Sampling rate conversion

Upsampling and downsampling

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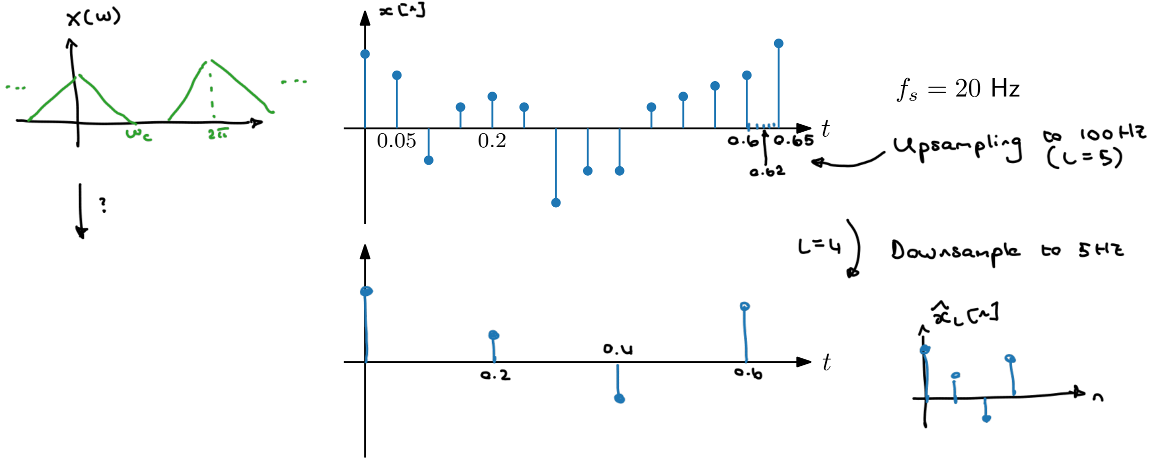
Upsampling and downsampling intuition



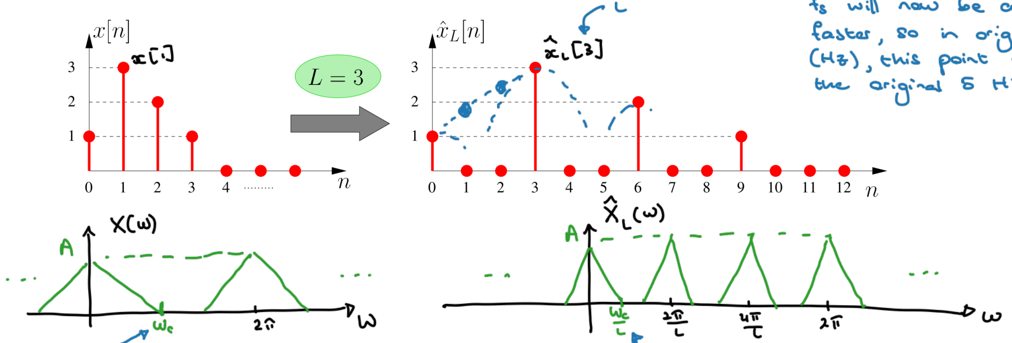
We only consider up- and downsampling by integer factors (principles can be extended to non-integer case)

Upsampling

Insert $L-1$ zeros between each sample of $x[n]$:

$$\hat{x}_L[n] = \begin{cases} x[n/L] & \text{when } n = kL \\ 0 & \text{otherwise} \end{cases}$$

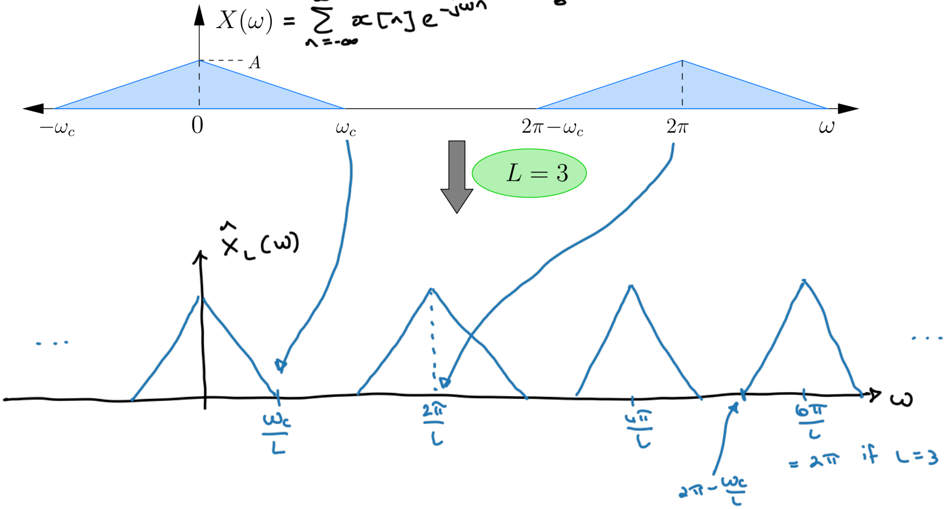
This now moved, but remember, f_s will now be a factor L faster, so in original frequency (Hz), this point will stay at the original 5 Hz.



E.g. 5 Hz when we sampled at $f_s = 20$ Hz

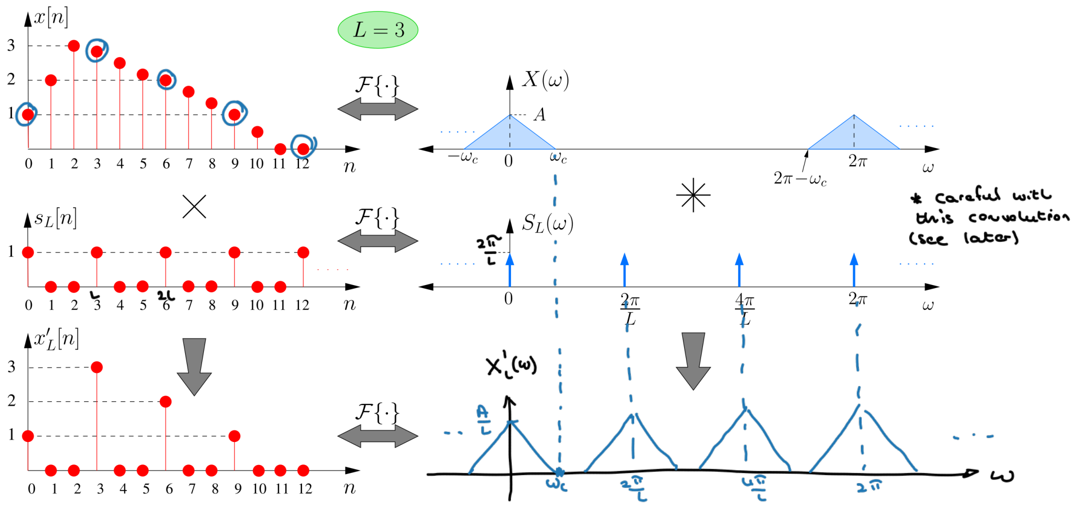
DTFT: $\hat{X}_L(\omega) = \sum_{n=-\infty}^{\infty} \hat{x}_L[n] e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n L}$ $\therefore \hat{X}_L(\omega) = X(\omega L)$

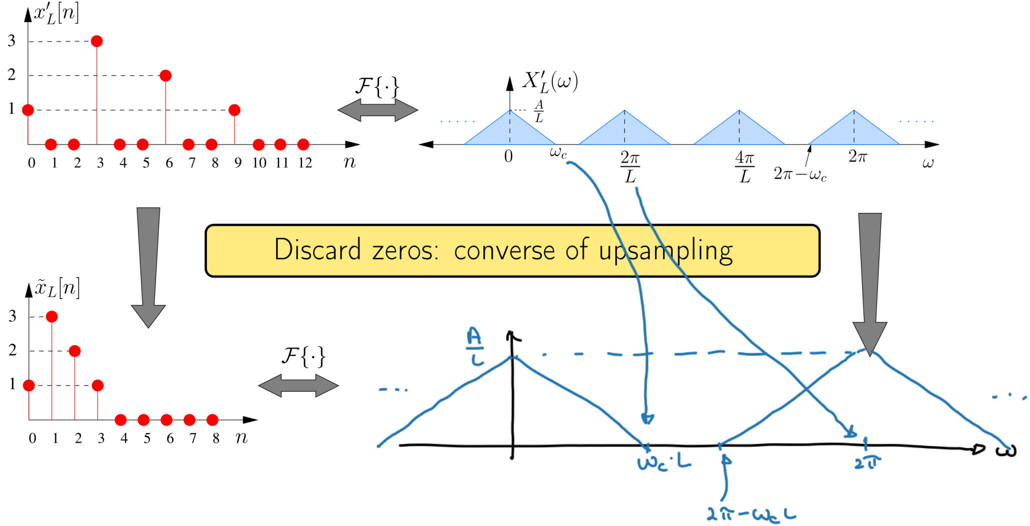
$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$



Downsampling

Keep each Lth sample: $\tilde{x}_L[n] = x[nL]$





$x'_L[n]$
$\mathcal{F}\{\cdot\}$
$X'_L(\omega)$
$\frac{A}{L}$
ω_c
$\frac{2\pi}{L}$
$\frac{4\pi}{L}$
$2\pi - \omega_c$
2π
ω
Discard zeros: converse of upsampling
$\tilde{x}_L[n]$
$\mathcal{F}\{\cdot\}$
n
$\frac{A}{L}$
$\omega_c \cdot L$
$2\pi - \omega_c L$
2π
ω

Being more precise about $X'_L(\omega)$

Why do we not get infinite sums of the periodic components of $X(\omega)$ after convolution?

Because, remember, convolution of sampled signals is different from continuous signals:

$$\mathcal{F}\{x_1[n] \cdot x_2[n]\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\lambda) \cdot X_2(\omega - \lambda) d\lambda$$

I.e. we only integrate over one period, not from $-\infty$ to ∞.

You can get the derived result from the impulse train DTFT: $S_L(\omega) = \frac{2\pi}{L\,\cancel{N}} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{L}\right)$

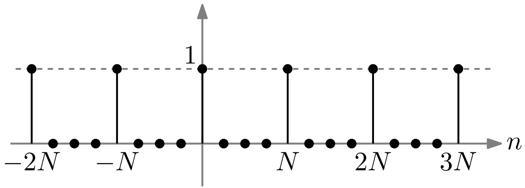
and then show that: $\frac{1}{2\pi} \int_{-\pi}^{\pi} X(\lambda) \cdot S_L(\omega - \lambda) d\lambda = \frac{1}{L} \sum_{k=-\infty}^{\infty} X\left(\omega - \frac{2\pi k}{L}\right)$

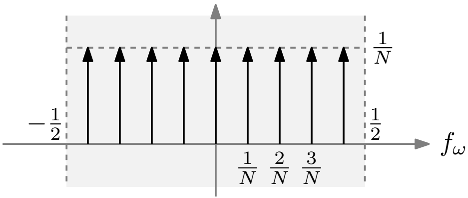
which is exactly what is visually derived.

The DTFT of the impulse train is a little strange (see next slide).

We made use of this DTFT pair:

$$h[n] = \sum_{k=-\infty}^{\infty} \delta[n - kN] \qquad\qquad H(f_\omega) = \frac{1}{N} \sum_{k=-\infty}^{\infty} \delta\left(f_\omega - \frac{k}{N}\right)$$





But note that, if we switch to angular frequency, the right-hand-side becomes:

$$H(\omega) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$$

This scaling by 2π when switching from f_ω to ω is not generally what happens! It is a peculiar result that comes from properties of the impulse. (Ask ChatGPT, but I won't examine this.) So there is actually a mistake on the slides where the amplitude of $1/L$ is used with the ω-axis.