27.04.23

$$S(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$$ D – област на сходимост

1) R радиус на сходимост

$$(a-R, a+R) \subset D \subset [a-R, a+R] \qquad R = \frac{1}{\limsup \sqrt[n]{|a_n|}}$$

2) $S: D \longrightarrow \mathbb{R}$ (Абел)
непрекъсната

3) $\sum_{n=1}^{\infty} a_n \cdot n (x-a)^{n-1}$ има същия радиус на сходимост $\sum_{n=0}^{\infty} a_n (x-a)^n$

$$S'(x) = \sum_{n=1}^{\infty} a_n \cdot n (x-a)^{n-1}$$

$$\int S(x)\, dx = \sum_{n=0}^{\infty} a_n \cdot \frac{x^{n+1}}{n+1} + C$$

в $(a-R, a+R)$

Развитие на Тейлър

1) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

2) $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

3) $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

в $(-\infty, +\infty)$

4) $(1+x)^a = \sum_{n=0}^{\infty} \binom{a}{n} x^n \rightarrow (-1, 1)$

$a = -1$

$$\binom{-1}{n} = \frac{(-1)(-2)\dots(-1-n+1)}{n!} = (-1)^n \cdot \frac{\cancel{1.2.\dots n}}{\cancel{n!}} = (-1)^n$$

$$(1+x)^{-1} = \sum_{n=0}^{\infty} (-1)^n x^n \quad \forall x (-1, 1)$$

$$= \sum_{n=0}^{\infty} (-x)^n = \frac{1}{1-(-x)}$$

$a = \frac{1}{2}$

$$= \frac{1-2n+2}{2} = \frac{-2n+3}{2} = -\frac{1}{2}(2n-3)$$

$$\binom{\frac{1}{2}}{n} = \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})\dots(\frac{1}{2}-n+1)}{n!} = \frac{(-1)^{n-1}}{2^n} \frac{1.1.3.5\dots(2n-3)}{n!} =$$

$$= \frac{(-1)^{n-1}}{2^n} \frac{(2n-1)!!}{2^n . n!} = (-1)^{n-1} \frac{(2n-3)!!}{(2n)!!}$$

$$\sqrt{1+x} = \sum_{n=0}^{\infty} (-1)^n \frac{(2n-3)!!}{(2n)!!} x^n$$

$$\sum_{n=0}^{\infty} \frac{(2n-3)!!}{(2n)!!} \qquad x=1 \qquad x=-1$$

Даламбер: $\frac{(2n-1)!!}{(2n+2)!!} \cdot \frac{(2n)!!}{(2n-3)!!} = \frac{2n-1}{2n+2} \xrightarrow[n\to\infty]{} 1$ няма резултат

Раабе - Дюамел: $n.\left(\frac{2n+2}{2n-1} - 1\right) = n. \frac{\cancel{2n}+2-\cancel{2n}+1}{2n-1} = \frac{3n}{2n-1} \xrightarrow[n\to\infty]{} \frac{3}{2}$ сходящ

5) $\underbrace{\ln(1+x)}_{def:\, x\in(-1,+\infty)} = \underbrace{\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}}_{x\in(-1,1]}$, $x\in(-1,1]$

$$(\ln(1+x))' = \frac{1}{1+x} = (1+x)^{-1} \underset{x\in(-1,1)}{=} \sum_{n=0}^{\infty} (-1)^n . x^n$$

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n . \frac{x^{n+1}}{n+1} + C = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} + C \qquad x=0 \quad \ln(1+0) = 0 + C \to C=0$$

$\ln(1+x)$ непр. в 1, $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ е непр. в 1 (Абел)

f, g непр. в $(-1,1]$

$f(x) = g(x)$ $\forall x \in (-1,1]$

$$f(1) \underset{f.\text{ непр в } 1}{=} \lim_{\substack{x\to 1\\ x<1}} f(x) = \lim_{\substack{x\to 1\\ x<1}} g(x) \underset{g.\text{ непр в } 1}{=} g(1) =$$

6) $\underbrace{\operatorname{arctg} x}_{def.\ \mathbb{R}} = \underbrace{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C}_{def.\ [-1,1]} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$, $x\in(-1,1)$

$$(\operatorname{arctg} x)' = \frac{1}{1+x^2} = (1+x^2)^{-1} \underset{y=x^2}{=} (1+y)^{-1} \underset{y\in(-1,1)}{=} \sum_{n=0}^{\infty} (-1)^n (y)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$x=0 \qquad \operatorname{arctg} 0 = C = 0$

$$7)\ \arcsin x = \sum_{n=0}^{\infty} \cdot \frac{(2n-1)!!}{(2n)!!} \cdot \frac{x^{2n+1}}{2n+1} \quad \text{в } (-1,1)$$

$$(\arcsin x)' = \frac{1}{\sqrt{1+x^2}} = (1+x^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{-1/2}{n} \cdot x^{2n}$$

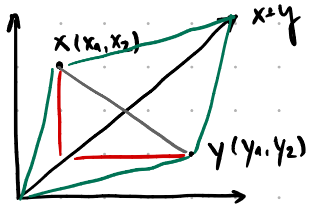
$$\binom{-1/2}{n} = \frac{(-\frac{1}{2})(-\frac{3}{2})\ldots(-\frac{1}{2}-n+1)}{n!} = \left(-\frac{1}{2}\right)^n \frac{(1.3.5.\ldots.(2n-1))}{n!} = \cdot\frac{(2n-1)!!}{(2n)!!}$$

$$\arcsin x = \sum_{n=0}^{\infty} \cdot \frac{(2n-1)!!}{(2n)!!} \cdot \frac{x^{2n+1}}{2n+1} + C$$

$x=0 \qquad \arcsin 0 = 0 + C \Rightarrow C=0$

$\mathbb{R}^n \{x=(x_1,x_2,\ldots,x_n) : x_i \in \mathbb{R}\ \forall i=\{1,\ldots,n\}\}$

$x \in \mathbb{R}^n$



$x=(x_1,x_2,\ldots,x_n)$

$y=(y_1,y_2,\ldots,y_n)$

$x+y=(x_1+y_1, x_2+y_2, \ldots, x_n+y_n)$

$\lambda \in \mathbb{R}$

$\lambda x = (\lambda x_1, \lambda x_2, \ldots, \lambda x_n)$

$$d(x,y) = \sqrt{\sum_{i=1}^{n}(x_i-y_i)^2} = \|y-x\|$$

евклидово разст. м/у x и y

произлиза от нормата

$\|x\| = \sqrt{\sum_{i=1}^{n} x_i^2}$ евклидова норма в $\mathbb{R}^n$

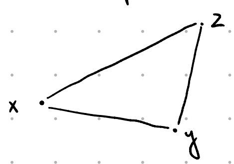
$\|.\| : \mathbb{R}^n \to [0,+\infty)$ е норма в $\mathbb{R}^n$, ако:

1) $\|x\| = 0 \Leftrightarrow x = \theta$

2) $\|\lambda x\| = |\lambda| . \|x\| \quad \forall \lambda \in \mathbb{R}\ \forall x \in \mathbb{R}^n$

3) $\|x+y\| = \|x\| + \|y\| \quad \forall x,y \in \mathbb{R}^n$

неравенство на Δ



$\|x-y\| = \|(x-z)+(z-y)\| \leq$

$\leq \|x-z\| + \|z-y\|$

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i \longrightarrow \|x\| = \sqrt{\langle x, x\rangle}$$

Неравенство на Коши-Буняковски

$$|\langle x, y\rangle| \le \|x\| \cdot \|y\| \quad \forall x, y \in \mathbb{R}^n$$

$$\left(\left| \sum_{i=1}^{\infty} x_i y_i \right| \le \sqrt{\sum_{i=1}^{\infty} x_i^2} \cdot \sqrt{\sum_{i=1}^{\infty} y_i^2} \right)$$

$\lambda \in \mathbb{R}$

$$0 \le \|x + \lambda y\|^2 = \langle x + \lambda y, x + \lambda y\rangle = \langle x, x\rangle + \lambda\langle y, x\rangle + \lambda\langle x, y\rangle + \lambda^2\langle y, y\rangle =$$

$$= \|x\|^2 + 2\lambda\langle x, y\rangle + \lambda^2\|y\|^2$$

$$(2\langle x, y\rangle)^2 - 4\|x\|^2\|y\|^2 \le 0$$

$$(\langle x, y\rangle)^2 \le \|x\|^2\|y\|^2$$

$$\|x + y\|^2 = \|x\| + 2\langle x, y\rangle + \|y\|^2 \le \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2$$

Равнинно-координатна норма

$$\|(x_1, x_2)\|_\infty = \max\{|x_1|, |x_2|\} \qquad \|(x_1, x_2)\|_p = (|x_1|^p + |x_2|^p)^{1/p} \quad p \ge 1$$

$$\|(x_1, x_2)\|_1 = |x_1| + |x_2|$$

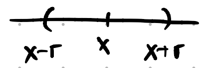
Топология в $\mathbb{R}^n$

$x \in \mathbb{R}^n$, $r > 0$

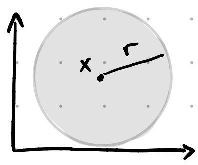
$B_r(x) := \{y \in \mathbb{R}^n : \|x - y\| < r\}$

отворено кълбо с център x и радиус r

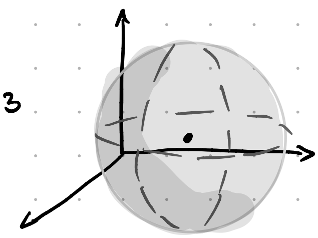
$n=1$



$n=2$



$n=3$



1) $U \subset \mathbb{R}^n$ се нарича околност на x, ако същ. $\varepsilon > 0$ такова, че $U \supset B_\varepsilon(x)$
2) $U \subset \mathbb{R}$ се нарича отворено, ако $\forall x \in U$ същ. $\varepsilon > 0$, такова, че $B_\varepsilon(x) \subset U$

Свойства

(а) $\emptyset$, $\mathbb{R}^n$ са отворени

(б) U_α отворено в $\mathbb{R}^n$ $\forall \alpha \in I \Rightarrow \bigcup_{\alpha \in I} U_\alpha$ е отворено

$\bigcup_{\alpha\in I} U_\alpha = \{x\in\mathbb{R}^n : \exists \alpha\in I, x\in U_\alpha\}$ $\qquad$ $x\in \bigcup_{\alpha\in I} U_\alpha \Rightarrow \exists \alpha_0\in I, x\in U_{\alpha_0}$ е отворено

$$\Rightarrow \exists \varepsilon>0,\ B_\varepsilon(x)\subset U_{\alpha_0}\subset \bigcup_{\alpha\in I} U_\alpha$$

(в) $U_1,\dots,U_k$ отворени в $\mathbb{R}^n \Rightarrow \bigcap_{i=1}^{k} U_i$ е отворено

$x\in \bigcap_{i=1}^{k} U_i$

$\forall i\in\{1,\dots,k\}: x\in U_i$ отв $\Rightarrow \forall i\in\{1,\dots,k\}\ \exists\, \varepsilon_i>0,\ B_{\varepsilon_i}(x)\subset U_i$

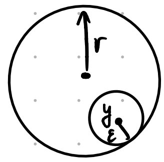
$\varepsilon=\min\{\varepsilon_1,\varepsilon_2,\dots,\varepsilon_k\}>0$ $\qquad$ $B_\varepsilon(x)\subset B_{\varepsilon_i}(x)\subset U_i \quad \forall i\in\{1,\dots,k\}$ $\Bigg| \Rightarrow B_\varepsilon(x)\subset \bigcap_{i=1}^{k} U_i$

Пример:

Отв. кълбо са отв. множества

$B_r(x)=\{y\in\mathbb{R}^n : \|y-x\|<r\}$, $r>0$ $\qquad$ $y\in B_r(x)$

$\varepsilon := r-\|y-x\|>0$



Искаме да проверим, че $B_\varepsilon(y)\subset B_r(x)$

$z\in B_\varepsilon(y)$ $\qquad$ $\|z-x\|\le\|z-y\|+\|y-x\|<\varepsilon+\|y-x\| = r-\|y-x\|+\|y-x\| = r$

3) $F\subset\mathbb{R}^n$ се нарича затворено, ако $\mathbb{R}^n\setminus F$ е отворено

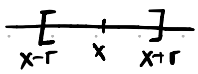
Свойства

(а) $\emptyset,\ \mathbb{R}^n$ са затворени

(б) F_α затворени $\forall \alpha\in I \Rightarrow \bigcap_{i=1}^{k} F_i$ затворено

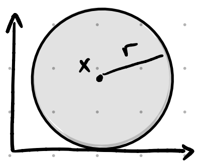
Док: $\underbrace{\mathbb{R}^n\setminus\Big(\bigcap_{\alpha\in I} F_\alpha\Big)}_{\text{затв.}} = \underbrace{\bigcup_{\alpha\in I}\big(\underbrace{\mathbb{R}^n\setminus F_\alpha}_{\text{отв.}}\big)}_{\text{отв.}}$

$\overline{B}_r(x) := \{y\in\mathbb{R}^n : \|y-x\|\le r\}$ $\qquad$ $n=1$



затворено кълбо

$n=2$



$n=3$