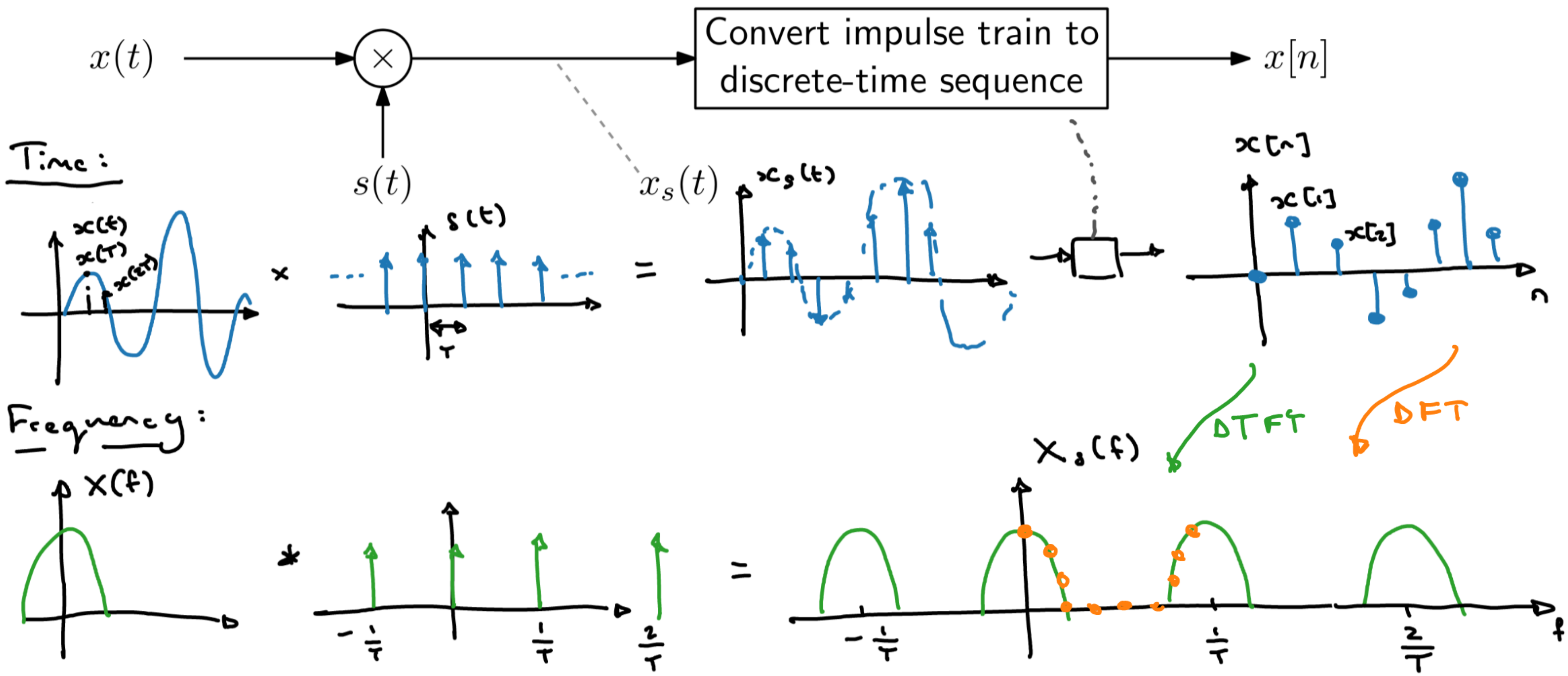
Discrete Fourier transform

Herman Kamper

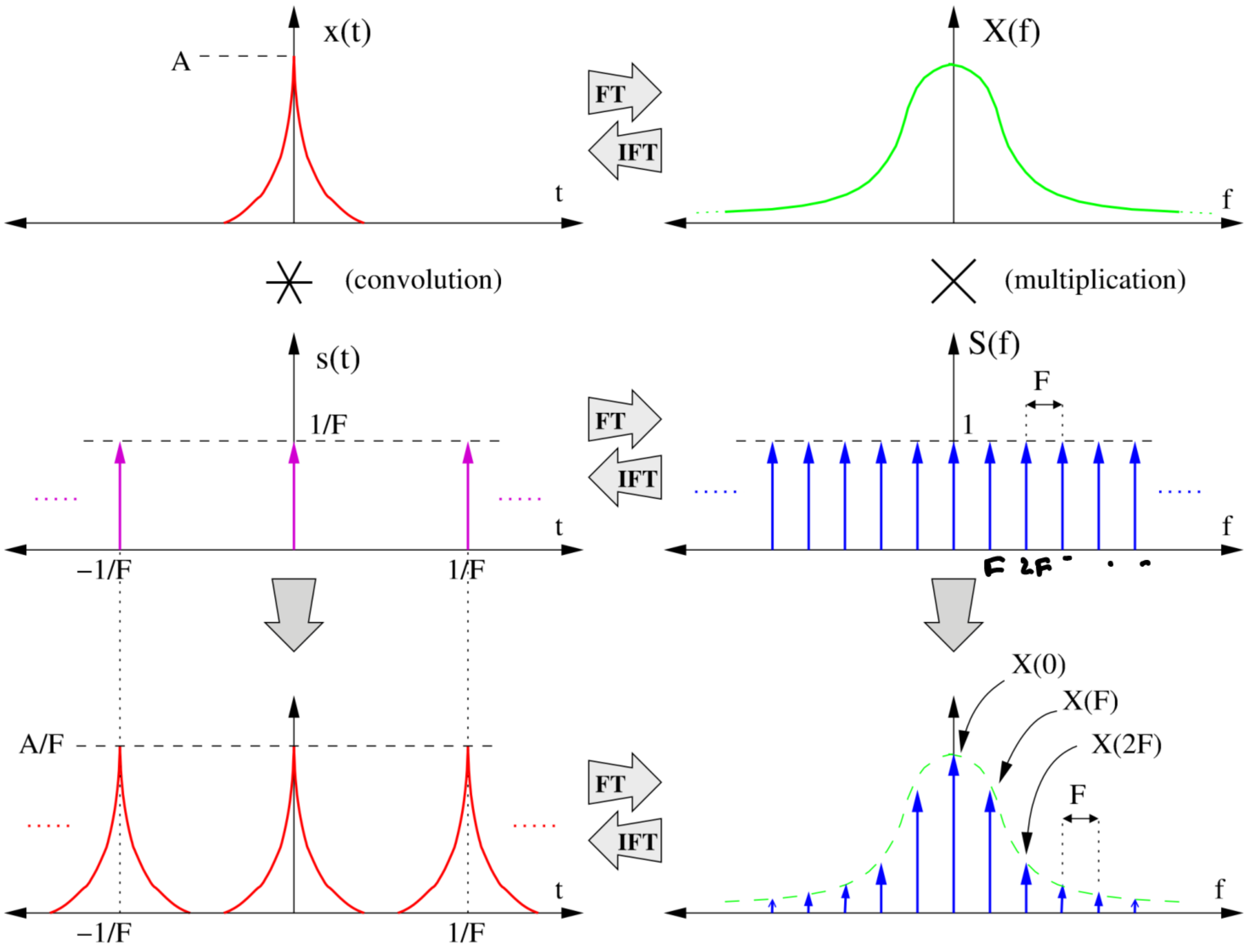
Sampling in time domain



Getting to the DFT

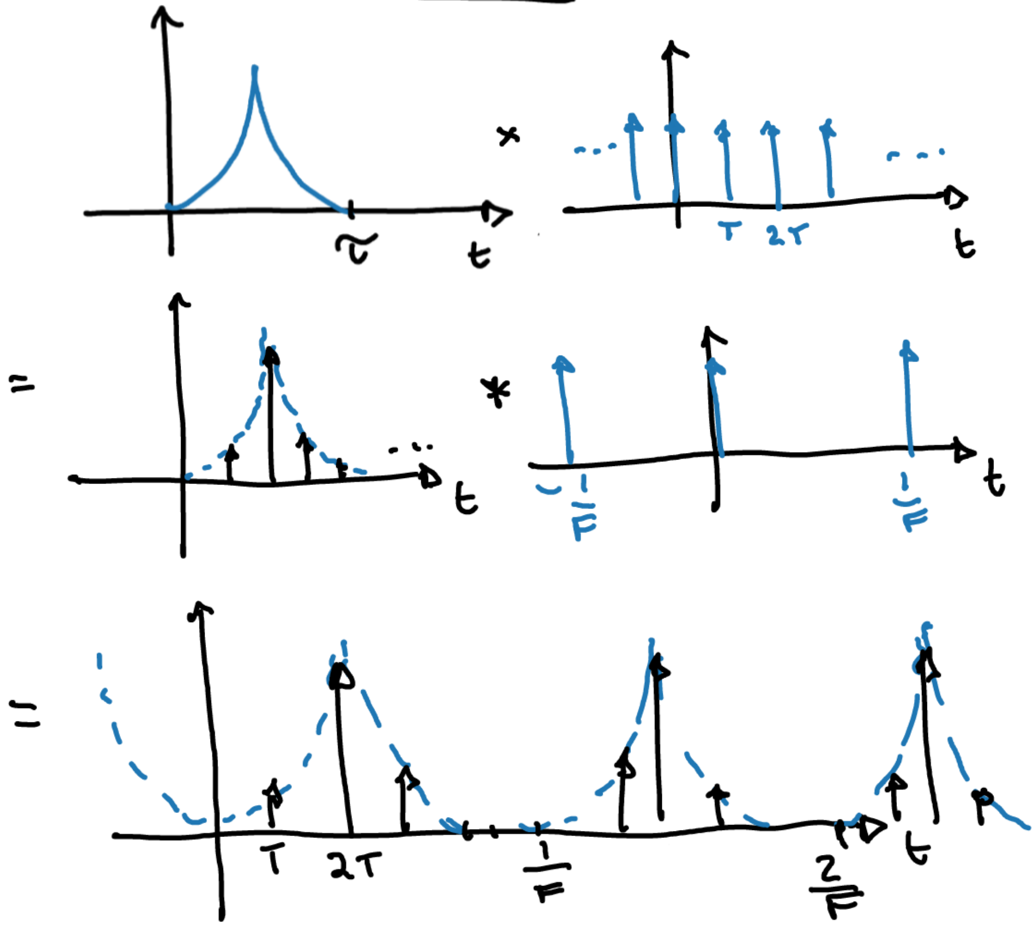
- Discrete signal: Periodic continuous spectrum (DTFT)
- Discrete spectrum: Periodic continuous signal
- Discrete periodic signal: Both signal and spectrum discrete
- But what if your signal is not periodic? Just hack it: window and repeat
- Discrete Fourier transform (DFT): Discrete time and spectrum for arbitrary signals

Sampling in frequency domain

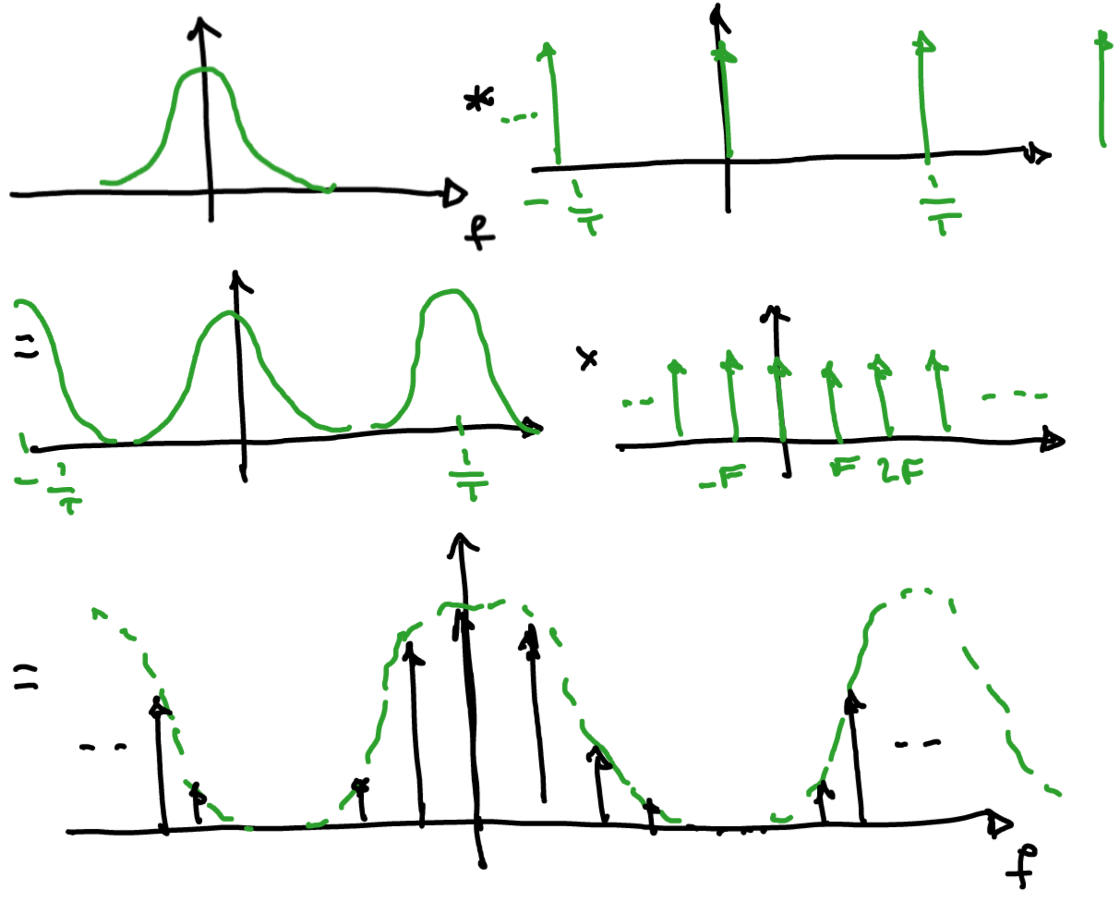


Sampling in time and frequency

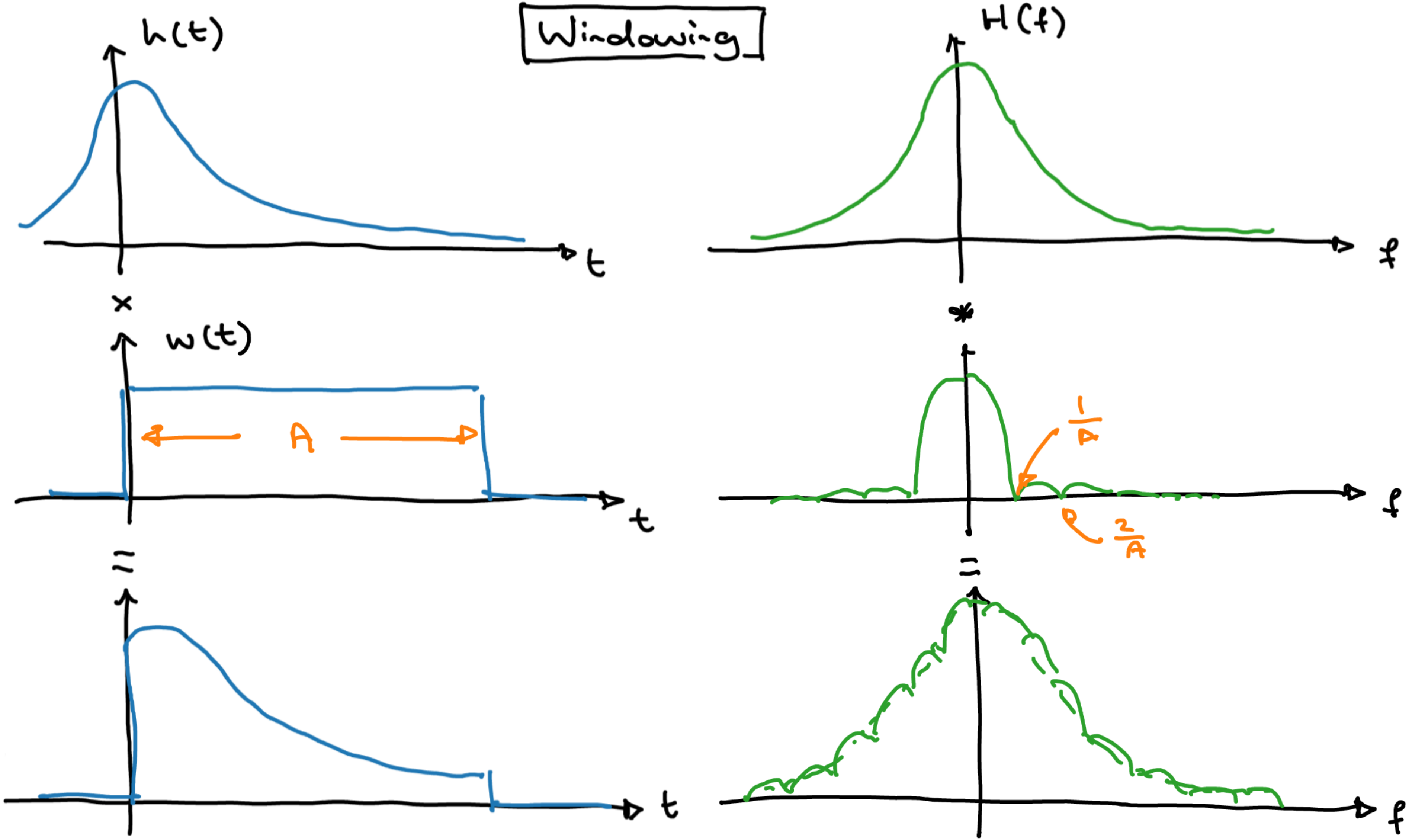
Time



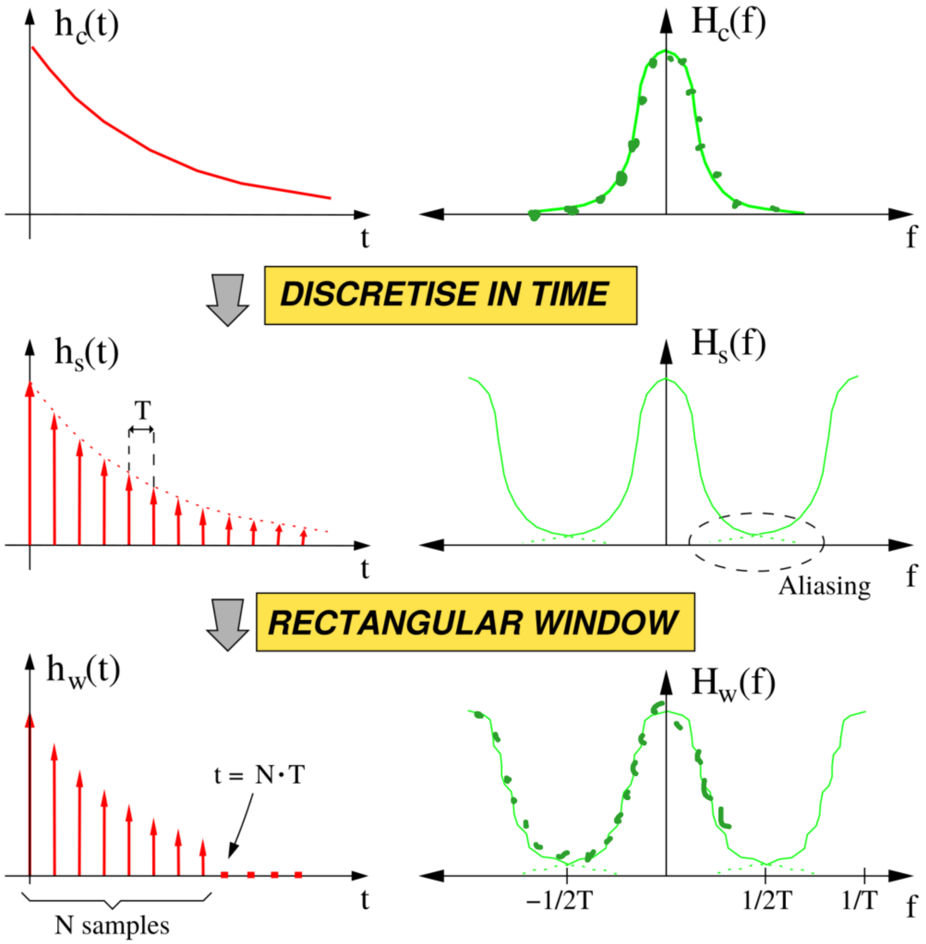
Frequency



Windowing



Discrete Fourier transform (DFT)



Sampling:

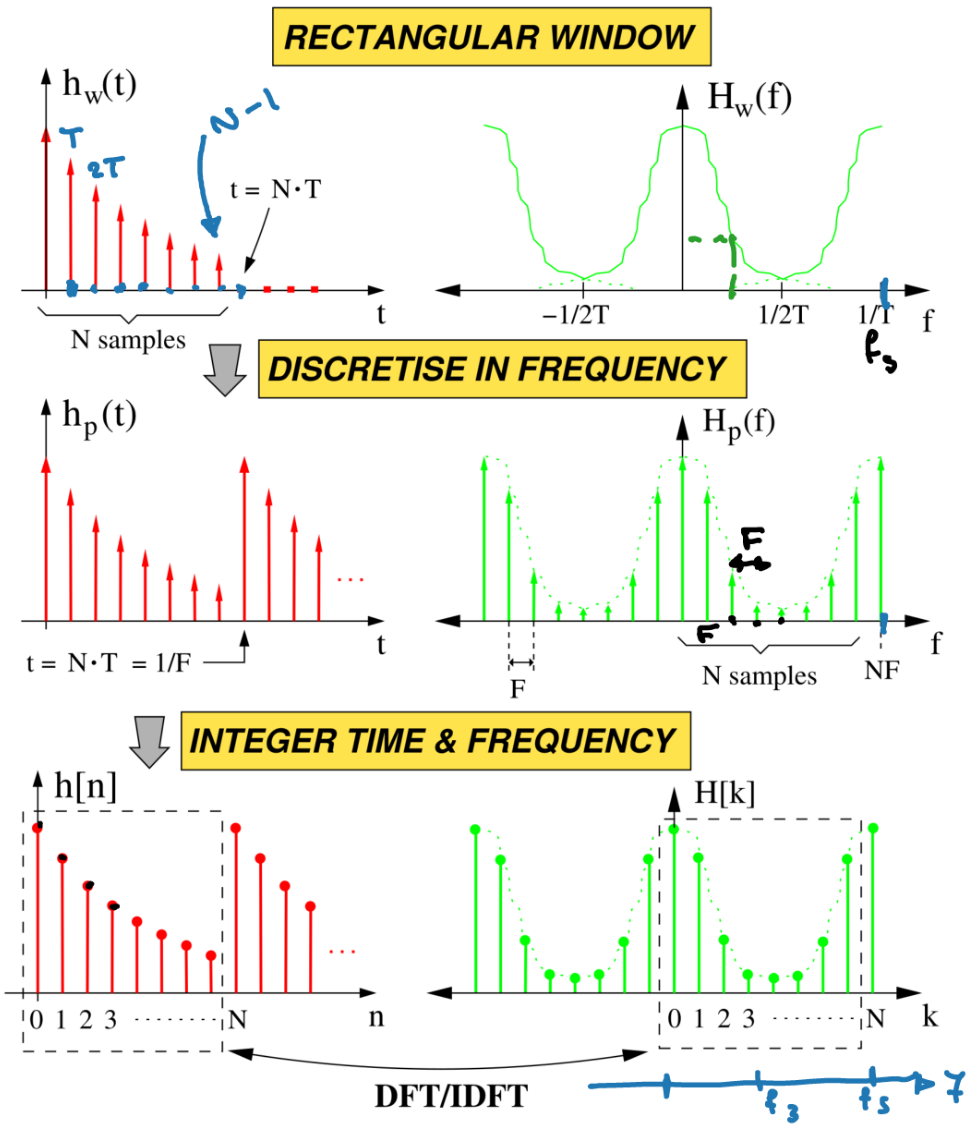
$$\text{DTFT: } H_s(f) = \sum_{n=-\infty}^{\infty} \overbrace{h(nT)}^{h[n]} \cdot e^{-j2\pi f nT}$$

Window:

$$H_w(f) = H_s(f) * W(f)$$

$$= \sum_{n=0}^{N-1} h(nT) \cdot e^{-j2\pi f nT}$$

Discrete Fourier transform (DFT)



Window:

$$H_w(f) = H_s(f) * W(f)$$

$$= \sum_{n=0}^{N-1} h(nT) \cdot e^{-j2\pi f nT}$$

$$\frac{1}{F} = NT$$

Periodic extension:

$$H_p(f) = H_w(f) \times \sum_{k=-\infty}^{\infty} \delta(f - kF)$$

Discrete Fourier transform:

$$h[n] = h(nT)$$

$$H[2] = H_p(2F)$$

$$H[k] = H_w(kF)$$

$$= \sum_{n=0}^{N-1} h[n] \cdot e^{-j2\pi kFnT}$$

$$kFT = k\frac{1}{N\cancel{T}}\cancel{T}$$

$$= \sum_{n=0}^{N-1} h[n] \cdot e^{-j2\pi \frac{k}{N} n}$$

IDFT:

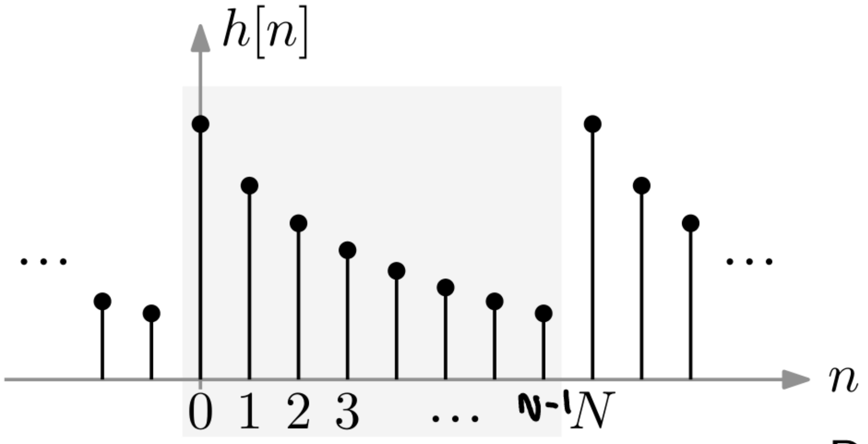
$$h[n] = \frac{1}{N}\sum_{k=0}^{N-1} H[k]e^{j2\pi kn/N}$$

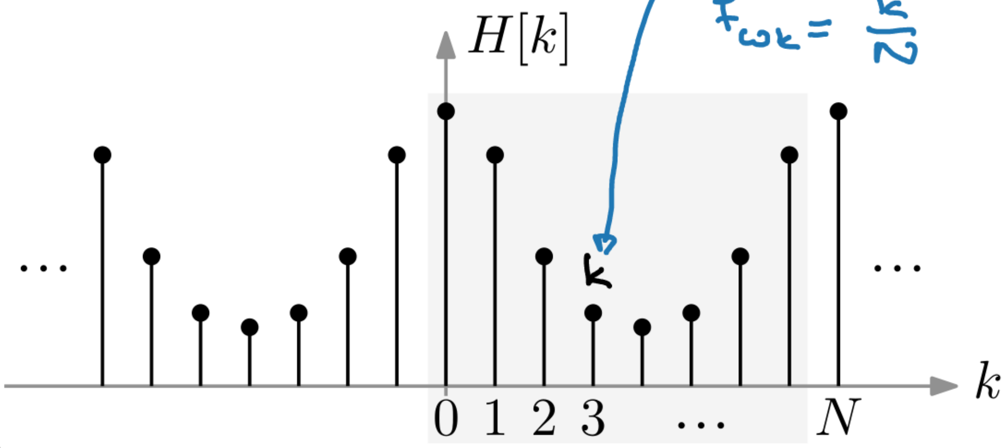
DFT:

$$H[k] = \sum_{n=0}^{N-1} h[n]e^{-j2\pi kn/N}$$

$$f_k = \frac{k}{N} f_s$$

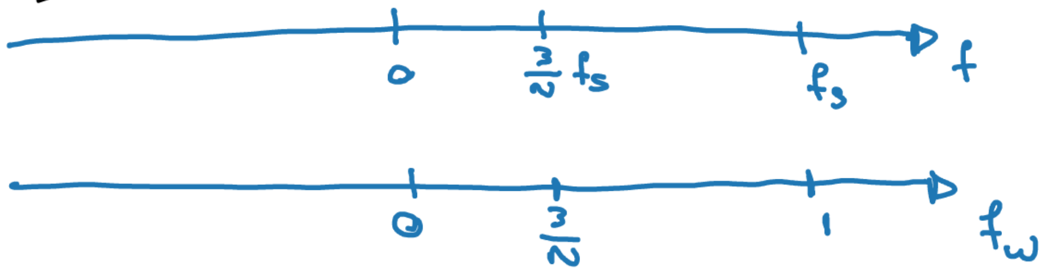
$$f_{\omega k} = \frac{k}{N}$$





DFT ⟶

⟵ IDFT



Inverse discrete Fourier transform (IDFT)

An N-point discrete periodic signal can be expressed as the sum of N complex exponentials with discrete-time frequencies $\omega_k = \frac{2\pi k}{N}$.

If you don't believe me: you've proved it already in the DFT. The discrete $H[k]$ is given by a sum of N complex exponentials. We are now flipping things.

So we know that we should be able to write:

$$h[n] = \sum_{k=0}^{N-1} c_k e^{j2\pi nk/N}$$

with c_k unkown. The goal is to find all the c_ks.

Multiply the above queation by $e^{-j2\pi nk_1/N}$ on both sides, and take the sum over k_1:

$$
\begin{aligned}
h[n]e^{-j2\pi nk_1/N} &= \left[\sum_{k=0}^{N-1} c_k e^{j2\pi nk/N}\right] e^{-j2\pi nk_1/N} \\
\sum_{k_1=0}^{N-1} h[n]e^{-j2\pi nk_1/N} &= \sum_{k_1=0}^{N-1}\sum_{k=0}^{N-1} c_k e^{j2\pi n(k-k_1)/N} \\
&= \sum_{k=0}^{N-1} c_k \sum_{k_1=0}^{N-1} e^{j2\pi n(k-k_1)/N} \\
&= c_{k_1} N
\end{aligned}
$$

That last step looks strange but it comes from

$$\sum_{n=0}^{N-1} e^{j2\pi kn/N} = \begin{cases} N & \text{if } k = 0, \pm N, \pm 2N, \ldots \\ 0 & \text{otherwise} \end{cases}$$

which itself comes from the geometric series formula

$$\sum_{n=0}^{N-1} a = \begin{cases} N & \text{if } a = 1 \\ \frac{1-a^N}{1-a} & \text{if } a \neq 1 \end{cases}$$

So we have

$$\sum_{k_1=0}^{N-1} h[n]e^{-j2\pi nk_1/N} = c_{k_1} N$$

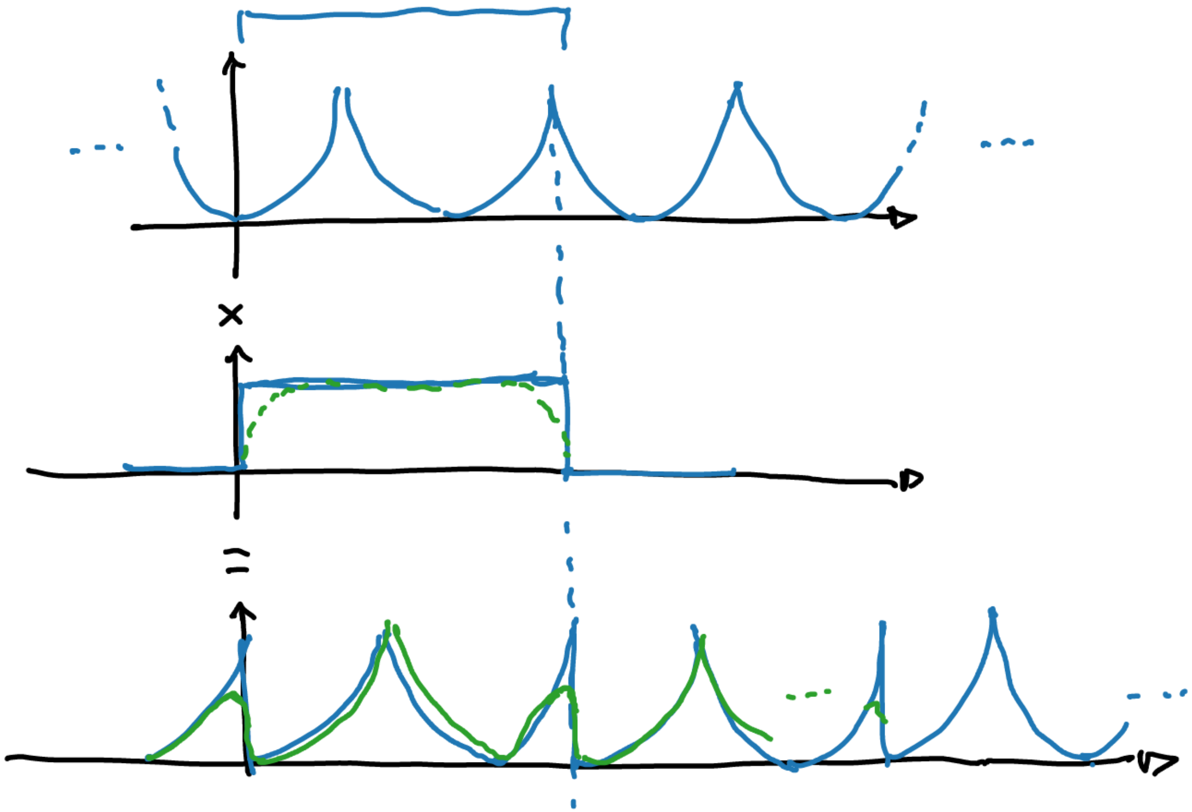
$$\begin{aligned} c_{k_1} &= \frac{1}{N} \sum_{k_1=0}^{N-1} h[n]e^{-j2\pi nk_1/N} \\ &= \frac{1}{N} H[k_1] \end{aligned}$$

If we plug this back into the equation where we started, we get the IDFT:

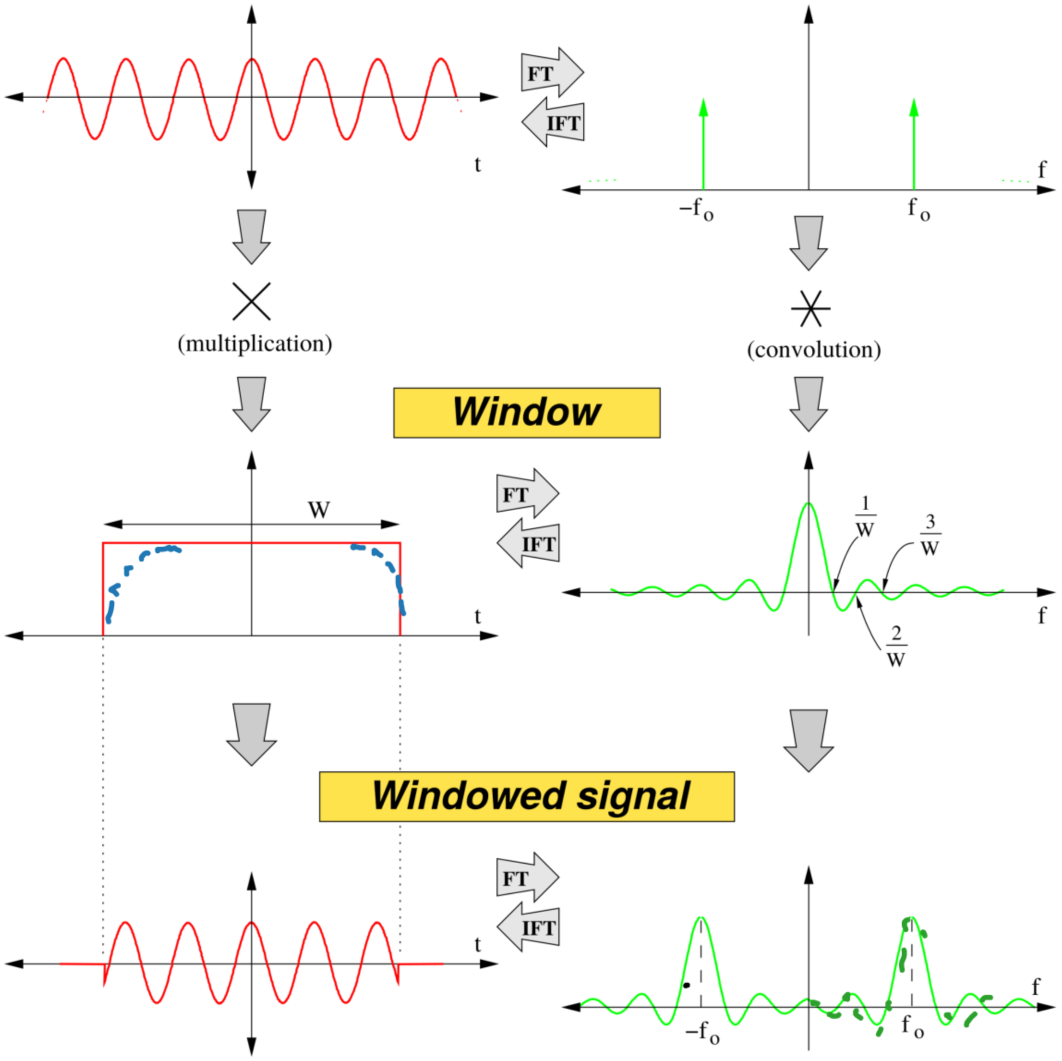
$$h[n] = \frac{1}{N} \sum_{k=0}^{N-1} H[k] e^{j2\pi nk/N}$$

How does the DFT mess up?

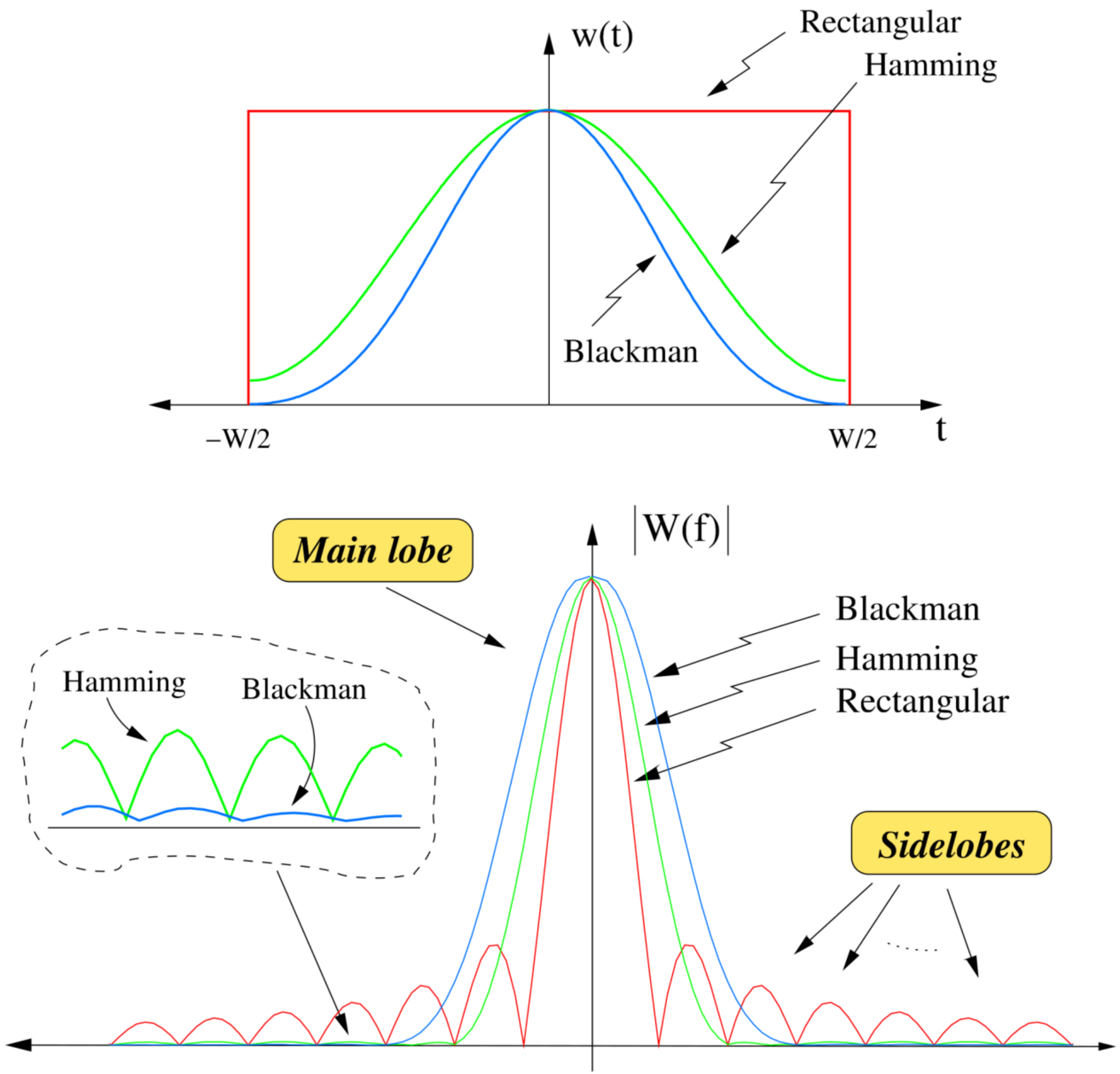
- Aliasing
- Windowing
- Periodicity



Windowing



FT
IFT
t
f
$-f_o$
f_o
(multiplication)
(convolution)
Window
W
$\frac{1}{W}$
$\frac{2}{W}$
$\frac{3}{W}$
Windowed signal


w(t)
Rectangular
Hamming
Blackman
–W/2
W/2
t
|W(f)|
Main lobe
Blackman
Hamming
Rectangular
Hamming
Blackman
Sidelobes