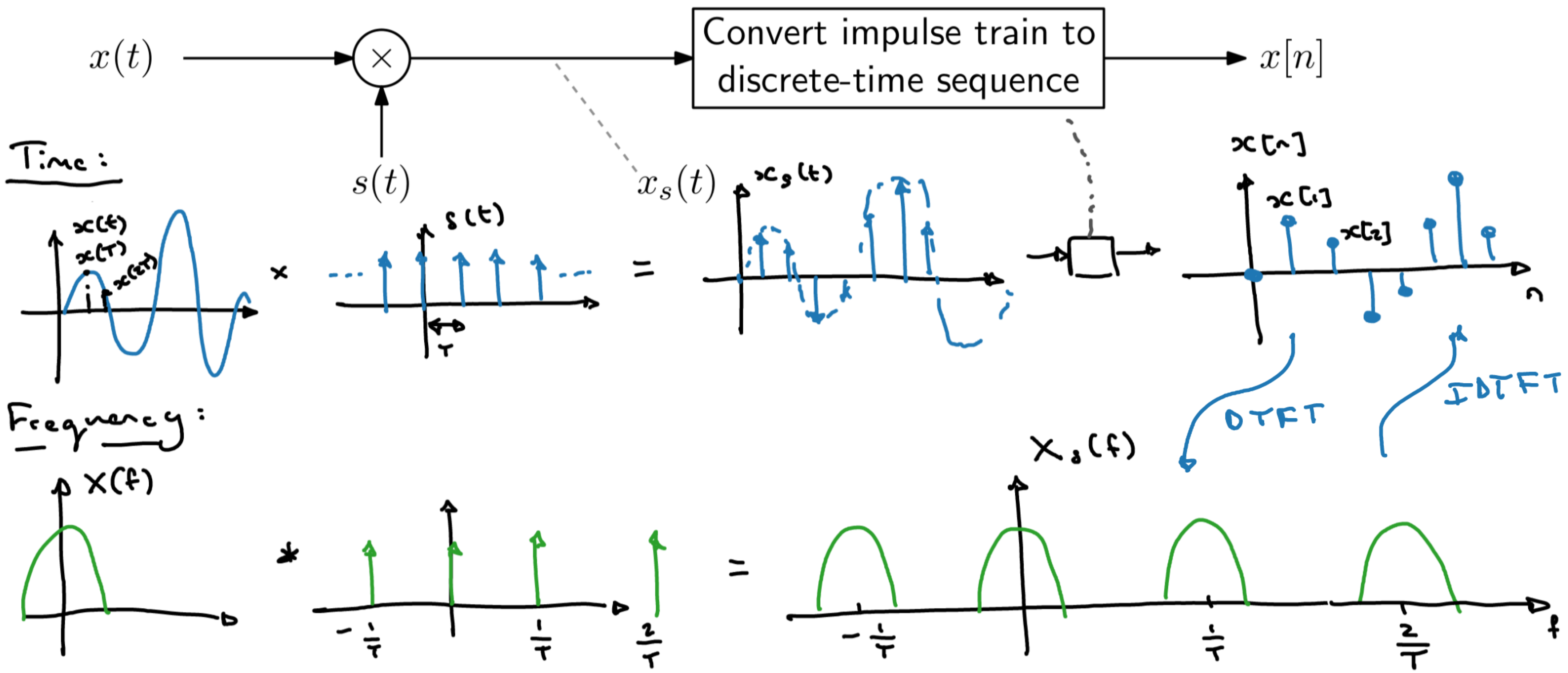
Discrete-time Fourier transform (DTFT)

And how it leads to aliasing and affects periodicity

Herman Kamper

FT $\rightarrow$ DTFT $\rightarrow$ DFT

Mathematical model of sampling



Discrete-time Fourier transform (DTFT)

$$x_s(t) = x(t) \cdot s(t) = x(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT) = \sum_{n=-\infty}^{\infty} x(nT) \cdot \delta(t - nT)$$

$$X_s(f) = \mathcal{F}\{x_s(t)\} = \int_{-\infty}^{\infty} x_s(t) \cdot e^{-j2\pi f t} \cdot dt$$

$$= \int_{-\infty}^{\infty} \left[\sum_{n=-\infty}^{\infty} x(nT) \cdot \delta(t - nT) \right] \cdot e^{-j2\pi f t} \cdot dt$$

$$= \sum_{n=-\infty}^{\infty} x(nT) \int_{-\infty}^{\infty} \delta(t - nT) \cdot e^{-j2\pi f t} \cdot dt$$

$$= \sum_{n=-\infty}^{\infty} \overbrace{x(nT)}^{x[n]} \cdot e^{-j2\pi f n T}$$

Define:

$$f_\omega \equiv f \underbrace{T}_{[\text{sec/sample}]} = \frac{f \;[\text{cycles/sec}]}{f_s \;[\text{samples/sec}]}$$

$$f_\omega : \left[\text{cycles/sample} \right]$$

DTFT:

$$\boxed{\begin{aligned} X(f_\omega) &= \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j2\pi n f_\omega} \\ X(\omega) &= \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\omega n}, \quad \underbrace{\omega}_{\text{radians/sample}} = 2\pi f_\omega \end{aligned}}$$

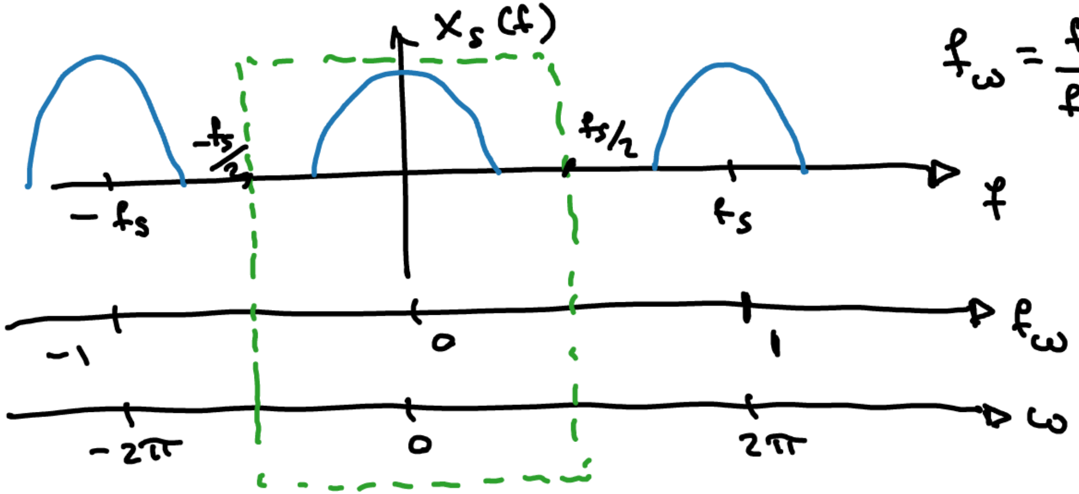
Periodicity:

$$X(f_\omega) = X(f_\omega + k)$$

$$X(\omega) = X(\omega + 2\pi k)$$

Inverse DTFT

$$\hat{X}_s(f) = \begin{cases} X_s(f) & \text{for } -\frac{f_s}{2} \le f \le \frac{f_s}{2} \\ 0 & \text{otherwise} \end{cases}$$



$$f_\omega = \frac{f}{f_s}$$

$$X_s(f) = \hat{X}_s(f) * \sum_{k=-\infty}^{\infty} \delta(f - kf_s)$$

$$\mathcal{F}^{-1}\{X_s(f)\} = \mathcal{F}^{-1}\{\hat{X}_s(f)\} \cdot \underbrace{\mathcal{F}^{-1}\left\{\sum_{k=-\infty}^{\infty} \delta(f - kf_s)\right\}}_{\frac{1}{f_s}\sum_{n=-\infty}^{\infty}\delta\left(t - \frac{n}{f_s}\right)}$$

$$\mathcal{F}^{-1}\{\hat{X}_s(f)\} = \int_{-\infty}^{\infty} \hat{X}_s(f) \cdot e^{j2\pi ft}\, df = \int_{-f_s/2}^{f_s/2} X_s(f) \cdot e^{j2\pi ft} \cdot df$$

$$x_s(t) = \left\{\left[\int_{-f_s/2}^{f_s/2} X_s(f) \cdot e^{j2\pi ft} \cdot df\right] \times \frac{1}{f_s}\right\} \left[\sum_{n=-\infty}^{\infty} \delta\Big(t - \overbrace{\tfrac{n}{f_s}}^{nT}\Big)\right]$$

$$x[n] = x(nT) = \frac{1}{f_s} \int_{-f_s/2}^{f_s/2} X_s(f) \cdot e^{j2\pi f nT} \cdot df$$

$$x[n] = \frac{1}{f_s} \int_{-f_s/2}^{f_s/2} X_s(f) \cdot e^{j2\pi f n T} \cdot df$$

$$f_\omega = fT = \frac{f}{f_s}$$

$$\frac{df_\omega}{df} = \frac{1}{f_s}$$

$$\therefore \; df_\omega = \frac{1}{f_s} df$$

f	f_ω
$-f_s/2$	$-1/2$
$f_s/2$	$1/2$

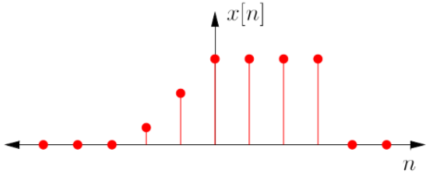
IDTFT

$$\boxed{x[n] = \int_{-1/2}^{1/2} X(f_\omega) e^{j2\pi n f_\omega} \cdot df_\omega}$$

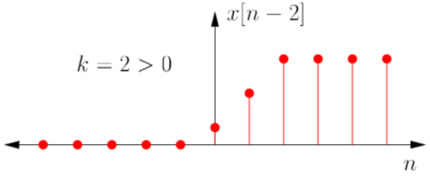
$$\omega = 2\pi f_\omega$$

Operations on discrete-time signals

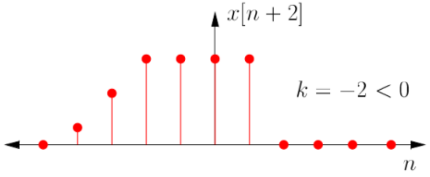
- **Time shift:** $x[n-k]$ is a version of $x[n]$ shifted by $|k|$ samples to the right if $k > 0$ or to the left if $k < 0$



Unshifted signal

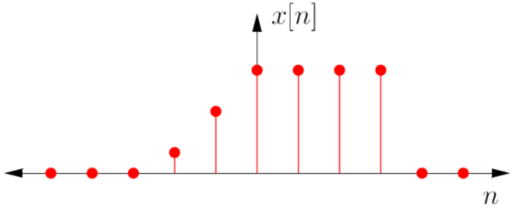


Delayed by 2 samples

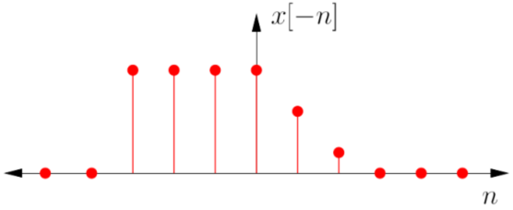


Advanced by 2 samples

- **Reflection about time origin** $x[-n]$ is reflection of $x[n]$ about $n = 0$

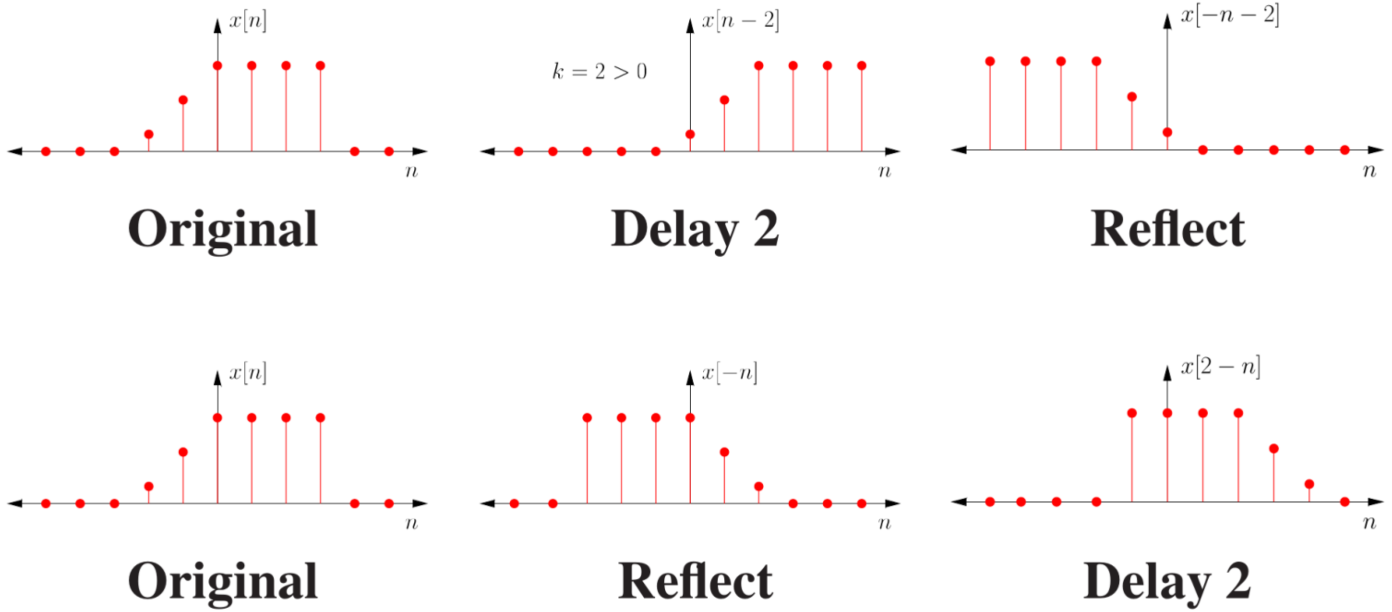


Unreflected signal



Reflected about $n = 0$

- Time-shifting and reflection about $n = 0$ are <u>not commutative</u>



Properties of the DTFT

- Linearity:

$$\mathcal{F}\{\alpha x_1[n] + \beta x_2[n]\} = \alpha X_1(\omega) + \beta X_2(\omega)$$

- Time shift:

$$\mathcal{F}\{x[n-k]\} = e^{-j\omega k} X(\omega)$$

- Time reversal and frequency reversal:

$$\mathcal{F}\{x[-n]\} = X(-\omega)$$

- Convolution:

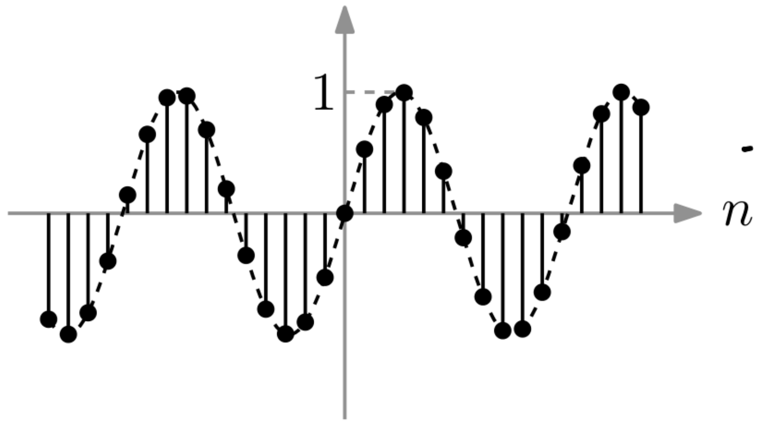
$$\mathcal{F}\{x_1[n] * x_2[n]\} = X_1(\omega) \cdot X_2(\omega)$$

- Windowing:

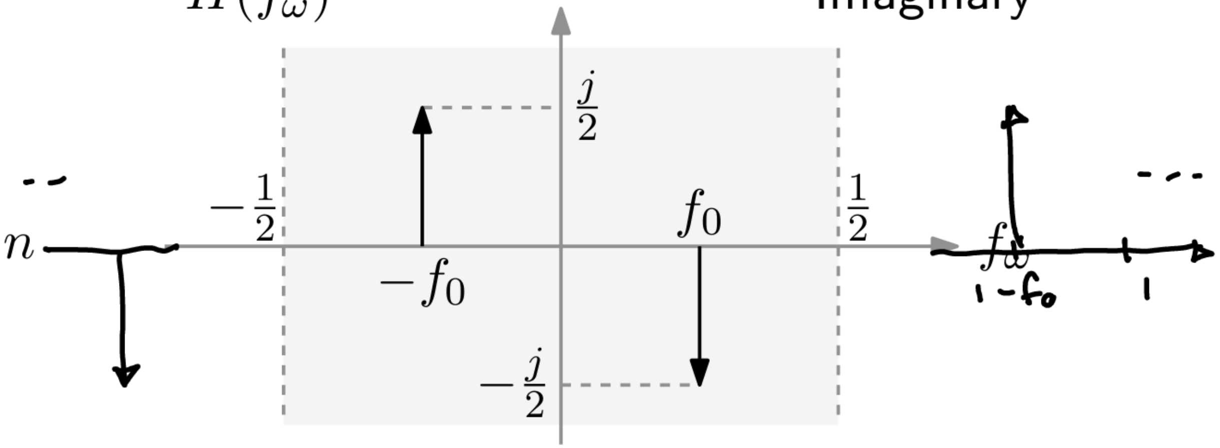
$$\mathcal{F}\{x_1[n] \cdot x_2[n]\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\lambda) \cdot X_2(\omega - \lambda) d\lambda$$

Discrete-time domain	Frequency domain

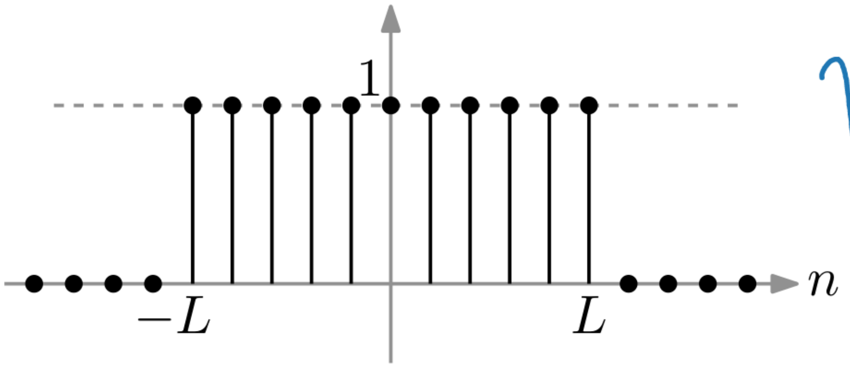
$h[n] = \sin(2\pi f_0 n)$



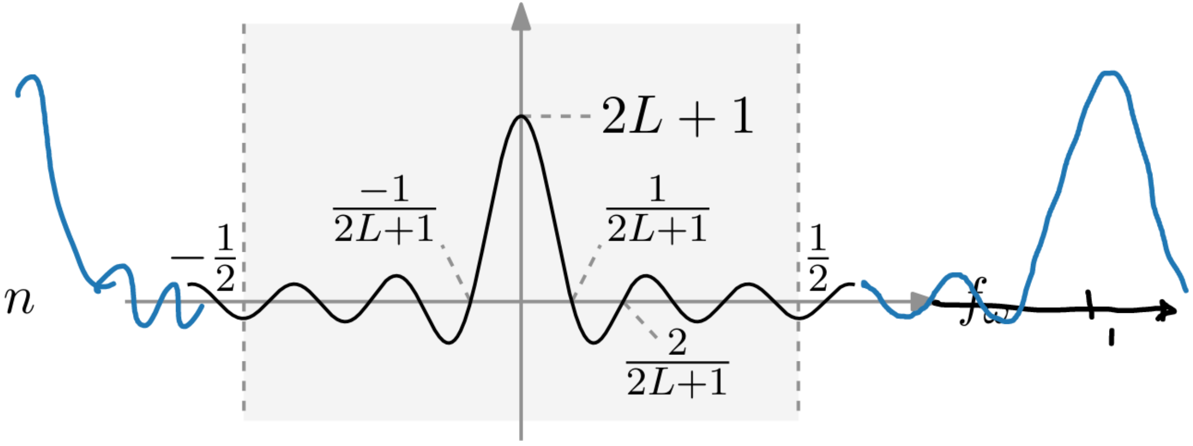
$H(f_\omega)$ Imaginary



$h[n]$



$H(f_\omega) = \frac{\sin(\pi(2L+1)f_\omega)}{\sin(\pi f_\omega)}$ Real

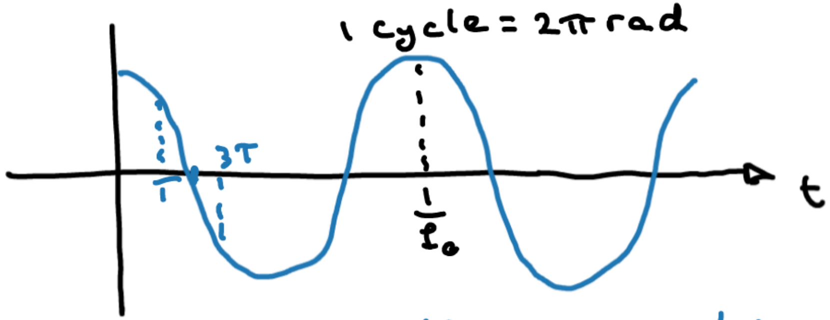


Frequency of continuous vs discrete time by looking at exponentials

$\omega_0 = 2\pi f_{\omega_0}$

Continuous

$$x(t) = e^{j2\pi f_0 t} = e^{j\Omega_0 t}$$



f_0 : continuous cycles/sec

Ω_0 : rad/sec

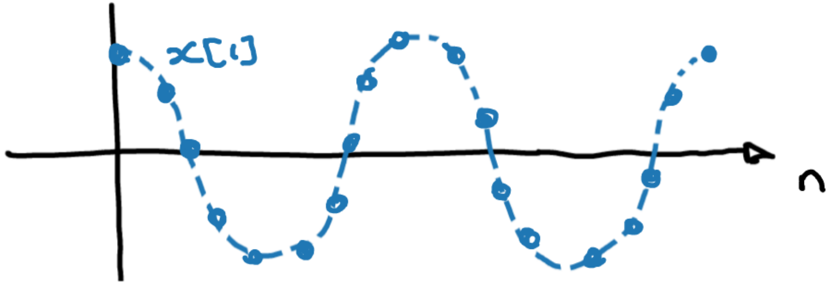
$$x[n] = x(nT)$$

$$e^{j2\pi f_{\omega_0} n} = e^{j2\pi f_0 nT}$$

$$\cancel{2}\cancel{\pi} f_{\omega_0} \cancel{n} = \cancel{2}\cancel{\pi} f_0 \cancel{n} T$$

Discrete

$$x[n] = e^{j2\pi f_{\omega_0} n} = e^{j2\pi \omega_0 n}$$



f_{ω_0} : continuous cycles/sample

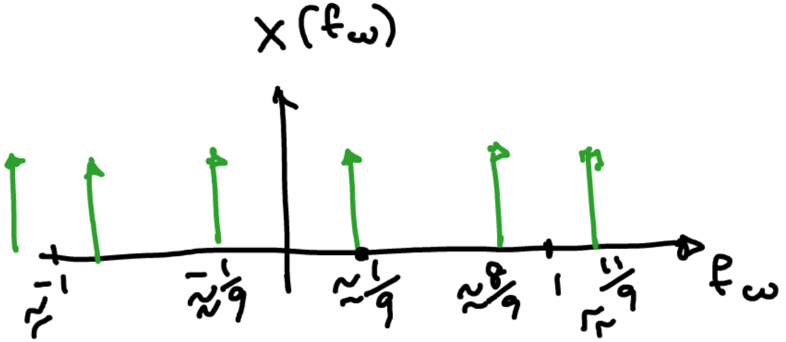
ω_0 : rad/sample

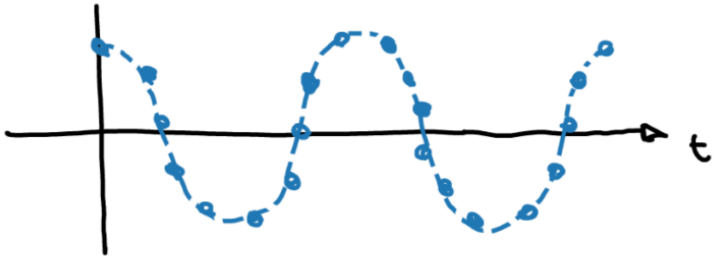
$$f_{\omega_0} = f_0 T = \frac{f_0}{f_s}$$

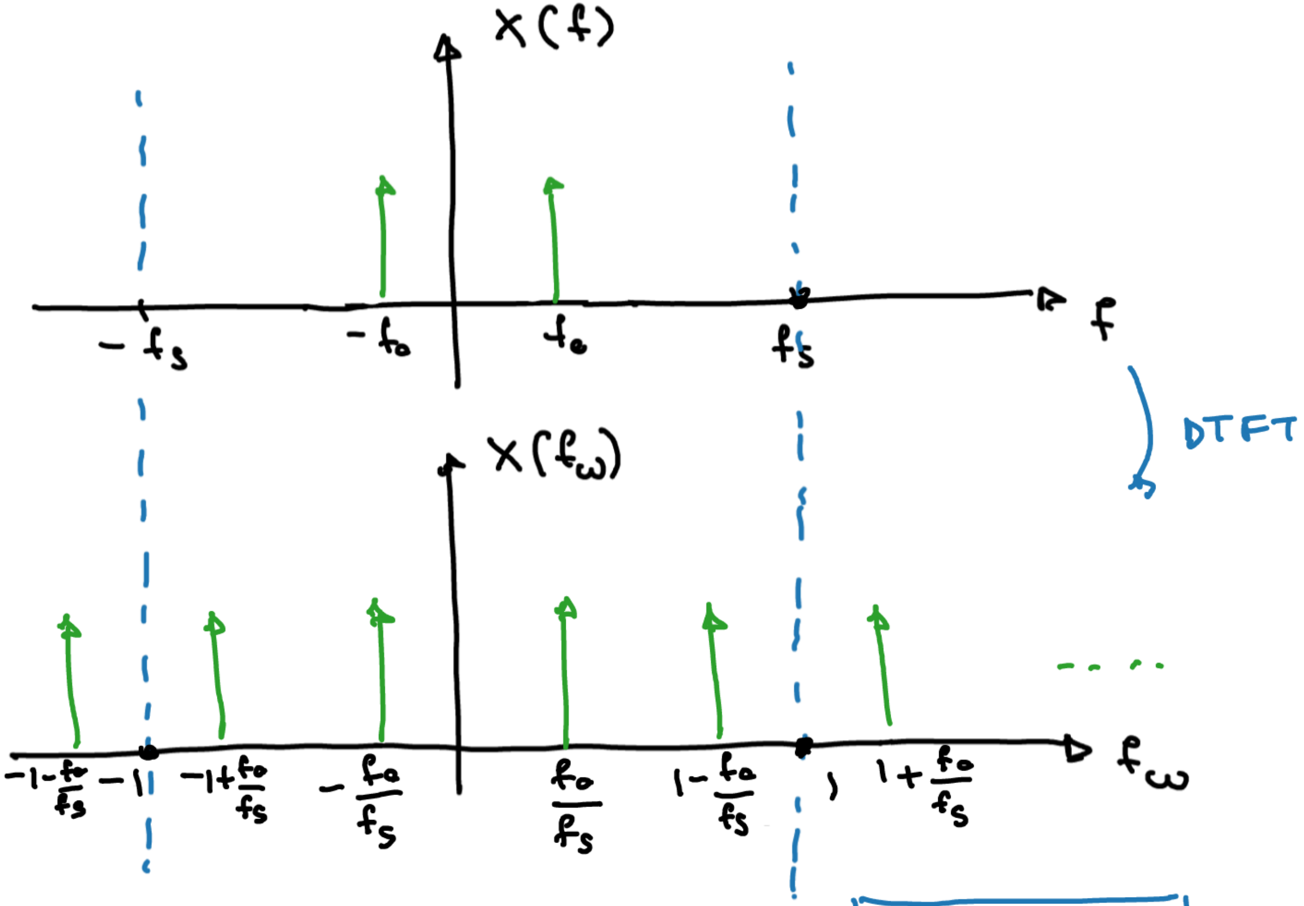
cycles/sec

samples/sec

cycles/sample







$$\boxed{f_\omega = \frac{f}{f_s}}$$

Periodicity of sampled exponentials

Discrete-time signal $x[n]$ periodic with N if: $x[n] = x[n+N]$ for all N

$$A e^{j(2\pi f_{\omega_0} n + \theta)} = A e^{j(2\pi f_{\omega_0} n + \underline{2\pi f_{\omega_0} N} + \theta)}$$

$$\Rightarrow \cancel{2\pi} f_{\omega_0} N = \cancel{2\pi} k$$

continuous cycles

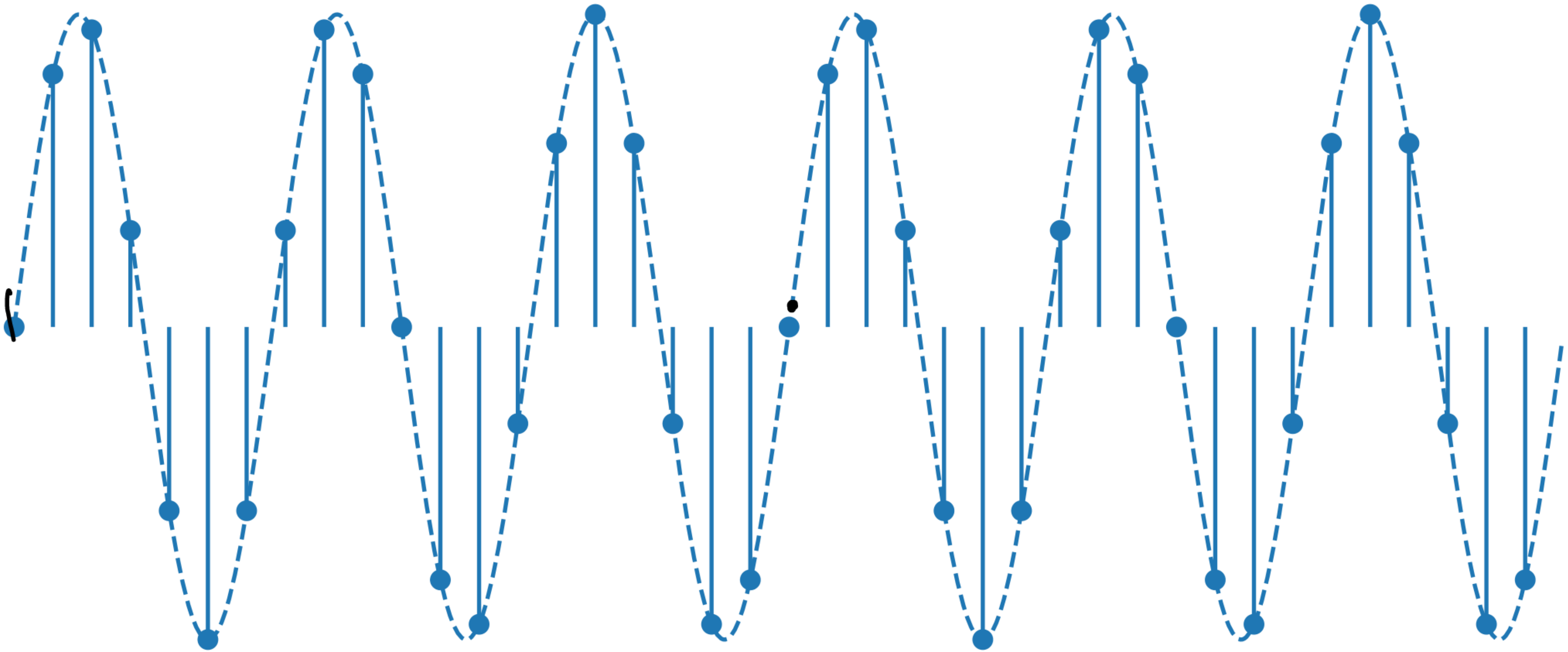
$$f_{\omega_0} = \frac{k}{N} = \frac{f_0}{f_s}$$

samples

~~300 Hz signal sampled at 2000 Hz:~~

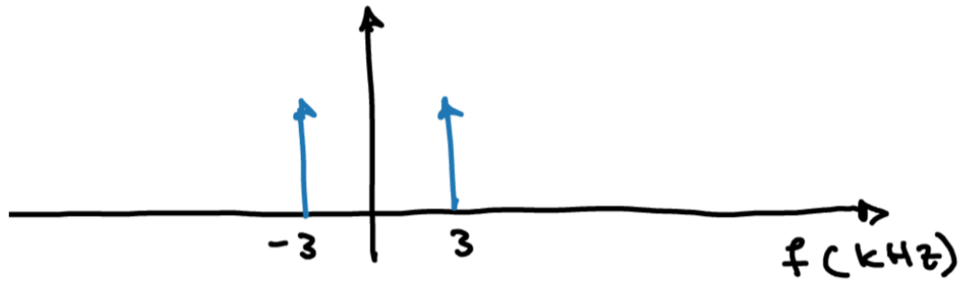
$$x[n] = \sin\left(2\pi \frac{3}{20} n\right)$$

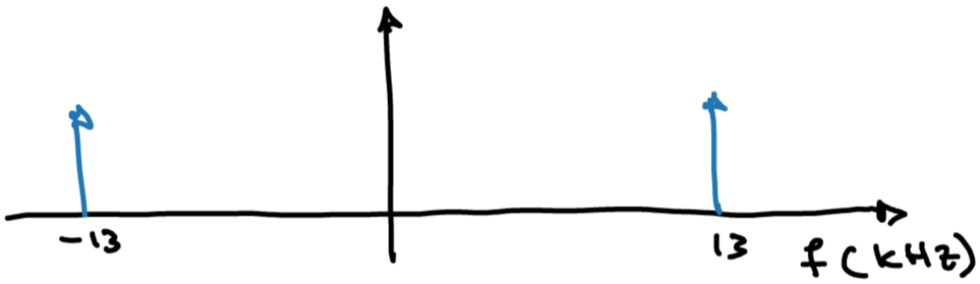
$$f_{wo} = \frac{f_o}{f_s} = \frac{3\cancel{0}\cancel{0}}{20\cancel{0}\cancel{0}} = \frac{3}{20}$$



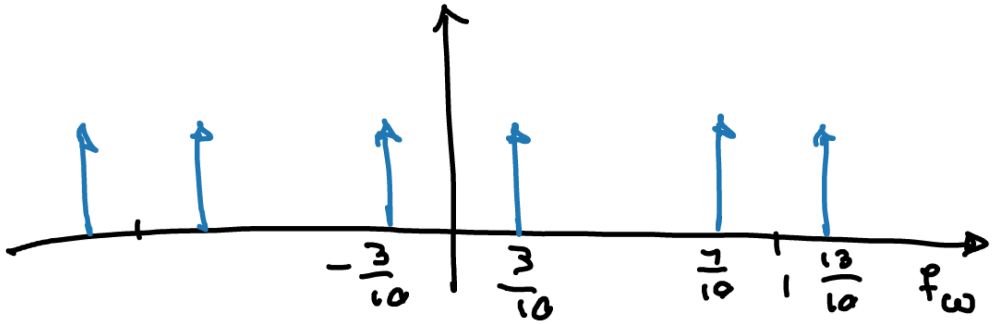
$$x[n] = \sin\left(2\pi \frac{3}{20} n\right)$$

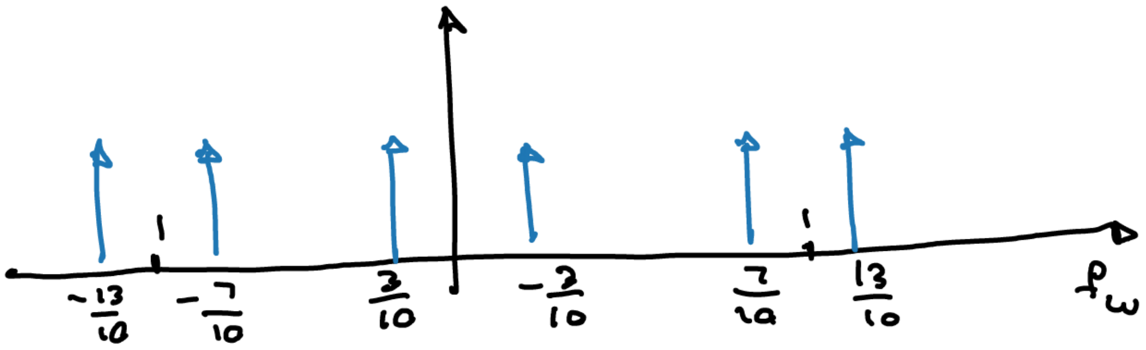
Aliasing of sinusoidal signals

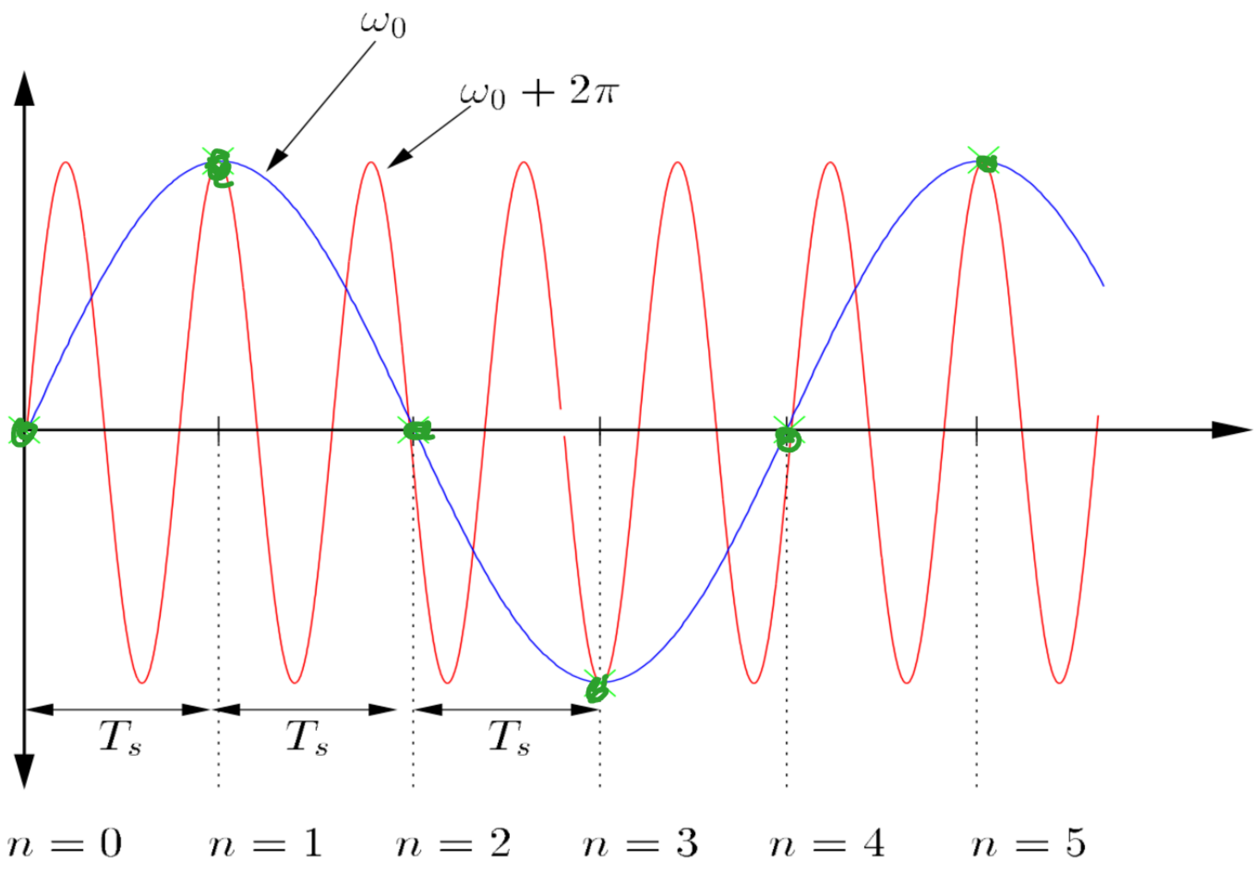




Sample at $f_s = 10$ kHz:






ω_0
$\omega_0 + 2\pi$
T_s
T_s
T_s
$n = 0$
$n = 1$
$n = 2$
$n = 3$
$n = 4$
$n = 5$