CHAPTER 5

BIHARMONIC EQUATION

We study the biharmonic equation as an example of higher order elliptic problems. It turns out that conforming finite element spaces V_h need to belong to C^1 which is more difficult to satisfy than $V_h \subset C^0$. In order to satisfy this smoothness requirement, higher order polynomials are needed. Indeed, in the two-dimensional case, at least fifth degree polynomials (Argyris triangle) are needed on a triangular mesh, see Ženíšek (1974b). In three dimensions, even higher degree polynomials are needed. A three-dimensional conforming finite element with 220 dof per cell based on ninth degree polynomials has been proposed in Ženíšek (1974a). Such higher order degree polynomials lead to high dimensional finite element spaces and high computational costs. Indeed, to compute the matrix entries we have to integrate the product of second order derivatives of polynomials of degree 9, thus polynomials of degree 14. An exact quadrature over tetrahedrons for such polynomials would need the evaluation of the integrand at 236 quadrature points.

One way to construct finite element spaces $V_h \subset C^1$ with a reduced number of dof per cell K is to use local spaces $\mathcal{P}_K$ of piecewise polynomials (or even a class of more general functions) instead of polynomials. This leads to the concept of composite finite elements. Another way is to relax the strong requirement $V_h \subset C^1$ by considering nonconforming finite elements. Note that even nonconforming finite element spaces $V_h \not\subset C^0$ can be used to approximate the solution of fourth order equations. Alternatively, we could also use mixed finite element methods that are based on the reformulation of fourth order problems as systems of two equations of second order. For details of this approach we refer to Chapter 10 of Boffi et al. (2013).

5.1. Deflection of a thin clamped plate

The domain in the thin clamped plate example is a flat elastic object subjected to a load in the transversal direction. The resulting transversal deflection can be modeled using the Kirchhoff plate theory. As the thickness of the plate is assumed to be very thin compared to the other two dimensions, it is sufficient to describe the deformation of the plate under the load f by

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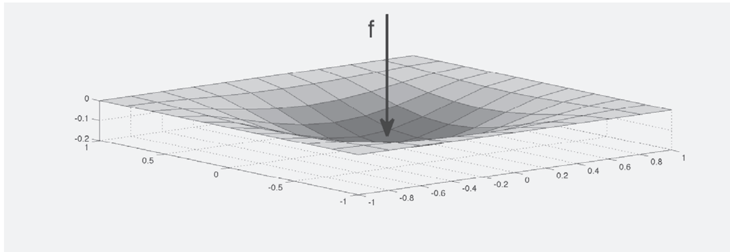

FIGURE 5.1 Deflection of a clamped plate under the load f.

its vertical deflection u (Figure 5.1) leading to the biharmonic equation with homogeneous boundary conditions

$$\Delta\Delta u = f \quad \text{in } \Omega, \qquad u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma = \partial\Omega. \tag{5.1}$$

Then, a classical solution of the biharmonic equation with data $f \in C(\Omega)$ is a function $u \in C^4(\Omega) \cap C^1(\overline{\Omega})$ satisfying (5.1) in the sense of continuous functions.

5.2. WEAK FORMULATION OF THE BIHARMONIC EQUATION

We multiply the biharmonic equation in (5.1) by a test function $v \in C_0^\infty(\Omega)$, integrate over Ω and apply the integration by parts formula twice to get

$$\begin{aligned}\int_\Omega (\Delta\Delta u) v \,\mathrm{d}x &= \int_\Gamma \frac{\partial \Delta u}{\partial n} v \,\mathrm{d}\gamma - \int_\Omega \nabla(\Delta u)\nabla v \,\mathrm{d}x \\ &= -\int_\Gamma \Delta u \frac{\partial v}{\partial n}\,\mathrm{d}\gamma + \int_\Omega \Delta u \Delta v \,\mathrm{d}x = \int_\Omega \Delta u \Delta v \,\mathrm{d}x.\end{aligned}$$

Here, we used the test function $v \in C_0^\infty(\Omega)$ and its derivatives vanish in a neighbourhood of the boundary Γ. As a result, we see that a classical solution of the biharmonic equation satisfies

$$\int_\Omega \Delta u \Delta v \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x \quad \text{for all } v \in C_0^\infty(\Omega).$$

Since $C_0^\infty(\Omega)$ is dense in $H_0^2(\Omega)$ and the mappings

$$v \mapsto \int_\Omega \Delta u \Delta v \,\mathrm{d}x, \qquad v \mapsto \int_\Omega f v \,\mathrm{d}x$$

are continuous on $H_0^2(\Omega)$, a classical solution satisfies also

$$\int_\Omega \Delta u \Delta v \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x \quad \text{for all } v \in H_0^2(\Omega).$$

Therefore, the weak formulation of the biharmonic equation reads

Find $u \in V := H_0^2(\Omega)$ such that

$$a(u,v) := \int_\Omega \Delta u \Delta v \, \mathrm{d}x = \int_\Omega fv \, \mathrm{d}x =: F(v) \quad \text{for all } v \in H_0^2(\Omega). \tag{5.2}$$

Contrary, let $f \in C(\Omega)$ and let the weak solution $u \in H_0^2(\Omega)$ of (5.2) belong to $C^4(\Omega) \cap C^1(\overline{\Omega})$. Then, considering (5.2) for all $v \in C_0^\infty(\Omega) \subset H_0^2(\Omega)$ and applying integration by parts we obtain

$$\int_\Omega \Delta \Delta u v \, \mathrm{d}x = \int_\Omega fv \, \mathrm{d}x \quad \text{for all } v \in C_0^\infty(\Omega) \tag{5.3}$$

from which $\Delta \Delta u = f$ in Ω follows by a density argument. Further, the regularity assumption on $u \in H_0^2(\Omega)$ enforces the boundary conditions. Roughly speaking, a classical solution of Problem (5.1) is always a weak solution and a weak solution being sufficiently regular is a classical solution.

In order to study the existence and uniqueness of weak solutions we check the assumptions of the Lax–Milgram theorem. It is not difficult to see that the forms $a : V \times V \to \mathbb{R}$ and $F : V \to \mathbb{R}$ are bilinear and linear, respectively. Further, the continuity follows by applying Cauchy–Schwarz inequality

$$|a(u,v)| \leq \|\Delta u\|_0 \, \|\Delta v\|_0 \leq C|u|_2 \, |v|_2 \leq C\|u\|_2 \, \|v\|_2 \quad \text{for all } u,v \in V,$$
$$|F(v)| \leq \|f\|_0 \, \|v\|_0 \leq C\|v\|_2 \quad \text{for all } v \in V.$$

In order to show the coercivity of $a : V \times V \to \mathbb{R}$ we need the following lemma.

Lemma 5.1. *For all $u,v \in H_0^2(\Omega)$ we have*

$$\int_\Omega \frac{\partial^2 u}{\partial x_i^2} \frac{\partial^2 v}{\partial x_j^2} \, \mathrm{d}x = \int_\Omega \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j} \, \mathrm{d}x \quad i,j = 1,\ldots,d.$$

Proof. Since $C_0^\infty(\Omega)$ is dense in $H_0^2(\Omega)$, it is enough to prove the statement for functions $u,v \in C_0^\infty(\Omega)$. Taking into consideration that u,v vanish in the neighbourhood of Γ, we get by integration by parts

$$\int_\Omega \frac{\partial^2 u}{\partial x_i^2} \frac{\partial^2 v}{\partial x_j^2} \, \mathrm{d}x = -\int_\Omega \frac{\partial^3 u}{\partial x_i^2 \partial x_j} \frac{\partial v}{\partial x_j} \, \mathrm{d}x = \int_\Omega \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j} \, \mathrm{d}x \quad \text{for } i,j = 1,\ldots,d.$$

∎

Lemma 5.2. *The bilinear form $a : V \times V \to \mathbb{R}$ is coercive, that is, there is a positive constant $m > 0$ such that*

$$a(v,v) \geq m\|v\|_2^2 \quad \textit{for all } v \in V.$$

Proof. Applying Lemma 5.1, we get

$$a(v,v) = \int_\Omega (\Delta v)^2\,\mathrm{d}x = \int_\Omega \left(\sum_{i=1}^d \frac{\partial^2 v}{\partial x_i^2}\right)\left(\sum_{j=1}^d \frac{\partial^2 v}{\partial x_j^2}\,\mathrm{d}x\right) = \int_\Omega \sum_{i,j=1}^d \frac{\partial^2 v}{\partial x_i^2}\frac{\partial^2 v}{\partial x_j^2}\,\mathrm{d}x$$
$$= \int_\Omega \sum_{i,j=1}^d \frac{\partial^2 v}{\partial x_i \partial x_j}\frac{\partial^2 v}{\partial x_i \partial x_j}\,\mathrm{d}x = |v|_2^2 \geq m\|v\|_2^2,$$

where the equivalence of the seminorm $|\cdot|_2$ to the norm $\|\cdot\|_2$ in V was used. ■

Theorem 5.1. *Let $f \in L^2(\Omega)$. Then, there is a unique solution of* (5.2).

Proof. Application of Theorem 2.1 (Lax–Milgram). ■

Remark 5.1. The necessary condition for minimizing the energy functional

$$\mathbb{E}(u) = \frac{1}{2}a(u,u) - F(u) = \int_\Omega \left[\frac{1}{2}(\Delta u)^2 - f\,u\right]\mathrm{d}x$$

over $V = H_0^2(\Omega)$ is just the weak formulation of the biharmonic equation. Due to the convexity of the energy functional, the minimization problem

$$\text{Find } u \in V := H_0^2(\Omega) \text{ with } \mathbb{E}(u) = \inf_{v\in V}\mathbb{E}(v)$$

is equivalent to the weak formulation of the biharmonic equation.

5.3. Conforming finite element methods

Let $V_h \subset V = H_0^2(\Omega)$ be a finite element space. Piecewise smooth functions belong to $H_0^2(\Omega)$ iff they are in $C^1(\overline{\Omega})$. For examples of C^1 elements, see Chapter 3. The discrete problem reads

$$\text{Find } u_h \in V_h \text{ such that } a(u_h, v_h) = F(v_h) \quad \text{for all } v_h \in V_h. \tag{5.4}$$

Theorem 5.2. *There is a unique solution $u_h \in V_h$ of the discrete problem* (5.4) *satisfying*

$$\|u - u_h\|_2 \leq C \inf_{v_h \in V_h} \|u - v_h\|_2.$$

Proof. Since $V_h \subset V$ all assumptions of Theorem 2.1 (Lax–Milgram) are satisfied and the error estimate follows from Lemma 2.1 (Cea's lemma). ■

As for second order problem, the discrete problem (5.4) is equivalent to a linear algebraic system of equations. Let Φ_i, $i = 1, \ldots, n_{\text{glob}}$, be a basis of V_h. Then, $u_h \in V_h$ can be represented as

$$u_h = \sum_{j=1}^{n_{\text{glob}}} U_j \Phi_j.$$

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Setting $v_h = \Phi_i$, $i = 1,\dots,n_{\text{glob}}$, we get the linear algebraic system of equations

$$\sum_{j=1}^{n_{\text{glob}}} a_{ij} U_j = f_i$$

for the unknown coefficients U_j, $j = 1,\dots,n_{\text{glob}}$. Here, the entries of the matrix $A = (a_{ij})$ and the entries of the right hand side vector $b = (f_i)$ are given by $a_{ij} = a(\Phi_j, \Phi_i)$ and $f_i = F(\Phi_i)$, $i, j = 1,\dots,n_{\text{glob}}$, respectively.

In the following, we present three examples of finite element spaces $V_h \subset C^1(\Omega)$ and discuss how homogeneous Dirichlet boundary condition can be realized by setting a selected set of dof equal to zero. We start with a simple rectangular element with 16 dof and polynomial local basis functions. Unfortunately, on general decompositions of Ω into triangles or tetrahedra the number of dof (and the polynomial degree of functions in $\mathcal{P}_K$) needed to guarantee C^1 continuity increases. One way to circumvent the use of these computationally expensive elements, for example, the 2d Argyris element (21 dof) or the 3d Ženíšek tetrahedral element (220 dof), is to use composite finite elements. The basic idea of composite finite elements is to relax the requirement that the local approximation space $\mathcal{P}_K$ consists of polynomials and getting more flexibility for constructing global C^1 finite element spaces. Here, we present examples of such a strategy in two and three space dimensions.

The Bogner–Fox–Schmit rectangle

Our first example of a conforming finite element method for solving the biharmonic equation is based on the Bogner–Fox–Schmit rectangle. Let $\Omega \subset \mathbb{R}^2$ be decomposed into axiparallel rectangles K. We denote the vertices of a single rectangle K by a_i, $i = 0,\dots,3$, as indicated in Figure 5.2.

As the local approximation space we choose the space of polynomials of degree less than or equal to three in each variable, more precisely $\mathcal{P}_K = Q_3(K)$ where $\dim \mathcal{P}_K = 16$. We define the

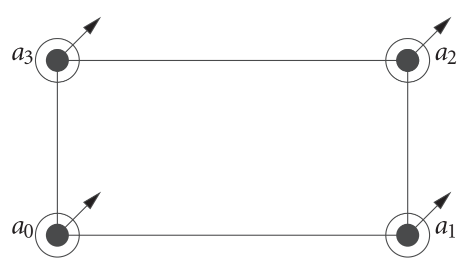

FIGURE 5.2 Degrees of freedom of the Bogner–Fox–Schmit rectangle. Values of the function, its first derivatives and its second mixed derivatives at the vertices.

set of dof Σ_K by 16 nodal functionals

$$N_i(v) := v(a_i), \qquad N_{4+i}(v) := \frac{\partial v}{\partial x_1}(a_i), \qquad N_{8+i}(v) := \frac{\partial v}{\partial x_2}(a_i),$$
$$N_{12+i}(v) := \frac{\partial^2 v}{\partial x_1\, \partial x_2}(a_i) \quad \text{for } i = 0, \ldots, 3.$$

Lemma 5.3. *The set of dof Σ_K in the Bogner–Fox–Schmit element is $\mathcal{P}_K$ unisolvent. The generated finite element space V_h belongs to $C^1(\overline{\Omega})$. We get $V_h \subset H_0^2(\Omega)$ by setting all dof on the boundary $\partial\Omega$ equal to zero.*

Proof. We first show that $v \in \mathcal{P}_K$ with $N_i(v) = 0$, $i = 0, \ldots, 15$, implies $v = 0$. The restriction of $v \in \mathcal{P}_K$ onto an edge of $K = (\alpha, \beta) \times (\gamma, \delta)$ is a polynomial of degree less than or equal three in one variable. The function and its tangential derivative vanishes at the two endpoints. The associated 1d Hermite interpolation is unique and equal to zero. This observation guarantees the continuity over the edge. Since the argument holds for any edge, $v \in \mathcal{P}_K$ with $N_i(v) = 0$, $i = 0, \ldots, 15$, vanishes along ∂K and can be represented as

$$v(x_1, x_2) = (x_1 - \alpha)(\beta - x_1)(x_2 - \gamma)(\delta - x_2)\, q(x_1, x_2), \quad q \in Q_1(K).$$

The function

$$p(x_1, x_2) = (x_1 - \alpha)(\beta - x_1)(x_2 - \gamma)(\delta - x_2)$$

and its first derivatives vanishes at the four vertices a_i, $i = 0, \ldots, 3$. The mixed second derivative of p does not vanish at the vertices, thus

$$0 = \frac{\partial^2 v}{\partial x_1\, \partial x_2}(a_i) = \frac{\partial^2 p}{\partial x_1\, \partial x_2}(a_i)\, q(a_i) \quad \Rightarrow \quad q(a_i) = 0, \quad i = 0, \ldots, 3.$$

Having in mind that the nodal functionals $N_i(q)$, $i = 0, \ldots, 3$, define a set of dof which is $Q_1(K)$ unisolvent, we end up with $q = 0$ and consequently $v = 0$, that is, the set of dof is $\mathcal{P}_K$ unisolvent.

Along the edge $x_2 = \delta$, $x_1 \in (\alpha, \beta)$, the normal derivative $\partial v / \partial x_2$ belongs to $P_3(\alpha, \beta)$ and vanishes together with its tangential derivatives at the endpoints of the edge. Again, the associated unique Hermite interpolation becomes zero. Applying this argument for each edge, we see that the normal derivatives are continuous over the edges and $V_h \subset C^1(\overline{\Omega})$.

Let E be a boundary edge connecting the vertices a_1 and a_2. Then, we require eight dof to be zero, that is,

$$u_h(a_i) = \frac{\partial u_h}{\partial x_1}(a_i) = \frac{\partial u_h}{\partial x_2}(a_i) = \frac{\partial^2 u_h}{\partial x_1\, \partial x_2}(a_i) = 0, \quad i = 1, 2.$$

Since $u_h|_E \in P_3(E)$ and $\frac{\partial u_h}{\partial n}|_E \in P_3(E)$, we conclude $u_h = \frac{\partial u_h}{\partial n} = 0$. ∎

The nodal functionals for the Bogner–Fox–Schmit rectangle can be replaced by the following ones (index for a_i is counted modulo 4)

$$\widetilde{N}_i(v) := v(a_i), \qquad \widetilde{N}_{4+i}(v) := Dv(a_i)(a_{i-1} - a_i),$$
$$\widetilde{N}_{8+i}(v) := Dv(a_i)(a_{i+1} - a_i), \qquad \widetilde{N}_{12+i}(v) := D^2 v(a_i)(a_{i-1} - a_i, a_{i+1} - a_i) \quad \text{with } 0 \le i \le 3.$$

The advantage of this new set is its affine equivalence to the reference cell. Indeed, let $F_K : \widehat{K} \to K$ with $F_K(\hat{x}) = B_K\hat{x} + b_K$, be the affine mapping from the reference cell to the general cell where $a_i = F_K(\hat{a}_i)$, $i = 0, \ldots, 3$. Then, applying the chain rule, we obtain for $\hat{v}(\hat{x}) = v(F_K(\hat{x}))$, for example,

$$Dv(a_i)(a_{i+1} - a_i) = \widehat{D}\hat{v}(\hat{a}_i)B_K^{-1}(a_{i+1} - a_i) = \widehat{D}\hat{v}(\hat{a}_{i+1} - \hat{a}_i).$$

As a consequence, the value of the (scaled) tangential derivative of v at a vertex equals to the value of tangential derivative of $\hat{v}$ at the associated vertex. We get similar expressions for the other dof. Finally, the space of local ansatz functions is transformed to

$$\mathcal{P}_K = Q_3(K) = \{p : K \to \mathbb{R} : p = \hat{p} \circ F_K^{-1}, \hat{p} \in Q_3(\widehat{K})\}.$$

Using Theorem 1.5 (Sobolev embedding), we are able to define point values of $D^\alpha v$, $|\alpha| \le 2$, provided that $v \in H^4(\Omega)$. Thus, we can define the interpolation operator $\Pi_K : H^4(K) \to \mathcal{P}_K$ locally by

$$\widetilde{N}_i(\Pi_K v) = \widetilde{N}_i(v), \quad v \in H^4(K), i = 0, \ldots, 15.$$

Since $\Pi_K p = p$ for all polynomials $p \in P_3(K)$ we have the interpolation estimates

$$|v - \Pi_K v|_{m,K} \le Ch_K^{4-m}|v|_{4,K}, \quad v \in H^4(K), 0 \le m \le 4. \tag{5.5}$$

The global interpolation $\Pi_h : H^4(\Omega) \to V_h \subset C^1(\overline{\Omega})$ is defined by

$$\Pi_h v|_K = \Pi_K(v|_K) \quad \text{for all } K \in \mathcal{T}_h.$$

Theorem 5.3. *Let the solution u of problem* (5.2) *belong to $H_0^2(\Omega) \cap H^4(\Omega)$ and u_h denote the solution of the discrete problem* (5.4) *with the Bogner–Fox–Schmit finite element space. Then,* $\|u - u_h\|_2 \le Ch^2|u|_4$.

Proof. Apply Theorem 5.2 and the interpolation error estimates (5.5). ∎

Remark 5.2. We could also use other $C^1(\overline{\Omega})$ elements like the Argyris triangle or the Bell's triangle with 21 and 18 dof, respectively. Note, however, that both elements are not affine equivalent and in order to obtain an analog to the interpolation error estimate (5.5), one has to modify the concept of affine equivalent elements. For details, see Ciarlet (2002).

Exercise 5.1. Let K be a triangle with vertices a_i, $i = 0, 1, 2$, and midpoints $a_{ij} = (a_i + a_j)/2$, $0 \le i < j \le 2$. Show that the set of dof

$$\Sigma_K := \{v(a_i), D^2v(a_i), i = 0, 1, 2, v(a_{ij}), 0 \le i < j \le 2\}$$

is $\mathcal{P}_K$ unisolvent, where $\mathcal{P}_K = P_4(K)$. Does this element yield the inclusion $V_h \subset C(\overline{\Omega})$ and $V_h \subset C^1(\overline{\Omega})$, respectively?

The Hsieh–Clough–Tocher (HCT) triangle

Let $\Omega \subset \mathbb{R}^2$ be decomposed into triangles K. We denote the vertices of a single triangle K by a_i, $i = 0, 1, 2$. Each triangle is subdivided into three sub-triangles where K_j, $j = 0, 1, 2$, are created by connecting an inner point a_K with the vertices a_{j+1} and a_{j+2} (counted modulo 3). The midpoint of an edge opposite to a_i is noted as b_i, $i = 0,1,2$, as indicated in Figure 5.3. As a local approximation space we chose C^1 continuous, piecewise polynomials of degree three, more precisely

$$\mathcal{P}_K = \{v \in C^1(K) : v|_{K_i} \in P_3(K_i),\ i = 0,1,2\}.$$

We define the set of dof Σ_K by the nodal functionals

$$N_i(v) := v(a_i), \qquad N_{3+i}(v) := \frac{\partial v}{\partial x_1}(a_i), \qquad N_{6+i}(v) := \frac{\partial v}{\partial x_2}(a_i),$$
$$N_{9+i}(v) := \frac{\partial v}{\partial n}(b_i), \quad i = 0,1,2.$$

Since $\dim P_3(K_i) = 10$, we need 30 equations to define the three polynomials $v|_{K_i}$, $i = 0,1,2$. The nodal functionals define 6 function values and 12 derivatives at the vertices. Moreover, 3 normal derivatives at the midpoint of edges complete the given 21 dof. For the C^1 continuity over the inner edges of the subdomains we add six equations to guarantee the C^1 continuity at a_K and three equations for the continuity of the normal derivative in the midpoints of inner edges of the subdomains. Indeed, the restrictions of functions $v \in P_3(K_{i+1})$ and $w \in P_3(K_{i+2})$ (counted modulo 3) on the inner edge $[a_K, a_i]$ are cubic functions in one variable for which the function values and the tangential derivatives coincide, that is, $v|_{[a_K,a_i]} = w|_{[a_K,a_i]}$ is the unique Hermite interpolation. Consequently, v and the tangential derivatives are continuous across $[a_K, a_i]$. In order to prove the continuity of the normal derivative across $[a_K, a_i]$ we note that $n \cdot \nabla v \in P_2([a_K, a_i])$ is continuous at a_K, a_i, and $(a_K + a_i)/2$ and thus uniquely defined on any inner edge. Finally, the C^1 continuity of the generated finite element space across edges of K can be shown in the same way.

Lemma 5.4. *The set of dof Σ_K in the HCT element is $\mathcal{P}_K$ unisolvent. The generated finite element space V_h belongs to $C^1(\overline{\Omega})$. We get $V_h \subset H_0^2(\Omega)$ by setting all dof on the boundary $\partial\Omega$ equal to zero.*

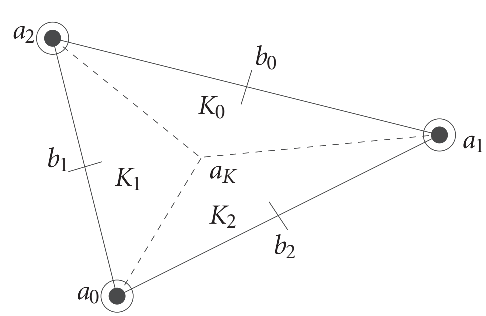

FIGURE 5.3 Degrees of freedom of the HCT triangle. Values of the function, its first derivatives at the vertices, and its normal derivatives at the midpoints.

Proof. See Section 6.1.3 of Ciarlet (2002) for the first two statements. We show $V_h \subset H_0^2(\Omega)$ provided all dof on the boundary are set to be zero. Let E be an edge on the boundary. Since $u_h|_E \in P_3(E)$ and vanishes with its tangential derivatives at the endpoints of E, we conclude $u_h = 0$ on E. Note that $(\partial u_h/\partial n)|_E \in P_2(E)$ vanishes at the mid and endpoints of E. Then, $\partial u_h/\partial n = 0$ on the boundary edge E. ■

Remark 5.3. The HTC element is not affine equivalent due to the presence of the normal derivatives at b_i, $i=0,1,2$, as dof. An additional reason for being not affine is that a_K may be allowed to vary inside of K. Nevertheless, the estimate (5.5) for the interpolation error holds true, for proof see Theorem 6.1 of Ciarlet (2002).

Remark 5.4. If we require that the normal derivative vary linearly along the edges of K we obtain the reduced HCT element. This results in three additional constraints and the dimension of the associated local approximation space

$$\mathcal{P}_K = \left\{ v \in C^1(K) : v|_{K_i} \in P_3(K_i),\ \frac{\partial v}{\partial n}\bigg|_{[a_i,a_{i+1}]} \in P_1([a_i,a_{i+1}]),\ i=0,1,2 \right\}$$

reduces to 9. The set of dof become the function values and the first derivatives at the vertices of K. For the HCT element it holds $P_3 \subset \mathcal{P}_K$ but for the reduced HCT element we only have $P_2 \subset \mathcal{P}_K$ and the estimate for the interpolation error becomes

$$|v - \Pi_K v|_{m,K} \le C h_K^{3-m} |v|_{3,K}, \quad v \in H^3(K),\ 0 \le m \le 3.$$

A C^1 tetrahedral finite element

A three-dimensional conforming finite element based on ninth degree polynomials has been constructed by Ženíšek (1974a). It has $\dim P_9(K) = 220$ dof. Recently, a composite element based on piecewise polynomials of degree 5 with 45 dof has been proposed by Walkington (2014). In the following, we explain the main features of this element.

Let $\Omega \subset \mathbb{R}^3$ be decomposed into tetrahedrons K. We denote the vertices of a single tetrahedron K by a_i, $i=0,1,2,3$. Then, each tetrahedron is subdivided into four sub-tetrahedrons where K_j, $j=0,1,2,3$, are created by connecting the barycentre a_K with three vertices a_{j+1}, a_{j+2}, a_{j+3} of K (indices are counted modulo 4). If $f \subset \partial K$ is a triangular face, then its centroid is denoted by b_f and the outer normal by n_f, see Figure 5.4. The local approximation space is given by piecewise polynomials of degree 5 with normal derivatives on the faces restricted to polynomials of degree 3, more precisely,

$$\mathcal{P}_K = \left\{ v \in C^1(K) \cap C^4(a_K) : v|_{K_i} \in P_5(K_i),\ \frac{\partial v}{\partial n_f}\bigg|_f \in P_3(f) \text{ for all } f \subset \partial K \right\},$$

where $v \in C^4(a_K)$ means that v and its derivatives up to order 4 are continuous at a_K. The set of dof Σ_K consists of

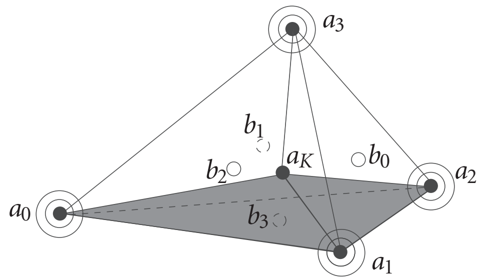

FIGURE 5.4 Degrees of freedom of the composite tetrahedron. Values of the function, its first and second derivatives at the vertices, and its normal derivatives at the centroids of faces.

(i) the function values and its derivative up to order 2 at the vertices (40 dof)

$$N_i^\alpha(v) := D^\alpha v(a_i), \quad 0 \le |\alpha| \le 2, \;\; i = 0,1,2,3, \tag{5.6}$$

(ii) the normal derivatives at the centroids of the faces (4 dof)

$$N_f(v) := n_f \cdot \nabla v(b_f), \quad f \subset \partial K, \tag{5.7}$$

(iii) the function value at the barycenter of the tetrahedron (1 dof)

$$N(v) := v(a_K). \tag{5.8}$$

Lemma 5.5. *The set of dof Σ_K in the composite tetrahedron is $\mathcal{P}_K$ unisolvent. The generated finite element space V_h belongs to $C^1(\overline{\Omega})$.*

Proof. See Lemma 2.3 of Walkington (2014). ■

As usual we consider the canonical interpolation operator Π_K defined by

$$N_i^\alpha(\Pi_K v) = N_i^\alpha(v), \qquad N_f(\Pi_K v) = N_f(v), \qquad N(\Pi_K v) = N(v),$$
$$\text{for all } 0 \le |\alpha| \le 2, i = 0,1,2,3, f \subset \partial K.$$

Theorem 1.5 (Sobolev embedding) tells us that $v \in W^{m,p}(\Omega) \subset C^2(\overline{\Omega})$ for $2 + 3/p < m$ such that the interpolant $\Pi_K : W^{m,p}(K) \to \mathcal{P}_K$ is well defined.

Remark 5.5. We can replace the set of dof associated with the vertices of the tetrahedron by

$$\widetilde{N}_i^\alpha(v) := v(a_i), \quad |\alpha| = 0, \qquad \widetilde{N}_i^\alpha(v) := \nabla v(a_i) \cdot (a_j - a_i), \quad |\alpha| = 1, j \ne i,$$
$$\widetilde{N}_i^\alpha(v) := (a_k - a_i)^T D^2 v(a_i)(a_j - a_i), \quad |\alpha| = 2, j, k \ne i, 0 \le j \le k \le 3.$$

Nevertheless, as for the HTC element, the face dof $N_f(v)$ are not invariant under affine maps. Thus, we cannot apply the classical concept of affine equivalent finite elements to get estimates for the interpolation error.

Theorem 5.4. *Let $v \in W^{m,p}(K)$ with $2+3/p \le m \le 5$. Then, the canonical interpolation $\Pi_K : W^{m,p}(\Omega) \to \mathcal{P}_K$ is well defined and satisfies*

$$|v - \Pi_K v|_{i,p,K} \le C h_K^{m-l} |v|_{m,p,K}, \qquad 0 \le l \le m.$$

Proof. Let $F_K : \widehat{K} \to K$ be the bijective affine mapping from the reference cell $\widehat{K}$ onto a cell $K \in \mathcal{T}_h$. The interpolation Π_K defines an interpolation $\widehat{\Pi}_K$ on the reference cell by

$$(\widehat{\Pi}_K \hat{v})(\hat{x}) := \left(\Pi_K \left(\hat{v} \circ F_K^{-1}\right)\right) \left(F_K(\hat{x})\right),$$

where $\widehat{\Pi}_K$ cannot depend on K for an affine equivalent family of elements. As a consequence, the uniform continuity of $\widehat{\Pi}_K : W^{m,p}(\widehat{K}) \to W^{l,p}(\widehat{K})$ with respect to $K \in \mathcal{T}_h$ can be established and the estimate follows by applying the Bramble-Hilbert lemma. Note that this classical argument with minor modification holds true provided that $\widehat{\Pi}_K$ is invariant under the translation and scaling of K and depends continuously on the Jacobian DF_K. For more details, see Theorem 2.5 of Walkington (2014). ■

Finally, we discuss the construction of the finite element space $V_h \cap H_0^2(\Omega)$ by setting appropriate dof equal to zero. The starting point is the observation that $v = 0$ on a face $f \subset \partial K \cap \partial\Omega$ of a cell K implies that the tangential derivatives of v vanish on f. Furthermore, we have to take into consideration the boundary condition $n_f \cdot \nabla v = 0$ on f. Thus, we require

$$D^\alpha v(a_i) = 0, \quad 0 \le \alpha \le 1, \qquad n_f \cdot \nabla v(b_j) = 0$$

at a vertex a_i on the boundary and at the centroid of the face $f \subset \partial\Omega$. It remains to set the second derivatives at a_i. We distinguish two cases:

(i) all boundary edges $E \subset \partial\Omega$ meeting at a_i are situated in the same plane (a_i is an inner point of a boundary face of the polyhedron Ω) and
(ii) there are three boundary edges $E \subset \partial\Omega$ meeting at a_i building a nondegenerated parallelepiped (a_i is a vertex or an inner point of an boundary edge of the polyhedron Ω).

Let τ_1 and τ_2 be the two unit vectors spanning the boundary face f with the normal n_f in the first case. Since $v = 0$ and $n_f \cdot \nabla v = 0$ on f, we conclude

$$D^2 v(\tau_j, \tau_k) = 0, \qquad D^2 v(n_f, \tau_j) = 0, \quad \text{on } f, \ \ 1 \le j \le k \le 2.$$

In the second case, we have two boundary faces spanned by τ_1, τ_2 and τ_2, τ_3, respectively. Therefore, we conclude that all second order derivatives have to vanish

$$D^2 v(\tau_j, \tau_k) = 0, \quad \text{on } f, \ \ 1 \le j \le k \le 3.$$

We complete our setting of dof in the first case by

$$\text{(i)} \quad D^2 v(a_i)(\tau_j, \tau_k) = 0, \qquad D^2 v(a_i)(n_f, \tau_j) = 0, \quad 1 \le j \le k \le 2,$$

letting $D^2 v(a_i)(n_f, n_f)$ free and in the second case by

$$\text{(ii)} \quad D^2 v(a_i)(\tau_j, \tau_k) = 0, \quad 1 \le j \le k \le 3.$$

We still have to show that the setting of dof above guarantees $u_h = 0$ and $n_f \cdot \nabla u_h = 0$ on f. Let E be an edge of f. We know that $w := n_f \cdot \nabla u_h|_f \in P_3(f)$, therefore $z := w|_E \in P_3(E)$. The setting above implies that z and its tangential derivatives vanish at the two endpoints of E. Thus, $z = 0$ on E. Repeating the arguments for another edge shows that $w = 0$ on ∂f. Recall that $w \in P_3(f)$ and vanishes at the barycenter of f. This gives $w = 0$ on f. Now we prove $u_h = 0$ on f. Consider f as Argyris triangle. Then, all dof at the three vertices of f (u_h and its tangential derivatives upto second order) vanish. It remains to show that the conormal derivative (normal to $E \subset \partial f$ and tangential to f) in the midpoints of the three edges vanish. For this we use the observation that on an edge $E = [a_i, a_j]$ with midpoint $a_{ij} = (a_i + a_j)/2$ it holds

$$P_3(E) = \{p \in P_4(E) : \Lambda(p) = 0\},$$

where

$$\Lambda(p) := \frac{1}{2}(p(a_i) + p(a_j)) - p(a_{ij}) + \frac{1}{8}(Dp(a_i) - Dp(a_j))(a_j - a_i).$$

Let E be the intersection of the two faces f and f' of K. Since $n_f \cdot \nabla u_h \in P_3(E)$, we can express $n_f \cdot \nabla u_h(a_{ij})$ as a linear combination of first and mixed second order derivatives at the endpoints of E. Thus, $n_f \cdot \nabla u_h(a_{ij}) = 0$. The same argument applied to $n_{f'} \cdot \nabla u_h \in P_3(E)$ leads to $n_{f'} \cdot \nabla u_h(a_{ij}) = 0$. Therefore, any first derivative of u_h in a direction perpendicular to E vanishes at a_{ij}, in particular, the conormal derivative mentioned above.

5.4. Nonconforming finite element methods

In order to avoid C^1elements with a large number of dof locally in each cell, we consider now the case $V_h \not\subset V = H_0^2(\Omega)$. Then, the bilinear form $a : H_0^2(\Omega) \times H_0^2(\Omega) \to \mathbb{R}$ is, in general, not defined on the finite element space V_h. Therefore, in a first step, we extend the bilinear form onto $(V + V_h) \times (V + V_h)$. A natural choice would be to compute the bilinear form cellwise

$$a_h^*(u_h, v_h) := \sum_{K \in \mathcal{T}_h} \int_K \Delta u_h \Delta v_h \, dx.$$

Of course $a_h^* : (V + V_h) \times (V + V_h) \to \mathbb{R}$ is an extension of $a : V \times V \to \mathbb{R}$ satisfying

$$a_h^*(u, v) = a(u, v) \quad \text{for all } u, v \in V. \tag{5.9}$$

However, for a piecewise linear functions $w_h \in V_h \not\subset V$ we get $\Delta w_h|_K = 0$ and thus $a_h^*(w_h, w_h) = 0$ without being $w_h = 0$. This shows that, in general, a_h^* is not coercive on a nonconforming space V_h. The reason is that we cannot apply Lemma 5.1 for functions from the discrete space V_h.

Now the idea is to reformulate the continuous bilinear form by means of Lemma 5.1 and then using some cellwise computed version of it which guarantees the coercivity on the discrete finite element space. We define

$$a_h(u_h, v_h) := \sum_{K \in \mathcal{T}_h} \int_K \Delta u_h \Delta v_h \,\mathrm{d}x + (1-\sigma) \sum_{K \in \mathcal{T}_h} \int_K \sum_{i,j=1}^{d} \left(\frac{\partial^2 u_h}{\partial x_i \partial x_j} \frac{\partial^2 v_h}{\partial x_i \partial x_j} - \frac{\partial^2 u_h}{\partial x_i^2} \frac{\partial^2 v_h}{\partial x_j^2} \right) \mathrm{d}x.$$

Note that the second term vanishes for $u_h, v_h \in H_0^2(\Omega)$ according to Lemma 5.1, thus a_h is an extension of a on $V + V_h$ satisfying (5.9). Now

$$\Delta u_h \Delta v_h = \sum_{i,j=1}^{d} \frac{\partial^2 u_h}{\partial x_i^2} \frac{\partial^2 v_h}{\partial x_j^2}$$

consequently, the bilinear form a_h can be written as

$$a_h(u_h, v_h) = \sum_{K \in \mathcal{T}_h} \int_K \left\{ \sigma \Delta u_h \Delta v_h + (1-\sigma) \sum_{i,j=1}^{d} \frac{\partial^2 u_h}{\partial x_i \partial x_j} \frac{\partial^2 v_h}{\partial x_i \partial x_j} \right\} \mathrm{d}x. \tag{5.10}$$

Theorem 5.5. *Let $0 \leq \sigma < 1$ and let the broken seminorm $|\cdot|_{2,h}$ be a norm on $V + V_h$. Then, the bilinear form $a_h : (V + V_h) \times (V + V_h) \to \mathbb{R}$ is coercive, that is,*

$$a_h(v, v) \geq (1-\sigma)|v|_{2,h}^2 \quad \textit{for all } v \in V + V_h.$$

Proof. Setting $u_h = v_h = v \in V + V_h$ in Equation (5.10), we obtain immediately

$$a_h(v, v) = \sum_{K \in \mathcal{T}_h} [\sigma \|\Delta v\|_{0,K}^2 + (1-\sigma)|v|_{2,K}^2] \geq (1-\sigma)|v|_{2,h}^2.$$

■

The continuity of the bilinear form is shown by applying Cauchy–Schwarz inequality (in case of sums and integrals), for $u, v \in V + V_h$ we have

$$|a_h(u, v)| \leq \sum_{K \in \mathcal{T}_h} \left[\sigma \|\Delta u\|_{0,K} \|\Delta v\|_{0,K} + (1-\sigma)|u|_{2,K}|v|_{2,K} \right] \leq C\|u\|_{2,h}\|v\|_{2,h}.$$

The discrete problem reads as

$$\text{Find } u_h \in V_h \text{ such that } a_h(u_h, v_h) = F(v_h) \text{ for all } v_h \in V_h. \tag{5.11}$$

Theorem 5.6. *Let $0 \leq \sigma < 1$ and let the broken seminorm $|\cdot|_{2,h}$ be a norm on $V + V_h$. Then, there is a unique solution $u_h \in V_h$ of the discrete problem* (5.11) *satisfying*

$$\|u - u_h\|_{2,h} \leq C \left(\inf_{v_h \in V_h} \|u - v_h\|_{2,h} + \sup_{w_h \in V_h} \frac{|a_h(u, w_h) - F(w_h)|}{\|w_h\|_{2,h}} \right).$$

Proof. The existence of a unique solution of the discrete problem follows from the Lax–Milgram theorem. Consider now the error estimate. Let $v_h \in V_h$ arbitrary. Then, the triangle inequality

$$\|u - u_h\|_{2,h} \leq \|u - v_h\|_{2,h} + \|v_h - u_h\|_{2,h}$$

and Theorem 5.5 give

$$\begin{aligned}
(1-\sigma)|v_h - u_h|_{2,h}^2 &\leq a_h(v_h - u_h, v_h - u_h) \\
&= a_h(u - u_h, v_h - u_h) + a_h(v_h - u, v_h - u_h) \\
&= a_h(u, v_h - u_h) - F(v_h - u_h) + a_h(v_h - u, v_h - u_h) \\
&\leq C\left(\sup_{w_h \in V_h} \frac{|a_h(u, w_h) - F(w_h)|}{\|w_h\|_{2,h}} + \|u - v_h\|_{2,h}\right)\|v_h - u_h\|_{2,h}.
\end{aligned}$$

Using the equivalence of the seminorm $|\cdot|_{2,h}$ to the norm $\|\cdot\|_{2,h}$ and combining the inequalities above, we get for all $v_h \in V_h$

$$\|u - u_h\|_{2,h} \leq C\left(\|u - v_h\|_{2,h} + \sup_{w_h \in V_h} \frac{|a_h(u, w_h) - F(w_h)|}{\|w_h\|_{2,h}}\right).$$

Taking the infimum over all $v_h \in V_h$ the statement follows. ■

Remark 5.6. As shown for the conforming case, the discrete problem (5.11) is equivalent to a linear algebraic system of equations. Let Φ_i, $i = 1, \ldots, n_{\text{glob}}$, be a basis of V_h, $U = (U_j)$ the unknown coefficient vector in the basis representation of u_h, $a_{ij} = a_h(\Phi_j, \Phi_i)$ the matrix entries of A, and $f_i = F(\Phi_i)$ the entries of vector b. Then, $AU = b$.

The rectangular Adini element

As a first example of a nonconforming finite element method for solving the biharmonic equation, the rectangular Adini element is considered. Let $\Omega \subset \mathbb{R}^2$ be decomposed into axiparallel rectangles K. We denote the vertices of a single rectangle K by a_i, $i = 0, \ldots, 3$ as indicated in Figure 5.5. An enrichment of the space of polynomials of degree less than or equal to three is taken as the local approximation space, more precisely

$$\mathcal{P}_K = P_3(K) \oplus \text{span}\{x_1 x_2^3, x_1^3 x_2\}.$$

The dimension is given by $\dim \mathcal{P}_K = 10 + 2 = 12$. The set of dof Σ_K is defined by the 12 nodal functionals

$$N_i(v) := v(a_i), \qquad N_{4+i}(v) := \frac{\partial v}{\partial x_1}(a_i), \qquad N_{8+i}(v) := \frac{\partial v}{\partial x_2}(a_i), \quad \text{for } i = 0, \ldots, 3.$$

Lemma 5.6. *The set of dof Σ_K in Adini's element is $\mathcal{P}_K$ unisolvent.*

Proof. We have to show that $v \in \mathcal{P}_K$ with $N_j(v) = 0$, $j = 0, \ldots, 11$, implies $v = 0$. The restriction of v on the segment $[a_1, a_2]$ is a polynomial with respect to x_1 of degree less than or equal three.

FIGURE 5.5 Degrees of freedom of the Adini's rectangular element. Values of the function and its first derivatives at the vertices.

This polynomial satisfies

$$v(a_1) = N_1(v) = v(a_2) = N_2(v) = \frac{\partial v}{\partial x_1}(a_1) = N_5(v) = \frac{\partial v}{\partial x_1}(a_2) = N_6(v) = 0.$$

From the 1d Hermite interpolation we conclude $v|_{[a_1,a_2]} = 0$. The same argument can be used to show that v vanishes on the other edges of $K = [\alpha,\beta] \times [\gamma,\delta]$. As a consequence, v admits a representation

$$v(x_1,x_2) = (x_1 - \alpha)(\beta - x_1)(x_2 - \gamma)(\delta - x_2)p(x_1,x_2)$$

with some polynomial p. Since, however, the function $(x_1,x_2) \mapsto x_1^2 x_2^2$ does not belong to $\mathcal{P}_K$, the polynomial p has to be identically zero. ■

The local basis functions φ_j defined by $N_i(\varphi_j) = \delta_{ij}$, with $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ii} = 1$ are given on the reference cell $\widehat{K} = (-1,+1)^2$ by

$$\varphi_0(x_1,x_2) = \frac{(1+x_1)(1+x_2)}{4}\left(1 + \frac{x_1+x_2}{2} - \frac{x_1^2+x_2^2}{2}\right),$$

$$\varphi_1(x_1,x_2) = \frac{(1-x_1)(1+x_2)}{4}\left(1 + \frac{x_2-x_1}{2} - \frac{x_1^2+x_2^2}{2}\right),$$

$$\varphi_2(x_1,x_2) = \frac{(1-x_1)(1-x_2)}{4}\left(1 - \frac{x_1+x_2}{2} - \frac{x_1^2+x_2^2}{2}\right),$$

$$\varphi_3(x_1,x_2) = \frac{(1+x_1)(1-x_2)}{4}\left(1 + \frac{x_1-x_2}{2} - \frac{x_1^2+x_2^2}{2}\right),$$

$$\varphi_4(x_1,x_2) = \frac{(1+x_2)(1+x_1)^2(x_1-1)}{8},$$
$$\varphi_5(x_1,x_2) = \frac{(1+x_2)(1-x_1)^2(x_1+1)}{8},$$
$$\varphi_6(x_1,x_2) = \frac{(1-x_2)(1-x_1)^2(x_1+1)}{8},$$
$$\varphi_7(x_1,x_2) = \frac{(1-x_2)(1+x_1)^2(x_1-1)}{8},$$
$$\varphi_8(x_1,x_2) = \frac{(1+x_1)(1+x_2)^2(x_2-1)}{8},$$
$$\varphi_9(x_1,x_2) = \frac{(1-x_1)(1+x_2)^2(x_2-1)}{8},$$
$$\varphi_{10}(x_1,x_2) = \frac{(1-x_1)(1-x_2)^2(x_2+1)}{8},$$
$$\varphi_{11}(x_1,x_2) = \frac{(1+x_1)(1-x_2)^2(x_2+1)}{8}.$$

We construct a finite element space V_h based on the Adini element. A function $v_h \in V_h$ is defined locally by its dof in all vertices of the decomposition of Ω into rectangles

$$\begin{aligned} V_h := \{v_h : \Omega \to \mathbb{R} : v_h|_K \in \mathcal{P}_K, D^\alpha v_h &\text{ continuous at the vertices,} \\ D^\alpha v_h &= 0 \text{ at boundary vertices, } |\alpha| \le 1\}. \end{aligned}$$

Note that the restriction of v_h onto an edge is a polynomial of degree less than or equal to three and depends on the two function values and the two values for the tangential derivatives at the endpoint of the edge. Therefore, $V_h \subset C(\overline{\Omega})$. We also see, however, that $V_h \not\subset C^1(\overline{\Omega})$. Indeed, along the edge $x_1 = 1$, we have

$$\frac{\partial \varphi_9}{\partial x_1}(1,x_2) = (1+x_2)^2(1-x_2)$$

but the nodal functionals along this edge

$$N_0(\varphi_9) = N_3(\varphi_9) = N_4(\varphi_9) = N_7(\varphi_9) = N_8(\varphi_9) = N_{11}(\varphi_9) = 0$$

vanish and allow $v = 0$ in the neighbouring cell, that is, the normal derivative is not continuous over this edge.

Lemma 5.7. *On the space V_h of Adini elements $v_h \mapsto |v_h|_{2,h}$ is a norm.*

Proof. Since $v_h \mapsto |v_h|_{2,h}$ is a seminorm we have only to show that $|v_h|_{2,h} = 0$ implies $v_h = 0$. Let $v_h \in V_h$ with $|v_h|_{2,h} = 0$. Then, the first derivatives are constant on each $K \in \mathcal{T}_h$. The first derivatives are continuous in the vertices, thus they are constant on $\overline{\Omega}$. The first derivatives at boundary nodes are zero and we conclude that they are identically zero. It follows that $v_h|_K$ is constant on each $K \in \mathcal{T}_h$. The continuity of v_h on $\overline{\Omega}$ and the homogenuous boundary conditions imply that this constant is equal to zero. ■

The family of Adini's rectangle can be modified to build an affine equivalent family by replacing its dof by

$$\widetilde{N}_i(v) = v(a_i), \qquad \widetilde{N}_{4+i}(v) = Dv(a_i)(a_{i-1} - a_i), \qquad \widetilde{N}_{8+i}(v) = Dv(a_i)(a_{i+1} - a_i),$$

where $i = 0, \ldots, 3$ and the index of a_i is counted modulo 4. Thus, assuming $u \in H_0^2(\Omega) \cap H^3(\Omega)$ the approximation error is bounded by

$$\inf_{v_h \in V_h} \|u - v_h\|_{2,h} \leq C \left(\sum_{K \in \mathcal{T}_h} \|u - \Pi_K u\|_{2,K}^2 \right)^{1/2} \leq Ch|u|_3.$$

Based on Theorem 5.6 it remains to estimate the consistency error

$$\sup_{w_h \in V_h} \frac{|a_h(u, w_h) - F(w_h)|}{\|w_h\|_{2,h}}.$$

Let $u \in H_0^2(\Omega) \cap H^3(\Omega)$ be a solution of problem (5.2). We show that

$$F(w_h) = -\int_\Omega \nabla(\Delta u) \cdot \nabla w_h \, \mathrm{d}x \qquad \text{for all } w_h \in V_h.$$

We recall that $V_h \subset C(\overline{\Omega}) \cap H_0^1(\Omega)$. Thus, there is a sequence $(w_h^k)_{k \in \mathbb{N}} \in C_0^\infty(\Omega)$ converging to w_h in $H_0^1(\Omega)$. Then, integrating by parts we obtain

$$\begin{aligned} F(w_h) &= \lim_{k \to \infty} F(w_h^k) = \lim_{k \to \infty} a(u, w_h^k) = \lim_{k \to \infty} \int_\Omega \Delta u \Delta w_h^k \, \mathrm{d}x \\ &= \lim_{k \to \infty} \left[-\int_\Omega \nabla(\Delta u) \cdot \nabla w_h^k \, \mathrm{d}x \right] = -\int_\Omega \nabla(\Delta u) \cdot \nabla w_h \, \mathrm{d}x. \end{aligned}$$

Now, integrating elementwise by parts

$$\begin{aligned} &\int_K \Delta u \Delta w_h \, \mathrm{d}x + (1-\sigma) \int_K \sum_{i,j=1}^2 \left(\frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 w_h}{\partial x_i \partial x_j} - \frac{\partial^2 u}{\partial x_i^2} \frac{\partial^2 w_h}{\partial x_j^2} \right) \mathrm{d}x \\ &\quad = -\int_K \nabla(\Delta u) \cdot \nabla w_h \, \mathrm{d}x + \int_{\partial K} \Delta u \frac{\partial w_h}{\partial n_K} \, \mathrm{d}\gamma \\ &\qquad + (1-\sigma) \int_{\partial K} \left(-\frac{\partial^2 u}{\partial \tau_K^2} \frac{\partial w_h}{\partial n_K} + \frac{\partial^2 u}{\partial \tau_K \partial n_K} \frac{\partial w_h}{\partial \tau_K} \right) \mathrm{d}\gamma \end{aligned}$$

and sum up over all cells K we get

$$a_h(u,w_h) = -\int_\Omega \nabla(\Delta u)\cdot\nabla w_h\,\mathrm{d}x + \sum_{K\in\mathcal{T}_h}\int_{\partial K}\Delta u\frac{\partial w_h}{\partial n_K}\,\mathrm{d}\gamma$$

$$+(1-\sigma)\sum_{K\in\mathcal{T}_h}\int_{\partial K}\left(-\frac{\partial^2 u}{\partial\tau_K^2}\frac{\partial w_h}{\partial n_K}+\frac{\partial^2 u}{\partial\tau_K\,\partial n_K}\frac{\partial w_h}{\partial\tau_K}\right)\mathrm{d}\gamma\,.$$

Here, we used the notation n_K for the outer normal along ∂K and τ_K for the (counter clockwise rotated n_K) unit vector tangent to ∂K, respectively. For an inner edge $E = \partial K\cap\partial K'$ we observe that $\tau_K = -\tau_{K'}$, thus the two integrals over E in the sum cancel. The tangential derivative of w_h along an edge $E\subset\Gamma$ vanishes due to $V_h\subset C(\overline{\Omega})$, consequently we end up with a new representation of the numerator within the consistency error

$$a_h(u,w_h) - F(w_h) = \sum_{K\in\mathcal{T}_h}\int_{\partial K}\left(\Delta u-(1-\sigma)\frac{\partial^2 u}{\partial\tau_K^2}\right)\frac{\partial w_h}{\partial n_K}\,\mathrm{d}\gamma\,. \tag{5.12}$$

Let $E_i = [a_i, a_{i+1}]$, $i = 0,\dots,3$, denote the edges of K where the indices are counted modulo 4. Then, (5.12) can be written as

$$\begin{aligned}
&a_h(u,w_h) - F(w_h)\\
&= \sum_{K\in\mathcal{T}_h}\left(\int_{E_0}\left(\Delta u-(1-\sigma)\frac{\partial^2 u}{\partial x_1^2}\right)\frac{\partial w_h}{\partial x_2}\,\mathrm{d}\gamma - \int_{E_2}\left(\Delta u-(1-\sigma)\frac{\partial^2 u}{\partial x_1^2}\right)\frac{\partial w_h}{\partial x_2}\,\mathrm{d}\gamma\right)\\
&\quad+\sum_{K\in\mathcal{T}_h}\left(\int_{E_3}\left(\Delta u-(1-\sigma)\frac{\partial^2 u}{\partial x_2^2}\right)\frac{\partial w_h}{\partial x_1}\,\mathrm{d}\gamma - \int_{E_1}\left(\Delta u-(1-\sigma)\frac{\partial^2 u}{\partial x_2^2}\right)\frac{\partial w_h}{\partial x_1}\,\mathrm{d}\gamma\right).
\end{aligned}$$

Replacing the first derivatives of w_h by a continuous function vanishing along boundary edges the above expression vanishes. Indeed, in the sum integrals over inner edges appear twice with opposite sign (and thus cancel) and the tangential derivative over boundary edges vanish. This observation allows us to subtract a continuous piecewise bilinear function without changing the value of the functional $w_h\mapsto a_h(u,w_h)-F(w_h)$. Taking into consideration that the first derivatives of w_h are continuous in the vertices of the decomposition, we can define the Q_1 interpolation Π_h^1, locally by

$$\Pi_K^1\frac{\partial w_h}{\partial x_2}(a_i) = \frac{\partial w_h}{\partial x_2}(a_i),\qquad \Pi_K^1\frac{\partial w_h}{\partial x_1}(a_i) = \frac{\partial w_h}{\partial x_1}(a_i),\quad i=0,\dots,3.$$

Now (5.12) is rewritten as

$$a_h(u,w_h) - F(w_h) = \sum_{K\in\mathcal{T}_h}\left[D_1^K\left(u,\frac{\partial w_h}{\partial x_2}\right) + D_2^K\left(u,\frac{\partial w_h}{\partial x_1}\right)\right],$$

where $D_1^K : H^3(K) \times \dfrac{\partial}{\partial x_2}\mathcal{P}_K \to \mathbb{R}$ and $D_2^K : H^3(K) \times \dfrac{\partial}{\partial x_1}\mathcal{P}_K \to \mathbb{R}$ are given by

$$D_1^K(u, z_h) = \left(\int_{E_0} - \int_{E_2}\right)\left(\Delta u - (1-\sigma)\frac{\partial^2 u}{\partial x_1^2}\right)\left(z_h - \Pi_K^1 z_h\right)\,\mathrm{d}\gamma\,,$$

$$D_2^K(u, z_h) = \left(\int_{E_3} - \int_{E_1}\right)\left(\Delta u - (1-\sigma)\frac{\partial^2 u}{\partial x_2^2}\right)\left(z_h - \Pi_K^1 z_h\right)\,\mathrm{d}\gamma\,.$$

In the following, we only consider $D_1^K(u, z_h)$ since the estimation of $D_2^K(u, z_h)$ is similar. The term to be estimated is a bilinear form $d : H^1(K) \times \frac{\partial}{\partial x_2}\mathcal{P}_K \to \mathbb{R}$, given by

$$d(\varphi, z_h) = \left(\int_{E_0} - \int_{E_2}\right)\varphi\left(z_h - \Pi_K^1 z_h\right)\,\mathrm{d}\gamma\,.$$

Transforming it to the reference cell $\widehat{K} = (0,1)^2$ with the edges $\widehat{E}_0, \ldots, \widehat{E}_3$ and using the continuity of the trace operator we obtain

$$\left|\frac{1}{h_1} d(\varphi, z_h)\right| = \left|\hat{d}(\hat{\varphi}, \hat{z})\right| \le C\|\hat{\varphi}\|_{1,\widehat{K}}\,\|\hat{z}\|_{1,\widehat{K}},$$

that is, the bilinear form $\hat{d}$ is continuous on $H^1(\widehat{K}) \times \frac{\partial}{\partial \hat{x}_2}P_{\widehat{K}}$. Since

$$\begin{aligned} \hat{d}(\hat{\varphi}, \hat{q}) &= 0 \quad \text{for all } \hat{\varphi} \in H^1(\widehat{K}),\ \hat{q} \in P_0(\widehat{K}), \\ \hat{d}(\hat{p}, \hat{z}) &= 0 \quad \text{for all } \hat{p} \in P_0(\widehat{K}),\ \hat{z} \in \frac{\partial}{\partial \hat{x}_2}P_{\widehat{K}}, \end{aligned} \tag{5.13}$$

the generalization of Bramble–Hilbert lemma shows that even

$$\left|\hat{d}(\hat{\varphi}, \hat{z})\right| \le C|\hat{\varphi}|_{1,\widehat{K}}\,|\hat{z}|_{1,\widehat{K}}.$$

The second property of $\hat{d}$ in Equation (5.13) needs to be explained in some more detail. It holds true iff

$$\int_{\widehat{E}_0}(\hat{z} - \widehat{\Pi}^1\hat{z})\,\mathrm{d}\gamma = \int_{\widehat{E}_2}(\hat{z} - \widehat{\Pi}^1\hat{z})\,\mathrm{d}\gamma \quad \text{for all } \hat{z} \in \frac{\partial}{\partial \hat{x}_2}P_{\widehat{K}}.$$

Now a function $\hat{z} \in \dfrac{\partial}{\partial \hat{x}_2}P_{\widehat{K}}$ can be represented as

$$\hat{z} = A(\hat{x}_2) + B(\hat{x}_2)\hat{x}_1 + C\hat{x}_1^2 + D\hat{x}_1^3,$$

where A and B are polynomials of degree less than or equal to two with respect to $\hat{x}_2$. Subtracting the $Q_1(\widehat{K})$ interpolation, we have

$$(\hat{z} - \widehat{\Pi}^1\hat{z})|_{\widehat{E}_0} = C\hat{x}_1(\hat{x}_1 - 1) + D\hat{x}_1(\hat{x}_1^2 - 1) = (\hat{z} - \widehat{\Pi}^1\hat{z})|_{\widehat{E}_2}.$$

Transforming back to the cell K and replacing φ and z_h by the corresponding values, we end up with

$$\left| D_1^K\left(u, \frac{\partial w_h}{\partial x_2}\right)\right| = |d(\varphi, z_h)| \leq Ch_K|\varphi|_{1,K}\,|z_h|_{1,K} \leq Ch_K|u|_{3,K}\,|w_h|_{2,K}.$$

Applying Cauchy–Schwarz inequality, we get

$$\left|\sum_{K\in\mathcal{T}_h} D_1^K\left(u, \frac{\partial w_h}{\partial x_2}\right)\right| \leq Ch\left(\sum_{K\in\mathcal{T}_h}|u|_{3,K}^2\right)^{1/2}\left(\sum_{K\in\mathcal{T}_h}|w_h|_{2,K}^2\right)^{1/2} \leq Ch|u|_3\,\|w_h\|_{2,h}.$$

The second sum over D_2^K can be estimated in the same way.

Theorem 5.7. *Let the solution u of problem* (5.2) *belong to $H_0^2(\Omega)\cap H^3(\Omega)$ and u_h denote the solution of the discrete problem* (5.11) *with the Adini's finite element. Then, $\|u-u_h\|_{2,h} \leq Ch|u|_3$.*

Proof. Collect the estimates for the approximation and consistency error, respectively, and apply Lemma 5.7 and Theorem 5.6. ■

The triangular Morley element

In the previous subsection we saw that the finite element space V_h based on the Adini element belongs to $C(\overline{\Omega})$ but not to $C^1(\overline{\Omega})$. We will now show that even finite element spaces which do not belong to $C(\overline{\Omega})$ can be used to approximate the solution of the biharmonic equation.

Let $\Omega \subset \mathbb{R}^2$ be decomposed into triangles $K \in \mathcal{T}_h$. We denote the vertices and midpoints of edges of a single triangle K by a_i and b_i, $i = 0, 1, 2$, respectively, as indicated in Figure 5.6. For the local approximation space we choose $\mathcal{P}_K = P_2(K)$ with $\dim \mathcal{P}_K = 6$. We define six degrees of freedom by the nodal functionals

$$N_i(v) := v(a_i), \qquad N_{i+3}(v) := \frac{\partial v}{\partial n}(b_i), \quad i = 0, 1, 2.$$

Lemma 5.8. *The set of dof Σ_K in the Morley element is $\mathcal{P}_K$ unisolvent.*

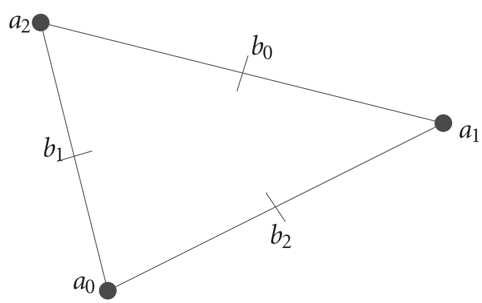

FIGURE 5.6 Degrees of freedom of the Morley triangle. Values of the function at the vertices and first normal derivatives at the midpoint of edges.

Proof. We show that $v \in \mathcal{P}_K$ with $N_i(v) = 0$, $i = 0, \dots, 5$, implies $v = 0$. As usual we use the convention that the indices are considered modulo 3, that is, $a_{i+3} = a_i$, $b_{i+3} = b_i$, for all i. The first derivative of $v \in P_2(K)$ belongs to $P_1(K)$, thus

$$\frac{\partial v}{\partial \tau}(b_i) = \frac{1}{|E_i|}\int_{E_i} \frac{\partial v}{\partial \tau}\,\mathrm{d}\gamma = \frac{1}{|E_i|}\left(v(a_{i+2}) - v(a_{i+1})\right) = 0,$$
$$\frac{\partial v}{\partial n}(b_i) = |E_i|\, N_i(v) = 0, \quad i = 0, 1, 2,$$

where τ denotes the tangent unit vector obtained by rotating the normal, n, counter clockwise. Since the first derivatives of v vanish at the three midpoints of edges and $Dv \in P_1^2(K)$ we conclude $Dv = 0$ on K which means $v = \text{const}$ on K, but v vanishes at the vertices which gives $v = 0$ on K. ∎

The local basis functions φ_j, defined by $N_i(\varphi_j) = \delta_{ij}$, are given on the reference triangle $\widehat{K}$ with the vertices $a_0 = (0,0)$, $a_1 = (1,0)$, $a_2 = (0,1)$ by

$$\varphi_0(x_1,x_2) = (1 - x_1 - x_2)^2 + x_1(1 - x_1) + x_2(1 - x_2),$$
$$\varphi_1(x_1,x_2) = x_1^2 - \frac{1}{2}(x_1 + x_2)(x_1 + x_2 - 1),$$
$$\varphi_2(x_1,x_2) = x_2^2 - \frac{1}{2}(x_1 + x_2)(x_1 + x_2 - 1),$$
$$\varphi_3(x_1,x_2) = \frac{1}{\sqrt{2}}(x_1 + x_2)(x_1 + x_2 - 1),$$
$$\varphi_4(x_1,x_2) = x_1(x_1 - 1),$$
$$\varphi_5(x_1,x_2) = x_2(x_2 - 1).$$

We construct a finite element space V_h based on the Morley element. A function $v_h \in V_h$ is defined locally by its dof in all vertices and midpoints of the edges of the decomposition of Ω into triangles, that is,

$$\begin{aligned} V_h := \{v_h : \Omega \to \mathbb{R} : {} & v_h|_K \in \mathcal{P}_K,\ v_h \text{ continuous at the inner vertices},\\ & \partial_n v_h \text{ continuous at the midpoint of inner edges},\ v_h(a_k) = 0,\\ & \partial_n v_h(b_k) = 0 \text{ for } a_k, b_k \in \Gamma\}. \end{aligned}$$

We see that $V_h \not\subset C(\overline{\Omega})$. Indeed, consider the local basis function φ_1 on the reference triangle $\widehat{K}$ along the edge $x_1 = 0$, we have

$$\varphi_1(0, x_2) = \frac{1}{2}x_2(1 - x_2)$$

but all nodal functional along this edge vanish,

$$N_0(\varphi_1) = N_2(\varphi_1) = N_4(\varphi_1) = 0$$

and allow $v = 0$ in the neighbouring cell.

Exercise 5.2. Consider the triangulation of Ω as indicated in Figure 5.7. Show that the function

$$v_h(x_1,x_2) := \begin{cases} x_1(1-x_1) - x_2(1-x_2) & \text{in } K_1 \\ x_1(1+x_1) + x_2(1-x_2) & \text{in } K_2 \\ -x_1(1+x_1) + x_2(1+x_2) & \text{in } K_3 \\ -x_1(1-x_1) - x_2(1+x_2) & \text{in } K_4 \end{cases}$$

belongs to V_h but is not continuous on Ω.

Although the space V_h of Morley elements does not belong to $C^1(\overline{\Omega})$ it satisfied a weakened continuity property, as shown in the next lemma.

Lemma 5.9. *The mean values along the edges of the first derivatives of a function in V_h are equal on both sides of an inner edge and equal to zero on a boundary edge.*

Proof. Let $E_i = [a_{i+1}, a_{i+2}]$ and τ be the tangential unit vector directed from a_{i+1} to a_{i+2}, $i = 0,1,2$, where the indices are counted modulo 3. Then, the mean value of the tangential derivative over the edge satisfies

$$\int_{E_i} \frac{\partial v_h}{\partial \tau}\, \mathrm{d}\gamma = v_h(a_{i+2}) - v_h(a_{i+1}), \quad i = 0,1,2.$$

The continuity of v_h at the vertices shows that the mean value along both sides of E_i of the tangential derivative are equal. In case that E_i is a boundary edge, the mean value vanishes due to $v_h(a_{i+1}) = v_h(a_{i+2}) = 0$. The midpoint rule is exact for polynomials of degree less than or equal to one, thus

$$\int_{E_i} \frac{\partial v_h}{\partial n}\, \mathrm{d}\gamma = |E_i| \frac{\partial v_h}{\partial n}(b_i) \quad \text{for all } v_h \in V_h,\ i = 0,1,2.$$

From the continuity of the normal derivatives at the midpoint of inner edges and its vanishing at boundary edges we obtain the stated result. ■

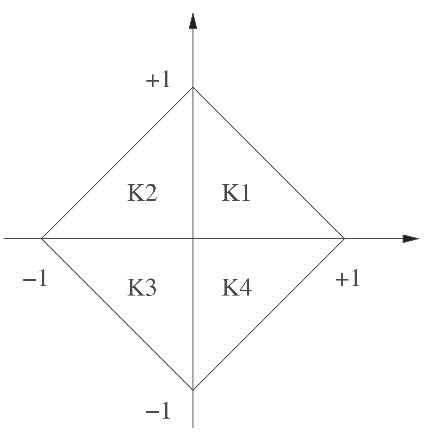

FIGURE 5.7 Triangulation of Ω for a finite element space generated by the Morley element.

Lemma 5.10. *On the space V_h of Morley elements $v_h \mapsto |v_h|_{2,h}$ is a norm.*

Proof. Since $v_h \mapsto |v_h|_{2,h}$ is a seminorm, we have only to show that $|v_h|_{2,h} = 0$ implies $v_h = 0$. Let $v_h \in V_h$ with $|v_h|_{2,h} = 0$. Then, the first order derivatives of v_h are constant on each cell $K \in \mathcal{T}_h$. Since the mean value of these derivatives are continuous at the inner cell edges and equal to zero along the boundary edges, the first order derivatives are equal to zero. This implies that v_h is constant on each cell $K \in \mathcal{T}_h$. Since v_h is continuous at the vertices of the cells and equal to zero at boundary vertices, we conclude $v_h = 0$. ■

Taking into consideration that for all $v \in \mathcal{P}_K$

$$\frac{\partial v}{\partial n}(b_i) = \frac{1}{|E_i|} \int_{E_i} \frac{\partial v}{\partial n} \, d\gamma \,, \quad i = 0, 1, 2,$$

we could replace the set of dof by

$$N_i^*(v) = v(a_i), \qquad N_{i+3}^*(v) = \frac{1}{|E_i|} \int_{E_i} \frac{\partial v}{\partial n} \, d\gamma \,, \quad i = 0, 1, 2.$$

Note that this replacement changes the domain of definition in the canonical interpolation Π_K defined by $N_i^*(\Pi_K v) = N_i^*(v)$, $i = 0, \ldots, 5$. The normal derivatives need not be continuous; it is enough if they are integrable over the edges. Unfortunately, both families of Morley triangles are not affine equivalent, however, we can replace the dof using the nodal functionals

$$\widetilde{N}_i(v) = v(a_i), \qquad \widetilde{N}_{i+3}(v) = \frac{1}{|E_i|} \int_{E_i} Dv\,(a_i - b_i) \, d\gamma \,, \quad i = 0, 1, 2$$

to get an affine equivalent family of elements. It turns out that the canonical interpolations based on N_i^* and $\widetilde{N}_i$, $i = 0, \ldots, 5$, are equal. Let us define the canonical local interpolations Π_K^* and $\widetilde{\Pi}_K$ by

$$N_i^*(\Pi_K^* v) = N_i^*(v), \qquad \widetilde{N}_i(\widetilde{\Pi}_K v) = \widetilde{N}_i(v), \quad i = 0, \ldots, 5.$$

We have

$$N_i^*(\widetilde{\Pi}_K v) = \widetilde{\Pi}_K v(a_i) = v(a_i) = \Pi_K^* v(a_i) = N_i^*(\Pi_K^* v), \quad i = 0, 1, 2.$$

From the representation

$$a_i - b_i = ((a_i - b_i) \cdot n)\, n + ((a_i - b_i) \cdot \tau)\, \tau$$

we get for $\alpha_i := (a_i - b_i) \cdot n$, $\beta_i := (a_i - b_i) \cdot \tau$, $i = 0, 1, 2$,

$$\begin{aligned} \frac{1}{|E_i|} \int_{E_i} Dv\,(a_i - b_i) \, d\gamma &= \frac{\alpha_i}{|E_i|} \int_{E_i} \frac{\partial v}{\partial n} \, d\gamma + \frac{\beta_i}{|E_i|} \int_{E_i} \frac{\partial v}{\partial \tau} \, d\gamma \\ &= \frac{\alpha_i}{|E_i|} \int_{E_i} \frac{\partial v}{\partial n} \, d\gamma + \frac{\beta_i}{|E_i|} \left(v(a_{i+2}) - v(a_{i+1}) \right). \end{aligned}$$

Since $(\widetilde{\Pi}_K v - v)(a_j) = 0$, we conclude

$$\alpha_i N_{i+3}^*(\widetilde{\Pi}_K v - v) = \widetilde{N}_{i+3}(\widetilde{\Pi}_K v - v) = 0.$$

Note that $\alpha_i \neq 0$, thus, we have shown

$$N_{i+3}^*(\widetilde{\Pi}_K v) = N_{i+3}^*(v) = N_{i+3}^*\left(\Pi_K^* v\right), \quad i = 0, 1, 2.$$

Now, $\widetilde{\Pi}_K v = \Pi_K^* v$ follows from the $\mathcal{P}_K$ unisolvence of the set of dof N_i^*, $i = 0, \ldots, 5$.

The interpolation operators $\widetilde{\Pi}_K = \Pi_K^*$ equal the identity on the subspace $\mathcal{P}_K = P_2(K)$. Thus, assuming $u \in H_0^2(\Omega) \cap H^3(\Omega)$ the approximation error is bounded by

$$\inf_{v_h \in V_h} \|u - v_h\|_{2,h} \leq C \left(\sum_{K \in \mathcal{T}_h} \|u - \widetilde{\Pi}_K u\|_{0,K}^2 \right)^{1/2} \leq Ch|u|_3.$$

Based on Theorem 5.6 it remains to estimate the consistency error

$$\sup_{w_h \in V_h} \frac{|a_h(u, w_h) - F(w_h)|}{\|w_h\|_{2,h}}.$$

In case of the Adini element we used Equation (5.7) for the numerator which holds true for $V_h \subset C(\overline{\Omega})$, however, in case of the Morley element we have $V_h \not\subset C(\overline{\Omega})$. Assuming $u \in H^4(\Omega)$ and $f \in L^2(\Omega)$ we obtain

$$F(w_h) = \int_\Omega f w_h \, \mathrm{d}x = \sum_{K \in \mathcal{T}_h} \left\{ \sigma \int_K \Delta\Delta u \, w_h \, \mathrm{d}x + (1 - \sigma) \int_K \Delta\Delta u \, w_h \, \mathrm{d}x \right\}.$$

Elementwise integration by parts yields

$$\int_K \Delta\Delta u \, w_h \, \mathrm{d}x = \int_K \Delta u \Delta w_h \, \mathrm{d}x + \int_{\partial K} \frac{\partial \Delta u}{\partial n_K} w_h \, \mathrm{d}\gamma - \int_{\partial K} \Delta u \frac{\partial w_h}{\partial n_K} \, \mathrm{d}\gamma,$$

$$\begin{aligned} \int_K \Delta\Delta u \, w_h \, \mathrm{d}x = & \int_K \sum_{i,j=1}^{2} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 w_h}{\partial x_i \partial x_j} \, \mathrm{d}x + \int_{\partial K} \frac{\partial \Delta u}{\partial n_K} w_h \, \mathrm{d}\gamma \\ & + \int_{\partial K} \left(\frac{\partial^2 u}{\partial \tau_K^2} - \Delta u \right) \frac{\partial w_h}{\partial n_K} \, \mathrm{d}\gamma - \int_{\partial K} \frac{\partial^2 u}{\partial n_K \partial \tau_K} \frac{\partial w_h}{\partial \tau_K} \, \mathrm{d}\gamma, \end{aligned}$$

where we used the notation n_K for the outer normal along ∂K and τ_K for the (counter clockwise rotated n_K) unit vector tangent to ∂K, respectively. Multiply the expressions by σ and $(1 - \sigma)$, and take into consideration the definition of the discrete bilinear form a_h given in (5.10), we obtain

$$F(w_h) = a_h(u, w_h) + R_1(u, w_h) + R_2(u, w_h) + R_3(u, w_h)$$

with

$$R_1(u, w_h) = \sum_{K \in \mathcal{T}_h} \int_{\partial K} \left((1-\sigma) \frac{\partial^2 u}{\partial \tau_K^2} - \Delta u \right) \frac{\partial w_h}{\partial n_K} \, d\gamma \,,$$

$$R_2(u, w_h) = -\sum_{K \in \mathcal{T}_h} \int_{\partial K} (1-\sigma) \frac{\partial^2 u}{\partial n_K \partial \tau_K} \frac{\partial w_h}{\partial \tau_K} \, d\gamma \,,$$

$$R_3(u, w_h) = \sum_{K \in \mathcal{T}_h} \int_{\partial K} \frac{\partial \Delta u}{\partial n_K} w_h \, d\gamma \,.$$

Before estimating R_i, $i = 1, 2, 3$, let us mention that $R_2(u, w_h) = R_3(u, w_h) = 0$ for $w_h \in C(\overline{\Omega})$ and $u \in H^4(\Omega)$. Indeed, for any boundary edge $E \subset \Gamma$ we have $w_h = 0$ from which $\partial w_h / \partial \tau_K = 0$ follows. Further, for an inner edge $E = \partial K \cap \partial K'$ the contribution associated with the integral over E appears twice with opposite sign since on E the outer normal of K equals the inner normal of K'.

In order to estimate R_1 we first use the property of the finite element space generated by the Morley element that the mean values along edges of the first derivatives of functions of V_h are equal on both sides of an inner edge and equal to zero on a boundary edge (see Lemma 5.9). Therefore, R_1 can be represented as

$$R_1(u, w_h) = \sum_{K \in \mathcal{T}_h} \int_{\partial K} \left((1-\sigma) \frac{\partial^2 u}{\partial \tau_K^2} - \Delta u \right) \left(\frac{\partial w_h}{\partial n_K} - \Pi_0 \frac{\partial w_h}{\partial n_K} \right) d\gamma \,,$$

where $\Pi_0(\partial w_h / \partial n_K)$ denotes the mean value of $\partial w_h / \partial n_K$ over the edge E which is well defined in Lemma 5.9. We mention that the continuous bilinear form $D : H^1(K) \times H^1(K) \to \mathbb{R}$ given by

$$D(\varphi, \psi) := \int_E \varphi (\psi - \Pi_0 \psi) \, d\gamma$$

has the properties

$$D(\varphi, q) = 0 \quad \text{for all } \varphi \in H^1(K),\, q \in P_0(K),$$
$$D(p, \psi) = 0 \quad \text{for all } p \in P_0(K),\, \psi \in H^1(K).$$

Using this observation on a reference cell, applying the generalization of Bramble–Hilbert lemma and summing up the local estimates, we can show

$$|R_1(u, w_h)| \le Ch |u|_3 \, \|w_h\|_{2,h}, \quad u \in H^3(\Omega),\, w_h \in V_h,$$

for details we refer to Lascaux and Lesaint (1975). The same inequality can be derived for $R_2(u, w_h)$. In order to estimate $R_3(u, w_h)$ we introduce the continuous piecewise linear interpolation Π_1 which is well defined on $H^2(\Omega) + V_h$. Since $\Pi_1 w_h$ is continuous over the inner edges and zero along the boundary edges, we can write

$$R_3(u, w_h) = \sum_{K \in \mathcal{T}_h} \int_{\partial K} \frac{\partial \Delta u}{\partial n_K} (w_h - \Pi_1 w_h) \, d\gamma \,, \quad u \in H^4(\Omega),\, w_h \in V_h.$$

Standard local interpolation estimates lead to

$$|R_3(u,w_h)| \le Ch(|u|_3 + h|u|_4)\,\|w_h\|_{2,h}, \quad u \in H^4(\Omega),\, w_h \in V_h.$$

Theorem 5.8. *Let the solution u of problem* (5.2) *belong to $H_0^2(\Omega) \cap H^4(\Omega)$ and u_h denote the solution of the discrete problem* (5.11) *with the Morley finite element space. Then,*

$$\|u - u_h\|_{2,h} \le Ch(|u|_3 + h|u|_4).$$

Proof. Combine the approximation error estimate with the estimates of the consistency error and apply Theorem 5.6. ∎

A nonconforming tetrahedral element

In the following subsection we describe a nonconforming tetrahedral element proposed by Ming and Xu (2007). Let $\Omega \subset \mathbb{R}^3$ be decomposed into tetrahedrons $K \in \mathcal{T}_h$. We denote the vertices and centroids of faces of a single tetrahedron K by a_i and b_i, $i = 0, \ldots, 3$, respectively. We choose the local approximation space $\mathcal{P}_K = P_3(K)$ with $\dim \mathcal{P}_K = 20$. We define 20 dof by the nodal functionals (see Figure 5.8)

$$N_i^\alpha(v) := D^\alpha v(a_i), \qquad N_i(v) := \frac{\partial v}{\partial n}(b_i) \quad 0 \le |\alpha| \le 1,\, i = 0, \ldots, 3.$$

Remark 5.7. Note that the three, first derivative dof at the vertex a_i, $i = 0, \ldots, 3$ can be replaced by the first derivatives in the direction of the edges $[a_j, a_i]$, $j \ne i$, that is, $Dv(a_i)(a_j - a_i)$, without changing the finite element space. Nevertheless, the associated family of elements is not affine equivalent due to the normal derivative dof.

Lemma 5.11. *The set of dof Σ_K in the Ming–Xu element is $\mathcal{P}_K$ unisolvent.*

Proof. The basis functions φ_i, ψ_i, $i = 0, \ldots, 3$ and φ_{ij}, $0 \le i \ne j \le 3$, are explicitly given in Ming and Xu (2007) for which the canonical interpolation $\Pi_K : C^1(K) \to \mathcal{P}_K$ reads

$$\Pi_K v := \sum_{i=0}^{3} v(a_i)\,\varphi_i + \sum_{i=0}^{3} \frac{\partial v}{\partial n}(b_i)\,\psi_i + \sum_{0 \le i \ne j \le 3} Dv(a_i)(a_j - a_i)\,\varphi_{ij}.$$

∎

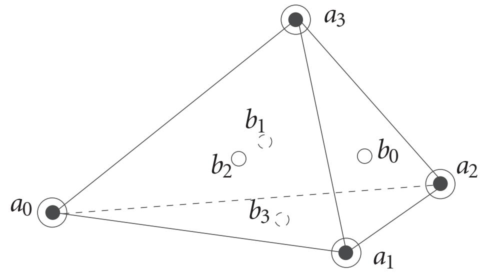

FIGURE 5.8 Degrees of freedom of the nonconforming tetrahedral element. Values of the function, its first derivatives at the vertices, and its normal derivatives at the barycenters of the faces.

Remark 5.8. A reduced element with 16 dof can be constructed by removing the normal derivative dof. Let f be a face associated with the three vertices a_i, a_j, a_k and the outer normal, n. Then, the normal derivative of dof at the centroid $a_{ijk} = (a_i + a_j + a_k)/3$ is replaced by

$$\frac{\partial v}{\partial n}(a_{ijk}) = \frac{1}{3} \sum_{l=i,j,k} \frac{\partial v}{\partial n}(a_l)$$

leading to the local approximation space $\mathcal{P}_K^{\text{red}}$ of dimension $\dim \mathcal{P}_K^{\text{red}} = 16$. Note that $P_2(K) \subset \mathcal{P}_K^{\text{red}}$.

We construct finite element spaces V_h and V_h^{red} based on the nonconforming tetrahedral and the reduced element, respectively. A function $v_h \in V_h$ is defined locally by its dof at all vertices and centroids of the faces of the decomposition of Ω into tetrahedrons, that is,

$$\begin{aligned} V_h := \{v_h : \Omega \to \mathbb{R} : \; & v_h|_K \in \mathcal{P}_K,\, D^\alpha v_h \text{ continuous at the inner vertices,} \\ & \partial_n v_h \text{ continuous at the centroids of inner faces, } D^\alpha v_h(a_k) = 0, \\ & \partial_n v_h(b_k) = 0 \text{ for } a_k,\, b_k \in \Gamma, 0 \le |\alpha| \le 1\}. \end{aligned}$$

Similarly the reduced finite element space is defined as

$$\begin{aligned} V_h^{\text{red}} := \{v_h : \Omega \to \mathbb{R} : \; & v_h|_K \in \mathcal{P}_K^{\text{red}},\, D^\alpha v_h \text{ continuous at the inner vertices,} \\ & D^\alpha v_h(a_k) = 0, \text{ for } a_k \in \Gamma, 0 \le |\alpha| \le 1\}. \end{aligned}$$

As in the Morley triangle we have the following result.

Lemma 5.12. *The mean values over faces of the first derivatives of a function of $v_h \in V_h$ are equal on both sides of an inner face and equal to zero on a boundary face. Moreover $v_h \mapsto |v_h|_{2,h}$ is a norm on V_h.*

Proof. For the first statement see Lemma 3.1 of Ming and Xu (2007). Since $v_h \mapsto |v_h|_{2,h}$ is a seminorm, we have to show that $|v_h|_{2,h} = 0$ implies $v_h = 0$. The arguments are same as in the proof of Lemma 5.10. ■

Finally, a careful study of the approximation and consistency error leads to the following theorem.

Theorem 5.9. *Let the solution u of problem* (5.2) *belong to $H_0^2(\Omega) \cap H^4(\Omega)$ and u_h denote the solution of the discrete problem* (5.11) *with the nonconforming tetrahedral or the reduced finite element space. Then,*

$$\|u - u_h\|_{2,h} \le Ch(|u|_3 + h|u|_4).$$