

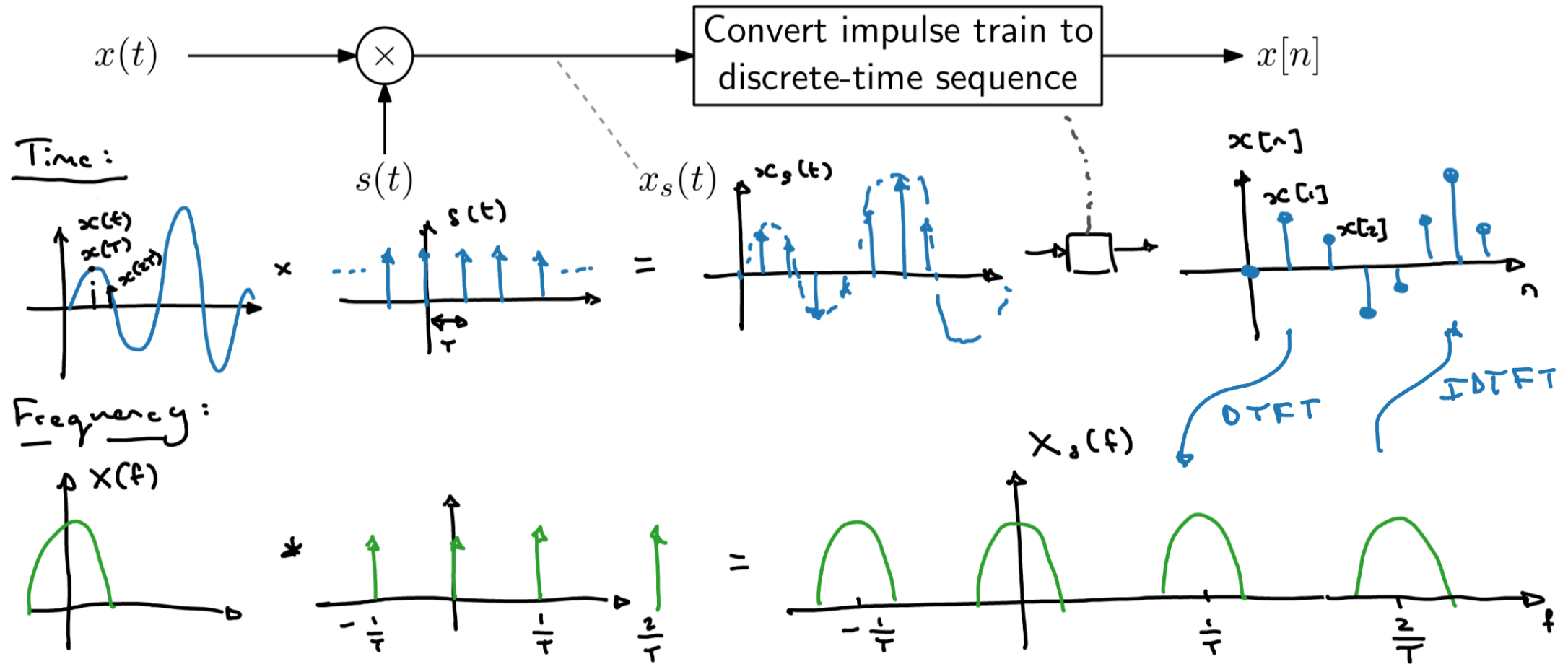
# Discrete-time Fourier transform (DTFT)

And how it leads to aliasing and affects periodicity

Herman Kamper

$FT \rightarrow DTFT \rightarrow DFT$

# Mathematical model of sampling



# Discrete-time Fourier transform (DTFT)

$$x_s(t) = x(t) \cdot s(t) = x(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT) = \sum_{n=-\infty}^{\infty} x(nT) \cdot \delta(t - nT)$$

$$X_s(f) = \mathcal{F}\{x_s(t)\} = \int_{-\infty}^{\infty} x_s(t) \cdot e^{-j2\pi ft} \cdot dt$$

$$= \int_{-\infty}^{\infty} \left[ \sum_{n=-\infty}^{\infty} x(nT) \cdot \delta(t - nT) \right] \cdot e^{-j2\pi ft} \cdot dt$$

$$= \sum_{n=-\infty}^{\infty} x(nT) \int_{-\infty}^{\infty} \delta(t - nT) \cdot e^{-j2\pi ft} \cdot dt$$

$$= \sum_{n=-\infty}^{\infty} \underbrace{x(nT)}_{x[n]} \cdot e^{-j2\pi fnT}$$

DTFT:

$$X(f_w) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j2\pi n f_w}$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{j\omega n}, \quad \omega = 2\pi f_w$$

radians/sample

Define:

$$f_w \equiv fT = \frac{f}{f_s}$$

[sec/sample] (pointing to  $T$ )  
 [cycles/sec] (pointing to  $f$ )  
 [samples/sec] (pointing to  $f_s$ )

$f_w$ : [cycles/sample]

Periodicity:

$$X(f_w) = X(f_w + k)$$

$$X(\omega) = X(\omega + 2\pi k)$$

# Inverse DTFT

$$\hat{X}_s(f) = \begin{cases} X_s(f) & \text{for } -\frac{f_s}{2} \leq f \leq \frac{f_s}{2} \\ 0 & \text{otherwise} \end{cases}$$

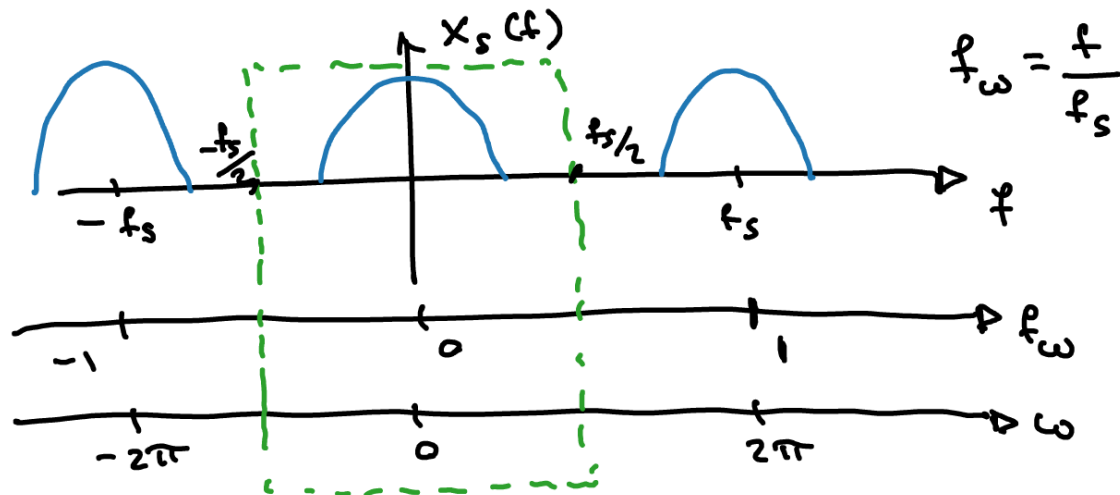
$$X_s(f) = \hat{X}_s(f) * \sum_{k=-\infty}^{\infty} \delta(f - kf_s)$$

$$\mathcal{F}^{-1}\{X_s(f)\} = \underbrace{\mathcal{F}^{-1}\{\hat{X}_s(f)\}} \cdot \underbrace{\mathcal{F}^{-1}\left\{\sum_{k=-\infty}^{\infty} \delta(f - kf_s)\right\}}$$

$$\mathcal{F}^{-1}\{\hat{X}_s(f)\} = \int_{-\infty}^{\infty} \hat{X}_s(f) \cdot e^{j2\pi ft} df = \int_{-f_s/2}^{f_s/2} X_s(f) \cdot e^{j2\pi ft} df$$

$$x_s(t) = \left[ \int_{-f_s/2}^{f_s/2} X_s(f) \cdot e^{j2\pi ft} df \right] \times \frac{1}{f_s} \left[ \sum_{n=-\infty}^{\infty} \delta\left(t - \frac{n}{f_s}\right) \right] \quad \text{sample period } T$$

$$x[n] = x(nT) = \frac{1}{f_s} \int_{-f_s/2}^{f_s/2} X_s(f) \cdot e^{j2\pi f nT} df$$



$$x[n] = \frac{1}{f_s} \int_{-f_s/2}^{f_s/2} X_0(f) \cdot e^{j2\pi f n T} \cdot df$$

IDTFT

$$x[n] = \int_{-1/2}^{1/2} X(f_\omega) e^{j2\pi n f_\omega} \cdot df_\omega$$

$$\omega = 2\pi f_\omega$$

$$f_\omega = fT = \frac{f}{f_s}$$

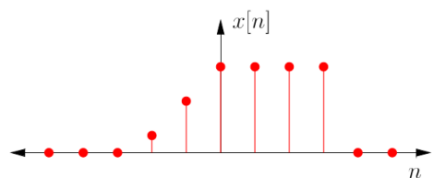
$$\frac{df_\omega}{df} = \frac{1}{f_s}$$

$$\therefore df_\omega = \frac{1}{f_s} df$$

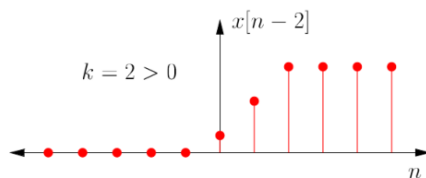
$f$	$f_\omega$
$-f_s/2$	$-1/2$
$f_s/2$	$1/2$

# Operations on discrete-time signals

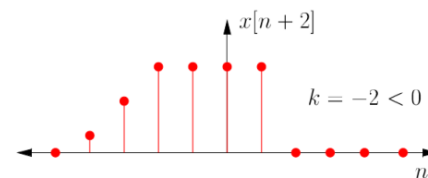
- **Time shift:**  $x[n - k]$  is a version of  $x[n]$  shifted by  $|k|$  samples to the right if  $k > 0$  or to the left if  $k < 0$



**Unshifted signal**

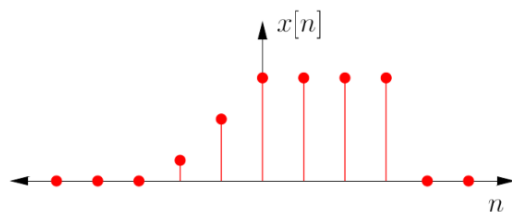


**Delayed by 2 samples**

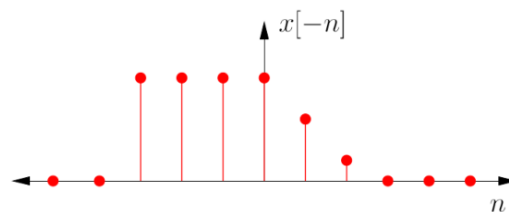


**Advanced by 2 samples**

- **Reflection about time origin**  $x[-n]$  is reflection of  $x[n]$  about  $n = 0$

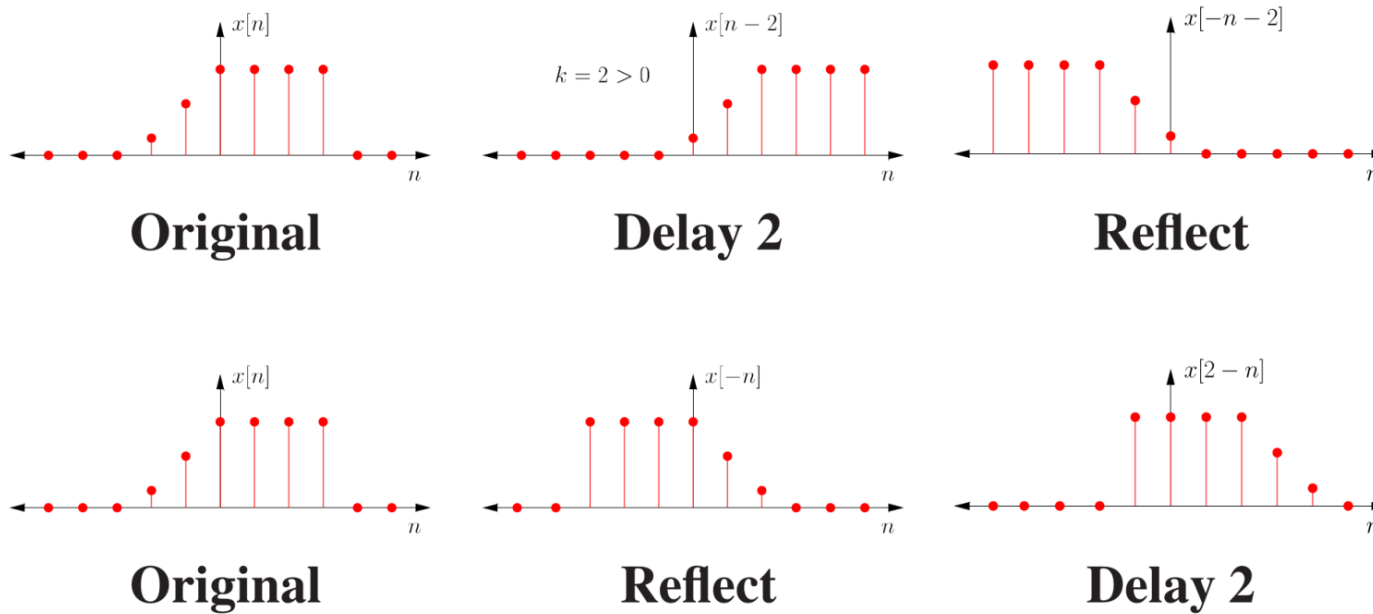


**Unreflected signal**



**Reflected about  $n = 0$**

- Time-shifting and reflection about  $n = 0$  are not commutative



# Properties of the DTFT

- Linearity:

$$\mathcal{F}\{\alpha x_1[n] + \beta x_2[n]\} = \alpha X_1(\omega) + \beta X_2(\omega)$$

- Time shift:

$$\mathcal{F}\{x[n - k]\} = e^{-j\omega k} X(\omega)$$

- Time reversal and frequency reversal:

$$\mathcal{F}\{x[-n]\} = X(-\omega)$$

- Convolution:

$$\mathcal{F}\{x_1[n] * x_2[n]\} = X_1(\omega) \cdot X_2(\omega)$$

- Windowing:

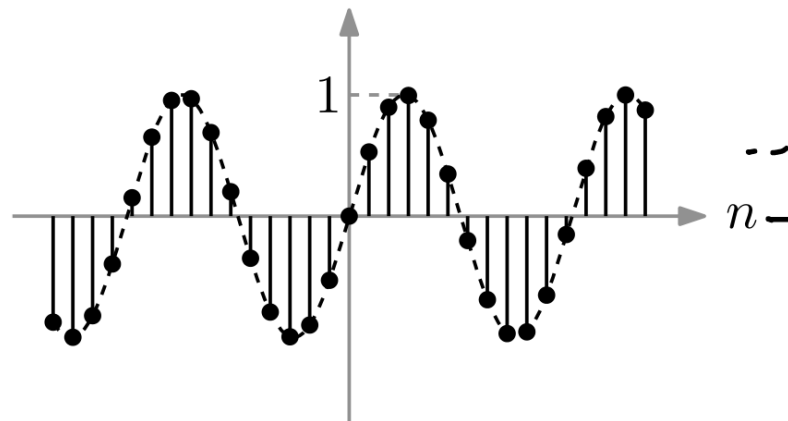
$$\mathcal{F}\{x_1[n] \cdot x_2[n]\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\lambda) \cdot X_2(\omega - \lambda) d\lambda$$



# Discrete-time domain

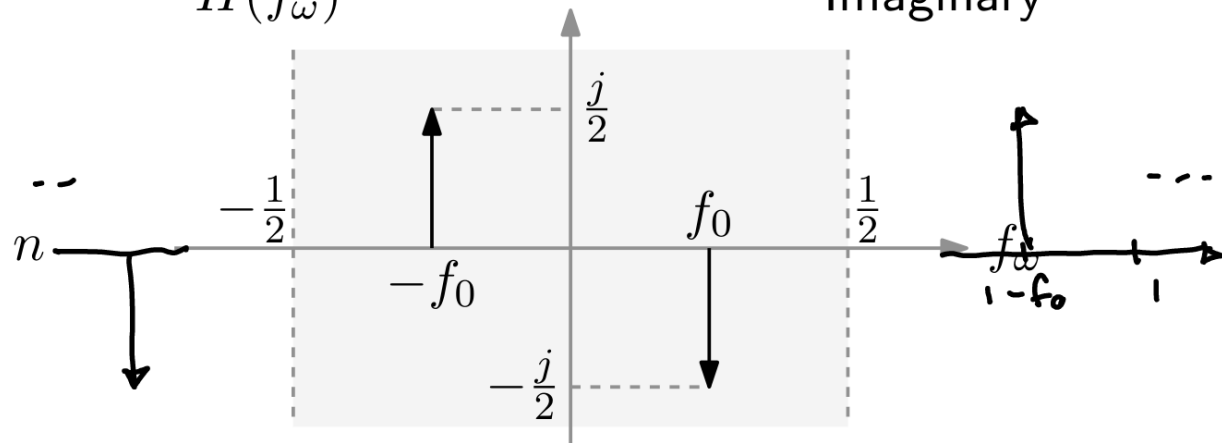
# Frequency domain

$$h[n] = \sin(2\pi f_0 n)$$

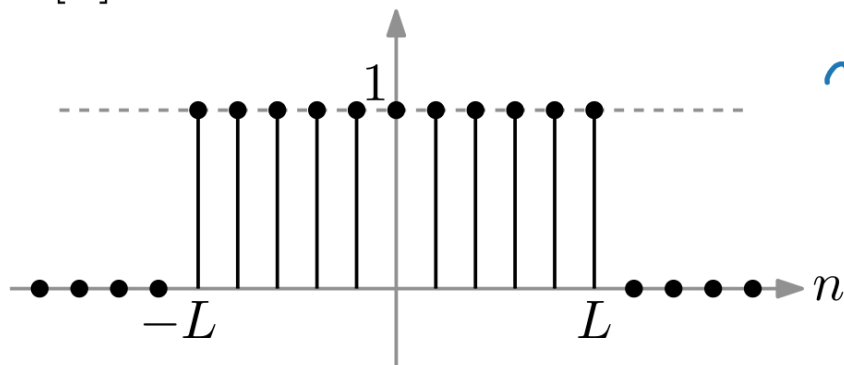


$$H(f_\omega)$$

Imaginary

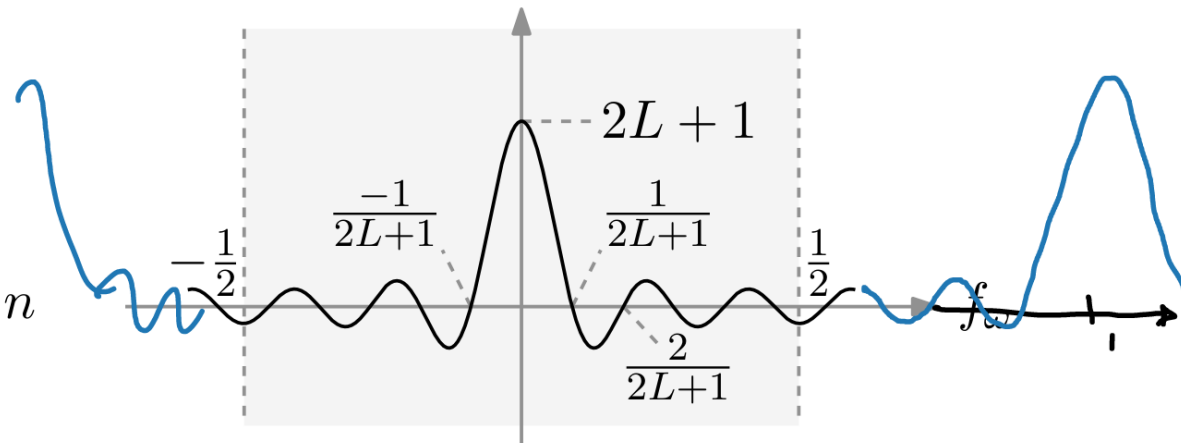


$$h[n]$$



$$H(f_\omega) = \frac{\sin(\pi(2L+1)f_\omega)}{\sin(\pi f_\omega)}$$

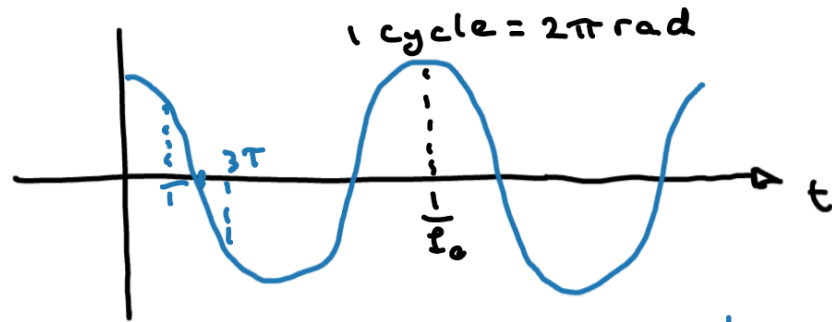
Real



# Frequency of continuous vs discrete time by looking at exponentials

Continuous

$$x(t) = e^{j2\pi f_0 t} = e^{j\Omega_0 t}$$



$f_0$  : continuous cycles/sec  
 $\Omega_0$  : rad/sec

$$x[n] = x(nT)$$

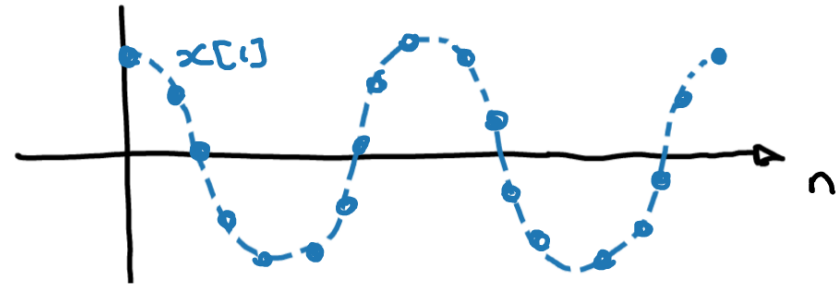
$$e^{j2\pi f_{\omega_0} n} = e^{j2\pi f_0 nT}$$

$$\cancel{2\pi} f_{\omega_0} n = \cancel{2\pi} f_0 nT$$

$$\omega_0 = 2\pi f_{\omega_0}$$

Discrete

$$x[n] = e^{j2\pi f_{\omega_0} n} = e^{j\omega_0 n}$$



$f_{\omega_0}$  : continuous cycles/sample  
 $\omega_0$  : rad/sample

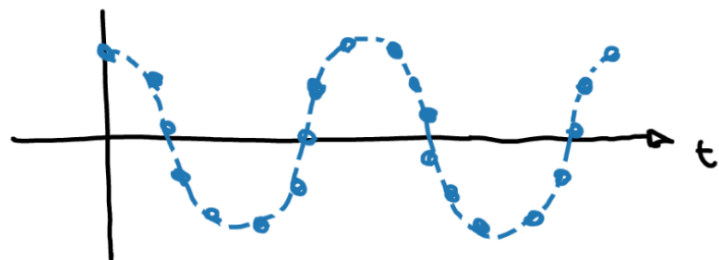
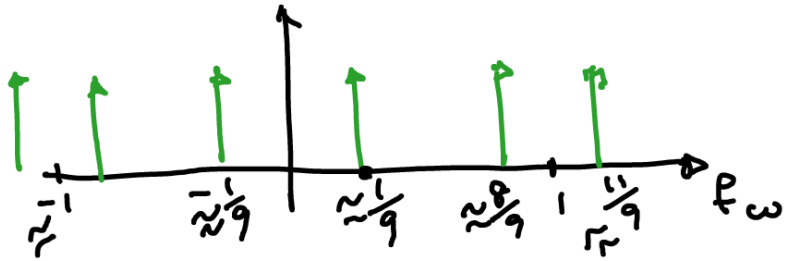
$$f_{\omega_0} = f_0 T = \frac{f_0}{f_s}$$

cycles/sec

samples/sec

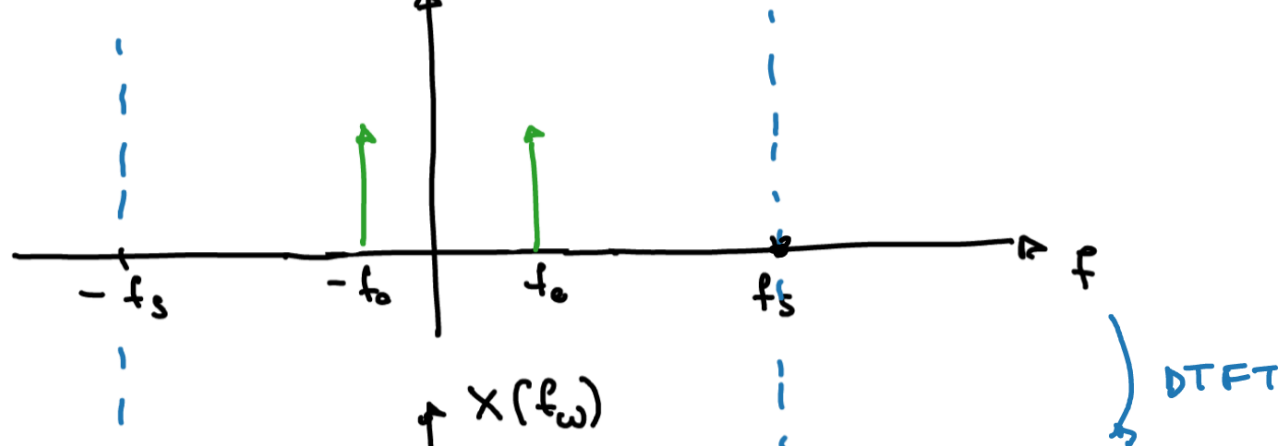
cycles/sample

$X(f\omega)$

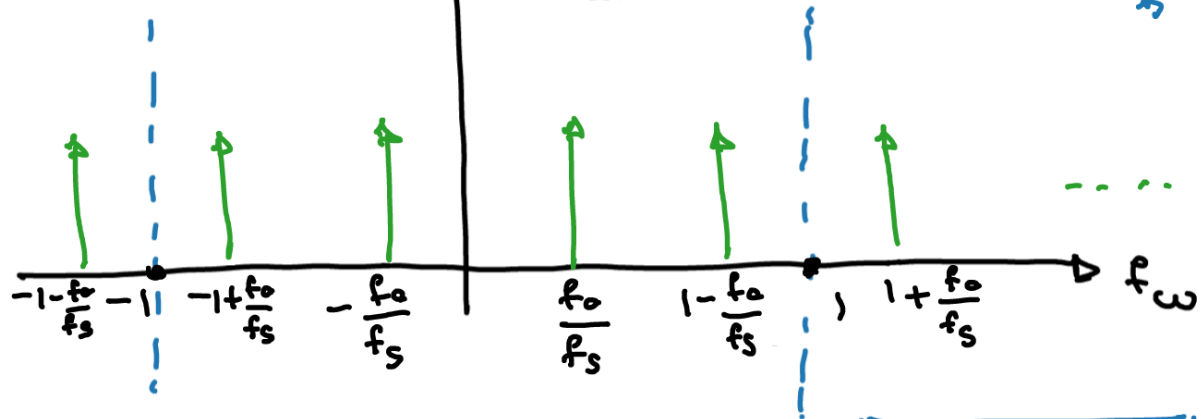


...

$X(f)$



$X(f\omega)$



$$f\omega = \frac{f}{f_s}$$

# Periodicity of sampled exponentials

Discrete-time signal  $x[n]$  periodic with  $N$  if:  $x[n] = x[n + N]$  for all  $N$

$$A e^{j(2\pi f_{\omega_0} n + \theta)} = A e^{j(2\pi f_{\omega_0} n + \underline{2\pi f_{\omega_0} N} + \theta)}$$

$$\Rightarrow \cancel{2\pi f_{\omega_0} N} = \cancel{2\pi} k$$
$$f_{\omega_0} = \frac{k}{N} = \frac{f_0}{f_s}$$

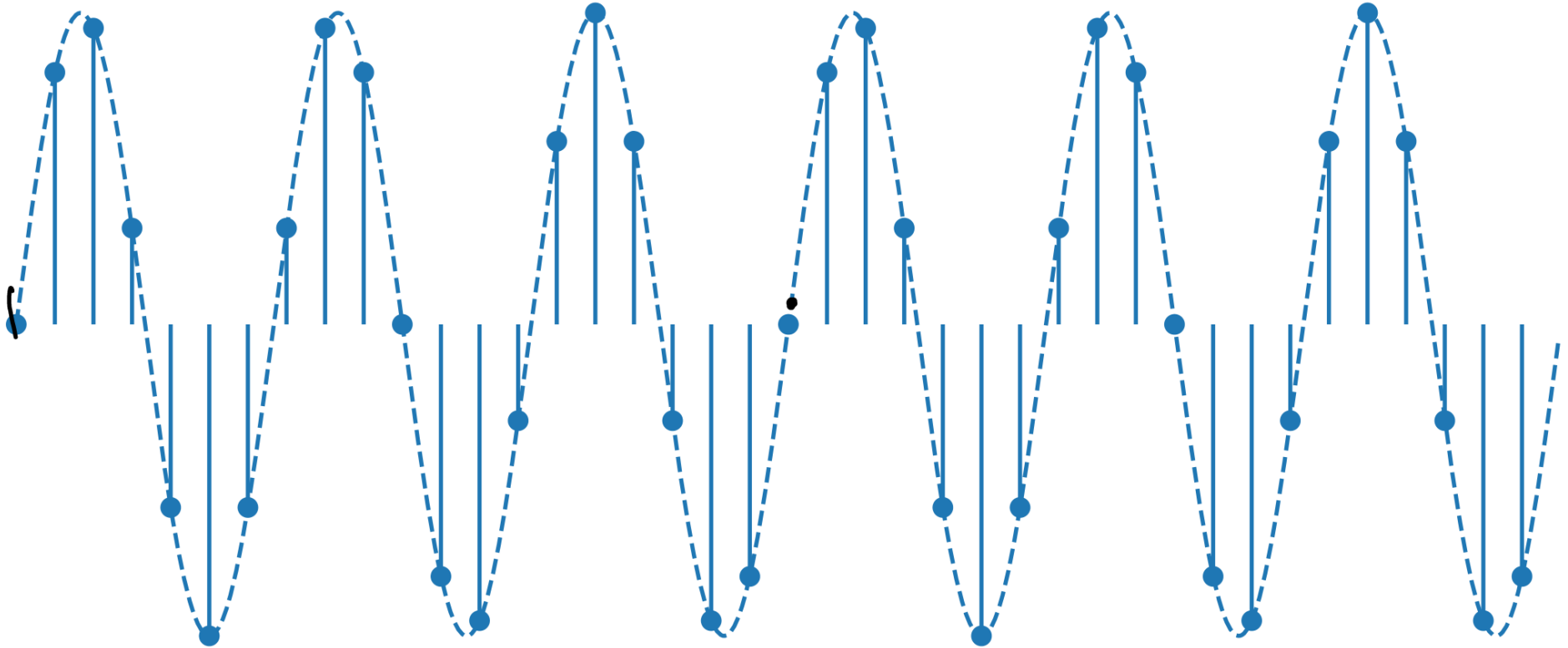
continuous cycles

samples

$$x[n] = \sin\left(2\pi \frac{3}{20} n\right)$$

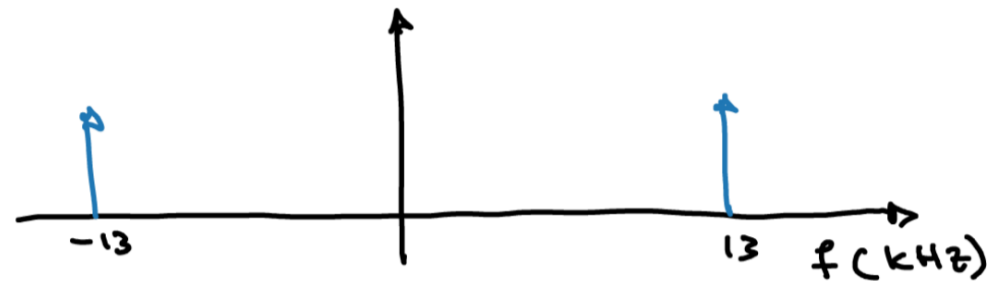
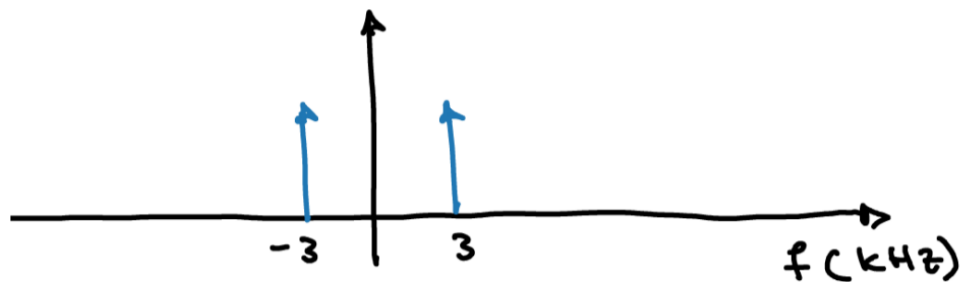
$$f_{\omega_0} = \frac{f_0}{f_s} = \frac{300}{2000} \\ = \frac{3}{20}$$

~~300 Hz signal sampled at 2000 Hz:~~

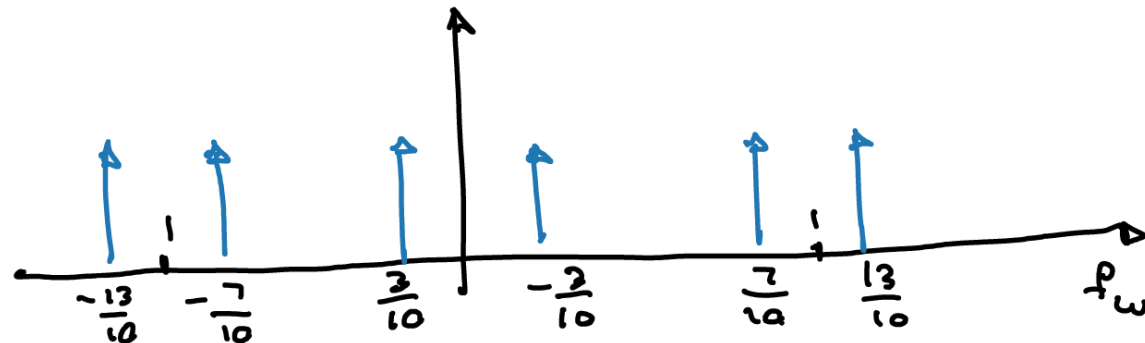
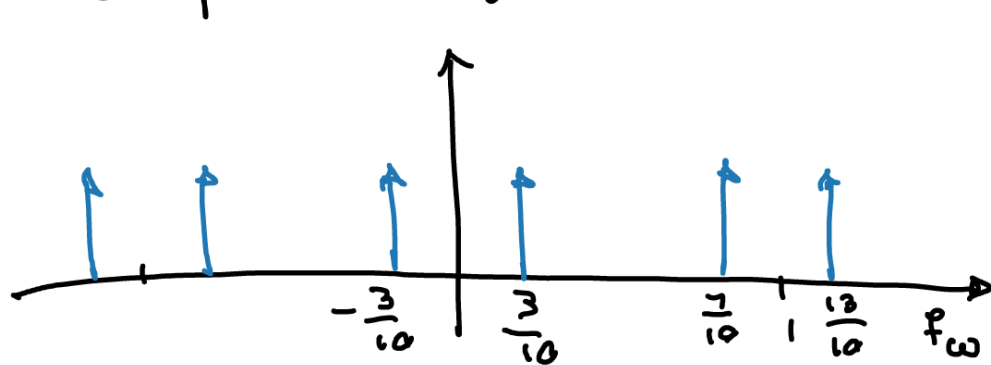


$$x[n] = \sin\left(2\pi \frac{3}{20} n\right)$$

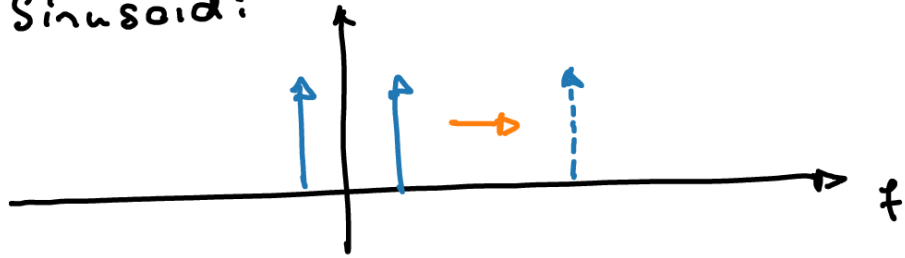
# Aliasing of sinusoidal signals



Sample at  $f_s = 10$  kHz:



Sinusoid:



Sampled sinusoid:

