Error estimates and the condition number

1. Find the solutions of two systems with close coefficients and explain the results.

$$\begin{cases} x + 3y = 4 \\ x + 3.00001y = 4.00001 \end{cases} \quad \text{и} \quad \begin{cases} x + 3y = 4 \\ x + 2.99999y = 4.00001 \end{cases}$$

2. Consider the system of linear equations
$$\begin{cases} 3x_1 + 2x_2 = 1, \\ 4x_1 + 3x_2 = 2. \end{cases}$$
The matrix elements can be changed by at most $\varepsilon_1 > 0$, while the elements of the right hand part can be changed by at most $\varepsilon_2 > 0$. Give bounds for the possible change of each of the solution coordinates for arbitrary small values of ε_1 and ε_2. Specify the bounds for $\varepsilon_1 = \varepsilon_2 = 0.001$.

Hint. Note that as $\delta x = |\Delta x|/|x|$. Then $|\Delta x| = \delta x|x| \approx \delta x|\hat{x}|$. One can apply this equality in the max-norm $|\cdot|_\infty$.

3. Consider the system of linear equations
$$\begin{cases} x_1 + x_2 = 2 \\ x_1 - x_2 = 4 \end{cases}$$
The main diagonal elements can be changed by at most $\varepsilon_1 > 0$, the elements of the right hand part can be changed by at most $\varepsilon_2 > 0$. Assume that the parameters $\varepsilon_1, \varepsilon_2$ are small. Give an upper bound for possible change of the solution vector in the Euclidean norm, if $\varepsilon_1 = 0.01, \varepsilon_2 = 0.04$.

4. Prove that the condition number κ_2 with respect to the Euclidean norm is not changed if the matrix is multiplied by an unitarian matrix from the left or from the right.

5. Prove that if the matrix U is unitarian then the condition number $\kappa_2(U)$ equals 1.

6. a) Prove that if $(A + \Delta A)\hat{x} = b + \Delta b, Ax = b, \Delta x = \hat{x} - x$, then the approximate inequality holds:

$$\delta x \leqslant \kappa(A)(\delta b + \delta A)$$

for small $\delta b, \delta A, \delta x$. Assume here that $\kappa(A)\delta A \approx 0$.

б) Recall and prove an estimate for δx in the case $0 \not\approx \kappa(A)\delta A < 1$.