1. Metric spaces. Normed vector spaces

Metric space

Definition: Metric $\rho(x, y)$

Let X be a set. A metric ρ is a function[a] $\rho: X \times X \to [0, \infty)$ such that

1. Symmetric
$$\rho(x, y) = \rho(y, x).$$
2. Positive definite
$$\rho(x, y) > 0, \quad x \neq y,$$
$$\rho(x, x) = 0.$$
3. Triangle Inequality
$$\rho(x, z) \leq \rho(x, y) + \rho(y, z).$$

[a] metric ρ could be also denoted by d. In that way we can call it by 'distance'

Definition: Metric space

A metric space is a set X on which a metric ρ is defined.

Example 1: For $X = \mathbb{R}^n, \mathbb{C}^n$, we could define following Euclidean metric

$$\rho_E(\vec{x}, \vec{y}) := \sqrt{(y_1 - x_1)^2 + \cdots + (y_n - x_n)^2},$$

where

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}.$$

Note that Euclidean metric could be expressed using dot (inner) product as follows

$$\rho_E(\vec{x}, \vec{y}) = \sqrt{(\vec{x} - \vec{y}, \vec{x} - \vec{y})}.$$

Example 2: For $X = \mathbb{R}$, we could define following Euclidean metric

$$\rho(x, y) = |e^x - e^y|.$$

Example 3: For $X = \mathbb{R}^n$, we could define following angle metric

$$\rho(\vec{x}, \vec{y}) = \widehat{x, y} \in [0, \pi].$$

Example 4: For discrete set X, we could define following discrete metric

$$\rho(x, y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}.$$

Example 5: For $X = \mathbb{R}_+ := (0, +\infty)$, we could define following metric

$$\rho(x, y) = |\ln(x) - \ln(y)|.$$

Example 6: For $X = \mathbb{R}^n \setminus \{\vec{0}\}$, we could define following cosine similarity metric

$$\rho(\vec{x}, \vec{y}) = \cos(\vec{x}, \vec{y}) = \frac{(\vec{x}, \vec{y})}{||\vec{x}||\,||\vec{y}||}.$$

Example 7:

For $X = C[0,1] :=$ {continuous functions $f : [0,1] \to \mathbb{R}$}, we could define following metrics

$$\rho_0(f,g) = \max_{x\in[0,1]} |f(x) - g(x)|,$$

$$\rho_1(f,g) = \int_0^1 |f(x) - g(x)|dx,$$

$$\rho(f,g) = \rho_0(f,g) + |f(1) - g(1)|.$$

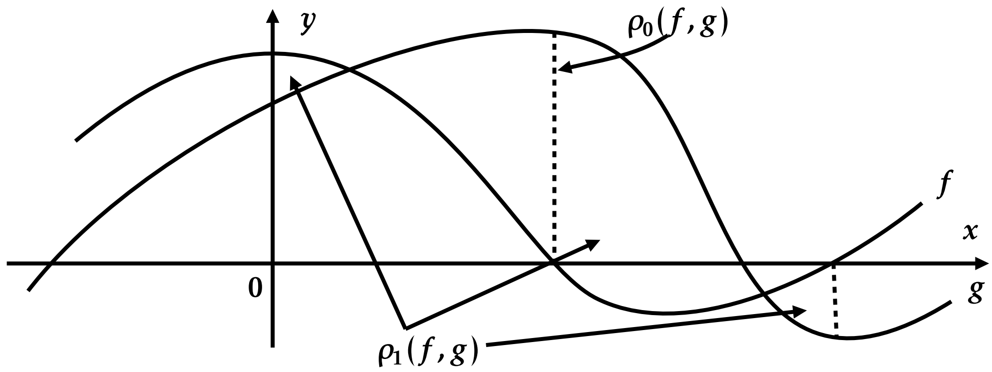

Figure 1: Geometric interpretation of metrics ρ_0 and ρ_1

Example 8: Let as consider weighted[1] multigraph[2] on finite set of nodes $X = \{A, B, C, D, E, F, G, H\}$ (see Figure 2). Let us define metric $\rho(x, y) :=$'length of shortest path between points x and y'. Then, we have

$$\rho(A, B) = 2,$$
$$\rho(A, A) = 0,$$
$$\rho(B, F) = 4 + 3 + 3 = 10.$$

[1] a weighted graph is a graph in which a number (the weight) is assigned to each edge. Such weights might represent for example costs, lengths or capacities, depending on the problem at hand. In our course we consider graphs only with non negatives weights

[2] a multigraph is a graph which is permitted to have multiple edges (also called parallel edges), that is, edges that have the same end nodes

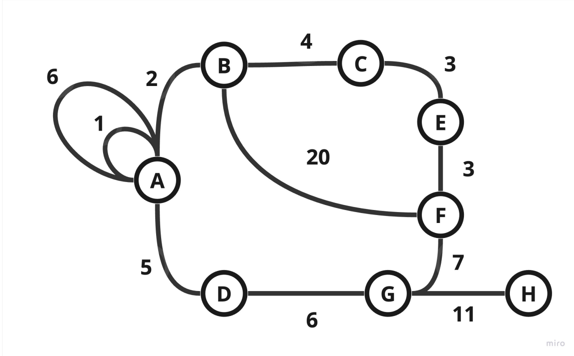

Figure 2: Weighted multigraph on X

Definition: Open ball $\overline{B}_R(x_0)$

An open ball of a radius $R \leq 0$ centered at a pont $c \in X$ is the set

$$\overline{B}_R(c) = \{x \in X \mid \rho(x,c) < R\}.$$

Definition: Closed ball $B_R(x_0)$

A closed ball of a radius $R \leq 0$ centered at a point $c \in X$ is the set

$$\overline{B}_R(c) = \{x \in X \mid f(x,c) \leq R\}.$$

Example 9: For the graph from the previous example (see Figure 2), we have

$$\overline{B}_{10}(A) = \{A, B, C, D, E\},$$
$$\overline{B}_9(B) = \{A, B, C, D, E\},$$
$$\overline{B}_{10}(B) = \{A, B, C, D, E, F\}.$$

Definition: Continuous function f

A function $f(x)$ is continuous at a point x_0 iff

$$\lim_{x \to x_0} f(x) = y_0.$$

That is $\forall \varepsilon > 0 \, \exists \delta > 0$

$$f\left(B_\delta(x_0)\right) \subset B_\varepsilon\left(f(x_0)\right).$$

Example 10: For $X = \mathbb{R}$, we could define non continuous metric

$$\rho(x,y) = \begin{cases} 1, & x - y \in \mathbb{Q} \\ 2, & x - y \notin \mathbb{Q} \\ 0, & x = y \end{cases}.$$

We see that function ρ is not continuous at any point. At the same time, function ρ is metric, since ρ is

1. Symmetric, because

$$x - y \in Q \Leftrightarrow y - x \in Q,$$
$$x - y \notin Q \Leftrightarrow y - x \notin Q.$$

2. Positive definite, because

$$\begin{aligned} &\rho(x,y) > 0, \quad x \neq y, \\ &\rho(x,x) = 0. \end{aligned}$$

3. Triangle Inequality, because
   (a) If $x \notin Q$ (similarly for $z \notin Q$) and $z \neq x$, then

$$\rho(x,z) \leq \rho(x,y) + \rho(y,z)$$
$$\Leftrightarrow$$
$$2 \leq 2 + \rho(y,z),$$

   which holds for any x and y.
   (b) If $x, z \in Q$ and $x \neq z$

$$\rho(x,z) \leq \rho(x,y) + \rho(y,z)$$
$$\Leftrightarrow$$
$$1 \leq \rho(x,y) + \rho(x,y),$$

   which holds for y.

Normed vector space

Definition: Norm

Let V be a vector space. A norm ν is a function[a] $\nu: V \to \mathbb{R}$, such that

1. Positive definite
$$\nu(\vec{x}) > 0, \quad x \neq \vec{0}.$$
2. Homogeneity
$$\nu(\alpha\vec{x}) = |\alpha|\nu(\vec{x}).$$
3. Triangle inequality
$$\nu(\vec{x}+\vec{y}) \leq \nu(\vec{x}) + \nu(\vec{y}).$$

[a] for $v \in V$ norm also could be denoted by $||v|| := \nu(v)$

Definition: Normed vector space

A normed vector space is a vector space V on which a norm ν is defined.

Definition: Euclidean norm $||\vec{x}||_2$

For $V = \mathbb{R}^n, \mathbb{C}^n$, we could define Euclidean norm

$$|\vec{x}|_2 := \rho_E(\vec{0}, \vec{x}) = \sqrt{\sum_{i=1}^{n} |x_i|^2}.$$

Lemma

Let ν be a norm, then $\nu\left(\vec{0}\right) = 0$.

Proof: $\nu\left(\vec{0}\right) = \nu\left(0 \cdot \vec{0}\right) = 0 \cdot \nu\left(\vec{0}\right) = 0.$ □

Prop

Let a normed vector space and it's norm be V and ν respectively. Then a function $\rho(\vec{x}, \vec{y}) := \nu(\vec{y} - \vec{x})$ is a metric.

Proof:

1. Positive definition

$$\rho(\vec{x}, \vec{y}) = \nu(\vec{y} - \vec{x}) > 0,$$
$$\rho(\vec{x}, \vec{x}) = \nu(\vec{x} - \vec{x}) = \nu\left(\vec{0}\right) \overset{\text{Lemma}}{=} 0.$$

2. Symmetric

$$\begin{aligned}\rho(\vec{x},\vec{y}) &= \nu(\vec{y}-\vec{x})\\ &= |-1|\nu(\vec{x}-\vec{y})\\ &= \nu(\vec{x}-\vec{y})\\ &= \rho(\vec{y},\vec{x}).\end{aligned}$$

3. Triangle inequality

$$\begin{aligned}\rho(\vec{x},\vec{y})+\rho(\vec{y},\vec{z}) &= \nu(\vec{y}-\vec{x})+\nu(\vec{z}-\vec{y})\\ &\geq \nu(\vec{y}-\vec{x}+\vec{z}-\vec{y})\\ &= \nu(\vec{z}-\vec{x})\\ &= \rho(\vec{x},\vec{z}).\end{aligned}$$

□

Note

Any normed space is a metric space.

Definition: Manhattan norm (or Taxicab norm) $||\vec{x}||_1$

For vector space $V = \mathbb{R}^n, \mathbb{C}^n$, we could define Manhattan norm (or Taxicab norm)

$$||\vec{x}||_1 := \sum_{i=1}^{n} |x_i|.$$

Proof: Let's prove that function $||x||_1$ is a norm.

1. Positive definite property: Let $x \in \mathbb{R}^n$ or $x \in \mathbb{C}^n$. Obviously $||x||_1 \geq 0$. Also $||x||_1 = 0$ iff $x = 0$.

2. Homogeneity property

$$\forall c \in \mathbb{R}: \ ||c \cdot x||_1 = \sum_{i=1}^{n} |c \cdot x_i| = |c| \cdot \sum_{i=1}^{n} |x_i| = |c| \cdot ||x||_1.$$

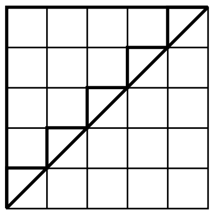

Figure 3: Geometric interpretation of Manhattan norm $||\vec{x}||_1$

3. Triangle inequality $\forall x, y \in \mathbb{R}^n$:

$$||x+y||_1 = \sum_{i=1}^{n} |x_i+y_i| \leq \sum_{i=1}^{n} (|x_i|+|y_i|) = \sum_{i=1}^{n} |x_i| + \sum_{i=1}^{n} |y_i| = ||x||_1 + ||y||_1.$$

□

Definition: Maximum norm (or Infinity norm) $||\vec{x}||_\infty$

For vector space $V = \mathbb{R}^n, \mathbb{C}^n$, we could define Maximum norm (or Infinity norm)

$$||\vec{x}||_\infty = \max_{1\le i\le n} |x_i|.$$

Proof: Let's prove that function $||\vec{x}||_\infty$ is a norm.

1. The function $||x||_\infty$ is positive since it is the maximum over a set of positive terms $|x_i|$.
2. Homogeneity property

$$||\alpha \cdot x||_\infty = \max_{1\le i\le n} |\alpha \cdot x_i| = \max_{1\le i\le n} |\alpha| \cdot |x_i| = |\alpha| \cdot \max_{1\le i\le n} = |\alpha| \cdot ||x||_\infty.$$

3. Triangle inequality

$$||x + y||_\infty = \max_{1\le i\le n} |x_i + y_i| \le \max_{1\le i\le n} (|x_i| + |y_i|) \le \max_{1\le i\le n} |x_i| + \max_{1\le i\le n} |y_i| = ||x||_\infty + ||y||_\infty.$$

□

Definition: Minkovskiy p−norm $||\vec{x}||_p$

For vector space of continuous functions $V = \mathbb{R}^n, \mathbb{C}^n$, we could define Minkovskiy p−norm

$$||\vec{x}||_p = \sqrt[p]{\sum_{i=1}^{n} |x_i|^p}, \quad p \ge 1.$$

The notations $||\cdot||_1, ||\cdot||_2$ and $||\cdot||_\infty$ are justified because of the fact that all these norms are special cases of the general Minkovskiy p−norm.

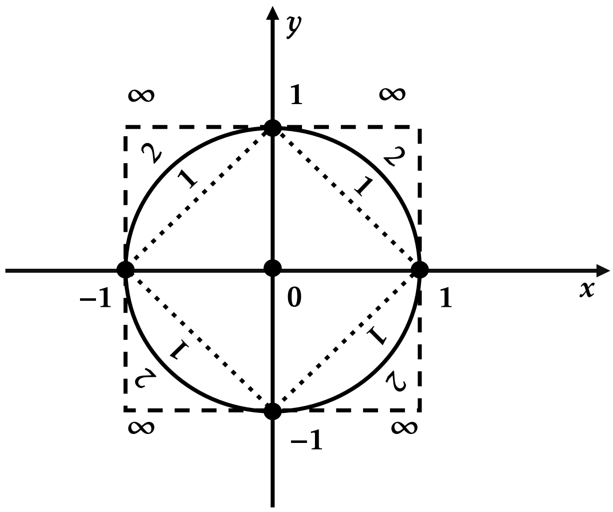

Figure 4: Unit balls for norms $||\cdot||_1, ||\cdot||_2$ and $||\cdot||_\infty$

Example 11: For the vector $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 5 \end{bmatrix}$, we have

$$||\vec{x}||_1 = 11, \quad ||\vec{x}||_2 = \sqrt{39}, \quad ||\vec{x}||_\infty = 5.$$

For the vector $\vec{y} = \begin{bmatrix} 1+i \\ 2-3i \\ 4 \end{bmatrix}$, we have

$$||\vec{y}||_1 = \sqrt{2} + \sqrt{13} + 4, \quad ||\vec{y}||_2 = \sqrt{31}, \quad ||\vec{y}||_\infty = 4.$$

Definition: p–norm $||\vec{x}||_p$ for $\mathcal{C}[a,b]$

For vector space $V = \mathcal{C}[a,b]$, we could define norms

$$||f||_1 = \int_a^b |f(x)|dx,$$
$$||f||_2 = \sqrt{\int_a^b f^2(x)dx},$$
$$||f||_\infty = \max_{x\in[a,b]} |f(x)|,$$
$$||f||_p = \sqrt[p]{\int_a^b |f(x)|^p dx}.$$

Example 12: For $V = \mathcal{C}[0,1]$, we have

$$||f||_p = \sqrt[p]{\int_0^1 |f(x)|^p dx},$$
$$||f||_1 = \int_0^1 |f(x)|dx,$$
$$||f||_\infty = ||f||_0 = \max_{x\in[0,1]} |f(x)|.$$

Definition: Weighted norms for $\mathcal{C}[0,1]$

For vector space $V = \mathcal{C}[0,1]$ and $\omega \geq 0$, we could define weighted norms

$$||f||_p^\omega = \sqrt[p]{\int_0^1 |f(x)|^p \cdot \omega(x)dx},$$

$$||f||_\infty^\omega = ||f||_0^\omega = \max_{x\in[0,1]} |f(x) \cdot \omega(x)|.$$

Balls in normed space

Let V be a normed vector space. Let $\rho(x,y) := \nu(x-y)$ be a metric induced by norm ν on V. Then,

$$B_R(c) = \{x \mid \rho(x,c) \leq R\} = \{x \mid \nu(x-c) \leq R\}$$

Prop

Let V be a normed space.

1. Any two balls $B_R(\vec{x})$ and $B_R(\vec{y})$, are congruent (same geometric figures). That is, there is a parallel translation $\vec{x} \mapsto \vec{x} + \vec{v}$, which maps one ball onto another. Specifically, we mean that

$$B_R(\vec{x}) + \vec{v} = B_R(\vec{y}).$$

2. For any two balls $B_R(\vec{c})$ and $B_r(\vec{c})$, there is a homothety $\vec{x} \mapsto \lambda\vec{x}$, which transfers one ball onto another. Specifically, we mean that

$$B_R(\vec{c}) = \lambda B_r(\vec{c}).$$

Proof:

1. Let $v = \vec{y} - \vec{x}$, then we have

$$\begin{aligned}
B_R(\vec{x}) + \vec{v} &= \{\vec{a} \mid \nu(\vec{a} - \vec{x}) \leq R\} + (\vec{y} - \vec{x}) \\
&= \left\{\vec{a} + \vec{y} - \vec{x} \mid \nu(\vec{a} - \vec{x}) \leq R\right\} \\
&= \left\{\vec{b} \mid \nu\left(\vec{b} - \vec{y}\right) \leq R\right\} \\
&= B_R(\vec{y}),
\end{aligned}$$

where $\vec{y} = \vec{b} - \vec{a} - \vec{x}$.

2. Let $\lambda = \dfrac{R}{r}$, then we have

$$\begin{aligned}
\lambda B_r(\vec{c}) &= \frac{R}{r} \cdot \{\vec{a} \mid \nu(\vec{a} - \vec{c}) \leq r\} \\
&= \left\{\frac{R}{r} \cdot \vec{a} \mid \nu(\vec{a}) \leq r\right\} \\
&= \left\{\vec{b} \mid \nu\left(\frac{r}{R} \cdot \vec{b}\right) \leq r\right\} \\
&= \left\{\vec{b} \mid \frac{r}{R} \cdot \nu\left(\vec{b}\right) \leq r\right\}
\end{aligned}$$

$$= \left\{\vec{b} \mid \nu\left(\vec{b}\right) \le \frac{r \cdot R}{r}\right\}$$
$$= \left\{\vec{b} \mid \nu\left(\vec{b}\right) \le R\right\}$$
$$= B_R(\vec{c}).$$

□

Definition: Inner product

An inner product on a real vector space V is a function

$$(\cdot,\cdot) : V \times V \to \mathbb{R},$$

such that $\forall \vec{x}, \vec{y}, \vec{z} \in V, \alpha, \beta \in \mathbb{R}$

1. Symmetric
$$(\vec{x}, \vec{y}) = (\vec{y}, \vec{x}).$$
2. Linear
$$(\alpha\vec{x} + \beta\vec{y}, \vec{z}) = \alpha(\vec{x}, \vec{z}) + \beta(\vec{y}, \vec{z}).$$
3. Positive-definite
$$(\vec{x}, \vec{x}) > 0, \quad \vec{x} \neq \vec{0}.$$

Theorem: Inner product and norm

1. If norm $||\cdot||$ is Euclidean, then parallelogram rule holds. That is,
$$2||\vec{a}||^2 + 2||\vec{b}||^2 = ||\vec{a} + \vec{b}||^2 + ||\vec{a} - \vec{b}||^2.$$
2. If parallelogram rule holds, then a norm $||\cdot||$ is Euclidean. That is,
$$(\vec{a}, \vec{b}) = \frac{||\vec{a} + \vec{b}||^2 - ||\vec{a}||^2 - ||\vec{b}||^2}{2},$$
defines inner product on vector space V.

Example 13: Let us consider vector space $\mathbb{R}^2$. Let us define following norm $||\cdot||$ on $\mathbb{R}^2$

$$\left\|\begin{bmatrix} x \\ y \end{bmatrix}\right\| = \sqrt{x^2 - 2xy + 5y^2}.$$

Then corresponding inner product $(\vec{a}, \vec{b})$ of arbitrary vectors $\vec{a} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \vec{b} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \in \mathbb{R}^2$ is the following

$$(\vec{a}, \vec{b}) = \frac{||\vec{a} + \vec{b}||^2 - ||\vec{a}||^2 - ||\vec{b}||^2}{2}$$
$$= \frac{(x_1 + x_2)^2 - 2(x_1 + x_2)(y_1 + y_2) + 5(y_1 + y_2)^2}{2} - \frac{x_1^2 - 2x_1y_1 + 5y_1^2}{2} - \frac{x_2^2 - 2x_2y_2 + 5y_2^2}{2}$$
$$= x_1x_2 - x_1y_2 - x_2y_1 + 10y_1y_2.$$