01.06.23

Теорема на Шварц

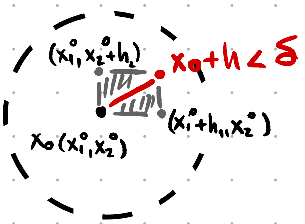
$U \subset \mathbb{R}^n$ отворено

$f: U \to \mathbb{R}$: $x_0 \in U$, $i,j \in \{1,\dots,n\}$, $i \neq j$ $\qquad \frac{df}{dx_i}, \frac{df}{dx_j}, \frac{d^2f}{dx_i dx_j}, \frac{d^2f}{dx_j dx_i}$ същ. в U

$\frac{d^2f}{dx_i dx_j}$ и $\frac{d^2f}{dx_j dx_i}$ са непрекъснати в x_0

Тогава $\frac{d^2f}{dx_i dx_j}(x_0) = \frac{d^2f}{dx_j dx_i}(x_0)$

Док: БОО $n=2$



$\delta > 0$ $\quad B_\delta(x_0) \subset U$ отворено, $h(h_1, h_2)$

$$\varphi(h_1,h_2) = f(x_1^0+h_1, x_2^0+h_2) + f(x_1^0, x_2^0) - f(x_1^0+h_1, x_2^0) - f(x_1^0, x_2^0+h_2)$$

$$a(t) = f(x_1^0+h_1, t) - f(x_1^0, t)$$

$$\varphi(h_1,h_2) = a(x_2^0+h_2) - a(x_2^0) \underset{\text{Лагранж}}{\overset{\text{Th.}}{=}} a'(x_2^0+\theta_1 . h_2).h_2 \qquad 0<\theta_1<1$$

Ако $t \in (-\delta, \delta)$ можем да диференцираме по t:

$$a'(t) = \frac{df}{dx_2}(x_1^0+h_1, t) - \frac{df}{dx_2}(x_1^0, t)$$

$$\varphi(h_1,h_2) = \left[\frac{df}{dx_2}(x_1^0+h_1, x_2^0+\theta_1 h_2) - \frac{df}{dx_2}(x_1^0, x_2^0+\theta_1 h_2)\right].h_2$$

t е между x_2^0 и x_2^0+h $\qquad \|(x_1^0+h_1, t) - (x_1^0, x_2^0)\| = \sqrt{h_1^2 + (t-x_2^0)^2} \leq \|h\| < \delta$

$$\alpha(t) := \frac{df}{dx_2}(t, x_2^0+\theta_1 h_2)$$

$$\alpha'(t) = \frac{d}{dx_1}\left(\frac{df}{dx_2}\left(t, x_2^0+\theta_1 h_2\right)\right) = \frac{d^2f}{dx_2 dx_1}(t, x_2^0+\theta_1 h_2)$$

$$\varphi(h_1,h_2) = (\alpha(x_1^0+h_1) - \alpha(x_1^0))h_2 \underset{\text{Лагранж}}{\overset{\text{Th.}}{=}} \alpha'(x_1^0+\theta_2 h_1)(\cancel{x_1^0}+h_1-\cancel{x_1^0})h_2$$

$$0<\theta_2<1$$

$$\varphi(h_1,h_2) = \frac{d^2f}{dx_2 dx_1}(x_1^0+\theta_2 h_1, x_2^0+\theta_1 h_2)h_1 h_2$$

$$b(t) = f(t, x_2^0+h_2) - f(t, x_2^0)$$

$$\varphi(h_1,h_2) = b(x_1^0+h_1) - b(x_1^0) = \underset{0<\theta_3<1}{b'}(x_1^0+\theta_3 h_1).h_1 = \left(\frac{df}{dx_1}(x_1^0+\theta_3 h_1, x_2^0+h_2) - \frac{df}{dx_1}(x_1^0+\theta_3 h_1, x_2^0)\right) h_1 =$$

$$= (\beta(x_2^0+h_2) - \beta(x_2^0)h_1 - \beta'(x_2^0+\theta_4 h_2).h_2 h_1) = \frac{d^2 f}{dx_1 x_2}(x_1^0+\theta_3 h_1, x_2^0+\theta_4 h_2)\, h_1 h_2$$

$$\beta(t) = \frac{df}{dx_1}(x_1^0+\theta_3 h_1, t)$$

$$\frac{d^2 f}{dx_2 dx_1}(x_1^0+\theta_2 h_1, x_2^0+\theta_1 h_2)\,\cancel{h_1 h_2} = \frac{d^2 f}{dx_1 x_2}(x_1^0+\theta_3 h_1, x_2^0+\theta_4 h_2)\,\cancel{h_1 h_2}$$

$$\|(x_1^0+\theta_2 h_1, x_2^0+\theta_1 h_2) - (x_1^0, x_2^0)\| = \sqrt{\theta_2^2 h_1^2 + \theta_1^2 h_2^2} < \|h\| \xrightarrow[h\to\theta]{} 0$$

От непрекъснатостта на $\frac{d^2 f}{dx_1 dx_2}$ и $\frac{d^2 f}{dx_2 dx_1}$ в x_0 следва, че: $\frac{d^2 f}{dx_1 dx_2}(x_0) = \frac{d^2 f}{dx_2 dx_1}(x_0)$

пример: $f(x,y) = \begin{cases} xy.\frac{x^2-y^2}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$ $\quad f: \mathbb{R}^2 \to \mathbb{R}$

$$f'_x(x,y) = y.\frac{x^2-y^2}{x^2+y^2} + xy.\frac{2x(x^2+y^2)-2x(x^2-y^2)}{(x^2+y^2)^2} = y.\frac{x^2-y^2}{x^2+y^2} + xy\frac{4xy^2}{(x^2+y^2)^2}$$

$$f'_x(0,0) = \lim_{t\to 0}\frac{f(t,0)-f(0,0)}{t} = \lim_{t\to 0}\frac{1}{t} = 0$$

$$f'_y(x,y) = x\frac{x^2-y^2}{x^2+y^2} + xy.\frac{-2y(x^2+y^2)-2y(x^2-y^2)}{(x^2+y^2)^2} = x.\frac{x^2-y^2}{x^2+y^2} + xy.\frac{-4x^2y}{(x^2+y^2)^2}$$

$$f'_y(0,0) = \lim_{t\to 0}\frac{f(0,t)-f(0,0)}{t} = 0$$

$$f''_{xy}(0,0) = \lim_{t\to 0}\frac{f'_x(0,t)-f'_x(0,0)}{t} = \lim_{t\to 0}\frac{1}{t}.t.\frac{-t^2}{t^2} = -1$$

$$f''_{yx}(0,0) = \lim_{t\to 0}\frac{f'_y(t,0)-f'_y(0,0)}{t} = \lim_{t\to 0}\frac{1}{t}.t.\frac{t^2-0^2}{t^2+0^2} = 1$$

Локални екстремуми на функции с повече от една променлива

$f: D \to \mathbb{R}$, $D \subset \mathbb{R}^n$, $x_0 \in D$

Казваме, че x_0 е точка на локален екстремум (мин/макс) за f, ако съществува $\delta > 0$ такова, че $B_\delta(x_0) \subset D$ и $f(x_0) \le f(x) / f(x_0) \ge f(x)$ $\forall x \in B_\delta(x_0)$

x_0 е строг локален минимум/максимум за f, ако съществува $\delta > 0$ такова, че

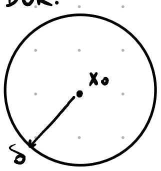
$B_\delta(x_0) \subset D$ и $f(x) > f(x_0)$ / $f(x) < f(x_0)$ $\forall x \in B_\delta(x_0) \setminus \{x_0\}$

Теорема на Ферма (НУ)

$f: D \to \mathbb{R}$, $D \subset \mathbb{R}^n$, $x_0 \in D$ точка на лок. екстремум за f

Нека $\frac{df}{dx_i}(x_0)$ $\forall i \in \{1, \dots, n\}$. Тогава $\frac{df}{dx_i}(x_0) = 0$ $\forall i \in \{1, \dots, n\}$ (x_0 е критична точка)

Док:



б.о.о. разгл. максимум

$B_\delta(x_0) \subset D$, $\varepsilon > 0$

$f(x_0) \geq f(x)$ $\forall x \in B_\delta(x_0)$

$\varphi_i(t) := f(x_0 + te_i)$, деф в $t \in (-\delta, \delta)$ $e_i = (0, \dots, 1, \dots 0)$ (1 на i-то място)

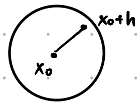
$\varphi_i(0) = f(x_0) \geq \varphi_i(t)$ $\forall t \in (-\delta, \delta)$ | Th. Ферма за $n=1$

$\varphi_i'(0) = \frac{df}{dx_i}(x_0)$ същ | $\frac{df}{dx_i}(x_0) = 0$ $i \in \{1, \dots, n\}$

Твърдение: $x_0 \in U \subset \mathbb{R}^n$, U отворено $f \in C^2(U)$.

Тогава $f(x_0 + h) = f(x_0) + \sum_{i=1}^{n} \frac{df}{dx_i}(x_0) h_i + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{d^2 f}{dx_i dx_j}(x_0 + \theta h) . h_i h_j$

Док: $\underset{\delta > 0}{U \supset B_\delta(x_0)}$ $\|h\| < \delta$ фикс. посока:



$\varphi(t) = f(x_0 + th)$ $t \in \underset{\text{отворен}}{\Delta} \supset [0,1]$

$\varphi'(t) = \langle \operatorname{grad} f(x_0 + th), h \rangle = \sum_{i=1}^{n} \frac{df}{dx_i}(x_0 + th) h_i$

$\varphi(t) = \frac{d}{dt}\left(f(x_1^0 + th_1, x_2^0 + th_2, \dots, x_n^0 + th_n) \right) = \frac{df}{dx_1}(x_0 + th) . h_1 + \frac{df}{dx_2}(x_0 + th) . h_2 + \dots + \frac{df}{dx_n}(x_0 + th) h_n$

$\varphi''(t) = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \frac{d}{dx_j} \left(\frac{df}{dx_i}(x_0 + th) . h_j \right) h_i \right) = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{d^2 f}{dx_i dx_j}(x_0 + th) h_i h_j$

$\varphi(t) = \varphi(0) + \varphi'(0) . 1 + \frac{\varphi''(\theta)}{2} . 1^2$

$0 < \theta < 1$

$f(x_0 + h) = f(x_0) + \sum_{i=1}^{n} \frac{df}{dx_i}(x_0) h_i + \frac{1}{2} \sum_{j,i=1}^{n} \frac{d^2 f}{dx_i dx_j}(x_0 + th) . h_i h_j$

def. Втори диференциал $d^2 f(x_0): \mathbb{R}^n \to \mathbb{R}$ – квадратична норма

$h \in \mathbb{R}^n \longmapsto d^2 f(x_0)(h) = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{d^2 f}{dx_i dx_j}(x_0) . h_i h_j =$

$$= \left\langle \underbrace{\begin{pmatrix} \frac{d^2 f(x_0)}{dx_1^2} & \frac{d^2 f(x_0)}{dx_1 dx_2} & \cdots & \frac{d^2 f(x_0)}{dx_1 dx_n} \\ \frac{d^2 f(x_0)}{dx_2 dx_1} & \frac{d^2 f(x_0)}{dx_2^2} & \cdots & \frac{d^2 f(x_0)}{dx_2 dx_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{d^2 f(x_0)}{dx_n dx_1} & \frac{d^2 f(x_0)}{dx_n dx_2} & \cdots & \frac{d^2 f(x_0)}{dx_n^2} \end{pmatrix}}_{\text{Хесиан}} \cdot h, h \right\rangle$$

симетрична от Шварц

$a(h) = \langle Ah, h \rangle = \sum_{i,j=1}^{n} a_{ij} h_i h_j \qquad A = (a_{ij})_{i,j=1}^{n}$ симетрична

1) a се нарича положително дефинитна $(a>0)$, ако $a(h)>0 \ \forall h \neq \theta$

2) a е отрицателно дефинитна, ако $a(h)<0 \ \forall h \neq \theta$

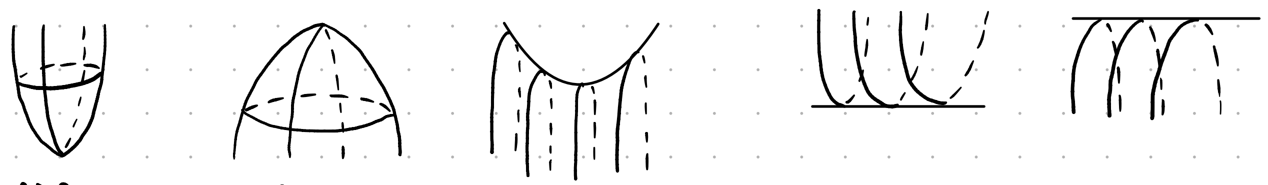
3) a не е дефинитна, ако приема както положителни, така и отрицателни стойности

4) a е положително полудефинитна, ако $a(h) \geq 0 \ \forall h \in \mathbb{R}^n$

5) a е отрицателно полудефинитна, ако $a(h) \leq 0 \ \forall h \in \mathbb{R}^n$

$h = (h_1, h_2)$

1) $h_1^2 + h_2^2$ 2) $-h_1^2 - h_2^2$ 3) $h_1^2 - h_2^2$ h_1^2



$a>0$ $a<0$ недеф

Критерий на Силвестър

A симетрична, $a(h) = \langle Ah, h \rangle$

$$A: \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$a>0 \iff a_{11}>0, \ \det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} > 0, \ldots, \det A > 0$

$a<0 \iff a_{11}<0, \ \det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} > 0, \ldots, \det A < 0$

Теорема

$x_0 \in U \subset \mathbb{R}^n$, U отворено $\quad f \in C^2(U)$

x_0 е критична точка, т.е. $\frac{df}{dx_i}(x_0) = 0 \quad \forall i \in \{1, \ldots, n\}$

1) Ако $d^2 f(x_0)$ е положително дефинитна квадратна форма, то x_0 е т. на строг локален минимум

2) Ако $d^2f(x_0)$ е отрицателно дефинитна квадратна форма, то x_0 е т. на строг локален максимум

3) Ако $d^2f(x_0)$ не е дефинитна квадратна форма, то x_0 не е точка на локален екстремум за f.

Док: 1): $f(x_0+h) = f(x_0) + \underbrace{\boxed{df(x_0)(h)}}_{=0} + \frac{1}{2} d^2f(x_0 + \theta_h h)(h)$ $\quad 0<\theta_h<1$

$$f(x_0+h) - f(x_0) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{d^2f}{dx_idx_j}(x_0+\theta_h h) h_i h_j = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{d^2f}{dx_idx_j}(x_0)h_ih_j + \frac{1}{2}\left(\sum_{i,j=1}^{n}\left(\frac{d^2f}{dx_idx_j}(x_0+\theta_h h) - \frac{d^2f}{dx_idx_j}(x_0)\right)h_ih_j\right) =$$

$$= \frac{1}{2}\|h\|^2\left(\underbrace{\sum_{i,j=1}^{n}\frac{d^2f}{dx_idx_j}(x_0)\cdot\frac{h_i}{\|h\|}\cdot\frac{h_j}{\|h\|}}_{d^2f(x_0)\left(\frac{h}{\|h\|}\right)} + \underbrace{\sum_{i,j=1}^{n}\left(\frac{d^2f}{dx_idx_j}(x_0+\theta_h h) - \frac{d^2f}{dx_idx_j}(x_0)\right)\frac{h_i}{\|h\|}\cdot\frac{h_j}{\|h\|}}_{<\frac{\alpha}{2}\ \forall h,\ \|h\|<\eta}\right)$$

$S := \{h\in\mathbb{R}^n,\ \|h\|=1\}$ $\qquad$ S компакт, $d^2f(x_0)$ непрекъсната (гладка)

$d^2f(x_0)(h)>0\ \forall h\in S$ $\qquad \Rightarrow d^2f(x_0)(\bar{h}) = \min_S d^2f(x) = \alpha > 0 \qquad \bar{h}\in S$

$$\left|\sum_{i,j=1}^{n}\left(\frac{d^2f}{dx_idx_j}(x_0+\theta_h h) - \frac{d^2f}{dx_idx_j}(x_0)\right)\frac{h_i}{\|h\|}\cdot\frac{h_j}{\|h\|}\right| \le \sum_{i,j=1}^{n}\left|\frac{d^2f}{dx_idx_j}(x_0+\theta_h h) - \frac{d^2f}{dx_idx_j}(x_0)\right| \xrightarrow[h\to 0]{} 0$$

Същ. $0<\eta\le\delta$ такова, че $\sum_{i,j=1}^{n}\left(\frac{d^2f}{dx_idx_j}(x_0+\theta_h h) - \frac{d^2f}{dx_idx_j}(x_0)\right)\frac{h_i}{\|h\|}\cdot\frac{h_j}{\|h\|} < \frac{\alpha}{2}$ $\quad\forall h,\ \|h\|<\eta$

$\Rightarrow f(x_0+h) - f(x_0) \ge \frac{1}{2}\|h\|^2\cdot\frac{\alpha}{2} \quad \forall h,\ \|h\|<\eta$

x_0 е строг локален минимум

3): Същ. $\bar{h}\in\mathbb{R}^n,\ d^2f(x_0)(\bar{h})>0$

$\tilde{\bar{h}}\in\mathbb{R}^n,\ d^2f(x_0)(\tilde{\bar{h}})<0$

$$f(x_0+t\bar{h}) = \frac{1}{2}d^2f(x_0)(t\bar{h}) + \frac{1}{2}\sum_{i,j=1}^{n}\left(\frac{d^2f}{dx_idx_j}(x_0+\theta_t t\bar{h}) - \frac{d^2f}{dx_idx_j}(x_0)\right)t\bar{h}_i\,t\bar{h}_j =$$

$$= \frac{1}{2}t^2\left(d^2f(x_0)(\bar{h}) + \frac{1}{2}\sum_{i,j=1}^{n}\left(\frac{d^2f}{dx_idx_j}(x_0+\theta_t t\bar{h}) - \frac{d^2f}{dx_idx_j}(x_0)\right)\bar{h}_i\bar{h}_j\right) > 0 \quad \forall t\in(-1,1)$$

$f(x_0+t\bar{h}) > f(x_0)\ \forall t\in(-1,1)$ $\qquad f(x_0+t\tilde{\bar{h}}) < f(x_0)\ \forall t\in(-\eta',\eta')$

пример

$f_1(x,y) = x^2+y^2$

$f_2(x,y) = x^2-y^2$

$d^2f_i(0,0)(h) = 2h_i^2$
$i=1,2$

Изследвайте за локални екстремуми $f(x,y) = x^3+y^3-3xy$

$f'_x = 3x^2-3y$ $\qquad x^2=y \qquad x^4=x \qquad x(x^3-1)=0 \qquad M_1(0,0)\ M_2(1,1)$

$f'_y = 3y^2-3x$ $\qquad y^2=x$

$f''_{xx} = 6x$

$f''_{xy} = -3$

$f''_{yy} = 6y$

$M_2 = \begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix}$ $\quad 6 > 0$, $\quad 36 - 9 > 0$

$d^2 f(M_2) > 0$

$\Rightarrow M_2$ строг лок. минимум

$M_1 \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}$ $\quad d^2(M_1)(h) = -6 h_1 h_2$

не е дефинитна

$\Rightarrow M_1$ не е т. на локален екстремум