Wavelets and Other Adaptive Methods

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Introduction

- A new **extending** mathematical tools.
- For studying the **local behavior**.
- A description of **two basic functions**.
- Infinite collection of **translated** and **scaled** versions.

Question...

Why we need **wavelets**?

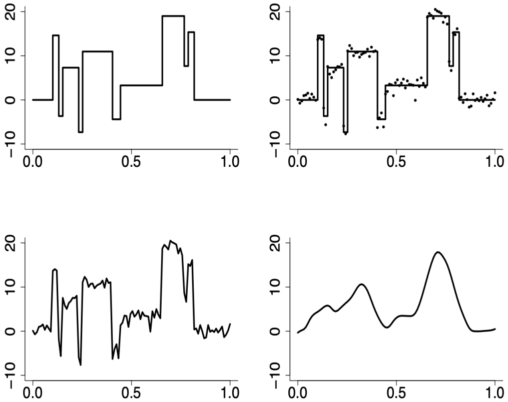

Figure: The blocks function (upper left plot) is inhomogenous. The upper right plot shows One-hundred data points. The lower left plot picks up the jumps but adds many wiggles. But the lower right represents smoothness with misses the jumps.

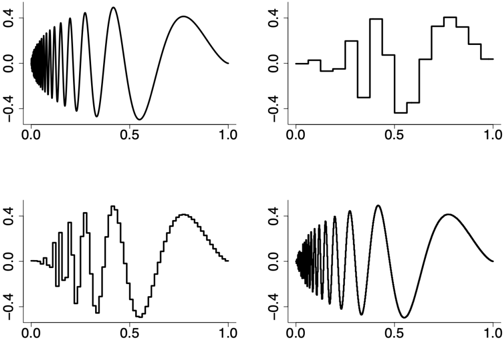

Figure: The Doppler signal (top left) and its construction $f_J(x) = \alpha\phi(x) + \sum_{j=0}^{J-1}\sum_k \beta_{jk}\psi_{jk}(x)$ based on J=3 (top right), J=5 (bottom left) and J=8 (bottom right).

Haar father wavelet

The **Harr father wavelet** or **Haar scaling function** is defined by

$$\phi(x) = \begin{cases} 1 & 0 \leq x < 1 \\ 0 & otherwise \end{cases}$$

Haar father wavelet

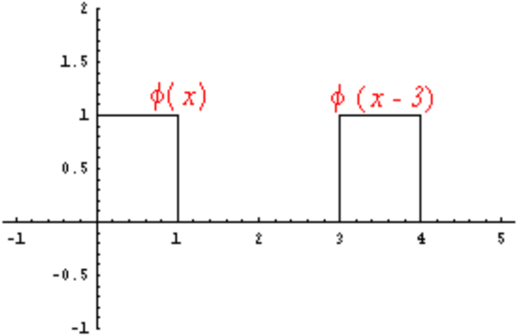

Figure: $\phi(x)$ and its translates

Haar father wavelet

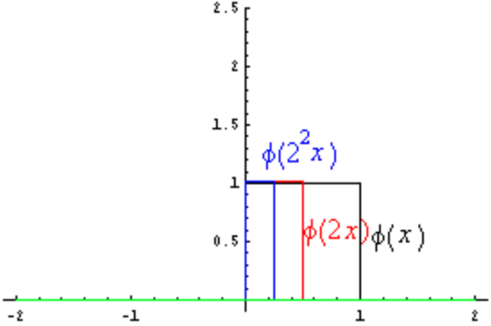

Figure: $\phi(x)$, $\phi(2x)$, and $\phi(2^2x)$

Mother Haar wavelet

The **mother Haar wavelet** is defined by

$$\psi(x) = \begin{cases} -1 & 0 \leq x \leq \frac{1}{2} \\ 1 & \frac{1}{2} < x \leq 1 \\ 0 & \textit{otherwise} \end{cases}$$

Mother Harr wavelet

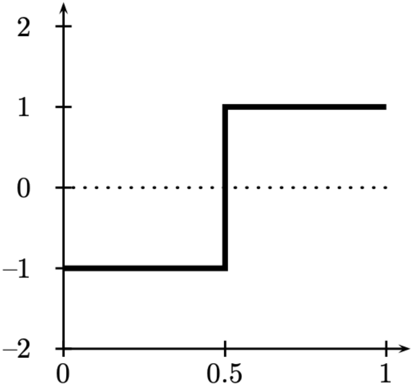

Figure: $\psi(x) = \psi_{0,0}(x)$

In general,

A wavelet system of $\boldsymbol{\phi(x)}$ and $\boldsymbol{\psi(x)}$ is defined by

$$\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k), \quad j, k \in \mathbb{Z}$$

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k), \quad j, k \in \mathbb{Z}$$

Mother Harr wavelet

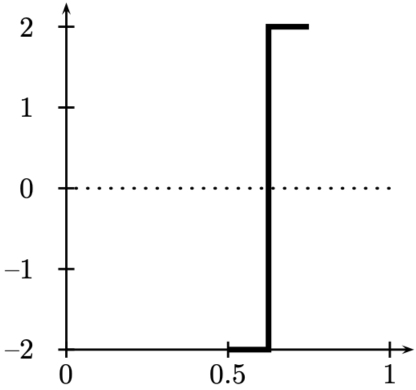

Figure: $\psi_{2,2}(x)$

Infinite collection of **translated** and **scaled** versions of Haar scaling function $\phi(x)$ and the details function $\psi(x)$.

Let

$$W_j = \{\psi_{jk}, \quad k = 0, 1, 2, ..., 2^j - 1\}$$

be the set of **rescaled** and **shifted** mother wavelets at resolution j.

Theorem

The set of functions

$$\{\phi, W_0, W_1, W_2, ..., \}$$

is an Orthonormal basis for $L_2(0,1)$.

$$f(x) = \alpha\phi(x) + \sum_{j=0}^{\infty}\sum_{k=0}^{2^j-1}\beta_{jk}\psi_{jk}(x)$$

We call the finite sum

$$f_J(x) = \alpha\phi(x) + \sum_{j=0}^{J-1}\sum_{k=0}^{2^j-1}\beta_{jk}\psi_{jk}(x)$$

Linear combinations of the $\phi(x-k)$

$$f(x) = 2\phi(x) + 3\phi(x-1) - 2\phi(x-2) + 4\phi(x-3)$$

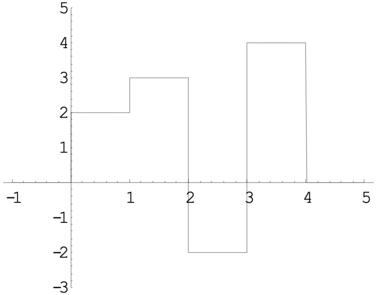

Figure: A linear combination of

$$\phi(x-k)$$

Given any function ϕ define

$$V_0 = \left\{ f : f(x) = \sum_{k \in \mathbb{Z}} c_k \phi(x-k), \sum_{k \in \mathbb{Z}} c_k^2 < \infty \right\}$$

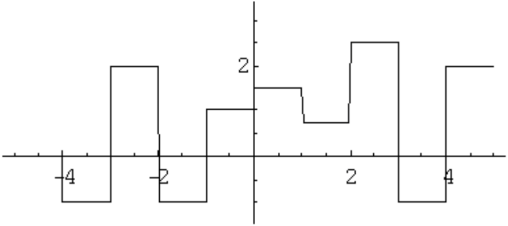

Figure: A function in V_0

$$V_1 = \{f(x) = g(2x) : g \in V_0\}$$

$$V_2 = \{f(x) = g(2x) : g \in V_1\}$$

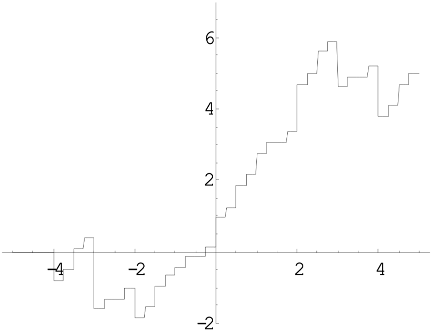

Figure: A function in V_2

In general, All square integrable functions which are constant on 2^{-j} length intervals

$$V_j = \left\{ f : f(x) = \sum_{k \in \mathbb{Z}} c_k \phi(2^j x - k), \sum_{k \in \mathbb{Z}} c_k^2 < \infty \right\}$$

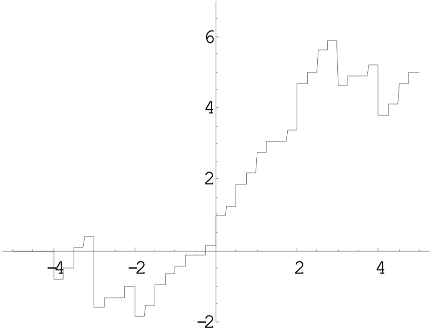

Figure: A function in $V_{j=2}$

$$...V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset V_3...$$

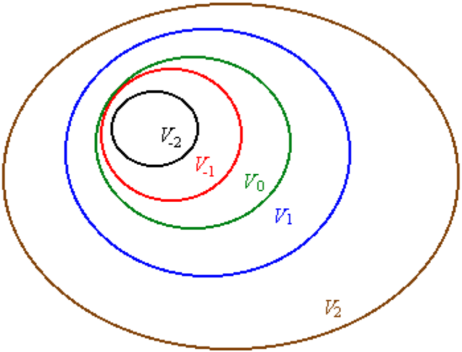

Figure: Relationship of nested Spaces V_j

Definition

Given a function ϕ and define V_0, V_1, V_2... We say that ϕ generates a **multiresolution analysis (MRA)** of $\mathbb{R}$ if

$$V_j \subset V_{j+1}, \quad j \geq 0$$

and

$$\bigcup_{j \geq 0} V_j \text{ is dense in } L_2(\mathbb{R})$$

We call ϕ the **father wavelet** or the **scaling function**.

Lemma

If $V_0, V_1, V_2...$ is an *MRA* generated by ϕ, then $\{\phi_{jk},\ k \in \mathbb{Z}\}$ is an orthonormal basis for V_j.

Theorem

Given coefficients $\{l_k,\ k \in \mathbb{Z}\}$, define a function

$$m_0(t) = \frac{1}{\sqrt{2}} \sum_k l_k e^{-itk}$$

Let

$$\phi^*(t) = \prod_{j=1}^{\infty} m_0\left(\frac{t}{2^j}\right)$$

and let ϕ be the inverse Fourier transform of ϕ^*.

Suppose that

$$\frac{1}{\sqrt{2}} \sum_{k=N_0}^{N_1} l_k = 1$$

for some $N_0 < N_1$, that

$$|m_0(t)|^2 + |m_0(t+\pi)|^2 = 1$$

$$|\phi(x)| \leq \Phi(|x|)$$

Then ϕ is a compactly supported **father wavelet** and ϕ is zero outside the interval $[N_0, N_1]$.

Question...

How can we construct W_k by using V_k?

Next, define W_k to be the orthogonal complement of V_k in V_{k+1}.

$$V_{k+1} = V_k \bigoplus W_k$$

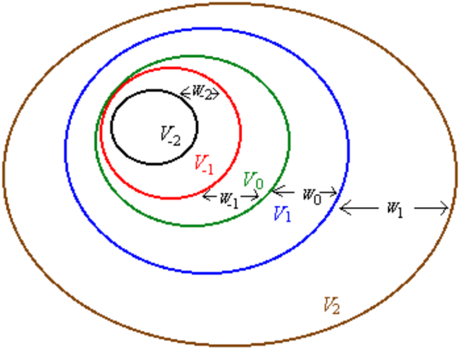

Figure: Relationship of V_j and W_j

In general

$$W_{j-1} = V_j \ominus V_{j-1}$$

Thus,

$$L_2(\mathbb{R}) = \overline{\bigcup_k V_k} = V_0 \bigoplus W_0 \bigoplus W_1 \bigoplus W_2 \ldots$$

Define the **mother wavelet** by

$$\psi(x) = \sqrt{2} \sum_k (-1)^{k+1} l_{1-k} \phi(2x - k)$$

Theorem

The functions $\{\psi_{jk},\ k \in \mathbb{Z}\}$ form a basis for W_j. The functions

$$\left\{\phi_k, \psi_{jk},\ k \in \mathbb{Z},\ j \in \mathbb{Z}_+\right\}$$

are an orthonormal basis for $L_2(\mathbb{R})$.

$$f(x) = \sum_k \alpha_{0k}\phi_{0k}(x) + \sum_{j=0}^{\infty}\sum_k \beta_{jk}\psi_{jk}(x)$$

Consider the regression problem

$$Y_i = r(x_i) + \sigma\epsilon_i$$

To estimate r using wavelets,

$$r(x) \approx r_n(x) = \sum_{k=0}^{2^{j_0}-1} \alpha_{j_0,k}\phi_{j_0,k}(x) + \sum_{j=j_0}^{J}\sum_{k=0}^{2^j-1} \beta_{jk}\psi_{jk}(x)$$

The **empirical scaling coefficients**

$$S_k = \frac{1}{n} \sum_i \phi_{j_0,k}(x_i) Y_i$$

and the **empirical detail coefficients**

$$D_{jk} = \frac{1}{n} \sum_i \psi_{jk}(x_i) Y_i$$

$$S_k \approx N(\alpha_{j_0,k} \, , \frac{\sigma^2}{n}) \text{ and } D_{jk} \approx N(\beta_{jk}, \frac{\sigma^2}{n})$$

For the scaling coefficients, we take

$$\hat{\alpha}_{j_0,k} = S_k$$

Finally, we put the estimates

$$\hat{r}_n(x) = \sum_{k=0}^{2^{j_0}-1} \hat{\alpha}_{j_0,k}\phi_{j_0,k}(x) + \sum_{j=j_0}^{J}\sum_{k=0}^{2^j-1} \hat{\beta}_{jk}\psi_{jk}(x)$$

What we have learned today?

- **Haar father** and **Mother haar** wavelets
- How to **construct** wavelets function.
- **wavelet regression**.
- Good tool because of **local behavior**.

Question

References

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