1. Matrix norms

Definition: Matrix norm

A norm $||\cdot||$ on the vector space of matrices $V = Mat_{n\times n}(\mathbb{C})$ is called matrix norm iff

1. $||\cdot||$ defines a norm
   - $||A|| > 0, \quad A \neq 0_{n\times n}$
   - $||\alpha A|| = |\alpha|\cdot||A||$
   - $||A+B|| \leq ||A|| + ||B||$
2. Submultiplicative property holds

$$||AB|| \leq ||A||\cdot||B||.$$

Lemma: 1

For any matrix norm $||\cdot||$ following holds

$$||I|| \geqslant 1.$$

Definition: Unit preserving matrix norm $||\cdot||$

A matrix norm $||\cdot||$ is called unit preserving matrix norm iff

$$||I|| = 1.$$

Definition: Matrix norm $||\cdot||$ compatible with a vector norm $|\cdot|$

A matrix norm $||\cdot||$ is called compatible with a vector norm $|\cdot|$ iff

$$||A\vec{x}|| \leq ||A||\cdot|\vec{x}|.$$

Lemma: 2

Any matrix norm $||\cdot||$ is compatible with some vector norm $|\cdot|$. In particular, a norm $||\cdot||_{\star,0}$ is compatible with

$$|v|_{\star,0} := \left\|\begin{array}{c|ccc} & 0 & \dots & 0 \\ & \vdots & \dots & \vdots \\ \vec{v} & 0 & \dots & 0 \\ & \vdots & \dots & \vdots \\ & 0 & \dots & 0 \end{array}\right\|_{\star,0}.$$

Lemma: 3

If $||\cdot||$ is a matrix norm compatible with $|\cdot|$, then $||A||' = \lambda||A||$ (where $\lambda > 1$) is another matrix norm compatible with $|\cdot|$.

Definition: Frobenius norm

Following norm is called Frobenius norm

$$\begin{aligned} ||A||_F &= \sqrt{\sum_{i,j} |a_{ij}|^2} \\ &= \sqrt{\mathrm{tr}(A^*A)} \\ &= \sqrt{\sigma_1^2 + \cdots + \sigma_n^2}, \end{aligned}$$

where σ_i are singular values of A.

Frobenius norm is unitary invariant, that is,

$$||AW||_F = ||WA||_F = ||A||_F, \quad W^* = (W)^{-1}.$$

Definition: Induced matrix norm

Let $|\cdot|_\star$ be a vector norm on $\mathbb{C}^n$, then

$$||A||_\star = \max_{\vec{x}\neq\vec{0}} \frac{|A\vec{x}|_\star}{|a|_\star}$$

is called a matrix norm which is induced by norm $|\cdot|_\star$.

Theorem: Induced matrix norm

1. The function $||\cdot||_\star$ is well defined (max exists)
2. The function $||\cdot||_\star$ is a matrix norm
3. The function $||\cdot||_\star$ is compatible with $|\cdot|_\star$
4. The function $||\cdot||_\star$ is unit preserving
5. For any matrix norm $||\cdot||_{\star,?}$ compatible with norm $|\cdot|_\star$ following holds

$$||A||_{\star,?} \geqslant ||A||_\star, \quad \forall A.$$

Theorem: Main matrix norms

The following are induced matrix norms for main vector norms

Vector norm $\|\cdot\|$	Induced matrix norm $\|\|\cdot\|\|_1$
$\|\vec{x}\|_1 = \|x_1\| + \cdots + \|x_n\|$	$\|\|A\|\|_1 = \max_j \|A^j\|_1$
$\|\vec{x}\|_2 = \sqrt{\|x_1\|^2 + \cdots + \|x_n\|^2}$	$\|\|A\|\|_2 = \sigma_1$
$\|\vec{x}\|_\infty = \max_i \|x_i\|$	$\|\|A\|\|_\infty = \max_i \|A_i\|_\infty$