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$\mathscr{C}^1$-curved finite elements with numerical integration for thin plate and thin shell problems, Part 1: Construction and interpolation properties of curved $\mathscr{C}^1$ finite elements

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The purpose of this paper is to construct curved finite elements of class $\mathscr{C}^1$, compatible with the elements of Argyris–Fried–Scharpf and Bell. We start by approximating the curved boundary and then we construct curved finite elements of class $\mathscr{C}^1$. We conclude by proving that the corresponding asymptotic interpolation error has the same order as the associate straight elements.

1. Introduction

Many structural problems are set on curved boundary plane domains and are modelled by fourth order partial differential equations or by systems of fourth order partial differential equations. In this way, let us mention:

(i) many thin plate problems whose middle surfaces are curved boundary plane domains. It is well known that the approximation of such problems by straight finite elements can produce some lack of convergence as reported in [1] and the bibliography of this work.

(ii) many thin shell problems: in [2], it is shown that a high accuracy approximation method can be easily developed when a general thin shell problem is formulated on a plane polygonal reference domain. This is generally obtained through the use of an exact or an approximated (for instance, by B-spline methods) mapping of the middle surface of the shell. But, in general, such plane reference domains have curved boundaries.

(iii) many structural problems that include junctions between different thin shells: for example let us mention junctions between circular cylinders or pipes, which are used in oil platform constructions. Each shell element can be associated with a plane reference domain and, due to the complexity of surface intersections, the part of the boundary associated with such intersections is generally curved. Corresponding open problems are mentioned in [3].

Then, the development of approximations of a high degree of accuracy by finite element methods requires the use of $\mathscr{C}^1$-curved elements. Thus, in this paper, we develop and analyze such methods. The whole paper comprises two parts:

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(i) the construction of curved finite elements of class $\mathscr{C}^1$ and the study of their interpolation properties;

(ii) the use of these curved $\mathscr{C}^1$-finite elements in order to approximate the solutions of plate or shell problems posed over plane reference domains with a curved boundary. This second part [4] will include the study of convergence and the obtention of asymptotic error estimates.

The detailed description of how to implement such curved $\mathscr{C}^1$-elements and some numerical results obtained from benchmark plate problems will be reported in an additional paper [5].

In this first part, we are only concerned with the construction of curved finite elements which have a connection of class $\mathscr{C}^1$ with some classical $\mathscr{C}^1$-elements such as the Argyris triangle (see [6]) or the Bell triangle (see [7]). In the sequel, we say that such elements are $\mathscr{C}^1$-compatible with Argyris or Bell triangles.

In the second part, we generalize the results of [8] relative to the approximation of solutions of thin shell problems posed on polygonal domains to the case of curved boundary domains. In this way, we analyze the simultaneous use of curved $\mathscr{C}^1$-elements, considered here, and of numerical integration techniques.

In addition to the introduction, this paper comprises three sections. In Section 2, we consider the approximation of the domain Ω. We first define an exact triangulation of the domain Ω. Then, to every triangle with a curved side located on the boundary, we associate an approximate triangle by interpolating this curved side. Thus, we obtain an approximate domain Ω_h whose constitutive triangles are in bijective correspondence with a reference triangle $\hat{K}$ through applications F_K, affine or not. Similar constructions were first analyzed by [9–11]; some modifications were considered in [12–17]. Let us also mention Lenoir [18], who obtained optimal error estimates for the finite element solution of second order elliptic problems by using isoparametric simplicial elements, Bernardi [19], who considered the case of the interpolation on curved domains of functions that are not smooth and Krizkova [20], who analyzed special exact curved finite elements useful for solving contact problems of second order in a domain with piecewise circular boundaries.

Section 3 is devoted to the definition of curved finite elements $\mathscr{C}^1$-compatible with Argyris or Bell triangles. Each curved finite element is in correspondence with a reference element on $\hat{K}$ which is obtained by constraining a suitable polynomial space $\hat{P}$. To our knowledge, the results concerning the first curved element, $\mathscr{C}^1$-compatible with the Argyris triangle, are new. For the second curved element, $\mathscr{C}^1$-compatible with the Bell triangle, we obtain, in an independent way, results comparable with some of [17]. In this direction, we also mention [21, 22]. In addition, other curved $\mathscr{C}^1$-elements were considered by different authors: when the plane curved domain Ω is the image of a plane polygonal domain ω through a parametric representation ψ, [23] introduces a curved version of the Argyris element, named TUBAC 6, which is the image of the associated straight element defined on straight triangles of the polygonal domain ω through the mapping ψ. This construction is really fruitful when such a mapping ψ can be explicit as for circular or elliptical domains for instance. In addition [24, 25] have considered curved $\mathscr{C}^1$-elements for which the finite dimensional space is polynomial over the current triangle. This is convenient for the approximation of a constant coefficient operator as in some plate problems. Nevertheless, for corresponding finite element methods, it seems difficult to analyze the effect of numerical integration. A significant improvement in the method, suggested in [24], is given in [26].

In Section 4, we give estimates of the asymptotic interpolation errors for every family of these curved $\mathscr{C}^1$-elements. These asymptotic error estimates are of the same order as the one obtained for the associate straight elements.

Throughout this paper, we use the notations of [27, 28]. In particular, let $W^{k,p}(\Omega)$ be the Sobolev space of real-valued functions which, together with all their partial distributional derivatives of order k or less, belong to $L^p(\Omega)$. We set $H^k(\Omega) = W^{k,2}(\Omega)$. On $W^{k,p}(\Omega)$ we shall use the norms and semi-norms defined by

$$\|u\|_{k,p,\Omega} = \Big(\sum_{|\alpha|\leqslant k} \|D^\alpha u\|^p_{L^p(\Omega)}\Big)^{1/p}, \qquad |u|_{k,p,\Omega} = \Big(\sum_{|\alpha| = k} \|D^\alpha u\|^p_{L^p(\Omega)}\Big)^{1/p}$$

for $1 \leqslant p \leqslant \infty$, with the standard modification for $p = +\infty$ (see for example [29–31]).

2. Approximation of the domain Ω

Let there be given the plane $\mathbb{R}^2$ referred to an orthonormal system $(0, e_1, e_2)$, the corresponding system of coordinates (x_1, x_2) and a bounded domain Ω in $\mathbb{R}^2$. We assume that the boundary Γ can be subdivided into a finite number of arcs, each of them admitting the following parametric representation:

$$x_1 = \chi_1(s), \qquad x_2 = \chi_2(s), \qquad s_m \leqslant s \leqslant s_M. \tag{2.1}$$

The functions $\chi_1(s), \chi_2(s)$ belong to C^{q+1}, q sufficiently large, and they are such that $[(\chi_1)']^2 + [(\chi_2)']^2$ is different from 0 on the interval $[s_m, s_M]$.

2.1. Exact triangulation of the domain Ω

Let us subdivide the set $\bar{\Omega}$ into a finite number of triangles K with straight of curvilinear sides. More precisely, given two distinct triangles of this triangulation $\mathscr{T}$, we assume that either they are disjoint or they have a common vertex or they have a common side. Moreover, we assume that every 'interior' triangle K (i.e. a triangle having at most one vertex on Γ) has only straight sides and that every 'boundary' triangle K_c has at most one curved side located over an arc of type (2.1) upon the boundary Γ. In the following, we shall denote $\tilde{K}$ the (rectilinear) triangle with the same vertices as the curvilinear triangle K_c.

From now on, we consider regular families of triangulations $\mathscr{T}$ of the domain Ω, i.e.:

(i) there exists a constant σ such that

$$\frac{h_K}{\rho_K} \leqslant \sigma \quad \forall \mathscr{T} \quad \forall K\,(\text{or } \tilde{K}) \in \mathscr{T}, \tag{2.2}$$

where $h_K = \operatorname{diam}(K)$ and $\rho_K = \sup\{\operatorname{diam}(C); C \text{ is a disk contained in } K\}$ (in the case of curved triangles K_c, we replace K_c by $\tilde{K}$);

(ii) the parameter

$$h = \max_{K \in \mathscr{T}} h_K \to 0. \tag{2.3}$$

REMARK 2.1. For h sufficiently small, it was proved in [10, Theorem 1] and [16, Theorem 1] that every triangle K_c is the image of a reference triangle $\hat{K}$ through a diffeomorphism of order q.

2.2. *Construction of an approximate triangulation of the domain* Ω

In order to construct finite elements of class $\mathscr{C}^1$, it is convenient to approach the curvilinear side of every triangle K_c by an arc parameterized with the help of polynomial functions. This amounts to associating with every curved triangle K_c, an approximate curved triangle K (see Fig. 1). Thus, the initial domain Ω is replaced by an approximate domain Ω_h whose corresponding triangulation is denoted by $\mathscr{T}_h$. Every triangulation $\mathscr{T}_h$ is the union
(i) of a triangulation $\mathscr{T}_h^1$ constituted by straight side triangles K in correspondence with a fixed reference triangle $\hat{K}$ through an affine mapping F_K;
(ii) of a triangulation $\mathscr{T}_h^2$ constituted by triangles K with two straight sides and a third curved side approximating an arc of the boundary Γ. These triangles K are the images of the reference triangle $\hat{K}$ through a nonlinear mapping F_K.

In both cases, we indicate how to construct an application $F_K : \hat{K} \rightarrow K$, for all $K \in \mathscr{T}_h$. Firstly, if the triangle $K \in \mathscr{T}_h^1$, then the application F_K is affine; there exists an invertible matrix B_K and a vector b_K of $\mathbb{R}^2$ such that

$$F_K : (\hat{x}_1, \hat{x}_2) \in \hat{K} \rightarrow F_K(\hat{x}_1, \hat{x}_2) = B_K \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + b_K \in K . \tag{2.4}$$

For clarity, using the notation of Fig. 2:

$$F_K(\hat{x}_1, \hat{x}_2) = \begin{cases} x_1 = x_{13} + (x_{11} - x_{13})\hat{x}_1 + (x_{12} - x_{13})\hat{x}_2 , \\ x_2 = x_{23} + (x_{21} - x_{23})\hat{x}_1 + (x_{22} - x_{23})\hat{x}_2 , \end{cases} \tag{2.5}$$

where $x_{\alpha i}$, $\alpha = 1, 2$, denote the coordinates of the vertices a_i, $i = 1, 2, 3$, of the triangle K.

Secondly, when the triangle $K \in \mathscr{T}_h^2$, the application $F_K : \hat{K} \rightarrow K$ is in general nonlinear and can be conveniently defined in two steps.

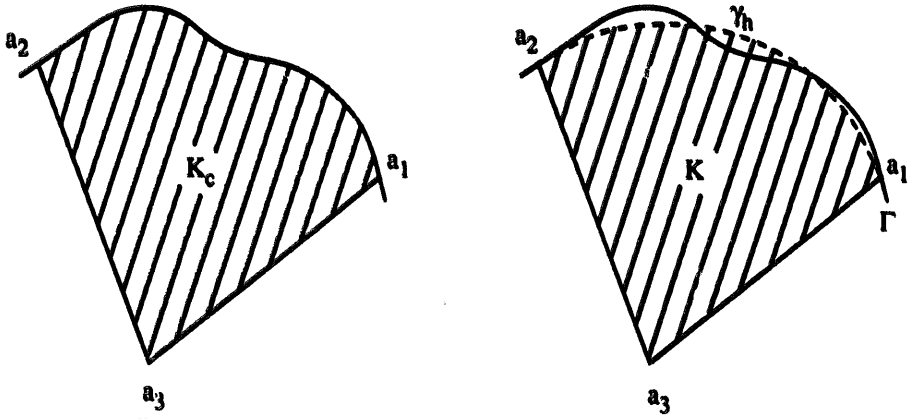


Fig. 1. Curved triangles, exact K_c and approximate K.

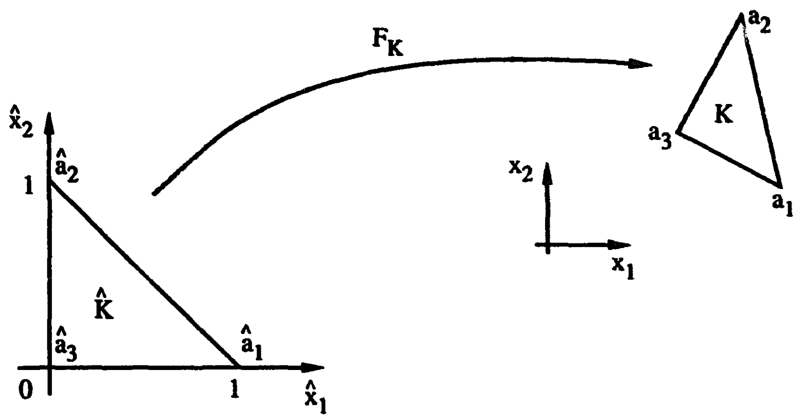


Fig. 2. Case of a triangle $K \in \mathscr{T}_h^1$.

STEP 1. Interpolation of the curved side a_1a_2 of the triangle K_c. Consider the new parameterization of the arc a_1a_2 by using variable $\hat{x}_2$:

$$x_1 = \psi_1(\hat{x}_2), \qquad x_2 = \psi_2(\hat{x}_2), \quad 0 \leq \hat{x}_2 \leq 1, \tag{2.6}$$

where

$$\psi_\alpha(\hat{x}_2) = \chi_\alpha(\underline{s} + (\bar{s} - \underline{s})\hat{x}_2), \quad \alpha = 1, 2.$$

Then, the approximate arc γ_h is defined by the parametric equations

$$x_1 = \psi_{1h}(\hat{x}_2), \qquad x_2 = \psi_{2h}(\hat{x}_2), \quad 0 \leq \hat{x}_2 \leq 1,$$

where the functions ψ_{1h} and ψ_{2h} satisfy the following hypothesis.

HYPOTHESIS 2.1. The functions $\psi_{\alpha h}$ are interpolation polynomials of Lagrange or Hermite type, with degree $n \geq 1$, of the functions ψ_α, $\alpha = 1, 2$, over the interval $[0, 1]$ and such that

$$\begin{aligned} &\psi_{\alpha h}(0) = \psi_\alpha(0), \qquad \psi_{\alpha h}(1) = \psi_\alpha(1), \quad \alpha = 1, 2, \\ &|\psi_\alpha - \psi_{\alpha h}|_{p,\infty} \leq c h_K^{n+1-p} |\psi_\alpha|_{n+1,\infty}, \quad p = 0, \dots, n+1, \end{aligned} \tag{2.7}$$

where c is a constant independent of h_K.

From Hypothesis 2.1, we immediately derive

$$\psi_{\alpha h}(\hat{x}_2) = x_{\alpha 1} + (x_{\alpha 2} - x_{\alpha 1})\hat{x}_2 + \begin{cases} 0, & \text{if } n = 1, \\ \hat{x}_2(1 - \hat{x}_2)P_{n-2;\alpha}(\hat{x}_2), & \text{if } n \geq 2, \end{cases} \tag{2.8}$$

where $P_{n-2;\alpha}$, $\alpha = 1, 2$, denotes polynomials of degree $n-2$ with respect to $\hat{x}_2$, completely determined by the choice of the interpolation method. Some examples are given in Section 2.4.

STEP 2. Definition of the application $F_K : \hat{K} \to K$. To any point $\hat{M}$ of the reference triangle $\hat{K}$, the application F_K associates the point M_h as follows:

$$OM_h = F_{K1}(\hat{x}_1, \hat{x}_2)e_1 + F_{K2}(\hat{x}_1, \hat{x}_2)e_2 , \tag{2.9}$$

where the functions $F_{K\alpha}$, $\alpha = 1, 2$, are defined by the relations

$$F_{K\alpha}(\hat{x}_1, \hat{x}_2) = x_{\alpha 3} + (x_{\alpha 1} - x_{\alpha 3})\hat{x}_1 + (x_{\alpha 2} - x_{\alpha 3})\hat{x}_2 + \begin{cases} 0 , & \text{if } n = 1 , \\ \frac{1}{2}\hat{x}_1\hat{x}_2[P_{n-2;\alpha}(1 - \hat{x}_1) + P_{n-2;\alpha}(\hat{x}_2)] , & \text{if } n \geq 2 \end{cases} \tag{2.10}$$

(where polynomials $P_{n-2;\alpha}$ are defined by (2.8)).

Let us note that these functions $F_{K\alpha}$ are constructed so that $F_{K\alpha}(1 - \hat{x}_2, \hat{x}_2) \equiv \psi_{\alpha h}(\hat{x}_2)$, i.e., the image of the side $\hat{a}_1\hat{a}_2$ of the reference triangle $\hat{K}$ through the application F_K, is the approximate arc γ_h. Moreover, if we denote by $\tilde{M}$ the point of barycentric coordinates $(\hat{x}_1, \hat{x}_2, 1 - \hat{x}_1 - \hat{x}_2)$ in the (straight) triangle $\tilde{K} = (a_1, a_2, a_3)$, (2.10) can be rewritten in the vectorial form

$$OM_h = O\tilde{M} + \tfrac{1}{2}(\tilde{M}M_h^1 + \tilde{M}M_h^2) . \tag{2.11}$$

The corresponding geometrical interpretation is illustrated in Fig. 3 and obtained from relations

$$\begin{aligned} &\tilde{M}M_h^1 = \frac{\hat{x}_2}{1 - \hat{x}_1}\tilde{P}^1P_h^1 , \quad \text{if } \hat{x}_1 \neq 1 , \qquad \tilde{M} \equiv M_h^1 \equiv a_1 , \qquad \text{if } \hat{x}_1 = 1 , \\ &\tilde{M}M_h^2 = \frac{\hat{x}_1}{1 - \hat{x}_2}\tilde{P}^2P_h^2 , \quad \text{if } \hat{x}_2 \neq 1 , \qquad \tilde{M} \equiv M_h^2 \equiv a_2 , \qquad \text{if } \hat{x}_2 = 1 , \end{aligned} \tag{2.12}$$

since

$$\tilde{P}^1P_h^1 = \sum_{\alpha=1}^{2} \hat{x}_1(1 - \hat{x}_1)P_{n-2;\alpha}(1 - \hat{x}_1)e_\alpha , \qquad \tilde{P}^2P_h^2 = \sum_{\alpha=1}^{2} \hat{x}_2(1 - \hat{x}_2)P_{n-2;\alpha}(\hat{x}_2)e_\alpha .$$

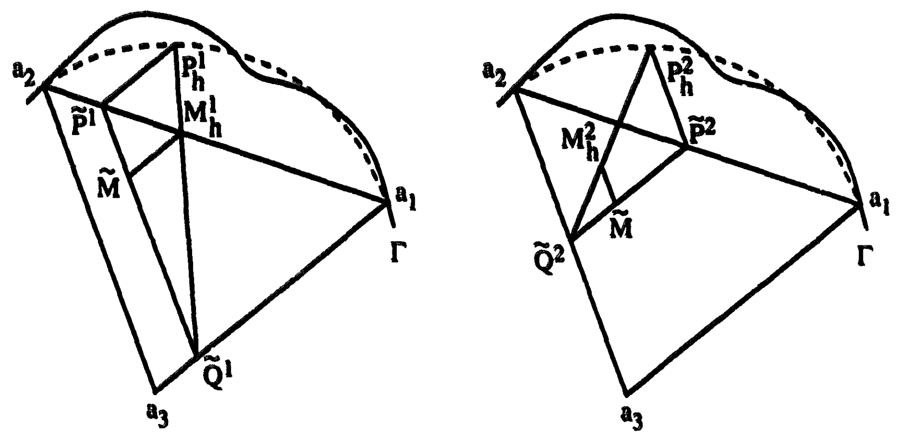


Fig. 3. Construction of $\tilde{M} \to M_h$ through $\tilde{M}M_h = \frac{1}{2}(\tilde{M}M_h^1 + \tilde{M}M_h^2)$ (for clarity, the constructions of $\tilde{M} \to M_h^1$ and $\tilde{M} \to M_h^2$ are displayed separately).

2.3. Some properties of the application F_K

In the following theorem, we prove directly some properties of the application F_K defined by (2.10). Some of these results have been obtained in [10, 11] by using properties of the diffeomorphism $\tilde{K} \to K_c$ mentioned in Remark 2.1.

We use the following notation:

$$J_{F_K}(\hat{x}) = \text{Jacobian of } F_K \text{ at point } \hat{x} = (\hat{x}_1, \hat{x}_2)\,, \tag{2.13}$$

$$J_{F_K^{-1}}(\hat{x}) = \text{Jacobian of } F_K^{-1} \text{ at point } x = (x_1, x_2)\,, \tag{2.14}$$

$$| F_K |_{l,\infty,\hat{K}} = \sup_{\hat{x} \in \hat{K}} \| D^l F_K(\hat{x}) \|\,, \tag{2.15}$$

$$| F_K^{-1} |_{l,\infty,K} = \sup_{x \in K} \| D^l F_K^{-1}(x) \|\,. \tag{2.16}$$

THEOREM 2.1. Let $\mathscr{T}$ be a regular family of triangulations of the domain Ω, i.e., satisfying conditions (2.2) and (2.3). To any curved triangle K_c of the 'exact' triangulation of the domain Ω, we associate the approximate curved triangle K obtained through the interpolation of the curved side located on the boundary Γ. We assume that this interpolation satisfies Hypothesis 2.1, and that the boundary Γ of the bounded domain $\Omega \subset \mathbb{R}^2$ is piecewise of class C^{q+1}, $q \geq n$ (n = degree of the components $F_{K\alpha}$). Then, for h_K sufficiently small, we have the following properties that are independent of the degree n of the components $F_{K\alpha}$:

(i) the application $F_K : \bar{\hat{K}} \to \bar{K}$, defined by (2.9), (2.10) is a $\mathscr{C}^\infty$-diffeomorphism from $\bar{\hat{K}}$ onto $\bar{K}$;

(ii) the application F_K and its inverse $F_K^{-1} : \bar{K} \to \bar{\hat{K}}$, satisfy the following estimates:

$$| F_K |_{l,\infty\hat{K}} \leq c h_K^l\,, \quad l = 0, 1, \ldots, \tag{2.17}$$

$$| F_K^{-1} |_{l,\infty,K} \leq c h_K^{-1}\,, \quad l = 1, 2, \ldots; \tag{2.18}$$

(iii) The Jacobians $J_{F_K}(\hat{x})$ and $J_{F_K^{-1}}(x)$ satisfy the following estimates:

$$c_1 h_K^2 \leq |J_{F_K}|_{0,\infty,\hat{K}} \leq c_2 h_K^2\,, \qquad |J_{F_K}|_{l,\infty,\hat{K}} \leq c h_K^{2+l}\,, \quad l = 0, \ldots, n\,, \tag{2.19}$$

$$\frac{c_1}{h_K^2} \leq |J_{F_K^{-1}}|_{0,\infty,K} \leq \frac{c_2}{h_K^2}\,, \tag{2.20}$$

where, in the inequalities (2.17)–(2.20), the letters c, c_1, c_2 denote strictly positive constants which are not necessarily the same from one inequality to the other.

PROOF. When $n = 1$ (F_K affine), these results are obvious. Henceforth, we assume $n \geq 2$. Then, the proof takes seven steps which can be summarized as follows (more details can be found in [32]).

STEP 1. F_K verifies estimates (2.17). From Hypothesis 2.1, we have

$$|\psi_\alpha - \psi_{\alpha h}|_{p,\infty} \leq c h_K^{n+1-p} |\psi_\alpha|_{n+1,\infty}, \quad p = 0, \ldots, n+1. \tag{2.21}$$

Taking into account the hypothesis $[(\chi_1)']^2 + [(\chi_2)']^2 \neq 0$ on $[s_m, s_M]$ and the definition (2.6), we obtain $|\psi_\alpha|_{p,\infty} = \mathrm{O}(h_K^p)$, $p = 0, \ldots, n+1$, so that

$$|\psi_{\alpha h}|_{p,\infty} \leq c h_K^p, \; p = 0, 1, \ldots \tag{2.22}$$

(note that $|\psi_{\alpha h}|_{p,\infty} = 0$ when $p \geq n+1$).

Substituting the estimate (2.22) into definition (2.8), we derive

$$\sup_{\hat{x}_2 \in [0,1]} |[P_{n-2;\alpha}(\hat{x}_2)]^{(m)}| \leq c h_K^{m+2}, \quad m = 0, 1, \ldots \tag{2.23}$$

and hence, with (2.10),

$$\sup_{(\hat{x}_1, \hat{x}_2) \in \hat{K}} \left| \frac{\partial^{i_1 + i_2} F_{K\alpha}}{(\partial \hat{x}_1)^{i_1} (\partial \hat{x}_2)^{i_2}} (\hat{x}_1, \hat{x}_2) \right| \leq c h_K^{i_1 + i_2}, \quad 0 \leq i_1 + i_2, \; \alpha = 1, 2. \tag{2.24}$$

Thus, we obtain

$$\|D^l F_K(\hat{x})\| \leq c \max_{|\beta| = l} |\partial^\beta F_K(\hat{x})| \leq c h_K^l,$$

so that the definition (2.15) involves estimates (2.17).

STEP 2. F_K verifies estimates (2.19). Expressions (2.10) and estimates (2.23) imply

$$|J_{F_K}(\hat{x})| = |\boldsymbol{a}_3\boldsymbol{a}_1 \times \boldsymbol{a}_3\boldsymbol{a}_2| + \mathrm{O}(h_K^3). \tag{2.25}$$

Then, the hypotheses (2.2) and (2.3) give the two first estimates (2.19). The last inequalities (2.19) are direct consequences of estimates (2.24).

STEP 3. For h_K sufficiently small, F_K is injective. We use a result from [33, Corollary 2] expressing that if $\mathcal{O}$ is an open set of $\mathbb{R}^n$, if M is a compact of $\mathcal{O}$ with a connected boundary ∂M, if $g : \mathcal{O} \to \mathbb{R}^n$ is of class $\mathscr{C}^1$ such that $J_g(x) > 0$ for all $x \in M$, and if $g|_{\partial M}$ is injective, then $g|_M$ is injective.

Here we take $M = \bar{\hat{K}}$, $g = F_K$ which can be extended to an open set $\mathcal{O}$ of $\mathbb{R}^n$ containing $\hat{K}$ without any difficulty. Since estimates (2.19) imply $J_{F_K}(\hat{x}) > 0$, $\forall \hat{x} \in \hat{K}$, it remains to prove that $F_K|_{\partial \hat{K}}$ is injective.

Firstly, the definition (2.10) implies immediately that the restriction of F_K to the sides $\hat{a}_3\hat{a}_1$ and $\hat{a}_3\hat{a}_2$ is injective. Now, it remains to prove that situations as illustrated in Fig. 4 cannot occur. Situation (i) is avoided if the restriction of F_K to the side $\hat{a}_1\hat{a}_2$ is injective. Since $F_{K\alpha}(1 - \hat{x}_2, \hat{x}_2) \equiv \psi_{\alpha h}(\hat{x}_2)$, we only need to prove that the system $\psi_{\alpha h}(X) = \psi_{\alpha h}(Y)$, $\alpha = 1, 2$, $X, Y \in [0, 1]$ has the unique solution $X = Y$. This property is easy to prove by using estimates

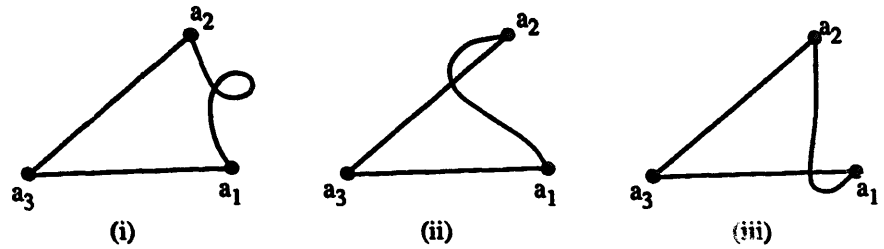


Fig. 4. Basic situations for which F_K is not injective.

(2.23) and by assuming h_K sufficiently small. In the same way, we prove that situations (ii) and (iii) cannot occur for h_K sufficiently small. For instance, situation (ii) leads to the system

$$x_{\alpha 3} - x_{\alpha 1} + (x_{\alpha 2} - x_{\alpha 3})X - (x_{\alpha 2} - x_{\alpha 1})Y - Y(1 - Y)P_{n-2;\alpha}(Y) = 0\,, \quad 0 \leq X, Y \leq 1\,,$$

which admits the unique solution $X = Y = 1$ (i.e., point a_2) when h_K is sufficiently small.

STEP 4. F_K is a homeomorphism from $\bar{\hat{K}}$ onto $\bar{K}$. Since F_K is continuous and injective, F_K is a homeomorphism from $\bar{\hat{K}}$ onto $F_K(\bar{\hat{K}})$. By construction, the image of the boundary $\partial\hat{K}$ is the boundary ∂K. Since a homeomorphism applies the interior of a Jordan curve onto the interior of its image (see [34, Sections 110–113]), the application F_K is a homeomorphism from $\bar{\hat{K}}$ onto $\bar{K}$.

STEP 5. F_K is a $\mathscr{C}^\infty$-diffeomorphism from $\bar{\hat{K}}$ onto $\bar{K}$. Application F_k is $\mathscr{C}^\infty$ and its Jacobian $J_{F_K}(\hat{x})$ is different from 0 over $\bar{\hat{K}}$. It follows that the inverse application of F_K is also C^∞ (see [35]).

STEP 6. Application F_K^{-1} verifies estimates (2.18). We give a proof by induction. Starting from identity $F_K \circ F_K^{-1} = I$ and using estimates (2.17) and (2.24), we obtain

$$|F_K^{-1}|_{1,\infty,K} = \sup_{x\in K} \|DF_K^{-1}(x)\| \leq c \sup_{x\in K} \max_{\alpha=1,2} \left|\frac{\partial F_K^{-1}}{\partial x_\alpha}(x)\right| \leq \frac{c}{h_K}\,,$$

i.e., the estimate (2.18) for $l = 1$.

Next, let us assume that estimate (2.18) is true for $l = 2, \ldots, m-1$ and let us show that (2.18) is true for $l = m$. For $m \geq 2$, we have (see [35, Section 7.5])

$$m! \sum_{l=1}^{m} \sum_{j\in J(l,m)} \frac{1}{l!} D^l F_K^{-1}(x_1, x_2)\Big[\frac{1}{j_1!} D^{j_1}F_K(\hat{x}_1, \hat{x}_2)(\xi_1, \ldots, \xi_{j_1}), \ldots,$$

$$\frac{1}{j_l!} D^{j_l}F_K(\hat{x}_1, \hat{x}_2)(\xi_{m-j_l+1}, \ldots, \xi_m)\Big] = 0\,, \qquad (2.26)$$

where

$$J(l, m) = \{j = (j_1, \ldots, j_l) \in \mathbb{N}^l;\ 1 \leq j_1, \ldots, j_l \leq m,\ j_1 + j_2 + \cdots + j_l = m\}\,.$$

Since the application F_K is a C^∞-diffeomorphism from $\bar{\hat{K}}$ onto $\bar{K}$, the application $DF_K(\hat{x}_1, \hat{x}_2) : \bar{\hat{K}} \to \bar{K}$ is bijective. Let $\eta_i = DF_K(\hat{x}_1, \hat{x}_2)\xi_i$, $i = 1, \ldots, m$ and, conversely, $\xi_i = DF_K^{-1}(x_1, x_2)\eta_i$. Relation (2.26) can be written as

$$\begin{aligned} D^m F_K^{-1}(x)(\eta_1, \ldots, \eta_m) = -m! \sum_{l=1}^{m-1} \sum_{j \in J(l,m)} \frac{1}{l!} D^l F_K^{-1}(x) \\ \times \left\{ \frac{1}{j_1!} D^{j_1} F_K(\hat{x})[DF_K^{-1}(x)\eta_1, \ldots, DF_K^{-1}(x)\eta_{j_1}], \ldots, \right. \\ \left. \frac{1}{j_l!} D^{j_l} F_K(\hat{x})[DF_K^{-1}(x)\eta_{m-j_l+1}, \ldots, DF_K^{-1}(x)\eta_m] \right\}. \end{aligned} \tag{2.27}$$

Then, estimates (2.17) and the hypothesis of the proof by induction give $\|D^m F_K^{-1}(x)\| \leq c/h_K$, hence $|F_K^{-1}|_{m,\infty,K} \leq ch_K^{-1}$, and thus estimates (2.18) are proved.

STEP 7. F_K^{-1} satisfies estimates (2.20). It suffices to combine equality $J_{F_K}(\hat{x})J_{F_K^{-1}}(x) = 1$, with the estimates (2.19). □

2.4. Examples

In this section, we give three examples of functions F_K which seem to be practically the most attractive. According to (2.8) and (2.10), it suffices to define the approximate arc γ_h.

EXAMPLE 2.1. Construction of γ_h by using polynomials of order 2. The degrees of freedom of the Lagrange-type interpolation are given by

$$\begin{aligned} &x_{\alpha 1} = \psi_\alpha(0) = \chi_\alpha(\underline{s}),\ x_{\alpha 2} = \psi_\alpha(1) = \chi_\alpha(\bar{s}), \\ &x_{\alpha 0} = \psi_\alpha(\tfrac{1}{2}) = \chi_\alpha(\tfrac{1}{2}(\underline{s} + \bar{s})), \quad \alpha = 1, 2, \end{aligned} \tag{2.28}$$

so that (2.8) and (2.10) lead to

$$\psi_{\alpha h}(\hat{x}_2) = x_{\alpha 1} + (x_{\alpha 2} - x_{\alpha 1})\hat{x}_2 + 4\hat{x}_2(1 - \hat{x}_2)[x_{\alpha 0} - \tfrac{1}{2}(x_{\alpha 1} + x_{\alpha 2})], \tag{2.29}$$

and then

$$F_{K_\alpha}(\hat{x}_1, \hat{x}_2) = x_{\alpha 3} + (x_{\alpha 1} - x_{\alpha 3})\hat{x}_1 + (x_{\alpha 2} - x_{\alpha 3})\hat{x}_2 + 4[x_{\alpha 0} - \tfrac{1}{2}(x_{\alpha 1} + x_{\alpha 2})]\hat{x}_1\hat{x}_2. \tag{2.30}$$

EXAMPLE 2.2. Construction of γ_h by using polynomials of order 3. The degrees of freedom of the Hermite-type interpolation are given by

$$\begin{aligned} &x_{\alpha 1} = \psi_\alpha(0) = \chi_\alpha(\underline{s}), \qquad x_{\alpha 2} = \psi_\alpha(1) = \chi_\alpha(\bar{s}), \\ &\psi'_\alpha(0) = (\bar{s} - \underline{s})\chi'_\alpha(\underline{s}), \qquad \psi'_\alpha(1) = (\bar{s} - \underline{s})\chi'_\alpha(\bar{s}), \quad \alpha = 1, 2, \end{aligned} \tag{2.31}$$

so that

$$\psi_{\alpha h}(\hat{x}_2) = x_{\alpha 1} + (x_{\alpha 2} - x_{\alpha 1})\hat{x}_2 \\ + \hat{x}_2(1-\hat{x}_2)\{[2(x_{\alpha 2} - x_{\alpha 1}) - (\bar{s} - \underline{s})[\chi'_\alpha(\underline{s}) + \chi'_\alpha(\bar{s})]]\hat{x}_2 \\ + x_{\alpha 1} - x_{\alpha 2} + (\bar{s} - \underline{s})\chi'_\alpha(\underline{s})\} , \tag{2.32}$$

and then

$$F_{K\alpha}(\hat{x}_1, \hat{x}_2) = x_{\alpha 3} + (x_{\alpha 1} - x_{\alpha 3})\hat{x}_1 + (x_{\alpha 2} - x_{\alpha 3})\hat{x}_2 \\ + \tfrac{1}{2}\hat{x}_1\hat{x}_2\{[2(x_{\alpha 2} - x_{\alpha 1}) - (\bar{s} - \underline{s})[\chi'_\alpha(\underline{s}) + \chi'_\alpha(\bar{s})]](\hat{x}_2 - \hat{x}_1) \\ + (\bar{s} - \underline{s})[\chi'_\alpha(\underline{s}) - \chi'_\alpha(\bar{s})]\} . \tag{2.33}$$

EXAMPLE 2.3. Construction of γ_h by using polynomials of order 5. Now, the degrees of freedom of the Hermite-type interpolation are given by

$$x_{\alpha 1} = \psi_\alpha(0) = \chi_\alpha(\underline{s}) , \qquad x_{\alpha 2} = \psi_\alpha(1) = \chi_\alpha(\bar{s}) , \\ \psi_\alpha^{(l)}(0) = (\bar{s} - \underline{s})^l \chi_\alpha^{(l)}(\underline{s}) , \qquad \psi_\alpha^{(l)}(1) = (\bar{s} - \underline{s})^l \chi_\alpha^{(l)}(\bar{s}) , \quad l = 1, 2,\ \alpha = 1, 2 , \tag{2.34}$$

so that

$$\psi_{\alpha h}(\hat{x}_2) = x_{\alpha 1} + (x_{\alpha 2} - x_{\alpha 1})\hat{x}_2 + \hat{x}_2(1-\hat{x}_2)[\beta_{\alpha 3}(\hat{x}_2)^3 + \beta_{\alpha 2}(\hat{x}_2)^2 + \beta_{\alpha 1}\hat{x}_2 + \beta_{\alpha 0}] , \tag{2.35}$$

where the coefficients $\beta_{\alpha l}$, $l = 0, 1, 2, 3$, are given by

$$\beta_{\alpha 0} = x_{\alpha 1} - x_{\alpha 2} + (\bar{s} - \underline{s})\chi'_\alpha(\underline{s}) , \\ \beta_{\alpha 1} = x_{\alpha 1} - x_{\alpha 2} + (\bar{s} - \underline{s})\chi'_\alpha(\underline{s}) + \tfrac{1}{2}(\bar{s} - \underline{s})^2\chi''_\alpha(\underline{s}) , \\ \beta_{\alpha 2} = 9(x_{\alpha 2} - x_{\alpha 1}) - (\bar{s} - \underline{s})[5\chi'_\alpha(\underline{s}) + 4\chi'_\alpha(\bar{s})] - \tfrac{1}{2}(\bar{s} - \underline{s})^2[2\chi''_\alpha(\underline{s}) - \chi''_\alpha(\bar{s})] , \\ \beta_{\alpha 3} = 6(x_{\alpha 1} - x_{\alpha 2}) + 3(\bar{s} - \underline{s})[\chi'_\alpha(\underline{s}) + \chi'_\alpha(\bar{s})] + \tfrac{1}{2}(\bar{s} - \underline{s})^2[\chi''_\alpha(\underline{s}) - \chi''_\alpha(\bar{s})] . \tag{2.36}$$

Hence, from (2.10),

$$F_{K\alpha}(\hat{x}_1, \hat{x}_2) = x_{\alpha 3} + (x_{\alpha 1} - x_{\alpha 3})\hat{x}_1 + (x_{\alpha 2} - x_{\alpha 3})\hat{x}_2 \\ + \tfrac{1}{2}\hat{x}_1\hat{x}_2[\beta_{\alpha 3}(\hat{x}_2)^3 + \beta_{\alpha 2}(\hat{x}_2)^2 + \beta_{\alpha 1}\hat{x}_2 + \beta_{\alpha 0} \\ + \tilde{\beta}_{\alpha 3}(\hat{x}_1)^3 + \tilde{\beta}_{\alpha 2}(\hat{x}_1)^2 + \tilde{\beta}_{\alpha 1}\hat{x}_1 + \tilde{\beta}_{\alpha 0}] , \tag{2.37}$$

where the coefficients $\tilde{\beta}_{\alpha l}$, $l = 0, 1, 2, 3$, are given by

$$\tilde{\beta}_{\alpha 0} = x_{\alpha 2} - x_{\alpha 1} - (\bar{s} - \underline{s})\chi'_\alpha(\bar{s}) , \\ \tilde{\beta}_{\alpha 1} = x_{\alpha 2} - x_{\alpha 1} - (\bar{s} - \underline{s})\chi'_\alpha(\bar{s}) + \tfrac{1}{2}(\bar{s} - \underline{s})^2\chi''_\alpha(\bar{s}) , \tag{2.38}$$

$$\tilde{\beta}_{\alpha 2} = 9(x_{\alpha 1} - x_{\alpha 2}) + (\bar{s} - \underline{s})[5\chi'_\alpha(\bar{s}) + 4\chi'_\alpha(\underline{s})] - \tfrac{1}{2}(\bar{s} - \underline{s})^2[2\chi''_\alpha(\bar{s}) - \chi''_\alpha(\underline{s})]\,,$$
$$\tilde{\beta}_{\alpha 3} = 6(x_{\alpha 2} - x_{\alpha 1}) - 3(\bar{s} - \underline{s})[\chi'_\alpha(\bar{s}) + \chi'_\alpha(\underline{s})] + \tfrac{1}{2}(\bar{s} - \underline{s})^2[\chi''_\alpha(\bar{s}) - \chi''_\alpha(\underline{s})]\,.$$

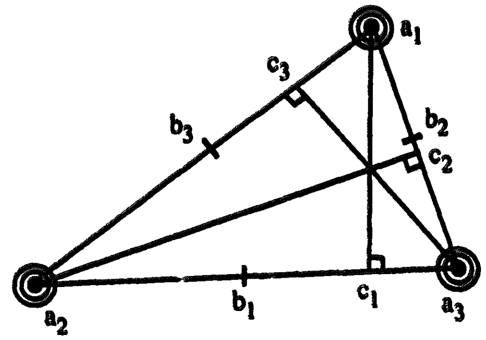
3. Definition of curved $\mathscr{C}^1$ finite elements

In this section, we define curved finite elements which have a connection of class $\mathscr{C}^1$ (we say $\mathscr{C}^1$-compatible) with Argyris and Bell triangles. We state the basic principles in Section 3.1. In Section 3.2, we detail this definition for a connection between a curved finite element and an Argyris triangle in the case of an approximate boundary parameterized by polynomials of degree five. Following a remark of [17], we indicate in Section 3.3 the simplifications that it is possible to realize when the boundary conditions are of homogeneous Dirichlet type; the approximate boundary γ_h can be parameterized by polynomials of degree three only. In both cases, we mention the modifications which permit to obtain a curved element which is $\mathscr{C}^1$-compatible with the Bell triangle. Finally, in Section 3.4, we define curved elements which have a connection of class $\mathscr{C}^0$ with a Hermite element of degree 3; these elements can be used for instance to approximate tangential components of the displacement for thin shell problems.

3.1. Basic principles

Figures 5 and 6 recall the definitions of the Argyris and Bell triangles as well as their interpolation properties. By P_K and Σ_K we denote, respectively, the functional space and the set of degrees of freedom of the finite element while π_K means the associate interpolation operator.

Among the principles we need to observe when constructing such curved finite elements, we shall consider the 'essential' conditions and the 'desirable' conditions.

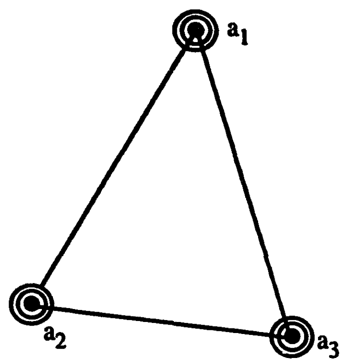


$P_K = P_5(K)\,, \qquad \dim P_K = 21\,,$

$\Sigma_K = \{p(a_i), Dp(a_i)(a_{i-1} - a_i), Dp(a_i)(a_{i+1} - a_i), 1 \leq i \leq 3;$
$\quad D^2p(a_i)(a_{j+1} - a_{j-1})^2, 1 \leq i, j \leq 3; Dp(b_i)(a_i - c_i), 1 \leq i \leq 3\}$

$\|v - \pi_K v\|_{m,K} \leq c h_K^{k+1-m} |v|_{k+1,K} \quad \forall v \in H^{k+1}(K), 3 \leq k \leq 5$ and $0 \leq m \leq k+1$.

Fig. 5. Argyris triangle [6].



$P_K = \{p \in P_5(K); \partial_n p \in P_3(K') \text{ for each side } K' \text{ of } K\}$

$\dim P_K = 18, \quad P_4(K) \subset P_K \subset P_5(K)$

$\Sigma_K = \{p(a_i), Dp(a_i)(a_{i-1} - a_i), Dp(a_i)(a_{i+1} - a_i), 1 \leq i \leq 3;$

$D^2p(a_i)(a_{j+1} - a_{j-1})^2, 1 \leq i, j \leq 3\}$;

$\|v - \pi_K v\|_{m,K} \leq c h_K^{k+1-m} |v|_{k+1,K} \quad \forall v \in H^{k+1}(K), k = 3, 4, \quad 0 \leq m \leq k+1.$

Fig. 6. Bell triangle [7].

3.1.1. '*Essential*' conditions

The connections between the curved elements, associated with curved triangles K of Fig. 1, and the adjacent (straight or curved) finite elements are realized through the straight sides a_3a_1 and a_3a_2, as mentioned in Fig. 7.

First, consider the connection of a curved finite element associated with the curved triangle K with the Argyris (or Bell) triangle associated with the triangle K_D of Fig. 7. Such a connection will be of class $\mathscr{C}^1$ if, and only if, the traces of functions $p \in P_K$ and of their normal derivatives $\partial p / \partial n_{31}$ along the side a_3a_1 coincide with their homologues of the adjacent finite element. In order to satisfy these conditions, it is sufficient to meet the following requirements:

the degrees of freedom of the curved finite element relative to the sides a_3a_α, $\alpha = 1, 2$, are identical to that of the adjacent finite element; (3.1)

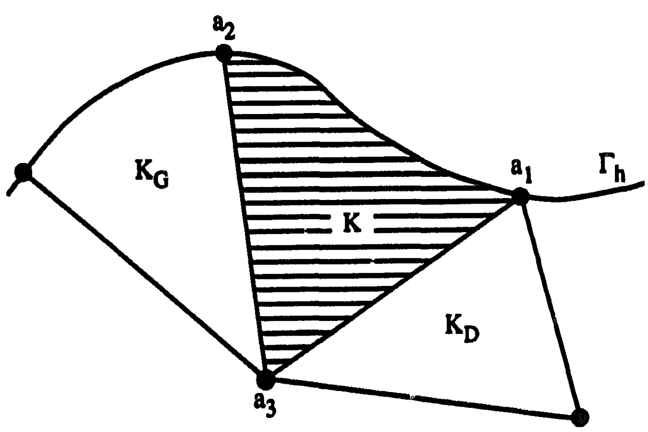


Fig. 7. Adjacent triangles to a curved triangle K.

$$\begin{array}{ll} \text{the traces } p|_{[a_3,a_\alpha]} \text{ (respectively } (\partial p/\partial n_{3\alpha})|_{[a_3,a_\alpha]}),\ \alpha = 1,2, \text{ of functions} & \\ p \in P_K \text{ associated with the curved triangle } K \text{ are one-variable} & \\ \text{polynomials of degree 5 (respectively 4 for Argyris triangle and 3} & (3.2) \\ \text{for Bell triangle), entirely determined by the degrees of} & \\ \text{freedom relative to the sides } a_3a_\alpha,\ \alpha = 1,2. & \end{array}$$

Likewise, these conditions ensure a $\mathscr{C}^1$ connection between two adjacent curved finite elements.

3.1.2. 'Desirable' conditions

The application F_K, studied in Section 2, associates the curved triangle K with the reference triangle $\hat{K}$. We use the same application F_K to associate with any function v defined over the triangle K, a function $\hat{v}$ defined over the triangle $\hat{K}$, i.e.,

$$v = \hat{v} \circ F_K^{-1}\,, \qquad \hat{v} = v \circ F_K\,. \tag{3.3}$$

Then, it is 'desirable' that the following condition is satisfied:

$$\begin{array}{ll} \text{to any function } p \in P_K, \text{ defined over the curved triangle } K\,, & \\ \text{the correspondence (3.3) associates a polynomial function } \hat{p} = p \circ F_K\,. & (3.4) \end{array}$$

The condition (3.4) is convenient for the study of the approximation error and to take into account the numerical integration and the boundary conditions.

On the other hand, this condition leads to the definition of reference finite elements which are more complicated than those associated with corresponding straight finite elements. Indeed, let $\hat{a}$ be any point of the side $\hat{a}_3\hat{a}_1$ of the triangle $\hat{K}$ (see Fig. 8) and set $a = F_K(\hat{a})$. Then, relation $\hat{p} = p \circ F_K$ involves

$$\frac{\partial \hat{p}}{\partial \hat{x}_2}(\hat{a}) = \left\langle \frac{\partial F_K}{\partial \hat{x}_2}(\hat{a}), t_{31} \right\rangle \frac{\partial p}{\partial t_{31}}(a) + \left\langle \frac{\partial F_K}{\partial \hat{x}_2}(\hat{a}), n_{31} \right\rangle \frac{\partial p}{\partial n_{31}}(a)\,, \tag{3.5}$$

and a similar relation for any point $\hat{a} \in \hat{a}_3\hat{a}_2$:

$$\frac{\partial \hat{p}}{\partial \hat{x}_1}(\hat{a}) = \left\langle \frac{\partial F_K}{\partial \hat{x}_1}(\hat{a}), t_{32} \right\rangle \frac{\partial p}{\partial t_{32}}(a) + \left\langle \frac{\partial F_K}{\partial \hat{x}_1}(\hat{a}), n_{32} \right\rangle \frac{\partial p}{\partial n_{32}}(a)\,, \tag{3.6}$$

where $\langle\ ,\ \rangle$ denotes the usual scalar product of $\mathbb{R}^2$.

By consideration of the hypotheses (3.2), we verify that

$$\begin{array}{ll} \text{the derivatives } \partial p/\partial t_{3\alpha}(a), \text{ for } a \in a_3a_\alpha,\ \alpha = 1,2, \text{ are polynomials} & \\ \text{of degree 4 with respect to } \hat{x}_\alpha \text{ (note that } F_K \text{ is affine along } \hat{a}_3\hat{a}_\alpha); & (3.7) \end{array}$$

$$\begin{array}{ll} \text{the derivatives } \partial p/\partial n_{3\alpha}(a), \text{ for } a \in a_3a_\alpha,\ \alpha = 1,2, \text{ are polynomials} & \\ \text{of degree 4 (Argyris) or 3 (Bell) with respect to } \hat{x}_\alpha. & (3.8) \end{array}$$

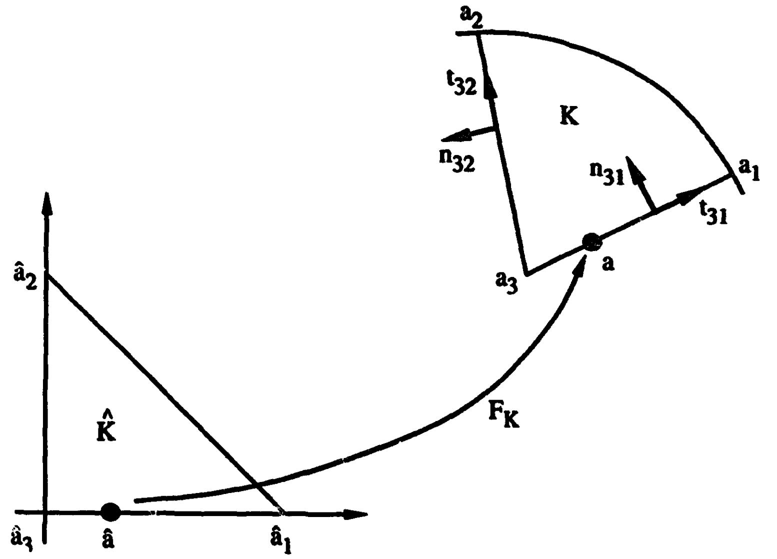


Fig. 8. Local reference systems along the straight sides ot the curved triangle K ($t_{3\alpha}$, $n_{3\alpha}$ are the unit tangential and normal vectors to the side a_3a_α).

Thus, (2.10), (3.5–3.8) involve that, for $\hat{a} \in \hat{a}_3\hat{a}_1$, $\partial\hat{p}/\partial\hat{x}_2(\hat{a})$ is a polynomial of degree $n+3$ with respect to $\hat{x}_1$, while for $\hat{a} \in \hat{a}_3\hat{a}_2$, $\partial\hat{p}/\partial\hat{x}_1(\hat{a})$ is a polynomial of degree $n+3$ with respect to $\hat{x}_2$. Let us denote

$$\hat{P}_K = \{\hat{p} : \hat{K} \to \mathbb{R};\ \hat{p} = p \circ F_K,\ p \in P_K\}$$

the space of functions defined over the reference triangle $\hat{K}$ from the space P_K and through the application F_K. Then, in order to satisfy condition (3.4), the above remarks show that we need to satisfy the inclusion

$$\hat{P}_K \subset P_{n+4}\,. \tag{3.9}$$

By using these considerations, we show in Section 3.2 that it is possible to define curved finite elements compatible with Argyris or Bell triangles in the case of $n = 5$, i.e., $F_K \in (P_5)^2$. In Section 3.3, we examine the more simple case, $n = 3$, which is sufficient for homogeneous Dirichlet boundary conditions.

3.2. Definition of curved finite elements $\mathscr{C}^1$-compatible with Argyris or Bell triangles when $F_K \in (P_5)^2$

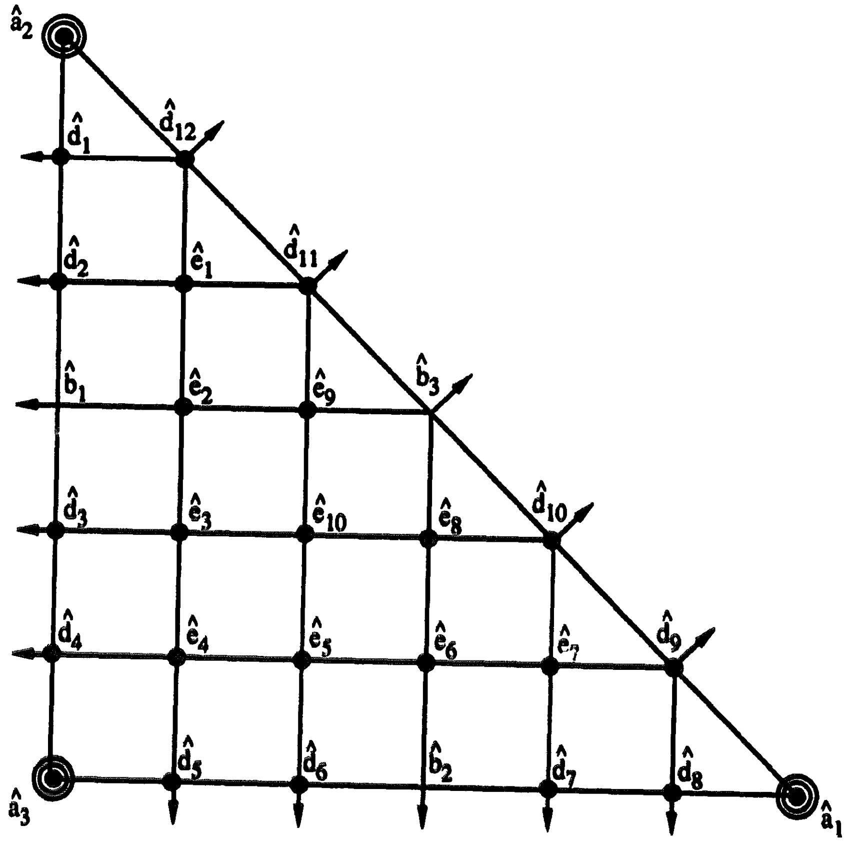
Subsequently, we detail the definition of a curved finite element $\mathscr{C}^1$-compatible with the Argyris triangle. Similarly a curved finite element $\mathscr{C}^1$-compatible with the Bell triangle can be constructed; the modifications are mentioned by 'respectively Bell: . . .'. Since all the curved elements under consideration satisfy conditions (3.1), (3.2) and (3.4), the $\mathscr{C}^1$-compatibility between two adjacent curved finite elements is automatically satisfied.

For our definition, we associate with the reference triangle $\hat{K}$ a basic finite element

$(\hat{K}, \hat{P}, \hat{\Sigma})$ described in Fig. 9. The choice of $\hat{P} = P_9$ takes into account inclusion (3.9). We say that this finite element is 'basic' by opposition to the notion of reference finite element (see the theory of affine finite elements). To obtain the reference finite element associated with the curved finite element under consideration, we will have to impose 24 (respectively Bell: 27) constraints to the set $\hat{P}$.

Considering Fig. 9, it remains to prove the following theorem.

THEOREM 3.1. The triple $(\hat{K}, \hat{P}, \hat{\Sigma})$ of Fig. 9 defines a finite element.



$\hat{K}$ = unit right-angled triangle;

$\hat{P} = P_9$, $\quad \dim \hat{P} = 55$;

$$\hat{\Sigma}(\hat{w}) = \Big\{ \hat{w}(\hat{a}_i),\ \frac{\partial \hat{w}}{\partial \hat{x}_1}(\hat{a}_i),\ \frac{\partial \hat{w}}{\partial \hat{x}_2}(\hat{a}_i),\ \frac{\partial^2 \hat{w}}{\partial \hat{x}_1^2}(\hat{a}_i),\ \frac{\partial^2 \hat{w}}{\partial \hat{x}_1 \partial \hat{x}_2}(\hat{a}_i),\ \frac{\partial^2 \hat{w}}{\partial \hat{x}_2^2}(\hat{a}_i),\ i = 1, 2, 3;$$

$$-\frac{\partial \hat{w}}{\partial \hat{x}_1}(\hat{b}_1);\ -\frac{\partial \hat{w}}{\partial \hat{x}_2}(\hat{b}_2);\ \frac{\sqrt{2}}{2}\Big(\frac{\partial \hat{w}}{\partial \hat{x}_1} + \frac{\partial \hat{w}}{\partial \hat{x}_2}\Big)(\hat{b}_3);\ \hat{w}(\hat{d}_i),\ i = 1, \ldots, 12;$$

$$-\frac{\partial \hat{w}}{\partial \hat{x}_1}(\hat{d}_i),\ i = 1, \ldots, 4;\ -\frac{\partial \hat{w}}{\partial \hat{x}_2}(\hat{d}_i),\ i = 5, \ldots, 8;$$

$$\frac{\sqrt{2}}{2}\Big(\frac{\partial \hat{w}}{\partial \hat{x}_1} + \frac{\partial \hat{w}}{\partial \hat{x}_2}\Big)(\hat{d}_i),\ i = 9, \ldots, 12;\ \hat{w}(\hat{e}_i),\ i = 1, \ldots, 10\Big\}.$$

Fig. 9. 'Basic' P_9-Hermite finite element.

PROOF. Since $\dim \hat{P} = \mathrm{card}(\hat{\Sigma})$, we need to prove that 0 is the unique element of $\hat{P}$ whose corresponding degrees of freedom $\hat{\Sigma}$ are zero. First we remark that such a function $\hat{p}$ is identically zero when restricted to the sides of $\hat{K}$ so that $\hat{p}(\hat{x}_1, \hat{x}_2) = [\hat{x}_1\hat{x}_2(1 - \hat{x}_1 - \hat{x}_2)]^2 \times \hat{q}(\hat{x}_1, \hat{x}_2)$, where $\hat{q} \in P_3$. Next, we have $\hat{q}(\hat{e}_i) = 0$, $i = 1, \ldots, 10$ so that $\hat{q} \equiv 0$. □

Construction of the interpolate function $v \to \pi_K v$

By using this basic finite element we are going to associate with any regular function v (for instance $v \in C^2(K)$) defined over the curved triangle K, its interpolate $\pi_K v$. It takes three steps.

STEP 1. Definition of the set Σ_K of degrees of freedom of the curved element. In Section 3.2, application F_K is that of Example 2.3, i.e., (2.37). Let us set

$$\begin{aligned} &a_i = F_K(\hat{a}_i)\,, \quad b_i = F_K(\hat{b}_i)\,, \quad i = 1, 2, 3\,,\\ &d_i = F_K(\hat{d}_i)\,, \quad i = 1, \ldots, 12\,,\\ &e_i = F_K(\hat{e}_i)\,, \quad i = 1, \ldots, 10\,, \end{aligned} \tag{3.10}$$

and $t_{3\alpha}$, $n_{3\alpha}$, $\alpha = 1, 2$, are unit vectors defined according to Fig. 8.

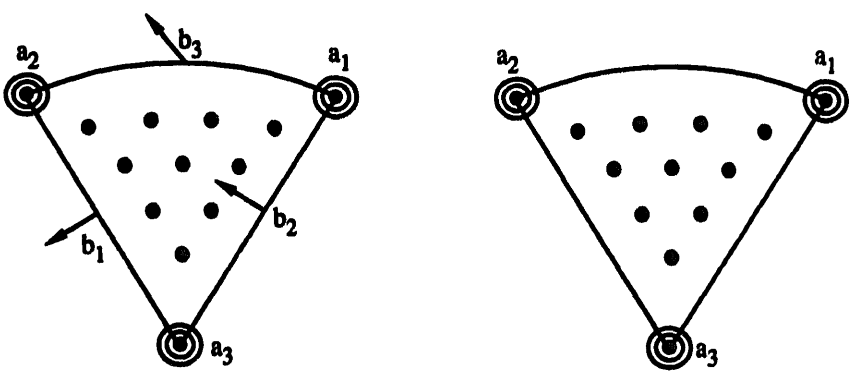
Then, the set $\Sigma_K(v)$ of values of degrees of freedom of v is given by (see Fig. 10)

$$\begin{aligned} \Sigma_K(v) = \{&(D^\alpha v(a_i), |\alpha| = 0, 1, 2), i = 1, 2, 3; Dv(b_2)n_{31}\,;\\ &Dv(b_1)n_{32}; Dv(b_3)DF_K(\hat{b}_3)(\hat{a}_3 - \hat{b}_3); v(e_i), i = 1, \ldots, 10\} \end{aligned} \tag{3.11}$$

(respectively Bell:

$$\Sigma_K(v) = \{(D^\alpha v(a_i), |\alpha| = 0, 1, 2), i = 1, 2, 3; v(e_i), i = 1, \ldots, 10\})\,. \tag{3.12}$$

Note that in addition to the usual degrees of freedom of the corresponding classical elements, we find ten additional degrees of freedom inside the triangle K. Also, note that the sets Σ_K given by (3.11), (3.12) satisfy condition (3.1).



(ARGYRIS) (BELL)

Fig. 10. Sets Σ_K for $\mathscr{C}^1$-compatible curved finite element ($F_K \in (P_5)^2$).

STEP 2. Definition of the set $\hat{\Delta}_K(v)$ from $\Sigma_K(v)$. Consider the following partition of the set $\hat{\Sigma}$: $\hat{\Sigma} = \hat{\Sigma}_1 \cup \hat{\Sigma}_2 \cup \hat{\Sigma}_3$, where

$$\begin{aligned}
&\hat{\Sigma}_1 = \{(D^\alpha \hat{p}(a_i), |\alpha| = 0, 1, 2), i = 1, 2, 3;\ \hat{p}(\hat{e}_i), i = 1, \ldots, 10\}\,, \\
&\hat{\Sigma}_2 = \Big\{ \hat{p}(\hat{d}_i), i = 1, \ldots, 8;\ -\frac{\partial \hat{p}}{\partial \hat{x}_1}(\hat{b}_1);\ -\frac{\partial \hat{p}}{\partial \hat{x}_1}(\hat{d}_i), i = 1, \ldots, 4; \\
&\qquad -\frac{\partial \hat{p}}{\partial \hat{x}_2}(\hat{b}_2);\ -\frac{\partial \hat{p}}{\partial \hat{x}_2}(\hat{d}_i), i = 5, \ldots, 8\Big\}\,, \\
&\hat{\Sigma}_3 = \Big\{ \hat{p}(\hat{d}_i), i = 9, \ldots, 12;\ \frac{\sqrt{2}}{2}\Big(\frac{\partial \hat{p}}{\partial \hat{x}_1} + \frac{\partial \hat{p}}{\partial \hat{x}_2}\Big)(\hat{b}_3); \\
&\qquad \frac{\sqrt{2}}{2}\Big(\frac{\partial \hat{p}}{\partial \hat{x}_1} + \frac{\partial \hat{p}}{\partial \hat{x}_2}\Big)(\hat{d}_i), i = 9, \ldots, 12\Big\}\,.
\end{aligned} \tag{3.13}$$

With the set of values $\Sigma_K(v)$, we are going to associate the sets of values $\hat{\Delta}_i(v)$ that we need to attribute to $\hat{\Sigma}_i$, $i = 1, 2, 3$, to obtain a suitable function $\hat{w} \in \hat{P}$. Let us note $\hat{\Delta}_K(v) = \hat{\Delta}_{K1}(v) \cup \hat{\Delta}_{K2}(v) \cup \hat{\Delta}_{K3}(v)$. By means of the application F_K we define the function

$$\hat{v} = v \circ F_K\,. \tag{3.14}$$

Then, from the set $\Sigma_K(v)$, we immediately derive

$$\hat{\Delta}_{K1}(v) = \{(D^\alpha \hat{v}(\hat{a}_i), |\alpha| = 0, 1, 2), i = 1, 2, 3;\ \hat{v}(\hat{e}_i), i = 1, \ldots, 10\}\,. \tag{3.15}$$

Now, consider the set $\hat{\Delta}_{K2}(v)$. We first examine the case of the degrees of freedom of $\hat{\Sigma}_2$ located on the side $\hat{a}_3\hat{a}_1$. In order to obtain an interpolate function $\pi_K v$ which satisfies conditions (3.2), we need that
(a) its trace $\pi_K v|_{[a_3,a_1]}$ coincides with the one-variable P_5-Hermite polynomial defined by the data of the following degrees of freedom:

$$\begin{aligned}
&\{v(a_1), v(a_3), Dv(a_1)(a_3 - a_1), Dv(a_3)(a_1 - a_3), D^2 v(a_1)(a_3 - a_1)^2, \\
&\ D^2 v(a_3)(a_1 - a_3)^2\}\,.
\end{aligned} \tag{3.16}$$

We parameterize the side $a_3 a_1$ by using $\hat{x}_1$, i.e.,

$$x_1 = x_{13} + (x_{11} - x_{13})\hat{x}_1\,, \qquad x_2 = x_{23} + (x_{21} - x_{23})\hat{x}_1\,, \tag{3.17}$$

and we denote

$$\hat{f}_1 \text{ (which will coincide with } (\pi_K v) \circ F_K|_{[\hat{a}_3,\hat{a}_1]})\,, \tag{3.18}$$

the so-defined Hermite polynomial;
(b) its normal derivative $\partial \pi_K v / \partial n_{31}$ has a trace $\partial \pi_K v / \partial n_{31}|_{[a_3,a_1]}$ which coincides with the

one-variable P_4-Hermite polynomial (respectively Bell: P_3), defined by the data of the following degrees of freedom:

$$\{Dv(a_i)n_{31}, D^2v(a_i)(n_{31}, t_{31}), i=1,3; Dv(b_2)n_{31}\} \tag{3.19}$$

$$(\text{respectively Bell: } \{Dv(a_i)n_{31}, D^2v(a_i)(n_{31}, t_{31}), i=1,3\})\,. \tag{3.20}$$

Similarly to (3.17) and (3.18), we denote

$$\hat{g}_1 \left(\text{which will coincide with } \left(\frac{\partial \pi_K v}{\partial n_{31}}\right) \circ F_K|_{[\hat{a}_3,\hat{a}_1]}\right) \tag{3.21}$$

$$\left(\text{respectively Bell: } \hat{h}_1 \left(\text{which will coincide with } \left(\frac{\partial \pi_K v}{\partial n_{31}}\right) \circ F_K|_{[\hat{a}_3,\hat{a}_1]}\right)\right), \tag{3.22}$$

the so-defined Hermite polynomial. Next, we define Hermite polynomials over the second straight side, i.e.,

$$x_1 = x_{13} + (x_{12} - x_{13})\hat{x}_2, \qquad x_2 = x_{23} + (x_{22} - x_{23})\hat{x}_2\,, \tag{3.23}$$

$$\hat{f}_2 \text{ (which will coincide with } (\pi_K v) \circ F_K|_{[\hat{a}_3,\hat{a}_2]})\,, \tag{3.24}$$

$$\hat{g}_2 \text{ (which will coincide with } \left(\frac{\partial \pi_K v}{\partial n_{32}}\right) \circ F_K|_{[\hat{a}_3,\hat{a}_2]}\Big) \tag{3.25}$$

$$\left(\text{respectively Bell: } \hat{h}_2 \left(\text{which will coincide with } \left(\frac{\partial \pi_K v}{\partial n_{32}}\right) \circ F_K|_{[\hat{a}_3,\hat{a}_2]}\right)\right). \tag{3.26}$$

At this stage, note that the polynomial functions $\hat{f}_\alpha$, $\hat{g}_\alpha$, $\hat{h}_\alpha$, $\alpha = 1, 2$, are only dependent on the values of the degrees of freedom of the function v, given in the set $\Sigma_K(v)$, and relative to the sides a_3a_1 and a_3a_2. Then, the set of values $\hat{\Delta}_{K2}(v)$ associated with the set of degrees of freedom $\hat{\Sigma}_2$ (see (3.13)) is given by

$$\begin{aligned}
\hat{\Delta}_{K2}(v) = \Big\{ & \hat{f}_2(\hat{d}_i), i = 1, \dots, 4;\ \hat{f}_1(\hat{d}_i), i = 5, \dots, 8\,; \\
& -\frac{1}{|a_3a_2|}\left\{\left\langle \frac{\partial F_K}{\partial \hat{x}_1}(\hat{b}_1), t_{32}\right\rangle \frac{\mathrm{d}\hat{f}_2}{\mathrm{d}\hat{x}_2}(\hat{b}_1) + |a_3a_2|\left\langle \frac{\partial F_K}{\partial \hat{x}_1}(\hat{b}_1), n_{32}\right\rangle \hat{g}_2(\hat{b}_1)\right\}; \\
& -\frac{1}{|a_3a_2|}\left\{\left\langle \frac{\partial F_K}{\partial \hat{x}_1}(\hat{d}_i), t_{32}\right\rangle \frac{\mathrm{d}\hat{f}_2}{\mathrm{d}\hat{x}_2}(\hat{d}_i) + |a_3a_2|\left\langle \frac{\partial F_K}{\partial \hat{x}_1}(\hat{d}_i), n_{32}\right\rangle \hat{g}_2(\hat{d}_i)\right\} \\
& (i = 1, \dots, 4); \\
& -\frac{1}{|a_3a_1|}\left\{\left\langle \frac{\partial F_K}{\partial \hat{x}_2}(\hat{b}_2), t_{31}\right\rangle \frac{\mathrm{d}\hat{f}_1}{\mathrm{d}\hat{x}_1}(\hat{b}_2) + |a_3a_1|\left\langle \frac{\partial F_K}{\partial \hat{x}_2}(\hat{b}_2), n_{31}\right\rangle \hat{g}_1(\hat{b}_2)\right\};
\end{aligned}$$

$$- \frac{1}{|a_3 a_1|} \left\{ \left\langle \frac{\partial F_K}{\partial \hat{x}_2}(\hat{a}_i), t_{31} \right\rangle \frac{\mathrm{d}\hat{f}_1}{\mathrm{d}\hat{x}_1}(\hat{a}_i) + |a_3 a_1| \left\langle \frac{\partial F_K}{\partial \hat{x}_2}(\hat{a}_i), n_{31} \right\rangle \hat{g}_3(\hat{a}_i) \right\}$$

$$(i = 5, \dots, 8) \Big\}$$

$$\text{(respectively Bell: replace } \hat{g}_\alpha \text{ by } \hat{h}_\alpha,\ \alpha = 1, 2) . \tag{3.27}$$

To find the expression of the ten last elements of $\hat{\Delta}_{K2}(v)$ we have used (3.5) and (3.6), by observing that from conditions (3.18) and (3.24), we obtain

$$\begin{aligned} &\frac{\mathrm{d}\hat{f}_2}{\mathrm{d}\hat{x}_2} \text{ will coincide with } |a_3 a_2| \left(\frac{\partial \pi_K v}{\partial t_{32}} \right) \circ F_K \Big|_{[\hat{a}_3, \hat{a}_2]} , \\ &\frac{\mathrm{d}\hat{f}_1}{\mathrm{d}\hat{x}_1} \text{ will coincide with } |a_3 a_1| \left(\frac{\partial \pi_K v}{\partial t_{31}} \right) \circ F_K \Big|_{[\hat{a}_3, \hat{a}_1]} . \end{aligned} \tag{3.28}$$

In a third set $\hat{\Delta}_{K3}(v)$, we give suitable values for the nine degrees of freedom of $\hat{\Sigma}_3$. First, let us observe that the correspondence $\hat{v} = v \circ F_K$ involves

$$D\hat{v}(\hat{b}_3)(\hat{a}_3 - \hat{b}_3) = Dv(b_3) DF_K(\hat{b}_3)(\hat{a}_3 - \hat{b}_3) . \tag{3.29}$$

The second member is known since it appears among the degrees of freedom of Σ_K (see (3.11)). Moreover, (3.15) contains the values of $D^\alpha \hat{v}(\hat{a}_i)$, $|\alpha| = 0, 1, 2$. The side $\hat{a}_1 \hat{a}_2$ of triangle $\hat{K}$ is parameterized by

$$\hat{x}_1 = \hat{x}_1 , \qquad \hat{x}_2 = 1 - \hat{x}_1 . \tag{3.30}$$

Then, we denote $\hat{f}_3(\hat{x}_1)$, the P_5-Hermite polynomial defined by the data of the degrees of freedom,

$$\begin{aligned} &\{\hat{v}(\hat{a}_1), \hat{v}(\hat{a}_2), D\hat{v}(\hat{a}_1)(\hat{a}_2 - \hat{a}_1), D\hat{v}(\hat{a}_2)(\hat{a}_1 - \hat{a}_2), D^2\hat{v}(\hat{a}_1)(\hat{a}_2 - \hat{a}_1)^2, \\ &D^2\hat{v}(\hat{a}_2)(\hat{a}_1 - \hat{a}_2)^2\} , \end{aligned} \tag{3.31}$$

and by $\hat{g}_3(\hat{x}_1)$ [respectively Bell: $\hat{h}_3(\hat{x}_1)$] the P_4-Hermite polynomial (respectively Bell: P_3), defined by the data of the degrees of freedom

$$\begin{aligned} &\{D\hat{v}(\hat{a}_\alpha)(\hat{a}_3 - \hat{b}_3), \alpha = 1, 2; D^2\hat{v}(\hat{a}_1)(\hat{a}_3 - \hat{b}_3, \hat{a}_2 - \hat{a}_1); \\ &D^2\hat{v}(\hat{a}_2)(\hat{a}_3 - \hat{b}_3, \hat{a}_1 - \hat{a}_2); D\hat{v}(\hat{b}_3)(\hat{a}_3 - \hat{b}_3)\} \end{aligned} \tag{3.32}$$

$$\begin{aligned} &\text{[respectively Bell: } \{D\hat{v}(\hat{a}_\alpha)(\hat{a}_3 - \hat{b}_3), \alpha = 1, 2; D^2\hat{v}(\hat{a}_1)(\hat{a}_3 - \hat{b}_3, \hat{a}_2 - \hat{a}_1); \\ &\qquad D^2\hat{v}(\hat{a}_2)(\hat{a}_3 - \hat{b}_3, \hat{a}_1 - \hat{a}_2)\}] . \end{aligned} \tag{3.33}$$

Then, we set

$$\hat{\Delta}_{K3}(v) = \{\hat{f}_3(\hat{d}_i), i = 9, \dots, 12; -\sqrt{2}\hat{g}_3(\hat{b}_3); -\sqrt{2}\hat{g}_3(\hat{d}_i), i = 9, \dots, 12\}$$
$$\text{[respectively Bell: replace } \hat{g}_3 \text{ by } \hat{h}_3] . \tag{3.34}$$

Here again, note that the set of values $\hat{\Delta}_{K3}(v)$ is only dependent on the values $\Sigma_K(v)$ of the degrees of freedom of the function v. Thus, the set $\hat{\Delta}_K(v) = \{\hat{\Delta}_{K1}(v), \hat{\Delta}_{K2}(v), \hat{\Delta}_{K3}(v)\}$ (see (3.15), (3.27) and (3.34)) specifies a value to each degree of freedom of $\hat{\Sigma}$ (see Fig. 9).

STEP 3. Definition of function $\pi_K v$ from the set $\hat{\Delta}_K(v)$. Let $\hat{w}$ be the function of $\hat{P}$ which takes the set of values $\hat{\Delta}_K(v)$ over the set of degrees of freedom $\hat{\Sigma}$ (see Fig. 9). Then, the function w is obtained through the mapping F_K^{-1}, i.e.,

$$w = \hat{w} \circ F_K^{-1} . \tag{3.35}$$

In Theorem 3.2, we prove that w interpolates the function v and verifies properties (3.1), (3.2) and (3.4), so that we are allowed to pose $w = \pi_K v$.

REMARK 3.1. With the function $\hat{v}$ defined by (3.14), the basic finite element $(\hat{K}, \hat{\Sigma}, \hat{P})$ (see Fig. 9) associates an interpolate function $\hat{\pi}\hat{v}$ which is generally different from $\hat{w}$. The difference $\hat{w} - \hat{\pi}\hat{v}$ is studied in Section 4.

Now, we check that the function w verifies the desirable interpolation properties.

THEOREM 3.2. The function w, defined by (3.35), is determined in a unique way by the data of the set $\Sigma_K(v)$ of the values of the degrees of freedom of the function v, i.e.,

$$\begin{aligned}\Sigma_K(v) = \{(D^\alpha v(a_i), |\alpha| = 0, 1, 2), i = 1, 2, 3;\ & Dv(b_2)n_{31};\ Dv(b_1)n_{32} \\ & Dv(b_3)DF_K(\hat{b}_3)(\hat{a}_3 - \hat{b}_3);\ v(e_i), i = 1, \dots, 10\}\end{aligned} \tag{3.36}$$

(respectively Bell: $\Sigma_K(v) = \{(D^\alpha v(a_i), |\alpha| = 0, 1, 2), i = 1, 2, 3;$
$v(e_i), i = 1, \dots, 10\})$.

Moreover the function w verifies the relations

$$D^\alpha w(a_i) = D^\alpha v(a_i) , \quad |\alpha| = 0, 1, 2, i = 1, 2, 3 , \tag{3.37}$$

$$Dw(b_2)n_{31} = Dv(b_2)n_{31} , \tag{3.38}$$

$$Dw(b_1)n_{32} = Dv(b_1)n_{32} , \tag{3.39}$$

$$Dw(b_3)DF_K(\hat{b}_3)(\hat{a}_3 - \hat{b}_3) = Dv(b_3)DF_K(\hat{b}_3)(\hat{a}_3 - \hat{b}_3) , \tag{3.40}$$

$$w(e_i) = v(e_i) , \quad i = 1, \dots, 10 \tag{3.41}$$

(respectively Bell: w verifies (3.37) and (3.41)) and w satisfies conditions (3.1), (3.2) and (3.4) so that we have $w = \pi_K v$.

PROOF. The functions $\hat{w}$ and w are only dependent on the set of values $\Sigma_K(v)$. By construction, the function $\hat{w}$ verifies (see (3.15))

$$D^\alpha \hat{w}(\hat{a}_i) = D^\alpha \hat{v}(\hat{a}_i)\,, \quad |\alpha| = 0,1,2,\ i = 1,2,3\,,$$
$$\hat{w}(\hat{e}_i) = \hat{v}(\hat{e}_i)\,, \quad i = 1,\ldots,10\,,$$

so that, by using (3.14) and (3.35), we obtain (3.37) and (3.41).

In the same way, by construction of $\hat{\Delta}_{K3}(v)$, we obtain

$$\begin{aligned} Dw(b_3)DF_K(\hat{b}_3)(\hat{a}_3 - \hat{b}_3) &= D\hat{w}(\hat{b}_3)(\hat{a}_3 - \hat{b}_3) = \hat{g}_3(\tfrac{1}{2}) = D\hat{v}(\hat{b}_3)(\hat{a}_3 - \hat{b}_3) \\ &= Dv(b_3)DF_K(\hat{b}_3)(\hat{a}_3 - \hat{b}_3)\,, \end{aligned}$$

and hence we obtain (3.40).

To prove (3.38), (3.39), we show that

$$\left[\frac{\partial w}{\partial n_{3\alpha}}\right] \circ F_K\big|_{[\hat{a}_3,\hat{a}_\alpha]} = \hat{g}_\alpha\,, \quad \alpha = 1,2\,. \tag{3.42}$$

Then, (3.38), (3.39) are direct consequences of definitions (3.19), (3.21) and (3.25) for interpolating functions $\hat{g}_\alpha$ which verify

$$\hat{g}_1(\hat{b}_2) = Dv(b_2)n_{31}\,, \qquad \hat{g}_2(\hat{b}_1) = Dv(b_1)n_{32}\,.$$

Therefore let us prove (3.42). First, by construction

$$\hat{w}\big|_{[\hat{a}_3,\hat{a}_\alpha]} = \hat{f}_\alpha\,. \tag{3.43}$$

Since application F_K is affine along the sides $\hat{a}_3\hat{a}_\alpha$, $\alpha = 1,2$, the traces $w|_{[a_3,a_\alpha]}$ are one-dimensional P_5-polynomials entirely determined by the degrees of freedom relative to the sides a_3a_α ($\alpha = 1,2$). Thus, the function $w|_{[a_3,a_\alpha]}$ realizes the first part of condition (3.2).

Now, we prove (3.42) for $\alpha = 1$. Relations (3.5) and $\hat{w} = w \circ F_K$ involve for any $\hat{a} \in [\hat{a}_3, \hat{a}_1]$,

$$D\hat{w}(\hat{a})e_2 = \left\langle \frac{\partial F_K}{\partial \hat{x}_2}(\hat{a}), t_{31}\right\rangle \frac{\partial w}{\partial t_{31}}(a) + \left\langle \frac{\partial F_K}{\partial \hat{x}_2}(\hat{a}), n_{31}\right\rangle \frac{\partial w}{\partial n_{31}}(a)\,. \tag{3.44}$$

The functions $\mathrm{d}\hat{f}_1/\mathrm{d}\hat{x}_1$, $\hat{g}_1$, $(\partial F_K/\partial \hat{x}_2)|_{[\hat{a}_3,\hat{a}_1]}$ are P_4-polynomials in $\hat{x}_1$. Then, the expression

$$\frac{1}{|a_3a_1|}\left\langle \frac{\partial F_K}{\partial \hat{x}_2}(\hat{a}), t_{31}\right\rangle \frac{\mathrm{d}\hat{f}_1}{\mathrm{d}\hat{x}_1}(\hat{a}) + \left\langle \frac{\partial F_K}{\partial \hat{x}_2}(\hat{a}), n_{31}\right\rangle \hat{g}_1(\hat{a})$$

is a P_8-polynomial in $\hat{x}_1$. This polynomial is identical to $D\hat{w}(\hat{a})e_2$, which is a P_8-polynomial in $\hat{x}_1$, since both polynomials take the same values over the set

$$\left\{\hat{p}(\hat{a}_3);\ \hat{p}(\hat{a}_1);\ \hat{p}(\hat{b}_2);\ \hat{p}(\hat{d}_i),\ i = 5,\ldots,8;\ \frac{\partial \hat{p}}{\partial \hat{x}_1}(\hat{a}_3),\ \frac{\partial \hat{p}}{\partial \hat{x}_1}(\hat{a}_1)\right\}.$$

In particular, we use the equalities

$$\frac{\mathrm{d}\hat{f}_1}{\mathrm{d}\hat{x}_1}(\hat{a}_i) = |a_3a_1| Dv(a_i)t_{31}\,, \qquad \hat{g}_1(\hat{a}_i) = Dv(a_i)n_{31}\,,$$

$$\frac{\mathrm{d}^2\hat{f}_1}{(\mathrm{d}\hat{x}_1)^2}(\hat{a}_i) = |a_3a_1|^2 D^2v(a_i)(t_{31}, t_{31})\,, \qquad \frac{\mathrm{d}\hat{g}_1}{\mathrm{d}\hat{x}_2}(\hat{a}_i) = |a_3a_1| D^2v(a_i)(n_{31}, t_{31})\,,$$
$$i = 1, 3\,,$$

which are themselves direct consequences of the definitions of $\hat{f}_1$ and $\hat{g}_1$. Thus for any $\hat{a} \in [\hat{a}_3, \hat{a}_1]$,

$$D\hat{w}(\hat{a})e_2 = \frac{1}{|a_3a_1|}\left\langle \frac{\partial F_K}{\partial \hat{x}_2}(\hat{a}), t_{31}\right\rangle \frac{\mathrm{d}\hat{f}_1}{\mathrm{d}\hat{x}_1}(\hat{a}) + \left\langle \frac{\partial F_K}{\partial \hat{x}_2}(\hat{a}), n_{31}\right\rangle \hat{g}_1(\hat{a})\,. \tag{3.45}$$

But, (3.43) imply

$$\frac{\partial w}{\partial t_{31}}(a) = \frac{1}{|a_3a_1|}\frac{\mathrm{d}\hat{f}_1}{\mathrm{d}\hat{x}_1}(\hat{a})\,, \qquad a = F_K(\hat{a}) \quad \forall \hat{a} \in \hat{a}_3\hat{a}_1\,.$$

Then, (3.44) and (3.45) involve (3.42) for $\alpha = 1$. Indeed, for h_K sufficiently small, we have

$$\left|\left\langle \frac{\partial F_K}{\partial \hat{x}_2}(\hat{a}), n_{31}\right\rangle\right| = \frac{1}{|a_3a_1|}|a_3a_1 \times a_3a_2| + \mathrm{O}(h_K^2) \neq 0\,.$$

Similarly, we could show that (3.42) is true for $\alpha = 2$. Since application F_K is affine along the sides $\hat{a}_3\hat{a}_\alpha$, $\alpha = 1, 2$, the traces $(\partial w/\partial n_{3\alpha})|_{[a_3, a_\alpha]}$ are one-variable P_4 (respectively Bell: P_3) polynomials entirely determined by the degrees of freedom relative to the sides a_3a_α, $\alpha = 1, 2$). Thus, the function $(\partial w/\partial n_{3\alpha})|_{[a_3, a_\alpha]}$ realizes the second part of condition (3.2).

By construction, $\hat{w}$ is a polynomial function. Hence, condition (3.4) is verified by $w = \hat{w} \circ F_K^{-1}$. Finally, the definition of Σ_K (see (3.11) (respectively Bell: (3.12))) ensures condition (3.1). □

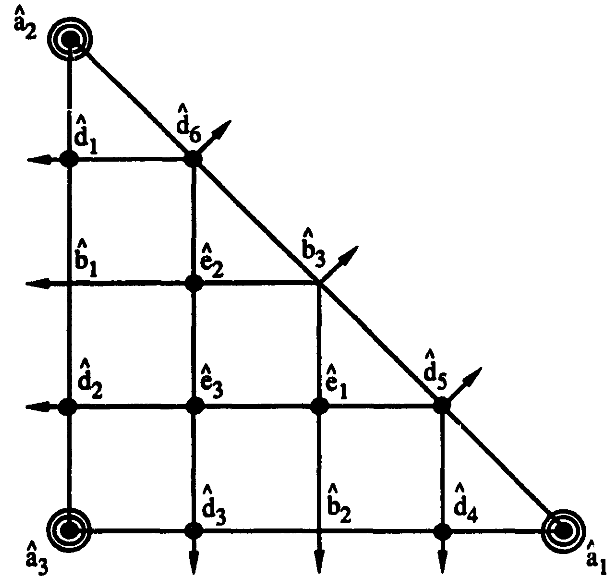
Thus, with the set $\Sigma_K(v)$ of values of degrees of freedom of a function v, the previous construction associates a suitable interpolate function w that we denote $\pi_K v$ in the sequel. It remains to study the interpolation error: this is the aim of Section 4.

3.3. *Definition of curved finite element $\mathscr{C}^1$-compatible with Argyris or Bell triangles when $F_K \in (P_3)^2$*

The construction of such elements follows the same lines as in the case $F_K \in (P_5)^2$. Thus we just indicate the main changes.

Now, the basic finite element is described in Fig. 11 (see also [36]) and by similarity with Theorem 3.1, we obtain the following theorem.

THEOREM 3.3. In Fig. 11, the triple $(\hat{K}, \hat{P}, \hat{\Sigma})$ defines a finite element.



$\hat{K}$ = unit right-angled triangle;

$\hat{P} = P_7$; dim $\hat{P} = 36$;

$$\hat{\Sigma}(\hat{w}) = \left\{ \hat{w}(\hat{a}_i),\ \frac{\partial \hat{w}}{\partial \hat{x}_1}(\hat{a}_i),\ \frac{\partial \hat{w}}{\partial \hat{x}_2}(\hat{a}_i),\ \frac{\partial^2 \hat{w}}{\partial \hat{x}_1^2}(\hat{a}_i),\ \frac{\partial^2 \hat{w}}{\partial \hat{x}_1 \partial \hat{x}_2}(\hat{a}_i),\ \frac{\partial^2 \hat{w}}{\partial \hat{x}_2^2}(\hat{a}_i),\ i = 1, 2, 3; \right.$$
$$-\frac{\partial \hat{w}}{\partial \hat{x}_1}(\hat{b}_1);\ -\frac{\partial \hat{w}}{\partial \hat{x}_2}(\hat{b}_2);\ \frac{\sqrt{2}}{2}\left(\frac{\partial \hat{w}}{\partial \hat{x}_1} + \frac{\partial \hat{w}}{\partial \hat{x}_2}\right)(\hat{b}_3);\ \hat{w}(\hat{d}_i),\ i = 1, \dots, 6;\ \hat{w}(\hat{e}_i),\ i = 1, 2, 3;$$
$$\left. -\frac{\partial \hat{w}}{\partial \hat{x}_1}(\hat{d}_i),\ i = 1, 2;\ -\frac{\partial \hat{w}}{\partial \hat{x}_2}(\hat{d}_i),\ i = 3, 4;\ \frac{\sqrt{2}}{2}\left(\frac{\partial \hat{w}}{\partial \hat{x}_1} + \frac{\partial \hat{w}}{\partial \hat{x}_2}\right)(\hat{d}_i),\ i = 5, 6 \right\}.$$

Fig. 11. 'Basic' P_7-Hermite finite element.

Construction of the interpolate function $v \dashrightarrow \pi_K v$

By using the basic finite element, we associate with any regular function v its interpolate $\pi_K v$. First, with the notation of Figs. 8 and 12 we define the set $\Sigma_K(v)$ of values of degrees of freedom of the curved element. Next, from the 24 (respectively Bell: 21) elements of the set $\Sigma_K(v)$, we associate a set $\Delta_K(v)$ with 36 elements from which we define a suitable interpolating function $\hat{w} \in \hat{P}$:

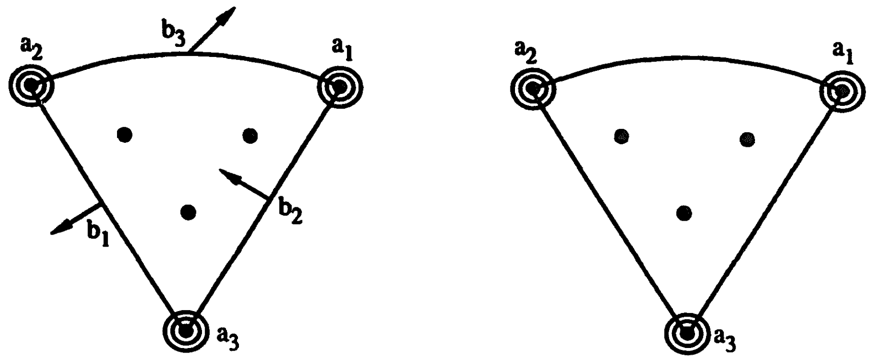
$$\Sigma_K(v) = \{(D^\alpha v(a_i), |\alpha| = 0, 1, 2),\ i = 1, 2, 3;\ Dv(b_2)n_{31};$$
$$Dv(b_1)n_{32};\ Dv(b_3)DF_K(\hat{b}_3)(\hat{a}_3 - \hat{b}_3);\ v(e_i),\ i = 1, 2, 3\} \tag{3.46}$$

(respectively Bell:

$$\Sigma_K(v) = \{(D^\alpha v(a_i),\ |\alpha| = 0, 1, 2),\ i = 1, 2, 3;\ v(e_i),\ i = 1, 2, 3\}). \tag{3.47}$$

Finally we set

$$w = \hat{w} \circ F_K^{-1} \tag{3.48}$$



(ARGYRIS) (BELL)

Fig. 12. Sets Σ_K for $\mathscr{C}^1$-compatible curved finite elements ($F_K \in (P_3)^2$).

and by similarity with Theorem 3.2, we prove in the next theorem that w satisfies the desirable properties.

THEOREM 3.4. The function w, defined in (3.48), is determined in a unique way by the data of the set $\Sigma_K(v)$ of the values of the degrees of freedom of the function v, i.e.,

$$\begin{aligned}\Sigma_K(v) = \{(D^\alpha v(a_i), |\alpha| = 0,1,2), i = 1,2,3;\ Dv(b_2)n_{31};\\ Dv(b_1)n_{32};\ Dv(b_3)DF_K(\hat{b}_3)(\hat{a}_3 - \hat{b}_3);\ v(e_i), i = 1,2,3\}\end{aligned} \tag{3.49}$$

(respectively Bell:

$$\Sigma_K(v) = \{(D^\alpha v(a_i), |\alpha| = 0,1,2), i = 1,2,3;\ v(e_i), i = 1,2,3\}).$$

Moreover the function w verifies the relations

$$D^\alpha w(a_i) = D^\alpha v(a_i), \quad |\alpha| = 0,1,2,\ i = 1,2,3, \tag{3.50}$$

$$Dw(b_2)n_{31} = Dv(b_2)n_{31}, \tag{3.51}$$

$$Dw(b_1)n_{32} = Dv(b_1)n_{32}, \tag{3.52}$$

$$Dw(b_3)DF_K(\hat{b}_3)(\hat{a}_3 - \hat{b}_3) = Dv(b_3)DF_K(\hat{b}_3)(\hat{a}_3 - \hat{b}_3), \tag{3.53}$$

$$w(e_i) = v(e_i), \quad i = 1,2,3 \tag{3.54}$$

(respectively Bell: w verifies (3.50) and (3.54)) and w satisfies conditions (3.1), (3.2) and (3.4)) so that we have $w = \pi_K v$.

3.4. Definition of curved finite element $\mathscr{C}^0$-compatible with Hermite triangles of type (3) when $F_K \in (P_n)^2$, $n = 3$ or 5

These elements will be used in the approximation of tangential components of the displacement for thin shell problems. They are defined by

$$K = F_K(\hat{K}), \quad F_K \in (P_n)^2,\ n = 3 \text{ (see Example 2.2) or } n = 5 \text{ (see Example 2.3)},$$
$$P_K = \{p : K \to \mathbb{R};\ p = \hat{p} \circ F_K^{-1};\ \hat{p} \in P_3\}, \tag{3.55}$$
$$\Sigma_K = \{D^\alpha p(a_i), |\alpha| \leq 1, i = 1,2,3;\ p(F_K(\hat{a}_0)),\quad \hat{a}_0 = \text{barycenter of } \hat{K}\}.$$

Thus, with any regular function v (for instance $v \in \mathscr{C}^1(K)$), we associate the interpolating function $\pi_K v$ defined by

$$\pi_K v = \hat{\pi}\hat{v} \circ F_K^{-1}, \qquad \hat{v} = v \circ F_K, \tag{3.56}$$

where $\hat{\pi}\hat{v}$ is the P_3-Hermite interpolating function of $\hat{v}$ over the reference triangle $\hat{K}$.

REMARK 3.2. When $F_K \in (P_3)^2$, the curved finite element defined in (3.56) is similar, but not the same, as the isoparametric Hermite triangle of type (3) studied in [37, Example 6]. Indeed, our definition of application F_K is different.

4. Estimates of the interpolation errors

In this section, we give estimates of the interpolation error for the curved finite elements of class $\mathscr{C}^1$ constructed in the previous paragraph. It is worth noting that
(i) these estimates have the same order as those obtained for the associate straight finite elements;
(ii) these estimates use the interpolate triangle K instead of the 'original' triangle K_e (see Fig. 1). The 'geometrical' approximation resulting from the substitution of triangle K_e by triangle K will be analyzed in Part 2.

4.1. Estimates of the interpolation error for curved finite elements $\mathscr{C}^1$-compatible with Argyris or Bell triangles when $F_K \in (P_5)^2$

We are going to extend to the associate curved elements, the interpolation error estimates obtained for the Argyris (respectively Bell) triangle in [27, Chapter 6].

THEOREM 4.1. There exists a constant c, independent of h_K, such that for all curved finite elements $\mathscr{C}^1$-compatible with Argyris (respectively Bell) triangles we have

$$|v - \pi_K v|_{m,K} \leq c h_K^{k+1-m} \|v\|_{k+1,K} \quad \forall v \in H^{k+1}(K),\quad k = 3, \ldots, 5,\quad 0 \leq m \leq k+1 \tag{4.1}$$

$$(\textit{respectively Bell: } |v - \pi_K v|_{m,K} \leq c h_K^{k+1-m} \|v\|_{k+1,K} \quad \forall v \in H^{k+1}(K),\quad k = 3, 4,\quad 0 \leq m \leq k+1), \tag{4.2}$$

where $\pi_K v$ is the interpolate function of v defined in Section 3.2.

PROOF. We prove estimates (4.1) (respectively Bell: (4.2)) in the most significant cases, i.e., $k=5$ (respectively Bell: $k=4$). The other cases can be obtained similarly. Let $\hat{v}$ and $\widehat{\pi_K v}$ be corresponding functions over the reference triangle $\hat{K}$, i.e.

$$\hat{v} = v \circ F_K , \qquad \widehat{\pi_K v} = \pi_K v \circ F_K . \tag{4.3}$$

The assumption $v \in H^6(K)$ involves $\hat{v} \in H^6(\hat{K})$. Moreover, note that function $\widehat{\pi_K v}$ (respectively $\pi_K v$) is denoted $\hat{w}$ (resptively w) in Section 3.2.

According to [37, (3.10)], there exists a constant c, independent of h_K such that

$$\begin{aligned}|v|_{k,p,K} \leq c|J_{F_K}|^{1/p}_{0,\infty,\hat{K}} \sum_{j=1}^{k} |\hat{v}|_{j,p,\hat{K}} \sum_{i \in I(j,k)} (|F_K^{-1}|^{i_1}_{1,\infty,K} |F_K^{-1}|^{i_2}_{2,\infty,K} \cdots |F_K^{-1}|^{i_k}_{k,\infty,K}) \\ \forall v \in W^{k,p}(K) ,\end{aligned} \tag{4.4}$$

where $|F_K^{-1}|_{m,\infty,K}$, $1 \leq m \leq k$, is defined by (2.16) and where

$$\begin{aligned} I(j,k) = \{ i = (i_1, \ldots, i_k) \in N^k ; i_1 + i_2 + \cdots + i_k = j, \\ i_1 + 2i_2 + \cdots + ki_k = k \} , \quad 1 \leq j \leq k . \end{aligned} \tag{4.5}$$

Thus, $\forall v \in H^m(K)$, we have for $1 \leq m \leq 6$ (respectively Bell: $1 \leq m \leq 5$)

$$|v - \pi_K v|_{m,K} \leq c|J_{F_K}|^{1/2}_{0,\infty,\hat{K}} \sum_{j=1}^{m} |\hat{v} - \widehat{\pi_K v}|_{j,\hat{K}} \sum_{i \in I(j,m)} (|F_K^{-1}|^{i_1}_{1,\infty,K} \cdots |F_K^{-1}|^{i_m}_{m,\infty,K}) . \tag{4.6}$$

We prove below the existence of a constant c such that for any $v \in H^6(K)$ (respectively Bell: $v \in H^5(K)$)

$$|\hat{v} - \widehat{\pi_K v}|_{j,\hat{K}} \leq c h_K^5 \|v\|_{6,K} , \quad 0 \leq j \leq 6 , \tag{4.7}$$

$$\text{(respectively Bell: } |\hat{v} - \widehat{\pi_K v}|_{j,\hat{K}} \leq c h_K^4 \|v\|_{5,K} , \quad 0 \leq j \leq 5) , \tag{4.8}$$

where $v = \hat{v} \circ F_K^{-1}$.

Moreover, from Theorem 2.1,

$$\begin{aligned} |F_K|_{l,\infty,\hat{K}} \leq c h_K^l , \quad l = 0,1,\ldots ; \qquad |F_K^{-1}|_{l,\infty,K} \leq c h_K^{-1} , \quad l = 1,2,\ldots ; \\ |J_{F_K}|_{0,\infty,\hat{K}} \leq c h_K^2 ; \qquad |J_{F_K^{-1}}|_{0,\infty,K} \leq c h_K^{-2} . \end{aligned} \tag{4.9}$$

Then, by combining estimates (4.7) (respectively Bell: (4.8)) and (4.9) with inequality (4.6), we obtain the result

$$\begin{aligned} |v - \pi_K v|_{m,K} \leq c h_K^{6-m} \|v\|_{6,K} \quad \forall v \in H^6(K), \quad 1 \leq m \leq 6 \\ \text{(respectively Bell: } |v - \pi_K v|_{m,K} \leq c h_K^{5-m} \|v\|_{5,K} \quad \forall v \in H^5(K), \quad 1 \leq m \leq 5) . \end{aligned} \tag{4.10}$$

Note that estimates (4.10) are still valid for $m = 0$. It suffices to replace inequality (4.6) by

$$|v - \pi_K v|_{0,K} \leq |J_{F_K}|^{1/2}_{0,\infty,\hat{K}} |\hat{v} - \widehat{\pi_K v}|_{0,\hat{K}} . \tag{4.11}$$

To complete the proof, it remains to obtain estimates (4.7) (respectively Bell: (4.8)).

Let $\hat{\pi}$ be the P_9-interpolation operator associated with the basic finite element of Fig. 9. Then, for any $0 \leq j \leq 6$ (respectively Bell: $0 \leq j \leq 5$), we have

$$|\hat{v} - \widehat{\pi_K v}|_{j,\hat{K}} \leq |\hat{v} - \hat{\pi}\hat{v}|_{j,\hat{K}} + |\hat{\pi}\hat{v} - \widehat{\pi_K v}|_{j,\hat{K}} . \tag{4.12}$$

In particular, polynomials of degree 5 (respectively Bell: 4) are invariant by $\hat{\pi}$; hence

$$|\hat{v} - \hat{\pi}\hat{v}|_{j,\hat{K}} \leq c|\hat{v}|_{6,\hat{K}} , \quad 0 \leq j \leq 6 \tag{4.13}$$

$$\text{(respectively Bell: } |\hat{v} - \hat{\pi}\hat{v}|_{j,\hat{K}} \leq c|\hat{v}|_{5,\hat{K}} , \quad 0 \leq j \leq 5) . \tag{4.14}$$

Similarly to (4.4), we obtain

$$|\hat{v}|_{k,p,\hat{K}} \leq c|J_{F_K^{-1}}|^{1/p}_{0,\infty,K} \sum_{j=1}^{k} |v|_{j,p,K} \sum_{i \in I(j,k)} (|F_K|^{i_1}_{1,\infty,\hat{K}} |F_K|^{i_2}_{2,\infty,\hat{K}} \cdots |F_K|^{i_k}_{k,\infty,\hat{K}})$$
$$\forall \hat{v} \in W^{k,p}(\hat{K}) . \tag{4.15}$$

From inequalities (4.9) and (4.15), we have

$$|\hat{v}|_{6,\hat{K}} \leq ch_K^5 \|v\|_{6,K} \quad \forall \hat{v} \in H^6(\hat{K}) \tag{4.16}$$

$$\text{(respectively Bell: } |\hat{v}|_{5,\hat{K}} \leq ch_K^4 \|v\|_{5,K} \quad \forall \hat{v} \in H^5(\hat{K})) , \tag{4.17}$$

so that, with (4.13), (4.16) (respectively Bell: (4.14), (4.17))

$$|\hat{v} - \hat{\pi}\hat{v}|_{j,\hat{K}} \leq ch_K^5 \|v\|_{6,K} \quad \forall \hat{v} \in H^6(\hat{K}), \quad 0 \leq j \leq 6 \tag{4.18}$$

$$\text{(respectively Bell: } |\hat{v} - \hat{\pi}\hat{v}|_{j,\hat{K}} \leq ch_K^4 \|v\|_{5,K} \quad \forall \hat{v} \in H^5(\hat{K}), \quad 0 \leq j \leq 5) . \tag{4.19}$$

Subsequently, we show (in three steps) that $\forall \hat{v} \in H^6(\hat{K})$ (respectively Bell: $\forall \hat{v} \in H^5(\hat{K})$), we have

$$|\hat{\pi}\hat{v} - \widehat{\pi_K v}|_{j,\hat{K}} \leq ch_K^5 \|v\|_{6,K} , \quad 0 \leq j \leq 6 \tag{4.20}$$

$$\text{(respectively Bell: } |\hat{\pi}\hat{v} - \widehat{\pi_K v}|_{j,\hat{K}} \leq ch_K^4 \|v\|_{5,K} , \quad 0 \leq j \leq 5) . \tag{4.21}$$

STEP 1 Method of estimation. The difference

$$\hat{\delta}(\hat{x}) = \hat{\pi}\hat{v}(\hat{x}) - \widehat{\pi_K v}(\hat{x}) \tag{4.22}$$

is a polynomial of degree 9, hence $\hat{\pi}\hat{\delta} = \hat{\delta}$. Let $\widehat{DL}_i$, $\hat{p}_i$, $1 \leq i \leq 55$, be the degrees of freedom and the corresponding basis polynomials of the basic element of Fig. 9, arranged according to (3.13). For any $\hat{x} \in \hat{K}$, we have

$$\hat{\pi}\hat{\delta}(\hat{x}) = \hat{\delta}(\hat{x}) = \sum_{i=1}^{55} \widehat{DL}_i(\hat{\delta})\hat{p}_i(\hat{x}) . \tag{4.23}$$

The sets of values of degrees of freedom used to construct $\widehat{\pi_K v} = \hat{w}$ are given by $\hat{\Delta}_{K1}(v) \cup \hat{\Delta}_{K2}(v) \cup \hat{\Delta}_{K3}(v)$ (see (3.15), (3.27) and (3.34)). Comparing with (3.13), we observe that $\hat{\Sigma}_1(\hat{v}) \equiv \hat{\Delta}_{K1}(v)$ so that

$$\widehat{DL}_i(\hat{\delta}) = 0 , \quad i = 1, \ldots, 28 . \tag{4.24}$$

STEP 2. Estimates of $\widehat{DL}_i(\hat{\delta})$, $i = 29, \ldots, 46$. Next, consider $\hat{\Sigma}_2(\hat{v})$ and $\hat{\Delta}_{K2}(v)$. We have from (3.13) and (3.27),

$$\widehat{DL}_{28+i}(\hat{\delta}) = \hat{v}(\hat{d}_i) - \hat{f}_2(\hat{d}_i) , \quad i = 1, \ldots, 4 , \tag{4.25}$$

$$\widehat{DL}_{28+i}(\hat{\delta}) = \hat{v}(\hat{d}_i) - \hat{f}_1(\hat{d}_i) , \quad i = 5, \ldots, 8 , \tag{4.26}$$

$$\begin{aligned} \widehat{DL}_{36+i}(\hat{\delta}) = & -\frac{\partial \hat{v}}{\partial \hat{x}_1}(\hat{q}_i) + \frac{1}{|a_3 a_2|}\left\langle \frac{\partial F_K}{\partial \hat{x}_1}(\hat{q}_i), t_{32}\right\rangle \frac{\mathrm{d}\hat{f}_2}{\mathrm{d}\hat{x}_2}(\hat{q}_i) \\ & + \left\langle \frac{\partial F_K}{\partial \hat{x}_1}(\hat{q}_i), n_{32}\right\rangle \hat{g}_2(\hat{q}_i) , \quad i = 1, \ldots, 5, \end{aligned} \tag{4.27}$$

where for convenience $\hat{q}_i$ means $\hat{b}_1$ for $i = 1$ and $\hat{d}_{i-1}$ for $i = 2, \ldots, 5$,

$$\begin{aligned} \widehat{DL}_{41+i}(\hat{\delta}) = & -\frac{\partial \hat{v}}{\partial \hat{x}_2}(\hat{q}_i) + \frac{1}{|a_3 a_1|}\left\langle \frac{\partial F_K}{\partial \hat{x}_2}(\hat{q}_i), t_{31}\right\rangle \frac{\mathrm{d}\hat{f}_1}{\mathrm{d}\hat{x}_1}(\hat{q}_i) \\ & + \left\langle \frac{\partial F_K}{\partial \hat{x}_2}(\hat{q}_i), n_{31}\right\rangle \hat{g}_1(\hat{q}_i) , \quad i = 1, \ldots, 5 , \end{aligned} \tag{4.28}$$

where for convenience $\hat{q}_i$ means $\hat{b}_2$ for $i = 1$ and $\hat{d}_{i+3}$ for $i = 2, \ldots, 5$ (respectively Bell: replace $\hat{g}_1$ and $\hat{g}_2$ by $\hat{h}_1$ and $\hat{h}_2$, respectively).

Relations (3.16)–(3.18) show that $\hat{f}_1$ is the P_5-Hermite polynomial which interpolates $\hat{v}$ over the side $\hat{a}_3\hat{a}_1$, from the given degrees of freedom

$$\begin{aligned} & \{\hat{v}(\hat{a}_1), \hat{v}(\hat{a}_3), D\hat{v}(\hat{a}_1)(\hat{a}_3 - \hat{a}_1), D\hat{v}(\hat{a}_3)(\hat{a}_1 - \hat{a}_3), D^2\hat{v}(\hat{a}_1)(\hat{a}_3 - \hat{a}_1)^2, \\ & D^2\hat{v}(\hat{a}_3)(\hat{a}_1 - \hat{a}_3)^2\} . \end{aligned} \tag{4.29}$$

In the sequel, we denote by the same letter the functions defined on a triangle K and their traces along the boundary ∂K. From the definition of function $\hat{f}_1$, we obtain for any $\hat{v} \in W^{4,\infty}(\hat{a}_3\hat{a}_1)$,

$$|\hat{v}(\hat{d}_i) - \hat{f}_1(\hat{d}_i)| \leq |\hat{v} - \hat{f}_1|_{0,\infty,\hat{a}_3\hat{a}_1} \leq c|\hat{v}|_{4,\infty,\hat{a}_3\hat{a}_1}, \quad i = 5, \ldots, 8,$$

and with Sobolev's theorem (see [29])

$$|\hat{v}(\hat{d}_i) - \hat{f}_1(\hat{d}_i)| \leq c\|\hat{v}\|_{4,\infty,\hat{K}} \leq c\|\hat{v}\|_{6,\hat{K}} \quad \forall \hat{v} \in H^6(\hat{K}), \quad i = 5, \ldots, 8.$$

Thus, the linear form

$$\mathscr{F} : \forall \hat{v} \in H^6(\hat{K}) \to \mathscr{F}(\hat{v}) = \max_{i=5,\ldots,8} |\hat{v}(\hat{d}_i) - \hat{f}_1(\hat{d}_i)|$$

is continuous. Since it vanishes for any $\hat{v} \in P_5$, the lemma of Bramble–Hilbert [38] implies the existence of a constant c such that

$$|\hat{v}(\hat{d}_i) - \hat{f}_1(\hat{d}_i)| \leq c|\hat{v}|_{6,\hat{K}} \quad \forall \hat{v} \in H^6(\hat{K}), \quad i = 5, \ldots, 8,$$

or, with (4.16),

$$|\widehat{DL}_{28+i}(\hat{\delta})| = |\hat{v}(\hat{d}_i) - \hat{f}_1(\hat{d}_i)| \leq ch_K^5\|v\|_{6,K} \quad \forall \hat{v} \in H^6(\hat{K}), \quad i = 5, \ldots, 8. \tag{4.30}$$

Similarly, we can prove that

(respectively Bell:

$$|\widehat{DL}_{28+i}(\hat{\delta})| = |\hat{v}(\hat{d}_i) - \hat{f}_1(\hat{d}_i)| \leq ch_K^4\|v\|_{5,K} \quad \forall \hat{v} \in H^5(\hat{K}), \quad i = 5, \ldots, 8), \tag{4.31}$$

$$|\widehat{DL}_{28+i}(\hat{\delta})| = |\hat{v}(\hat{d}_i) - \hat{f}_2(\hat{d}_i)| \leq ch_K^5\|v\|_{6,K} \quad \forall \hat{v} \in H^6(\hat{K}), \quad i = 1, \ldots, 4, \tag{4.32}$$

(respectively Bell:

$$|\widehat{DL}_{28+i}(\hat{\delta})| = |\hat{v}(\hat{d}_i) - \hat{f}_2(\hat{d}_i)| \leq ch_K^4\|v\|_{5,K} \quad \forall \hat{v} \in H^5(\hat{K}), \quad i = 1, \ldots, 4). \tag{4.33}$$

Now, let us examine the case of degrees of freedom (4.28). The correspondences $\hat{v} = v \circ F_K$ and $q_i = F_K(\hat{q}_i)$ imply

$$\begin{aligned}
\frac{\partial \hat{v}}{\partial \hat{x}_2}(\hat{q}_i) &= D\hat{v}(\hat{q}_i)e_2 = Dv(q_i)DF_K(\hat{q}_i)e_2 = Dv(q_i)\frac{\partial F_K}{\partial \hat{x}_2}(\hat{q}_i) \\
&= Dv(q_i)\Big[\Big\langle \frac{\partial F_K}{\partial \hat{x}_2}(\hat{q}_i), t_{31}\Big\rangle t_{31} + \Big\langle \frac{\partial F_K}{\partial \hat{x}_2}(\hat{q}_i), n_{31}\Big\rangle n_{31}\Big] \\
&= \frac{1}{|a_3a_1|}\Big\langle \frac{\partial F_K}{\partial \hat{x}_2}(\hat{q}_i), t_{31}\Big\rangle \frac{\partial \hat{v}}{\partial \hat{x}_1}(\hat{q}_i) + \Big\langle \frac{\partial F_K}{\partial \hat{x}_2}(\hat{q}_i), n_{31}\Big\rangle Dv(q_i)n_{31},
\end{aligned}$$

and hence

$$\widehat{DL}_{41+i}(\hat{\delta}) = \frac{1}{|a_3a_1|}\left\langle \frac{\partial F_K}{\partial \hat{x}_2}(\hat{q}_i), t_{31}\right\rangle\left[\frac{\mathrm{d}\hat{f}_1}{\mathrm{d}\hat{x}_1}(\hat{q}_i) - \frac{\partial \hat{v}}{\partial \hat{x}_1}(\hat{q}_i)\right]$$
$$+\left\langle \frac{\partial F_K}{\partial \hat{x}_2}(\hat{q}_i), n_{31}\right\rangle(\hat{g}_1(\hat{q}_i) - Dv(q_i)n_{31}), \quad i = 1, \dots, 5. \tag{4.34}$$

A proof similar to that of estimate (4.32) shows that

$$\left|\frac{1}{|a_3a_1|}\left\langle \frac{\partial F_K}{\partial \hat{x}_2}(\hat{q}_i), t_{31}\right\rangle\left[\frac{\mathrm{d}\hat{f}_1}{\mathrm{d}\hat{x}_1}(\hat{q}_i) - \frac{\partial \hat{v}}{\partial \hat{x}_1}(\hat{q}_i)\right]\right| \le ch_K^5\|v\|_{6,K} \quad \forall \hat{v} \in H^6(\hat{K}). \tag{4.35}$$

Now, let us consider the last term of (4.34). First according to the proof of Theorem 3.2, we have, since $\hat{q}_1$ means $\hat{b}_2$,

$$\hat{g}_1(\hat{q}_1) - Dv(q_1)n_{31} = \hat{g}_1(\hat{b}_2) - Dv(b_2)n_{31} = 0. \tag{4.36}$$

The next term is $\hat{g}_1(\hat{q}_2) - Dv(q_2)n_{31} = \hat{g}_1(\hat{d}_5) - Dv(d_5)n_{31}$, where, according to (3.17), (3.19) and (3.21), the function $g_1 = \hat{g}_1 \circ F_K^{-1}$ is the one-dimensional P_4-interpolate of the function $z(\cdot) = Dv(\cdot)n_{31}$ along the side a_3a_1 by using the set of degrees of freedom,

$$\{z(a_i),\ Dz(a_i)t_{31},\ i = 1, 3;\ z(b_2)\}.$$

Then the interpolation properties and Sobolev's theorem imply

$$|z(d_5) - g_1(d_5)| \le |z - g_1|_{0,\infty,a_3a_1} = |\hat{z} - \hat{g}_1|_{0,\infty,\hat{a}_3\hat{a}_1} \le c|\hat{z}|_{3,\infty,\hat{a}_3\hat{a}_1}$$
$$\le c\|\hat{z}\|_{3,\infty,\hat{K}} \le c\|\hat{z}\|_{5,\hat{K}} \quad \forall \hat{z} = z \circ F_K \in H^5(\hat{K}).$$

Thus, the linear form

$$\mathscr{G} : \forall \hat{z} \in H^5(K) \mapsto |\hat{z}(\hat{d}_5) - \hat{g}_1(\hat{d}_5)| \quad \text{is continuous}.$$

Since it vanishes for any $\hat{z} \in P_4$, the Bramble–Hilbert lemma implies the existence of a constant c such that

$$|\hat{z}(\hat{d}_5) - \hat{g}_1(\hat{d}_5)| = |z(d_5) - g_1(d_5)| \le c|\hat{z}|_{5,\hat{K}} \quad \forall \hat{z} \in H^5(\hat{K}). \tag{4.37}$$

By analogy with (4.15) and (4.16), we obtain

$$|\hat{z}|_{5,\hat{K}} \le Ch_K^4\|z\|_{5,K} \quad \forall \hat{z} \in H^5(\hat{K}) \tag{4.38}$$

and since $z(\cdot) = Dv(\cdot)n_{31}$, we obtain

$$\|z\|_{5,K} \le C\|v\|_{6,K}, \tag{4.39}$$

so that (4.37)–(4.39) imply

$$|\hat{g}_1(\hat{a}_5) - Dv(d_5)n_{31}| \leq Ch_K^4\|v\|_{6,K} \quad \forall v \in H^6(K)\,, \tag{4.40}$$

and three other relations at points $\hat{a}_6$, $\hat{a}_7$ and $\hat{a}_8$. Then, (2.17), (4.34), (4.35), (4.36) and (4.40) imply

$$|\widehat{DL}_{41+i}(\hat{\delta})| \leq Ch_K^5\|v\|_{6,K} \quad \forall v \in H^6(K), \quad i = 1, \ldots, 5\,. \tag{4.41}$$

In the same way, we could obtain

$$\text{(respectively Bell: } |\widehat{DL}_{41+i}(\hat{\delta})| \leq ch_K^4\|v\|_{5,K} \quad \forall v \in H^5(K), \quad i = 1, \ldots, 5)\,, \tag{4.42}$$

$$|\widehat{DL}_{36+i}(\hat{\delta})| \leq ch_K^5\|v\|_{6,K} \quad \forall v \in H^6(K), \quad i = 1, \ldots, 5 \tag{4.43}$$

$$\text{(respectively Bell: } |\widehat{DL}_{36+i}(\hat{\delta})| \leq ch_K^4\|v\|_{5,K} \quad \forall v \in H^5(K), \quad i = 1, \ldots, 5)\,. \tag{4.44}$$

STEP 3. Estimate of $DL_i(\hat{\delta})$, $i = 47, \ldots, 55$. From definitions (3.13) and (3.34), we obtain

$$\widehat{DL}_{46+i}(\hat{\delta}) = \hat{v}(\hat{a}_{8+i}) - \hat{f}_3(\hat{a}_{8+i})\,, \quad i = 1, \ldots, 4\,, \tag{4.45}$$

$$\widehat{DL}_{51}(\hat{\delta}) = \frac{\sqrt{2}}{2}\left[\frac{\partial\hat{v}}{\partial\hat{x}_1} + \frac{\partial\hat{v}}{\partial\hat{x}_2}\right](\hat{b}_3) + \sqrt{2}\hat{g}_3(\hat{b}_3)\,, \tag{4.46}$$

$$\widehat{DL}_{51+i}(\hat{\delta}) = \frac{\sqrt{2}}{2}\left[\frac{\partial\hat{v}}{\partial\hat{x}_1} + \frac{\partial\hat{v}}{\partial\hat{x}_2}\right](\hat{a}_{8+i}) + \sqrt{2}\hat{g}_3(\hat{a}_{8+i})\,, \quad i = 1, \ldots, 4\,. \tag{4.46}$$

By using the definition (3.31) of $(\hat{f}_3)$, a proof similar to that of (4.30) shows that

$$|\widehat{DL}_{46+i}(\hat{\delta})| = |\hat{v}(\hat{a}_{8+i}) - \hat{f}_3(\hat{a}_{8+i})| \leq ch_K^5\|v\|_{6,K} \quad \forall v \in H^6(K), \quad i = 1, \ldots, 4\,, \tag{4.47}$$

(resectively Bell:

$$|\widehat{DL}_{46+i}(\hat{\delta})| = |\hat{v}(\hat{a}_{8+i}) - \hat{f}_3(\hat{a}_{8+i})| \leq ch_K^4\|v\|_{5,K} \quad \forall v \in H^5(K), \quad i = 1, \ldots, 4)\,. \tag{4.48}$$

Now, consider the first term (4.46), i.e., $\widehat{DL}_{51}(\hat{\delta}) = -\sqrt{2}[\hat{g}_3(\hat{b}_3) - D\hat{v}(\hat{b}_3)(\hat{a}_3 - \hat{b}_3)]$. With definition (3.32) of $\hat{g}_3$, Sobolev's theorem and the Bramble–Hilbert lemma, we obtain

$$\begin{aligned}
|\widehat{DL}_{51}(\hat{\delta})| &\leq \sqrt{2}|\hat{g}_3(\cdot) - D\hat{v}(\cdot)(\hat{a}_3 - \hat{b}_3)|_{0,\infty,\hat{a}_1\hat{a}_2} \\
&\leq c|D\hat{v}(\cdot)(\hat{a}_3 - \hat{b}_3)|_{3,\infty,\hat{a}_1\hat{a}_2} \\
&\leq c|\hat{v}|_{4,\infty,\hat{a}_1\hat{a}_2} \leq c\|\hat{v}\|_{6,\hat{K}} \leq ch_K^5\|v\|_{6,K} \quad \forall v \in H^6(K)\,.
\end{aligned}$$

And, by similarity, we finally obtain

$$|\widehat{DL}_{50+i}(\hat{\delta})| \leq ch_K^5 \|v\|_{6,K} \quad \forall v \in H^6(K), \quad i = 1, \dots, 5, \tag{4.49}$$

$$\text{(respectively Bell: } |\widehat{DL}_{50+i}(\hat{\delta})| \leq ch_K^4 \|v\|_{5,K} \quad \forall v \in H^5(K), \quad i = 1, \dots, 5). \tag{4.50}$$

Note that for a curve finite element $\mathscr{C}^1$-compatible with an Argyris triangle, we have in fact $\widehat{DL}_{51}(\hat{\delta}) = 0$. This property is not generally true for a curved finite element $\mathscr{C}^1$-compatible with a Bell triangle.

Finally, (4.22), (4.23) imply

$$|\hat{\delta}|_{j,\hat{K}} \leq |\hat{\pi}\hat{v} - \widehat{\pi_K v}|_{j,\hat{K}} \leq c \sum_{i=1}^{55} |\widehat{DL}_i(\hat{\delta})| .$$

Then, estimates (4.24), (4.30), (4.32), (4.41), (4.43), (4.47) and (4.49) [respectively Bell: (4.24), (4.31), (4.33), (4.42), (4.44), (4.48) and (4.50)] give estimate (4.20) [respectively Bell: (4.21)]. We obtain estimates (4.7) [respectively Bell: (4.8)] by combining estimates (4.18) and (4.20) [respectively Bell: (4.19) and (4.21)]. □

4.2. Estimates of the interpolation error for curved finite elements $\mathscr{C}^1$-compatible with Argyris or Bell triangles in the case of $F_K \in (P_3)^2$

In this case, we can prove similar results as in Theorem 4.1.

THEOREM 4.2. There exists a constant c, independent of h_K, such that for all curved finite elements $\mathscr{C}^1$-compatible with Argyris (respectively Bell) triangles, we have

$$|v - \pi_K v|_{m,K} \leq ch_K^{k+1-m} \|v\|_{k+1,K} \quad \forall v \in H^{k+1}(K), \quad k = 3, \dots, 5, \quad 0 \leq m \leq k+1 \tag{4.51}$$

$$\text{(respectively Bell: } |v - \pi_K v|_{m,K} \leq ch_K^{k+1-m} \|v\|_{k+1,K} \quad \forall v \in H^{k+1}(K), \quad k = 3, 4, \quad 0 \leq m \leq k+1), \tag{4.52}$$

where $\pi_K v$ is the interpolating function of v defined in Section 3.3.

4.3. Estimates of the interpolation error for curved finite elements $\mathscr{C}^0$-compatible with Hermite triangle of type (3) ($F_K \in (P_n)^2$, $n = 3$ or 5)

For the curved finite elements introduced in Section 3.4, it is easy to prove the following theorem.

THEOREM 4.3. There exists a constant c, independent of h_K, such that for all curved finite elements $\mathscr{C}^0$-compatible with Hermite triangle of type (3), we have

$$|v - \pi_K v|_{m,K} \leq c h_K^{4-m} \|v\|_{4,K}, \quad 0 \leq m \leq 4, \quad \forall v \in H^4(K),$$

where $\pi_K v$ is the interpolating function of v defined in Section 3.4.

5. Concluding remarks

To conclude, it is worth emphasizing some very interesting properties of these interpolation methods on curved triangles:

(i) they allow the construction of $\mathscr{C}^1$-interpolating functions from just a set of degrees of freedom which is completely compatible with those of the associate straight finite elements. In this way, they use a mapping $F_K : \hat{K} \to K$ which is entirely symmetric and polynomial;

(ii) corresponding functions on the reference triangle are of polynomial type so that the study of the effect of numerical integration is straightforward (see Part 2, i.e. [4]);

(iii) for a given degree of regularity, the asymptotic interpolation error estimates have the same order for the straight and for the associate curved finite elements;

(iv) thus, these curved finite elements constitute a really powerful tool to approximate fourth order problems set on plane curved boundary domains.

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