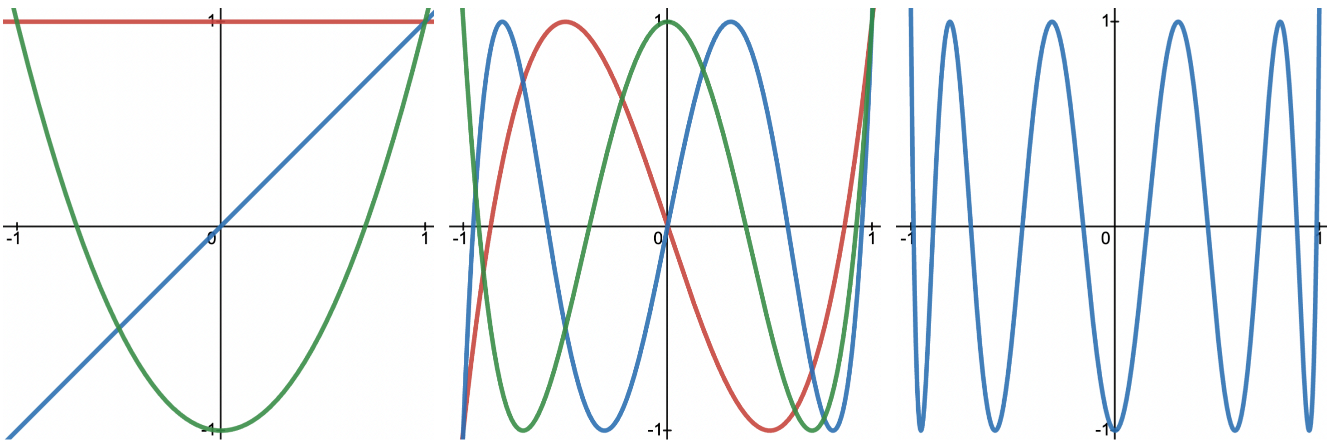
1. Chebyshev polynomials

Definition: Chebyshev polynomials of the first kind

$$
\begin{aligned}
T_0(x) &:= 1, \\
T_1(x) &:= x, \\
T_n(x) &:= 2xT_{n-1}(x) - T_{n-2}(x), \quad n \geqslant 2.
\end{aligned}
$$

Example 1:

n	$T_n(x)$
0	1
1	x
2	$2x - 1$
3	$4x^3 - 3x$
4	$8x^4 - 8x^2 + 1$
5	$16x^5 - 20x^3 + 5x$
6	$32x^6 - 48x^4 + 18x^2 - 1$
7	$64x^7 - 112x^5 + 56x^3 - 7x$
8	$128x^8 - 256x^6 + 160x^4 - 32x^2 + 1$
9	$256x^9 - 576x^7 + 432x^5 - 120x^3 + 9x$
10	$512x^{10} - 1280x^8 + 1120x^6 - 400x^4 + 50x^2 - 1$



Plots of $T_0, T_1, \dots, T_5$ and T_{10} (red,green, blue ordering).

Theorem

$$
\begin{aligned}
T_n(\cos\phi) &= \cos n\phi, \\
T_n(x) &= \cos(\arccos x), \quad |x| \le 1 \\
T_n(x) &= \frac{\left(x+\sqrt{x^2-1}\right)^n + \left(x-\sqrt{x^2-1}\right)^n}{2}, \quad |x| \geqslant 1.
\end{aligned}
$$

<u>**Proof:**</u> The proof is by induction. Let $x = \cos\phi \in [-1,1] \Leftrightarrow \phi = \cos^{-1}(x) \in [0,\pi]$. Then, for $n = 1$, we have

$$T_n(x) = 1 = \cos(0\cdot\phi), \quad T_1(x) = \cos\phi.$$

For $n \geqslant 2$, we have

$$
\begin{aligned}
T_n(x) &= 2xT_{n-1}(x) - T_{n-2}(x) \\
&= 2\cos\phi\cdot\cos((n-1)\phi) - \cos((n-2)\phi) \\
&= \cos(n\phi) + \cos((n-2)\phi) - \cos((n-2)\phi) \\
&= \cos n\phi.
\end{aligned}
$$

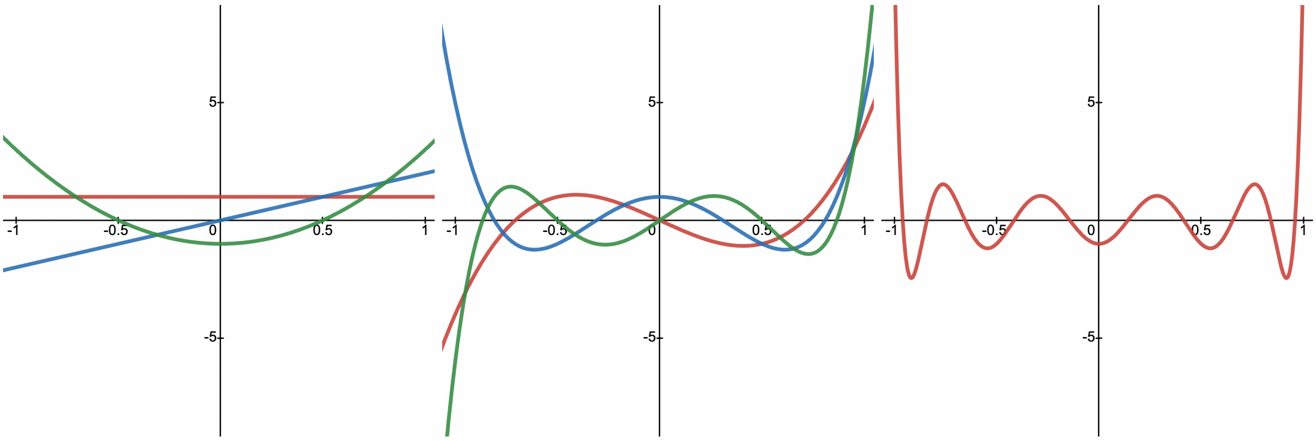
$\square$

Definition: Chebyshev polynomials of the second kind

$$U_n(x) := \frac{1}{n+1}T'_{n+1}(x), \quad n \geqslant 0.$$

Example 2:

n	$U_n(x)$
0	1
1	$2x$
2	$4x^2-1$
3	$8x^3-4x$
4	$16x^4-12x^2+1$
5	$32x^5-32x^3+6x$
6	$64x^6-80x^4+24x^2-1$
7	$128x^7-192x^5+80x^3-8x$
8	$256x^8-448x^6+240x^4-40x^2+1$
9	$512x^9-1024x^7+672x^5-160x^3+10x$
10	$1024x^{10}-2304x^8+1792x^6-560x^4+60x^2-1$



Plots of $U_0, U_1, \dots, T_5$ and U_{10} (red,green, blue ordering).

Theorem

$$U_n(\cos\phi)\sin\phi = \sin n\phi,$$
$$U_n(x) = \frac{\left(x+\sqrt{x^2-1}\right)^{n+1} - \left(x-\sqrt{x^2-1}\right)^{n+1}}{2\sqrt{x^2-1}}, \quad |x| \geqslant 1.$$

Theorem

1. The leading term of $T_n(x)$ and $U_n(x)$ is 2^{n-1} and 2^n respectively.
2. For $n \leq 1$, $T_n(x)$ has exactly n roots on $[-1,1]$, namely, $\cos\left(\frac{(2k-1)\pi}{2n}\right), k = 1, \dots, n$.
3. For $n \leq 1$, $U_n(x)$ has exactly n roots on $[-1,1]$, namely, $\cos\left(\frac{\pi k}{n+1}\right), k = 1, \dots, n$.

Proof: Let $x = \cos\phi \in [-1,1]$. Then, we have

$$T_n(x) = 0 \Leftrightarrow \cos(n\phi) = 0 \Leftrightarrow \phi_k = \frac{(2k-1)\pi}{2n}, k = 1, 2, \dots, n.$$

That means that $x_k = \cos\phi_k = \cos\left(\frac{(2k-1)\pi}{2n}\right), k = 1, \dots, n$ are the roots of $T_n(x)$. □

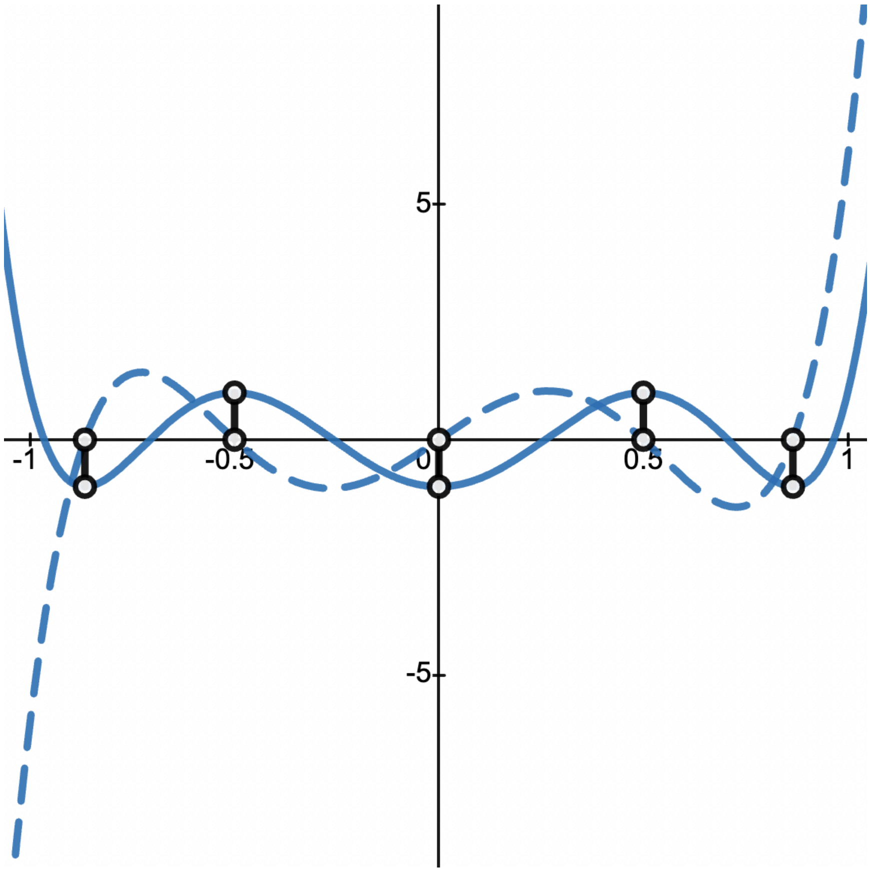
Prop

1. $T_n(x) = 2^{n-1}(x - \cos\frac{\pi}{2n})(x - \cos\frac{3\pi}{2n}) \dots (x - \cos\frac{(2n-1)\pi}{2n})$.
2. $U_n(x) = 2^n(x - \cos\frac{\pi}{n+1})(x - \cos\frac{2\pi}{n+1}) \dots (x - \cos\frac{\pi}{n+1})$.

Proof: Simple corollary from previous theorem. □

Theorem

Polynomial $T_n(x)$ on the segment $[0,1]$ reaches its extreme values, 1 and -1, at $n+1$ points including the ends of the segment.



Plot of extreme points of T_6. T_6 and U_5 is bold and dashed respectively.

Definition: Least deviating from zero polynomial

Let $||\cdot||$ be a norm on the space of continuous functions. A polynomial $f(x) = x^n + ...$ of degree n with the leading coefficient 1 is called the least deviating from zero with respect to the given norm if for any other polynomial $g(x) = x^n + ...$ the following holds

$$||f|| \leq ||g||.$$

Theorem: Chebyshev

The least deviating from zero polynomial on the segment $[-1,1]$ with respect to the Chebyshev norm (maximum of the function's absolute value on the segment)

$$||f||_0 = \max_{[-1,1]} |f(x)|$$

is

$$\widetilde{T}_n(x) := \frac{1}{2^{n-1}} T_n(x).$$

Example 3: The deviation from zero of the polynomial $\widetilde{T}_3(x) = 4T_3(x) = x^3 - \frac{3}{4}x$ with respect to the Chebyshev norm $||f||_0$ is equal to

$$||\frac{1}{4}T_3(x)||_0 = \frac{1}{4}||T3(x)||_0 = \frac{1}{4},$$

and, for example, the deviation from zero of the polynomial x^3 is $||x^3||_0 = 1$.

Example 4: Given $f(x) = x^3$, let us find $\tilde{f}(x) = ax^2 + bx + c$ such that

$$||f(x) - \tilde{f}(x)||_0 = \max_{[-1,1]} |f(x) - \tilde{f}(x)| \to \min.$$

Using Chebyshev Theorem, we get

$$f(x) - \tilde{f}(x) = \widetilde{T}_3(x),$$

which means that

$$\tilde{f}(x) = x^3 - \frac{1}{4}(4x^3 - 3x) = \frac{3}{4}x.$$

Prop: Chebyshev

The least deviating from zero polynomial on the segment $[a, b]$ with respect to the Chebyshev norm (maximum of the function's absolute value on the segment)

$$||f||_0 = \max_{[a,b]} |f(x)|$$

is

$$\overline{T}_n(x) := \frac{(b-a)^n}{2^{n-1}} T_n\left(\frac{2x - (b+a)}{b-a}\right).$$

Proof: The result is obtained by replacing variables in the previous theorem. □

Theorem: Korkin-Zolotarev

The least deviating from zero polynomial on the segment $[-1, 1]$ with respect to the norm (the area under the curve on the segment)

$$||f||_1 = \int_{-1}^{1} |f(x)| dx$$

is

$$\widetilde{U}_n(x) := \frac{1}{2^n} U_n(x).$$

Theorem: Chebyshev Equioscillation Theorem

A polynomial $q(x)$ of degree $\leq n$ is an algebraic polynomial of best approximation (w.r.t. the norm $||\cdot||_0$) for a continuous function f on $[a, b]$ if and only if there are least $n+2$ points $a \leq x_1 \leq \cdots \leq x_{n+2} \leq b$ (alternate points) such that

$$f(x_i) - q(x_i) = \alpha(-1)^i ||f - q||_0,$$

where $\alpha = -1$ or $\alpha = 1$ simultaneously for all i.

Theorem: The best approximation of a function f by polynomial of degree $\leq n$

A scalar product of continuous functions on the segment $[-1,1]$

$$\langle f,g\rangle = \int_{-1}^{1} \frac{f(x)g(x)}{\sqrt{1-x^2}}dx.$$

The corresponding norm is

$$||f|| = \sqrt{\langle f,f\rangle} = \sqrt{\int_{-1}^{1} \frac{f(x)^2}{\sqrt{1-x^2}}dx}.$$

The orthogonality relations

$$\langle T_k, T_m\rangle = \begin{cases} 0, & k \neq m \\ \frac{\pi}{2}, & k = m \neq 0 \\ \pi, & k = m = 0 \end{cases}.$$

The best approximation of a function f by polynomial of degree $\leq n$ is

$$\tilde{f}(x) = \sum_{i=0}^{n} \frac{\langle T_i, f\rangle}{\langle T_i, T_i\rangle} T_i(x).$$