1. Norms in finite dimension vector spaces

Theorem: Minkowski

A subset $B \subset \mathbb{R}^n$ is the unit ball for some norm μ

$$B = B_1^{\mu}(\vec{0}) = \{\vec{x} \mid \mu(\vec{x}) \leq 1\}$$

iff the following properties hold

1. B is centrally symmetric
$$\vec{x} \in B \Rightarrow -\vec{x} \in B.$$
2. B is convex
$$\vec{x}, \vec{y} \in B \Rightarrow [\vec{x}, \vec{y}] \subset B.$$
3. B is bounded by Euclidean norm $||\cdot||_2$
$$B \subset B_R^{||\cdot||_2}(\vec{0}), \quad \text{for some } R > 0.$$
4. B contains some Euclidean ball $B_r^{||\cdot||_2}(\vec{0})$
$$B \supset B_r^{||\cdot||_2}(\vec{0}), \quad \text{for some } r > 0.$$
5. B is closed (contains all limit points)
$$\{\vec{x}_n\} \to \vec{x}, \ \ \vec{x}_n \in B \Rightarrow \vec{x} \in B.$$

Lemma: 1

B contains $B_r(\vec{0})^{||\cdot||_0}$ for some $r > 0$.

Lemma: 2

The function $\mu : \mathbb{R}^n \to \mathbb{R}$ is continuous.

Lemma: 3

B bounded and closed.

Definition: Equivalent norms

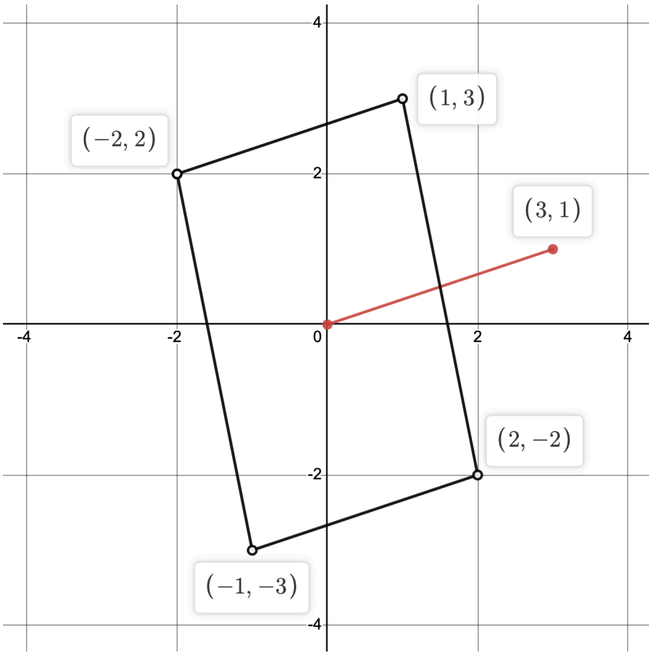
The norms μ and ν are equivalent iff for $c_1, c_2 > 0$

$$\begin{cases} \mu(x) \leq c_1 \nu(x) \\ \nu(x) \leq c_2 \mu(x) \end{cases}.$$

Prop All norms in $\mathbb{R}^n$ are equivalent.

Example 1: Points $P_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $P_2 = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ define unit ball in $\mathbb{R}^2$ with respect to some norm μ. Given $\vec{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, find $\mu(\vec{u})$.

First using centrally symmetric and convex properties from Minkowski Theorem let us plot respective unit ball.



Plot of unit ball B^μ and $\vec{u}$.

Now, let us stretch unit ball α times, so $\vec{u}$ lies on the stretched ball.

$$\begin{aligned} \alpha A(t) &= \vec{v} \\ (1-t)A_1 + tA_2 &= \frac{1}{\alpha}\vec{v} \\ \begin{bmatrix} 1+t \\ 3-5t \end{bmatrix} &= \frac{1}{\alpha}\begin{bmatrix} 3 \\ 1 \end{bmatrix} \\ \alpha = 2, \quad t &= \frac{1}{2}, \end{aligned} \tag{1}$$

which means that

$$\mu(\vec{u}) = \alpha = 2$$

Example 2: Find all $a \in \mathbb{R}$ such that following set B defines a unit ball with respect to some norm μ

$$B = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \; x^2 + 6axy + 9y^2 \leq 100 \right\}.$$

We see that B defines conics in B. Since hyperbola and parabola are unbounded and ellipses is bounded and symmetric, B defines a unit ball with respect to some norm μ iff B ellipses. So, we need to find all $a \in \mathbb{R}$

set B defines ellipses.

$$B = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \; x^2 + 6axy + 9y^2 \leq 100 \right\}$$
$$= \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \; (x + 3ay)^2 + 9(1 - a^2)y^2 \leq 100 \right\},$$

which means that $1 - a^2 > 0 \Leftrightarrow a \in (-1, 1)$.

Example 3: Following set B defines a unit ball with respect to some norm μ

$$B = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \; x^2 + 6axy + 9y^2 \leq 100, \quad a \in (-1, 1) \right\}.$$

Find $\mu \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\mu \begin{pmatrix} x \\ y \end{pmatrix}$.

Let us squeeze vector $\begin{bmatrix} x \\ y \end{bmatrix}$ α times, so it lies on unit ball B.

$$\begin{cases} (x, y) = \frac{1}{\alpha}(x, y) \\ x^2 + 6axy + 9y^2 = 100 \end{cases} \Rightarrow \begin{cases} (x, y) = \frac{1}{\alpha}(x, y) \\ \dfrac{x^2 + 6axy + 9y^2}{\alpha^2} = 100 \end{cases} \Rightarrow \alpha = \frac{\sqrt{x^2 + 6axy + 9y^2}}{10},$$

which means that

$$\mu \begin{pmatrix} x \\ y \end{pmatrix} = \frac{\sqrt{x^2 + 6axy + 9y^2}}{10}, \quad \mu \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \frac{3\sqrt{2 + 2a}}{10}.$$