

# 1. Norms in finite dimension vector spaces

## Theorem: Minkowski

A subset  $B \subset \mathbb{R}^n$  is the unit ball for some norm  $\mu$

$$B = B_1^\mu(\vec{0}) = \{\vec{x} \mid \mu(\vec{x}) \leq 1\}$$

iff the following properties hold

1.  $B$  is centrally symmetric

$$\vec{x} \in B \Rightarrow -\vec{x} \in B.$$

2.  $B$  is convex

$$\vec{x}, \vec{y} \in B \Rightarrow [\vec{x}, \vec{y}] \subset B.$$

3.  $B$  is bounded by Euclidean norm  $\|\cdot\|_2$

$$B \subset B_R^{\|\cdot\|_2}(\vec{0}), \quad \text{for some } R > 0.$$

4.  $B$  contains some Euclidean ball  $B_r^{\|\cdot\|_2}(\vec{0})$

$$B \supset B_r^{\|\cdot\|_2}(\vec{0}), \quad \text{for some } r > 0.$$

5.  $B$  is closed (contains all limit points)

$$\{\vec{x}_n\} \rightarrow \vec{x}, \quad \vec{x}_n \in B \Rightarrow \vec{x} \in B.$$

### Lemma: 1

$B$  contains  $B_r^{\|\cdot\|_0}(\vec{0})$  for some  $r > 0$ .

### Lemma: 2

The function  $\mu : \mathbb{R}^n \rightarrow \mathbb{R}$  is continuous.

### Lemma: 3

$B$  bounded and closed.

### Definition: Equivalent norms

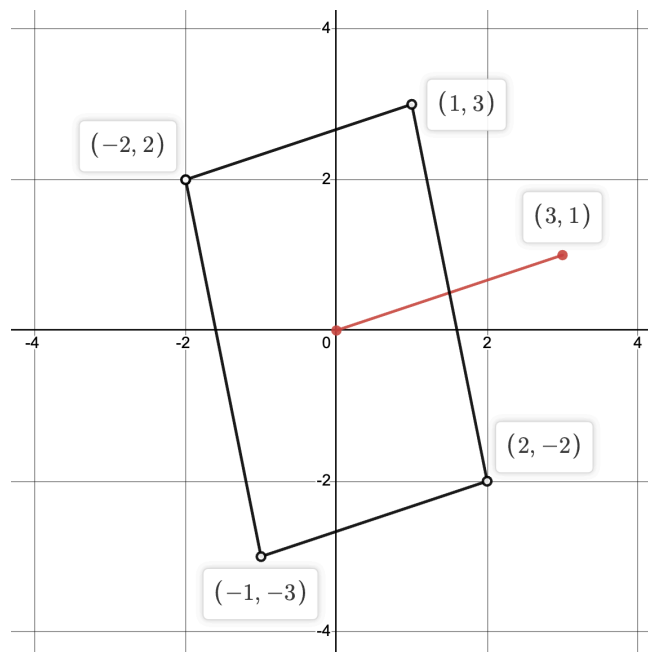
The norms  $\mu$  and  $\nu$  are equivalent iff for  $c_1, c_2 > 0$

$$\begin{cases} \mu(x) \leq c_1 \nu(x) \\ \nu(x) \leq c_2 \mu(x) \end{cases}.$$

All norms in  $\mathbb{R}^n$  are equivalent.

**Example 1:** Points  $P_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$  and  $P_2 = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$  define unit ball in  $\mathbb{R}^2$  with respect to some norm  $\mu$ . Given  $\vec{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ , find  $\mu(\vec{u})$ .

First using centrally symmetric and convex properties from Minkowski Theorem let us plot respective unit ball.



Plot of unit ball  $B^\mu$  and  $\vec{u}$ .

Now, let us stretch unit ball  $\alpha$  times, so  $\vec{u}$  lies on the stretched ball.<sup>1</sup>

$$\begin{aligned} \alpha A(t) &= \vec{u} \\ (1-t)P_1 + tP_2 &= \frac{1}{\alpha} \vec{u} \\ \begin{bmatrix} 1+t \\ 3-5t \end{bmatrix} &= \frac{1}{\alpha} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \\ \alpha &= 2, \quad t = \frac{1}{2}, \end{aligned} \tag{1}$$

which means that

$$\mu(\vec{u}) = \alpha = 2.$$

**Example 2:** Find all  $a \in \mathbb{R}$  such that following set  $B$  defines a unit ball with respect to some norm  $\mu$

$$B = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid x^2 + 6axy + 9y^2 \leq 100 \right\}.$$

<sup>1</sup>Here we use parametric function  $A(t)$  that defines side of unit ball that starts at the point  $P_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$  and ends at the point  $P_2 = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ . We choose this side in accordance with the plot of unit ball  $B^\mu$  and  $\vec{u}$

We see that  $B$  defines conics in  $B$ . Since hyperbola and parabola are unbounded and ellipses is bounded and symmetric,  $B$  defines a unit ball with respect to some norm  $\mu$  iff  $B$  ellipses. So, we need to find all  $a \in \mathbb{R}$  set  $B$  defines ellipses.

$$\begin{aligned} B &= \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid x^2 + 6axy + 9y^2 \leq 100 \right\} \\ &= \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid (x + 3ay)^2 + 9(1 - a^2)y^2 \leq 100 \right\}, \end{aligned}$$

which means that  $1 - a^2 > 0 \Leftrightarrow a \in (-1, 1)$ .

**Example 3:** Following set  $B$  defines a unit ball with respect to some norm  $\mu$

$$B = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid x^2 + 6axy + 9y^2 \leq 100, \quad a \in (-1, 1) \right\}.$$

Find  $\mu \begin{pmatrix} 3 \\ 1 \end{pmatrix}$  and  $\mu \begin{pmatrix} x \\ y \end{pmatrix}$ .

Let us denote a vector that lies on unit ball  $B$  by  $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ . Let us squeeze vector  $\begin{bmatrix} x \\ y \end{bmatrix}$   $\alpha$  times, so it lies on unit ball  $B$

$$\begin{cases} (x_0, y_0) = \frac{1}{\alpha}(x, y) \\ x_0^2 + 6ax_0y_0 + 9y_0^2 = 100 \end{cases} \Rightarrow \begin{cases} (x_0, y_0) = \frac{1}{\alpha}(x, y) \\ \frac{x^2 + 6axy + 9y^2}{\alpha^2} = 100 \end{cases} \Rightarrow \alpha = \frac{\sqrt{x^2 + 6axy + 9y^2}}{10},$$

which means that

$$\mu \begin{pmatrix} x \\ y \end{pmatrix} = \frac{\sqrt{x^2 + 6axy + 9y^2}}{10}, \quad \mu \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \frac{3\sqrt{2+2a}}{10}.$$