А) $f(0,0)=\lim\limits_{(x,y)\to(0,0)} f(x,y)$

$$\lim_{(x,y)\to(0,0)} x(x-y)\sin\frac{1}{x^2+y^2} = \lim_{r\to0} r\cos\varphi\,(r\cos\varphi - r\sin\varphi)\cdot\sin\frac{1}{r^2} =$$

$$\left|\begin{array}{ll} x=r\cos\varphi & r\geq 0 \\ y=r\sin\varphi & 0\leq\varphi<2\pi \end{array}\right. \qquad = \lim_{r\to0} \underset{\downarrow\atop 0}{r^2}\underbrace{\cos\varphi(\cos\varphi-\sin\varphi)}_{\text{огр.}}\cdot\underbrace{\sin\frac{1}{r^2}}_{\text{огр.}} = 0 \qquad f(0,0)=0$$

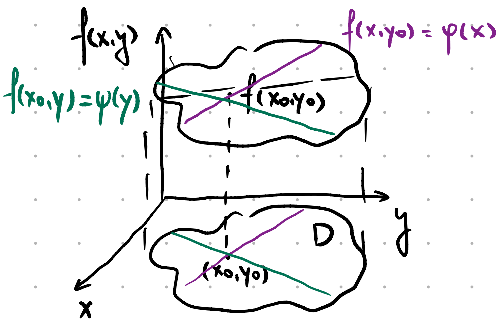
Б) $$f'_x(0,0)=\lim_{h\to0}\frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to0}\frac{\cancel{h}(h-0)\sin\frac{1}{h^2+0^2}-0}{\cancel{h}} =$$

$$=\lim_{h\to0} \underbrace{h\cdot\sin\frac{1}{h^2}}_{\text{огр}} = 0$$

$$f'_y(0,0)=\lim_{h\to0}\frac{f(0,0+h)-f(0,0)}{h} = \lim_{h\to0}\frac{0\cdot(0-h)\sin\frac{1}{0+h^2}-0}{h} =$$

$$=\lim_{h\to0} 0 = 0$$

Диференцируемост



$\mathbb{R}^1$: $\lim\limits_{h\to0}\dfrac{f(x_0+h)-f(x_0)}{h} = f'(x_0)$

$$\lim_{h\to0}\frac{f(x_0+h)-f(x_0)-f'(x_0)\cdot h}{h}=0$$

$\mathbb{R}^n$: същ. лин. опер. $df(x_0)$:

$x_0\in\mathbb{R}^n$, $h\in\mathbb{R}^n$ $\qquad \lim\limits_{h\to0}\dfrac{f(x_0+h)-f(x_0)-df(x_0)h}{\|h\|}=0$

Ако $\exists\, df(x_0)$, то $df(x_0)h=\sum\limits_{i=1}^{n} f'_{x_i}(x_0)\,h_i$

$\mathbb{R}^2$: $\quad f(x,y)$ е диференцируема в (x_0,y_0), ако:

$(x,y)\in\mathbb{R}^2$

$h=(h_1,h_2)\in\mathbb{R}^2$ $\qquad \lim\limits_{(h_1,h_2)\to(0,0)}\dfrac{f(x_0+h_1,y_0+h_2)-f(x_0,y_0)-f'_x(x_0,y_0)\cdot h_1-f'_y(x_0,y_0)\,h_2}{\sqrt{h_1^2+h_2^2}}=0$

① Диференцируема ли е в т. $(0,0)$ функцията $f(x,y)=\begin{cases}\dfrac{x^3+y^3}{x^2+y^2}, & x^2+y^2\neq0\\ 0, & x^2+y^2=0\end{cases}$

$$f'_x(0,0)=\lim_{h\to0}\frac{f(0+h,0)-f(0,0)}{h}=\lim_{h\to0}\frac{\frac{h^3+0^3}{h^2+0^2}-0}{h}=\lim_{h\to0}\frac{h^3}{h^3}=1$$

$$f'_y(0,0)=\lim_{h\to0}\frac{f(0,0+h)-f(0,0)}{h}=\lim_{h\to0}\frac{\frac{0^3+h^3}{0^2+h^2}-0}{h}=\lim_{h\to0}\frac{h^3}{h^3}=1$$

$$\lim_{(h_1,h_2)\to(0,0)} \frac{f(0+h_1,0+h_2)-f(0,0)-f'_x(0,0)h_1-f'_y(0,0)h_2}{\sqrt{h_1^2+h_2^2}} = \lim_{(h_1,h_2)\to(0,0)} \frac{\frac{h_1^3+h_2^3}{h_1^2+h_2^2}-0-1.h_1-1h_2}{\sqrt{h_1^2+h_2^2}} =$$

$$= \lim_{(h_1,h_2)\to(0,0)} \frac{\cancel{h_1^3}+\cancel{h_2^3}-h_1(\cancel{h_1^2}+h_2^2)-h_2(h_1^2+\cancel{h_2^2})}{(h_1^2+h_2^2)\sqrt{h_1^2+h_2^2}} = \lim_{(h_1,h_2)\to(0,0)} \frac{-h_1h_2(h_1+h_2)}{(h_1^2+h_2^2)\sqrt{h_1^2+h_2^2}} =$$

пол. коорд. $h_1 = r\cos\varphi \quad r\geq 0$
$h_2 = r\sin\varphi \quad 0\leq\varphi<2\pi$

$$= \lim_{r\to 0} \frac{-\cancel{r^2}\cos\varphi\sin\varphi(\cancel{r}\sin\varphi+\cancel{r}\cos\varphi)}{\cancel{r^2}\sqrt{\cancel{r^2}}} = \lim_{r\to 0}(-\cos\varphi\sin\varphi)(\sin\varphi+\cos\varphi) = (-\cos\varphi\sin\varphi)(\sin\varphi+\cos\varphi)$$

$\varphi = 0 \qquad 0$

$\varphi = \frac{\pi}{4} \qquad -\frac{1}{2}\cdot 2\cdot\frac{\sqrt{2}}{2} \neq 0$

$\Rightarrow$ границата не същ.

$\Rightarrow$ функцията $f(x,y)$ не е диференцируема в $(0,0)$

② Диференцируема ли е в $(0,0)$ функцията $f(x,y)=\sqrt{x^2+xy+y^2}\,.\,\sin(x+y)$

$$f'_x(0,0) = \lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{\sqrt{h^2+h.0+0^2}\,.\sin(h+0)-0}{h} = \lim_{h\to 0}\frac{\sqrt{h^2}.\sin h}{h} = 0$$

$$f'_y(0,0) = \lim_{h\to 0}\frac{f(0,0+h)-f(0,0)}{h} = \lim_{h\to 0}\frac{\sqrt{0^2+0.h+h^2}\,.\sin(0+h)-0}{h} = \lim_{h\to 0}\frac{\sqrt{h^2}.\sin h}{h} = 0$$

$$\lim_{(h_1,h_2)\to(0,0)} \frac{f(0+h_1,0+h_2)-f(0,0)-f'_x(0,0)h_1-f'_y(0,0)h_2}{\sqrt{h_1^2+h_2^2}} =$$

$$\lim_{(h_1,h_2)\to(0,0)} \frac{\sqrt{h_1^2+h_1h_2+h_2^2}\,.\,\sin(h_1+h_2)-0-0.h_1-0.h_2}{\sqrt{h_1^2+h_2^2}} =$$ пол. коорд. $h_1 = r\cos\varphi \quad r\geq 0$
$h_2 = r\sin\varphi \quad 0\leq\varphi<2\pi$

$$\lim_{r\to 0}\frac{\sqrt{r^2\cos^2\varphi+r^2\cos\varphi\sin\varphi+r^2\sin^2\varphi}\,.\,\sin(r\cos\varphi+r\sin\varphi)}{\sqrt{r^2}} =$$

$$\lim_{r\to 0}\frac{\cancel{r}\overbrace{(1+\cos\varphi\sin\varphi)}^{\text{огр.}}.\sin \overset{\to 0}{r(\cos\varphi+\sin\varphi)}}{\cancel{r}} = 0$$

$\Rightarrow f(x,y)$ е диференцируема в $(0,0)$

Диференцируемост на композиция

$\mathbb{R}^1$: $g: D\to D_1 \qquad D, D_1 \subseteq \mathbb{R} \qquad F(x) = f(g(x))$

$f: D_1 \to \mathbb{R} \qquad F'(x) = f'(g(x)).\,g'(x)$

$\mathbb{R}^n$: $g: D \to D_1$ $\quad D \subseteq \mathbb{R}^n$ $\quad g(x_1 \dots x_n) = \begin{pmatrix} g_1(x_1 \dots x_n) \\ g_2(x_1 \dots x_n) \\ \dots \\ g_m(x_1 \dots x_n) \end{pmatrix}$ $\quad f(y_1, \dots y_m) \in \mathbb{R}$

$f: D_1 \to \mathbb{R}$ $\quad D_1 \subseteq \mathbb{R}^m$

$$F(x_1, \dots x_n) = f \circ g = f(g_1(x_1, \dots x_n), g_2(x_1, \dots, x_n), \dots, g_m(x_1, \dots x_n))$$

$$F'_{x_i} = \sum_{j=1}^{m} f'_{y_j} \cdot g'_{j x_i} = \sum_{j=1}^{m} \frac{df}{dy_i} \cdot \frac{dg_j}{dx_i}$$

$$F'_{x_i}(x_1, \dots x_n) = \sum_{j=1}^{m} \frac{df(g_1, \dots g_m)}{dg_j} \frac{dg_j}{dx_i}(x_1, \dots x_n)$$

частен случай $\quad f(s,t) \quad s = s(x,y) \quad g(x,y) = \begin{pmatrix} s(x,y) \\ t(x,y) \end{pmatrix} \quad F(x,y) = f(s(x,y), t(x,y))$

$f: \mathbb{R}^2 \to \mathbb{R} \quad t = t(x,y) \quad g: \mathbb{R}^2 \to \mathbb{R}^2$

$$F'_x(x,y) = f'_s \cdot s'_x + f'_t \cdot t'_x$$

$$F'_y(x,y) = f'_s \cdot s'_y + f'_t \cdot t'_y$$

③ $f(s,t) = \dfrac{s^2+t}{s-t} \qquad s = xy, \quad t = x^y$

$$F(x,y) = f(s(x,y), t(x,y)) = \frac{(xy)^2 - 2xy \cdot x^y - x^y}{(xy - x^y)^2} \cdot y + \frac{xy + (xy)^2}{(xy - x^y)^2} \cdot y \cdot x^{y-1}$$

$$F'_x(x,y) = f'_s \cdot s'_x + f'_t \cdot t'_x = \dots$$

$$F'_y(x,y) = f'_s \cdot s'_y + f'_t \cdot t'_y$$

$$f'_s(s,t) = \frac{2s(s-t) - (s^2+t) \cdot 1}{(s-t)^2} = \frac{s^2 - 2st - t}{(s-t)^2}$$

$$f'_t(s,t) = \frac{1(s-t) - (s^2+t) \cdot (-1)}{(s-t)^2} = \frac{s+s^2}{(s-t)^2}$$

$s'_x = y \qquad t'_x = y \cdot x^{y-1}$

$s'_y = x \qquad t'_y = x^y \cdot \ln y$

④ Нека $f: \mathbb{R}^2 \to \mathbb{R}$ е достатъчно гладка и $\varphi(x,y,z) = f(\overset{s}{\frac{x}{y}}, \overset{t}{x^2 + y - z^2})$

Намерете $A = 2xz \cdot \varphi'_x + 2yz \cdot \varphi'_y + 2(x^2+y)\varphi'_z$

Нека $s = \frac{x}{y} \quad t = x^2 + y - z^2$

$$\varphi(x,y,z) = f(s,t)$$

$$\varphi'_x = f'_s \cdot s'_x + f'_t \cdot t'_x = f'_s \cdot \frac{1}{y} + f'_t \cdot 2x$$

$\varphi'_y = f'_s \cdot s'_y + f'_t \cdot t'_y = f'_s \cdot \left(-\frac{x}{y^2}\right) + f'_t \cdot 1$

$\varphi'_z = f'_s \cdot s'_z + f'_t \cdot t'_z = f'_s \cdot 0 + f'_t \cdot (-2z)$

$s'_x = \frac{1}{y} \quad s'_y = -\frac{x}{y^2} \quad s'_z = 0$

$t'_x = 2x \quad t'_y = 1 \quad t'_z = -2z$

$A = 2xz \cdot \left(\frac{1}{y} \cdot f'_s + 2x f'_t\right) + 2yz\left(-\frac{x}{y^2} \cdot f'_s + f'_t\right) + (2x^2+y)(-2z) \cdot f'_t =$

$= f'_s \cdot \left(\frac{2xz}{y} - \frac{2yzx}{y^{\cancel{2}}}\right) + f'_t\left(\cancel{4x^2z} + \cancel{2yz} - \cancel{4x^2z} - \cancel{2yz}\right) = 0$

⑤ Нека $f: \mathbb{R} \to \mathbb{R}$ и $g: \mathbb{R}^2 \to \mathbb{R}$ имат непрекъснати (частни) производни до втори ред и

$u(x,y,z) = y \cdot f(x+z) + g(xy, yz)$

ДСД, че $(x-z) \cdot u''_{xz} + y(u''_{xy} - u''_{yz}) = x u''_{xx} - z u''_{zz} + u'_x - u'_z$

Нека $s = xy$ и $t = yz$

$u(x,y,z) = y \cdot f(x+z) + g(s(x,y), t(x,y))$

$s'_x = y \quad t'_x = 0$
$s'_y = x \quad t'_y = z$
$s'_z = 0 \quad t'_z = y$

$u'_x = y \cdot f'(x+z) \cdot (x+z)'_x + g'_s \cdot s'_x + g'_t \cdot \underbrace{t'_x}_{=0} =$

$= y \cdot f'(x+z) + y \cdot g'_s$

$u''_{xx} = (u'_x)'_x = y \cdot f''(x+z) \cdot 1 + y \cdot (g''_{ss} \cdot s'_x + g''_{st} \cdot t'_x) =$

$= y \cdot f''(x+z) + y^2 g''_{ss}$

$u''_{xy} = (u'_x)'_y = f'(x+z) + 1 \cdot g'_s + y \cdot (g''_{ss} \cdot s'_y + g''_{st} \cdot t'_y) =$

$= f'(x+z) + g'_s + xy \cdot g''_{ss} + yz\, g''_{st}$

$u''_{xz} = (u'_x)'_z = y \cdot f''(x+z) \cdot 1 + y \cdot (g''_{ss} \cdot \underbrace{s'_z}_{=0} + g''_{st} \cdot t'_z) =$

$= y \cdot f''(x+z) + y^2 g''_{st}$

$u'_z = y \cdot f'(x+z) \cdot 1 + g'_s \cdot \underbrace{s'_z}_{=0} + g'_t \cdot t'_z = y \cdot f'(x+z) + y \cdot g'_t$

$u''_{zz} = (u'_z)'_z = y f''(x+z) \cdot 1 + y \cdot (g''_{ts} \cdot \underbrace{s'_z}_{=0} + g''_{tt} \cdot t'_z) = y f''(x+z) + y^2 g''_{tt}$

$u''_{yz} = u''_{zy} = (u'_z)'_y = f'(x+z) + 1 \cdot g'_t + y \cdot (g''_{ts} \cdot s'_y + g''_{tt} \cdot t'_y) = f'(x+z) + g'_t + xy\, g''_{ts} + yz \cdot g''_{tt}$

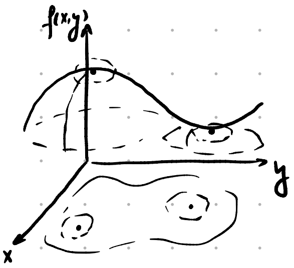
от непр. на частни произв.

Локални екстремуми на ф-ции на повече от една пром.

$\mathbb{R}^n$: $f(x)$ има локален минимум (максимум) в т. x_0 – вътрешна за D, ако
$x\in\mathbb{R}^n$

съществува околност на т. x_0 $U_\varepsilon(x_0)$ такава, че:

$$f(x)\ge f(x_0)\quad \forall x\in U_\varepsilon(x_0)\qquad \left(f(x)\le f(x_0)\quad \forall x\in U_\varepsilon(x_0)\right)$$



НУ: Нека f има локален екстремум в т. x_0 и $\underset{i=1,\dots,n}{f'_{x_i}(x_0)}$ съществуват. Тогава $\underset{i=1,\dots,n}{f'_{x_i}(x_0)=0}$

$$f(x)=f(x_0)+\underbrace{df(x_0).h}_{=0}+o(\|h\|)\qquad h=x-x_0\in\mathbb{R}^n$$

$$df(x_0)h=\sum_{i=1}^{n}\underbrace{f'_{x_i}(x)}_{=0}.h_i$$

$$f(x)=f(x_0)+df(x_0)h+\frac{1}{2}d^2f(x_0)(h)+o(\|h\|^2)$$

$$d^2f(x_0)(h)=h^T H(x_0)h\qquad \underset{\text{хесиян}}{H(x_0)}=\left(f''_{x_ix_j}\right)_{i,j=1}^{n}(x_0)=\begin{pmatrix} f''_{x_1x_1} & f''_{x_1x_2} & \dots & f''_{x_1x_n}\\ f''_{x_2x_1} & f''_{x_2x_2} & \dots & f''_{x_2x_n}\\ \dots & \dots & \dots & \dots\\ f''_{x_nx_1} & f''_{x_nx_2} & \dots & f''_{x_nx_n}\end{pmatrix}(x_0)$$

ДУ: Нека f има частни производни от втори ред и $f'_{x_i}(x_0)=0$ $i=1,\dots,n$
Тогава, ако:

1) $H(x_0)$ е положително дефинитна ($h^T H(x_0)h>0\ \forall h\in\mathbb{R}^n$), то f има лок. минимум в т. x_0

2) $H(x_0)$ е отрицателно дефинитна ($h^T H(x_0)h<0\ \forall h\in\mathbb{R}^n$), то f има лок. максимум в т. x_0

3) $H(x_0)$ не е дефинитна ($h^T H(x_0)h$ приема $+$ и $-$ ст-сти), то f няма екстремум в т. x_0

Критерий на Силвестър

A- симетрична
$$A=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n}\\ a_{21} & a_{22} & \dots & a_{2n}\\ \dots & \dots & & \dots\\ a_{n1} & a_{n2} & \dots & a_{nn}\end{pmatrix}$$

$A_1=a_{11}$

$$A_2=\begin{vmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}$$

$\dots$

$$A_n=\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n}\\ a_{21} & a_{22} & \dots & a_{2n}\\ \dots & \dots & & \dots\\ a_{n1} & a_{n2} & \dots & a_{nn}\end{vmatrix}$$

1) Ако $A_1>0,\ A_2>0 \dots A_n>0$, то A е пол. дефинитна

2) Ако $A_1<0,\ A_2>0 \dots (-1)^n A_n>0$, то A е отр. дефинитна

3) Ако $A_{2k}<0$ за някое $k\in\mathbb{N}$, то A не е дефинитна

$\mathbb{R}^2$: 1) критични точки $\left| \begin{array}{l} f'_x = 0 \\ f'_y = 0 \end{array} \right. \longrightarrow M_i$

2) $H(M_i) = \begin{pmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{pmatrix} (M_i)$

Ако $f''_{xx}(M_i) > 0$, $|H(M_i)| > 0$, то f има лок. минимум в M_i

Ако $f''_{xx}(M_i) < 0$, $|H(M_i)| > 0$, то f има лок. максимум в M_i

Ако $|H(M_i)| < 0$, то f няма екстремум в M_i

Ако $f''_{xx}(M_i) = 0$, но $|H(M_i)| \geq 0$ или $|H(M_i)| = 0$, то трябват доп. изследвания

① DCH лок. екстремуми на

а) $f(x,y) = x^2 + xy + y^2 - 6x - 1$

крит. точки $\left| \begin{array}{l} f'_x = 0 \\ f'_y = 0 \end{array} \right.$ $\left| \begin{array}{l} 2x + y - 6 = 0 \\ x + 2y = 0 \end{array} \right.$ $\left| \begin{array}{l} -4y + y - 6 = 0 \\ x = -2y \end{array} \right.$ $\begin{array}{l} y = -2 \\ x = 4 \end{array}$ $M(4;-2)$

$H(x,y) = \begin{pmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{pmatrix} (x,y) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

$H(4;-2) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ $\begin{array}{l} H_1 = 2 > 0 \\ H_2 = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 = 3 > 0 \end{array}$ $\Rightarrow f(x,y)$ има лок. минимум в $(4;-2)$

б) $f(x,y) = x^3 + y^3 - 3xy$

крит. точки $\left| \begin{array}{l} f'_x(x,y) = 0 \\ f'_y(x,y) = 0 \end{array} \right.$ $\left| \begin{array}{l} 3x^2 - 3y = 0 \\ 3y^2 - 3x = 0 \end{array} \right.$ $\left| \begin{array}{l} x^2 - y = 0 \\ y^2 - x = 0 \end{array} \right.$ $\left| \begin{array}{l} y = x^2 \\ x^4 - x = 0 \end{array} \right.$ $(x-1)(x^2+x+1) = 0$

$x_1 = 0 \quad x_2 = 1$

$y_1 = 0 \quad y_2 = 1$

$M_1(0,0) \quad M_2(1,1)$

$M(x,y) = \begin{pmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{pmatrix} (x,y) = \begin{pmatrix} 6x & -3 \\ -3 & 6y \end{pmatrix}$

$H(1,1) = \begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix}$

$H_1(1,1) = 6 > 0$

$H_2(1,1) = \begin{vmatrix} 6 & -3 \\ -3 & 6 \end{vmatrix} = 36 - 9 > 0$ $\Rightarrow f(x,y)$ има лок. минимум в $(1,1)$

$H(0,0) = \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}$

$H_1(0,0) = 0 \quad H_2(0,0) = -9 < 0$ $\Rightarrow f(x,y)$ няма екстремум в $(0,0)$

в) $f(x,y) = x^4 + y^4 - x^2 - y^2$

крит. точки $\begin{cases} f'_x = 0 \\ f'_y = 0 \end{cases}$ $\begin{cases} 4x^3 - 2x = 0 \\ 4y^3 - 2y = 0 \end{cases}$ $\begin{cases} 2x(2x^2-1) = 0 & x = 0, \pm \frac{1}{\sqrt{2}} \\ 2y(2y^2-1) = 0 & y = 0, \pm \frac{1}{\sqrt{2}} \end{cases}$

$M_1(0,0)$ $M_{2,3}\left(0, \pm \frac{1}{\sqrt{2}}\right)$ $M_{4,5}\left(\pm \frac{1}{\sqrt{2}}, 0\right)$ $M_{6,7,8,9}\left(\pm \frac{1}{\sqrt{2}}, \pm \frac{1}{\sqrt{2}}\right)$

$$H(x,y) = \begin{pmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{pmatrix}(x,y) = \begin{pmatrix} 12x^2-2 & 0 \\ 0 & 12y^2-2 \end{pmatrix}$$

$H(0,0) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$ $H_1(0,0) = -2 < 0$, $H_2(0,0) = 4 - 0 > 0$ $\Rightarrow$ f има лок. максимум в $(0,0)$

$M\left(0, \pm \frac{1}{\sqrt{2}}\right) = \begin{pmatrix} -2 & 0 \\ 0 & 4 \end{pmatrix}$ $H_1\left(0, \pm \frac{1}{\sqrt{2}}\right) = -2 < 0$, $H_2\left(0, \pm \frac{1}{\sqrt{2}}\right) = -8 < 0$ $\Rightarrow$ няма екстремум

$M\left(\pm \frac{1}{\sqrt{2}}, 0\right) = \begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}$ $H_1 = 4 > 0$, $H_2 = -8 < 0$ $\Rightarrow$ няма екстремум

$M\left(\pm \frac{1}{\sqrt{2}}, \pm \frac{1}{\sqrt{2}}\right) = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$ $H_1 = 4 > 0$, $H_2 = 16 > 0$ $\Rightarrow$ лок. минимум в т. $\left(\pm \frac{1}{\sqrt{2}}, \pm \frac{1}{\sqrt{2}}\right)$