25.04.24

① $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(\sqrt{2})^n}\sqrt[4]{\binom{2n+1}{n+1}}$

$\sum_{n=1}^{\infty} a_n$ е абс. сх., ако $\sum_{n=1}^{\infty} |a_n|$ е сх.

$\sum_{n=1}^{\infty} a_n$ е усл. сх, ако $\sum_{n=1}^{\infty} a_n$ е сх и $\sum_{n=1}^{\infty} |a_n|$ е разх.

$a_n = (-1)^{n-1}\frac{1}{(\sqrt{2})^n}\sqrt{\frac{(2n+1)!}{n!\,(n+1)!}}$ $\quad \sum_{n=1}^{\infty} |a_n|$

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\frac{1}{(\sqrt{2})^{n+1}}\sqrt[4]{\frac{\overset{(2n+3)(2n+2)}{\cancel{(2n+3)!}}}{\underset{(n+2)(n+1)}{\cancel{(n+2)!}\,\cancel{(n+1)!}}}}\,(\cancel{\sqrt{2}})^n\cdot\sqrt[4]{\frac{\cancel{(n+1)!}\,\cancel{n!}}{\cancel{(2n+1)!}}} =$$

$$= \lim_{n\to\infty}\frac{1}{\sqrt{2}}\sqrt[4]{\frac{2\cancel{(n+1)}(2n+3)}{\cancel{(n+1)}(n+2)}} = \lim_{n\to\infty}\sqrt[4]{\frac{\cancel{2}(2n+3)}{4(n+2)}} = \lim_{n\to\infty}\sqrt[4]{\frac{2n+3}{2n+4}} = 1$$

Раабе-Дюамел $\lim_{n\to\infty} n\cdot\left(\frac{|a_n|}{|a_{n+1}|}-1\right) = \lim_{n\to\infty} n\cdot\left(\sqrt[4]{\frac{2n+3}{2n+4}}-1\right) = \lim_{n\to\infty} n\left(\left(1+\frac{1}{2n+3}\right)^{1/4}-1\right) =$

$$= \lim_{n\to\infty} n\cdot\left(\cancel{1}+\frac{1}{4}\cdot\frac{1}{2n+3}+o\left(\frac{1}{2n+3}\right)-\cancel{1}\right) = \lim_{n\to\infty} n\cdot\left(\frac{1}{8n+12}+o\left(\frac{1}{n}\right)\right) = \quad (1+y)^\alpha = 1+\alpha y+o(y)$$

$$= \lim_{n\to\infty}\frac{n}{8n+12}+\frac{o(\frac{1}{n})}{\frac{1}{n}} = \frac{1}{8}+0 = \frac{1}{8}<1 \Rightarrow \sum_{n=1}^{\infty}|a_n| \text{ е разх.}$$

$\frac{o\left(\frac{1}{2n+3}\right)}{\frac{1}{n}} \xrightarrow[n\to\infty]{} 0$, $\quad \frac{\frac{1}{n}}{\frac{1}{2n+3}} \to 2$

$\frac{1}{8}>0 \xRightarrow[\text{Лайбниц}]{\text{лема}}$ a_n е сходящ $\Rightarrow \sum_{n=1}^{\infty} a_n$ е усл. сходящ

② $\sum_{n=1}^{\infty} (-4)^{3n}\frac{(n!)^6}{((2n+1)!)^3}\cdot\sqrt[5]{n^4}$

$$\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty} 4^{3n}\frac{(n!)^6}{((2n+1)!)^3}\cdot\sqrt[5]{n^4}$$

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} 4^{3n+3}\cdot\frac{((n+1)!)^6}{\underset{(2n+2)(2n+3)}{(\cancel{(2n+3)!})^3}}\cdot\sqrt[5]{(n+1)^4}\cdot\frac{1}{4^{3n}}\cdot\frac{(\cancel{(2n+1)!})^3}{\cancel{n!^6}}\cdot\frac{1}{\sqrt[5]{n^4}} =$$

$$= \lim_{n\to\infty}\frac{\overset{2^3}{\cancel{4^3}}(n+1)^{\cancel{6}}\sqrt[5]{(n+1)^4}}{\cancel{2^3}(n+1)^3(2n+3)^3\sqrt[5]{n^4}} = \lim_{n\to\infty}\frac{8(n+1)^3\sqrt[5]{(n+1)^4}}{(2n+3)^3\sqrt[5]{n^4}} = 1$$

$$\lim_{n\to\infty} n\cdot\left(\frac{|a_n|}{|a_{n+1}|}-1\right) = \lim_{n\to\infty} n\left(\frac{(2n+3)^3\sqrt[5]{n^4}}{8(n+1)^3\sqrt[5]{(n+1)^4}}\right) = \lim_{n\to\infty} n\left(\left(\frac{2n+3}{2n+2}\right)^3\sqrt[5]{\left(\frac{n}{n+1}\right)^4}-1\right) =$$

$$= \lim_{n\to\infty} n\left(\left(1+\frac{1}{2n+2}\right)^3\left(1-\frac{1}{n+1}\right)^{4/5}-1\right) = \lim_{n\to\infty} n\left(\left(1+3\cdot\frac{1}{2n+2}+o\left(\frac{1}{2n+2}\right)\right)\left(1-\frac{4}{5}\frac{1}{n+1}+o\left(\frac{1}{n+1}\right)\right)-1\right)$$

$$= \lim_{n\to\infty} n\cdot\left(1-\frac{4}{5}\cdot\frac{1}{n+1}+3\underbrace{\frac{1}{2n+2}}_{\frac{1}{2(n+1)}}+o\left(\frac{1}{n}\right)\right) = \lim_{n\to\infty} n\left(\frac{1}{n+1}\cdot\left(\frac{3}{2}-\frac{4}{5}\right)+o\left(\frac{1}{n}\right)\right) =$$

$$= \lim_{n\to\infty}\frac{n}{n+1}\cdot\frac{7}{10}+\frac{o(\frac{1}{n})}{\frac{1}{n}} = \frac{7}{10}+0<1 \Rightarrow \sum_{n=1}^{\infty}|a_n| \text{ е разходящ}$$

$\frac{7}{10} > 0 \xRightarrow[\text{Лайбниц}]{\text{лема}} \sum_{n=1}^{\infty} a_n$ е сходящ $\Rightarrow \sum_{n=1}^{\infty} a_n$ е усл. сходящ

Степенни редове

$\sum_{n=1}^{\infty} a_n$ $\quad a_1, a_2, a_3, \ldots, a_n, \ldots \in \mathbb{R}$

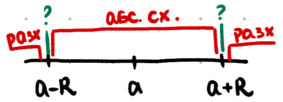
$f_1(x), f_2(x), f_3(x), \ldots, f_n(x), \ldots \quad f_n : M \to \mathbb{R}$

$\sum_{n=1}^{\infty} f_n(x)$ функционален ред

$f_n(x) = a_n(x-a)^n \quad a_n \in \mathbb{R}, a \in \mathbb{R}$

$\sum_{n=0}^{\infty} a_n(x-a)^n$ - степенен ред $\quad x = x_0 \quad \sum_{n=0}^{\infty} a_n(x-a)^n$

$D = \{x \in \mathbb{R} : \sum_{n=0}^{\infty} a_n(x-a)^n$ е сходящ$\}$ област на сходимост



R-радиус на сходимост $\quad \sum_{n=1}^{\infty} a_n(x-a)^n$ сходящ $x \in (a-R, a+R)$

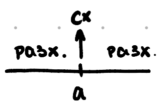
разходящ $x \notin [a-R, a+R]$

$R \in \mathbb{R}^+ \cup \{0\} \cup \{+\infty\}$

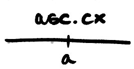
$x = a-R$?

$x = a+R$

$R = 0$



$R = \infty$



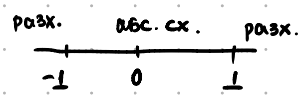
Формули за радиуса

Даламбер $\quad R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}$

Коши-Адамар $\quad R = \frac{1}{\limsup \sqrt[n]{|a_n|}}$ $\quad$ lim sup - най-дясната точка на сгъстяване

Намерете областта на сходимост

① $\sum_{n=1}^{\infty} \frac{x^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n} \cdot (x-0)^n \quad a_n = \frac{1}{n} \quad R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n+1}} = 1$

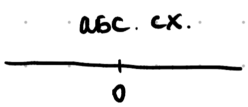


крайни точки: $x = 1 \quad \sum_{n=1}^{\infty} \frac{1}{n} \cdot 1^n$ разходящ

$\Rightarrow D = [-1, 1)$

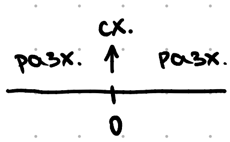
$x = -1 \quad \sum_{n=1}^{\infty} \frac{1}{n} \cdot (-1)^n$ сходящ

② $\sum_{n=1}^{\infty} \frac{x^n}{n!} = \sum_{n=1}^{\infty} \frac{1}{n!} (x-0)^n \quad a_n = \frac{1}{n!} \quad R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{1}{\cancel{n!}} \cdot (n+1)! = \lim_{n\to\infty} n+1 = \infty$



$\Rightarrow D = (-\infty, +\infty)$

③ $\sum_{n=0}^{\infty} n^n x^n = \sum_{n=0}^{\infty} n^n (x-0)^n \quad a_n = n^n \quad R = \frac{1}{\limsup \sqrt[n]{n^n}} = \frac{1}{\limsup n} = 0$



$\Rightarrow D = \{0\}$

④ $\sum\limits_{n=1}^{\infty} \frac{x^n}{7^n n(n+1)(n+2)} = \sum\limits_{n=1}^{\infty} \frac{1}{7^n n(n+1)(n+2)} \cdot (x-0)^n \qquad a_n = \frac{1}{7^n n(n+1)(n+2)}$

$R = \lim\limits_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim\limits_{n\to\infty} \frac{1}{\cancel{7^n} n \cancel{(n+1)} \cancel{(n+2)}} \cdot \cancel{7}^{(n+1)} \cancel{(n+1)} \cancel{(n+2)} (n+3) = \lim\limits_{n\to\infty} \frac{7(n+3)}{n} = 7$

разх. ↑ сх. абс сх. ↑ сх. разх

$-7 \quad 0 \quad 7$

крайни точки: $x = 7 \quad \sum\limits_{n=1}^{\infty} \frac{\cancel{7^n}}{\cancel{7^n} n(n+1)(n+2)} = \sum\limits_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} \sim \sum\limits_{n=1}^{\infty} \frac{1}{n^3}$ сх.

$D = [-7, 7]$

$x = -7 \quad \sum\limits_{n=1}^{\infty} \frac{(-\cancel{7})^n}{\cancel{7^n} n(n+1)(n+2)} = \sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)(n+2)}$ сходящ (Лайбниц / абс сх.)

⑤ $\sum\limits_{n=1}^{\infty} \frac{(x-1)^n}{n\sqrt[4]{3n+1}} = \sum\limits_{n=1}^{\infty} \frac{1}{n\sqrt[4]{3n+1}} (x-1)^n$ разх. абс сх разх. $0 \quad 1 \quad 2$ $\qquad a_n = \frac{1}{n\sqrt[4]{3n+1}}$

$R = \lim\limits_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim\limits_{n\to\infty} \frac{1}{n\sqrt[4]{3n+1}} \cdot (n+1)\sqrt[4]{3(n+1)+1} = 1$

крайни точки: $x = 2 \quad \sum\limits_{n=1}^{\infty} \frac{1}{n\sqrt[4]{3n+1}} (2-1)^n = \sum\limits_{n=1}^{\infty} \frac{1}{n\sqrt[4]{3n+1}} \sim \sum\limits_{n=1}^{\infty} \frac{1}{n^{1+1/4}}$ сходящ

$x = 0 \quad \sum\limits_{n=1}^{\infty} \frac{1}{n\sqrt[4]{3n+1}} (0-1)^n = \sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n\sqrt[4]{3n+1}}$ – сходящ (от абс. сход.)

$\Rightarrow D = [0, 2]$

⑥ $\sum\limits_{n=1}^{\infty} \frac{7^n + (-3)^n}{n} x^n = \sum\limits_{n=1}^{\infty} \frac{7^n + (-3)^n}{n} (x-0)^n \qquad a_n = \frac{7^n + (-3)^n}{n}$ разх. абс сх. разх. $-\frac{1}{7} \quad 0 \quad \frac{1}{7}$

$R = \lim\limits_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim\limits_{n\to\infty} \frac{7^n + (-3)^n}{n} \cdot \frac{(n+1)}{7^{n+1} + (-3)^{n+1}} = \lim\limits_{n\to\infty} \frac{\cancel{7^n}(1 - \underbrace{(\frac{3}{7})}_{\to 0})^n}{\cancel{7^{n+1}}(1 - \underbrace{(\frac{3}{7})}_{\to 0})^{n+1}} \cdot \underbrace{\frac{n+1}{n}}_{\to 1} = \frac{1}{7}$

крайни точки: $x = \frac{1}{7} \quad \sum\limits_{n=1}^{\infty} \frac{7^n + (-3)^n}{n} \cdot \left(\frac{1}{7}\right)^n = \sum\limits_{n=1}^{\infty} \frac{\cancel{7^n}}{n} \cdot \frac{1}{\cancel{7^n}} + \frac{(-3)^n}{n} \cdot \frac{1}{7^n} = \sum\limits_{n=1}^{\infty} \Big(\underbrace{\frac{1}{n}}_{\text{разх.}} + \underbrace{\frac{(-\frac{3}{7})^n}{n}}_{\text{сх.}} \Big)$

$\Rightarrow$ разх.

$x = -\frac{1}{7} \quad \sum\limits_{n=1}^{\infty} \frac{7^n + (-3)^n}{n} \cdot \left(-\frac{1}{7}\right)^n = \sum\limits_{n=1}^{\infty} \frac{\cancel{7^n}}{n} \cdot \frac{(-1)^n}{\cancel{7^n}} + \frac{(-3)^n}{n} \cdot \frac{(-1)^n}{7^n} =$

$= \sum\limits_{n=1}^{\infty} \Big(\underbrace{\frac{(-1)^n}{n}}_{\text{сх.}} + \underbrace{\frac{(-\frac{3}{7})^n}{n}}_{\text{сх.}} \Big) \Rightarrow$ сх.

$\Rightarrow D = [-\frac{1}{7}, \frac{1}{7}]$

⑦ $\sum\limits_{n=0}^{\infty} \frac{3^{-\sqrt{n}}}{\sqrt{n^2+n+2}} x^n \qquad R = \lim\limits_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim\limits_{n\to\infty} \frac{3^{-\sqrt{n}}}{\sqrt{n^2+n+2}} \cdot \frac{\sqrt{(n+1)^2+(n+1)+2}}{3^{-\sqrt{n+1}}} = \lim\limits_{n\to\infty} 3^{\sqrt{n+1}-\sqrt{n}} = 1$

$\to 1 \qquad = 3^0$

разх. абс сх. разх.

$-1 \quad 0 \quad 1$

$(\sqrt{n+1} - \sqrt{n}) \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}} \xrightarrow[n\to\infty]{} 0$

крайни точки $x = 1 \quad \sum\limits_{n=1}^{\infty} \frac{1}{3^{\sqrt{n}}\sqrt{n^2+n+2}} 1^n \to$ сходящ

$$\frac{1}{3^{\sqrt{n}}\sqrt{n^2+n+2}} < \frac{1}{n\sqrt{n^2+n+2}} \qquad \sum_{n=1}^{\infty}\frac{1}{n\sqrt{n^2+n+2}} \sim \sum_{n=1}^{\infty}\frac{1}{n^2} \text{ сходящ}$$

$x=-1 \quad \sum_{n=1}^{\infty}\frac{1}{3^{\sqrt{n}}\sqrt{n^2+n+2}}(-1)^n$ сходящ (Лайбниц / абс. сх.) $\quad \Rightarrow D=[-1,1]$

⑧ $$\sum_{n=1}^{\infty}(2x)^{n^2} = \sum_{n=1}^{\infty}2^{n^2}(x-0)^{n^2} = \sum_{k=1}^{\infty}a_k x^k \qquad a_k=\begin{cases}2^k, & k=n^2\\ 0, & k\neq n^2\end{cases}$$

разх. — $-\frac{1}{2}$ — абс. сх. (0) — $\frac{1}{2}$ — разх.

$$= 2x+2^4x^4+2^9x^9+2^{16}x^{16}+\dots$$

$$R=\frac{1}{\limsup\sqrt[k]{|a_k|}}=\frac{1}{2}$$

крайни точки $\quad x=\frac{1}{2} \quad \sum_{n=1}^{\infty}\left(2\cdot\frac{1}{2}\right)^{n^2}=\sum_{n=1}^{\infty}1^{n^2}$ разх.

$$\sqrt[k]{|a_k|}=\begin{cases}\sqrt[k]{2^k}=2, & k=n^2\\ 0=0, & k\neq n^2\end{cases}$$

$2>0$

$x=-\frac{1}{2} \quad \sum_{n=1}^{\infty}\left(2\cdot\left(-\frac{1}{2}\right)\right)^{n^2}=\sum_{n=1}^{\infty}(-1)^{n^2}$ разх.

$$\Rightarrow D=\left(-\frac{1}{2},\frac{1}{2}\right)$$

⑨ $$\sum_{n=1}^{\infty}\sqrt{\binom{2n}{n}}\frac{x^n}{\sqrt[6]{n+1}}\cdot(-1)^n=\sum_{n=1}^{\infty}\sqrt{\frac{(2n)!}{n!\,n!}}\frac{(-1)^n}{\sqrt[6]{n+1}}(x-0)^n$$

разх. — $-\frac{1}{2}$ — абс. сх. (0) — $\frac{1}{2}$ — разх.

$$\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|=\lim_{n\to\infty}\sqrt{\frac{\cancel{(2n)!}}{\cancel{n!}\,\cancel{n!}}}\cdot\frac{1}{\sqrt[6]{n+1}}\cdot\sqrt[6]{n+2}\cdot\sqrt{\frac{\overset{(n+1)\,(n+1)}{\cancel{(n+1)!}\,\cancel{(n+1)!}}}{\underset{\underset{2(\cancel{n+1})}{(2n+1)\,(2n+2)}}{\cancel{(2n+2)!}}}}=$$

$$=\lim_{n\to\infty}\sqrt[6]{\frac{n+2}{n+1}}\cdot\sqrt{\frac{(n+1)}{2(2n+1)}}=\frac{1}{2}$$

крайни точки $\quad x=\frac{1}{2} \quad \sum_{n=1}^{\infty}\overbrace{\sqrt{\frac{(2n)!}{n!\,n!}}\cdot\frac{1}{2^n}\cdot\frac{1}{\sqrt[6]{n+1}}\cdot(-1)^n}^{c_n} \Rightarrow$ сходящ

$(-1)^{2n}=1$

$$x=-\frac{1}{2} \quad \sum_{n=1}^{\infty}\sqrt{\frac{(2n)!}{n!\,n!}}\cdot\frac{(-1)^n}{2^n}\cdot\frac{1}{\sqrt[6]{n+1}}\cdot(-1)^n=\sum_{n=1}^{\infty}\underbrace{\sqrt{\frac{(2n)!}{n!\,n!}}\cdot\frac{1}{2^n\sqrt[6]{n+1}}}_{b_n}=$$

$$\lim_{n\to\infty}\frac{b_{n+1}}{b_n}=\lim_{n\to\infty}\sqrt{\frac{\overset{\overset{2(\cancel{n+1})}{(2n+1)\,(2n+2)}}{\cancel{(2n+2)!}}}{\underset{(n+1)\,(\cancel{n+1})}{\cancel{(n+1)!}\,\cancel{(n+1)!}}}}\cdot\frac{1}{\cancel{2^{n+1}}\sqrt[6]{n+2}}\cdot\sqrt{\frac{\cancel{n!}\,\cancel{n!}}{\cancel{(2n)!}}}\cdot\cancel{2^n}\cdot\sqrt[6]{n+1}=$$

$$=\lim_{n\to\infty}\sqrt{\frac{2(2n+1)}{(n+1)}}\cdot\frac{1}{2}\cdot\sqrt[6]{\frac{n+1}{n+2}}=1$$

$$\lim_{n\to\infty}n\cdot\left(\frac{b_n}{b_{n+1}}-1\right)=\lim_{n\to\infty}n\cdot\left(2\cdot\sqrt{\frac{n+1}{4n+2}}\cdot\sqrt[6]{\frac{n+2}{n+1}}-1\right)=$$

$$=\lim_{n\to\infty}n\cdot\left(\frac{\cancel{2}}{\cancel{2}}\cdot\left(1+\frac{\frac{1}{2}}{n+1}\right)^{\frac{1}{2}}\left(1+\frac{1}{n+2}\right)^{1/6}\right)-1\Big)=$$

$$= \lim_{n\to\infty} n \cdot \left(\left(1 + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{n+\frac{1}{2}} \cdot o\left(\frac{1}{n+\frac{1}{2}}\right)\right)\left(1 + \frac{1}{6} \cdot \frac{1}{n+2} \cdot o\left(\frac{1}{n+2}\right)\right) - 1 \right) =$$

$$= \lim_{n\to\infty} \frac{1}{6} \cdot \frac{n}{n+1} + \frac{1}{4} \cdot \frac{n}{n+\frac{1}{2}} + \frac{o\left(\frac{1}{n}\right)}{\frac{1}{n}} = \frac{1}{6} + \frac{1}{4} = \frac{5}{12} \in (0,1)$$

$\frac{5}{12} < 1 \Rightarrow$ Вп разх.

$\frac{5}{12} > 0 \overset{\text{лема}}{\Rightarrow}$ ал сход.

$\Rightarrow D = \left(-\frac{1}{2}, \frac{1}{2}\right]$