1. Approximate systems

Consider a system of linear equation $Ax = b$ and it's approximate version $\hat{A}\hat{x} = \hat{b}$ such that

$$\Delta A = \hat{A} - A \approx 0, \quad \Delta b = \hat{b} - b \approx 0, \quad \Delta x = \hat{x} - x \approx 0.$$

Definition: Absolute errors

Let $|\cdot|$ be compatible norm to matrix norm $||\cdot||$, then absolute errors are

$$||\Delta A||, \quad |\Delta b|, \quad |\Delta x|.$$

Definition: Relative errors

Let $|\cdot|$ be compatible norm to matrix norm $||\cdot||$, then relative errors are

$$\delta A = \frac{||\Delta A||}{||A||}, \quad \delta b = \frac{|\Delta b|}{|b|}, \quad \delta x = \frac{|\Delta x|}{|x|}.$$

In many applications system A and b with small δx is considered to be "good"system and "bad"system otherwise. So, naturally emerges following problem. Given bounds of δA and δb find bound for δx.

Example 1: Let us give an example of a good system. Consider

$$\begin{cases} x + 0.99y = 1 \\ x - 1.01y = 0.99 \end{cases}, \quad A = \begin{bmatrix} 1 & 0.99 \\ 1 & -1.01 \end{bmatrix}, \quad b = \begin{bmatrix} 1.01 \\ 0.99 \end{bmatrix}.$$

The system has following solution

$$x = \begin{bmatrix} 1.0001 \\ 0.001 \end{bmatrix}.$$

Now let us round the system and find solution of the rounded (approximate) system

$$\hat{A} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \hat{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The approximate system has following solution

$$\hat{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

We see that in the considered example little change in the coefficients makes the little change in the solution (i.e. $x \approx \hat{x}$). It means that the system is good.

Example 2: Let us give an example of bad system. Consider

$$\begin{cases} x + 0.99y = 1 \\ x + 1.01y = 0.99 \end{cases}, \quad A = \begin{bmatrix} 1 & 0.99 \\ 1 & 1.01 \end{bmatrix}, \quad b = \begin{bmatrix} 1.01 \\ 0.99 \end{bmatrix}.$$

The system has following solution

$$x = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Now let us round system and find solution of the rounded (approximate) system. For example, we could consider following rounding

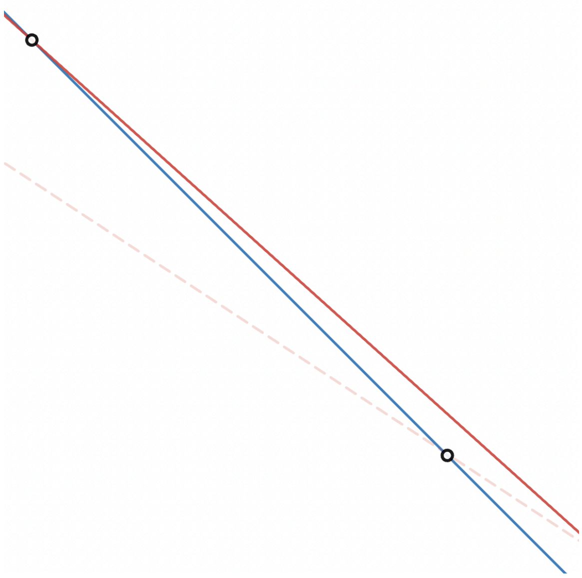
$$A = \begin{bmatrix} 1 & 0.99 \\ 1 & 1.01 \end{bmatrix}, \quad \hat{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The approximate system does not have solution.

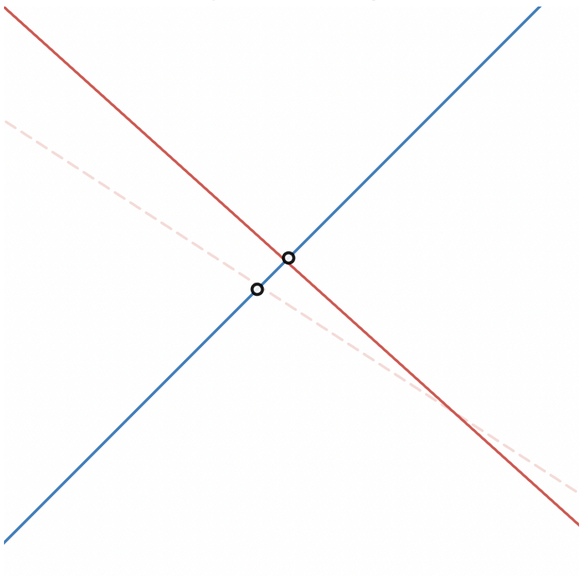
Alternatively we could consider

$$\hat{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \hat{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The approximate system has infinitely many solution.



Example of bad system



Example of good system

Definition: Condition number

A condition number of a matrix A for a norm $||\cdot||$ is

$$\kappa(A) = ||A|| \cdot ||A^{-1}||$$

Theorem: Bounds of δx (Special case)

If $A = \widehat{A}$, $b \neq \widehat{b}$ and A is not degenerate, then

$$\frac{1}{\kappa(A)}\delta b \leq \delta x \leq \kappa(A)\delta b.$$

Proof:

$$\begin{cases} Ax = b \\ A\widehat{x} = \widehat{b} \end{cases} \Leftrightarrow \begin{cases} Ax = b \\ A\Delta x = \Delta b \end{cases} \Leftrightarrow \begin{cases} x = A^{-1}b \\ \Delta x = A^{-1}\Delta b \end{cases},$$

which means

$$\begin{cases} |x| \geqslant \dfrac{|b|}{||A||} \\ |\Delta x| \geqslant \dfrac{|\Delta b|}{||A||} \end{cases} \Leftrightarrow \begin{cases} |x| \leq ||A^{-1}|| \cdot |b| \\ |\Delta x| \leq ||A^{-1}|| \cdot |\Delta b| \end{cases}.$$

Hence

$$\delta x = \frac{|\Delta x|}{|x|} \leq ||A^{-1}|| \cdot |\Delta b| \cdot \frac{||A||}{|b|} = \kappa(A)\delta b,$$

$$\delta x = \frac{|\Delta x|}{|x|} \geqslant \frac{|\Delta b|}{||A||} \cdot \frac{1}{||A^{-1}|| \cdot |b|} = \frac{1}{\kappa(A)}\delta b.$$

□

Theorem: Upper bound of δx (General case)

If $\widehat{A} \neq A$ and $\widehat{b} \neq b$, then

$$\delta x \leq \frac{\kappa(A)}{1 - \kappa(A)\delta A}(\delta A + \delta b).$$

Prop: Properties of $\kappa(A)$

1. $\kappa(A) \geqslant 1$
2. $\kappa(A) = \kappa(A^{-1})$
3. $\kappa(AB) \leq \kappa(A)\kappa(B)$
4. For a matrix A and matrix norm $||\cdot||_2$, we have $\kappa(A) = \dfrac{\sigma_1}{\sigma_n}$
5. For a matrix A and an arbitrary matrix norm $||\cdot||$, we have $\kappa(A) \geqslant \dfrac{|\lambda_1|}{|\lambda_n|}$

Theorem: Upper bound of δA^{-1}

Suppose $\widehat{A}$ and A are not degenerate, then

$$\delta A^{-1} \leq \frac{\kappa(A)}{1 - \kappa(A)\delta A}\delta A.$$