1. Spectral radius

Definition: Spectral radius

A spectral radius $\rho(A)$ of a matrix $A \in Mat_{n\times n}$ is maximal eigenvalue of matrix A

$$\rho(A) := \max_i(|\lambda_i(A)|) = |\lambda_1|,$$

where

$$|\lambda_1| \geqslant |\lambda_2| \geqslant \cdots \geqslant |\lambda_n| \geqslant 0.$$

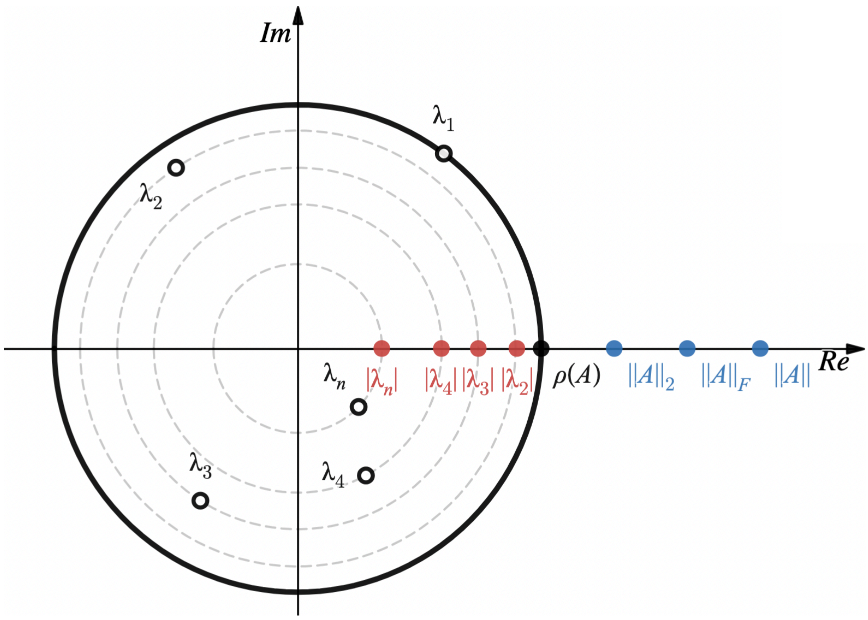

Illustration of $\rho(A)$ with respect to matrix norms, singular values and eigenvalues of matrix A

Prop: 1

For any matrix norm $||\cdot||$, we have

$$\rho(A) \leq ||A||.$$

Example 1: For following matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix},$$

we have $\rho(A) = \lambda_1 \approx 16.116$ and $||A||_2 = 16.84$, which illustrates the fact that

$$\rho(A) \leq ||A||_2.$$

Prop: 2

For any matrix A and $\epsilon > 0$ there is a matrix norm $||\cdot||$ such that

$$\rho(A) \leq ||A|| \leq \rho(A) + \epsilon.$$

Note

Spectral radius $\rho(A)$ is not a matrix norm. Consider, for example

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \rho(A) = 0.$$

Definition: Singular radius

A singular radius $\sigma(A)$ of a matrix A is the maximal singular value of A

$$\sigma(A) := \sigma_1,$$

where

$$|\lambda_1| \geqslant |\lambda_2| \geqslant \cdots \geqslant |\lambda_n| \geqslant 0.$$

Note

For any matrix A, we have

$$|A||_2 = \sigma_1 \geqslant \lambda_1 = \rho(A).$$

Prop: 3

Suppose a matrix A is symmetric Hermitian matrix (i.e. $A = A^*$), then

$$\rho(A) = \sigma(A) = ||A||_2.$$

Note

For any matrix A symmetric Hermitian matrix and any matrix norm $||\cdot||$, we have

$$||A||_2 \leq ||A||.$$

Example 2: Consider symmetric matrix

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

We have

$$\chi_A(\lambda) = \lambda^2 - 5\lambda - 1, \quad \lambda_{1,2} = 2 \pm \sqrt{5}$$

Then

$$\begin{aligned}
||A||_1 &= \max(3, 5) = 5, \\
||A||_\infty &= 5, \\
||A||_F &= \sqrt{1^2 + 2^2 + 2^2 + 3^2} = \sqrt{18} \approx 4.242, \\
||A||_2 &= \sigma_1 = 2 + \sqrt{5} \approx 4.236, \\
\rho(A) &= \lambda_1 = 2 + \sqrt{5} \approx 4.236.
\end{aligned}$$

2. Low rank approximation

Problem 1. Low Rank approximation (Informal)

Given matrix $A \in Mat_{n\times n}$ and $r > 0$, find a matrix $X \approx A$ such that $\operatorname{rank} X \leq \operatorname{rank} A$.

Problem 2. Low rank approximation (Formal)

Given matrix $A \in Mat_{m\times n}$ and $r > 0$ and a matrix norm $||\cdot||$. Find matrix X such that

$$||X - A|| \to \min, \quad \operatorname{rank} X \leq \operatorname{rank} A.$$

Theorem: (Eckart-Young-Mirsky)

Suppose that $||\cdot||$ is a unitary invariant matrix norm (for example, $||A||_2$ or $||A||_F$). Let us denote SVD of a matrix A

$$A = U\Sigma V^T, \quad \Sigma = \operatorname{diag}(\sigma_1, \sigma_2, \dots, \sigma_n), \quad \sigma_1 \geqslant \sigma_2 \geqslant \dots \geqslant \sigma_n.$$

Then following matrix X is a solution of the Low rank approximation problem

$$X = U_k \Sigma_k V_k^T,$$

where

$$U_k = \begin{pmatrix} | & | & \cdots & | \\ u^1 & u^2 & \cdots & u^k \\ | & | & \cdots & | \end{pmatrix}, \quad \Sigma_k = \operatorname{diag}(\sigma_1, \sigma_2, \dots, \sigma_k), \quad V_k = \begin{pmatrix} | & | & \cdots & | \\ v^1 & v^2 & \cdots & v^k \\ | & | & \cdots & | \end{pmatrix}.$$

Example 3: Let us find Low rank approximation of the following matrix

$$A = \begin{pmatrix} 6 & 0 & 6 \\ 0 & 12 & 6 \\ 6 & 6 & 9 \end{pmatrix}.$$

Let us find $\chi_A(\lambda)$

$$\chi_A(\lambda) = -\lambda(\lambda - 9)(\lambda - 18),$$

which means that

$$\Sigma = \operatorname{diag}(18, 9, 0).$$

Since $\operatorname{rank} A = 2$, we have following low rank approximations

$$A_3 = A_2 = A.$$

Now let us find rank one approximation A_1. Let us find normed eigenvector $\vec{v}_1$ of eigenvalue $\lambda_1 = 18$

$$(A - 18E)\vec{v}_1 = 0 \Leftrightarrow \vec{v}_1 = \frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

Since $X^T = X$, normed vector column of the matrix V is the following

$$\vec{u}_1 = \vec{v}_1,$$

which by Eckart-Young-Mirsky Theorem

$$A_1 = \vec{u}_1 \Sigma_1 \vec{u}_1^T = \frac{1}{3}\begin{pmatrix}1\\2\\2\end{pmatrix}\begin{pmatrix}18\end{pmatrix}\frac{1}{3}\begin{pmatrix}1 & 2 & 2\end{pmatrix} = \begin{pmatrix}2 & 4 & 4\\4 & 8 & 8\\4 & 8 & 8\end{pmatrix}.$$