1. Iteration methods

Theorem: Iteration process convergence criteria

Iteration process

$$x_{n+1} = Px_n + b$$

is convergent for any initial approximation x_0 iff

$$\rho(P) = |\lambda_{\max}(P)| < 1.$$

Definition: Number correct significant decimal digits

Number of correct decimal digits we have after k-steps is

$$\xi(x_k) := -c \log_{10} |x_k - x_*|,$$

where $c \in \mathbb{R}$ is some costant that depends on the vector norm $|\cdot|$.

Note

For $|\cdot| = |\cdot|_\infty$, we have

$$\xi(x_k) := -\log_{10} |x_k - x_*|.$$

I.e. $c = 1$.

Prop: 1

$$\xi(x_{k+1}) - \xi(x) \geqslant -c \log_{10} \rho(P).$$

Prop: 2

$$|x_k - x| \leq \frac{|x_{k+1} - x_k|}{1 - ||P||}, \quad ||P|| \leq 1.$$

Lemma: 2

$$|x_k - x_*| \leq \frac{||P||^k}{1 - ||P||} |x_1 - x_0|, \quad ||P|| \leq 1.$$

Prop: 3

$$A = A^*, \quad \lambda_i(A) \leq 0$$

$$\Rightarrow$$

$$\rho(E - \tau A) = \frac{\lambda_{\max}(A) - \lambda_{\min}(A)}{\lambda_{\max}(A) + \lambda_{\min}(A)} < 1, \quad \tau = \frac{2}{\lambda_{\max}(A) + \lambda_{\min}(A)}.$$

Definition

Any matrix A has following decomposition

$$A = L + D + R,$$

where

$$L := \begin{bmatrix} 0 & 0 & \cdots & 0 \\ a_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & 0 \end{bmatrix}, \quad D := \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}, \quad R := \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & 0 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

1. Jacobi method

$$D(x_{k+1} - x_k) + Ax_k = b, \quad a_{ii} \neq 0$$

or

$$Dx_{k+1} = -(L + R)x_k + b, \quad a_{ii} \neq 0.$$

Convergent if (suffient but not neccesary)

$$|a_{ii}| > \sum_{i \neq j} |a_{ij}|.$$

2. Gauss-Zeidal method

$$(D + \tau L)(x_{k+1} - x_k) + Ax_k = b$$

or

$$(D + \tau L)x_{k+1} = -Rx_k + b.$$

3. SOR method

$$Dx_{k+1} = -(L + R)x_k + b, \quad a_{ii} \neq 0$$

or

$$(\frac{1}{\tau}D + L)x_{k+1} = (\frac{1}{\tau}D + L - A)x_k + b.$$