURED_COMPOSITE(FEM1, S)"						
Degree	dimension	d.o.f. number	class	vectorial	τ-equivalent	Polynomial
degree of FEM1	dimension of FEM1	variable	variable	No ($Q=1$)	If FEM1 is	piecewise

It is important to use a corresponding composite integration method.

1.4.3 Hierarchical composite elements

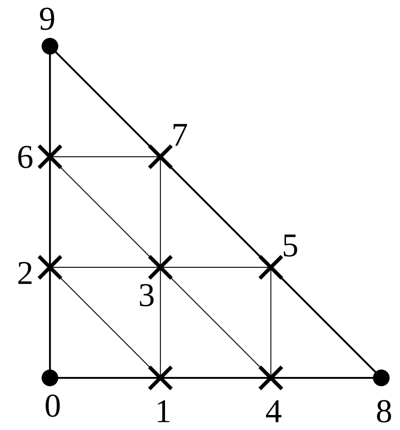


Figure 11: *hierarchical composite element* "FEM_PK_HIERARCHICAL_COMPOSITE(2,1,3)"

hierarchical composition of a P_K finite element method on a simplex with S subdivisions "FEM_PK_HIERARCHICAL_COMPOSITE(P,K,S)"						
Degree	dimension	d.o.f. number	class	vectorial	τ-equivalent	Polynomial
K	P	$\frac{(SK+P)!}{(SK)!P!}$	variable	No ($Q=1$)	If FEM1 is	piecewise

hierarchical composition of a hierarchical P_K finite element method on a simplex with S subdivisions "FEM_PK_FULL_HIERARCHICAL_COMPOSITE(P,K,S)"						
Degree	dimension	d.o.f. number	class	vectorial	τ-equivalent	Polynomial
K	P	$\frac{(SK+P)!}{(SK)!P!}$	variable	No ($Q=1$)	If FEM1 is	piecewise

Other constructions are possible thanks to "FEM_GEN_HIERARCHICAL(FEM1, FEM2)" and "FEM_STRUCTURED_COMPOSITE(FEM1, S)"

It is important to use a corresponding composite integration method.

1.5 Classical vectorial elements

1.5.1 Raviart-Thomas 0 elements

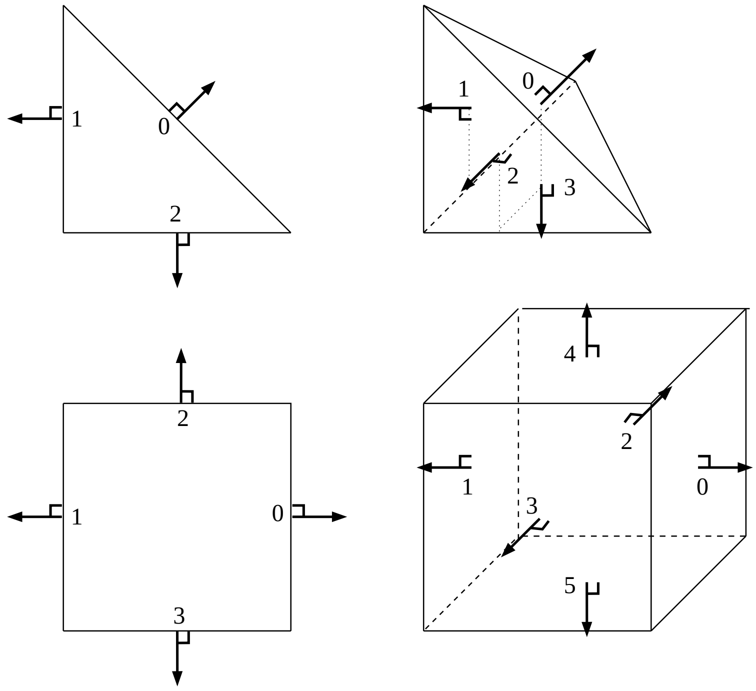


Figure 12: *RT0 elements in dimension two and three. (P+1 dof, H(div))*

Raviart-Thomas 0 element on simplices "FEM_RT0(P)"						
Degree	dimension	d.o.f. number	class	vectorial	τ-equivalent	Polynomial
1	P	$P+1$	$H(div)$	Yes ($Q=P$)	No	Yes

Raviart-Thomas 0 element on parallelepipeds (quadrilaterals, hexahedrals) "FEM_RT0Q(P)"						
Degree	dimension	d.o.f. number	class	vectorial	τ-equivalent	Polynomial
1	P	$2P$	$H(div)$	Yes ($Q=P$)	No	Yes

1.5.2 Nedelec (or Whitney) edge elements

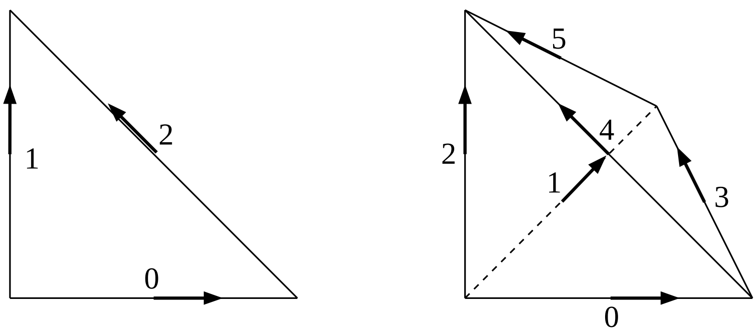


Figure 13: *Nedelec edge element in dimension two and three. (P(P+1)/2 dof, H(rot))*

Nedelec (or Whitney) edge element "FEM_NEDELEC(P)"						
Degree	dimension	d.o.f. number	class	vectorial	τ-equivalent	Polynomial
1	P	$P(P+1)/2$	$H(rot)$	Yes ($Q=P$)	No	Yes

1.6 Specific elements in dimension 1

1.6.1 GaussLobatto element

The 1D GaussLobatto P_K element is similar to the classical P_K fem on the segment, but the nodes are given by the Gauss-Lobatto-Legendre quadrature rule of order $2K-1$. This FEM is known to lead to better conditioned linear systems, and can be used with the correspounding quadrature to perform mass-lumping (on segments or parallelepipeds).

The polynomials coefficients have been pre-computed with Maple (they require the inversion of an ill-conditionned system), hence they are only available for the following values
of K: $1,2,3,4,5,6,7,8,9,10,11,12,13,14,16,24,32$. Note that for $K=1$ and $K=2$, this is the classical $P1$ and $P2$ fem.

GaussLobatto P_K element on the segment "FEM_PK_GAUSSLOBATTO1D(K)"						
Degree	dimension	d.o.f. number	class	vectorial	τ-equivalent	Polynomial
K	1	$K+1$	C^0	No ($Q=1$)	Yes	Yes

1.6.2 Hermite element

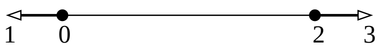


Figure 14: P_3 *Hermite element on a segment, 4 d.o.f.,* C^1

Base functions on the reference element

$$\hat{\varphi}_0 = (2x+1)(x-1)^2, \qquad \hat{\varphi}_1 = x(x-1)^2,$$
$$\hat{\varphi}_2 = x^2(3-2x), \qquad \hat{\varphi}_3 = x^2(x-1).$$

This element is close to be τ-equivalent but it is not. On the real element the value of the gradient on vertices will be multiplied by the gradient of the geometric transformation. The matrix $\tilde{M}$ is not equal to identity but is still diagonal.

Hermite element on the segment "FEM_HERMITE(1)"						
Degree	dimension	d.o.f. number	class	vectorial	τ-equivalent	Polynomial
3	1	4	C^1	No ($Q=1$)	No	Yes

1.6.3 Lagrange element with an additional bubble function

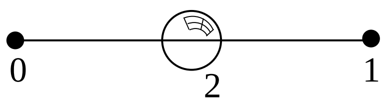
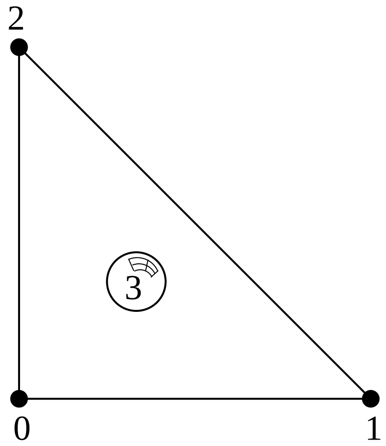


Figure 15: P_1 *Lagrange element on a segment with additional internal bubble function, 3 d.o.f.,* C^0

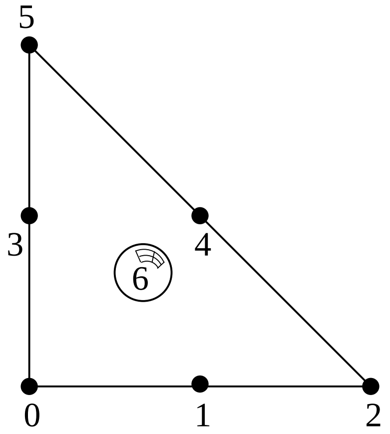
Lagrange P_1 element with an additional internal bubble function "FEM_PK_WITH_CUBIC_BUBBLE(1, 1)"						
Degree	dimension	d.o.f. number	class	vectorial	τ-equivalent	Polynomial
2	1	3	C^0	No ($Q=1$)	Yes	Yes

1.7 Specific elements in dimension 2

1.7.1 Elements with additional bubble functions



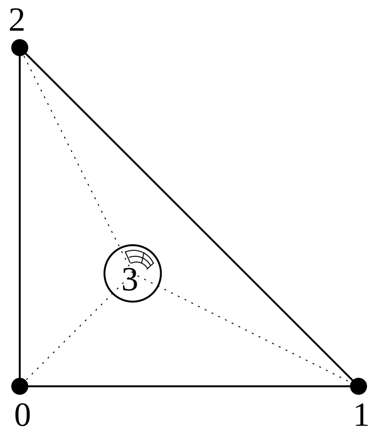
P_1 with additional bubble function, 4 d.o.f., C^0



P_2 with additional bubble function, 7 d.o.f., C^0

Figure 16: *Lagrange element on a triangle with additional internal bubble function*

Lagrange P_1 or P_2 element with an additional internal bubble function "FEM_PK_WITH_CUBIC_BUBBLE(2, K)"						
Degree	dimension	d.o.f. number	class	vectorial	τ-equivalent	Polynomial
3	2	4 or 7	C^0	No ($Q=1$)	Yes	Yes



P_1 with additional bubble function, 4 d.o.f., C^0

Figure 17: P_1 *Lagrange element on a triangle with additional internal piecewise linear bubble function*

Lagrange P_1 with an additional internal piecewise linear bubble function "FEM_P1_PIECEWISE_LINEAR_BUBBLE"						
Degree	dimension	d.o.f. number	class	vectorial	τ-equivalent	Polynomial
1	2	4	C^0	No ($Q=1$)	Yes	No

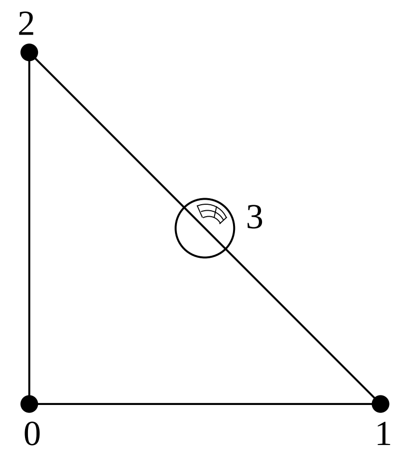


Figure 18: P_1 *Lagrange element on a triangle with additional bubble function on face 0, 4 d.o.f.,* C^0

Lagrange P_1 element with an additional bubble function on face 0 `"FEM_P1_BUBBLE_FACE(2)"`						
Degree	dimension	d.o.f. number	class	vectorial	τ-equivalent	Polynomial
2	2	4	C^0	No ($Q = 1$)	Yes	Yes

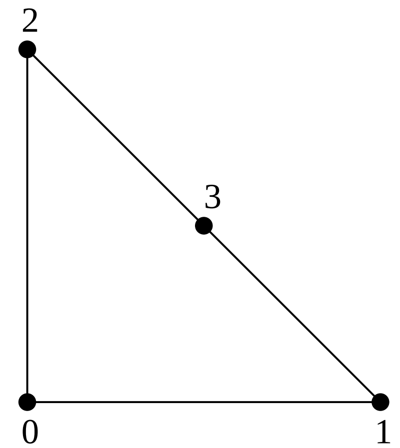


Figure 19: P_1 *Lagrange element on a triangle with additional d.o.f on face 0, 4 d.o.f.,* C^0

P_1 Lagrange element on a triangle with additional d.o.f on face 0 `"FEM_P1_BUBBLE_FACE_LAG"`						
Degree	dimension	d.o.f. number	class	vectorial	τ-equivalent	Polynomial
2	2	4	C^0	No ($Q = 1$)	Yes	Yes

1.7.2 Non-conforming P_1 element

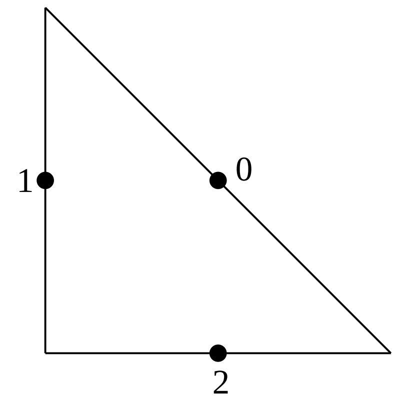


Figure 20: P_1 *non-conforming element on a triangle, 3 d.o.f., discontinuous*

P_1 non-conforming element on a triangle "FEM_P1_NONCONFORMING"						
Degree	dimension	d.o.f. number	class	vectorial	τ-equivalent	Polynomial
1	2	3	discon-tinuous	No ($Q = 1$)	Yes	Yes

1.7.3 Hermite element

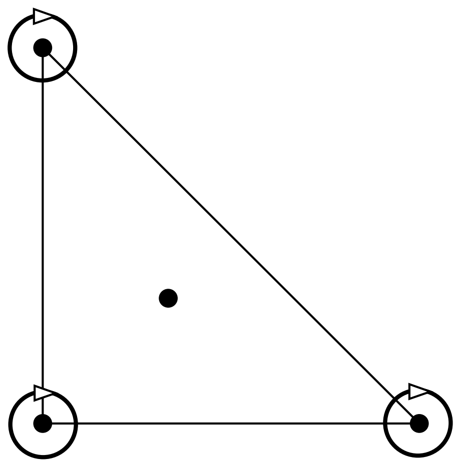


Figure 21: *Hermite element on a triangle, P_3, 10 d.o.f., C^0*

Base functions on the reference element:

$$
\begin{array}{ll}
\hat{\varphi}_0 = (1-x-y)(1+x+y-2x^2-2y^2-11xy), & (\hat{\varphi}_0(0,0)=1), \\
\hat{\varphi}_1 = x(1-x-y)(1-x-2y), & (\partial_x\hat{\varphi}_1(0,0)=1), \\
\hat{\varphi}_2 = y(1-x-y)(1-2x-y), & (\partial_y\hat{\varphi}_2(0,0)=1), \\
\hat{\varphi}_3 = -2x^3+7x^2y+7xy^2+3x^2-7xy, & (\hat{\varphi}_3(1,0)=1), \\
\hat{\varphi}_4 = x^3-2x^2y-2xy^2-x^2+2xy, & (\partial_x\hat{\varphi}_4(1,0)=1), \\
\hat{\varphi}_5 = xy(y+2x-1), & (\partial_y\hat{\varphi}_5(1,0)=1), \\
\hat{\varphi}_6 = 7x^2y+7xy^2-2y^3+3y^2-7xy, & (\hat{\varphi}_6(0,1)=1), \\
\hat{\varphi}_7 = xy(x+2y-1), & (\partial_x\hat{\varphi}_7(0,1)=1), \\
\hat{\varphi}_8 = y^3-2x^2y-2xy^2-y^2+2xy, & (\partial_y\hat{\varphi}_8(0,1)=1), \\
\hat{\varphi}_9 = 27xy(1-x-y), & (\hat{\varphi}_9(1/3,1/3)=1),
\end{array}
$$

This element is not τ-equivalent (The matrix $\tilde{M}$ is not equal to identity). On the real element linear combinaisons of $\hat{\varphi}_4$ and $\hat{\varphi}_7$ are used to match the gradient on the corresponding vertex. Idem for the two couples ($\hat{\varphi}_5$, $\hat{\varphi}_8$) and ($\hat{\varphi}_6$, $\hat{\varphi}_9$) for the two other vertices.

Hermite element on a triangle "FEM_HERMITE(2)"						
Degree	dimension	d.o.f. number	class	vectorial	τ-equivalent	Polynomial
3	2	10	C^0	No ($Q = 1$)	No	Yes

1.7.4 Morley element

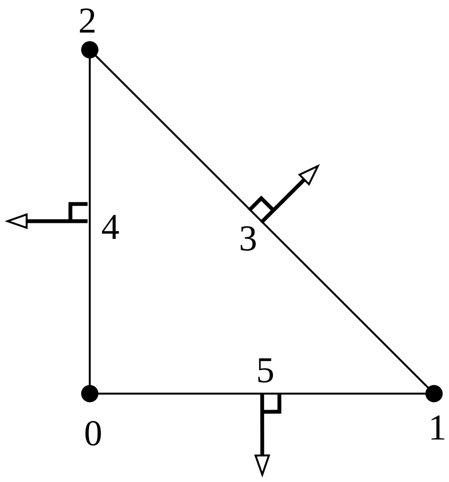


Figure 22: *tiangle Morley element, P_2, 6 d.o.f., C^0*

This element is not τ-equivalent (The matrix $\tilde{M}$ is not equal to identity). In particular, it can be used for non-conforming discretization of fourth order problems, despite the fact that it is not $\mathcal{C}^0$.

Morley element on a triangle "FEM_MORLEY"						
Degree	dimension	d.o.f. number	class	vectorial	τ-equivalent	Polynomial
2	2	6		No ($Q = 1$)	No	Yes

1.7.5 Argyris element

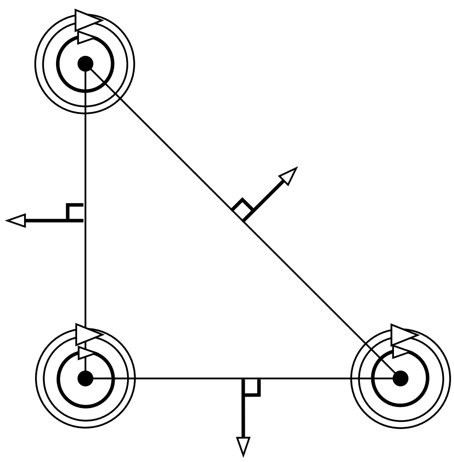


Figure 23: *Argyris element, P_5, 21 d.o.f., C^1*

The base functions on the reference element are:

$$
\begin{array}{ll}
\hat{\varphi}_0(x,y) = 1 - 10x^3 - 10y^3 + 15x^4 - 30x^2y^2 + 15y^4 - 6x^5 + 30x^3y^2 + 30x^2y^3 - 6y^5, & (\hat{\varphi}_0(0,0) = 1), \\
\hat{\varphi}_1(x,y) = x - 6x^3 - 11xy^2 + 8x^4 + 10x^2y^2 + 18xy^3 - 3x^5 + x^3y^2 - 10x^2y^3 - 8xy^4, & (\partial_x\hat{\varphi}_1(0,0) = 1), \\
\hat{\varphi}_2(x,y) = y - 11x^2y - 6y^3 + 18x^3y + 10x^2y^2 + 8y^4 - 8x^4y - 10x^3y^2 + x^2y^3 - 3y^5, & (\partial_y\hat{\varphi}_2(0,0) = 1), \\
\hat{\varphi}_3(x,y) = 0.5x^2 - 1.5x^3 + 1.5x^4 - 1.5x^2y^2 - 0.5x^5 + 1.5x^3y^2 + x^2y^3, & (\partial^2_{xx}\hat{\varphi}_3(0,0) = 1), \\
\hat{\varphi}_4(x,y) = xy - 4x^2y - 4xy^2 + 5x^3y + 10x^2y^2 + 5xy^3 - 2x^4y - 6x^3y^2 - 6x^2y^3 - 2xy^4, & (\partial^2_{xy}\hat{\varphi}_4(0,0) = 1), \\
\hat{\varphi}_5(x,y) = 0.5y^2 - 1.5y^3 - 1.5x^2y^2 + 1.5y^4 + x^3y^2 + 1.5x^2y^3 - 0.5y^5, & (\partial^2_{yy}\hat{\varphi}_5(0,0) = 1), \\
\hat{\varphi}_6(x,y) = 10x^3 - 15x^4 + 15x^2y^2 + 6x^5 - 15x^3y^2 - 15x^2y^3, & (\hat{\varphi}_6(1,0) = 1), \\
\hat{\varphi}_7(x,y) = -4x^3 + 7x^4 - 3.5x^2y^2 - 3x^5 + 3.5x^3y^2 + 3.5x^2y^3, & (\partial_x\hat{\varphi}_7(1,0) = 1), \\
\hat{\varphi}_8(x,y) = -5x^2y + 14x^3y + 18.5x^2y^2 - 8x^4y - 18.5x^3y^2 - 13.5x^2y^3, & (\partial_y\hat{\varphi}_8(1,0) = 1), \\
\hat{\varphi}_9(x,y) = 0.5x^3 - x^4 + 0.25x^2y^2 + 0.5x^5 - 0.25x^3y^2 - 0.25x^2y^3, & (\partial^2_{xx}\hat{\varphi}_9(1,0) = 1), \\
\hat{\varphi}_{10}(x,y) = x^2y - 3x^3y - 3.5x^2y^2 + 2x^4y + 3.5x^3y^2 + 2.5x^2y^3, & (\partial^2_{xy}\hat{\varphi}_{10}(1,0) = 1), \\
\hat{\varphi}_{11}(x,y) = 1.25x^2y^2 - 0.75x^3y^2 - 1.25x^2y^3, & (\partial^2_{yy}\hat{\varphi}_{11}(1,0) = 1), \\
\hat{\varphi}_{12}(x,y) = 10y^3 + 15x^2y^2 - 15y^4 - 15x^3y^2 - 15x^2y^3 + 6y^5, & (\hat{\varphi}_{12}(0,1) = 1), \\
\hat{\varphi}_{13}(x,y) = -5xy^2 + 18.5x^2y^2 + 14xy^3 - 13.5x^3y^2 - 18.5x^2y^3 - 8xy^4, & (\partial_x\hat{\varphi}_{13}(0,1) = 1), \\
\hat{\varphi}_{14}(x,y) = -4y^3 - 3.5x^2y^2 + 7y^4 + 3.5x^3y^2 + 3.5x^2y^3 - 3y^5, & (\partial_y\hat{\varphi}_{14}(0,0) = 1), \\
\hat{\varphi}_{15}(x,y) = 1.25x^2y^2 - 1.25x^3y^2 - 0.75x^2y^3, & (\partial^2_{xx}\hat{\varphi}_{15}(0,1) = 1), \\
\hat{\varphi}_{16}(x,y) = xy^2 - 3.5x^2y^2 - 3xy^3 + 2.5x^3y^2 + 3.5x^2y^3 + 2xy^4, & (\partial^2_{xy}\hat{\varphi}_{16}(0,1) = 1), \\
\hat{\varphi}_{17}(x,y) = 0.5y^3 + 0.25x^2y^2 - y^4 - 0.25x^3y^2 - 0.25x^2y^3 + 0.5y^5, & (\partial^2_{yy}\hat{\varphi}_{17}(0,1) = 1), \\
\hat{\varphi}_{18}(x,y) = \sqrt{2}(-8x^2y^2 + 8x^3y^2 + 8x^2y^3), & (\sqrt{0.5}(\partial_x\hat{\varphi}_{18}(0.5,0.5) + \partial_y\hat{\varphi}_{18}(0.5,0.5)) = 1), \\
\hat{\varphi}_{19}(x,y) = -16xy^2 + 32x^2y^2 + 32xy^3 - 16x^3y^2 - 32x^2y^3 - 16xy^4, & (-\partial_x\hat{\varphi}_{19}(0,0.5) = 1), \\
\hat{\varphi}_{20}(x,y) = -16x^2y + 32x^3y + 32x^2y^2 - 16x^4y - 32x^3y^2 - 16x^2y^3, & (-\partial_y\hat{\varphi}_{20}(0.5,0) = 1),
\end{array}
$$

This element is not τ-equivalent (The matrix $\tilde{M}$ is not equal to identity). On the real element linear combinaisons of the transformed base functions $\hat{\varphi}_i$ are used to match the gradient, the second derivatives and the normal derivatives on the faces. Note that the use of the matrix $\tilde{M}$ (see also the documentation on the finite element kernel [3]) allows to define Argyris element even with nonlinear geometric transformations (for instance to treat curved boundaries).

Argyris element on a triangle "FEM_ARGYRIS"						
Degree	dimension	d.o.f. number	class	vectorial	τ-equivalent	Polynomial
5	2	21	C^1	No ($Q = 1$)	No	Yes

1.7.6 Hsieh-Clough-Tocher element

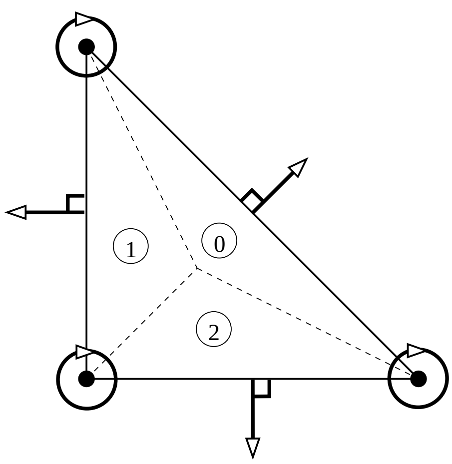


Figure 24: *Hsieh-Clough-Tocher (HCT) element, P_3, 12 d.o.f., C^1*

This element is not τ-equivalent. This is a composite element. Polynomial of degree 3 on each of the three sub-triangles (see figure 24 and [1]). It is strongly advised to use a `IM_HCT_COMPOSITE` integration method with this finite element. The numeration of the dof is the following : 0, 3 and 6 for the lagrange dof on the first second and third vertex respectively; 1, 4, 7 for the derivative with respects to the first variable; 2, 5, 8 for the derivative with respects to the second variable and 9, 10, 11 for the normal derivatives on face 0, 1, 2 respectively.

HCT element on a triangle "FEM_HCT_TRIANGLE"						
Degree	dimension	d.o.f. number	class	vectorial	τ-equivalent	Polynomial
3	2	12	C^1	No ($Q = 1$)	No	composite

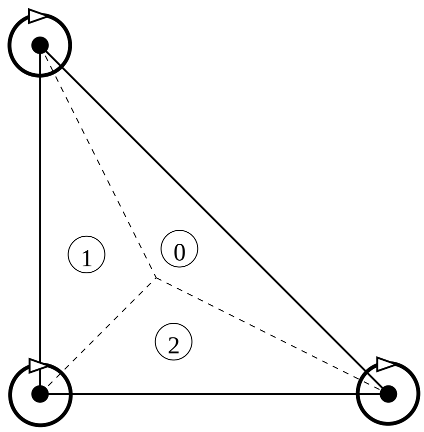


Figure 25: *Reduced Hsieh-Clough-Tocher (reduced HCT) element, P_3, 9 d.o.f., C^1*

This element exists also in its reduced form, where the normal derivatives is assumed to be polynomial of degree one on each edge (see figure 25)

Reduced HCT element on a triangle "FEM_REDUCED_HCT_TRIANGLE"						
Degree	dimension	d.o.f. number	class	vectorial	τ-equivalent	Polynomial
3	2	9	C^1	No ($Q = 1$)	No	composite

1.7.7 A composite C^1 element on quadrilaterals

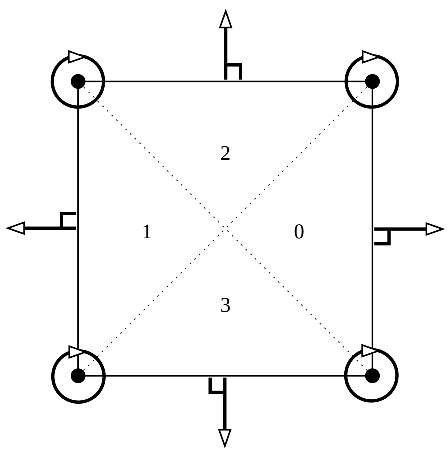


Figure 26: *Composite element on quadrilaterals, piecewise P_3, 16 d.o.f., C^1*

This element is not τ-equivalent. This is a composite element. Polynomial of degree 3 on each of the four sub-triangles (see figure 26). At least on the reference element it correponds to the Fraeijs de Veubeke-Sander element (see [1]). It is strongly advised to use a `IM_QUADC1_COMPOSITE` integration method with this finite element.

HCT element on a triangle "FEM_QUADC1_COMPOSITE"						
Degree	dimension	d.o.f. number	class	vectorial	τ-equivalent	Polynomial
3	2	16	C^1	No ($Q = 1$)	No	composite

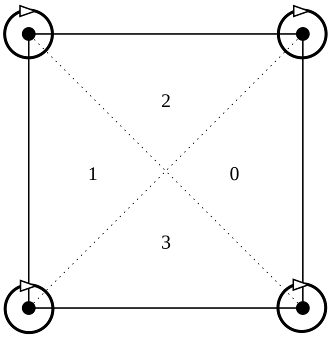


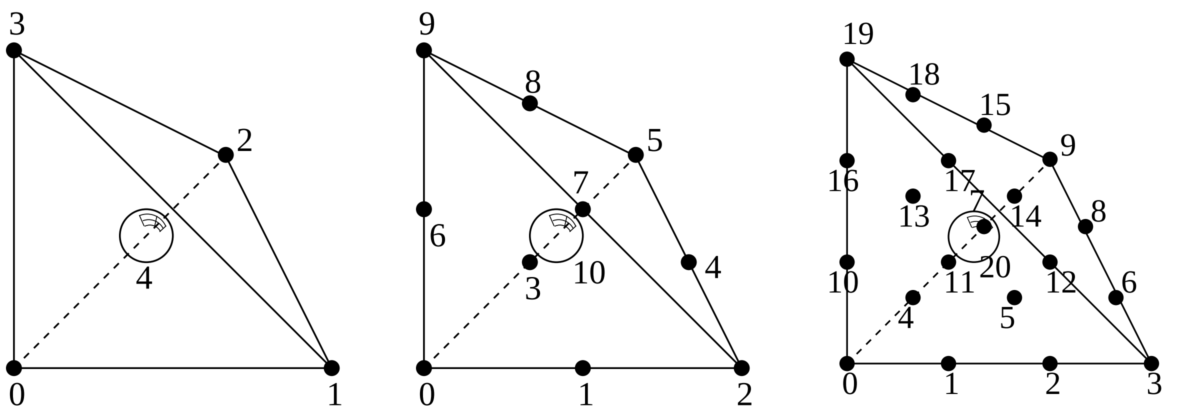
Figure 27: *Reduced composite element on quadrilaterals, piecewise P_3, 12 d.o.f., C^1*

This element exists also in its reduced form, where the normal derivatives is assumed to be polynomial of degree one on each edge (see figure 27)

Reduced HCT element on a triangle "FEM_REDUCED_QUADC1_COMPOSITE"						
Degree	dimension	d.o.f. number	class	vectorial	τ-equivalent	Polynomial
3	2	12	C^1	No ($Q = 1$)	No	composite

1.8 Specific elements in dimension 3

1.8.1 Elements with additional bubble functions



P_1 with additional bubble function, 5 d.o.f., C^0 — P_2 with additional bubble function, 11 d.o.f., C^0 — P_3 with additional bubble function, 21 d.o.f., C^0

Figure 28: *Lagrange element on a tetrahedron with additional internal bubble function.*

P_K **Lagrange element with an additional internal bubble function** `"FEM_PK_WITH_CUBIC_BUBBLE(3, K)"`

Degree	dimension	d.o.f. number	class	vectorial	τ-equivalent	Polynomial
4	3	5, 11 or 21	C^0	No ($Q = 1$)	Yes	Yes

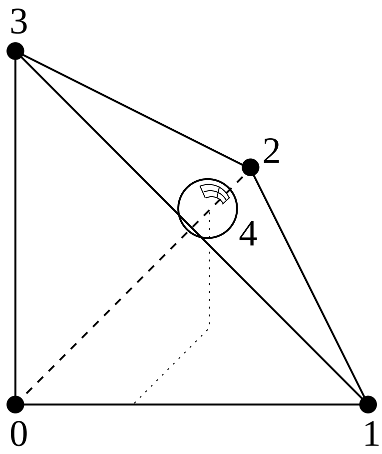


Figure 29: *P_1 Lagrange element on a tetrahedron with additional bubble function on face 0, 5 d.o.f., C^0*

Lagrange P_1 element with an additional bubble function on face 0 "FEM_P1_BUBBLE_FACE(3)"						
Degree	dimension	d.o.f. number	class	vectorial	τ-equivalent	Polynomial
3	3	5	C^0	No ($Q = 1$)	Yes	Yes

1.8.2 Hermite element

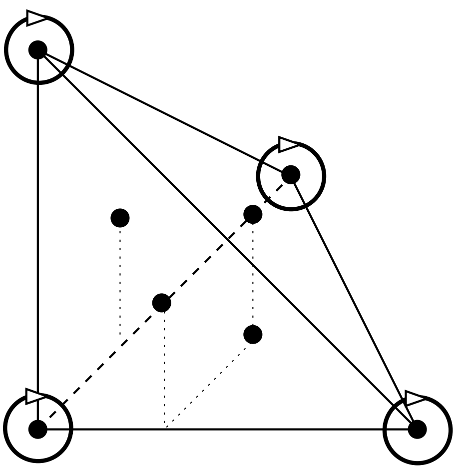


Figure 30: *Hermite element on a tetrahedron, P_3, 20 d.o.f., C^0*

Base functions on the reference element:

$$
\begin{aligned}
\hat{\varphi}_0(x,y) &= 1-3x^2-13xy-13xz-3y^2-13yz-3z^2+2x^3+13x^2y+13x^2z \\
&\quad +13xy^2+33xyz+13xz^2+2y^3+13y^2z+13yz^2+2z^3, && (\hat{\varphi}_0(0,0,0)=1), \\
\hat{\varphi}_1(x,y) &= x-2x^2-3xy-3xz+x^3+3x^2y+3x^2z+2xy^2+4xyz+2xz^2, && (\partial_x\hat{\varphi}_1(0,0,0)=1), \\
\hat{\varphi}_2(x,y) &= y-3xy-2y^2-3yz+2x^2y+3xy^2+4xyz+y^3+3y^2z+2yz^2, && (\partial_y\hat{\varphi}_2(0,0,0)=1), \\
\hat{\varphi}_3(x,y) &= z-3xz-3yz-2z^2+2x^2z+4xyz+3xz^2+2y^2z+3yz^2+z^3, && (\partial_z\hat{\varphi}_3(0,0,0)=1), \\
\hat{\varphi}_4(x,y) &= 3x^2-7xy-7xz-2x^3+7x^2y+7x^2z+7xy^2+7xyz+7xz^2, && (\hat{\varphi}_4(1,0,0)=1), \\
\hat{\varphi}_5(x,y) &= -x^2+2xy+2xz+x^3-2x^2y-2x^2z-2xy^2-2xyz-2xz^2, && (\partial_x\hat{\varphi}_5(1,0,0)=1), \\
\hat{\varphi}_6(x,y) &= -xy+2x^2y+xy^2, && (\partial_y\hat{\varphi}_6(1,0,0)=1), \\
\hat{\varphi}_7(x,y) &= -xz+2x^2z+xz^2, && (\partial_z\hat{\varphi}_7(1,0,0)=1), \\
\hat{\varphi}_8(x,y) &= -7xy+3y^2-7yz+7x^2y+7xy^2+7xyz-2y^3+7y^2z+7yz^2, && (\hat{\varphi}_8(0,1,0)=1), \\
\hat{\varphi}_9(x,y) &= -xy+x^2y+2xy^2, && (\partial_x\hat{\varphi}_9(0,1,0)=1), \\
\hat{\varphi}_{10}(x,y) &= 2xy-y^2+2yz-2x^2y-2xy^2-2xyz+y^3-2y^2z-2yz^2, && (\partial_y\hat{\varphi}_{10}(0,1,0)=1), \\
\hat{\varphi}_{11}(x,y) &= -yz+2y^2z+yz^2, && (\partial_z\hat{\varphi}_{11}(0,1,0)=1), \\
\hat{\varphi}_{12}(x,y) &= -7xz-7yz+3z^2+7x^2z+7xyz+7xz^2+7y^2z+7yz^2-2z^3, && (\hat{\varphi}_{12}(0,0,1)=1), \\
\hat{\varphi}_{13}(x,y) &= -xz+x^2z+2xz^2, && (\partial_x\hat{\varphi}_{13}(0,0,1)=1), \\
\hat{\varphi}_{14}(x,y) &= -yz+y^2z+2yz^2, && (\partial_y\hat{\varphi}_{14}(0,0,1)=1), \\
\hat{\varphi}_{15}(x,y) &= 2xz+2yz-z^2-2x^2z-2xyz-2xz^2-2y^2z-2yz^2+z^3, && (\partial_z\hat{\varphi}_{15}(0,0,1)=1), \\
\hat{\varphi}_{16}(x,y) &= 27xyz, && (\hat{\varphi}_{16}(1/3,1/3,1/3)=1), \\
\hat{\varphi}_{17}(x,y) &= 27yz-27xyz-27y^2z-27yz^2, && (\hat{\varphi}_{17}(0,1/3,1/3)=1), \\
\hat{\varphi}_{18}(x,y) &= 27xz-27x^2z-27xyz-27xz^2, && (\hat{\varphi}_{18}(1/3,0,1/3)=1), \\
\hat{\varphi}_{19}(x,y) &= 27xy-27x^2y-27xy^2-27xyz, && (\hat{\varphi}_{19}(1/3,1/3,0)=1),
\end{aligned}
$$

This element is not τ-equivalent (The matrix $\tilde{M}$ is not equal to identity). On the real element linear combinaisons of $\hat{\varphi}_8$, $\hat{\varphi}_{12}$ and $\hat{\varphi}_{16}$ are used to match the gradient on the corresponding vertex. Idem on the orther vertices.

Hermite element on a tetrahedron "FEM_HERMITE(3)"						
Degree	dimension	d.o.f. number	class	vectorial	τ-equivalent	Polynomial
3	3	20	C^0	No ($Q=1$)	No	Yes

1.9 Interpolation of elements on different meshes

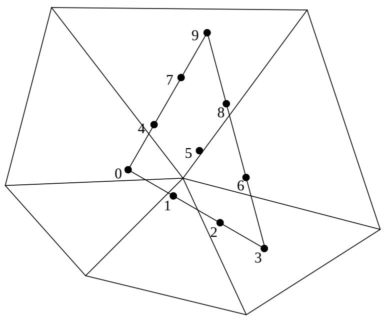


Figure 31: *Element which intepolates a finite element method defined on another mesh. The element has as many d.o.f. as the union of d.o.f. of elements of the other mesh having an intersection with it. The interpolation is made on Gauss points of the integration method.*

To increase the precision, it is not necessary to raise the order of the integration method. It is recommended to keep the normal order and use composite integration methods (see below).

Element which interpolates an element defined on another mesh

```
getfem::virtual_link_fem(getfem::mesh_fem mf1, getfem::mesh_fem mf2,
                         getfem::pintegration_method pim)
          getfem::virtual_link_fem_with_gradient(getfem::mesh_fem mf1,
          getfem::mesh_fem mf2, getfem::pintegration_method pim)
```

2 Integration methods

2.1 Integration methods description

The integration methods are of two kinds. Exact integrations of polynomials and approximated integrations (cubature formulas) of any function. The exact integration can only be used if all the elements are polynomial and if the geometric transformation is linear.

A descriptor on an integration method is available thanks to the function

```
ppi = getfem::int_method_descriptor("name of method");
```

where `"name of method"` is a string to be choosen among the existing methods.

The program `integration` located in the `tests` directory lists and checks the degree of each integration method.

2.2 Exact Integration methods

The list of available Exact integration methods is the following

`"IM_NONE()"`	Dummy integration method (new in getfem++-1.7).
`"IM_EXACT_SIMPLEX(n)"`	Description of the exact integration of polynomials on the simplex of reference of dimension `n`.
`"IM_PRODUCT(a, b)"`	Description of the exact integration on the convex which is the direct product of the convex in `a` and in `b`.
`"IM_EXACT_PARALLELEPIPED(n)"`	Description of the exact integration of polynomials on the parallelepiped of reference of dimension `n`
`"IM_EXACT_PRISM(n)"`	Description of the exact integration of polynomials on the prism of reference of dimension `n`

Even though a description of exact integration method exists on parallelepipeds or prisms, most of the time the geometric transformations on such elements are not linear and the exact integration cannot be used.

Beware : In fact a lot of computation cannot be done with exact integration methods. So, it is recommended to use cubature formulas instead.

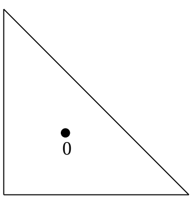
2.3 Newton cotes Integration methods

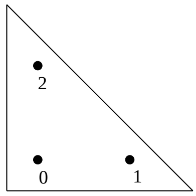
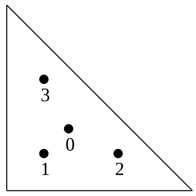
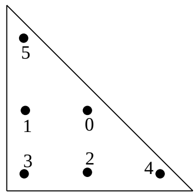
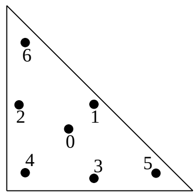
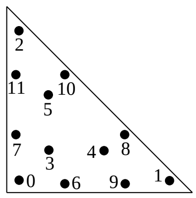
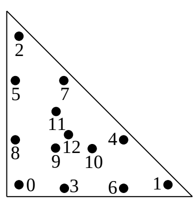
use `"IM_NC(N,K)"`, `"IM_NC_PARALLELEPIPED(N,K)"` and `"IM_NC_PRISM(N,K)"` to have the Newton cotes integration of order `K` respectively on simplices, parallelepipeds and prisms.

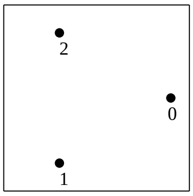
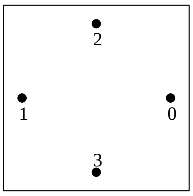
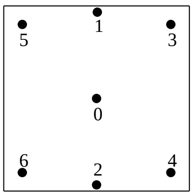
2.4 Gauss Integration methods on dimension 1

use `"IM_GAUSS1D(K)"` to have the Gauss-Legendre integration on the segment of order `K` (with `K`/2 + 1 points), and `"IM_GAUSSLOBATTO1D(K` to have the Gauss-Lobatto-Legendre integration on the segment of order `K` (with `K`/2 + 1 points). The latter integration method is only available for odd values of K. The Gauss-Lobatto integration method can be used in conjunction with `"FEM_PK_GAUSSLOBATTO1D(K/2)"` to perform mass-lumping.

2.5 Gauss Integration methods on dimension 2

graphic	coordinates x	y	weights	function to call / order
	1/3	1/3	1/2	`"IM_TRIANGLE(1)"` 1 point, order 1.

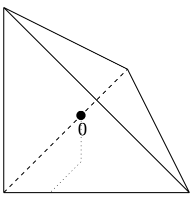
	1/6 2/3 1/6	1/6 1/6 2/3	1/6 1/6 1/6	"IM_TRIANGLE(2)" 3 points, order 2.
	1/3 1/5 3/5 1/5	1/3 1/5 1/5 3/5	-27/96 25/96 25/96 25/96	"IM_TRIANGLE(3)" 4 points, order 3.
	a $1-2a$ a b $1-2b$ b	a a $1-2a$ b b $1-2b$	c c c d d d	"IM_TRIANGLE(4)" 6 points, order 4, a = 0.445948490915965, b = 0.091576213509771, c = 0.111690794839005, d = 0.054975871827661.
	1/3 a $1-2a$ a b $1-2b$ b	1/3 a a $1-2a$ b b $1-2b$	9/80 c c c d d d	"IM_TRIANGLE(5)" 7 points, order 5, $a = \frac{6+\sqrt{15}}{21}$, $b = 4/7 - a$, $c = \frac{155+\sqrt{15}}{2400}$, $d = 31/240 - c$.
	a $1-2a$ a b $1-2b$ b c d $1-c-d$ $1-c-d$ c d	a a $1-2a$ b b $1-2b$ d c c d $1-c-d$ $1-c-d$	e e e f f f g g g g g g	"IM_TRIANGLE(6)" 12 points, order 6, $a = 0.063089104491502$, $b = 0.249286745170910$, $c = 0.310352451033785$, $d = 0.053145049844816$, $e = 0.025422453185103$, $f = 0.058393137863189$, $g = 0.041425537809187$.
	a b a c d e d c e f g f 1/3	a a b e c d e d c f f g 1/3	h h h i i i i i i j j j k	"IM_TRIANGLE(7)" 13 points, order 7, $a = 0.0651301029022$, $b = 0.8697397941956$, $c = 0.3128654960049$, $d = 0.6384441885698$, $e = 0.0486903154253$, $f = 0.2603459660790$, $g = 0.4793080678419$, $h = 0.0266736178044$, $i = 0.0385568804451$, $j = 0.0878076287166$, $k = -0.0747850222338$.
				"IM_TRIANGLE(8)" (see [6]) 16 points, order 8

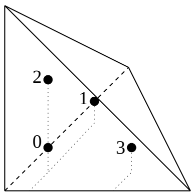
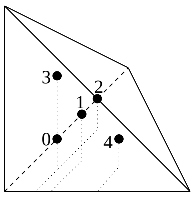
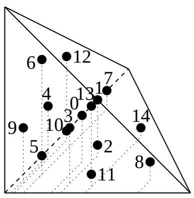
				`"IM_TRIANGLE(9)"` (see [6]) 19 points, order 9
				`"IM_TRIANGLE(10)"` (see [6]) 25 points, order 10
				`"IM_TRIANGLE(13)"` (see [6]) 37 points, order 13
	$1/2+\sqrt{1/6}$ $1/2-\sqrt{1/24}$	$1/2$ $1/2\pm\sqrt{1/8}$	1/3 1/3	`"IM_QUAD(2)"` 3 points, order 2.
	$1/2\pm\sqrt{1/6}$ $1/2$	$1/2$ $1/2\pm\sqrt{1/6}$	1/4 1/4	`"IM_QUAD(3)"` 4 points, order 3.
	$1/2$ $1/2\pm\sqrt{7/30}$ $1/2\pm\sqrt{1/12}$	$1/2$ $1/2$ $1/2\pm\sqrt{3/20}$	2/7 5/63 5/36	`"IM_QUAD(5)"` 7 points, order 5.
				`"IM_QUAD(7)"` 12 points, order 7
				`"IM_QUAD(9)"` 20 points, order 9
				`"IM_QUAD(17)"` 70 points, order 17

There is also the `IM_GAUSS_PARALLELEPIPED(n,k)` which is a direct product of 1D gauss integrations.

Important note: do not forget that `IM_QUAD(k)` is exact for polynomials up to degree k, and that a Q_k polynomial has a degree of $2*k$. For example, `IM_QUAD(7)` cannot integrate exactly the product of two Q_2 polynomials. On the other hand, `IM_GAUSS_PARALLELEPIPED(2,4)` can integrate exactly that product...

2.6 Gauss Integration methods on dimension 3

graphic	coordinates x	y	z	weights	function to call / order
	1/4	1/4	1/4	1/6	`"IM_TETRAHEDRON(1)"` 1 point, order 1.

	a a a b	a b a a	a a b a	1/24 1/24 1/24 1/24	"IM_TETRAHEDRON(2)" 4 points, order 2 $a = \frac{5-\sqrt{5}}{20}$, $b = \frac{5+3\sqrt{5}}{20}$.
	1/4 1/6 1/6 1/6 1/2	1/4 1/6 1/2 1/6 1/6	1/4 1/6 1/6 1/2 1/6	-2/15 3/40 3/40 3/40	"IM_TETRAHEDRON(3)" 5 points, order 3
	1/4 a a a c b b b d e e f e f f	1/4 a a c a b b d b e f e f e f	1/4 a c a a b d b b f e e f f e	8/405 h h h h i i i i 5/567 5/567 5/567 5/567 5/567 5/567	"IM_TETRAHEDRON(5)" 15 points, order 5 $a = \frac{7+\sqrt{15}}{34}$, $b = \frac{7-\sqrt{15}}{34}$, $c = \frac{13+3\sqrt{15}}{34}$, $d = \frac{13-3\sqrt{15}}{34}$, $e = \frac{5-\sqrt{15}}{20}$, $f = \frac{5+\sqrt{15}}{20}$, $h = \frac{2665-14\sqrt{15}}{226800}$, $i = \frac{2665+14\sqrt{15}}{226800}$,

Others methods are:

name	convex type	nb of points
`IM_TETRAHEDRON(6)`	3D simplex	24
`IM_TETRAHEDRON(8)`	3D simplex	43
`IM_SIMPLEX4D(3)`	4D simplex	6
`IM_HEXAHEDRON(5)`	3D parallelepipeded	14
`IM_HEXAHEDRON(9)`	3D parallelepipeded	58
`IM_HEXAHEDRON(11)`	3D parallelepipeded	90
`IM_CUBE4D(5)`	4D parallelepipeded	24
`IM_CUBE4D(9)`	4D parallelepipeded	145

2.7 Direct product of integration methods

You can use `"IM_PRODUCT(IM1, IM2)"` to produce integration methods on quadrilateral or prisms. It gives the direct product of two integration mathods. For instance `IM_GAUSS_PARALLELEPIPED(2,k)` is an alias for `IM_PRODUCT(IM_GAUSS1D(2,k),IM_GAUSS1D(2` and ca be use instead of the `IM_QUAD` integrations.

2.8 Composite integration methods

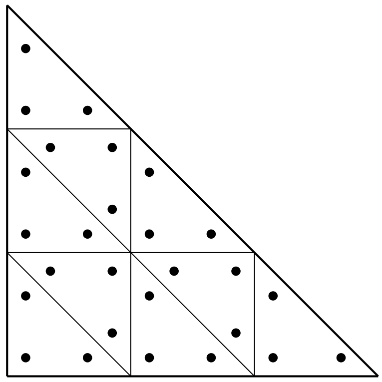


Figure 32: *composite method* "IM_STRUCTURED_COMPOSITE(IM_TRIANGLE(2), 3)"

use "IM_STRUCTURED_COMPOSITE(IM1, S)" to copy IM1 on an element with S subdivisions. The resulting integration method has the same order but with more points. This could be more stable to use composite method rather than to improve the order of the method. Those methods have to be used also with composite elements. Most of the time for composite element, it is preferable to choose the basic method IM1 with no points on the boundary (because the gradient coulb be not defined on the boundary of sub-elements).

For the HCT element, it is advised to use the IM_HCT_COMPOSITE(im) composite integration (which split the original triangle into 3 sub-triangles).

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