Low rank approximation

1. Suppose that A is a matrix of order 2 with singular values $\sigma_1 \geq \sigma_2 > 0$. Find $||A^{-1}||_2$ and $||A^{-1}||_F$.

2. Find the closest rank one approximations of the matrices

$$P = \begin{pmatrix} 10 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}, Q = \begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix}, R = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

3. Find the closest rank 2 approximation B_2 with respect to the Frobenius norm for

$$B = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 4 & 2 \\ 2 & 2 & 3 \end{pmatrix}$$

and the approximation error $||B_2 - B||_F$.

4. Find the closest rank one approximation B_1 for B in the Euclidean norm $||\cdot||_2$ and $||B - B_1||_2$, where

$$B = \begin{pmatrix} 6 & 30 & -21 \\ 17 & 10 & -22 \end{pmatrix}$$

5. Prove that for the norm $||A||_\infty = \max_i |A_i|_1$ Eckart–Young–Mirsky theorem does not hold even for the matrices of order 2.

Hint. One can consider the matrix $\begin{pmatrix} 3 & 0 \\ 4 & 5 \end{pmatrix}$

6. Prove that the norm $||A||_\infty$ is not unitary invariant.

7. Describe all matrices of rank at most 3 such that:

а) $||A - A_1||_2 = ||A - A_2||_2$;

б) $||A - A_1||_F = ||A - A_2||_F$.