1. Matrix norms

Definition: Matrix norm

A norm $||\cdot||$ on the vector space of matrices $V = Mat_{n\times n}(\mathbb{C})$ is called matrix norm iff

1. $||\cdot||$ defines a norm
   - $||A|| > 0, \quad A \neq 0_{n\times n}$
   - $||\alpha A|| = |\alpha| \cdot ||A||$
   - $||A + B|| \leq ||A|| + ||B||$
2. Submultiplicative property holds
$$||AB|| \leq ||A|| \cdot ||B||.$$

Lemma: 1

For any matrix norm $||\cdot||$ following holds
$$||I|| \geqslant 1.$$

Definition: Unit preserving matrix norm $||\cdot||$

A matrix norm $||\cdot||$ is called unit preserving matrix norm iff
$$||I|| = 1.$$

Definition: Matrix norm $||\cdot||$ compatible with a vector norm $|\cdot|$

A matrix norm $||\cdot||$ is called compatible with a vector norm $|\cdot|$ iff
$$||A\vec{x}|| \leq ||A|| \cdot |\vec{x}|.$$

Lemma: 2

Any matrix norm $||\cdot||$ is compatible with some vector norm $|\cdot|$. In particular, a norm $||\cdot||_{\star,0}$ is compatible with
$$|v|_{\star,0} := \left\| \begin{array}{c|ccc} & 0 & \dots & 0 \\ & \vdots & \dots & \vdots \\ \vec{v} & 0 & \dots & 0 \\ & \vdots & \dots & \vdots \\ & 0 & \dots & 0 \end{array} \right\|_{\star,0}.$$

Lemma: 3

If $||\cdot||$ is a matrix norm compatible with $|\cdot|$, then $||A||' = \lambda||A||$ (where $\lambda > 1$) is another matrix norm compatible with $|\cdot|$.

Definition: Frobenius norm

Following norm is called Frobenius norm

$$\|A\|_F = \sqrt{\sum_{i,j} |a_{ij}|^2} = \sqrt{\operatorname{tr}(A^* A)} = \sqrt{\sigma_1 + \dots + \sigma_n},$$

where σ_i are singular values of A.

Frobenius norm is unitary invariant, that is,

$$\|AW\|_F = \|WA\|_F = \|A\|_F, \quad W^* = (W)^{-1}.$$

Definition: Induced matrix norm

Let $|\cdot|_\bigstar$ be a vector norm on $\mathbb{C}^n$, then

$$\|A\|_\bigstar = \max_{\vec{x} \neq \vec{0}} \frac{|A\vec{x}|_\bigstar}{|a|_\bigstar}$$

is called a matrix norm which is induced by norm $|\cdot|_\bigstar$.

Theorem: Induced matrix norm

1. The function $\|\cdot\|_\bigstar$ is well defined (max exists)
2. The function $\|\cdot\|_\bigstar$ is a matrix norm
3. The function $\|\cdot\|_\bigstar$ is compatible with $|\cdot|_\bigstar$
4. The function $\|\cdot\|_\bigstar$ is unit preserving
5. For any matrix norm $\|\cdot\|_{\bigstar,?}$ compatible with norm $|\cdot|_\bigstar$ following holds

$$\|A\|_{\bigstar,?} \geqslant \|A\|_\bigstar, \quad \forall A.$$

Theorem: Main matrix norms

The following are induced matrix norms for main vector norms

Vector norm $\vert\cdot\vert$	Induced matrix norm $\Vert\cdot\Vert$
$\vert\vec{x}\vert = \vert x_1\vert + \dots + \vert x_n\vert$	$\Vert A\Vert_1 = \max_j \vert A^j\vert_1$
$\vert\vec{x}\vert_2 = \sqrt{\vert x_1\vert^2 + \dots + \vert x_n\vert^2}$	$\Vert A\Vert_2 = \sigma_1$
$\vert\vec{x}\vert_\infty = \max_i \vert x_i\vert$	$\Vert A\Vert_\infty = \max_i \vert A_i\vert_\infty$