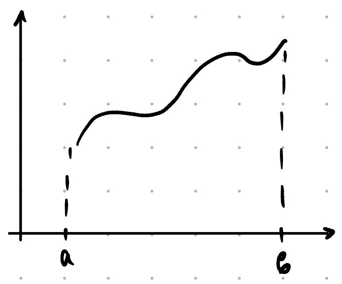
Несобствени интеграли



$f: [a,b] \to \mathbb{R}$ ограничена

Несобствен интеграл от I род

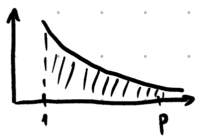
$f: [a,+\infty) \to \mathbb{R}$ f интегруема в $[a,p]$, $p \in [a,+\infty)$

Тогава $\int_a^{+\infty} f(x)\,dx$ е сходящ, ако $\exists \lim\limits_{p\to\infty} \int_a^p f(x)\,dx$, в прот случай е разходящ

$$\int_a^{+\infty} f(x)\,dx = \lim_{p\to\infty} \int_a^p f(x)\,dx$$

Аналог., ако $f: (-\infty,a] \to \mathbb{R}$, f интегруема в $[p,a]$, $p \in (-\infty,a]$, то $\int_{-\infty}^a f(x)\,dx = \lim\limits_{p\to-\infty} \int_p^a f(x)dx$

① $\int_1^{+\infty} \frac{1}{x}\,dx = \lim\limits_{p\to+\infty} \int_1^p \frac{1}{x}\,dx = \lim\limits_{p\to+\infty} \ln x \Big|_1^p = \lim\limits_{p\to+\infty} \ln p - \ln 1 = +\infty$ разх.



② $\int_2^{+\infty} \frac{1}{x\ln x}\,dx = \lim\limits_{p\to+\infty} \int_2^p \frac{1}{x\ln x}\,dx = \lim\limits_{p\to+\infty} \int_2^p \frac{1}{\ln x}\,d\ln x = \lim\limits_{p\to+\infty} \ln|\ln x| \Big|_2^p = \lim\limits_{p\to\infty} \ln|\ln p| - \ln|\ln 2| = +\infty$ разх.

③ $\int_0^{\infty} e^{-\alpha x}\,dx = \lim\limits_{p\to+\infty} \int_0^p e^{-\alpha x}\,dx \underset{\alpha\neq 0}{=\!=} \lim\limits_{p\to+\infty} \int_0^p \frac{e^{-\alpha x}}{-\alpha}\,d(-\alpha x) = \lim\limits_{p\to+\infty} -\frac{1}{\alpha} \cdot e^{-\alpha x} \Big|_0^p =$

$\alpha \in \mathbb{R}$

$$\lim_{p\to+\infty} -\frac{1}{\alpha} e^{-\alpha p} + \underbrace{\frac{1}{\alpha} e^0}_{= \frac{1}{\alpha}} = \begin{cases} \frac{1}{\alpha} & \alpha > 0 \quad \text{сходящ} \\ +\infty & \alpha < 0 \quad \text{разходящ} \end{cases}$$

$-\frac{1}{\alpha} = -\infty$ $e^{-\alpha p} = +\infty \Rightarrow -\frac{1}{\alpha} e^{-\alpha p} = 0$

$\alpha = 0$ $\lim\limits_{p\to+\infty} \int_0^p e^0\,dx = \lim\limits_{p\to+\infty} x \Big|_0^p = +\infty$ разходящ

④ $\int_0^{\pi/2} \frac{1}{2-\sin^2 x}\,dx = \int_0^{\pi/2} \frac{1}{\frac{2\cos^2 x + 2\sin^2 x - \sin^2 x}{\cos^2 x}} \cdot \frac{1}{\cos^2 x}\,dx = \int_0^{\pi/2} \frac{1}{2+\mathrm{tg}^2 x}\,d\,\mathrm{tg}\,x = \int_0^{+\infty} \frac{1}{2+t^2}\,dt =$

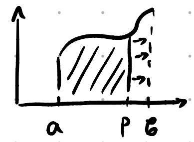
$t = \mathrm{tg}\,x$ $\quad x=0 \quad t=0$
$\quad x=\frac{\pi}{2} \quad t=+\infty$

$= \lim\limits_{p\to+\infty} \int_0^p \frac{1}{2+t^2}\,dt = \lim\limits_{p\to+\infty} \int_0^p \frac{1}{\cancel{2}\left(1+\left(\frac{t}{\sqrt{2}}\right)^2\right)} \frac{\cancel{\sqrt{2}}\,dt}{\sqrt{2}} = \lim\limits_{p\to+\infty} \frac{1}{\sqrt{2}} \cdot \mathrm{arctg}\,\frac{t}{\sqrt{2}} \Big|_0^p = \lim\limits_{p\to+\infty} \frac{1}{\sqrt{2}} \left(\mathrm{arctg}\,\frac{p}{\sqrt{2}} - \mathrm{arctg}\,0 \right) =$

$= \frac{\pi}{2\sqrt{2}}$ сходящ

Несобствен интеграл от II род

$f: [a,b) \to \mathbb{R}$ и f е интегруема в $[a,p]$, $p \in [a,b)$



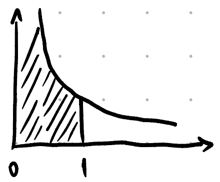
$\int_a^b f(x)\,dx$ е сходящ, ако $\exists \lim_{p\to b^-} \int_a^p f(x)\,dx$, в прот. случай е разх.

$$\int_a^b f(x)\,dx = \lim_{p\to b^-} \int_a^p f(x)\,dx$$

Аналог., ако $f: (a,b] \to \mathbb{R}$ е интегруема в $[p,b]$, $p \in (a,b]$, то

$$\int_a^b f(x)\,dx = \lim_{p\to a^+} \int_p^b f(x)\,dx$$

⑤ $\int_{\underline{\underline{0}}}^1 \frac{1}{x}\,dx = \lim_{p\to 0^+} \int_p^1 \frac{1}{x}\,dx = \lim_{p\to 0^+} \ln x \Big|_p^1 = \lim_{p\to 0^+} \underset{0}{\underset{\shortparallel}{\ln 1}} - \ln p = +\infty$ разходящ



⑥ $\int_0^{\underline{\underline{1}}} \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{p\to 1^-} \int_0^p \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{p\to 1^-} \arcsin x \Big|_0^p = \lim_{p\to 1^-} \arcsin p - \underset{0}{\underset{\shortparallel}{\arcsin 0}} = \frac{\pi}{2}$ сходящ

⑦ $\int_0^{\underline{\underline{1}}} \frac{1}{(2-x)\sqrt{1-x}}\,dx = \int_1^0 \frac{1}{(t^2+1)\cancel{t}} \cdot (-2\cancel{t})\,dt = \int_1^0 \frac{-2}{t^2+1}\,dt = -2\int_1^0 \frac{1}{1+t^2}\,dt = -2\,\mathrm{arctg}\,t \Big|_1^0 =$

$t = \sqrt{1-x}$ $\quad x=0 \quad t=1$

$t^2 = 1-x$ $\quad x=1 \quad t=0$

$x = 1-t^2 \quad dx = -2t\,dt$

$= -2\,\mathrm{arctg}\,0 + 2\,\mathrm{arctg}\,1 = \frac{\pi}{4}$ сходящ

⑧ $\int_{\underline{\underline{-1}}}^{\underline{\underline{1}}} \frac{x}{1-x^2}\,dx = \underbrace{\int_{\underline{\underline{-1}}}^0 \frac{x}{1-x^2}\,dx}_{I_1} + \underbrace{\int_0^1 \frac{x}{1-x^2}\,dx}_{I_2}$ $\quad I_2$ е разх $\Rightarrow I$ е разходящ

$I_2 = \int_0^{\underline{\underline{1}}} \frac{x}{1-x^2}\,dx = \lim_{p\to 1^-} \int_0^p \frac{x}{1-x^2}\,dx = \lim_{p\to 1^-} \int_0^p \frac{-1}{1-x^2}\,d(1-x^2) = \lim_{p\to 1^-} -\frac{1}{2} \ln|1-x^2| \Big|_0^p =$

$= \lim_{p\to 1^-} -\frac{1}{2} \underset{+\infty}{\underset{\shortparallel}{\ln|1-p|}} + \frac{1}{2} \underset{0}{\underset{\shortparallel}{\ln|1-0|}} = +\infty$

Критерии за сравнение

$f, g: [a, +\infty) \to \mathbb{R}$, f, g - интегруема в $[a, p]$, $p \in [a, +\infty)$

$0 \le f(x) \le g(x) \quad \forall x \in [a, +\infty)$

Тогава: 1) ако $\int_a^{+\infty} f(x)\,dx$ е разх., то $\int_a^{+\infty} g(x)\,dx$ е разх.

2) ако $\int_a^{+\infty} g(x)\,dx$ е сх., то $\int_a^{+\infty} f(x)\,dx$ е сх.

⑨ $\int_1^{+\infty} \frac{\cos^2 5x}{\sqrt{x^5+1}}\,dx$ сходящ ли е? $\quad 0 \le \frac{\cos^2 5x}{\sqrt{x^5+1}} \le \frac{1}{\sqrt{x^5}}$

$$\int_1^{+\infty} g(x)\,dx = \int_1^{+\infty} x^{-5/2}\,dx = \lim_{p \to +\infty} \int_1^p x^{-5/2}\,dx = \lim_{p\to+\infty} x^{-3/2} \cdot -\frac{2}{3}\Big|_1^p =$$

$$= \lim_{p\to+\infty} -\frac{2}{3}\left(\frac{1}{\sqrt{p^3}} - 1\right) = \frac{2}{3} \Rightarrow \text{сходящ} \Rightarrow \int_1^{+\infty} f(x)\,dx \text{ сх.}$$

⑩ сх? $\int_2^{+\infty} \frac{2-\sin^2 x}{\sqrt[3]{x^3-1}}\,dx$ $\qquad \int_2^{+\infty} f(x)\,dx = \int_2^{+\infty} \frac{1}{x}\,dx \to$ разходящ

$$0 \le \underbrace{\frac{1}{\sqrt[3]{x^2}}}_{f(x)} \le \frac{1}{\sqrt[3]{x^3-1}} \le \underbrace{\frac{2-\sin^2 x}{\sqrt[3]{x^3-1}}}_{g(x)} \qquad \Rightarrow \int_2^{+\infty} g(x)\,dx \text{ разходящ}$$

$0 \le \sin^2 x \le 1$

$1 \le 2 - \sin^2 x \le 2$

Следствие (гранична форма на критерий за сравн.)

$f, g: [a, +\infty) \to \mathbb{R}$, f, g - интегр. в $[a, p]$, $p \in [a, +\infty)$

$f(x) \ge 0$, $g(x) > 0 \quad \forall x \in [a, +\infty)$

Нека $l = \lim_{x \to +\infty} \frac{f(x)}{g(x)}$

1) Ако $l \in \mathbb{R}, l \ne 0$, то $\int_a^{+\infty} f(x)\,dx$ и $\int_a^{+\infty} g(x)\,dx$ са едновр. сх. или разх.

2) Ако $l = 0$ и $\int_a^{+\infty} g(x)\,dx$ е сх $\Rightarrow \int_a^{+\infty} f(x)\,dx$ е сх.

3) Ако $l = +\infty$ и $\int_a^{+\infty} f(x)\,dx$ е сх. $\Rightarrow \int_a^{+\infty} g(x)\,dx$ е сх.

$$\int_1^{+\infty} \frac{1}{x^\alpha}\,dx = \lim_{p\to+\infty} \int_1^p \frac{1}{x^\alpha}\,dx \underset{\alpha\neq 1}{=} \lim_{p\to+\infty} \frac{x^{-\alpha+1}}{-\alpha+1}\Big|_1^p =$$

$$= \lim_{p\to+\infty} \frac{1}{-\alpha+1}\cdot\frac{1}{p^{\alpha-1}} - \frac{1}{1-\alpha} = \begin{cases} -\dfrac{1}{1-\alpha} & \alpha-1>0 \iff \alpha>1 \text{ сх.} \\ +\infty & \alpha-1<0 \iff \alpha<1 \text{ разх.} \end{cases}$$

$\boxed{\int_1^{+\infty} \frac{1}{x^\alpha}\,dx \quad \text{сходящ } \alpha>1, \quad \text{разходящ } \alpha<1}$

$\boxed{\int_0^1 \frac{1}{x^\alpha}\,dx \quad \text{сходящ } \alpha<1, \quad \text{разходящ } \alpha>1}$

$$\int_{\underline{0}}^1 \frac{1}{x^\alpha} = \lim_{p\to 0^+} \frac{1}{1-\alpha}\cdot\frac{1}{x^{\alpha-1}}\Big|_p^1 = \lim_{p\to 0^+} \frac{1}{1-\alpha} - \frac{1}{1-\alpha}\cdot\frac{1}{p^{\alpha-1}} \begin{cases} +\infty & \alpha-1>0 \iff \alpha>1 \text{ разх.} \\ \dfrac{1}{1-\alpha} & \alpha-1<0 \iff \alpha<1 \text{ сх.} \end{cases}$$

$$\int_1^{+\infty} \frac{1}{(x-a)^\alpha}\,dx = \int_0^{b-a} \frac{1}{t^\alpha}\,dt \qquad \int_0^1 \frac{1}{(b-x)^\alpha}\,dx = -\int_a^0 \frac{1}{t^\alpha}\,dx = \int_0^a \frac{1}{t^\alpha}\,dx$$

$t = x-a \qquad\qquad t = b-x$

(11) Сходящи ли са?

А) $\int_{\underline{0}}^{\pi/2} \frac{\sin x}{x^2}\,dx \qquad f(x) = \frac{\sin x}{x^2} \qquad g(x) = \frac{1}{x^\alpha}$

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{\frac{\sin x}{x^2}}{\frac{1}{x^\alpha}} = \lim_{x\to 0} \frac{\sin x}{x^2}\cdot x^\alpha = \lim_{x\to 0} \underbrace{\frac{\sin x}{x}}_{=1}\cdot\frac{x^\alpha}{x} = \lim_{x\to 0} x^{\alpha-1} = 1 \text{ ако } \alpha=1$$

$$\int_0^{\pi/2} \frac{1}{x}\,dx \text{ е разх.} \Rightarrow \int_0^{\pi/2} \frac{\sin x}{x^2}\,dx \text{ е разх.}$$

Б) $\int_0^1 \frac{\sqrt{x}}{tg^2(2x)}\,dx \qquad f(x) = \frac{\sqrt{x}}{tg^2(2x)} \qquad g(x) = \frac{1}{x^\alpha}$

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \frac{\sqrt{x}}{tg^2(2x)}\cdot x^\alpha = \lim_{x\to 0} \frac{\sqrt{x}}{4x}\cdot x^\alpha = \frac{1}{4}\lim_{x\to 0} x^{\alpha-\frac{3}{2}} = \frac{1}{4}\cdot 1 \text{ при } \alpha = \frac{3}{2}$$

$$\int_0^1 f(x)\,dx \sim \int_0^1 \frac{1}{x^{3/2}}\,dx \qquad 3/2 > 1 \Rightarrow \text{разх.}$$

в) $\int_1^{+\infty} \frac{1}{\sqrt[3]{4x^2+\ln x}}\,dx \qquad f(x)=\frac{1}{\sqrt[3]{4x^2+\ln x}} \qquad g(x)=\frac{1}{x^{2/3}}$

$$\lim_{x\to+\infty}\frac{f(x)}{g(x)}=\lim_{x\to+\infty}\sqrt[3]{\frac{x^2}{4x^2+\ln x}}=\lim_{x\to+\infty}\sqrt[3]{\frac{\cancel{x^2}}{4\cancel{x^2}\left(1+\frac{\ln x}{x^2}\right)}}=\frac{1}{\sqrt[3]{4}}$$

$\int_1^{+\infty} f(x) \sim \int_1^{+\infty}\frac{1}{x^{2/3}} \qquad \frac{2}{3}<1 \Rightarrow$ разх.

г) $\int_{\underline{\underline{0}}}^{\pi/2} tg^7 x\,(x^{17}+1)\sin^3 x\,dx = \int_0^{\pi/2}\frac{\sin^7 x}{\cos^7 x}(x^{17}+1)\sin^3 x\,dx = -\int_{\pi/2}^{0}\frac{\sin^{10}(\pi/2-t)}{\cos^7(\pi/2-t)}\left(\left(\frac{\pi}{2}-t\right)^{17}+1\right)dt =$

$t=\frac{\pi}{2}-x \qquad x=0 \quad t=\frac{\pi}{2}$

$\qquad\qquad\qquad x=\frac{\pi}{2} \quad t=0$

$dt=-dx$

$$=\int_{\underline{\underline{0}}}^{\pi/2}\underbrace{\frac{\cos^{10}t}{\sin^7 t}\left[\left(\frac{\pi}{2}-t\right)^{17}+1\right]}_{f(t)}dt \qquad g(x)=\frac{1}{x^\alpha}$$

$$\lim_{x\to0^+}\frac{f(x)}{g(x)}=\lim_{x\to0^+}\frac{\cos^{10}x\left(\left(\pi/2-x\right)^{17}+1\right)}{\sin^7 x}\cdot x^\alpha=\frac{\pi}{2}^{17}+1 \quad \text{при } \alpha=7 \quad \left(\frac{x}{\sin x}\xrightarrow[x\to\infty]{}1\right)$$

$\int_0^{\pi/2} f(t)\,dt \sim \int_0^{\pi/2}\frac{1}{x^7}\,dx \qquad 7>1 \Rightarrow$ разх.

д) $\int_{\underline{\underline{0}}}^{\underline{\underline{+\infty}}}\frac{1}{x^3+\sqrt[4]{x}}\,dx = \underbrace{\int_{\underline{\underline{0}}}^{1}\frac{1}{x^3+\sqrt[4]{x}}}_{I_2}+\underbrace{\int_1^{\underline{\underline{+\infty}}}\frac{1}{x^3+\sqrt[4]{x}}}_{I_1}$

$\boxed{\begin{array}{l}\text{в } \infty \text{ за сравн. е водеща } \uparrow \text{ степен}\\ \text{в } 0 \text{ за сравн. е водеща } \downarrow \text{ степен}\end{array}}$

$I_1=\int_1^{\underline{\underline{+\infty}}}\frac{1}{x^3+\sqrt[4]{x}}\,dx \qquad g(x)=\frac{1}{x^3}$

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{1}{x^3+\sqrt[4]{x}}\cdot x^3=\lim_{x\to\infty}\frac{\cancel{x^3}}{\cancel{x^3}\Big(1+\underbrace{\frac{\sqrt[4]{x}}{x^3}}_{\to0}\Big)}=1$$

$I_1\sim\int_1^{+\infty}\frac{1}{x^3}\,dx \qquad 3>1$ сходящ

$I_2=\int_{\underline{\underline{0}}}^{1}\frac{1}{x^3+\sqrt[4]{x}} \qquad g(x)=\frac{1}{x^{1/4}}$

$$\lim_{x\to0^+}\frac{f(x)}{g(x)}=\lim_{x\to0^+}\frac{1}{x^3+\sqrt[4]{x}}\cdot\sqrt[4]{x}=\lim_{x\to0^+}\frac{\cancel{\sqrt[4]{x}}}{\cancel{\sqrt[4]{x}}\Big(\underbrace{\frac{x^3}{\sqrt[4]{x}}}_{\to0}+1\Big)}=1$$

$$I_2 \sim \int_0^1 \frac{1}{x^{1/4}}\,dx \qquad 1/4 < 1 \quad \text{сходящ}$$

$\Rightarrow I = I_1 + I_2$ е сходящ

(12) Изследвайте за сходимост в зав. от парам. $\lambda \in \mathbb{R}$ $\displaystyle\int_0^{+\infty} \frac{\operatorname{arctg} 2x}{x^{\lambda}}\,dx$

$$\int_0^{+\infty} \frac{\operatorname{arctg} 2x}{x^{\lambda}}\,dx = \underbrace{\int_0^{1} \frac{\operatorname{arctg} 2x}{x^{\lambda}}\,dx}_{I_2} + \underbrace{\int_1^{+\infty} \frac{\operatorname{arctg} 2x}{x^{\lambda}}\,dx}_{I_1}$$

$$I_1 = \int_1^{+\infty} \frac{\operatorname{arctg} 2x}{x^{\lambda}}\,dx \qquad g_1(x) = \frac{1}{x^{\alpha}}$$

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = \lim_{x\to+\infty} \frac{\overbrace{\operatorname{arctg} 2x}^{\to \frac{\pi}{2}}}{x^{\lambda}} \cdot x^{\alpha} = \lim_{x\to+\infty} \frac{\pi}{2}\,\frac{x^{\alpha}}{x^{\lambda}} = \frac{\pi}{2} \text{ за } \lambda = \alpha$$

$I_1 \sim \dfrac{1}{x^{\alpha}}$ — сходящ $\lambda > 1$, разходящ $\lambda \le 1$

$$I_2 = \int_0^{1} \frac{\operatorname{arctg} 2x}{x^{\lambda}}\,dx \qquad g_2(x) = \frac{1}{x^{\alpha}} = \frac{1}{x^{\lambda-1}}$$

$$\lim_{x\to 0^+} \frac{f(x)}{g(x)} = \lim_{x\to 0^+} \frac{\operatorname{arctg} 2x}{x^{\lambda}} \cdot x^{\alpha} = \lim_{x\to 0^+} \frac{2x \cdot x^{\alpha}}{x^{\lambda}} = \lim_{x\to 0^+} 2 \cdot \frac{x^{\alpha}}{x^{\lambda-1}} = 2, \text{ ако } \alpha = \lambda - 1$$

$I_2 \sim \displaystyle\int_0^1 \frac{1}{x^{\lambda-1}}\,dx$ — сходящ $\lambda < 2$, разходящ $\lambda \ge 2$

$I = I_1 + I_2$ е сходящ за $1 < \lambda < 2$, в прот. сл. разходящ