Computational Methods in Applied Mathematics
Vol. 12 (2012), No. 4, pp. 486–499

Doi: 10.2478/cmam-2012-0029

DE GRUYTER

A Simplified Calculation of Reduced HCT–Basis Functions in a Finite Element Context

Arnd Meyer

In memory of Sergei Nepomnyaschikh - a pioneer in domain decomposition methods

Abstract — We present simple formulas for the definition of the basis functions on a reduced Hsieh-Clough-Tocher (rHCT) element. These formulas use the P_3-biorthogonal basis in the master triangle and form the resulting basis with the help of the edge vectors of the triangle only. This allows for a simple and efficient algorithm to compute the stiffness matrices.

2010 Mathematical subject classification: 65N30; 74S05; 74K20.

Keywords: finite elements; higher order elements; stable calculations .

1. Introduction

The conforming finite element approximation of the Kirchhoff-plate equation or of similar biharmonic equations leads to very expansive calculations caused by complicated polynomials for the C^1 continuity of the ansatz functions.

The simplest of those finite element functions are the reduced HCT elements [3] with 3 degrees of freedom at each node, namely the values of the finite element function and its derivatives. A speciality of this element is the definition of its shape functions as piecewise cubic polynomials in a partition of the given triangle into 3 subtriangles. The calculation of element matrices with these functions has to respect this split, e.g., in the Gaussian integration process over each subtriangle. This paper proposes a certain representation of the HCT basis functions which allows for a highly efficient implementation.

There are different approaches in the literature to define these ansatz functions. Mainly, the barycentric coordinates within the given triangle or within the subtriangles are used, (compare [1],[4],[6]) and the functions are based on Bernstein-Bezier polynomials. Our basis functions on the master triangle fulfill the bi-orthogonality to the pointwise functionals

$$u(\hat{a}_i) \qquad \text{and} \qquad \frac{\partial}{\partial \hat{x}_1} u(\hat{a}_i), \quad \frac{\partial}{\partial \hat{x}_2} u(\hat{a}_i)$$

for the corner nodes $\hat{a}_i, i = 1, \ldots, 3$.

We present a cheap algorithm that defines the HCT-basis functions in three steps from the basis functions in the master element. We consider the split of the triangle $\mathcal{T}$ under

Arnd Meyer
Department of Mathematics, TU Chemnitz, 09107 Chemnitz, Germany
E-mail: a.meyer@mathematik.tu-chemnitz.de.

consideration from the definition of the center of gravity

$$a_0 = \frac{1}{3}(a_1 + a_2 + a_3)$$

and connecting each vertex a_i with a_0. This definition simplifies slightly the formulas in the following, but it is by no means necessary. As in [6] it could be generalized to an arbitrary internal point $a_0 \in \mathcal{T}$ without much higher effort of the calculation.

2. rHCT Element – Basic Definitions

We have to construct nine basis functions fulfilling the properties 1 to 3 as indicated later. Throughout this paper we use triples of functions written as a row vector i.e.,

$$\Psi_j(x) = (\psi_j^{(0)}(x),\ \psi_j^{(1)}(x),\ \psi_j^{(2)}(x))$$

for $x \in \mathcal{T}$ which "belong" to the corner node a_j of the given triangle $\mathcal{T}$. Hence, the full rHCT-basis will be obtained as

$$\Psi = (\Psi_1 \vdots \Psi_2 \vdots \Psi_3).$$

This description as row vectors of basis functions allows a simple use of matrix notation for linear combinations (or change of basis) by matrix-multiply from right. For instance

$$\Phi_j M_j \text{ with a } (3 \times 3)\text{-matrix } M_j$$

would transform the 3 functions in Φ_j to 3 new functions.

In the same way linear operators (such as gradient operator) are applied from left, so with $\nabla = (\partial/\partial x_1,\ \partial/\partial x_2)^T$

$$\nabla\Phi_j = (\nabla\varphi_j^{(0)},\ \nabla\varphi_j^{(1)},\ \nabla\varphi_j^{(2)})$$

is a (2×3)-matrix of functions (derivatives).

Let $\mathcal{T}$ be the triangle of consideration of a given triangulation of a 2D-domain, having the 3 corner nodes a_j. By definition of

$$a_0 = \frac{1}{3}(a_1 + a_2 + a_3)$$

we subdivide $\mathcal{T}$ into three subtriangles $\mathcal{T}_k$.

We need the outer edge vectors

$$\begin{aligned} &\boldsymbol{E}_k \in \mathbb{R}^2,\ k = 1 \dots 3 \quad \text{as} \\ &\boldsymbol{E}_1 = a_3 - a_2, \quad \boldsymbol{E}_2 = a_1 - a_3, \quad \boldsymbol{E}_3 = a_2 - a_1, \end{aligned}$$

So, throughout this paper all indices run between 1 and 3 and $k \pm 1$ is meant modulo 3.

$$\boldsymbol{E}_k = a_{k-1} - a_{k+1}, \quad k = 1 \dots 3$$

We use k as an index of one subtriangle or referring to one of the 3 outer edges $\boldsymbol{E}_k$. In opposite to that we will use j as an index referring to a corner node a_j resp. to one of the basis functions associated with a_j. The subtriangle $\mathcal{T}_k$ has $\boldsymbol{E}_k$ as one of its edges opposite to a_0 and the other two edges are $\boldsymbol{f}_{k-1}$ and $\boldsymbol{f}_{k+1}$ with

$$\boldsymbol{f}_j = a_j - a_0,$$

hence,

$$\boldsymbol{E}_k = \boldsymbol{f}_{k-1} - \boldsymbol{f}_{k+1}.$$

Note that a_0 could be any interior point in $\mathcal{T}$, as considered in [6]. The definition of a_0 as center of gravity simplifies only the values of the Jacobian determinants of $\mathcal{T}_k$, which are $\mu = \frac{1}{6}\mathrm{meas}\mathcal{T}$ in our case, but $\mu = \det J_k$ in general.

Let $\mathcal{T}_k$ be mapped onto the unit triangle

$$\hat{\mathcal{T}} = \{\hat{x} = (\hat{x}_1, \hat{x}_2)^T : \hat{x}_i \in [0,1],\ \hat{x}_1 + \hat{x}_2 \leqslant 1\}$$

by

$$x = J_k \hat{x} + a_0 \quad \in \mathcal{T}_k.$$

Note, that a_0 is mapped to $\hat{x} = (0,0)$, so $\boldsymbol{f}_{k+1}$ is mapped to the $\hat{x}_1-$axis and $\boldsymbol{f}_{k-1}$ to the $\hat{x}_2-$axis, resp. Therefore, the (2×2)-Jacobian matrix is

$$J_k = (\boldsymbol{f}_{k+1} \vdots \boldsymbol{f}_{k-1}).$$

Furthermore, the vectors $\boldsymbol{N}_k$ and $\boldsymbol{n}_j$ orthogonal to $\boldsymbol{E}_k$ and $\boldsymbol{f}_j$ are defined as

$$\boldsymbol{N}_k = R\ \boldsymbol{E}_k, \quad \boldsymbol{n}_j = R\ \boldsymbol{f}_j$$

with the rotation matrix

$$R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Obviously, the matrix J_k^{-T} has the columns

$$J_k^{-T} = \frac{1}{\mu}\left(-\boldsymbol{n}_{k-1} \vdots \boldsymbol{n}_{k+1}\right)$$

with $\mu = \det J_k = \frac{1}{6}\mathrm{meas}\mathcal{T}$.

3. The Desired Basis Functions

We will construct the rHCT-basis functions in $\mathcal{T}$ as piecewise cubic polynomials fulfilling the following propositions:

Proposition 1: The function triples Ψ_j are cubic polynomials in each subtriangle $\mathcal{T}_k$, are continuous functions within $\mathcal{T}$ and fulfill

$$\Psi_j(a_i) = (1,0,0) \cdot \delta_{ij}$$
$$\nabla\Psi_j(a_i) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \delta_{ij} \quad \forall i,j = 1 \ldots 3$$

Proposition 2: Along the outer edges $\boldsymbol{E}_k$ the normal derivatives of all functions are linear w. r. t. the local line coordinate.

Proposition 3: The functions fulfill C^1-continuity inside $\mathcal{T}$, i.e., there are no jumps in normal derivatives along internal edges $\boldsymbol{f}_j$.

4. The Construction of these rHCT-Basis Functions Based on Master Functions

We start with the space $P_3(\hat{\mathcal{T}})$ spanned by special basis functions with properties similar to Proposition 1:

$$\begin{aligned}
\hat{\Phi}_0(\hat{x}) &= (1-\hat{x}_1-\hat{x}_2)^2 \cdot (\ 1+2\hat{x}_1+2\hat{x}_2 \ \vdots \ \hat{x}_1 \ \vdots \ \hat{x}_2 \ ) \\
\hat{\Phi}_1(\hat{x}) &= (\ \hat{x}_1^2(3-2\hat{x}_1) \ \vdots \ \hat{x}_1^2(\hat{x}_1-1) \ \vdots \ \hat{x}_1^2\hat{x}_2 \ ) \\
\hat{\Phi}_2(\hat{x}) &= (\ \hat{x}_2^2(3-2\hat{x}_2) \ \vdots \ \hat{x}_2^2\hat{x}_1 \ \vdots \ \hat{x}_2^2(\hat{x}_2-1) \ ) \\
\hat{\beta}(\hat{x}) &= \hat{x}_1\hat{x}_2(1-\hat{x}_1-\hat{x}_2).
\end{aligned}$$

Obviously, these 10 functions span the space $P_3(\hat{\mathcal{T}})$.

We will denote with

$$\hat{\nabla} = (\frac{\partial}{\partial \hat{x}_1}, \frac{\partial}{\partial \hat{x}_2})^T$$

the formal derivative w. r. t. $\hat{x}$ and with ∇ the true gradient operator

$$\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})^T.$$

Then the basis Ψ is formed in two steps after the transformation of these master functions onto each subtriangle $\mathcal{T}_k$.

Step 1: With these functions of $\hat{x}$ and the map $x = J_k\hat{x} + a_0$, we define an initial basis Φ_j^{init} $(j = 1, \ldots, 3)$ and auxiliary functions Φ_0 within each subtriangle $\mathcal{T}_k$ as:

$$\begin{aligned}
\Phi_{k+1}^{\text{init}}(x) &= \hat{\Phi}_1(\hat{x})\, H_k \\
\Phi_{k-1}^{\text{init}}(x) &= \hat{\Phi}_2(\hat{x})\, H_k \\
\Phi_0(x) &= \hat{\Phi}_0(\hat{x})\, H_k
\end{aligned}$$

and $\beta_k(x) = \hat{\beta}(\hat{x})$ (the cubic bubble of $\mathcal{T}_k$). Here, H_k is a (3×3)-matrix with

$$H_k = \left(\begin{array}{ccc} 1 & \vdots & 0 \ \ 0 \\ \cdots & \cdots & \cdots\cdots \\ 0 & \vdots & \\ 0 & & J_k^T \end{array} \right).$$

This is done for each $k = 1, .., 3$ which defines the functions Φ_j^{init} (with support in 2 triangles $\mathcal{T}_{j-1}, \ \mathcal{T}_{j+1}$) and Φ_0 (with support in whole $\mathcal{T}$).

A simple calculation proves Proposition 1 for Φ_j^{init}. For instance in $\mathcal{T}_k$:

$$\begin{aligned}
\nabla\Phi_{k+1}^{\text{init}}(a_{k+1}) &= J_k^{-T}\left[\hat{\nabla}\hat{\Phi}_1\begin{pmatrix}1\\0\end{pmatrix}\right] H_k \\
&= J_k^{-T}\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} H_k \\
&= \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}
\end{aligned}$$

Step 2: We have to add multiples of the bubble functions in each triangle $\mathcal{T}_k$ to fulfill Proposition 2:

With special chosen vectors $b_k^j \in \mathbb{R}^3$, let

$$\Phi_j = \Phi_j^{\text{init}} + \beta_k(x) \quad (b_k^j)^T$$

within $\mathcal{T}_k$ for $j = k+1$ and $j = k-1$. Note that each of the 3 functions in Φ_j^{init} is corrected by a special multiple of the one bubble function $\beta_k(x)$. The 3 factors are the entries of (b_k^j). The vectors (b_k^{k+1}) and (b_k^{k-1}) are calculated from

$$\frac{\partial}{\partial \boldsymbol{N}_k}\Phi_j = \frac{\partial}{\partial \boldsymbol{N}_k}\Phi_j^{\text{init}} + (\frac{\partial}{\partial \boldsymbol{N}_k}\beta_k) \ (b_k^j)^T$$

to be linear along $\boldsymbol{E}_k$:

$$\begin{aligned}\frac{\partial}{\partial \boldsymbol{N}_k}\beta_k &= \boldsymbol{N}_k^T J_k^{-T}(\hat{\nabla}\hat{\beta}) \\ &= \boldsymbol{N}_k^T J_k^{-T}\begin{pmatrix} -1 \\ -1 \end{pmatrix}(\hat{s}(1-\hat{s}))\end{aligned}$$

(Note that $\boldsymbol{E}_k$ is mapped to $\begin{pmatrix} \hat{s} \\ 1-\hat{s} \end{pmatrix} \in \hat{\mathcal{T}}, \ \hat{s} \in [0,1]$).

So,

$$\begin{aligned}\frac{\partial}{\partial \boldsymbol{N}_k}\beta_k &= -\hat{s}(1-\hat{s}) \quad \boldsymbol{N}_k^T \frac{1}{\mu}(\boldsymbol{n}_{k+1} - \boldsymbol{n}_{k-1}) \\ &= \hat{s}(1-\hat{s})\frac{1}{\mu}|\boldsymbol{E}_k|^2\end{aligned}$$

On the other hand, for $j = k+1$, we have

$$\frac{\partial}{\partial \boldsymbol{N}_k}\Phi_{k+1}^{\text{init}} = \boldsymbol{N}_k^T J_k^{-T}(\hat{\nabla}\hat{\Phi}_1)H_k$$

The matrix $\hat{\nabla}\hat{\Phi}_1$ arises as

$$\hat{s}(1-\hat{s})\begin{pmatrix} 6 & -3 & 2 \\ 0 & 0 & -1 \end{pmatrix} + (1-\hat{s})\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

hence, we obtain the desired linear behavior (w. r. t. $\hat{s}$) of the functions Φ_{k+1} for the choice

$$\begin{aligned}(b_k^{k+1})^T &= -\boldsymbol{N}_k^T J_k^{-T}\begin{pmatrix} 6 & -3 & 2 \\ 0 & 0 & -1 \end{pmatrix} H_k \cdot \frac{\mu}{|\boldsymbol{E}_k|^2} \\ &= (6\boldsymbol{E}_k^T \boldsymbol{f}_{k-1} \ \vdots \ 3\mu \boldsymbol{N}_k^T + 2|\boldsymbol{E}_k|^2 \boldsymbol{f}_{k-1}^T) \ / \ |\boldsymbol{E}_k|^2. \qquad (4.1)\end{aligned}$$

Analogously for $j = k-1$ we obtain

$$(b_k^{k-1})^T = (-6\boldsymbol{E}_k^T \boldsymbol{f}_{k+1} \ \vdots \ 3\mu \boldsymbol{N}_k^T + 2|\boldsymbol{E}_k|^2 \boldsymbol{f}_{k+1}^T) \ / \ |\boldsymbol{E}_k|^2. \qquad (4.2)$$

Note that the bubble functions β_k are identical zero along $\boldsymbol{f}_j$, so they cannot destroy the continuity of Φ_j^{init} inside $\mathcal{T}$. Proposition 2 guarantees the C^1-continuity of functions which are spanned by Φ_j in $\mathcal{T}$ along the edges $\boldsymbol{E}_k$ to the neighboring element.

The auxiliary functions Φ_0 inside $\mathcal{T}$ vanish together with their gradients completely along $\boldsymbol{E}_k$. So the resulting Step 3 cannot destroy these Proposition 2 when linear combinations of Φ_0 are added to Φ_j to obtain the desired HCT-basis Ψ_j.

Step 3: Let $\quad \Psi_j = \Phi_j + \Phi_0 \; M_j \quad$ with (3×3)-matrices M_j

such that the jumps of $\frac{\partial}{\partial \boldsymbol{n}_i}\Psi_j$ vanish along all $\boldsymbol{f}_i$. Note that Φ_j as well as Φ_0 have jumping normal derivatives along all three interior edges $\boldsymbol{f}_i$. The nine values in M_j can be chosen such that all these jumps disappear:

Here, it is simply seen that all jumps of $\frac{\partial}{\partial \boldsymbol{n}_i}\Phi_j$ and of $\frac{\partial}{\partial \boldsymbol{n}_i}\Phi_0$ are quadratic functions

$$\hat{s}(1-\hat{s}) \; (t_i^{(j)})^T$$

or

$$\hat{s}(1-\hat{s}) \; s_i^T$$

resp., w. r. t. the local coordinate $\hat{s}$ representing $\boldsymbol{f}_i$. From this observation, we define the matrices T_j having the rows $(t_i^{(j)})^T$ and the matrix S with rows s_i^T. Then

$$M_j = -S^{-1} T_j$$

are the desired (3×3)-matrices yielding the correct C^1-basis functions Ψ_j.

To calculate the jumps of $\frac{\partial}{\partial \boldsymbol{n}_i}\Phi_j$ we start with $i = j+1$. The edge $\boldsymbol{f}_{j+1}$ separates the two subtriangles $\mathcal{T}_j$ and $\mathcal{T}_{j-1}$. In $\mathcal{T}_j$ the functions Φ_j vanish completely and

$$\nabla \Phi_j^{\text{init}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

along $\boldsymbol{f}_{j+1}$ (from $\mathcal{T}_{j-1}$ side). So, the jump in $\frac{\partial}{\partial \boldsymbol{n}_{j+1}}\Phi_j$ stems from the bubble function only:

$$\text{jump}_{|\boldsymbol{f}_{j+1}} \left[\frac{\partial}{\partial \boldsymbol{n}_{j+1}} \Phi_j \right] = -\boldsymbol{n}_{j+1}^T J_{j-1}^{-T} \left[\hat{\nabla} \hat{\beta} \begin{pmatrix} 0 \\ \hat{s} \end{pmatrix} \right] \cdot (b_{j-1}^j)^T$$

We have used the minus sign, because $\boldsymbol{f}_{j+1}$ is mapped to the $\hat{x}_2-$axis $(0, \hat{s}) \in \hat{\mathcal{T}}$ in $\mathcal{T}_{j-1}$ (later we use the plus sign when an edge is mapped onto the $\hat{x}_1-$axis).

From

$$\left[\hat{\nabla} \hat{\beta} \begin{pmatrix} 0 \\ \hat{s} \end{pmatrix} \right] = \hat{s}(1-\hat{s}) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

we obtain

$$\left(t_{j+1}^{(j)} \right)^T = \frac{|\boldsymbol{f}_{j+1}|^2}{\mu} (b_{j-1}^j)^T$$

with the vectors b_j^i defined in step 2. Analogously, for $i = j-1$ we have

$$\left(t_{j-1}^{(j)} \right)^T = \frac{|\boldsymbol{f}_{j-1}|^2}{\mu} (b_{j+1}^j)^T$$

from the same calculation within $\mathcal{T}_{j+1}$. The case $i = j$ requires a longer treatment. Here,

$$\mathrm{jump}_{|\boldsymbol{f}_j}\left[\frac{\partial}{\partial \boldsymbol{n}_j}\Phi_j\right] = \boldsymbol{n}_j^T\left(J_{j-1}^{-T}\left[\hat{\nabla}\hat{\Phi}_1\begin{pmatrix}\hat{s}\\0\end{pmatrix}\right]H_{j-1} - J_{j+1}^{-T}\left[\hat{\nabla}\hat{\Phi}_2\begin{pmatrix}0\\\hat{s}\end{pmatrix}\right]H_{j+1}\right)$$
$$+\boldsymbol{n}_j^T\left(J_{j-1}^{-T}\left[\hat{\nabla}\hat{\beta}\begin{pmatrix}\hat{s}\\0\end{pmatrix}\right](b_{j-1}^j)^T - J_{j+1}^{-T}\left[\hat{\nabla}\hat{\beta}\begin{pmatrix}0\\\hat{s}\end{pmatrix}\right](b_{j+1}^j)^T\right).$$

The difference of the bubble part yields

$$\hat{s}(1-\hat{s})\frac{|\boldsymbol{f}_j|^2}{\mu}(b_{j-1}^j + b_{j+1}^j)^T$$

and the part from Φ_j^{init} yields

$$\hat{s}(1-\hat{s})\boldsymbol{n}_j^T\left(J_{j-1}^{-T}\begin{pmatrix}6 & -3 & 0\\ 0 & 0 & -1\end{pmatrix}H_{j-1} - J_{j+1}^{-T}\begin{pmatrix}6 & -1 & 0\\ 0 & 0 & -3\end{pmatrix}H_{j+1}\right),$$

while a linear in $\hat{s}$ part vanishes. This leads to the end result

$$(t_j^{(j)})^T = \frac{|\boldsymbol{f}_j|^2}{\mu}(6 \,\vdots\, -2\boldsymbol{f}_j^T) + \frac{|\boldsymbol{f}_j|^2}{\mu}(b_{j-1}^j + b_{j+1}^j)^T.$$

It remains to continue these calculations for $\frac{\partial}{\partial \boldsymbol{n}_j}\Phi_0$ (analogously to $\frac{\partial}{\partial \boldsymbol{n}_j}\Phi_j$)

$$\mathrm{jump}_{|\boldsymbol{f}_j}\left[\frac{\partial}{\partial \boldsymbol{n}_j}\Phi_0\right] = \boldsymbol{n}_j^T\left(J_{j-1}^{-T}\left[\hat{\nabla}\Phi_0\begin{pmatrix}\hat{s}\\0\end{pmatrix}\right]H_{j-1} - J_{j+1}^{-T}\left[\hat{\nabla}\hat{\Phi}_0\begin{pmatrix}0\\\hat{s}\end{pmatrix}\right]H_{j+1}\right).$$

Again some linear in $\hat{s}$ parts vanish and

$$s_j^T = \frac{|\boldsymbol{f}_j|^2}{\mu}(-18 \,\vdots\, -6\boldsymbol{f}_j^T).$$

Note that all i-th rows of T_j and of S are multiplied with $|\boldsymbol{f}_i|^2/\mu$ which will cancel out in the linear system

$$M_j = (-S)^{-1}T_j, \tag{4.3}$$

so we renew the definition of the rows in T_j and S to:

$$(-S) = 6\cdot\begin{pmatrix}3 & \vdots & \boldsymbol{f}_1^T\\ 3 & \vdots & \boldsymbol{f}_2^T\\ 3 & \vdots & \boldsymbol{f}_3^T\end{pmatrix} \tag{4.4}$$

and T_j has row i as

$$(t_i^{(j)})^T = \begin{cases}(b_{j+1}^j)^T & i = j-1\\ (b_{j-1}^j)^T & i = j+1\\ (b_{j-1}^j + b_{j+1}^j)^T + (6 \,\vdots\, -2\boldsymbol{f}_j^T) & i = j.\end{cases} \tag{4.5}$$

All these entries of T_j and S are simply the vectors b_j^i defined above or the internal edge vectors. Nearly no additional operations are required.

These 3 steps are to consider as a preparation for each $\mathcal{T}$. Then the integration routine runs first over the 3 subtriangles $\mathcal{T}_k$, next over all Gaussian points $\hat{g} \in \hat{\mathcal{T}}$ and will form the values of Ψ_j or $\nabla\Psi_j$ or $\nabla^2\Psi_j$ at $\hat{g}$ following

$$\Psi_k = \hat{\Phi}_0(\hat{g})\ H_k\ M_k \tag{4.6}$$

$$\Psi_{k+1} = \hat{\Phi}_1(\hat{g})\ H_k + \hat{\beta}(\hat{g})\ (b_k^{k+1})^T + \hat{\Phi}_0(\hat{g})\ H_k\ M_{k+1} \tag{4.7}$$

$$\Psi_{k-1} = \hat{\Phi}_2(\hat{g})\ H_k + \hat{\beta}(\hat{g})\ (b_k^{k-1})^T + \hat{\Phi}_0(\hat{g})\ H_k\ M_{k-1} \tag{4.8}$$

with the predefined vectors b_k^j and matrices M_j which are now constant over all Gaussian points and are defined following (4.1) until (4.5). The resulting algorithm for deriving the element stiffness matrix is considered in the following Section on the example of the Kirchhoff plate equation.

5. The Calculation of the Stiffness Matrix

We consider the Kirchhoff plate equation, written in the following form. For a scalar function $w(x)$ let $\nabla^2 w$ denote the 3-vector of second derivatives

$$\nabla^2 w = (\frac{\partial^2 w}{\partial x_1^2}\ ,\ \frac{\partial^2 w}{\partial x_2^2}\ ,\ \frac{\partial^2 w}{\partial x_1 x_2}\ )^T,$$

then the weak form of the plate equation is:

$$\begin{aligned}&\text{find } w \in \mathbb{H}_0^2(\Omega) \text{ with}\\ &\int_\Omega (\nabla^2 v)^T C(\nabla^2 w)\ d\Omega = \int_\Omega f \cdot v\ d\Omega \quad \forall v \in \mathbb{H}_0^2(\Omega).\end{aligned}$$

Here, the material matrix $C \in \mathbb{R}^{(3\times 3)}$ (for the simplest case of isotropic material) is

$$C = \frac{E}{1-\nu^2}\begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 2(1-\nu) \end{pmatrix}$$

and the function f denotes the force distribution orthogonal to the Ω–plane.

Using the shape functions $\Psi = (\Psi_1 \vdots \Psi_2 \vdots \Psi_3)$ of one element $\mathcal{T}$ as defined above we have to calculate the element stiffness matrix from the Gaussian integration process over $\mathcal{T}$ as

$$\begin{aligned} K_\mathcal{T} &= \int_\mathcal{T} (\nabla^2\Psi)^T C(\nabla^2\Psi)\ d\Omega \\ &= \sum_{k=1}^{3} \int_{\mathcal{T}_k} (\nabla^2\Psi)^T C(\nabla^2\Psi)\ d\Omega \end{aligned}$$

Hence, for each $k = 1 \ldots 3$ we have to calculate the (3×9)–matrix

$$D = \nabla^2\Psi(\hat{g})$$

for each Gaussian integration point $\hat{g} \in \hat{\mathcal{T}}$ and will sum up

$$K_\mathcal{T} := K_\mathcal{T} + (\mu\hat{\omega})D^T C D \tag{5.1}$$

with the integration weight $\hat{\omega}$ belonging to $\hat{g}$.

First, we may use $\hat{D} = \hat{\nabla}^2\Psi(\hat{g})$ from (4.6,4.7,4.8) with formal derivatives w.r.t. $\hat{x}$. That is, the matrix $\hat{D} = (\hat{\nabla}^2\Psi_1(\hat{g}) \vdots \hat{\nabla}^2\Psi_2(\hat{g}) \vdots \hat{\nabla}^2\Psi_3(\hat{g}))$ has its three parts (in $\mathcal{T}_k$) as:

$$\begin{aligned}
\hat{\nabla}^2\Psi_k &= \hat{\nabla}^2\hat{\Phi}_0(\hat{g})\ H_k\ M_k \\
\hat{\nabla}^2\Psi_{k+1} &= \hat{\nabla}^2\hat{\Phi}_1(\hat{g})\ H_k + \hat{\nabla}^2\hat{\beta}(\hat{g})\ (b_k^{k+1})^T + \hat{\nabla}^2\hat{\Phi}_0(\hat{g})\ H_k\ M_{k+1} \\
\hat{\nabla}^2\Psi_{k-1} &= \hat{\nabla}^2\hat{\Phi}_2(\hat{g})\ H_k + \hat{\nabla}^2\hat{\beta}(\hat{g})\ (b_k^{k-1})^T + \hat{\nabla}^2\hat{\Phi}_0(\hat{g})\ H_k\ M_{k-1}.
\end{aligned}$$

Note that all the (3×3)–matrices $\hat{\nabla}^2\hat{\Phi}_0(\hat{g})$, $\hat{\nabla}^2\hat{\Phi}_1(\hat{g})$ and $\hat{\nabla}^2\hat{\Phi}_2(\hat{g})$ are given from the initial definition at the beginning of Section 4, compare Appendix B.

After all we have to carry out the back-transformation of $\hat{D}$ with derivatives w.r.t. $\hat{x}$ onto D with the derivatives w.r.t. the world x. Therefore, we define the (3×3)–matrix G_k with

$$\nabla^2 w = G_k \hat{\nabla}^2 w \iff \begin{pmatrix} \frac{\partial^2 w}{\partial x_1^2} & \frac{\partial^2 w}{\partial x_1 x_2} \\ \frac{\partial^2 w}{\partial x_1 x_2} & \frac{\partial^2 w}{\partial x_2^2} \end{pmatrix} = J_k^{-T} \begin{pmatrix} \frac{\partial^2 w}{\partial \hat{x}_1^2} & \frac{\partial^2 w}{\partial \hat{x}_1 \hat{x}_2} \\ \frac{\partial^2 w}{\partial \hat{x}_1 \hat{x}_2} & \frac{\partial^2 w}{\partial \hat{x}_2^2} \end{pmatrix} J_k^{-1}$$

and end up with

$$D = G_k \hat{D}$$

for use in (5.1). The entries of G_k are simple products of entries of the Jacobian matrix J_k. Let

$$J_k = \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_3 & \tau_4 \end{pmatrix}$$

then

$$G_k = \begin{pmatrix} \tau_4^2 & \tau_3^2 & -2\tau_3\tau_4 \\ \tau_2^2 & \tau_1^2 & -2\tau_1\tau_2 \\ -\tau_2\tau_4 & -\tau_1\tau_3 & \tau_2\tau_3+\tau_1\tau_4 \end{pmatrix} \cdot \mu^{-2}. \tag{5.2}$$

6. Stable Calculations for Adaptive Finite Element Methods

If this technique is used inside an adaptive procedure, we have to avoid cancellations of leading digits in nodal differences. That is, we never will subtract nodal coordinates of neighboring nodes, such as

$$\boldsymbol{E}_k = a_{k-1} - a_{k+1},$$

because these nodes coincide in more and more leading digits after drastic refinement at some parts of the mesh. This has been addressed in [5] for simple linear and quadratic elements. Here, the same solution as in [5] can be used to guarantee a stable Jacobian matrix of $\mathcal{T}$ by inheriting the Jacobian during the refinement.

Hence, we consider

$$J = (a_2 - a_1 \vdots a_3 - a_1) = (\boldsymbol{E}_3 \vdots -\boldsymbol{E}_2)$$

to be given as input belonging to $\mathcal{T}$. Here, the nodal differences are only calculated in this direct way on the coarsest mesh. Then for the procedure in Section 4, we have to form

$$\boldsymbol{E}_1 = -\boldsymbol{E}_3 - \boldsymbol{E}_2$$

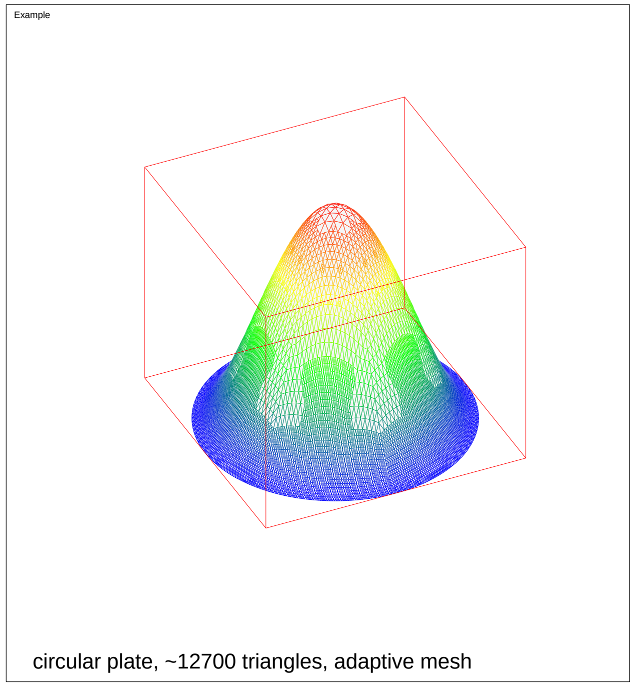

Figure 7.1. Solution on 10th level of adaptive mesh refinement

and

$$\boldsymbol{f}_j = \frac{1}{3}(\boldsymbol{E}_{j+1} - \boldsymbol{E}_{j-1})$$

which is a stable calculation, if both rows of J have been given stably.

7. Numerical Example

For demonstrating the applicability of the technique presented above we consider the solution of a plate equation on a circle of radius $R = 1$. Here, the correct analytical solution is known for a constant force term, the finite element approximation with adaptive refinement and rHCT-elements is seen in Fig. 7.1.

For sake of simplicity we have chosen $E = 3/2$ and $\nu = 1/2$ (yielding $2\mu = \lambda = 1$ as Lamé coefficients) and a constant force term $f = 128$ leads to the solution

$$w(x_1, x_2) = (R^2 - x_1^2 - x_2^2)^2.$$

We have started with a coarse mesh of 6 triangles and approximate the circle after the subsequent mesh refinements until about 12 thousand of elements. The adaptive procedure for the refinement control is based on a–posteriori error indicators similar to the well–known residual based error estimators.

The refinement history is given in table 7.1, where the time for calculating the new element matrices, the time and iterations for the PCG solver and the approximated value of $1 = w(0, 0)$ are presented together with the development of the number of nodes and elements. The running times are obtained on intel(R)Core(TM)2CPU6600 2.4GHz.

# nodes	# elements	time for element matrices	# iterations PCG solver	time PCG solver [sec]	approx. max($w(x)$)
19	6	0.00	2	0.00	.53
61	24	0.00	6	0.00	.88
217	96	0.01	18	0.01	.97
685	312	0.04	30	0.01	.99
1993	924	0.08	42	0.03	.999
2587	1230	0.06	47	0.05	1.00
4177	1968	0.10	54	0.09	1.00
7141	3408	0.17	63	0.18	1.00
10309	4992	0.19	74	0.32	1.00
16117	7692	0.35	81	0.59	1.00
26209	12732	0.61	94	1.12	1.00

Table 7.1. The adaptive mesh history

8. Conclusion

The construction of the element shape functions of the reduced HCT-element has been addressed in some papers in the prior literature. We could avoid rather complicate formulas for these basis functions from a 3 step procedure starting with simple cubic basis functions of the master triangle. The necessary additional calculations use some geometrically simply defined entries of the given triangle, such as its edge vectors.

References

[1] M.Bernadou, K.Hassan, *Basis functions for general Hsieh-Cough-Tocher triangles, complete or reduced*, I.N.R.I.A., 1980, pp. 784–789.

[2] P.G.Ciarlet,*The Finite Element Method for Elliptic Problems*, 530p., North-Holland, Amsterdam, 1978.

[3] R.W.Clough, J.L.Tocher, *Finite element stiffness matrices for analysis of plates in bending*, Proc.Conf on Matrix Meth., Wright Patterson A.F.B Ohio, 1965.

[4] G.Dupuis, J.-J.Goël, *Finite element with high degree of regularity*, Int.J.Num.Meth.Engng., **2** (1970), pp. 563–577.

[5] A.Meyer, *Stable Evaluation of Jacobian Matrices on Highly Refined Finite Element Meshes*, Computing, **70** (2003), no. 4, pp. 359–373.

[6] A.S.Peker, J.E.Lavery, S.-C.Fang, *A reduced Hsieh-Cough-Tocher element with splitting based on an arbitrary point*, J.Math.Anal.Appl., **333** (2007), pp. 500–504.

A. Pseudocode for calculating the element matrix

A.1. Input

The (2×2)-Jacobian matrix J of the element $\mathcal{T}$ defining the 2 edge vectors $\boldsymbol{E}_3$ and $\boldsymbol{E}_2$ from

$$J = (\boldsymbol{E}_3 \vdots - \boldsymbol{E}_2),$$

as well as the (3×3)-material matrix C and the Gaussian Points together with the weights $\hat{g}, \hat{\omega}$.

A.2. Declarations

We need to declare:

- Vectors of $\mathbb{R}^2$: $\boldsymbol{E}_1, \boldsymbol{E}_2, \boldsymbol{E}_3, \boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3, \boldsymbol{N}_1, \boldsymbol{N}_2, \boldsymbol{N}_3$
  (to be calculated from geometrical input)
- Vectors of $\mathbb{R}^3$: $Dbeta,\ b_k^j,\ t_j$ for $k, j = 1\ldots3$,
- Matrix $R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
- (3×3)-Matrices: $DPhi0, DPhi1, DPhi2, S, G, H$ and
  T_j, M_j, D_j for $j = 1\ldots3$,
- let $D = (D_1 \vdots D_2 \vdots D_3)$ a (3×9)-Matrix as well as D2 is (3×9),
- the output element matrix K_{el} is (9×9)
- a routine " rotateindex " :

  $rotateindex(i, ip, im)$ creates

  $ip := i+1,\ im := i-1$ (cyclically, hence:)

  <u>if</u> $i = 3$ <u>then</u> $ip := 1$; <u>if</u> $i = 1$ <u>then</u> $im := 3$

  <u>end</u> rotateindex

A.3. Preparation step

create the vectors b_i^j and the matrices M_j :

$\boldsymbol{E}_3, \boldsymbol{E}_2$ given from input J, then:
$\boldsymbol{E}_1 := -\boldsymbol{E}_3 - \boldsymbol{E}_2$ and $\mu := |\det J|/3$

<u>for</u> $j = 1$ <u>to</u> 3 <u>do</u>
    $rotateindex(j, jp, jm)$
    $\boldsymbol{N}_j := R * \boldsymbol{E}_j$
    $\boldsymbol{f}_j := (\boldsymbol{E}_{jp} - \boldsymbol{E}_{jm})/3$
<u>endfor</u>

<u>for</u> k=1 <u>to</u> 3 <u>do</u>
    $rotateindex(k, kp, km)$
    $normEk :=$ scapr $(2, \boldsymbol{E}_k, \boldsymbol{E}_k)$
    $b_k^{kp}(1) := 6 *$ scapr $(2, \boldsymbol{E}_k, \boldsymbol{f}_{km})/normEk$
    $b_k^{kp}(2\ldots3) := 2 * \boldsymbol{f}_{km} + (3 * \mu/normEk) * \boldsymbol{N}_k$
    $b_k^{km}(1) := -6 *$ scapr $(2, \boldsymbol{E}_k, \boldsymbol{f}_{kp})/normEk$
    $b_k^{km}(2\ldots3) := 2 * \boldsymbol{f}_{kp} + (3 * \mu/normEk) * \boldsymbol{N}_k$
<u>endfor</u>

form

$$S := 6 * \begin{pmatrix} 3 & \vdots & \boldsymbol{f}_1^T \\ 3 & \vdots & \boldsymbol{f}_2^T \\ 3 & \vdots & \boldsymbol{f}_3^T \end{pmatrix}$$

<u>for</u> j=1 <u>to</u> 3 <u>do</u>

  $rotateindex(j, jp, jm)$

  define the rows of T_j as:

  $t_{jm} := b^j_{jp}, \qquad t_{jp} := b^j_{jm}$

  $t_j := (b^j_{jp} + b^j_{jm}) + \begin{pmatrix} 6 \\ -2 * \boldsymbol{f}_j \end{pmatrix}$

  $T_j :=$ transpose $(t_1 \vdots t_2 \vdots t_3)$

  $M_j := S^{-1} T_j$

<u>endfor</u>

A.4. Forming the element matrix

Perform the Gaussian integration over the 3 subtriangles:

$K_{el} := \mathbb{O}$

<u>for</u> k=1 <u>to</u> 3 <u>do</u>

  $rotateindex(k, km, kp)$

  form $J = (\boldsymbol{f}_{kp} \vdots \boldsymbol{f}_{km})$ (local Jacobian)

  form

$$H := \begin{pmatrix} 1 & \vdots & 0 \;\; 0 \\ \cdots & \cdots & \cdots\cdots \\ \begin{matrix} 0 \\ 0 \end{matrix} & \vdots & J^T \end{pmatrix}$$

  form G from J according to (5.2)

  <u>for</u> $i = 1$ <u>to</u> $allGaussianPoints$ <u>do</u>

    give $\hat{g} \in \mathbb{R}^2$ (Gaussian Point) and $\hat{\omega}$ the integration weight of $\hat{g}$

    give $DPhi0, Dphi1, DPhi2, Dbeta$ at $\hat{g}$ according to Appendix B

    $D_k := DPhi0 * H * M_k$

    $D_{kp} := DPhi1 * H + DPhi0 * H * M_{kp} + Dbeta *$ transpose(b_k^{kp})

    $D_{km} := DPhi2 * H + DPhi0 * H * M_{km} + Dbeta *$ transpose(b_k^{km})

    $D := (D_1 \vdots D_2 \vdots D_3)$

    $D2 := G * D$

    $D := C * D2$ (with given material matrix C according to Section 5)

    $K_{el} := K_{el} + (\hat{\omega}\mu) *$ transpose$(D2) * D$

  <u>endfor</u>

<u>endfor</u>

B. Matrices of the second derivatives

From the definition of the ansatz functions $\hat{\Phi}_k$ in Section 4, we obtain the following matrices containing the second derivatives at a Gaussian point $\hat{g} = (\hat{x}_1, \hat{x}_2)^T \in \hat{\mathcal{T}}$ for use in Appendix A.

$$DPhi0(\hat{g}) = \begin{pmatrix} -6+12\hat{x}_1+12\hat{x}_2 & -4+6\hat{x}_1+4\hat{x}_2 & 2\hat{x}_2 \\ -6+12\hat{x}_1+12\hat{x}_2 & 2\hat{x}_1 & -4+4\hat{x}_1+6\hat{x}_2 \\ -6+12\hat{x}_1+12\hat{x}_2 & -2+4\hat{x}_1+2\hat{x}_2 & -2+2\hat{x}_1+4\hat{x}_2 \end{pmatrix}$$

$$DPhi1(\hat{g}) = \begin{pmatrix} 6-12\hat{x}_1 & -2+6\hat{x}_1 & 2\hat{x}_2 \\ 0 & 0 & 0 \\ 0 & 0 & 2\hat{x}_1 \end{pmatrix}$$

$$DPhi2(\hat{g}) = \begin{pmatrix} 0 & 0 & 0 \\ 6-12\hat{x}_2 & 2\hat{x}_1 & -2+6\hat{x}_2 \\ 0 & 2\hat{x}_2 & 0 \end{pmatrix}$$

$$Dbeta(\hat{g}) = \begin{pmatrix} -2\hat{x}_2 \\ -2\hat{x}_1 \\ 1-2\hat{x}_1-2\hat{x}_2 \end{pmatrix}$$