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Curved finite elements of class C^1: Implementation and numerical experiments. Part 1: Construction and numerical tests of the interpolation properties*

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The aim of this work is to show how to implement some of the methods of curved finite elements of class C^1 introduced and analyzed in a previous work by the first author. These methods use curved finite elements which are C^1-compatible with the Argyris triangle. A careful description of the implementation is completed by some numerical experiments which show the very high degree of accuracy of the associated interpolation. Subsequently, in a second part, we will show the efficiency of such methods for solving thin plate or thin shell problems set on curved boundary domains.

Introduction

The conforming approximation of thin plate problems with curved boundaries requires the use of curved C^1-finite elements which are compatible with some classical C^1-straight finite elements. A similar situation occurs for the conforming approximation of thin shells or for the conforming approximation of junctions between thin shells (see [1]).

We have introduced and analyzed such curved elements in [2, 3] so that they are C^1-compatible with the Argyris triangle [4–6]. These methods are particularly interesting to realize a very accurate approximation of thin plate or thin shell problems [2, 3].

In this paper, we record the main steps of the construction of curved finite elements which are C^1-compatible with the Argyris triangle, and then we give a very detailed description of how to implement such methods. More precisely, Section 1 is dedicated to the construction of the approximated domain Ω_h of the given curved boundary domain Ω, while Section 2 contains the main steps of the definition of the considered curved C^1 finite element. Next, in

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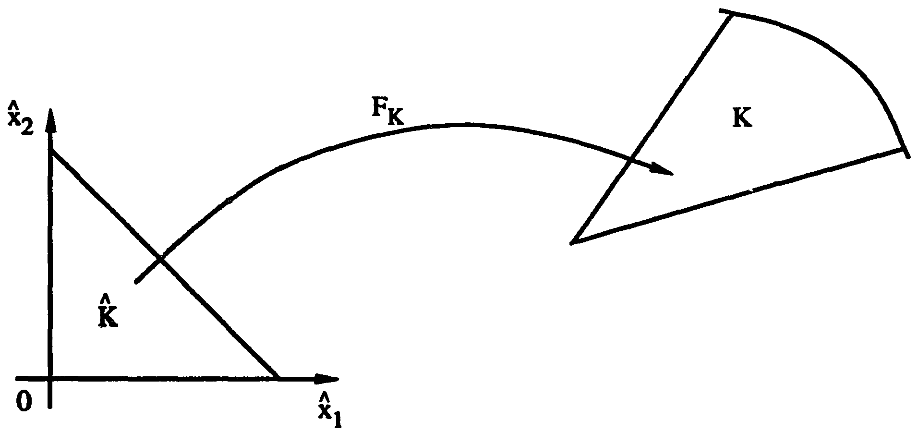


Fig. 1.

Section 3, we detail the matrix decompositions which allow the interpolation modules to be realized. Finally we report some numerical experiments which prove the efficiency and the very high accuracy of such interpolation methods.

Let us add that all the relations have been checked by using the formal computational tools provided by Mathematica [7].

Summary of the construction of these curved C^1-elements

Let v be the function to be interpolated on the curved triangle K. The construction of these curved $\mathscr{C}^1$-elements takes several steps:

(i) triangulation of the curved boundary domain Ω;
(ii) interpolation of the curved triangular sides located on the boundary;
(iii) definition of the mapping F_K (see Fig. 1);
(iv) computation of the set of values of the degrees of freedom $\Sigma_K(v)$ of function v;
(v) from $\Sigma_K(v)$, computation of the set of values of the degrees of freedom $\hat{\Delta}_K(v)$;
(iv) from the set $\hat{\Delta}_K(v)$, computation of the interpolate function $\hat{w}$;
(vii) computation of the interpolate function $\pi_K v = w = \hat{w} \circ F_K^{-1}$.

Notations

For more details concerning curved C^1 finite elements and for notation, we refer to [2, 3] and, more generally, the finite element notation is that used by [5, 6].

1. Approximation of the curved boundary domain Ω

This section records some useful results obtained in [2, 3].

1.1. 'Exact' triangulation of the domain Ω

Let Ω be a plane domain with a curved boundary Γ. We realize partitions of this domain by using regular families of triangulations $\mathscr{T}_h$ including (see Fig. 2)

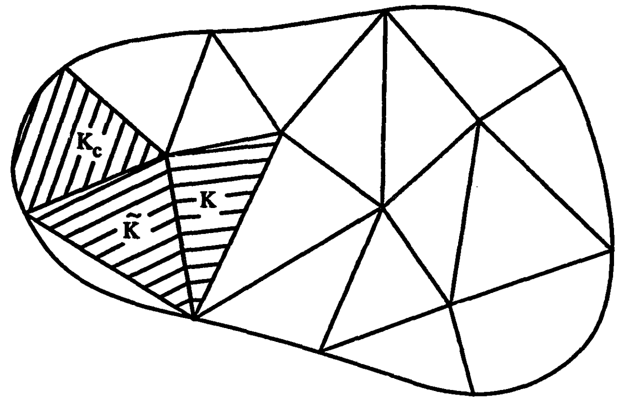


Fig. 2. 'Exact' triangulation of the domain Ω made up of straight triangles K and one-curved side triangles K_c.

(i) straight triangles, on the one hand;
(ii) triangles with one curved side located on Γ, on the other hand.
In this way, triangulations $\mathcal{T}_h$ can be split as follows: $\mathcal{T}_h = \mathcal{T}_h^1 \cup \mathcal{T}_h^2$, where $\mathcal{T}_h^1$ collects all the straight triangles K while $\mathcal{T}_h^2$ collects all the triangles K_c with one curved side on Γ. According to Fig. 2, we note $\tilde{K}$ the straight triangles associated with a one-curved side triangle K_c.

1.2. *'Approximate' triangulation of the domain Ω*

Any one-curved side triangle K_c is approximated by a triangle K. In this way, the curved side of K_c is approximated by an arc γ_h (see Fig. 3). Then the union of all straight triangles $K \in \mathcal{T}_h^1$ and of all one-curved side triangles $K \in \mathcal{T}_h^2$ gives the approximate domain Ω_h. Such a construction can be split into two steps:

Step 1: Interpolation of the curved side a_1a_2 of the triangle K_c. Assume that $\mathbb{R}^2$ is referred to an orthonormal system (O, e_1, e_2) whose associated coordinates are denoted as (x_1, x_2) and that Ω is a bounded domain. Moreover, assume that the boundary Γ can be subdivided into a finite

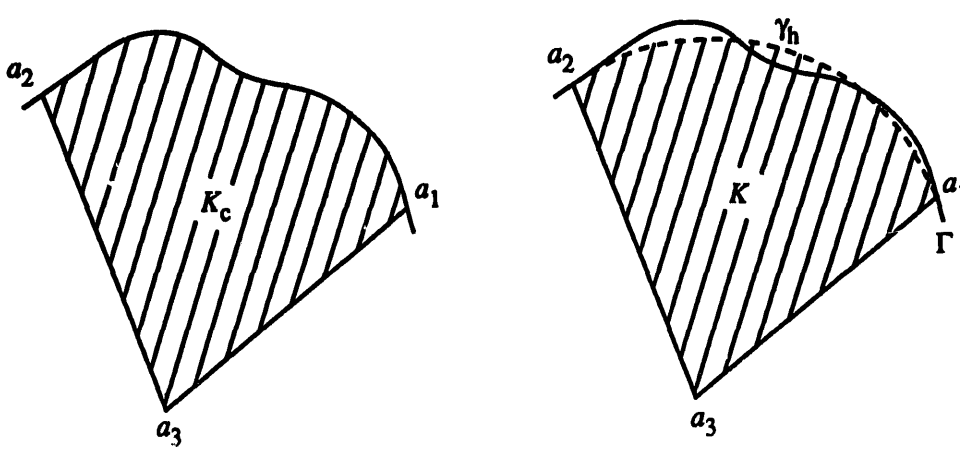


Fig. 3. 'Exact' and 'approximate' one-curved side triangles K_c and K.

number of arcs, each having a sufficiently smooth parametric representation of the following type:

$$x_1 = \chi_1(s)\,, \quad x_2 = \chi_2(s)\,, \quad \underline{s} \leq s \leq \bar{s}\,. \tag{1.2.1}$$

Subsequently, we also use another parameterization of the arc a_1a_2:

$$x_1 = \psi_1(\hat{x}_2)\,, \quad x_2 = \psi_2(\hat{x}_2)\,, \quad 0 \leq \hat{x}_2 \leq 1\,, \tag{1.2.2}$$

where

$$\psi_\alpha(\hat{x}_2) = \chi_\alpha(\underline{s} + (\bar{s} - \underline{s})\hat{x}_2)\,, \quad \alpha = 1, 2\,. \tag{1.2.3}$$

Then, every component $\psi_\alpha(\hat{x}_2)$ is interpolated by a polynomial function $\psi_{\alpha h}$ of degree $n \geq 2$, so that

$$\psi_{\alpha h}(0) = \psi_\alpha(0)\,, \quad \psi_{\alpha h}(1) = \psi_\alpha(1)\,, \quad \alpha = 1, 2\,. \tag{1.2.4}$$

Thus, for $n \geq 2$, we obtain

$$\psi_{\alpha h}(\hat{x}_2) = x_{\alpha 1} + (x_{\alpha 2} - x_{\alpha 1})\hat{x}_2 + \hat{x}_2(1 - \hat{x}_2)P_{n-2;\alpha}(\hat{x}_2)\,, \tag{1.2.5}$$

where $P_{n-2;\alpha}(\hat{x}_2)$ refers to polynomials of degree $n-2$ with respect to $\hat{x}_2$. These relations (1.2.5) define the approximate arc γ_h. With the notation of Fig. 4, observe that relation (1.2.5) can be geometrically interpretated as follows:

$$\boldsymbol{OP}_h = \boldsymbol{O\tilde{P}} + \boldsymbol{\tilde{P}P}_h\,. \tag{1.2.6}$$

Step 2: Definition of the application $F_K : \hat{K} \to K$. Let $\hat{K}$ be a given reference triangle, for instance the unit right-angled triangle $\hat{a}_1\hat{a}_2\hat{a}_3$ with $\hat{a}_1 = (1, 0)$, $\hat{a}_2 = (0, 1)$, $\hat{a}_3 = (0, 0)$. Then, with any point $\hat{M}$ of the reference triangle $\hat{K}$, the application F_K associates the point M_h:

$$\boldsymbol{OM}_h = F_{K1}(\hat{x}_1, \hat{x}_2)\boldsymbol{e}_1 + F_{K2}(\hat{x}_1, \hat{x}_2)\boldsymbol{e}_2\,, \tag{1.2.7}$$

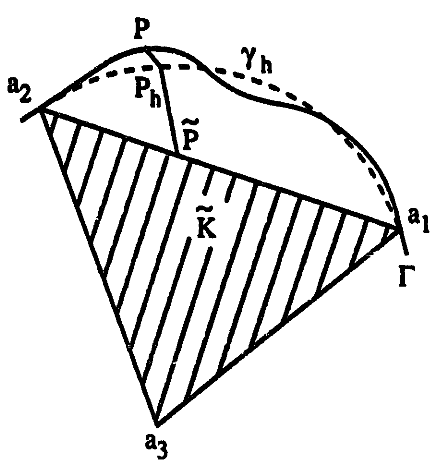


Fig. 4. Construction of the approximate arc γ_h.

where the functions $F_{K\alpha}$, $\alpha = 1, 2$, are defined for $n \geq 2$ by the relations

$$\begin{aligned} F_{K\alpha}(\hat{x}_1, \hat{x}_2) = x_{\alpha 3} + (x_{\alpha 1} - x_{\alpha 3})\hat{x}_1 + (x_{\alpha 2} - x_{\alpha 3})\hat{x}_2 \\ + \tfrac{1}{2}\hat{x}_1\hat{x}_2[P_{n-2;\alpha}(1 - \hat{x}_1) + P_{n-2;\alpha}(\hat{x}_2)] . \end{aligned} \tag{1.2.8}$$

These components are symmetrical with respect to $\hat{x}_1$ and $\hat{x}_2$, and they can be geometrically interpretated in the vectorial form (see Fig. 5)

$$OM_h = O\tilde{M} + \tfrac{1}{2}\tilde{M}M_h^1 + \tfrac{1}{2}\tilde{M}M_h^2 . \tag{1.2.9}$$

The properties of this application $F_K : \hat{K} \to K$ are detailed in [2].

1.3. Examples

Here, we give two examples of applications $\psi_{\alpha h}$ (see (1.2.5)) which define an approximate arc γ_h interpolating a given arc a_1a_2. These examples correspond to interpolations by polynomials of degree $n = 3$ or 5 which are the most interesting in practice. For each case, we indicate the expression of the applications $\psi_{\alpha h}$ from which we deduce the expression of the components $F_{K\alpha}$, $\alpha = 1, 2$. Afterwards, these expressions are permanently used; in order to make a distinction between them, we denote F_K as the one that is associated with Example 1.3.1 (i.e., $F_K \in (P_3)^2$) and F_K^* corresponds to Example 1.3.2 (i.e., $F_K^* \in (P_5)^2$).

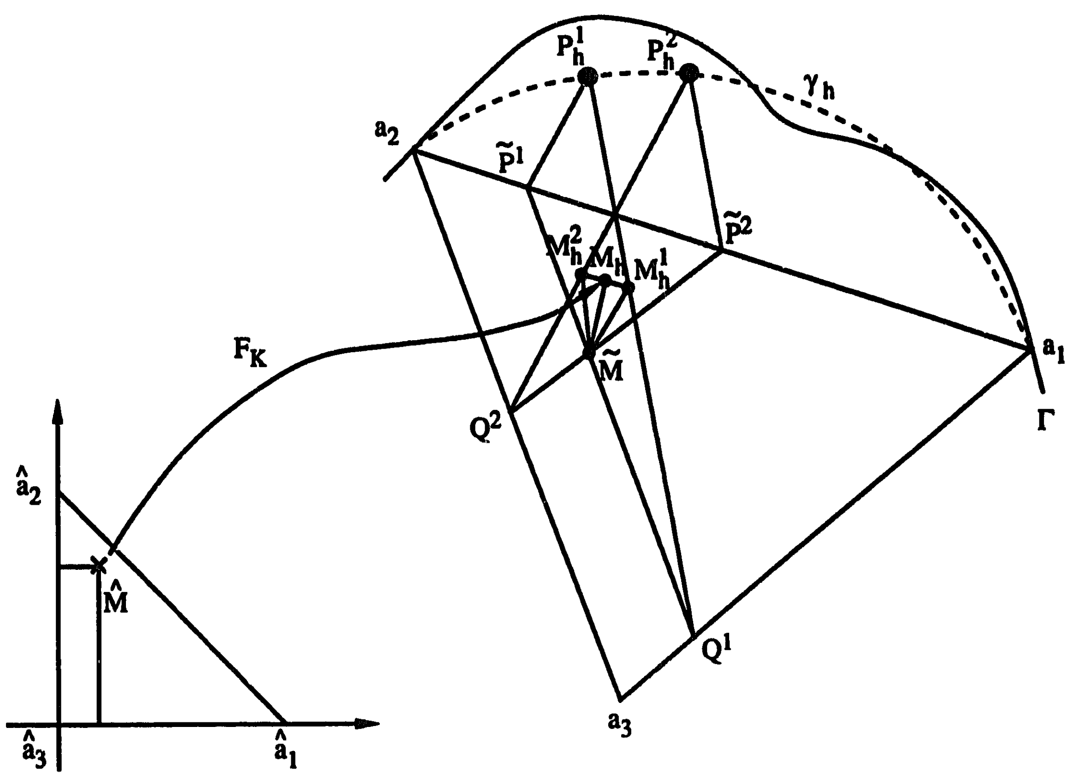


Fig. 5. Transformation $\tilde{M} \to M_h$ (triangles $Q^1\tilde{M}M_h^1$ and $Q^1\tilde{P}^1P_h^1$ are homothetic in the ratio $\hat{x}_2/(1 - \hat{x}_1)$ and triangles $Q^2\tilde{M}M_h^2$ and $Q^2\tilde{P}^2P_h^2$ are homothetic in the ratio $\hat{x}_1/(1 - \hat{x}_2)$).

EXAMPLE 1.3.1. Construction of the approximate arc γ_h by using polynomials of degree 3 (Fig. 6). Expressions (1.2.5) give

$$\begin{aligned}\psi_{\alpha h}(\hat{x}_2) = x_{\alpha 1} + (x_{\alpha 2} - x_{\alpha 1})\hat{x}_2 + \hat{x}_2(1-\hat{x}_2)\{[2(x_{\alpha 2} - x_{\alpha 1}) \\ -(\bar{s}-\underline{s})(\chi'_\alpha(\underline{s}) + \chi'_\alpha(\bar{s}))]\hat{x}_2 + x_{\alpha 1} - x_{\alpha 2} + (\bar{s}-\underline{s})\chi'_\alpha(\underline{s})\}\,,\end{aligned} \tag{1.3.1}$$

so that from (1.2.8), we obtain

$$\begin{aligned}F_{K\alpha}(\hat{x}_1, \hat{x}_2) = x_{\alpha 3} + (x_{\alpha 1} - x_{\alpha 3})\hat{x}_1 + (x_{\alpha 2} - x_{\alpha 3})\hat{x}_2 + \tfrac{1}{2}\hat{x}_1\hat{x}_2\{[2(x_{\alpha 2} - x_{\alpha 1}) \\ -(\bar{s}-\underline{s})(\chi'_\alpha(\underline{s}) + \chi'_\alpha(\bar{s}))][\hat{x}_2 - \hat{x}_1] + (\bar{s}-\underline{s})[\chi'_\alpha(\underline{s}) - \chi'_\alpha(\bar{s})]\}\,.\end{aligned} \tag{1.3.2}$$

EXAMPLE 1.3.2. Construction of the approximate arc γ_h^* by using polynomials of degree 5. We interpolate the functions ψ_α, $\alpha = 1, 2$, on the interval [0, 1] by using Hermite polynomials of degree 5. The degrees of freedom of the interpolation are the values of the functions ψ_α, ψ'_α and ψ''_α at points $\hat{x}_2 = 0$ or $\hat{x}_2 = 1$, i.e.,

$$\begin{aligned}&x_{\alpha 1} = \psi_\alpha(0) = \chi_\alpha(\underline{s})\,, \qquad x_{\alpha 2} = \psi_\alpha(1) = \chi_\alpha(\bar{s})\,, \\ &\psi_\alpha^{(l)}(0) = (\bar{s}-\underline{s})^l\chi_\alpha^{(l)}(\underline{s})\,, \qquad \psi_\alpha^{(l)}(1) = (\bar{s}-\underline{s})^l\chi_\alpha^{(l)}(\bar{s})\,, \quad l = 1, 2\,.\end{aligned} \tag{1.3.3}$$

From the expression (1.2.5), we obtain (we recall that all the results related to Example 1.3.2 are indexed with a start (*) in order to make the distinction with similar results relating to Example 1.3.1)

$$\psi^*_{\alpha h}(\hat{x}_2) = x_{\alpha 1} + (x_{\alpha 2} - x_{\alpha 1})\hat{x}_2 + \hat{x}_2(1-\hat{x}_2)[\beta_{\alpha 3}(\hat{x}_2)^3 + \beta_{\alpha 2}(\hat{x}_2)^2 + \beta_{\alpha 1}\hat{x}_2 + \beta_{\alpha 0}]\,, \tag{1.3.4}$$

where the coefficients $\beta_{\alpha l}$, $l = 0, 1, 2, 3$, are given by the relations

$$\begin{aligned}&\beta_{\alpha 0} = x_{\alpha 1} - x_{\alpha 2} + (\bar{s}-\underline{s})\chi'_\alpha(\underline{s})\,, \\ &\beta_{\alpha 1} = x_{\alpha 1} - x_{\alpha 2} + (\bar{s}-\underline{s})\chi'_\alpha(\underline{s}) + \tfrac{1}{2}(\bar{s}-\underline{s})^2\chi''_\alpha(\underline{s})\,,\end{aligned}$$

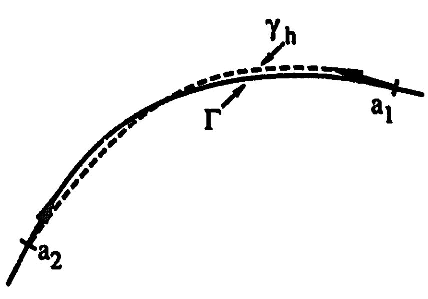


Fig. 6. Boundary interpolation with polynomials of degree 3.

$$\begin{aligned}\beta_{\alpha 2} &= 9(x_{\alpha 2} - x_{\alpha 1}) - (\bar{s} - \underline{s})[5\chi'_\alpha(\underline{s}) + 4\chi'_\alpha(\bar{s})] - \tfrac{1}{2}(\bar{s} - \underline{s})^2[2\chi''_\alpha(\underline{s}) - \chi''_\alpha(\bar{s})]\,,\\ \beta_{\alpha 3} &= 6(x_{\alpha 1} - x_{\alpha 2}) + 3(\bar{s} - \underline{s})[\chi'_\alpha(\underline{s}) + \chi'_\alpha(\bar{s})] + \tfrac{1}{2}(\bar{s} - \underline{s})^2[\chi''_\alpha(\underline{s}) - \chi''_\alpha(\bar{s})]\,.\end{aligned} \tag{1.3.5}$$

From expressions (1.2.8) and (1.3.4), we deduce

$$\begin{aligned}F^*_{K\alpha}(\hat{x}_1, \hat{x}_2) = x_{\alpha 3} &+ (x_{\alpha 1} - x_{\alpha 3})\hat{x}_1 + (x_{\alpha 2} - x_{\alpha 3})\hat{x}_2\\ &+ \tfrac{1}{2}\hat{x}_1\hat{x}_2[\beta_{\alpha 3}(\hat{x}_2)^3 + \beta_{\alpha 2}(\hat{x}_2)^2 + \beta_{\alpha 1}\hat{x}_2 + \beta_{\alpha 0}\\ &\qquad + \tilde{\beta}_{\alpha 3}(\hat{x}_1)^3 + \tilde{\beta}_{\alpha 2}(\hat{x}_1)^2 + \tilde{\beta}_{\alpha 1}\hat{x}_1 + \tilde{\beta}_{\alpha 0}]\,,\end{aligned} \tag{1.3.6}$$

where the coefficients $\beta_{\alpha l}$, $l = 0, \ldots, 3$, are given by relation (1.3.5) and where the coefficients $\tilde{\beta}_{\alpha l}$, $l = 0, \ldots, 3$, are given by

$$\begin{aligned}\tilde{\beta}_{\alpha 0} &= x_{\alpha 2} - x_{\alpha 1} - (\bar{s} - \underline{s})\chi'_\alpha(\bar{s})\,,\\ \tilde{\beta}_{\alpha 1} &= x_{\alpha 2} - x_{\alpha 1} - (\bar{s} - \underline{s})\chi'_\alpha(\bar{s}) + \tfrac{1}{2}(\bar{s} - \underline{s})^2\chi''_\alpha(\bar{s})\,,\\ \tilde{\beta}_{\alpha 2} &= 9(x_{\alpha 1} - x_{\alpha 2}) + (\bar{s} - \underline{s})[5\chi'_\alpha(\bar{s}) + 4\chi'_\alpha(\underline{s})] - \tfrac{1}{2}(\bar{s} - \underline{s})^2[2\chi''_\alpha(\bar{s}) - \chi''_\alpha(\underline{s})]\,,\\ \tilde{\beta}_{\alpha 3} &= 6(x_{\alpha 2} - x_{\alpha 1}) - 3(\bar{s} - \underline{s})[\chi'_\alpha(\bar{s}) + \chi'_\alpha(\underline{s})] + \tfrac{1}{2}(\bar{s} - \underline{s})^2[\chi''_\alpha(\bar{s}) - \chi''_\alpha(\underline{s})]\,.\end{aligned} \tag{1.3.7}$$

2. Definition of curved finite elements, C^1-compatible with the Argyris triangle

In this section, we record the definitions of *two curved finite elements* which have a connection of class C^1 with the classical Argyris triangle.

(i) the first corresponds to the interpolation of the boundary considered in Example 1.3.1. This interpolation is realized by using polynomials of degree 3 and it turns out to be sufficient for the approximation of fourth-order problems with homogeneous Dirichlet boundary conditions (see [3]);

(ii) the second corresponds to the interpolation of the boundary described in Example 1.3.2. This interpolation is realized by using polynomials of degree 5 and it can be used for more general boundary conditions (see [3, Remark 3.1]).

Both constructions are fully detailed in [2, 3] where, in addition, one can find necessary justifications and corresponding interpolation error estimates. The implementation of these methods is detailed in matrix form in Section 3.

2.1. *Basic principles*

We consider separately *essential* and *desirable* conditions.

2.1.1. *Essential conditions*

According to Fig. 3, the connections between Argyris triangles and curved elements are made along the straight sides a_3a_1 and a_3a_2. Let (K, P_K, Σ_K) be the approximate curved triangle, its associated functional space and its corresponding set of degrees of freedom. To obtain a connection of class C^1, it is essential to satisfy (see Fig. 7)

the degrees of freedom of the curved finite elements related to the sides a_3a_1 and a_3a_2 are identical to those of the Argyris triangle; (2.1.1.)

the traces $p|_{[a_3,a_\alpha]}$, $\alpha = 1, 2$ (respectively $Dp(\cdot)(a_2 - c_2)|_{[a_3,a_1]}$; $Dp(\cdot)(a_1 - c_1)|_{[a_3,a_2]}$) of the functions $p \in P_K$ defined on the curved triangle K, are one-variable polynomials of degree 5 (respectively 4), entirely determined by the degrees of freedom related to the sides a_3a_α, $\alpha = 1, 2$. (2.1.2)

2.1.2. Desirable conditions

The application F_K, introduced in Section 1, associates the curved triangle K with the reference triangle $\hat{K}$.

We also use this application F_K to associate with any function v defined on the triangle K, a function $\hat{v}$ defined on the triangle $\hat{K}$, i.e.,

$$v = \hat{v} \circ (F_K)^{-1}, \qquad \hat{v} = v \circ F_K. \tag{2.1.3}$$

Consequently, it is 'desirable' that the following condition is satisfied:

To any function $p \in P_K$, defined on the curved triangle K, the correspondence (2.1.3) associates a *polynomial* function $\hat{p} = p \circ F_K$. (2.1.4)

This condition (2.1.4) is convenient to study the approximation error and the effect of a numerical integration scheme and to take into account the boundary conditions. But this condition leads to a definition of reference finite elements that is more complicated than that associated with corresponding straight side finite elements. Indeed, with the notation of Fig. 7, we obtain for $a \in [a_3a_1]$

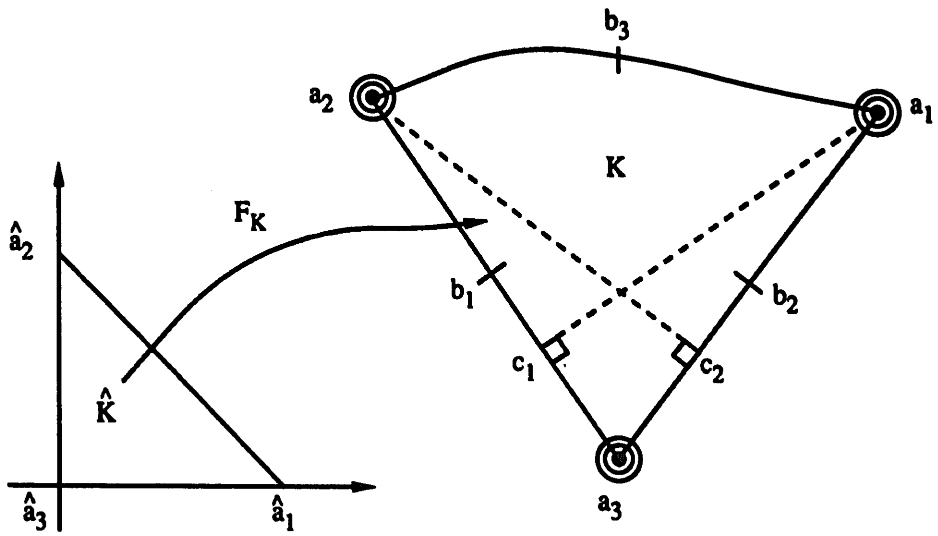


Fig. 7. The approximate curved triangle K.

$$\frac{\partial \hat{p}}{\partial \hat{x}_2}(\hat{a}) = D\hat{p}(\hat{a})e_2 = Dp(a)DF_K(\hat{a})e_2 = Dp(a)\frac{\partial F_K}{\partial \hat{x}_2}(\hat{a})$$

$$= \left\langle \frac{\partial F_K}{\partial \hat{x}_2}(\hat{a}), \quad \frac{a_1 - a_3}{|a_1 - a_3|^2} Dp(a)(a_1 - a_3) + \frac{a_2 - c_2}{|a_2 - c_2|^2} Dp(a)(a_2 - c_2) \right\rangle, \tag{2.1.5}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in $\mathbb{R}^2$. Likewise for $a \in [a_3 a_2]$, we obtain

$$\frac{\partial \hat{p}}{\partial \hat{x}_1}(\hat{a}) = \left\langle \frac{\partial F_K}{\partial \hat{x}_1}(\hat{a}), \quad \frac{a_2 - a_3}{|a_2 - a_3|^2} Dp(a)(a_2 - a_3) + \frac{a_1 - c_1}{|a_1 - c_1|^2} Dp(a)(a_1 - c_1) \right\rangle. \tag{2.1.6}$$

Relations (2.1.5) and (2.1.6) prove that $(\partial \hat{p}/\partial \hat{x}_2)(\hat{a})$ (respectively $(\partial \hat{p}/\partial \hat{x}_1)(\hat{a})$) is a polynomial of degree $n + 3$ with respect to $\hat{x}_1$ (respectively $\hat{x}_2$) for any $\hat{a} \in [\hat{a}_3 \hat{a}_1]$ (respectively $\hat{a} \in [\hat{a}_3 \hat{a}_2]$), with $n = 3$ or 5 depending on whether F_K is of degree 3 or 5 (see Examples 1.3.1 and 1.3.2). Thus with $\hat{K}$, we have to associate a finite element $(\hat{K}, \hat{P}, \hat{\Sigma})$ such that

$$\hat{P}_K \subset P_{n+4} \quad (n = 3 \text{ or } 5), \qquad \hat{P}_K = \{\hat{p} : \hat{K} \to \mathbb{R};\ \hat{p} = p \circ F_K,\ p \in P_K\}. \tag{2.1.7}$$

2.2. *Definition of curved finite elements C^1-compatible with the Argyris triangle for $F_K \in (P_3)^2$*

In this section, we consider the application F_K defined in Example 1.3.1. Then, relation (2.1.7) gives $\hat{P}_K \subset P_7$.

2.2.1. *The basic finite element $(\hat{K}, \hat{P}, \hat{\Sigma})$*

This finite element is described in Fig. 8 and we refer the reader to [8] for more details. Corresponding basis functions are given by relation (3.1.43).

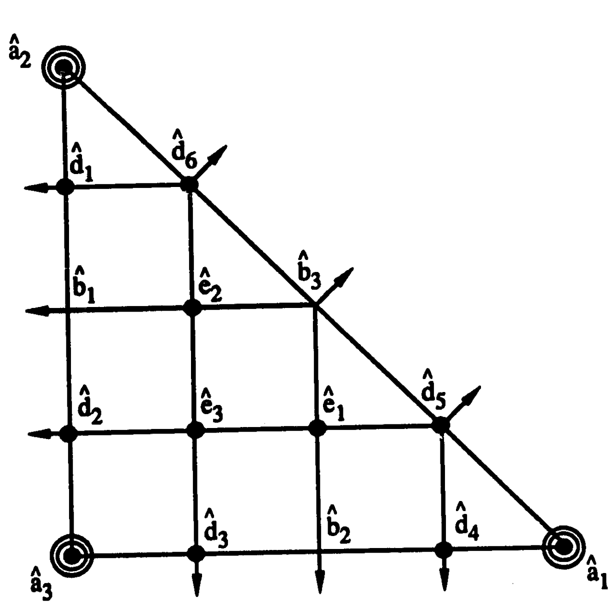
2.2.2. *Construction of the interpolating function $v \to \pi_K v$*

Now we use this basic finite element in order to associate an interpolate function $\pi_K v$ with any function $v \in C^2(\bar{K})$. This construction takes three steps:

STEP 1: Definition of the set Σ_K of the degrees of freedom of the curved element. Set (see Figs. 8 and 9)

$$\begin{aligned} &a_i = F_K(\hat{a}_i), \quad b_i = F_K(\hat{b}_i), \quad e_i = F_K(\hat{e}_i), \quad i = 1, 2, 3; \\ &c_\alpha = \text{orthogonal projection of } a_\alpha \text{ on the side } a_3 a_\alpha,\ \alpha = 1, 2. \end{aligned} \tag{2.2.1}$$

Then, the set $\Sigma_K(v)$ of degrees of freedom of the function $v \in C^2(K)$ is given, in its local version $[DLLC(v)]$, by



$\hat{K}$ = unit right-angled triangle; $\hat{P} = P_7$; $\dim \hat{P} = 36$

$$\hat{\Sigma}(\hat{p}) = \Big[\hat{p}(\hat{a}_i);\ \frac{\partial \hat{p}}{\partial \hat{x}_1}(\hat{a}_i);\ \frac{\partial \hat{p}}{\partial \hat{x}_2}(\hat{a}_i);\ \frac{\partial^2 \hat{p}}{\partial \hat{x}_1^2}(\hat{a}_i);\ \frac{\partial^2 \hat{p}}{\partial \hat{x}_1 \partial \hat{x}_2}(\hat{a}_i);\ \frac{\partial^2 \hat{p}}{\partial \hat{x}_2^2}(\hat{a}_i),\ i = 1, 2, 3;\ -\frac{\partial \hat{p}}{\partial \hat{x}_1}(\hat{b}_1);$$

$$-\frac{\partial \hat{p}}{\partial \hat{x}_2}(\hat{b}_2);\ \frac{\sqrt{2}}{2}\left(\frac{\partial \hat{p}}{\partial \hat{x}_1} + \frac{\partial \hat{p}}{\partial \hat{x}_2}\right)(\hat{b}_3);\ \hat{p}(\hat{d}_i),\ i = 1, \ldots, 6;\ -\frac{\partial \hat{p}}{\partial \hat{x}_1}(\hat{d}_i),\ i = 1, 2;$$

$$-\frac{\partial \hat{p}}{\partial \hat{x}_2}(\hat{d}_i),\ i = 3, 4;\ \frac{\sqrt{2}}{2}\left(\frac{\partial \hat{p}}{\partial \hat{x}_1} + \frac{\partial \hat{p}}{\partial \hat{x}_2}\right)(\hat{d}_i),\ i = 5, 6;\ \hat{p}(\hat{e}_i),\ i = 1, 2, 3 \Big]$$

Fig. 8. Basic finite element for the construction of a curved finite element C^1-compatible with the Argyris triangle ($F_K \in (P_3)^2$).

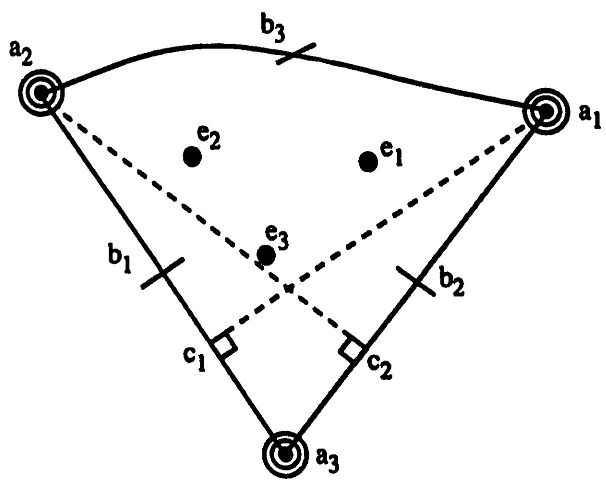


Fig. 9. The set of degrees of freedom Σ_K for the curved finite elements C^1-compatible with the Argyris triangle ($F_K \in (P_3)^2$).

$$\begin{aligned}\Sigma_k(v) = {} & [DLLC(v)]_{1\times 24} \\ = {} & [v(a_1);\ v(a_2);\ v(a_3);\ Dv(a_1)(a_3-a_1);\ (\bar{s}-\underline{s})Dv(a_1)\chi'(\underline{s}); \\ & (\underline{s}-\bar{s})Dv(a_2)\chi'(\bar{s});\ Dv(a_2)(a_3-a_2);\ Dv(a_3)(a_2-a_3); \\ & Dv(a_3)(a_1-a_3);\ D^2v(a_1)(a_3-a_1)^2;\ (\bar{s}-\underline{s})^2D^2v(a_1)(\chi'(\underline{s}))^2; \\ & (\underline{s}-\bar{s})^2D^2v(a_2)(\chi'(\bar{s}))^2;\ D^2v(a_2)(a_3-a_2)^2;\ D^2v(a_3)(a_2-a_3)^2; \\ & D^2v(a_3)(a_1-a_3)^2;\ D^2v(a_1)(a_2-a_3)^2;\ D^2v(a_2)(a_3-a_1)^2; \\ & (\bar{s}-\underline{s})^2D^2v(a_3)((\chi'(\underline{s}), \chi'(\bar{s}));\ Dv(b_1)(a_1-c_1);\ Dv(b_2)(a_2-c_2); \\ & Dv(b_3)DF_K(\hat{b}_3)-(\hat{a}_3-\hat{b}_3);\ v(e_1);\ v(e_2);\ v(e_3)]\,. \end{aligned} \tag{2.2.2}$$

STEP 2: Transition from $\Sigma_K(v)$ to $\hat{\Delta}_K(v)$. Starting from the set $\Sigma_K(v)$ of 24 elements, we introduce the set $\hat{\Delta}_K(v)$ of 36 values that we need to attribute to the set $\hat{\Sigma}$ of degrees of freedom in order to obtain a suitable interpolated function $\hat{w} \in \hat{P}$. In this way, it is convenient to introduce the following partition of the set $\hat{\Sigma}$:

$$\hat{\Sigma}(\hat{p}) = \hat{\Sigma}_1(\hat{p}) \cup \hat{\Sigma}_2(\hat{p}) \cup \hat{\Sigma}_3(\hat{p})\,,$$

where

$$\begin{aligned}\hat{\Sigma}_1(\hat{p}) &= \{(D^\alpha\hat{p}(\hat{a}_i); |\alpha| = 0,1,2; i = 1,2,3); \hat{p}(\hat{e}_i), i = 1,2,3\}\,, \\ \hat{\Sigma}_2(\hat{p}) &= \Big\{\hat{p}(\hat{d}_i), i = 1,\dots,4; -\frac{\partial\hat{p}}{\partial\hat{x}_1}(\hat{b}_1); -\frac{\partial\hat{p}}{\partial\hat{x}_1}(\hat{d}_i), i = 1,2; \\ &\qquad -\frac{\partial\hat{p}}{\partial\hat{x}_2}(\hat{b}_2); -\frac{\partial\hat{p}}{\partial\hat{x}_2}(\hat{d}_i), i = 3,4\Big\}\,, \\ \hat{\Sigma}_3(\hat{p}) &= \Big\{\hat{p}(\hat{d}_i), i = 5,6; \frac{\sqrt{2}}{2}\Big(\frac{\partial\hat{p}}{\partial\hat{x}_1} + \frac{\partial\hat{p}}{\partial\hat{x}_2}\Big)(\hat{b}_3); \frac{\sqrt{2}}{2}\Big(\frac{\partial\hat{p}}{\partial\hat{x}_1} + \frac{\partial\hat{p}}{\partial\hat{x}_2}\Big)(\hat{d}_i), i = 5,6\Big\}\,, \end{aligned} \tag{2.2.3}$$

and the associated partition of $\hat{\Delta}_K(v)$,

$$\hat{\Delta}_K(v) = \hat{\Delta}_{K1}(v) \cup \hat{\Delta}_{K2}(v) \cup \hat{\Delta}_{K3}(v)\,. \tag{2.2.4}$$

Then, from the knowledge of the set $\Sigma_K(v)$ of values of the degrees of freedom of v, we construct the corresponding set of values $\hat{\Delta}_{Ki}(v)$, $i = 1, 2, 3$.

CONSTRUCTION of $\hat{\Delta}_{K1}(v)$. The knowledge of the application F_K and of the set of values $\{(D^\alpha v(a_i),\ |\alpha| = 0,\ 1,\ 2;\ v(e_i)),\ i = 1,\ 2,\ 3\}$ leads immediately to $\hat{\Delta}_{K1}(v)$. It suffices to use relations (2.1.3) so that

$$\hat{\Delta}_{K1}(v) = \{(D^\alpha\hat{v}(\hat{a}_i), |\alpha| = 0,1,2; i = 1,2,3); \hat{v}(\hat{e}_i), i = 1,2,3\}\,. \tag{2.2.5}$$

This correspondence is detailed in Section 3.1.

CONSTRUCTION of $\hat{\Delta}_{K2}(v)$. The construction of this set of values is much more tricky. Firstly, let us examine the case of the degrees of freedom of $\hat{\Sigma}_2$ located on the side $\hat{a}_3\hat{a}_1$. To verify conditions (2.1.2), the expected interpolating function $\pi_K v$ has to satisfy:

(a) On the one hand, its trace $\pi_K v|_{[a_3,a_1]}$ coincides with the one-variable Hermite polynomial of degree 5, determined by the data of the degrees of freedom

$$\{v(a_1);\ v(a_3);\ Dv(a_1)(a_1-a_3);\ Dv(a_3)(a_1-a_3);\ D^2v(a_1)(a_1-a_3)^2;\ D^2v(a_3)(a_1-a_3)^2\}\,. \tag{2.2.6}$$

We will use $\hat{x}_1$ as a parameterization variable of the side a_3a_1, i.e.,

$$x_1 = x_{13} + (x_{11} - x_{13})\hat{x}_1\,, \qquad x_2 = x_{23} + (x_{21} - x_{23})\hat{x}_1\,, \tag{2.2.7}$$

where $x_{\alpha i}$ ($\alpha = 1, 2$; $i = 1, 2, 3$) notes the αth-coordinate of vertex a_i. The corresponding Hermite polynomial is named

$$\hat{f}_1 \text{ (which has to coincide with } (\pi_K v) \circ F_K|_{[\hat{a}_3,\hat{a}_1]})\,. \tag{2.2.8}$$

(b) On the other hand, the trace $(D\pi_K v(\cdot)(a_2 - c_2)) \circ F_K|_{[\hat{a}_3,\hat{a}_1]}$ of its normal derivative $D\pi_K v(\cdot)(a_2 - c_2)$ coincides with the one variable Hermite polynomial of degree 4 determined by the data of the degrees of freedom

$$\{Dv(a_i)(a_2 - c_2);\ D^2v(a_i)(a_2 - c_2, a_1 - a_3),\ i \in \{1,3\};\ Dv(b_2)(a_2 - c_2)\}\,. \tag{2.2.9}$$

In a similar way to (2.2.7) and (2.2.8), the corresponding Hermite polynomial is named

$$\hat{g}_1 \text{ (which has to coincide with } (D\pi_K v(\cdot)(a_2 - c_2)) \circ F_K|_{[\hat{a}_3,\hat{a}_1]})\,. \tag{2.2.10}$$

By analogy with the above definitions, we could define the following Hermite polynomials over the second straight side a_3a_2 of the triangle K, i.e.:

$$x_1 = x_{13} + (x_{12} - x_{13})\hat{x}_2\,, \qquad x_2 = x_{23} + (x_{22} - x_{23})\hat{x}_2\,, \tag{2.2.11}$$

$$\hat{f}_2 \text{ (which has to coincide with } (\pi_K v) \circ F_K|_{[\hat{a}_3,\hat{a}_2]})\,, \tag{2.2.12}$$

$$\hat{g}_2 \text{ (which has to coincide with } (D\pi_K v(\cdot)(a_1 - c_1)) \circ F_K|_{[\hat{a}_3,\hat{a}_2]})\,. \tag{2.2.13}$$

Then, the set of values $\hat{\Delta}_{K2}(v)$ associated with the set of degrees of freedom $\hat{\Sigma}_2$ (see (2.1.5), (2.1.6), (2.2.3)) is given by

$$\hat{\Delta}_{K2}(v) = \Big[\hat{f}_2(\hat{d}_i),\ i = 1, 2;\ \hat{f}_1(\hat{d}_i),\ i = 3, 4;$$
$$-\Big\langle \frac{\partial F_K}{\partial \hat{x}_1}(\hat{b}_1),\ \frac{a_2 - a_3}{|a_2 - a_3|^2}\,\frac{d\hat{f}_2}{d\hat{x}_2}(\hat{b}_1) + \frac{a_1 - c_1}{|a_1 - c_1|^2}\,\hat{g}_2(\hat{b}_1)\Big\rangle\,;$$

$$-\left\langle \frac{\partial F_K}{\partial \hat{x}_1}(\hat{d}_i), \frac{a_2 - a_3}{|a_2 - a_3|^2} \frac{\mathrm{d}\hat{f}_2}{\mathrm{d}\hat{x}_2}(\hat{d}_i) + \frac{a_1 - c_1}{|a_1 - c_1|^2} \hat{g}_2(\hat{d}_i) \right\rangle, i = 1, 2 ;$$

$$-\left\langle \frac{\partial F_K}{\partial \hat{x}_2}(\hat{b}_2), \frac{a_1 - a_3}{|a_1 - a_3|^2} \frac{\mathrm{d}\hat{f}_1}{\mathrm{d}\hat{x}_1}(\hat{b}_2) + \frac{a_2 - c_2}{|a_2 - c_2|^2} \hat{g}_1(\hat{b}_2) \right\rangle ;$$

$$-\left\langle \frac{\partial F_K}{\partial \hat{x}_2}(\hat{d}_i), \frac{a_1 - a_3}{|a_1 - a_3|^2} \frac{\mathrm{d}\hat{f}_1}{\mathrm{d}\hat{x}_1}(\hat{d}_i) + \frac{a_2 - c_2}{|a_2 - c_2|^2} \hat{g}_1(\hat{d}_i) \right\rangle, i = 3, 4 \Big] . \quad (2.2.14)$$

Note that the definition of the last six elements uses relations (2.1.6) and (2.1.5), specifications (2.2.12), (2.2.13), (2.2.8), (2.2.10), and

$$\begin{aligned} &\frac{\mathrm{d}\hat{f}_2}{\mathrm{d}\hat{x}_2} \text{ which will have to coincide with } (D\pi_K v(\cdot)(a_2 - a_3)) \circ F_K|_{[\hat{a}_3, \hat{a}_2]} , \\ &\frac{\mathrm{d}\hat{f}_1}{\mathrm{d}\hat{x}_1} \text{ which will have to coincide with } (D\pi_K v(\cdot)(a_1 - a_3)) \circ F_K|_{[\hat{a}_3, \hat{a}_1]} . \end{aligned} \quad (2.2.15)$$

CONSTRUCTION of $\hat{\Delta}_{K3}(v)$. It remains to compute the values that we have to attribute to the five degrees of freedom of $\hat{\Sigma}_3$. Firstly, the correspondence $\hat{v} = v \circ F_K$ leads to

$$D\hat{v}(\hat{b}_3)(\hat{a}_3 - \hat{b}_3) = Dv(b_3) DF_K(\hat{b}_3)(\hat{a}_3 - \hat{b}_3) , \quad (2.2.16)$$

which is known since this value is among the degrees of freedom of Σ_K (see (2.2.2)).

Next, the values $D^\alpha \hat{v}(\hat{a}_i)$, $|\alpha| = 0, 1, 2$, $i = 1, 2, 3$, are listed in (2.2.5), while the side $\hat{a}_1\hat{a}_2$ of the triangle $\hat{K}$ can be parameterized by

$$\hat{x}_1 = \hat{x}_1 , \qquad \hat{x}_2 = 1 - \hat{x}_1 . \quad (2.2.17)$$

Then, let $\hat{f}_3(\hat{x}_1)$ be the Hermite polynomial of degree 5, defined by the data of the degrees of freedom

$$\begin{aligned} &\{\hat{v}(\hat{a}_1); \hat{v}(\hat{a}_2); D\hat{v}(\hat{a}_1)(\hat{a}_1 - \hat{a}_2); D\hat{v}(\hat{a}_2)(\hat{a}_1 - \hat{a}_2); D^2\hat{v}(\hat{a}_1)(\hat{a}_1 - \hat{a}_2)^2; \\ &\quad D^2\hat{v}(\hat{a}_2)(\hat{a}_1 - \hat{a}_2)^2\} , \end{aligned} \quad (2.2.18)$$

and let $\hat{g}_3(\hat{x}_1)$ be the Hermite polynomial of degree 4, determined by the data of the degrees of freedom

$$\begin{aligned} &\{D\hat{v}(\hat{a}_\alpha)(\hat{a}_3 - \hat{b}_3), \alpha = 1, 2; D^2\hat{v}(\hat{a}_1)(\hat{a}_3 - \hat{b}_3, \hat{a}_1 - \hat{a}_2); \\ &\quad D^2\hat{v}(\hat{a}_2)(\hat{a}_3 - \hat{b}_3, \hat{a}_1 - \hat{a}_2); D\hat{v}(\hat{b}_3)(\hat{a}_3 - \hat{b}_3)\} . \end{aligned} \quad (2.2.19)$$

Then, we set

$$\hat{\Delta}_{K3}(v) = \{\hat{f}_3(\tfrac{3}{4}); \hat{f}_3(\tfrac{1}{4}); -\sqrt{2}\hat{g}_3(\tfrac{1}{2}); -\sqrt{2}\hat{g}_3(\tfrac{3}{4}); -\sqrt{2}\hat{g}_3(\tfrac{1}{4})\} . \quad (2.2.20)$$

Thus, only from the knowledge of the values $\Sigma_K(v)$ of the degrees of freedom of the function v, the relations (2.2.5), (2.2.14) and (2.2.20) assign one and only one value to every degree of freedom of $\hat{\Sigma}$. In view of the implementation, we give a matrix presentation of these correspondences in Section 3.1.

STEP 3: Transition from the set of values $\hat{\Delta}_K(v)$ to the interpolate function $w = \pi_K v$. Let $\hat{w}$ be the function of $\hat{P}$ which takes the set of values $\hat{\Delta}_K(v)$ on the set of degrees of freedom $\hat{\Sigma}$ (see Fig. 8). Then, with $\hat{w}$, defined on the reference triangle $\hat{K}$, relations (2.1.3) associate the function

$$\pi_K v = w = \hat{w} \circ (F_K)^{-1}, \tag{2.2.21}$$

where the function $(F_K)^{-1}$ is the inverse of the function $F_K: \hat{K} \to K$ given by relation (1.3.2).

Finally, according to [2], this function w is exactly the expected function $\pi_K v$, i.e., $w = \pi_K v$, since:

THEOREM 2.2.1. The function w, defined by (2.2.21), is determined in a unique way by the data (2.2.2) of the local version $[DLLC(v)]$ of the set $\Sigma_K(v)$ of the values of the degrees of freedom of the function v to be interpolated. Moreover, the function w satisfies

$$\begin{aligned}
&D^\alpha w(a_i) = D^\alpha v(a_i), \qquad |\alpha| = 0, 1, 2, \quad i = 1, 2, 3, \\
&Dw(b_1)(a_1 - c_1) = Dv(b_1)(a_1 - c_1), \\
&Dw(b_2)(a_2 - c_2) = Dv(b_2)(a_2 - c_2), \\
&Dw(b_3)DF_K(\hat{b}_3)(\hat{a}_3 - \hat{b}_3) = Dv(b_3)DF_K(\hat{b}_3)(\hat{a}_3 - \hat{b}_3), \\
&w(e_i) = v(e_i), \quad i = 1, 2, 3,
\end{aligned} \tag{2.2.22}$$

and the conditions (2.1.1), (2.1.2) and (2.1.4). Thus $w = \pi_K v$.

2.3. *Definition of curved finite elements C^1-compatible with the Argyris triangle for $F_K^* \in (P_5)^2$*

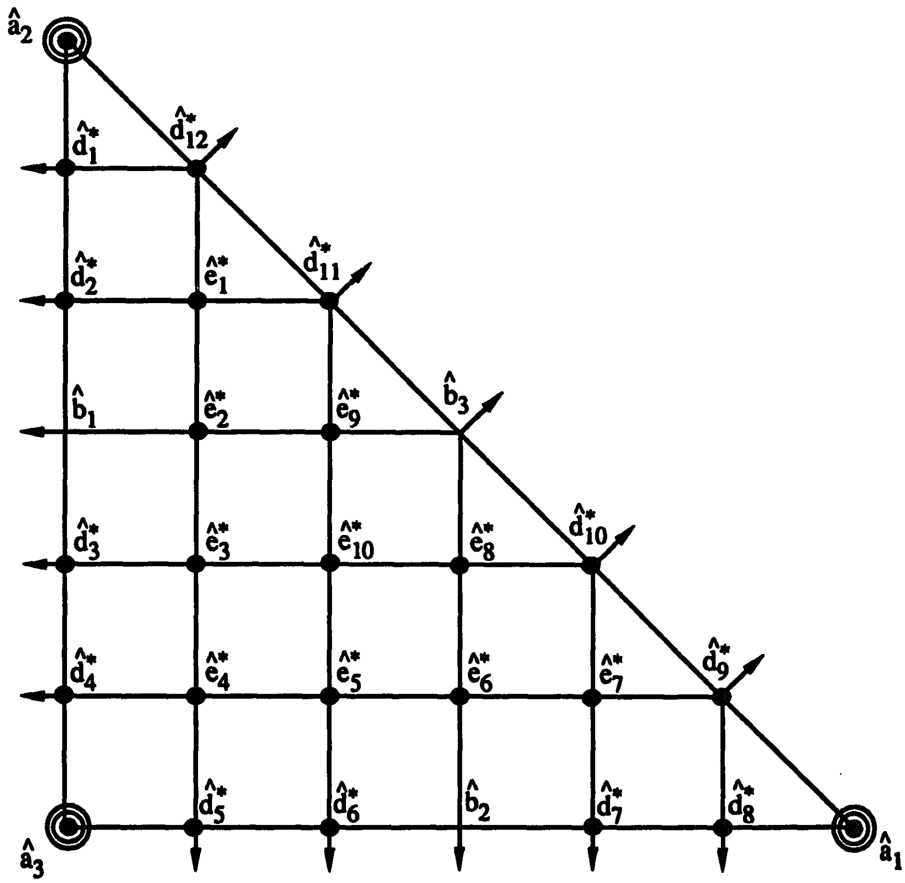
Since the construction is very similar to the case $F_K \in (P_3)^2$, we only detail the new aspects. Firstly, relation (2.1.7) involves $\hat{P}_K^* \subset P_9$ (let us record that a star (*) makes the distinction between the cases $F_K \in (P_3)^2$ and $F_K^* \in (P_5)^2$).

2.3.1. *The basic finite element $(\hat{K}, \hat{P}^*, \hat{\Sigma}^*)$*

This element is described in Fig. 10 and basis functions are given by (3.2.25).

2.3.2. *Construction of the interpolating function $v \to \pi_K^* v$*

As in Section 2.2, this construction can be split into three steps.



$\hat{K}$ = unit right-angled triangle; $\hat{P}^* = P_9$; $\dim \hat{P}^* = 55$

$$\hat{\Sigma}^*(\hat{p}) = \Big\{ \hat{p}(\hat{a}_i);\ \frac{\partial \hat{p}}{\partial \hat{x}_1}(\hat{a}_i);\ \frac{\partial \hat{p}}{\partial \hat{x}_2}(\hat{a}_i);\ \frac{\partial^2 \hat{p}}{\partial \hat{x}_1^2}(\hat{a}_i);\ \frac{\partial^2 \hat{p}}{\partial \hat{x}_1\,\partial \hat{x}_2}(\hat{a}_i);\ \frac{\partial^2 \hat{p}}{\partial \hat{x}_2^2}(\hat{a}_i);\ i = 1, 2, 3;$$

$$-\frac{\partial \hat{p}}{\partial \hat{x}_1}(\hat{b}_1);\ -\frac{\partial \hat{p}}{\partial \hat{x}_2}(\hat{b}_2);\ \frac{\sqrt{2}}{2}\Big(\frac{\partial \hat{p}}{\partial \hat{x}_1} + \frac{\partial \hat{p}}{\partial \hat{x}_2}\Big)(\hat{b}_3);\ \hat{p}(\hat{d}_i^*),\ i = 1, \ldots, 12;$$

$$-\frac{\partial \hat{p}}{\partial \hat{x}_1}(\hat{d}_i^*),\ i = 1, \ldots, 4;\ -\frac{\partial \hat{p}}{\partial \hat{x}_2}(\hat{d}_i^*),\ i = 5, \ldots, 8;$$

$$\frac{\sqrt{2}}{2}\Big(\frac{\partial \hat{p}}{\partial \hat{x}_1} + \frac{\partial \hat{p}}{\partial \hat{x}_2}\Big)(\hat{d}_i^*),\ i = 9, \ldots, 12;\ \hat{p}(\hat{e}_i^*),\ i = 1, \ldots, 10 \Big\}$$

Fig. 10. Basic finite element for the construction of a curved finite element C^1-compatible with the Argyris triangle ($F_K^* \in (P_5)^2$).

STEP 1: Definition of the set Σ_K^* of degrees of freedom of the curved element. Set (see Figs. 10 and 11)

$$a_i = F_K^*(\hat{a}_i)\,,\quad b_i = F_K^*(\hat{b}_i)\,,\quad i = 1, 2, 3;\qquad e_i^* = F_K^*(\hat{e}_i^*)\,,\quad i = 1, \ldots, 10\,, \tag{2.3.1}$$

while c_α is still the orthogonal projection of a_α on the side a_3a_α, $\alpha = 1, 2$. Then the set $\Sigma_K^*(v)$ of degrees of freedom of the function $v \in C^2(K)$ is given, in its local version [$DLLC^*(v)$], by

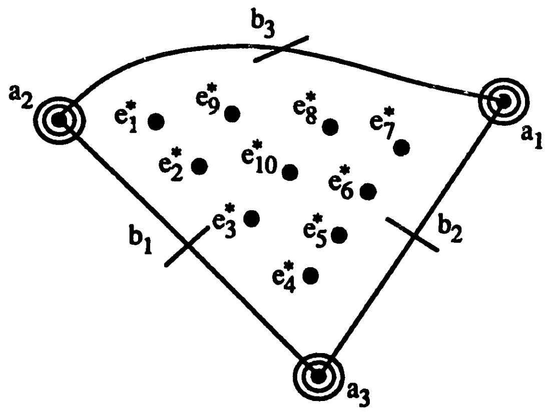


Fig. 11. The set of degrees of freedom Σ_K for the curved finite elements C^1-compatible with the Argyris triangle ($F_K^* \in (P_5)^2$).

$$\begin{aligned}\Sigma_K^*(v) = [DLLC^*(v)]_{1\times 31} = [&v(a_1);\; v(a_2);\; v(a_3);\; Dv(a_1)(a_3-a_1);\\ &(\bar{s}-\underline{s})Dv(a_1)\chi'(\underline{s});\; (\underline{s}-\bar{s})Dv(a_2)\chi'(\bar{s});\; Dv(a_2)(a_3-a_2);\\ &Dv(a_3)(a_2-a_3);\; Dv(a_3)(a_1-a_3);\; D^2v(a_1)(a_3-a_1)^2;\\ &(\bar{s}-\underline{s})^2D^2v(a_1)(\chi'(\underline{s}))^2;\; (\underline{s}-\bar{s})^2D^2v(a_2)(\chi'(\bar{s}))^2;\; D^2v(a_2)(a_3-a_2)^2;\\ &D^2v(a_3)(a_2-a_3)^2;\; D^2v(a_3)(a_1-a_3)^2;\; D^2v(a_1)(a_2-a_3)^2;\\ &D^2v(a_2)(a_3-a_1)^2;\; (\bar{s}-\underline{s})^2D^2v(a_3)((\chi'(\underline{s}),\chi'(\bar{s}));\; Dv(b_1)(a_1-c_1);\\ &Dv(b_2)(a_2-c_2);\; Dv(b_3)DF_K^*(\hat{b}_3)(\hat{a}_3-\hat{b}_3);\; v(e_1^*)\dots v(e_{10}^*)]\,. \end{aligned} \tag{2.3.2}$$

STEP 2: Transition from $\Sigma_K^*(v)$ to $\hat{\Delta}_K^*(v)$. Starting from the set $\Sigma_K^*(v)$ including 31 elements, we define the set $\hat{\Delta}_K^*(v)$ of 55 values that we need to attribute to the set $\hat{\Sigma}^*$ of degrees of freedom in order to obtain a suitable interpolated function $\hat{w} \in \hat{P}$. In this way, we consider the following partition of the set $\hat{\Sigma}^*$:

$$\hat{\Sigma}^*(\hat{p}) = \hat{\Sigma}_1^*(\hat{p}) \cup \hat{\Sigma}_2^*(\hat{p}) \cup \hat{\Sigma}_3^*(\hat{p})\,,$$

where

$$\begin{aligned}\hat{\Sigma}_1^*(\hat{p}) &= \{(D^\alpha\hat{p}(\hat{a}_i);\, |\alpha| = 0,1,2;\, i=1,2,3);\; \hat{p}(\hat{e}_i^*),\, i=1,\dots,10\}\,,\\ \hat{\Sigma}_2^*(\hat{p}) &= \Big\{\hat{p}(\hat{d}_i^*),\, i=1,\dots,8;\; -\frac{\partial\hat{p}}{\partial\hat{x}_1}(\hat{b}_1);\; -\frac{\partial\hat{p}}{\partial\hat{x}_1}(\hat{d}_i),\, i=1,\dots,4;\\ &\qquad -\frac{\partial\hat{p}}{\partial\hat{x}_2}(\hat{b}_2);\; -\frac{\partial\hat{p}}{\partial\hat{x}_2}(\hat{d}_i),\, i=5,\dots,8\Big\}\,, \qquad\qquad (2.3.3)\\ \hat{\Sigma}_3^*(\hat{p}) &= \Big\{\hat{p}(\hat{d}_i^*),\, i=9,\dots,12;\; \frac{\sqrt{2}}{2}\Big(\frac{\partial\hat{p}}{\partial\hat{x}_1}+\frac{\partial\hat{p}}{\partial\hat{x}_2}\Big)(\hat{b}_3)\,;\\ &\qquad \frac{\sqrt{2}}{2}\Big(\frac{\partial\hat{p}}{\partial\hat{x}_1}+\frac{\partial\hat{p}}{\partial\hat{x}_2}\Big)(\hat{d}_i^*),\, i=9,\dots,12\Big\}\,.\end{aligned}$$

As in Section 2.2, with $\Sigma_K^*(v)$ we associate the following partition of the set $\hat{\Delta}_K^*(v)$:

$$\hat{\Delta}_K^*(v) = \hat{\Delta}_{K1}^*(v) \cup \hat{\Delta}_{K2}^*(v) \cup \hat{\Delta}_{K3}^*(v)\,, \tag{2.3.4}$$

where each $\hat{\Delta}_{Ki}^*$ is associated with the corresponding $\hat{\Sigma}_i^*$, $i = 1, 2, 3$. We obtain successively:

CONSTRUCTION of $\hat{\Delta}_{K1}^*(v)$. The knowledge of F_K^* and of the set of values $\{(D^\alpha v(a_i), |\alpha| = 0, 1, 2, i = 1, 2, 3); v(e_i^*), i = 1, \dots, 10\}$ and relation $\hat{v} = v \circ F_K^*$ immediately involve the knowledge of

$$\hat{\Delta}_{K1}^*(v) = \{(D^\alpha \hat{v}(\hat{a}_i), |\alpha| = 0,1,2; i = 1,2,3); \hat{v}(\hat{e}_i^*), i = 1, \dots, 10\}\,. \tag{2.3.5}$$

CONSTRUCTION of $\hat{\Delta}_{K2}^*(v)$. The functions $\hat{f}_1$, $\hat{g}_1$, $\hat{f}_2$, $\hat{g}_2$ determined in Section 2.2 play exactly the same role here and, since $F_K|_{[\hat{a}_3,\hat{a}_\alpha]} \equiv F_K^*|_{[\hat{a}_3,\hat{a}_\alpha]}$, $\alpha = 1,2$, are affine, they are identical. Then, by similarity, we obtain

$$\begin{aligned}\hat{\Delta}_{K2}^*(v) = \Big[\, & \hat{f}_2(\hat{d}_i^*), i = 1, \dots, 4; \hat{f}_1(\hat{d}_i^*), i = 5, \dots, 8\,;\\
& -\Big\langle \frac{\partial F_K^*}{\partial \hat{x}_1}(\hat{b}_1), \frac{a_2 - a_3}{|a_2 - a_3|^2}\frac{\mathrm{d}\hat{f}_2}{\mathrm{d}\hat{x}_2}(\hat{b}_1) + \frac{a_1 - c_1}{|a_1 - c_1|^2}\hat{g}_2(\hat{b}_1)\Big\rangle\,;\\
& -\Big\langle \frac{\partial F_K^*}{\partial \hat{x}_1}(\hat{d}_i^*), \frac{a_2 - a_3}{|a_2 - a_3|^2}\frac{\mathrm{d}\hat{f}_2}{\mathrm{d}\hat{x}_2}(\hat{d}_i^*) + \frac{a_1 - c_1}{|a_1 - c_1|^2}\hat{g}_2(\hat{d}_i^*)\Big\rangle, i = 1, \dots, 4;\\
& -\Big\langle \frac{\partial F_K^*}{\partial \hat{x}_2}(\hat{b}_2), \frac{a_1 - a_3}{|a_1 - a_3|^2}\frac{\mathrm{d}\hat{f}_1}{\mathrm{d}\hat{x}_1}(\hat{b}_2) + \frac{a_2 - c_2}{|a_2 - c_2|^2}\hat{g}_1(\hat{b}_2)\Big\rangle\,;\\
& -\Big\langle \frac{\partial F_K^*}{\partial \hat{x}_2}(\hat{d}_i^*), \frac{a_1 - a_3}{|a_1 - a_3|^2}\frac{\mathrm{d}\hat{f}_1}{\mathrm{d}\hat{x}_1}(\hat{d}_i^*) + \frac{a_2 - c_2}{|a_2 - c_2|^2}\hat{g}_1(\hat{d}_i^*)\Big\rangle, i = 5, \dots, 8\Big]\,.\end{aligned} \tag{2.3.6}$$

CONSTRUCTION of $\hat{\Delta}_{K3}^*(v)$. Since the restrictions to $[\hat{a}_1, \hat{a}_2]$ of F_K^* and F_K are different, the associated functions $\hat{f}_3^*$ and $\hat{g}_3^*$ are generally different from $\hat{f}_3$ and $\hat{g}_3$. We obtain

$$\hat{\Delta}_{K3}^*(v) = \{\hat{f}_3^*(\hat{d}_i^*), i = 9, \dots, 12; -\sqrt{2}\hat{g}_3^*(\hat{b}_3); -\sqrt{2}\hat{g}_3^*(\hat{d}_i^*), i = 9, \dots, 12\}\,. \tag{2.3.7}$$

STEP 3: Transition from $\hat{\Delta}_K^*(v)$ *to the function* $w^* = \pi_K^* v$. Let $\hat{w}^* \in \hat{P}^*$ be the polynomial of degree 9 which takes the values $\hat{\Delta}_K^*(v)$ in the set of degrees of freedom $\hat{\Sigma}^*$. Then

$$\pi_K^* v = w^* = \hat{w}^* \circ (F_K^*)^{-1}\,. \tag{2.3.8}$$

In [2], we have proved that this function $w^* = \pi_K^* v$ satisfies all the expected results.

THEOREM 2.3.1. The function w^*, defined by (2.3.8), is determined in a unique way by the data of the set $\Sigma_K^*(v)$ of the values of the degrees of freedom of the function v. Moreover, the function w^* satisfies

$$\begin{aligned}
&D^\alpha w^*(a_i) = D^\alpha v(a_i)\,, \quad |\alpha| = 0,1,2, \quad i=1,2,3\,,\\
&Dw^*(b_1)(a_1 - c_1) = Dv(b_1)(a_1 - c_1)\,,\\
&Dw^*(b_2)(a_2 - c_2) = Dv(b_2)(a_2 - c_2)\,,\\
&Dw^*(b_3)DF_K^*(\hat b_3)(\hat a_3 - \hat b_3) = Dv(b_3)DF_K^*(\hat b_3)(\hat a_3 - \hat b_3)\,,\\
&w(e_i^*) = v(e_i^*)\,, \quad i=1,\dots,10\,,
\end{aligned} \tag{2.3.9}$$

and the conditions (2.1.1), (2.1.2) and (2.1.4). Thus $w^* = \pi_K^* v$.

3. Implementation of curved finite elements C^1-compatible with the Argyris triangle

In this section, we detail the matrix decompositions which allow us to easily realize the interpolation modules related to both curved finite elements C^1 compatible with the Argyris triangle.

3.1. Interpolation modules associated with the curved finite element C^1-compatible with the Argyris triangle when $F_K \in (P_3)^2$

Let v be a function of $C^2(\bar K)$. By using the construction detailed in Section 2.2, we are able to define its interpolate $\pi_K v$, i.e.,

$$\pi_K v(x) = w(x) = \hat w(\hat x)\,. \tag{3.1.1}$$

Subsequently we will use this function $\hat w$ on $\hat K$ instead of w on K. This is usual in finite element implementation; in particular this allows us to use numerical integration schemes on $\hat K$. Nevertheless, note that in [9] we took advantage of the linearity of F_K (we only considered straight triangles), of the properties of barycentric coordinates and of the eccentricity parameters to work directly on K.

Now it remains to transcribe the relation (3.1.1) in matrix expressions. As in Section 2.2, we consider three steps:

STEP 1: $v \to \Sigma_K(v) = [DLLC(v)]$. The application F_K is given in Example 1.3.1, while $\Sigma_K(v)$ is the following set of values (see (2.2.2)):

$$\begin{aligned}
[DLLC(v)]_{1\times 24} = [&v(a_1);\ v(a_2);\ v(a_3);\ Dv(a_1)(a_3 - a_1);\ (\bar s - \underline{s})Dv(a_1)\chi'(\underline{s});\\
&-(\bar s - \underline{s})Dv(a_2)\chi'(\bar s);\ Dv(a_2)(a_3 - a_2);\ Dv(a_3)(a_2 - a_3);\\
&Dv(a_3)(a_1 - a_3);\ D^2v(a_1)(a_3 - a_1)^2;\ (\bar s - \underline{s})^2 D^2 v(a_1)(\chi'(\underline{s}))^2;\\
&(\bar s - \underline{s})^2 D^2 v(a_2)(\chi'(\bar s))^2;\ D^2 v(a_2)(a_3 - a_2)^2;\ \ D^2 v(a_3)(a_2 - a_3)^2;
\end{aligned}$$

$$D^2v(a_3)(a_1-a_3)^2;\ D^2v(a_1)(a_2-a_3)^2;\ D^2v(a_2)(a_3-a_1)^2;$$
$$(\bar{s}-\underline{s})^2D^2v(a_3)((\chi'(\underline{s}),\chi'(\bar{s}));\ Dv(b_1)(a_1-c_1);\ Dv(b_2)(a_2-c_2);$$
$$Dv(b_3)DF_K(\hat{b}_3)(\hat{a}_3-\hat{b}_3);\ v(e_1);\ v(e_2);\ v(e_3)]\,. \tag{3.1.2}$$

When the side a_1a_2 is straight, note that the first 20 degrees of freedom become identical to the first 20 local degrees of freedom of the Argyris element (see [9, (3.1.18)]) since

$$\chi'(s)=\frac{1}{\bar{s}-\underline{s}}\sum_{\alpha=1}^{2}(\chi_\alpha(\bar{s})-\chi_\alpha(\underline{s}))e_\alpha=\frac{1}{\bar{s}-\underline{s}}(a_2-a_1)\,. \tag{3.1.3}$$

With the set (3.1.2), we associate the set of global degrees of freedom

$$\begin{aligned}[DLGL(v)]_{1\times 24}=[&v(a_1);\ v(a_2);\ v(a_3);\ \frac{\partial v}{\partial x_1}(a_1);\ \frac{\partial v}{\partial x_2}(a_1);\ \frac{\partial v}{\partial x_1}(a_2);\ \frac{\partial v}{\partial x_2}(a_2);\\ &\frac{\partial v}{\partial x_1}(a_3);\ \frac{\partial v}{\partial x_2}(a_3);\ \frac{\partial^2 v}{\partial x_1^2}(a_1);\ \frac{\partial^2 v}{\partial x_1\partial x_2}(a_1);\ \frac{\partial^2 v}{\partial x_2^2}(a_1);\ \frac{\partial^2 v}{\partial x_1^2}(a_2);\\ &\frac{\partial^2 v}{\partial x_1\,\partial v_2}(a_2);\ \frac{\partial^2 v}{\partial x_2^2}(a_2);\ \frac{\partial^2 v}{\partial x_1^2}(a_3);\ \frac{\partial^2 v}{\partial x_1\,\partial x_2}(a_3);\ \frac{\partial^2 v}{\partial x_2^2}(a_3);\ \frac{\partial v}{\partial \nu_1}(b_1);\\ &\frac{\partial v}{\partial \nu_2}(b_2);\ Dv(b_3)DF_K(\hat{b}_3)(\hat{a}_3-\hat{b}_3);\ v(e_1);\ v(e_2);\ v(e_3)]\,.\end{aligned} \tag{3.1.4}$$

Local and global degrees of freedom are linked by the relation

$$[DLLC(v)]_{1\times 24}=[DLGL(v)]_{1\times 24}[\tilde{D}]_{24\times 24}\,, \tag{3.1.5}$$

with

$$\tilde{D}=\begin{bmatrix} I_3 & & & & & & & \\ & \tilde{d}_1 & & & & & & \\ & & \tilde{d}_2 & & & 0 & & \\ & & & \tilde{d}_3 & & & & \\ & & & & \tilde{d}_4 & & & \\ & 0 & & & & n_1 & & \\ & & & & & & n_2 & \\ & & & & & & & I_4 \end{bmatrix}, \tag{3.1.6}$$

where, by using relations (1.3.3) and by noting $a_i(x_{1i}, x_{2i})$, we have

$$\tilde{d}_1 = \begin{bmatrix} X_{31} & \psi_1'(0) \\ Y_{31} & \psi_2'(0) \end{bmatrix}, \qquad \tilde{d}_2 = \begin{bmatrix} -\psi_1'(1) & X_{32} \\ -\psi_2'(1) & Y_{32} \end{bmatrix}, \qquad \tilde{d}_3 = \begin{bmatrix} X_{23} & X_{13} \\ Y_{23} & Y_{13} \end{bmatrix}, \tag{3.1.7}$$

$$\hat{d}_4 = \begin{bmatrix} (X_{31})^2 & (\psi_1'(0))^2 & 0 & 0 & 0 & 0 & (X_{32})^2 & 0 & 0 \\ 2X_{31}Y_{31} & 2\psi_1'(0)\psi_2'(0) & 0 & 0 & 0 & 0 & 2X_{32}Y_{32} & 0 & 0 \\ (Y_{31})^2 & (\psi_2'(0))^2 & 0 & 0 & 0 & 0 & (Y_{32})^2 & 0 & 0 \\ 0 & 0 & (\psi_1'(1))^2 & (X_{32})^2 & 0 & 0 & 0 & (X_{31})^2 & 0 \\ 0 & 0 & 2\psi_1'(1)\psi_2'(1) & 2X_{32}Y_{32} & 0 & 0 & 0 & 2X_{31}Y_{31} & 0 \\ 0 & 0 & (\psi_2'(1))^2 & (Y_{32})^2 & 0 & 0 & 0 & (Y_{31})^2 & 0 \\ 0 & 0 & 0 & 0 & (X_{32})^2 & (X_{31})^2 & 0 & 0 & \psi_1'(0)\psi_1'(1) \\ 0 & 0 & 0 & 0 & 2X_{32}Y_{32} & 2X_{31}Y_{31} & 0 & 0 & \psi_1'(0)\psi_2'(1)+\psi_2'(0)\psi_1'(1) \\ 0 & 0 & 0 & 0 & (Y_{32})^2 & (Y_{31})^2 & 0 & 0 & \psi_2'(0)\psi_2'(1) \end{bmatrix}, \tag{3.1.8}$$

with $X_{ij} = x_{1i} - x_{1j}$, $Y_{ij} = x_{2i} - x_{2j}$, $1 \leqslant i, j \leqslant 3$, and n_1 and n_2 as given in [9, (3.1.25)].

STEP 2: Transition from $\Sigma_K(v) = [DLLC(v)]$ to $\hat{\Delta}_K(v) = [DL(\hat{w})]$. In the fixed reference triangle $\hat{K}$, i.e., the unit right-angled triangle, we introduce only one set of degrees of freedom (see Fig. 8),

$$\begin{aligned}
[DL(\hat{w})]_{1\times 36} = \Big[& \hat{w}(\hat{a}_1);\ \hat{w}(\hat{a}_2);\ \hat{w}(\hat{a}_3);\ \frac{\partial \hat{w}}{\partial \hat{x}_1}(\hat{a}_1);\ \frac{\partial \hat{w}}{\partial \hat{x}_2}(\hat{a}_1);\ \frac{\partial \hat{w}}{\partial \hat{x}_1}(\hat{a}_2);\ \frac{\partial \hat{w}}{\partial \hat{x}_2}(\hat{a}_2); \\
& \frac{\partial \hat{w}}{\partial \hat{x}_1}(\hat{a}_3);\ \frac{\partial \hat{w}}{\partial \hat{x}_2}(\hat{a}_3);\ \frac{\partial^2 \hat{w}}{\partial \hat{x}_1^2}(\hat{a}_1);\ \frac{\partial^2 \hat{w}}{\partial \hat{x}_1\, \partial \hat{x}_2}(\hat{a}_1);\ \frac{\partial^2 \hat{w}}{\partial \hat{x}_2^2}(\hat{a}_1);\ \frac{\partial^2 \hat{w}}{\partial \hat{x}_1^2}(\hat{a}_2); \\
& \frac{\partial^2 \hat{w}}{\partial \hat{x}_1\, \partial \hat{x}_2}(\hat{a}_2);\ \frac{\partial^2 \hat{w}}{\partial \hat{x}_2^2}(\hat{a}_2);\ \frac{\partial^2 \hat{w}}{\partial \hat{x}_1^2}(\hat{a}_3);\ \frac{\partial^2 \hat{w}}{\partial \hat{x}_1\, \partial \hat{x}_2}(\hat{a}_3);\ \frac{\partial^2 \hat{w}}{\partial \hat{x}_2^2}(\hat{a}_3); \\
& -\frac{\partial \hat{w}}{\partial \hat{x}_1}(\hat{b}_1);\ -\frac{\partial \hat{w}}{\partial \hat{x}_2}(\hat{b}_2);\ \frac{\sqrt{2}}{2}\Big(\frac{\partial \hat{w}}{\partial \hat{x}_1} + \frac{\partial \hat{w}}{\partial \hat{x}_2}\Big)(\hat{b}_3); \\
& \hat{w}(\hat{d}_1);\ \hat{w}(\hat{d}_2);\ \hat{w}(\hat{d}_3);\ \hat{w}(\hat{d}_4);\ \hat{w}(\hat{d}_5);\ \hat{w}(\hat{d}_6); \\
& -\frac{\partial \hat{w}}{\partial \hat{x}_1}(\hat{d}_1);\ -\frac{\partial \hat{w}}{\partial \hat{x}_1}(\hat{d}_2);\ -\frac{\partial \hat{w}}{\partial \hat{x}_2}(\hat{d}_3);\ -\frac{\partial \hat{w}}{\partial \hat{x}_2}(\hat{d}_4); \\
& \frac{\sqrt{2}}{2}\Big(\frac{\partial \hat{w}}{\partial \hat{x}_1} + \frac{\partial \hat{w}}{\partial \hat{x}_2}\Big)(\hat{d}_5);\ \frac{\sqrt{2}}{2}\Big(\frac{\partial \hat{w}}{\partial \hat{x}_1} + \frac{\partial \hat{w}}{\partial \hat{x}_2}\Big)(\hat{d}_6);\ \hat{w}(\hat{e}_1);\ \hat{w}(\hat{e}_2);\ \hat{w}(\hat{e}_3) \Big].
\end{aligned} \tag{3.1.9}$$

Then, Step 2 of Section 2.2 leads to

$$[DL(\hat{w})]_{1\times 36} = [DLLC(v)]_{1\times 24}[B]_{24\times 36}, \tag{3.1.10}$$

where the matrix B is piecewise constructed below. Then relations (3.1.5) and (3.1.10) give

$$[DL(\hat{w})]_{1\times 36} = [DLGL(v)]_{1\times 24}[\tilde{D}]_{24\times 24}[B]_{24\times 36}\,. \tag{3.1.11}$$

By consideration of relations (3.1.9) and (3.1.10), it is natural to realize the following partition of matrix B:

$$[B]_{24\times 36} = [B_1 \quad B_2 \quad B_3 \quad B_4 \quad B_5 \quad B_6 \quad B_7]\,, \tag{3.1.12}$$

where submatrices B_i, $i = 1, \ldots, 7$, have 24 lines and 3, 6, 9, 3, 6, 6 and 3 columns, respectively, and are determined as follows.

CONSTRUCTION of submatrix B_1. From (2.2.21) and (2.2.22), we obtain $\hat{w}(\hat{a}_i) = v(a_i)$, $i = 1, 2, 3$, so that

$$^tB_1 = [I_3; 0_{3\times 21}]\,. \tag{3.1.13}$$

CONSTRUCTION of submatrix B_2. Since $\hat{w} = w \circ F_K$, we obtain

$$\frac{\partial \hat{w}}{\partial \hat{x}_\alpha}(\hat{x}) = D\hat{w}(\hat{x})\hat{e}_\alpha = Dw(x)DF_K(\hat{x})\hat{e}_\alpha = Dw(x)\frac{\partial F_K}{\partial \hat{x}_\alpha}(\hat{x})\,,$$

so that relation (2.2.22) involves

$$\frac{\partial \hat{w}}{\partial \hat{x}_\alpha}(\hat{a}_i) = Dv(a_i)\frac{\partial F_K}{\partial \hat{x}_\alpha}(\hat{a}_i)\,, \quad i = 1, 2, 3\,, \tag{3.1.14}$$

and with (1.3.2)

$$\begin{aligned}
&\frac{\partial \hat{w}}{\partial \hat{x}_1}(\hat{a}_1) = Dv(a_1)(a_1 - a_3)\,, && \frac{\partial \hat{w}}{\partial \hat{x}_2}(\hat{a}_1) = Dv(a_1)[a_1 - a_3 + (\bar{s} - \underline{s})\chi'(\underline{s})]\,,\\
&\frac{\partial \hat{w}}{\partial \hat{x}_1}(\hat{a}_2) = Dv(a_2)[a_2 - a_3 - (\bar{s} - \underline{s})\chi'(\bar{s})]\,, && \frac{\partial \hat{w}}{\partial \hat{x}_2}(\hat{a}_2) = Dv(a_2)(a_2 - a_3)\,,\\
&\frac{\partial \hat{w}}{\partial \hat{x}_1}(\hat{a}_3) = Dv(a_3)(a_1 - a_3)\,, && \frac{\partial \hat{w}}{\partial \hat{x}_2}(\hat{a}_3) = Dv(a_3)(a_2 - a_3)\,.
\end{aligned} \tag{3.1.15}$$

From these expressions and relations (3.1.2), (3.1.9), (3.1.10) and (3.1.12), we obtain

$$^tB_2 = [0_{6\times 3}; {}^t(b_2)_{6\times 6}; 0_{6\times 15}]\,, \tag{3.1.16}$$

with

$$b_2 = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & -1 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}. \tag{3.1.17}$$

CONSTRUCTION of submatrix B_3. Relations $\hat{w} = w \circ F_K$ and (2.2.22) involve, for $\alpha = 1, 2$,

$$\frac{\partial^2 \hat{w}}{\partial \hat{x}_\alpha\, \partial \hat{x}_\beta}(\hat{x}) = Dw(x)\, \frac{\partial^2 F_K}{\partial \hat{x}_\alpha\, \partial \hat{x}_\beta}(\hat{x}) + D^2 w(x)\left(\frac{\partial F_K}{\partial \hat{x}_\alpha}(\hat{x}), \frac{\partial F_K}{\partial \hat{x}_\beta}(\hat{x})\right), \tag{3.1.18}$$

so that with (1.3.2), we obtain

$$\begin{aligned}
&\frac{\partial^2 \hat{w}}{\partial \hat{x}_1^2}(\hat{a}_1) = D^2 v(a_1)(a_1 - a_3)^2\,,\\
&\frac{\partial^2 \hat{w}}{\partial \hat{x}_1\, \partial \hat{x}_2}(\hat{a}_1) = D^2 v(a_1)[a_1 - a_3, a_1 - a_3 + (\bar{s} - \underline{s})\chi'(\underline{s})]\\
&\qquad\qquad + Dv(a_1)[2(a_1 - a_2) + \tfrac{1}{2}(\bar{s} - \underline{s})(3\chi'(\underline{s}) + \chi'(\bar{s}))]\,,\\
&\frac{\partial^2 \hat{w}}{\partial \hat{x}_2^2}(\hat{a}_1) = D^2 v(a_1)[a_1 - a_3 + (\bar{s} - \underline{s})\chi'(\underline{s})]^2\\
&\qquad\qquad - Dv(a_1)[2(a_1 - a_2) + (\bar{s} - \underline{s})(\chi'(\underline{s}) + \chi'(\bar{s}))]\,,\\
&\frac{\partial^2 \hat{w}}{\partial \hat{x}_1^2}(\hat{a}_2) = D^2 v(a_2)[a_2 - a_3 - (\bar{s} - \underline{s})\chi'(\bar{s})]^2\\
&\qquad\qquad - Dv(a_2)[2(a_2 - a_1) - (\bar{s} - \underline{s})(\chi'(\underline{s}) + \chi'(\bar{s}))]\,,\\
&\frac{\partial^2 \hat{w}}{\partial \hat{x}_1\, \partial \hat{x}_2}(\hat{a}_2) = D^2 v(a_2)[a_2 - a_3 - (\bar{s} - \underline{s})\chi'(\bar{s}), a_2 - a_3]\\
&\qquad\qquad + Dv(a_2)[2(a_2 - a_1) - \tfrac{1}{2}(\bar{s} - \underline{s})(3\chi'(\bar{s}) + \chi'(\underline{s}))]\,,\\
&\frac{\partial^2 \hat{w}}{\partial \hat{x}_2^2}(\hat{a}_2) = D^2 v(a_2)[a_2 - a_3]^2\,,\\
&\frac{\partial^2 \hat{w}}{\partial \hat{x}_1^2}(\hat{a}_3) = D^2 v(a_3)[a_1 - a_3]^2\,,\\
&\frac{\partial^2 \hat{w}}{\partial \hat{x}_1\, \partial \hat{x}_2}(\hat{a}_3) = D^2 v(a_3)[a_1 - a_3, a_2 - a_3] - \tfrac{1}{2}(\bar{s} - \underline{s}) Dv(a_3)[\chi'(\bar{s}) - \chi'(\underline{s})]\,,\\
&\frac{\partial^2 \hat{w}}{\partial \hat{x}_2^2}(\hat{a}_3) = D^2 v(a_3)[a_2 - a_3]^2\,.
\end{aligned} \tag{3.1.19}$$

From these expressions and relations (3.1.2), (3.1.9), (3.1.10) and (3.1.12), we deduce

$${}^t B_3 = [0_{9\times 3};\, {}^t(b_{31})_{9\times 6};\, {}^t(b_{32})_{9\times 9};\, 0_{9\times 6}]\,, \tag{3.1.20}$$

where matrices b_{31} and b_{32} are given by relations (3.1.21), (3.1.22).

$$b_{31} = \begin{bmatrix} 0 & (2\tilde{a}^1 + \frac{1}{2}\tilde{\tilde{a}}^1) & -(2\tilde{a}^1 + \tilde{\tilde{a}}^1) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (\frac{3}{2} + 2\tilde{a}^2 + \frac{1}{2}\tilde{\tilde{a}}^2) & -(1 + 2\tilde{a}^2 + \tilde{\tilde{a}}^2) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -(1 + 2\tilde{b}^1 - \tilde{\tilde{b}}^1) & (\frac{3}{2} + 2\tilde{b}^1 - \frac{1}{2}\tilde{\tilde{b}}^1) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -(2\tilde{b}^2 - \tilde{\tilde{b}}^2) & (2\tilde{b}^2 - \frac{1}{2}\tilde{\tilde{b}}^2) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(\tilde{c}^1 + \tilde{\tilde{c}}^1) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(\tilde{c}^2 + \tilde{\tilde{c}}^2) & 0 \end{bmatrix}, \tag{3.1.21}$$

$$b_{32} = \begin{bmatrix} 1 & \left(1 + \dfrac{1 + \tilde{a}^1}{2\tilde{a}^2}\right) & \left(1 + \dfrac{1 + \tilde{a}^1}{\tilde{a}^2}\right) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \dfrac{\tilde{a}^2}{2(1 + \tilde{a}^1)} & \left(1 + \dfrac{\tilde{a}^2}{1 + \tilde{a}^1}\right) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \left(1 + \dfrac{\tilde{b}^1}{1 + \tilde{b}^2}\right) & \dfrac{\tilde{b}^1}{2(1 + \tilde{b}^2)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \left(1 + \dfrac{1 + \tilde{b}^2}{\tilde{b}^1}\right) & \left(1 + \dfrac{1 + \tilde{b}^2}{2\tilde{b}^1}\right) & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\dfrac{\tilde{c}^1\tilde{\tilde{c}}^1}{\tilde{c}^1\tilde{\tilde{c}}^2 + \tilde{c}^2\tilde{\tilde{c}}^1} & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\dfrac{\tilde{c}^2\tilde{\tilde{c}}^2}{\tilde{c}^1\tilde{\tilde{c}}^2 + \tilde{c}^2\tilde{\tilde{c}}^1} & 0 \\ 0 & \dfrac{-1}{2\tilde{a}^2(1 + \tilde{a}^1)} & \dfrac{-1}{\tilde{a}^2(1 + \tilde{a}^1)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{-1}{\tilde{b}^1(1 + \tilde{b}^2)} & \dfrac{-1}{2\tilde{b}^1(1 + \tilde{b}^2)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dfrac{-1}{\tilde{c}^1\tilde{\tilde{c}}^2 + \tilde{c}^2\tilde{\tilde{c}}^1} & 0 \end{bmatrix}. \tag{3.1.22}$$

It remains to indicate how to obtain the coefficients of the submatrices b_{31} and b_{32}. The second and third columns of b_{31} include terms in $Dv(a_1)$ that we have to express by using the fourth and fifth degrees of freedom of the matrix (3.1.2). Set

$$\boldsymbol{A}_1 = \boldsymbol{a}_3 - \boldsymbol{a}_1 \quad \text{and} \quad \boldsymbol{A}_2 = (\bar{s} - \underline{s})\boldsymbol{\chi}'(\underline{s})$$

and assume that these vectors are linearly independent. Then $\tilde{a}^\alpha$ and $\tilde{\tilde{a}}^\alpha$ are defined by

$$\boldsymbol{a}_1 - \boldsymbol{a}_2 = \tilde{a}^\alpha \boldsymbol{A}_\alpha \,, \qquad (\bar{s} - \underline{s})\boldsymbol{\chi}'(\bar{s}) = \tilde{\tilde{a}}^\alpha \boldsymbol{A}_\alpha \,,$$

so that, if we denote $\boldsymbol{e}_3 = \boldsymbol{e}_1 \times \boldsymbol{e}_2$,

$$\tilde{a}^1 = \frac{[(\boldsymbol{a}_1 - \boldsymbol{a}_2) \times \boldsymbol{A}_2] \cdot \boldsymbol{e}_3}{(\boldsymbol{A}_1 \times \boldsymbol{A}_2) \cdot \boldsymbol{e}_3} \,, \qquad \tilde{a}^2 = \frac{[\boldsymbol{A}_1 \times (\boldsymbol{a}_1 - \boldsymbol{a}_2)] \cdot \boldsymbol{e}_3}{(\boldsymbol{A}_1 \times \boldsymbol{A}_2) \cdot \boldsymbol{e}_3} \,,$$

$$\tilde{\tilde{a}}^1 = (\bar{s} - \underline{s}) \frac{[\boldsymbol{\chi}'(\bar{s}) \times \boldsymbol{A}_2] \cdot \boldsymbol{e}_3}{(\boldsymbol{A}_1 \times \boldsymbol{A}_2) \cdot \boldsymbol{e}_3} \,, \qquad \tilde{\tilde{a}}^2 = (\bar{s} - \underline{s}) \frac{[\boldsymbol{A}_1 \times \boldsymbol{\chi}'(\bar{s})] \cdot \boldsymbol{e}_3}{(\boldsymbol{A}_1 \times \boldsymbol{A}_2) \cdot \boldsymbol{e}_3} \,.$$

Similarly, set

$$\boldsymbol{B}_1 = -(\bar{s} - \underline{s})\boldsymbol{\chi}'(\bar{s}) \,, \qquad \boldsymbol{B}_2 = \boldsymbol{a}_3 - \boldsymbol{a}_2 \,, \qquad \boldsymbol{a}_2 - \boldsymbol{a}_1 = \tilde{b}^\alpha \boldsymbol{B}_\alpha \,,$$

$$(\bar{s} - \underline{s})\boldsymbol{\chi}'(\underline{s}) = \tilde{\tilde{b}}^\alpha \boldsymbol{B}_\alpha \,,$$

$$C_1 = a_2 - a_3\,, \qquad C_2 = a_1 - a_3\,, \qquad (\bar{s} - \underline{s})\chi'(\underline{s}) = \tilde{c}^\alpha C_\alpha\,;$$
$$-(\bar{s} - \underline{s})\chi'(\bar{s}) = \tilde{\tilde{c}}^\alpha C_\alpha\,.$$

From these definitions, we finally obtain

$$\begin{aligned}
&\tilde{a}^1 = \frac{(B_2 \times A_2)\cdot e_3}{(A_1 \times A_2)\cdot e_3} - 1\,, \qquad \tilde{a}^2 = \frac{(A_1 \times B_2)\cdot e_3}{(A_1 \times A_2)\cdot e_3}\,,\\
&\tilde{\tilde{a}}^1 = -\frac{(B_1 \times A_2)\cdot e_3}{(A_1 \times A_2)\cdot e_3}\,, \qquad \tilde{\tilde{a}}^2 = -\frac{(A_1 \times B_1)\cdot e_3}{(A_1 \times A_2)\cdot e_3}\,,\\
&\tilde{b}^1 = \frac{(A_1 \times B_2)\cdot e_3}{(B_1 \times B_2)\cdot e_3}\,, \qquad \tilde{b}^2 = \frac{(B_1 \times A_1)\cdot e_3}{(B_1 \times B_2)\cdot e_3} - 1\,,\\
&\tilde{\tilde{b}}^1 = \frac{(A_2 \times B_2)\cdot e_3}{(B_1 \times B_2)\cdot e_3}\,, \qquad \tilde{\tilde{b}}^2 = \frac{(B_1 \times A_2)\cdot e_3}{(B_1 \times B_2)\cdot e_3}\,,\\
&\tilde{c}^1 = -\frac{(A_1 \times A_2)\cdot e_3}{(A_1 \times B_2)\cdot e_3} = -\frac{1}{\tilde{a}^2}\,, \qquad \tilde{c}^2 = -\frac{(A_2 \times B_2)\cdot e_3}{(A_1 \times B_2)\cdot e_3} = -\frac{\tilde{\tilde{b}}^1}{\tilde{b}^1}\,,\\
&\tilde{\tilde{c}}^1 = -\frac{(A_1 \times B_1)\cdot e_3}{(A_1 \times B_2)\cdot e_3} = \frac{\tilde{\tilde{a}}^2}{\tilde{a}^2}\,, \qquad \tilde{\tilde{c}}^2 = -\frac{(B_1 \times B_2)\cdot e_3}{(A_1 \times B_2)\cdot e_3} = -\frac{1}{\tilde{b}^1}\,.
\end{aligned} \tag{3.1.23}$$

CONSTRUCTION of submatrices B_i, $i = 4, 5, 6$. We have seen in Section 2.2 that it is convenient to introduce the functions $\hat{f}_1$ (see (2.2.8)), $\hat{f}_2$ (see (2.2.12)), $\hat{f}_3$ (see (2.2.18)), $\hat{g}_1$ (see (2.2.10)), $\hat{g}_2$ (see (2.2.13)) and $\hat{g}_3$ (see (2.2.19)). Relations (1.3.2), (3.1.2), $\hat{v} = v \circ F_K$ and some technical computations prove that these functions $\hat{f}_i$, $\hat{g}_i$, $i = 1, 2, 3$, can be written as

$$\begin{aligned}
&[\hat{f}_1(\hat{x}_1);\, \hat{f}_2(\hat{x}_2);\, \hat{f}_3(\hat{x}_1)] = [DLLC(v)]_{1\times 24}\,[\hat{\mathscr{F}}_1(\hat{x}_1);\, \hat{\mathscr{F}}_2(\hat{x}_2);\, \hat{\mathscr{F}}_3(\hat{x}_1)]_{24\times 3}\,,\\
&[\hat{g}_1(\hat{x}_1);\, \hat{g}_2(\hat{x}_2);\, \hat{g}_3(\hat{x}_1)] = [DLLC(v)]_{1\times 24}\,[\hat{\mathscr{G}}_1(\hat{x}_1);\, \hat{\mathscr{G}}_2(\hat{x}_2);\, \hat{\mathscr{G}}_3(\hat{x}_1)]_{24\times 3}\,,
\end{aligned} \tag{3.1.24}$$

with

$$\begin{aligned}
{}^t[\hat{\mathscr{F}}_1(\hat{x}_1)] = [&\hat{x}_1^3(6\hat{x}_1^2 - 15\hat{x}_1 + 10); 0; (1 - \hat{x}_1)^3(6\hat{x}_1^2 + 3\hat{x}_1 + 1)\,;\\
&\hat{x}_1^3(1 - \hat{x}_1)(4 - 3\hat{x}_1); 0\,0\,0\,0; \hat{x}_1(1 - \hat{x}_1)^3(1 + 3\hat{x}_1)\,;\\
&\tfrac{1}{2}\hat{x}_1^3(1 - \hat{x}_1)^2; 0\,0\,0\,0; \tfrac{1}{2}\hat{x}_1^2(1 - \hat{x}_1)^3; 0\,0\,0\,0\,0\,0\,0\,0\,0]\,;
\end{aligned} \tag{3.1.25}$$

$$\begin{aligned}
{}^t[\hat{\mathscr{F}}_2(\hat{x}_2)] = [&0; \hat{x}_2^3(6\hat{x}_2^2 - 15\hat{x}_2 + 10); (1 - \hat{x}_2)^3(6\hat{x}_2^2 + 3\hat{x}_2 + 1)\,;\\
&0\,0\,0; \hat{x}_2^3(1 - \hat{x}_2)(4 - 3\hat{x}_2); \hat{x}_2(1 - \hat{x}_2)^3(1 + 3\hat{x}_2); 0\,0\,0\,0\,;\\
&\tfrac{1}{2}\hat{x}_2^3(1 - \hat{x}_2)^2; \tfrac{1}{2}\hat{x}_2^2(1 - \hat{x}_2)^3; 0\,0\,0\,0\,0\,0\,0\,0\,0\,0]\,;
\end{aligned} \tag{3.1.26}$$

$$
\begin{aligned}
{}^t[\hat{\mathscr{F}}_3(\hat{x}_1)] = [&\hat{x}_1^3(6\hat{x}_1^2 - 15\hat{x}_1 + 10); (1-\hat{x}_1)^3(6\hat{x}_1^2 + 3\hat{x}_1 + 1); 0 ;\\
&-(3\tilde{a}^1 + \tilde{\tilde{a}}^1)\hat{x}_1^3(1-\hat{x}_1)^2; \hat{x}_1^3(1-\hat{x}_1)\{2 - \hat{x}_1 - (1-\hat{x}_1)(3\tilde{a}^2 + \tilde{\tilde{a}}^2)\} ;\\
&+\hat{x}_1(1-\hat{x}_1)^3\{1 + \hat{x}_1 - \hat{x}_1(3\tilde{b}^1 - \tilde{\tilde{b}}^1)\}; -\hat{x}_1^2(1-\hat{x}_1)^3(3\tilde{b}^2 - \tilde{\tilde{b}}^2) ;\\
&0\,0\,0; \tfrac{1}{2}\hat{x}_1^3(1-\hat{x}_1)^2; \tfrac{1}{2}\hat{x}_1^2(1-\hat{x}_1)^3; 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0] ;
\end{aligned}
\tag{3.1.27}
$$

$$
\begin{aligned}
{}^t[\hat{\mathscr{G}}_1(\hat{x}_1)] = \Big[\,&0\,0\,0; -\tfrac{1}{2}(1 - \eta_2 + 2\tilde{a}^1)\hat{x}_1^2(2\hat{x}_1 - 1)(5 - 4\hat{x}_1) ;\\
&-\tilde{a}^2\hat{x}_1^2(2\hat{x}_1 - 1)(5 - 4\hat{x}_1); 0\,0; (1-\hat{x}_1)^2(1 - 2\hat{x}_1)(1 + 4\hat{x}_1) ;\\
&-\tfrac{1}{2}(1 + \eta_2)(1-\hat{x}_1)^2(1 - 2\hat{x}_1)(1 + 4\hat{x}_1); \tfrac{1}{2}(\tilde{a}^1 - \eta_2)\hat{x}_1^2(1-\hat{x}_1)(1 - 2\hat{x}_1) ;\\
&-\frac{1}{2}\,\frac{(\tilde{a}^2)^2}{1 + \tilde{a}^1}\,\hat{x}_1^2(1-\hat{x}_1)(1 - 2\hat{x}_1); 0\,0; \frac{-\tilde{c}^1\tilde{\tilde{c}}^1}{\tilde{c}^1\tilde{\tilde{c}}^2 + \tilde{c}^2\tilde{\tilde{c}}^1}\,\hat{x}_1(1-\hat{x}_1)^2(1 - 2\hat{x}_1) ;\\
&-\left\{\tfrac{1}{2}(1 + \eta_2) + \frac{\tilde{c}^2\tilde{\tilde{c}}^2}{\tilde{c}^1\tilde{\tilde{c}}^2 + \tilde{c}^2\tilde{\tilde{c}}^1}\right\}\hat{x}_1(1-\hat{x}_1)^2(1 - 2\hat{x}_1); \frac{\hat{x}_1(1-\hat{x}_1)(1 - 2\hat{x}_1)}{2(1 + \tilde{a}^1)} ;\\
&0 ; \frac{-1}{\tilde{c}^1\tilde{\tilde{c}}^2 + \tilde{c}^2\tilde{\tilde{c}}^1}\,\hat{x}_1(1-\hat{x}_1)^2(1 - 2\hat{x}_1); 0; 16\hat{x}_1^2(1-\hat{x}_1)^2; 0\,0\,0\,0\,\Big] ;
\end{aligned}
\tag{3.1.28}
$$

$$
\begin{aligned}
{}^t[\hat{\mathscr{G}}_2(\hat{x}_2)] = \Big[\,&0\,0\,0\,0\,0; -\tilde{b}^1\hat{x}_2^2(2\hat{x}_2 - 1)(5 - 4\hat{x}_2) ;\\
&-\tfrac{1}{2}(2\tilde{b}^2 + 1 + \eta_1)\hat{x}_2^2(2\hat{x}_2 - 1)(5 - 4\hat{x}_2);\\
&-\tfrac{1}{2}(1 - \eta_1)(1-\hat{x}_2)^2(1 - 2\hat{x}_2)(1 + 4\hat{x}_2) ;\\
&(1-\hat{x}_2)^2(1 - 2\hat{x}_2)(1 + 4\hat{x}_2); 0\,0; -\frac{1}{2}\,\frac{(\tilde{b}^1)^2}{1 + \tilde{b}^2}\,\hat{x}_2^2(1-\hat{x}_2)(1 - 2\hat{x}_2) ;\\
&\tfrac{1}{2}(\tilde{b}^2 + \eta_1)\hat{x}_2^2(1-\hat{x}_2)(1 - 2\hat{x}_2);\\
&-\frac{1}{2}\left(\frac{2\tilde{c}^1\tilde{\tilde{c}}^1}{\tilde{c}^1\tilde{\tilde{c}}^2 + \tilde{c}^2\tilde{\tilde{c}}^1} + 1 - \eta_1\right)\hat{x}_2(1-\hat{x}_2)^2(1 - 2\hat{x}_2) ;\\
&-\frac{\tilde{c}^2\tilde{\tilde{c}}^2}{\tilde{c}^1\tilde{\tilde{c}}^2 + \tilde{c}^2\tilde{\tilde{c}}^1}\,\hat{x}_2(1-\hat{x}_2)^2(1 - 2\hat{x}_2); 0; \frac{1}{2}\,\frac{1}{1 + \tilde{b}^2}\,\hat{x}_2^2(1-\hat{x}_2)(1 - 2\hat{x}_2) ;\\
&-\frac{1}{\tilde{c}^1\tilde{\tilde{c}}^2 + \tilde{c}^2\tilde{\tilde{c}}^1}\,\hat{x}_2(1-\hat{x}_2)^2(1 - 2\hat{x}_2); 16(1-\hat{x}_2)^2\hat{x}^2; 0\,0\,0\,0\,0\,\Big] ;
\end{aligned}
\tag{3.1.29}
$$

$$
\begin{aligned}
{}^t[\hat{\mathscr{G}}_3(\hat{x}_1)] = \Big[\,&0\,0\,0; \hat{x}_1^2(2\hat{x}_1 - 1)\{5 - 4\hat{x}_1 + (1-\hat{x}_1)(\tilde{a}^1 + \tfrac{1}{2}\tilde{\tilde{a}}^1)\} ;\\
&\hat{x}_1^2(2\hat{x}_1 - 1)\{-2 + \tfrac{3}{2}\hat{x}_1 + (1-\hat{x}_1)(\tilde{a}^2 + \tfrac{1}{2}\tilde{\tilde{a}}^2)\} ;
\end{aligned}
$$

$$- \tfrac{1}{2}(1-\hat{x}_1)^2(1-2\hat{x}_1)\{(1+3\hat{x}_1) - (2\tilde{b}^1 - \tilde{\tilde{b}}^1)\hat{x}_1\} ;$$
$$(1-\hat{x}_1)^2(1-2\hat{x}_1)\{1+4\hat{x}_1 + \tfrac{1}{2}(2\tilde{b}^2 - \tilde{\tilde{b}}^2)\hat{x}_1\}; 0\,0 ;$$
$$\frac{1+\tilde{a}^1}{2\tilde{a}^2}\hat{x}_1^2(1-\hat{x}_1)(1-2\hat{x}_1);\ \frac{1}{2}\Big(1+\frac{\tilde{a}^2}{1+\tilde{a}^1}\Big)\hat{x}_1^2(1-\hat{x}_1)(1-2\hat{x}_1) ;$$
$$-\frac{1}{2}\Big(1+\frac{\tilde{b}^1}{1+\tilde{b}^2}\Big)\hat{x}_1(1-\hat{x}_1)^2(1-2\hat{x}_1);\ -\frac{1+\tilde{b}^2}{2\tilde{b}^1}\hat{x}_1(1-\hat{x}_1)^2(1-2\hat{x}_1) ;$$
$$0\,0;\ -\frac{1}{2(1+\tilde{a}^1)\tilde{a}^2}\hat{x}_1^2(1-\hat{x}_1)(1-2\hat{x}_1) ;$$
$$\frac{1}{2\tilde{b}^1(1+\tilde{b}^2)}\hat{x}_1(1-\hat{x}_1)^2(1-2\hat{x}_1); 0\,0\,0; 16(1-\hat{x}_1)^2\hat{x}_1^2; 0\,0\,0\Big] . \quad (3.1.30)$$

To express $\hat{\mathscr{G}}_1(\hat{x}_1)$ and $\hat{\mathscr{G}}_2(\hat{x}_2)$, we have used the eccentricity parameters η_1 and η_2 of the triangle $a_1a_2a_3$ which are given by (see [9, p. 69])

$$a_1 - c_1 = a_1 - a_2 - \tfrac{1}{2}(1+\eta_1)(a_3 - a_2);\ a_2 - c_2 = a_2 - a_1 - \tfrac{1}{2}(1-\eta_2)(a_3 - a_1) .$$

Then we are able to construct the submatrices B_i, $i = 4, 5, 6$.

CONSTRUCTION of submatrix B_4. Relations (2.2.3) and (2.2.14) involve

$$-\frac{\partial \hat{w}}{\partial \hat{x}_1}(\hat{b}_1) = -\Big\langle \frac{\partial F_K}{\partial \hat{x}_1}(\hat{b}_1), \frac{a_2 - a_3}{|a_2 - a_3|^2}\frac{\mathrm{d}\hat{f}_2}{\mathrm{d}\hat{x}_2}\Big(\frac{1}{2}\Big) + \frac{a_1 - c_1}{|a_1 - c_1|^2}\hat{g}_2\Big(\frac{1}{2}\Big)\Big\rangle , \quad (3.1.31)$$

where $\langle\cdot,\cdot\rangle$ denotes the usual scalar product in $\mathbb{R}^2$. Hence relations (3.1.24), (3.1.26), (3.1.28) and (3.1.31) give

$$-\frac{\partial \hat{w}}{\partial \hat{x}_1}(\hat{b}_1) = [DLLC(v)]_{1\times 24}\Big[E^1\frac{\mathrm{d}}{\mathrm{d}\hat{x}_2}\hat{\mathscr{F}}_2\Big(\frac{1}{2}\Big) + E^2\hat{\mathscr{G}}_2\Big(\frac{1}{2}\Big)\Big]_{24\times 1} , \quad (3.1.32)$$

where we have set

$$E^1 = \Big\langle a_3 - a_1 + \tfrac{1}{4}(a_1 - a_2) + \tfrac{1}{8}(\bar{s} - \underline{s})\{3\chi'(\bar{s}) - \chi'(\underline{s})\}, \frac{a_2 - a_3}{|a_2 - a_3|^2}\Big\rangle ,$$
$$E^2 = \Big\langle a_3 - a_1 + \tfrac{1}{4}(a_1 - a_2) + \tfrac{1}{8}(\bar{s} - \underline{s})\{3\chi'(\bar{s}) - \chi'(\underline{s})\}, \frac{a_1 - c_1}{|a_1 - c_1|^2}\Big\rangle , \quad (3.1.33)$$

since (1.3.2) involves

$$-\frac{\partial F_K}{\partial \hat{x}_1}(\hat{b}_1) = a_3 - a_1 + \tfrac{1}{4}(a_1 - a_2) + \tfrac{1}{8}(\bar{s} - \underline{s})\{3\chi'(\bar{s}) - \chi'(\underline{s})\} .$$

By similarity, we obtain

$$-\frac{\partial \hat{w}}{\partial \hat{x}_2}(\hat{b}_2) = [DLLC(v)]_{1\times 24}\left[F^1 \frac{\mathrm{d}}{\mathrm{d}\hat{x}_1}\hat{\mathscr{F}}_1\left(\frac{1}{2}\right) + F^2\hat{\mathscr{G}}_1\left(\frac{1}{2}\right)\right]_{24\times 1}, \tag{3.1.34}$$

where

$$\begin{aligned} F^1 &= \left\langle a_3 - a_2 + \tfrac{1}{4}(a_2 - a_1) - \tfrac{1}{8}(\bar{s} - \underline{s})\{3\chi'(\underline{s}) - \chi'(\bar{s})\}, \frac{a_1 - a_3}{|a_1 - a_3|^2}\right\rangle, \\ F^2 &= \left\langle a_3 - a_2 + \tfrac{1}{4}(a_2 - a_1) - \tfrac{1}{8}(\bar{s} - \underline{s})\{3\chi'(\underline{s}) - \chi'(\bar{s})\}, \frac{a_2 - c_2}{|a_2 - c_2|^2}\right\rangle. \end{aligned} \tag{3.1.35}$$

Moreover,

$$\begin{aligned} \frac{\sqrt{2}}{2}\left(\frac{\partial \hat{w}}{\partial \hat{x}_1} + \frac{\partial \hat{w}}{\partial \hat{x}_2}\right)(\hat{b}_3) &= -\sqrt{2}D\hat{w}(\hat{b}_3)(\hat{a}_3 - \hat{b}_3) \\ &= [DLLC(v)]_{1\times 24}{}^t[0_{1\times 20}; -\sqrt{2}; 0\,0\,0]. \end{aligned} \tag{3.1.36}$$

Thus, relations (3.1.32), (3.1.34) and (3.1.36) lead to

$$B_4 = \left[E^1 \frac{\mathrm{d}\hat{\mathscr{F}}_2}{\mathrm{d}\hat{x}_2}\left(\frac{1}{2}\right) + E^2\hat{\mathscr{G}}_2\left(\frac{1}{2}\right); F^1 \frac{\mathrm{d}\hat{\mathscr{F}}_1}{\mathrm{d}\hat{x}_1}\left(\frac{1}{2}\right) + F^2\hat{\mathscr{G}}_1\left(\frac{1}{2}\right); \begin{pmatrix} 0_{20\times 1} \\ -\sqrt{2} \\ 0_{3\times 1}\end{pmatrix}\right]. \tag{3.1.37}$$

CONSTRUCTION of submatrix B_5. Relations (2.2.3), (2.2.14), (2.2.20) and (3.1.24) give

$$B_5 = \left[\hat{\mathscr{F}}_2\left(\frac{3}{4}\right); \hat{\mathscr{F}}_2\left(\frac{1}{4}\right); \hat{\mathscr{F}}_1\left(\frac{1}{4}\right); \hat{\mathscr{F}}_1\left(\frac{3}{4}\right); \hat{\mathscr{F}}_3\left(\frac{3}{4}\right); \hat{\mathscr{F}}_3\left(\frac{1}{4}\right)\right]_{24\times 6}. \tag{3.1.38}$$

CONSTRUCTION of submatrix B_6. By similarity with the derivation of (3.1.32), (3.1.34) and (3.1.36), we obtain

$$\begin{aligned} B_6 = \Big[& G^1 \frac{\mathrm{d}\hat{\mathscr{F}}_2}{\mathrm{d}\hat{x}_2}\left(\frac{3}{4}\right) + G^2\hat{\mathscr{G}}_2\left(\frac{3}{4}\right); H^1 \frac{\mathrm{d}\hat{\mathscr{F}}_2}{\mathrm{d}\hat{x}_2}\left(\frac{1}{4}\right) + H^2\hat{\mathscr{G}}_2\left(\frac{1}{4}\right); J^1 \frac{\mathrm{d}\hat{\mathscr{F}}_1}{\mathrm{d}\hat{x}_1}\left(\frac{1}{4}\right) + J^2\hat{\mathscr{G}}_1\left(\frac{1}{4}\right); \\ & K^1 \frac{\mathrm{d}\hat{\mathscr{F}}_1}{\mathrm{d}\hat{x}_1}\left(\frac{3}{4}\right) + K^2\hat{\mathscr{G}}_1\left(\frac{3}{4}\right); -\sqrt{2}\hat{\mathscr{G}}_3\left(\frac{3}{4}\right); -\sqrt{2}\hat{\mathscr{G}}_3\left(\frac{1}{4}\right)\Big]_{24\times 6}, \end{aligned} \tag{3.1.39}$$

where we have set

$$\begin{aligned} G^1 &= -\left\langle a_1 - a_3 + \tfrac{9}{16}(a_2 - a_1) + \tfrac{3}{32}(\bar{s} - \underline{s})\{\chi'(\underline{s}) - 7\chi'(\bar{s})\}, \frac{a_2 - a_3}{|a_2 - a_3|^2}\right\rangle, \\ G^2 &= -\left\langle a_1 - a_3 + \tfrac{9}{16}(a_2 - a_1) + \tfrac{3}{32}(\bar{s} - \underline{s})\{\chi'(\underline{s}) - 7\chi'(\bar{s})\}, \frac{a_1 - c_1}{|a_1 - c_1|^2}\right\rangle, \\ H^1 &= -\left\langle a_1 - a_3 + \tfrac{1}{16}(a_2 - a_1) + \tfrac{1}{32}(\bar{s} - \underline{s})\{3\chi'(\underline{s}) - 5\chi'(\bar{s})\}, \frac{a_2 - a_3}{|a_2 - a_3|^2}\right\rangle, \end{aligned}$$

$$H^2 = -\left\langle a_1 - a_3 + \tfrac{1}{16}(a_2 - a_1) + \tfrac{1}{32}(\bar{s} - \underline{s})\{3\chi'(\underline{s}) - 5\chi'(\bar{s})\}, \frac{a_1 - c_1}{|a_1 - c_1|^2} \right\rangle,$$

$$J^1 = -\left\langle a_2 - a_3 + \tfrac{1}{16}(a_1 - a_2) - \tfrac{1}{32}(\bar{s} - \underline{s})\{3\chi'(\bar{s}) - 5\chi'(\underline{s})\}, \frac{a_1 - a_3}{|a_1 - a_3|^2} \right\rangle,$$

$$J^2 = -\left\langle a_2 - a_3 + \tfrac{1}{16}(a_1 - a_2) - \tfrac{1}{32}(\bar{s} - \underline{s})\{3\chi'(\bar{s}) - 5\chi'(\underline{s})\}, \frac{a_2 - c_2}{|a_2 - c_2|^2} \right\rangle,$$

$$K^1 = -\left\langle a_2 - a_3 + \tfrac{9}{16}(a_1 - a_2) - \tfrac{3}{32}(\bar{s} - \underline{s})\{\chi'(\bar{s}) - 7\chi'(\underline{s})\}, \frac{a_1 - a_3}{|a_1 - a_3|^2} \right\rangle,$$

$$K^2 = -\left\langle a_2 - a_3 + \tfrac{9}{16}(a_1 - a_2) - \tfrac{3}{32}(\bar{s} - \underline{s})\{\chi'(\bar{s}) - 7\chi'(\underline{s})\}, \frac{a_2 - c_2}{|a_2 - c_2|^2} \right\rangle. \tag{3.1.40}$$

CONSTRUCTION of submatrix B_7. Since $\hat{w}(\hat{e}_i) = w(e_i) = v(e_i)$, $i = 1, 2, 3$, we obtain

$$'B_7 = [0_{3\times 21}; I_{3\times 3}]. \tag{3.1.41}$$

STEP 3: Transition from $\hat{\Delta}_K(v)$ to the function $\hat{w}(\hat{x}) = w(x) = \pi_K v(x)$. By using the finite element described in Fig. 8, we construct the function $\hat{w} \in \hat{P} = P_7$ which takes the values $\hat{\Delta}_K(v)$ on the set of degrees of freedom $\hat{\Sigma}$ so that, with relation (3.1.9),

$$\hat{w} = [DL(\hat{w})]_{1\times 36}[p]_{36\times 1}, \tag{3.1.42}$$

where $[p]$ denotes the column matrix of basis polynomials. They can be written

$$[p]_{36\times 1} = [A]_{36\times 36}[m7]_{36\times 1}, \tag{3.1.43}$$

where $[m7]_{36\times 1}$ collects the 36 basis monomials of degrees less than or equal to 7 with respect to $\hat{x}_1$ and $\hat{x}_2$. From (3.1.42), we have

$$[m7]_{36\times 1} = [DL(m7)]_{36\times 36}[p]_{36\times 1} \text{ so that } [A]_{36\times 36} = [DL(m7)]^{-1}.$$

These basis polynomials and the matrix $[A]$ are detailed in the annex of [10].

Finally, we obtain the interpolating function $\pi_K v$ of the function v by collecting relations (3.1.1), (3.1.11), (3.1.42) and (3.1.43),

$$\pi_K v(x) = \hat{w}(\hat{x}) = [DLGL(v)]_{1\times 24}[\tilde{D}]_{24\times 24}[B]_{24\times 36}[A]_{36\times 36}[m7]_{36\times 1}. \tag{3.1.44}$$

3.2. Interpolation modules associated with the curved finite element C^1-compatible with the Argyris triangle when $F_K^* \in (P_5)^2$

The matrix decompositions are entirely similar to that of Section 3.1 so we just give the results here and we refer the reader back to the above section for more details. Now

$$\pi_K^* v(x) = w^*(x) , \tag{3.2.1}$$

which is constructed in three steps from the application $F_K^* \in (P_5)^2$ defined by (1.3.6).

STEP 1: $v \rightarrow \Sigma_K^*(v) = [DLLC^*(v)]$. From (2.3.2), we obtain

$$\begin{aligned}[DLLC^*(v)]_{1\times 31} = [&v(a_1); v(a_2); v(a_3); Dv(a_1)(a_3 - a_1); (\bar{s} - \underline{s})Dv(a_1)\chi'(\underline{s});\\ &-(\bar{s} - \underline{s})Dv(a_2)\chi'(\bar{s}); Dv(a_2)(a_3 - a_2); Dv(a_3)(a_2 - a_3);\\ &Dv(a_3)(a_1 - a_3); D^2v(a_1)(a_3 - a_1)^2; (\bar{s} - \underline{s})^2 D^2v(a_1)(\chi'(\underline{s}))^2;\\ &(\underline{s} - \bar{s})^2 D^2v(a_2)(\chi'(\bar{s}))^2; D^2v(a_2)(a_3 - a_2)^2; D^2v(a_3)(a_2 - a_3)^2;\\ &D^2v(a_3)(a_1 - a_3)^2; D^2v(a_1)(a_2 - a_3)^2; D^2v(a_2)(a_3 - a_1)^2;\\ &(\bar{s} - \underline{s})^2 D^2v(a_3)((\chi'(\underline{s}), \chi'(\bar{s})); Dv(b_1)(a_1 - c_1);\\ &Dv(b_2)(a_2 - c_2); Dv(b_3)DF_K^*(\hat{b}_3)(\hat{a}_3 - \hat{b}_3); v(e_1^*) \ldots v(e_{10}^*)] ,\end{aligned} \tag{3.2.2}$$

while the set of global degrees of freedom is given by

$$\begin{aligned}[DLGL^*(v)]_{1\times 31} = \Big[&v(a_1);\ v(a_2);\ v(a_3);\ \frac{\partial v}{\partial x_1}(a_1);\ \frac{\partial v}{\partial x_2}(a_1);\ \frac{\partial v}{\partial x_1}(a_2);\\ &\frac{\partial v}{\partial x_2}(a_2);\ \frac{\partial v}{\partial x_1}(a_3);\ \frac{\partial v}{\partial x_2}(a_3);\ \frac{\partial^2 v}{\partial x_1^2}(a_1);\ \frac{\partial^2 v}{\partial x_1\,\partial x_2}(a_1);\ \frac{\partial^2 v}{\partial x_2^2}(a_1);\\ &\frac{\partial^2 v}{\partial x_1^2}(a_2);\ \frac{\partial^2 v}{\partial x_1\,\partial x_2}(a_2);\ \frac{\partial^2 v}{\partial x_2^2}(a_2);\ \frac{\partial^2 v}{\partial x_1^2}(a_3);\ \frac{\partial^2 v}{\partial x_1\,\partial x_2}(a_3);\\ &\frac{\partial^2 v}{\partial x_2^2}(a_3);\ \frac{\partial v}{\partial \nu_1}(b_1);\ \frac{\partial v}{\partial \nu_2}(b_2);\ Dv(b_3)DF_K^*(\hat{b}_3)(\hat{a}_3 - \hat{b}_3);\\ &v(e_1^*) \ldots v(e_{10}^*)\Big] .\end{aligned} \tag{3.2.3}$$

Then, with (3.1.6), we obtain

$$[DLLC^*(v)]_{1\times 31} = [DLGL^*(v)]_{1\times 31}[\tilde{D}^*]_{31\times 31} , \qquad \tilde{D}^* = \begin{bmatrix} \tilde{D} & 0 \\ 0 & I_7 \end{bmatrix} . \tag{3.2.4}$$

STEP 2: Transition from $\Sigma_K^*(v) = [DLLC^*(v)]$ *to* $\hat{\Delta}_K^*(v) = [DL^*(\hat{w}^*)]$. According to Fig. 10, we set

$$\begin{aligned}[DL^*(\hat{w}^*)]_{1\times 55} = \Big[& \hat{w}^*(\hat{a}_1);\ \hat{w}^*(\hat{a}_2);\ \hat{w}^*(\hat{a}_3);\ \frac{\partial \hat{w}^*}{\partial \hat{x}_1}(\hat{a}_1);\ \frac{\partial \hat{w}^*}{\partial \hat{x}_2}(\hat{a}_1);\\
& \frac{\partial \hat{w}^*}{\partial \hat{x}_1}(\hat{a}_2);\ \frac{\partial \hat{w}^*}{\partial \hat{x}_2}(\hat{a}_2);\ \frac{\partial \hat{w}^*}{\partial \hat{x}_1}(\hat{a}_3);\ \frac{\partial \hat{w}^*}{\partial \hat{x}_2}(\hat{a}_3);\ \frac{\partial^2 \hat{w}^*}{\partial \hat{x}_1^2}(\hat{a}_1);\\
& \frac{\partial^2 \hat{w}^*}{\partial \hat{x}_1 \partial \hat{x}_2}(\hat{a}_1);\ \frac{\partial^2 \hat{w}^*}{\partial \hat{x}_2^2}(\hat{a}_1);\ \frac{\partial^2 \hat{w}^*}{\partial \hat{x}_1^2}(\hat{a}_2);\ \frac{\partial^2 \hat{w}^*}{\partial \hat{x}_1 \partial \hat{x}_2}(\hat{a}_2);\\
& \frac{\partial^2 \hat{w}^*}{\partial \hat{x}_2^2}(\hat{a}_2);\ \frac{\partial^2 \hat{w}^*}{\partial \hat{x}_1^2}(\hat{a}_3);\ \frac{\partial^2 \hat{w}^*}{\partial \hat{x}_1 \partial \hat{x}_2}(\hat{a}_3);\ \frac{\partial^2 \hat{w}^*}{\partial \hat{x}_2^2}(\hat{a}_3);\ -\frac{\partial \hat{w}^*}{\partial \hat{x}_1}(\hat{b}_1);\\
& -\frac{\partial \hat{w}^*}{\partial \hat{x}_2}(\hat{b}_2);\ \frac{\sqrt{2}}{2}\left(\frac{\partial \hat{w}^*}{\partial \hat{x}_1} + \frac{\partial \hat{w}^*}{\partial \hat{x}_2}\right)(\hat{b}_3);\ \{\hat{w}^*(\hat{d}_i^*), i = 1, \ldots, 12\};\\
& \left\{-\frac{\partial \hat{w}^*}{\partial \hat{x}_1}(\hat{d}_i^*), i = 1, \ldots, 4\right\};\ \left\{-\frac{\partial \hat{w}^*}{\partial \hat{x}_2}(\hat{d}_i^*), i = 5, \ldots, 8\right\};\\
& \left\{\frac{\sqrt{2}}{2}\left(\frac{\partial \hat{w}^*}{\partial \hat{x}_1} + \frac{\partial \hat{w}^*}{\partial \hat{x}_2}\right)(\hat{d}_i^*), i = 9, \ldots, 12\right\};\\
& \{\hat{w}(\hat{e}_i^*), i = 1, \ldots, 10\}\Big]. \end{aligned} \tag{3.2.5}$$

It remains to determine the transition matrix $[B^*]$, i.e.,

$$[DL^*(\hat{w}^*)]_{1\times 55} = [DLLC^*(v)]_{1\times 31}[B^*]_{31\times 55}, \tag{3.2.6}$$

so that

$$[DL^*(\hat{w}^*)]_{1\times 55} = [DLGL^*(v)]_{1\times 31}[\tilde{D}^*]_{31\times 31}[B^*]_{31\times 55}. \tag{3.2.7}$$

To specify B^*, it is convenient to consider the following partition:

$$[B^*]_{31\times 55} = [B_1^*\ \ B_2^*\ \ B_3^*\ \ B_4^*\ \ B_5^*\ \ B_6^*\ \ B_7^*\ \ B_8^*\ \ B_9^*], \tag{3.2.8}$$

where matrices B_i^*, $i = 1, \ldots, 9$ have 31 lines and 3, 6, 9, 3, 12, 4, 4, 4, 10 columns.

CONSTRUCTION of submatrix B_1^* and B_9^*. Relations (2.3.8) and (2.3.9) involve $\hat{w}^*(\hat{a}_i) = w^*(a_i) = v(a_i)$, $i = 1, \ldots, 3$, and $\hat{w}^*(\hat{e}_i^*) = w^*(e_i) = v(e_i)$, $i = 1, \ldots, 10$, so that

$${}^tB_1^* = [I_{3\times 3}; 0_{3\times 28}], \qquad {}^tB_9^* = [0_{10\times 21}; I_{10\times 10}]. \tag{3.2.9}$$

CONSTRUCTION of submatrix B_2^*. Relations (3.1.14) are unchanged, except for the star (*). Then, from (1.3.6), relation (3.1.16) remains valid so that with (3.1.17), we have

$${}^tB_2^* = [0_{6\times 3}; {}^t(b_2)_{6\times 6}; 0_{6\times 22}]. \tag{3.2.10}$$

CONSTRUCTION of submatrix B_3^*. Relations (2.3.9) and (3.1.18) involve

$$\frac{\partial^2 \hat{w}^*}{\partial \hat{x}_\alpha\, \partial \hat{x}_\beta}(\hat{a}_i) = D^2 v(a_i)\Big(\frac{\partial F_K^*}{\partial \hat{x}_\alpha}(\hat{a}_i), \frac{\partial F_K^*}{\partial \hat{x}_\beta}(\hat{a}_i)\Big) + Dv(a_i)\, \frac{\partial^2 F_K^*}{\partial \hat{x}_\alpha\, \partial \hat{x}_\beta}(\hat{a}_i)\,, \tag{3.2.11}$$

so that, with (1.3.6),

$$\begin{aligned}
&\frac{\partial^2 \hat{w}^*}{\partial \hat{x}_1^2}(\hat{a}_1) = D^2 v(a_1)(a_1 - a_3)^2\,,\\
&\frac{\partial^2 \hat{w}^*}{\partial \hat{x}_1\, \partial \hat{x}_2}(\hat{a}_1) = D^2 v(a_1)[a_1 - a_3, a_1 - a_3 + (\bar{s} - \underline{s})\chi'(\underline{s})]\\
&\qquad\qquad + \tfrac{1}{2} Dv(a_1)[a_1 - a_2 + (\bar{s} - \underline{s})\chi'(\underline{s}) - \tfrac{1}{2}(\bar{s} - \underline{s})^2 \chi''(\underline{s})]\,,\\
&\frac{\partial^2 \hat{w}^*}{\partial \hat{x}_2^2}(\hat{a}_1) = D^2 v(a_1)[a_1 - a_3 + (\bar{s} - \underline{s})\chi'(\underline{s})]^2\\
&\qquad\qquad + Dv(a_1)[a_1 - a_2 + (\bar{s} - \underline{s})\chi'(\underline{s}) + \tfrac{1}{2}(\bar{s} - \underline{s})^2 \chi''(\underline{s})]\,,\\
&\frac{\partial^2 \hat{w}^*}{\partial \hat{x}_1^2}(\hat{a}_2) = D^2 v(a_2)[a_2 - a_3 - (\bar{s} - \underline{s})\chi'(\bar{s})]^2\\
&\qquad\qquad + Dv(a_2)[a_2 - a_1 - (\bar{s} - \underline{s})\chi'(\bar{s}) + \tfrac{1}{2}(\bar{s} - \underline{s})^2 \chi''(\bar{s})]\,,\\
&\frac{\partial^2 \hat{w}^*}{\partial \hat{x}_1\, \partial \hat{x}_2}(\hat{a}_2) = D^2 v(a_2)[a_2 - a_3 - (\bar{s} - \underline{s})\chi'(\bar{s}), a_2 - a_3]\\
&\qquad\qquad + \tfrac{1}{2} Dv(a_2)[a_2 - a_1 - (\bar{s} - \underline{s})\chi'(\bar{s}) - \tfrac{1}{2}(\bar{s} - \underline{s})^2 \chi''(\bar{s})]\,,\\
&\frac{\partial^2 \hat{w}^*}{\partial \hat{x}_2^2}(\hat{a}_2) = D^2 v(a_2)[a_2 - a_3]^2\,,\\
&\frac{\partial^2 \hat{w}^*}{\partial \hat{x}_1^2}(\hat{a}_3) = D^2 v(a_3)[a_1 - a_3]^2\,,\\
&\frac{\partial^2 \hat{w}^*}{\partial \hat{x}_1\, \partial \hat{x}_2}(\hat{a}_3) = D^2 v(a_3)[a_1 - a_3, a_2 - a_3] + \tfrac{1}{2}(\bar{s} - \underline{s}) Dv(a_3)[\chi'(\underline{s}) - \chi'(\bar{s})]\,,\\
&\frac{\partial^2 \hat{w}^*}{\partial \hat{x}_2^2}(\hat{a}_3) = D^2 v(a_3)[a_2 - a_3]^2\,.
\end{aligned} \tag{3.2.12}$$

Then

$${}^tB_3^* = [0_{9\times 3};\, {}^t(b_{31}^*)_{9\times 6};\, {}^t(b_{32})_{9\times 9};\, 0_{9\times 13}]\,, \tag{3.2.13}$$

where matrix b_{32} is defined by (3.1.22) while b_{31}^* is given by (3.2.14).

$$b_{31}^{*}=\begin{bmatrix}0 & \frac{1}{4}(2\tilde{a}^1-\underset{\sim}{a}^1) & (\tilde{a}^1+\frac{1}{2}\underset{\sim}{a}^1) & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & \frac{1}{4}(2+2\tilde{a}^2-\underset{\sim}{a}^2) & 1+\tilde{a}^2+\frac{1}{2}\underset{\sim}{a}^2 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & (1+\tilde{b}^{i}+\frac{1}{2}\underset{\sim}{b}^1) & \frac{1}{4}(2+2\tilde{b}^1-\underset{\sim}{b}^1) & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & (\tilde{b}^2+\frac{1}{2}\underset{\sim}{b}^2) & \frac{1}{4}(2\tilde{b}^2-\underset{\sim}{b}^2) & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(\tilde{c}^1+\tilde{\tilde{c}}^1) & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(\tilde{c}^2+\tilde{\tilde{c}}^2) & 0\end{bmatrix}. \tag{3.2.14}$$

In this expression, parameters $\tilde{a}^\alpha$, $\tilde{b}^\alpha$, $\tilde{c}^\alpha$ and $\tilde{\tilde{c}}^\alpha$ are given by relations (3.1.23), while parameters $\underset{\sim}{a}^\alpha$ and $\underset{\sim}{b}^\alpha$ are defined by

$$\begin{aligned}
&\underset{\sim}{a}^1=\frac{(\boldsymbol{D}_1\times \boldsymbol{A}_2)\cdot \boldsymbol{e}_3}{(\boldsymbol{A}_1\times \boldsymbol{A}_2)\cdot \boldsymbol{e}_3}, \qquad \underset{\sim}{a}^2=\frac{(\boldsymbol{A}_1\times \boldsymbol{D}_1)\cdot \boldsymbol{e}_3}{(\boldsymbol{A}_1\times \boldsymbol{A}_2)\cdot \boldsymbol{e}_3},\\
&\underset{\sim}{b}^1=\frac{(\boldsymbol{D}_2\times \boldsymbol{B}_2)\cdot \boldsymbol{e}_3}{(\boldsymbol{B}_1\times \boldsymbol{B}_2)\cdot \boldsymbol{e}_3}, \qquad \underset{\sim}{b}^2=\frac{(\boldsymbol{B}_1\times \boldsymbol{D}_2)\cdot \boldsymbol{e}_3}{(\boldsymbol{B}_1\times \boldsymbol{B}_2)\cdot \boldsymbol{e}_3},\\
&\boldsymbol{D}_1=(\bar{s}-\underline{s})^2\boldsymbol{\chi}''(\underline{s})=\underset{\sim}{a}^\alpha \boldsymbol{A}_\alpha, \qquad \boldsymbol{D}_2=(\bar{s}-\underline{s})^2\boldsymbol{\chi}''(\bar{s})=\underset{\sim}{b}^\alpha \boldsymbol{B}_\alpha.
\end{aligned} \tag{3.2.15}$$

CONSTRUCTION of submatrix B_4^*. Since F_K and F_K^* are affine along the sides $[\hat{a}_3^*, \hat{a}_\alpha^*]$, $\alpha=1,2$, functions $\hat{f}_\alpha$, $\hat{g}_\alpha$, $\alpha=1,2$, are unchanged. By using (1.3.6), we obtain

$$\begin{aligned}
&[\hat{f}_1(\hat{x}_1)\quad \hat{f}_2(\hat{x}_2)\quad \hat{f}_3^*(\hat{x}_1)]=[DLLC^*(v)]_{1\times 31}[\hat{\mathscr{F}}_1^*(\hat{x}_1)\quad \hat{\mathscr{F}}_2^*(\hat{x}_2)\quad \hat{\mathscr{F}}_3^*(\hat{x}_1)]_{31\times 3},\\
&[\hat{g}_1(\hat{x}_1)\quad \hat{g}_2(\hat{x}_2)\quad \hat{g}_3^*(\hat{x}_1)]=[DLLC^*(v)]_{1\times 31}[\hat{\mathscr{G}}_1^*(\hat{x}_1)\quad \hat{\mathscr{G}}_2^*(\hat{x}_2)\quad \hat{\mathscr{G}}_3^*(\hat{x}_1)]_{31\times 3},
\end{aligned} \tag{3.2.16}$$

with

$$\{{}^t[\hat{\mathscr{F}}_\alpha^*(\hat{x}_\alpha)]={}^t[\hat{\mathscr{F}}_\alpha(\hat{x}_\alpha);0_{1\times 7}];\ {}^t[\hat{\mathscr{G}}_\alpha^*(\hat{x}_\alpha)]={}^t[\hat{\mathscr{G}}_\alpha(\hat{x}_\alpha);0_{1\times 7}],\quad \alpha=1,2, \tag{3.2.17}$$

$$\begin{aligned}
{}^t[\hat{\mathscr{F}}_3^*(\hat{x}_1)]=[&(\hat{x}_1)^3\{6\hat{x}_1^2-15\hat{x}_1+10\};(1-\hat{x}_1)^3\{6\hat{x}_1^2+3\hat{x}_1+1\};\\
&0;\tfrac{1}{2}\hat{x}_1^3(1-\hat{x}_1)^2\underset{\sim}{a}^1;\hat{x}_1^3(1-\hat{x}_1)\{4-3\hat{x}_1+\tfrac{1}{2}(1-\hat{x}_1)\underset{\sim}{a}^2\};\\
&\hat{x}_1(1-\hat{x}_1)^3\{1+3\hat{x}_1+\tfrac{1}{2}\hat{x}_1\underset{\sim}{b}^1\};\tfrac{1}{2}\hat{x}_1^2(1-\hat{x}_1)^3\underset{\sim}{b}^2;0\,0\,0;\\
&\tfrac{1}{2}\hat{x}_1^3(1-\hat{x}_1)^2;\tfrac{1}{2}\hat{x}_1^2(1-\hat{x}_1)^3;0_{1\times 19}],
\end{aligned} \tag{3.2.18}$$

$$\begin{aligned}
{}^t[\hat{\mathscr{G}}_3^*(\hat{x}_1)]=\Big[&0\,0\,0;\hat{x}_1^2(2\hat{x}_1-1)\{5-4\hat{x}_1-\tfrac{1}{2}(1-\hat{x}_1)(\tilde{a}^1+\tfrac{1}{2}\underset{\sim}{a}^1)\};\\
&\hat{x}_1^2(2\hat{x}_1-1)\Big(-3+\frac{5\hat{x}_1}{2}-\tfrac{1}{2}(1-\hat{x}_1)(\tilde{a}^2+\tfrac{1}{2}\underset{\sim}{a}^2)\Big);\\
&-\tfrac{1}{2}(1-\hat{x}_1)^2(1-2\hat{x}_1)\{1+5\hat{x}_1+\hat{x}_1(\tilde{b}^1+\tfrac{1}{2}\underset{\sim}{b}^1)\};\\
&(1-\hat{x}_1)^2(1-2\hat{x}_1)\{1+4\hat{x}_1-\tfrac{1}{2}\hat{x}_1(\tilde{b}^2+\tfrac{1}{2}\underset{\sim}{b}^2)\};0\,0;\\
&\frac{1+\tilde{a}^1}{2\tilde{a}^2}\hat{x}_1^2(1-\hat{x}_1)(1-2\hat{x}_1);\frac{1}{2}\Big(1+\frac{\tilde{a}^2}{1+\tilde{a}^1}\Big)\hat{x}_1^2(1-\hat{x}_1)(1-2\hat{x}_1);\\
&-\frac{1}{2}\Big(1+\frac{\tilde{b}^1}{1+\tilde{b}^2}\Big)\hat{x}_1(1-\hat{x}_1)^2(1-2\hat{x}_1);
\end{aligned}$$

$$- \frac{1+\tilde{b}^2}{2\tilde{b}^1}\, \hat{x}_1(1-\hat{x}_1)^2(1-2\hat{x}_1); 0\,0\,;$$

$$- \frac{1}{2(1+\tilde{a}^1)\tilde{a}^2}\, \hat{x}_1^2(1-\hat{x}_1)(1-2\hat{x}_1);$$

$$\frac{1}{2\tilde{b}^1(1+\tilde{b}^2)}\, \hat{x}_1(1-\hat{x}_1)^2(1-2\hat{x}_1)\,;$$

$$0\,0\,0; 16(1-\hat{x}_1)^2\hat{x}_1^2; 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\Big]. \tag{3.2.19}$$

Then, by similarity with relation (3.1.37), we obtain

$$B_4^* = \left[\tilde{E}^1 \frac{\mathrm{d}\hat{\mathscr{F}}_2^*}{\mathrm{d}\hat{x}_2}\left(\frac{1}{2}\right) + \tilde{E}^2 \hat{\mathscr{G}}_2^*\left(\frac{1}{2}\right); \tilde{F}^1 \frac{\mathrm{d}\hat{\mathscr{F}}_1^*}{\mathrm{d}\hat{x}_1}\left(\frac{1}{2}\right) + \tilde{F}^2 \hat{\mathscr{G}}_1^*(\tfrac{1}{2}); \begin{pmatrix} 0_{20\times 1} \\ -\sqrt{2} \\ 0_{10\times 3} \end{pmatrix} \right]_{31\times 3}, \tag{3.2.20}$$

with

$$\tilde{E}^1 = \Big\langle a_3 - a_1 - \tfrac{1}{4}(a_2 - a_1) - \tfrac{1}{32}(\bar{s} - \underline{s})[5\boldsymbol{\chi}'(\underline{s}) - 13\boldsymbol{\chi}'(\bar{s})]$$
$$- \tfrac{1}{64}(\bar{s} - \underline{s})^2[\boldsymbol{\chi}''(\underline{s}) + \boldsymbol{\chi}''(\bar{s})], \frac{a_2 - a_3}{|a_2 - a_3|^2} \Big\rangle,$$

$$\tilde{E}^2 = \Big\langle a_3 - a_1 - \tfrac{1}{4}(a_2 - a_1) - \tfrac{1}{32}(\bar{s} - \underline{s})[5\boldsymbol{\chi}'(\underline{s}) - 13\boldsymbol{\chi}'(\bar{s})]$$
$$- \tfrac{1}{64}(\bar{s} - \underline{s})^2[\boldsymbol{\chi}''(\underline{s}) + \boldsymbol{\chi}''(\bar{s})], \frac{a_1 - c_1}{|a_1 - c_1|^2} \Big\rangle,$$

$$\tilde{F}^1 = \Big\langle a_3 - a_2 - \tfrac{1}{4}(a_1 - a_2) + \tfrac{1}{32}(\bar{s} - \underline{s})(5\boldsymbol{\chi}'(\bar{s}) - 13\boldsymbol{\chi}'(\underline{s}))$$
$$- \tfrac{1}{64}(\bar{s} - \underline{s})^2[\boldsymbol{\chi}''(\underline{s}) + \boldsymbol{\chi}''(\bar{s})], \frac{a_1 - a_3}{|a_1 - a_3|^2} \Big\rangle,$$

$$\tilde{F}^2 = \Big\langle a_3 - a_2 - \tfrac{1}{4}(a_1 - a_2) + \tfrac{1}{32}(\bar{s} - \underline{s})(5\boldsymbol{\chi}'(\bar{s}) - 13\boldsymbol{\chi}'(\underline{s}))$$
$$- \tfrac{1}{64}(\bar{s} - \underline{s})^2[\boldsymbol{\chi}''(\underline{s} + \boldsymbol{\chi}''(\bar{s})], \frac{a_2 - c_2}{|a_2 - c_2|^2} \Big\rangle.$$

CONSTRUCTION of submatrix B_5^*. Relations (2.3.3), (2.3.6), (2.3.7) and (3.2.16) give

$$B_5^* = \Big[\hat{\mathscr{F}}_2^*\left(\frac{5}{6}\right); \hat{\mathscr{F}}_2^*\left(\frac{2}{3}\right); \hat{\mathscr{F}}_2^*\left(\frac{1}{3}\right); \hat{\mathscr{F}}_2^*\left(\frac{1}{6}\right); \hat{\mathscr{F}}_1^*\left(\frac{1}{6}\right); \hat{\mathscr{F}}_1^*\left(\frac{1}{3}\right);$$
$$\hat{\mathscr{F}}_1^*\left(\frac{2}{3}\right); \hat{\mathscr{F}}_1^*\left(\frac{5}{6}\right); \hat{\mathscr{F}}_3^*\left(\frac{5}{6}\right); \hat{\mathscr{F}}_3^*\left(\frac{2}{3}\right); \hat{\mathscr{F}}_3^*\left(\frac{1}{3}\right); \hat{\mathscr{F}}_3^*\left(\frac{1}{6}\right)\Big]_{31\times 12}. \tag{3.2.21}$$

CONSTRUCTION of submatrices B_6^*, B_7^* and B_8^*. From relations (2.3.3), (2.3.6), (2.3.7) and (3.2.16), we obtain

$$\begin{aligned} B_6^* = \Big[& -\Big\langle \frac{\partial F_K^*}{\partial \hat{x}_1}(\hat{d}_1^*), \frac{a_2 - a_3}{|a_2 - a_3|^2} \frac{\mathrm{d}\hat{\mathscr{F}}_2^*}{d\hat{x}_2}\Big(\frac{5}{6}\Big) + \frac{a_1 - c_1}{|a_1 - c_1|^2} \hat{\mathscr{G}}_2^*\Big(\frac{5}{6}\Big)\Big\rangle ; \\ & -\Big\langle \frac{\partial F_K^*}{\partial \hat{x}_1}(\hat{d}_2^*), \frac{a_2 - a_3}{|a_2 - a_3|^2} \frac{\mathrm{d}\hat{\mathscr{F}}_2^*}{d\hat{x}_2}\Big(\frac{2}{3}\Big) + \frac{a_1 - c_1}{|a_1 - c_1|^2} \hat{\mathscr{G}}_2^*\Big(\frac{2}{3}\Big)\Big\rangle ; \\ & -\Big\langle \frac{\partial F_K^*}{\partial \hat{x}_1}(\hat{d}_3^*), \frac{a_2 - a_3}{|a_2 - a_3|^2} \frac{\mathrm{d}\hat{\mathscr{F}}_2^*}{d\hat{x}_2}\Big(\frac{1}{3}\Big) + \frac{a_1 - c_1}{|a_1 - c_1|^2} \hat{\mathscr{G}}_2^*\Big(\frac{1}{3}\Big)\Big\rangle ; \\ & -\Big\langle \frac{\partial F_K^*}{\partial \hat{x}_1}(\hat{d}_4^*), \frac{a_2 - a_3}{|a_2 - a_3|^2} \frac{\mathrm{d}\hat{\mathscr{F}}_2^*}{d\hat{x}_2}\Big(\frac{1}{6}\Big) + \frac{a_1 - c_1}{|a_1 - c_1|^2} \hat{\mathscr{G}}_2^*\Big(\frac{1}{6}\Big)\Big\rangle \Big]_{31\times 4}, \end{aligned} \tag{3.2.22}$$

$$\begin{aligned} B_7^* = \Big[& -\Big\langle \frac{\partial F_K^*}{\partial \hat{x}_2}(\hat{d}_5^*), \frac{a_1 - a_3}{|a_1 - a_3|^2} \frac{\mathrm{d}\hat{\mathscr{F}}_1^*}{\mathrm{d}\hat{x}_1}\Big(\frac{1}{6}\Big) + \frac{a_2 - c_2}{|a_2 - c_2|^2} \hat{\mathscr{G}}_1^*\Big(\frac{1}{6}\Big)\Big\rangle ; \\ & -\Big\langle \frac{\partial F_K^*}{\partial \hat{x}_2}(\hat{d}_6^*), \frac{a_1 - a_3}{|a_1 - a_3|^2} \frac{\mathrm{d}\hat{\mathscr{F}}_1^*}{\mathrm{d}\hat{x}_1}\Big(\frac{1}{3}\Big) + \frac{a_2 - c_2}{|a_2 - c_2|^2} \hat{\mathscr{G}}_1^*\Big(\frac{1}{3}\Big)\Big\rangle ; \\ & -\Big\langle \frac{\partial F_K^*}{\partial \hat{x}_2}(\hat{d}_7^*), \frac{a_1 - a_3}{|a_1 - a_3|^2} \frac{\mathrm{d}\hat{\mathscr{F}}_1^*}{\mathrm{d}\hat{x}_1}\Big(\frac{2}{3}\Big) + \frac{a_2 - c_2}{|a_2 - c_2|^2} \hat{\mathscr{G}}_1^*\Big(\frac{2}{3}\Big)\Big\rangle ; \\ & -\Big\langle \frac{\partial F_K^*}{\partial \hat{x}_2}(\hat{d}_8^*), \frac{a_1 - a_3}{|a_1 - a_3|^2} \frac{\mathrm{d}\hat{\mathscr{F}}_1^*}{\mathrm{d}\hat{x}_1}\Big(\frac{5}{6}\Big) + \frac{a_2 - c_2}{|a_2 - c_2|^2} \hat{\mathscr{G}}_1^*\Big(\frac{5}{6}\Big)\Big\rangle \Big], \end{aligned} \tag{3.2.23}$$

$$B_8^* = \Big[-\sqrt{2}\hat{\mathscr{G}}_3^*\Big(\frac{5}{6}\Big); -\sqrt{2}\hat{\mathscr{G}}_3^*\Big(\frac{2}{3}\Big); -\sqrt{2}\hat{\mathscr{G}}_3^*\Big(\frac{1}{3}\Big); -\sqrt{2}\hat{\mathscr{G}}_3^*\Big(\frac{1}{6}\Big)\Big]. \tag{3.2.24}$$

STEP 3: Transition from $\hat{\Delta}_K^*(v)$ *to* $\pi_K^* v(x) = w^*(x) = \hat{w}^*(\hat{x})$. Let $[m9]_{55\times 1}$ be the 55 basis monomials of degree less than or equal to 9 in $\hat{x}_1$ and $\hat{x}_2$ and $[A^*]_{55\times 55} = [DL^*(m9)]^{-1}$. Corresponding basis polynomials are detailed in the annex of [10]. Then

$$\pi_K^* v(x) = \hat{w}^*(\hat{x}) = [DLGL^*(v)]_{1\times 31}[\tilde{D}^*]_{31\times 31}[B^*]_{31\times 55}[A^*]_{55\times 55}[m9]_{55\times 1}. \tag{3.2.25}$$

3.3. *Some numerical tests of interpolation properties*

It is important to test the interpolation properties numerically. We compute the asymptotic order of the interpolation error for the following two cases:
(1) an exact representation of the curved boundary;
(2) an approximate representation of the curved boundary.

3.3.1. *Tests on the asymptotic order of interpolation error (generalities)*

From [2, (4.1) and (4.51)], we have to numerically verify that

$$\left.\begin{aligned}&|v-\pi_K v|_{m,K}\leq ch_K^{6-m}\|v\|_{6,K}\,,\\&|v-\pi_K^* v|_{m,K}\leq ch_K^{6-m}\|v\|_{6,K}\,,\end{aligned}\right\} \quad m=0,\dots,6 \quad \forall v\in H^6(K)\,. \tag{3.3.1}$$

For simplicity, since it can be proved that the asymptotic error estimates in norms $|\cdot|_{L^2(K)}$ and $|\cdot|_{L^\infty(K)}$ have the same order, we will just check that

$$|v-\pi_K v|_{0,\infty,K}=O(h_K^6)\,,\qquad |v-\pi_K^* v|_{0,\infty,K}=O(h_K^6)\quad \forall v\in W^{6,\infty}(K)\,, \tag{3.3.2}$$

or, more easily, that the asymptotic behaviour of $|v-\pi_K v|$ (respectively $|v-\pi_K^* v|$) at a given point $F_{K_h}(\hat{x}_0,\hat{x}_0)$ (respectively $F_{K_h}^*(\hat{x}_0,\hat{x}_0)$) is in $O(h^6)$.

In this way, for each example under consideration (see Sections 3.3.2 and 3.3.3), we successively

(i) consider a series of nested triangles whose sizes decrease as h, $\frac{1}{2}h$, $\frac{1}{4}h, \dots$;
(ii) interpolate given regular functions v;
(iii) illustrate the results graphically displaying $-\operatorname{Log} h$ on the abcissa and $-\operatorname{Log}|v-\pi_K v|$ (respectively $-\operatorname{Log}|v-\pi_K^* v|$) on the ordinate and check that

$$\operatorname{Log}|v-\pi_K v| = C + 6\operatorname{Log} h\,,\qquad \operatorname{Log}|v-\pi_K^* v| = C+6\operatorname{Log} h\,. \tag{3.3.3}$$

3.3.2. Numerical tests for an 'exact' interpolation of the boundary

Firstly it seems interesting to make some numerical experiments for an exact representation of the curved boundary so that it only remains the interpolation error of the function.

EXAMPLE 3.3.1. Unit right-angled triangle.

This is the most simple case which is associated with the following representation of the curved (here straight!) side a_1a_2:

$$x_1=\chi_1(s)=1-s\,,\quad x_2=\chi_2(s)=s\,,\quad \text{with}\ \underline{s}=0\leq s\leq \bar{s}=1\,. \tag{3.3.4}$$

This test is interesting since $F_K=I$ and $F_K^*=I$ involves

$$\pi_K v(x)\equiv v(x)\,,\quad \pi_K^* v(x)=v(x)\quad \forall v\in P_5 \tag{3.3.5}$$

or, equivalently, by using (3.1.44) and (3.2.25)

$$\left.\begin{aligned}&x_1^p x_2^q=[DLGL(x_1^p x_2^q)][\tilde{D}][B][A][m7]\quad \forall p,q\in\mathbb{N}\ \text{and}\ p+q=5\,,\\&x_1^p x_2^q=[DLGL^*(x_1^p x_2^q)][\tilde{D}^*][B^*][A^*][m9]\quad \forall p,q\in\mathbb{N}\ \text{and}\ p+q=5\,.\end{aligned}\right. \tag{3.3.6}$$

The numerical results show that identities (3.3.5) are exactly satisfied. We have also checked the estimates (3.3.3); in this way, we have considered a sequence of right-angled triangles deduced from the unit right-angled triangle by homothetic transformations centred at point $(\frac{1}{2},\frac{1}{2})$ and whose ratios are $h=\frac{1}{2},\frac{1}{4},\frac{1}{8},\dots$ (see Fig. 12). More generally, we have

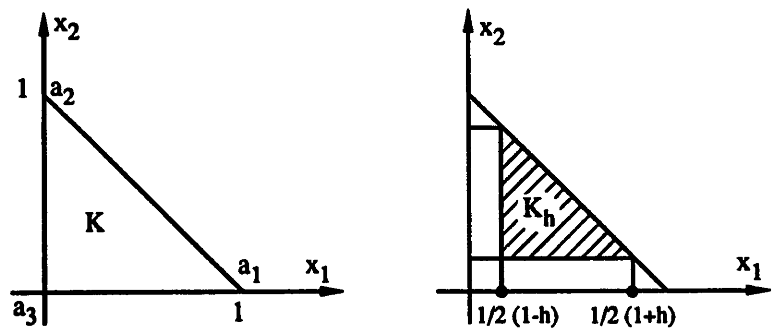


Fig. 12. The unit right-angled triangle and the associate sequence of triangles K_h.

successfully checked the same properties for any straight triangle K (note that the associate mapping F_K is still affine in this case).

EXAMPLE 3.3.2. A curved unit right-angled triangle (*case* $F_K \in (P_3)^2$). Now the straight side a_1a_2 is replaced by the curved side whose parametric equations are (see Fig. 13)

$$x_1 = \chi_1(s) = 1 - s\,, \qquad x_2 = \chi_2(s) = \tfrac{1}{2}s[s^2 - 3s + 4]\,, \quad \underline{s} = 0 \leqslant s \leqslant \bar{s} = 1\,, \tag{3.3.7}$$

so that its interpolation by the mapping (1.3.2) is exact. The associate application F_K is given by (see (1.3.2))

$$F_{K1}(\hat{x}_1, \hat{x}_2) = \hat{x}_1\,, \qquad F_{K2}(\hat{x}_1, \hat{x}_2) = \hat{x}_2 + \tfrac{1}{4}\hat{x}_1\hat{x}_2[3 + \hat{x}_1 - \hat{x}_2]\,. \tag{3.3.8}$$

Now, since the mapping F_K is no longer affine, the identities (3.3.5) are no longer satisfied. In order to test the estimate $(3.3.2)_1$, we have to introduce a sequence of triangle K_h whose diameter h is decreasing to 0 and whose curved sides coincide (for this particular example) with the curved boundary of the domain. In this way, we use the triangles K_h whose vertices (a_{1h}, a_{2h}, a_{3h}) are given by

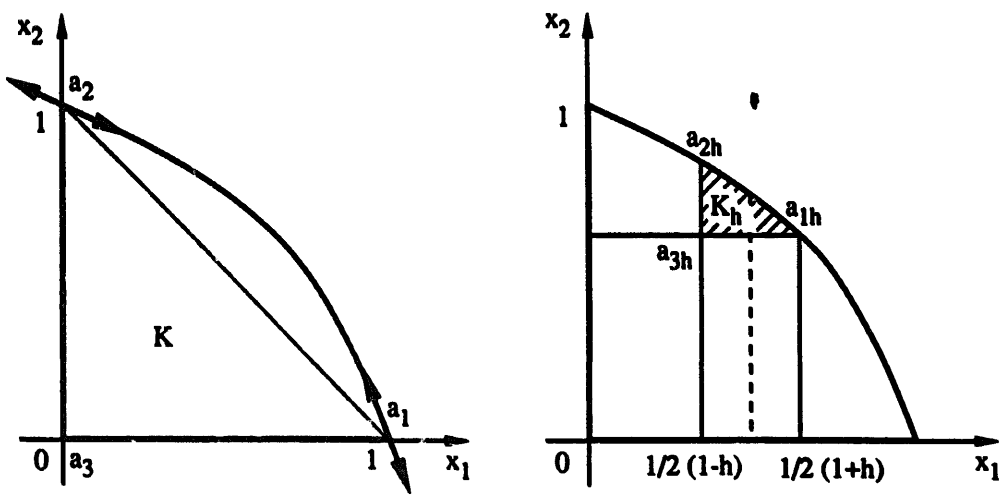


Fig. 13. The curved unit right-angled triangle and the associate sequence of triangles K_h.

$$
\begin{aligned}
&\text{Vertex } a_{1h}: \quad \underline{s}_h = \tfrac{1}{2}(1-h) \Rightarrow (x_{11} = \tfrac{1}{2}(1+h); x_{21} = \tfrac{1}{16}(1-h)(11+4h+h^2)), \\
&\text{Vertex } a_{2h}: \quad \bar{s}_h = \tfrac{1}{2}(1+h) \Rightarrow (x_{12} = \tfrac{1}{2}(1-h); x_{22} = \tfrac{1}{16}(1+h)(11-4h+h^2)), \\
&\text{Vertex } a_{3h}: \quad (x_{13} = \tfrac{1}{2}(1-h); x_{23} = \tfrac{1}{16}(1-h)(11+4h+h^2)),
\end{aligned} \tag{3.3.9}
$$

so that the associate application F_{K_h} is given by relation (1.3.2), i.e.,

$$
\begin{aligned}
&F_{K1_h}(\hat{x}_1, \hat{x}_2) = \tfrac{1}{2}(1-h) + h\hat{x}_1, \\
&F_{K2_h}(\hat{x}_1, \hat{x}_2) = \tfrac{1}{16}(1-h)(11+4h+h^2) + \tfrac{1}{8}h(7+h^2)\hat{x}_2 + \tfrac{1}{4}\hat{x}_1\hat{x}_2h^2[3 + h(\hat{x}_1 - \hat{x}_2)].
\end{aligned} \tag{3.3.10}
$$

Then, to check estimate $(3.3.2)_1$, we graphically represent $(3.3.3)_1$ for given functions v, for instance $v(x_1, x_2) = x_1^6x_2^4$, and $v(x_1, x_2) = \cos(3x_1)\sin(2x_2)$, at the image of point ($\hat{x}_1 = \frac{3}{8}$, $\hat{x}_2 = \frac{3}{8}$) by the application F_{K_h}, i.e.,

$$
x_1 = F_{K1_h}\left(\frac{3}{8}, \frac{3}{8}\right) = \frac{1}{2} - \frac{h}{8}, \qquad x_2 = F_{K2_h}\left(\frac{3}{8}, \frac{3}{8}\right) = \frac{11}{16} - \frac{7h}{64} - \frac{21h^2}{256} - \frac{h^3}{64}, \tag{3.3.11}
$$

for $h = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ Thus, Fig. 14 shows that the slope is effectively 6 according to the theoretical prediction (3.3.3).

EXAMPLE 3.3.3. A curved unit right-angled triangle (case $F_K^* \in (P_5)^2$) and the associate sequence of triangles K_h. Consider the curved side a_1a_2 whose parametric equations are (see Fig. 15)

$$
x_1 = \chi_1(s) = 1 - s, \quad x_2 = \chi_2(s) = \frac{s}{10}[4s^4 + 13s^2 - 27s + 20], \quad \underline{s} = 0 \leq s \leq \bar{s} = 1, \tag{3.3.12}
$$

so that its interpolation by the mapping (1.3.4) is exact. The associate application F_K^* is given by (see (1.3.6))

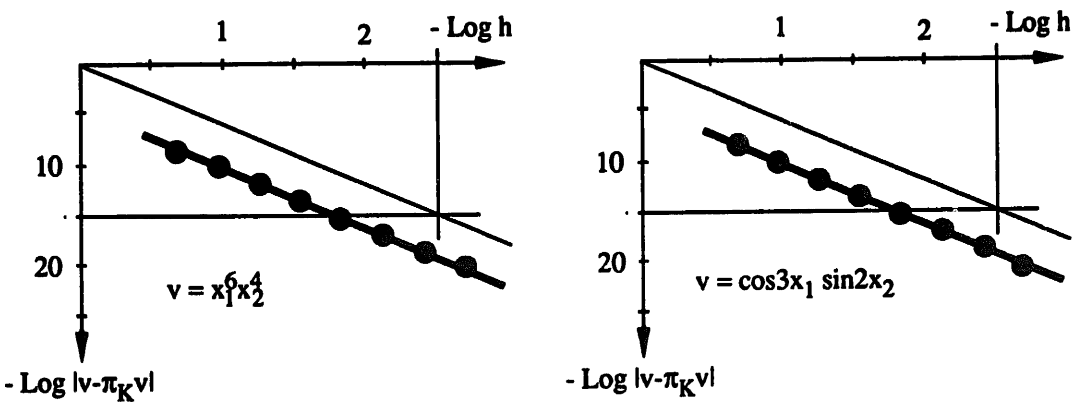


Fig. 14. Asymptotic behaviour of $|v - \pi_K v|$ at point $F_{K_h}(\frac{3}{8}, \frac{3}{8})$ when $h \to 0$.

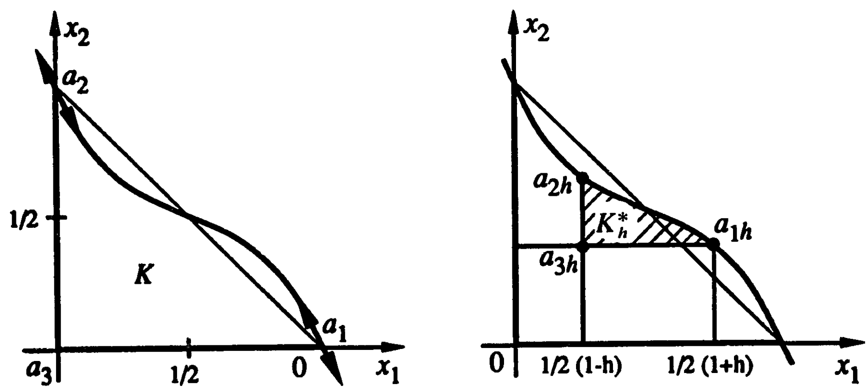


Fig. 15. The curved right-angled triangle and the associate sequence of triangles K_h^*.

$$F_{K1}^*(\hat{x}_1, \hat{x}_2) = \hat{x}_1 ,$$
$$F_{K2}^*(\hat{x}_1, \hat{x}_2) = \hat{x}_2 + \tfrac{1}{20}\hat{x}_1\hat{x}_2[-4(\hat{x}_2)^3 - 4(\hat{x}_2)^2 - 17\hat{x}_2 - 5 + 4(\hat{x}_1)^3 - 16(\hat{x}_1)^2 + 37\hat{x}_1] , \tag{3.3.13}$$

Here again we consider the sequence of triangles K_h whose vertices are given by

$$\begin{aligned}
&\text{Vertex } a_{1h}: \quad (x_{11} = \tfrac{1}{2}(1+h); x_{21} = \tfrac{1}{80}(1-h)(40+24h+19h^2-4h^3+h^4)) ,\\
&\text{Vertex } a_{2h}: \quad (x_{12} = \tfrac{1}{2}(1-h); x_{22} = \tfrac{1}{80}(1+h)(40-24h+19h^2+4h^3+h^4)) ,\\
&\text{Vertex } a_{3h}: \quad (x_{13} = \tfrac{1}{2}(1-h); x_{23} = \tfrac{1}{80}(1-h)(40+24h+19h^2-4h^3+h^4)) ,
\end{aligned} \tag{3.3.14}$$

so that the application $F_{K_h}^*$ is given by relation (1.3.6):

$$F_{K1_h}^*(\hat{x}_1, \hat{x}_2) = \tfrac{1}{2}(1-h) + h\hat{x}_1 ,$$
$$\begin{aligned}
F_{K2_h}^*(\hat{x}_1, \hat{x}_2) = {}& \frac{1}{80}(1-h)(40+24h+19h^2-4h^3+h^4) + \frac{h}{40}(16+23h^2+h^4)\hat{x}_2 \\
&+ \frac{1}{2}\hat{x}_1\hat{x}_2\Big[\frac{h^2}{2}(1-2h^2) + \frac{h^3}{10}(23+10h+4h^2)\hat{x}_1 + \frac{h^3}{10}(-23+10h-4h^2)\hat{x}_2 \\
&- \frac{h^4}{5}(5+3h)(\hat{x}_1)^2 - \frac{h^4}{5}(5-3h)(\hat{x}_2)^2 + \frac{2h^5}{5}(\hat{x}_1)^3 - \frac{2h^5}{5}(\hat{x}_2)^3\Big] .
\end{aligned} \tag{3.3.15}$$

Now we obtain Fig. 16 which again shows an $O(h^6)$ error estimate.

3.3.3. Numerical tests for an approximate interpolation of the boundary

Previous examples are such that applications $F_K \in (P_3)^2$ or $F_K^* \in (P_5)^2$ give an exact representation of the curved side a_1a_2. Now let us consider some examples for which the curved side is only approximated by the applications F_K or F_K^*.

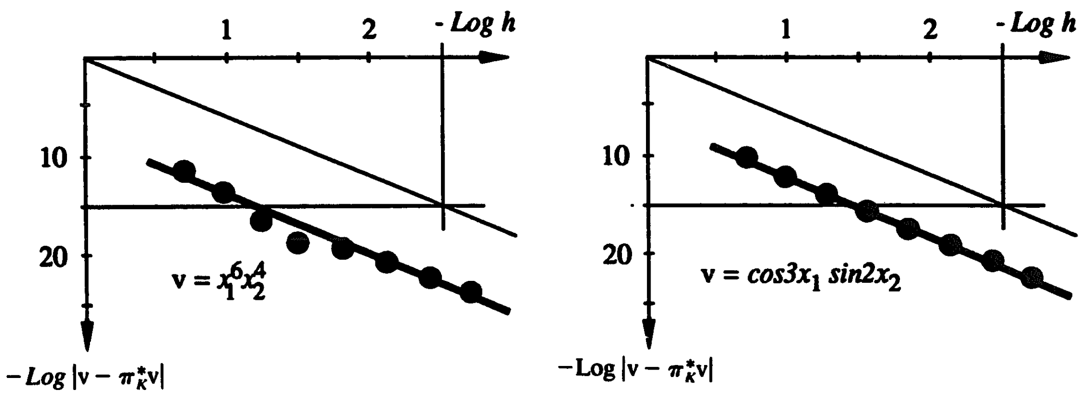


Fig. 16. Asymptotic behaviour of $|v - \pi^*_K v|$ at point $F^*_{K_h}(\frac{1}{4}, \frac{1}{4})$ when $h \to 0$.

EXAMPLE 3.3.4. The $(P_3)^2$ approximation of the curved unit right-angled triangle studied in Example 3.3.3. In Example 3.3.3, we considered a sequence of triangles K^*_h such that the curved side $a_{1h}a_{2h}$ coincided with the initial boundary. In this way, we have used the application $F^*_{K_h} \in (P_5)^2$ defined by relation (3.3.15). Now, starting from the same sequence of curved triangle $a_{1h}a_{2h}a_{3h}$, we consider the $(P_3)^2$-approximation of the curved side. Thus, by substituting relations (3.3.12), (3.3.14) into relations (1.3.2), we obtain the application F_{K_h},

$$
\begin{aligned}
F_{K1_h}(\hat{x}_1, \hat{x}_2) &= \tfrac{1}{2}(1 - h) + h\hat{x}_1 , \\
F_{K2_h}(\hat{x}_1, \hat{x}_2) &= \frac{1}{80}(1 - h)(40 + 24h + 19h^2 - 4h^3 + h^4) \\
&\quad + \frac{h}{40}(16 + 23h^2 + h^4)\hat{x}_2 + \frac{h^2}{20}\hat{x}_1\hat{x}_2[5 - 10h^2 + h(23 + 2h^2)(\hat{x}_1 - \hat{x}_2)] .
\end{aligned}
\tag{3.3.16}
$$

Here again Fig. 17 shows an $O(h^6)$ error estimate.

EXAMPLE 3.3.5. Case of an elliptic edge. The curved side of triangle K is represented by (see Fig. 18)

$$
x_1 = \chi_1(s) = \cos s , \quad x_2 = \chi_2(s) = R \sin s , \quad 0 < R < 1, \; \underline{s} = \frac{\pi}{6} \leqslant s \leqslant \bar{s} = \frac{\pi}{3} , \tag{3.3.17}
$$

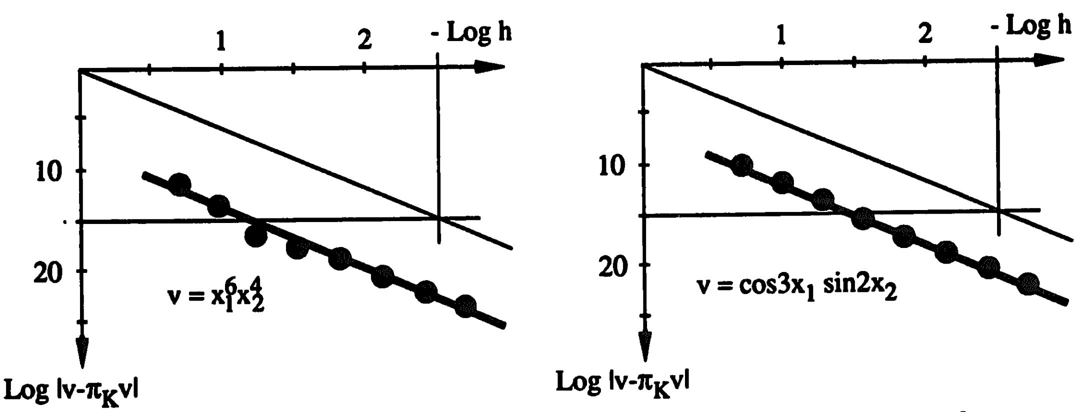


Fig. 17. Asymptotic behaviour of $|v - \pi_K v|$ at point $F_{K_h}(\frac{3}{8}, \frac{3}{8})$ when $h \to 0$.

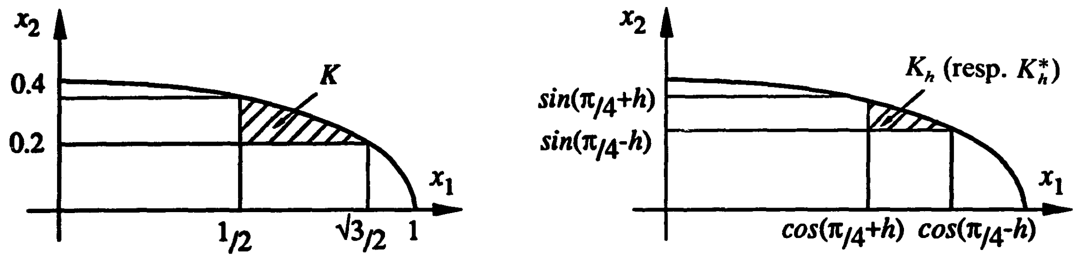


Fig. 18. The 'curved elliptic' triangle K and the associate sequence of triangles K_h (or K_h^*) when $R = 0.4$.

and it can be approximated with the help of $F_K \in (P_3)^2$ or $F_K^* \in (P_5)^2$, respectively defined by relations (1.3.2) or (1.3.6).

Here again we introduce the sequence K_h (or K_h^*) of triangles whose vertices are given by $(0 < h < \pi/12)$:

$$
\begin{aligned}
&\text{Vertex } a_{1h}\text{:} \quad \underline{s}_h = \frac{\pi}{4} - h \;\Rightarrow\; \left(x_1 = \cos\left(\frac{\pi}{4} - h\right); x_2 = R\sin\left(\frac{\pi}{4} - h\right)\right),\\
&\text{Vertex } a_{2h}\text{:} \quad \bar{s}_h = \frac{\pi}{4} + h \;\Rightarrow\; \left(x_1 = \cos\left(\frac{\pi}{4} + h\right); x_2 = R\sin\left(\frac{\pi}{4} + h\right)\right),\\
&\text{Vertex } a_{3h}\text{:} \quad \left(x_1 = \cos\left(\frac{\pi}{4} + h\right), x_2 = R\sin\left(\frac{\pi}{4} - h\right)\right),
\end{aligned}
\qquad (3.3.18)
$$

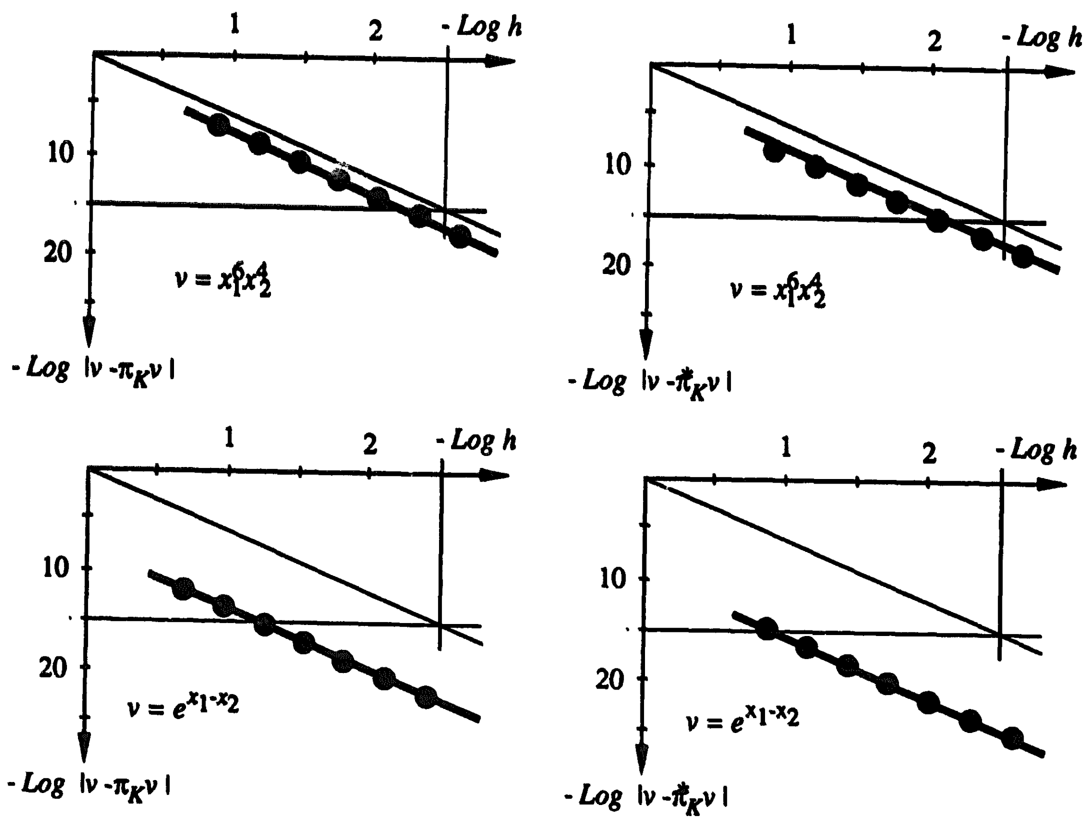


Fig. 19. Asymptotic behaviour of $|v - \pi_K v|$ (respectively $|v - \pi_K^* v|$) at point $F_{K_h}(\frac{3}{8}, \frac{3}{8})$ (respectively $F_{K_h}^*(\frac{1}{4}, \frac{1}{4})$) when $h \to 0$.

while the computation of the error is made at an internal point of triangles K_h (or K_h^*). The associated results are displayed in Fig. 19. Here again, we find an asymptotic error in $O(h^6)$.

4. Concluding remarks

These examples show the great efficiency of these curved C^1 elements and they illustrate the relevance of the theoretical study of the order of the approximation. These methods will be used to approximate thin plate and thin shell problems in the second part of this work [11].

Acknowledgment

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