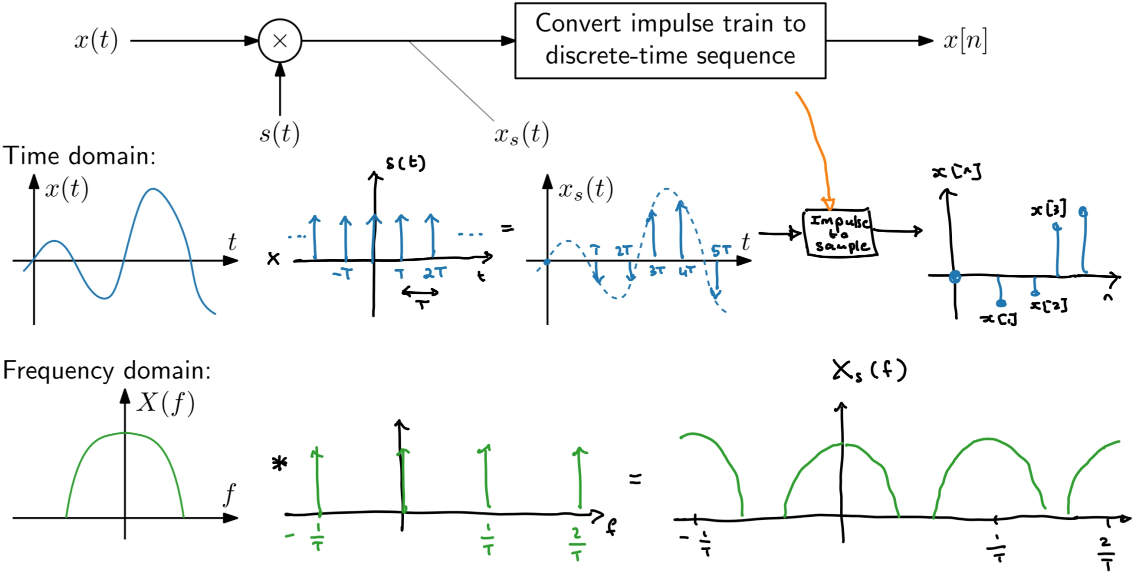
Discrete-time Fourier transform (DTFT)

And a case study on sampled sinusoids

Herman Kamper

Fourier transform → Discrete-time Fourier transform (DTFT) → Discrete Fourier transform (DFT)

Mathematical model of sampling



Discrete-time Fourier transform (DTFT)

$$x_s(t) = x(t)\cdot s(t) = x(t)\cdot \sum_{n=-\infty}^{\infty} \delta(t-nT) = \sum_{n=-\infty}^{\infty} x(nT)\cdot\delta(t-nT)$$

$$X_s(f) = \mathcal{F}\{x_s(t)\} = \int_{-\infty}^{\infty} x_s(t)\cdot e^{-j2\pi ft}\cdot dt$$

$$= \int_{-\infty}^{\infty}\left[\sum_{n=-\infty}^{\infty} x(nT)\cdot\delta(t-nT)\right]\cdot e^{-j2\pi ft}\cdot dt$$

$$= \sum_{n=-\infty}^{\infty} x(nT)\int_{-\infty}^{\infty}\delta(t-nT)\cdot e^{-j2\pi ft}\cdot dt$$

$$= \sum_{n=-\infty}^{\infty} x(nT)\cdot e^{-j2\pi fnT} = \sum_{n=-\infty}^{\infty} x[n]\cdot e^{-j2\pi fnT}$$

Define:

$$f_\omega \equiv f\underbrace{T}_{[\text{sec/sample}]} = \frac{\overbrace{f}^{[\text{cycles/sec}]}}{\underbrace{f_s}_{[\text{samples/sec}]}}$$

f_ω : [cycles/sample]

DTFT:

$$\boxed{\begin{aligned} X(f_\omega) &= \sum_{n=-\infty}^{\infty} x[n]\cdot e^{-j2\pi f_\omega n} \\ X(\omega) &= \sum_{n=-\infty}^{\infty} x[n]\cdot e^{-j\omega n} \quad \text{with } \underbrace{\omega}_{[\text{rad/sample}]} = 2\pi f_\omega \end{aligned}}$$

Periodicity:

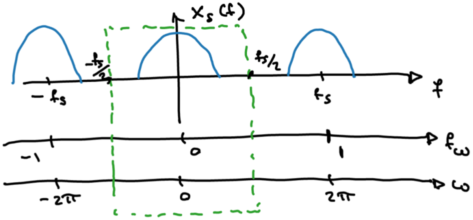
$$X(f_\omega) = X(f_\omega + \underbrace{k}_{\text{integer}})$$

$$X(\omega) = X(\omega + 2\pi k)$$

Inverse discrete-time Fourier transform (IDTFT)

Define $\hat{X}_s(f)$ to correspond to a single period of $X_s(f)$:

$$\hat{X}_s(f) = \begin{cases} X_s(f) & \text{for } -\frac{f_s}{2} \leq f \leq \frac{f_s}{2} \\ 0 & \text{otherwise} \end{cases}$$



The original $X_s(f)$ can be recovered by convolving $\hat{X}_s(f)$ with an impulse train:

$$X_s(f) = \hat{X}_s(f) * \sum_{k=-\infty}^{\infty} \delta(f - kf_s)$$

Therefore we can write the inverse Fourier transform of $X_s(f)$ as

$$\mathcal{F}^{-1}\{X_s(f)\} = \mathcal{F}^{-1}\{\hat{X}_s(f)\} \cdot \mathcal{F}^{-1}\left\{\sum_{k=-\infty}^{\infty} \delta(f - kf_s)\right\}$$

The first term in the last equation is

$$\mathcal{F}^{-1}\{\hat{X}_s(f)\} = \int_{-\infty}^{\infty} \hat{X}_s(f)\, e^{j2\pi f t}\, \mathrm{d}f = \int_{-f_s/2}^{f_s/2} X_s(f)\, e^{j2\pi f t}\, \mathrm{d}f$$

and the second term is

$$\mathcal{F}^{-1}\left\{\sum_{k=-\infty}^{\infty} \delta(f - k f_s)\right\} = \frac{1}{f_s} \sum_{n=-\infty}^{\infty} \delta\left(t - \frac{n}{f_s}\right)$$

Combining these, we obtain:

$$x_s(t) = \mathcal{F}^{-1}\{X_s(f)\} = \left[\int_{-f_s/2}^{f_s/2} X_s(f)\, e^{j2\pi f t}\, \mathrm{d}f\right] \cdot \left[\frac{1}{f_s} \sum_{n=-\infty}^{\infty} \delta\left(t - \frac{n}{f_s}\right)\right]$$

This equation describes a continuous function sampled by multiplication with an impulse train with an impulse every $1/f_s = T$ seconds. The sequence $x[n]$ is the strength of these impulses:

$$x[n] = x(nT) = \frac{1}{f_s} \int_{-f_s/2}^{f_s/2} X_s(f)\, e^{j2\pi f nT}\, \mathrm{d}f$$

We therefore have:

$$x[n] = x(nT) = \frac{1}{f_s} \int_{-f_s/2}^{f_s/2} X_s(f)\, e^{j2\pi f n T}\, \mathrm{d}f$$

Converting the integration over variable f so that we instead integrate over $f_\omega = fT$, we obtain the inverse DTFT:

$$\boxed{\begin{aligned} x[n] &= \int_{-1/2}^{1/2} X(f_\omega) e^{j2\pi f_\omega n}\, \mathrm{d}f_\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n}\, \mathrm{d}\omega \end{aligned}}$$

$f_\omega = fT = \frac{f}{f_s}$

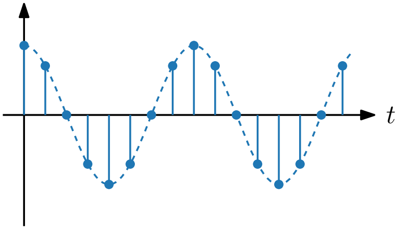
$\frac{df_\omega}{df} = \frac{1}{f_s}$

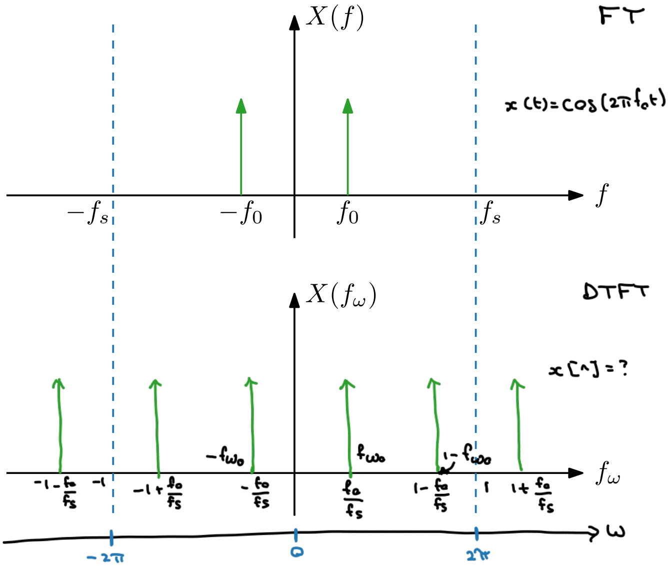
$\therefore df_\omega = \frac{1}{f_s} df$

f	f_ω
$-f_s/2$	$-1/2$
$f_s/2$	$1/2$

The second version is just expressed in terms of discrete angular frequency $\omega = 2\pi f_\omega$.

Case study: DTFT of a sinusoid



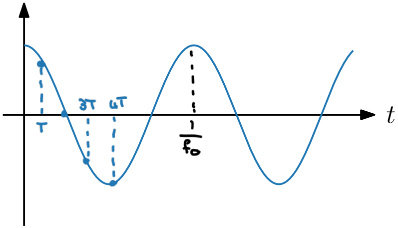


$$f_\omega = \frac{f}{f_s}$$

$$\omega = 2\pi f_\omega$$

Continuous vs discrete-time frequency

$x(t) = \cos(2\pi f_0 t)$



$$x(t) = \cos(2\pi f_0 t)$$

$$x(nT) = \cos(2\pi f_0 nT)$$

$$= \cos\left(2\pi \frac{f_0}{f_s} n\right)$$

$$\therefore \quad x[n] = \cos\left(2\pi \frac{f_0}{f_s} n\right)$$

$$f_{\omega_0} = \frac{f_0}{f_s}$$

$x[n] = \cos(2\pi f_{\omega_0} n)$

Continuous

f_0 : continuous cycles/sec

Ω_0 : rad/sec

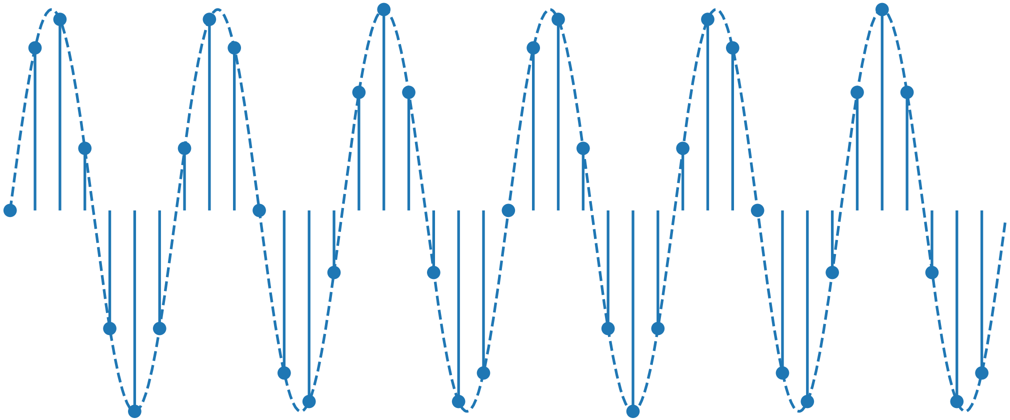
Discrete

f_{ω_0} : continuous cycles/sample

ω_0 : rad/sample

300 Hz signal sampled at 2000 Hz:

$$f_{\omega_0} = \frac{f_0}{f_s} = \frac{300}{2000} = \frac{3}{20} \quad \therefore \; x[n] = \sin\left(2\pi \cdot \frac{3}{20} n\right)$$



But I could have also read it directly from the plot:

3 continuous cycles in 20 samples $\therefore \; f_{\omega_0} = \frac{3}{20}$ cycles/sample

Periodicity of sampled exponentials

Discrete-time signal $x[n]$ periodic with N if: $x[n] = x[n+N]$ for all $\cancel{N}$ n

$$A e^{j(2\pi f_{\omega_0} n + \theta)} \stackrel{?}{=} A e^{j(2\pi f_{\omega_0}(n+N) + \theta)}$$

$$= A e^{j(2\pi f_{\omega_0} n + 2\pi f_{\omega_0} N + \theta)}$$

For RHS to equal LHS :

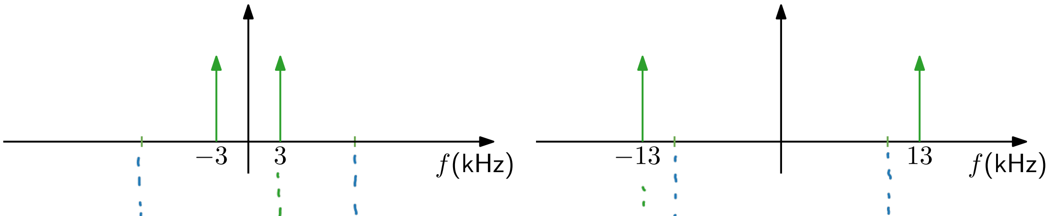
$$2\pi f_{\omega_0} N = 2\pi k$$

integer

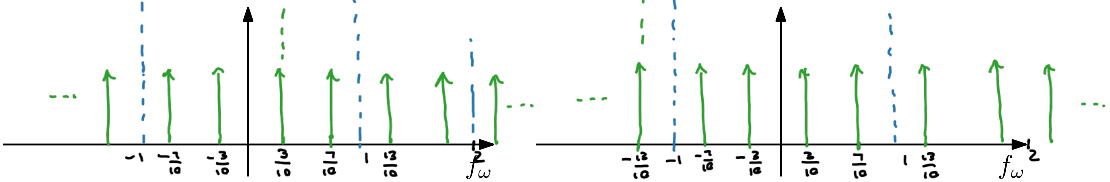
$$\therefore f_{\omega_0} = \frac{k}{N} = \frac{f_0}{f_s}$$

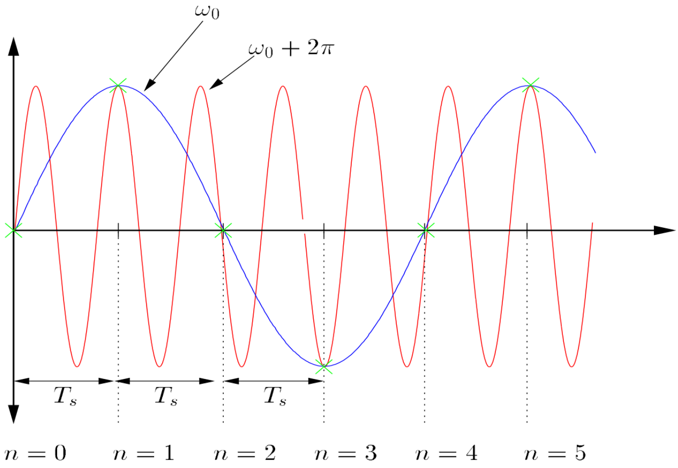
Aliasing of sinusoidal signals

Two sinusoidal signals:



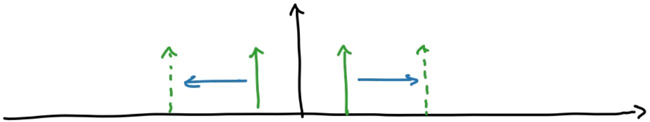
Sample both at $f_s = 10$ kHz:



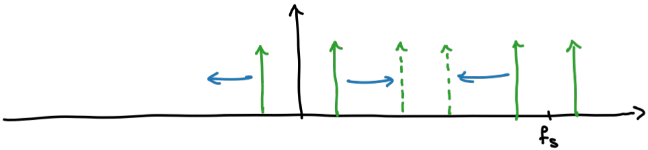


ω_0
$\omega_0 + 2\pi$
T_s
T_s
T_s
$n = 0$
$n = 1$
$n = 2$
$n = 3$
$n = 4$
$n = 5$

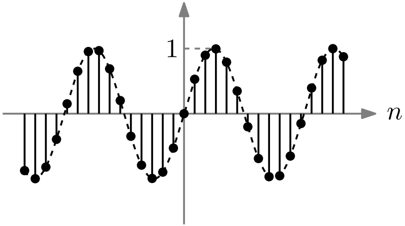
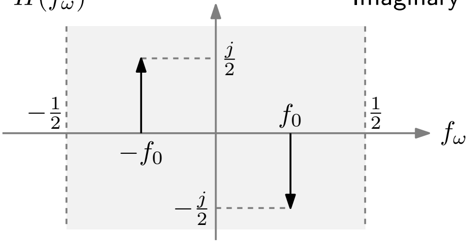
Sinusoid:

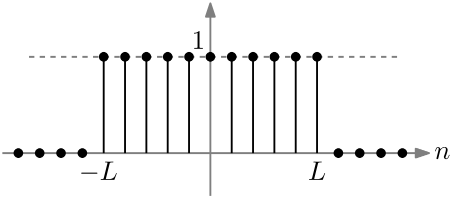
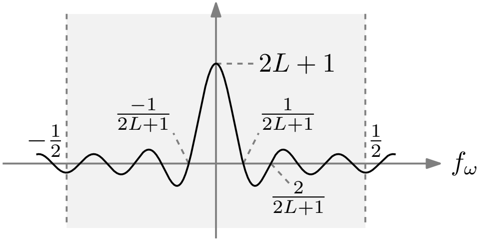
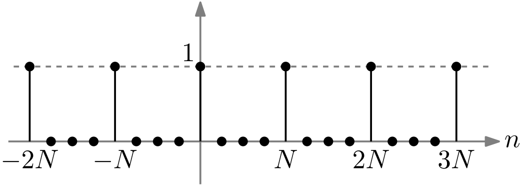
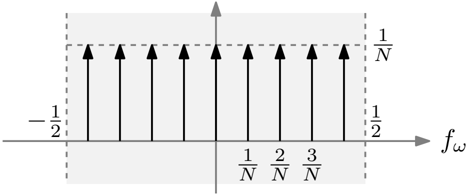


Sampled sinusoid:



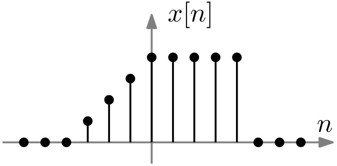
DTFT pairs

Discrete-time domain	Frequency domain
$h[n] = \sin(2\pi f_0 n)$ 	$H(f_\omega)$ Imaginary 

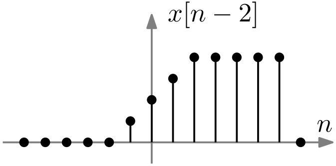
Discrete-time domain	Frequency domain
$h[n]$ 	$H(f_\omega) = \frac{\sin(\pi(2L+1)f_\omega)}{\sin(\pi f_\omega)}$ Real 
$h[n] = \sum_{k=-\infty}^{\infty} \delta[n-kN]$ 	$H(f_\omega) = \frac{1}{N} \sum_{k=-\infty}^{\infty} \delta\left(f_\omega - \frac{k}{N}\right)$ 

Operations on discrete-time signals

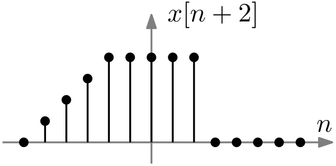
Time shift:



Unshifted signal

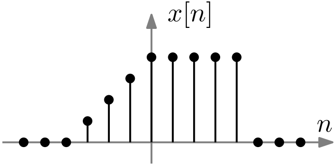


Delayed by 2 samples

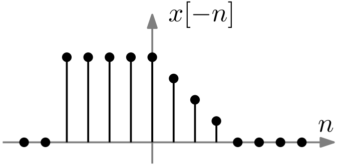


Advanced by 2 samples

Reflection:

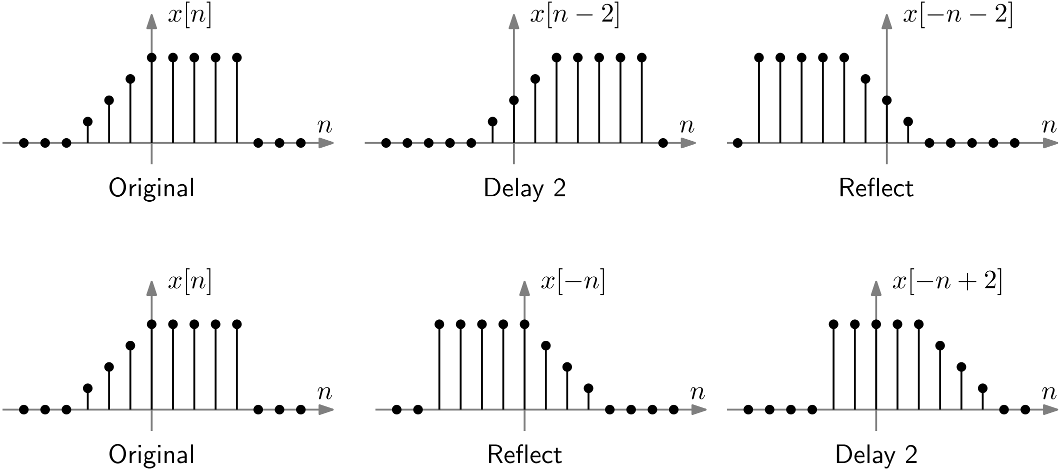


Unreflected signal



Reflected about $n = 0$

Time-shifting and reflection are not commutative:



Properties of the DTFT

- Linearity:

$$\mathcal{F}\{\alpha x_1[n] + \beta x_2[n]\} = \alpha X_1(\omega) + \beta X_2(\omega)$$

- Time shift:

$$\mathcal{F}\{x[n-k]\} = e^{-j\omega k} X(\omega)$$

- Time reversal and frequency reversal:

$$\mathcal{F}\{x[-n]\} = X(-\omega)$$

- Convolution:

$$\mathcal{F}\{x_1[n] * x_2[n]\} = X_1(\omega) \cdot X_2(\omega)$$

- Windowing:

$$\mathcal{F}\{x_1[n] \cdot x_2[n]\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\lambda) \cdot X_2(\omega - \lambda) \, \mathrm{d}\lambda$$