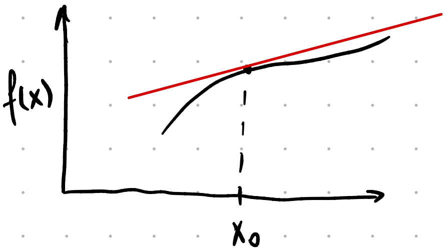
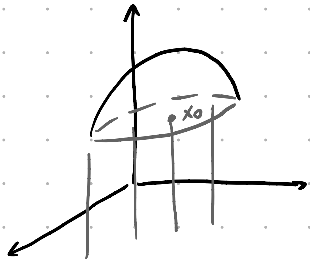
Диференцируема функция





$f(x_1, x_2)$

$(x_1, x_2) \in U$ ок. на $x_0 = (x_1^0, x_2^0)$

$$f(x) = f(x_0) + f'(x_0).(x - x_0) + R(x, x_0)$$

$$\frac{R(x, x_0)}{|x - x_0|} \xrightarrow[x \to x_0]{} 0$$

$$f(x_1, x_2) = f(x_1^0, x_2^0) + A(x_1 - x_1^0, x_2 - x_2^0) + R(x, x_0)$$

$A: \mathbb{R}^2 \to \mathbb{R}$ линеен оператор $\qquad \dfrac{R(x, x_0)}{\|x - x_0\|} \xrightarrow[x \to x_0]{} 0$

def. U отворено множество в $\mathbb{R}^n$ $\quad x_0 \in U \quad f: U \to \mathbb{R}$

Казваме, че f е диференцируема, ако съществува линеен оператор $df(x_0)$ (нарича се диференциал на f в т. x_0) такъв, че

$$f(x) = f(x_0) + df(x_0)(x - x_0) + o(\|x - x_0\|) \quad \text{или} \quad \lim_{x \to x_0} \frac{f(x) - f(x_0) - df(x_0)(x - x_0)}{\|x - x_0\|} = 0$$

$$f(x_0 + h) = f(x_0) + df(x_0)h + o(\|h\|) \quad \text{или} \quad \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0) - df(x_0)h}{\|h\|} = 0$$

$df(x_0): \mathbb{R}^n \to \mathbb{R}$ $\qquad h = (h_1, h_2, \dots, h_n)$

$e_1 = (1, 0, \dots, 0)$

$e_2 = (0, 1, \dots, 0)$

$\dots$

$e_i = (0, 0, \dots, 0, 1, 0, \dots, 0)$ (1 – i-та поз.)

$$df(x_0)h = df(x_0)\left(\sum_{i=1}^{n} h_i e_i\right) = \sum_{i=1}^{n} h_i\, df(x_0)e_i$$

$$\langle (df(x_0)(e_1), df(x_0)(e_2), \dots, df(x_0)(e_n)), h \rangle$$

скаларно произв.

$i \in \{1, 2, \dots, n\}$

$\{x_0 + \lambda e_i : \lambda \in \mathbb{R}\}$ права през x_0, успоредна на e_i

$$\lim_{\lambda \to 0} \frac{f(x_0 + \lambda e_i) - f(x_0)}{\lambda} =: \frac{df}{dx_i}(x_0)$$ частна производна на f в т. x_0 по x_i

$\{\lambda \in \mathbb{R} : x_0 + \lambda e_i \in U\}$ околност на 0 в $\mathbb{R}$

по x_1:

$$\frac{df}{dx_1}(x_0) = \lim_{\lambda \to 0} \frac{f(x_0 + \lambda e_1) - f(x_0)}{\lambda} = \lim_{\lambda \to 0} \frac{f(x_1^0 + \lambda, x_2^0, \dots, x_n^0) - f(x_1^0, \dots, x_n^0)}{\lambda} =$$

Твърдение Ако f е диференцируема в x_0, то $\frac{df}{dx_i}(x_0)$ същ. $\forall i \in \{1, 2, \dots, n\}$.

При това $df(x_0)(e_i) = \frac{df}{dx_i}(x_0)$

$$\lim_{\lambda \to 0} \frac{f(x_0 + \lambda e_i) - f(x_0)}{\lambda} = \lim \frac{f(x_0 + \lambda e_i) - f(x_0) - df(x_0)(\lambda e_i) + df(x_0)(\lambda e_i)}{\lambda} =$$

$$= \lim \left[\frac{f(x_0 + \lambda e_i) - f(x_0) - df(x_0)(\lambda e_i)}{\|\lambda e_i\|} \cdot \operatorname{sgn} \lambda + df(x_i)(e_i) \right] =$$

$$= 0 + df(x_i)(e_i)$$

От (*)

$$\operatorname{grad} f(x_0) = \left(\frac{\partial f}{\partial x_1}(x_0), \frac{\partial f}{\partial x_2}(x_0), \dots, \frac{\partial f}{\partial x_n}(x_0)\right)$$

$\downarrow$

градиент

$$df(x_0)(h) = \langle \operatorname{grad} f(x_0), h \rangle$$

Пример.

$$f(x_1,x_2) = \begin{cases} \dfrac{x_1+x_2}{x_1^2+x_2^2}, & (x_1,x_2) \neq (0,0) \\ 0, & (x_1,x_2) = (0,0) \end{cases}$$

$\downarrow$

Прекъсната в $(0,0)$.

$\frac{\partial f}{\partial x_i}$ съществува в $\mathbb{R}^2$ $\quad \lim\limits_{(x_1,x_2)\to(0,0)} \frac{x_1 x_2}{(x_1^2+x_2^2)^{3/2}} \nexists$

$(i=1,2)$

$$\frac{\partial f}{\partial x_1}(0,0) = \lim_{\lambda\to 0} \frac{f(0+\lambda,0) - f(0,0)}{\lambda} = 0$$

$$\frac{\partial f}{\partial x_2}(0,0) = \lim_{\lambda\to 0} \frac{f(0,0+\lambda) - f(0,0)}{\lambda} = 0$$

Твърдение: $\underset{\text{отв.}}{U \subset \mathbb{R}^n}$, $f: U \to \mathbb{R}$

f диф. в $x_0 \in U$

Тогава f е непрекъсната в x_0.

Доказателство:

$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0}\left(f(x) - f(x_0)\right) + f(x_0) =$$

$$= f(x_0) + \lim_{x\to x_0}\left(\frac{f(x) - df(x_0)(x-x_0)}{\|x-x_0\|} \cdot \|x-x_0\|\right) + \lim_{x\to x_0} df(x_0)(x-x_0) =$$

$= f(x_0) + 0 + 0$

$|df(x_0)(x-x_0)| \leq \|grad\, f(x_0)\| \cdot \|x-x_0\| \xrightarrow[x \to 0]{} 0$

Пример:

$$~~f(x_1, x_2)~~ \quad f(x) = \begin{cases} \|x\|^2 \cdot \sin \frac{1}{\|x\|^2}, & x \neq \theta \\ 0, & x = \theta \end{cases}$$

$x = (x_1, x_2) \neq (0,0)$

$f(x_1, x_2) = (x_1^2 + x_2^2) \cdot \sin \frac{1}{x_1^2 + x_2^2}$

$$\frac{\partial f}{\partial x_1}(0,0) = \lim_{\lambda \to 0} \frac{f(\lambda, 0) - f(0,0)}{\lambda} =$$

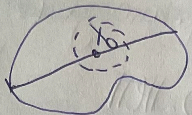
$$= \lim_{\lambda \to 0} \frac{\lambda^{\cancel{2}} \cdot \sin \frac{1}{\lambda^2}}{\cancel{\lambda}} = 0$$

$$\frac{\partial f}{\partial x_2}(0,0) = \lim_{\lambda \to 0} \frac{f(0,\lambda) - f(0,0)}{\lambda} = \lim_{\lambda \to 0} \lambda \cdot \sin \frac{1}{\lambda^2} = 0$$

$$\lim_{x \to 0} \frac{f(x) - \overset{0}{\overset{\|}{f(\theta)}} - \overset{0}{\overset{\|}{df(\theta)}}(x-\theta)}{\|x\|} = \lim_{x \to 0} \frac{\|x\|^{\cancel{2}} \cdot \sin \frac{1}{\|x\|^2}}{\cancel{\|x\|}} = 0$$

Производна по направление

$U \subset \mathbb{R}^2$, U отворено, $x_0 \in U$, $f: U \to \mathbb{R}$



$\ell \in \mathbb{R}^n$

$\{x_0+\lambda\ell : \lambda\in\mathbb{R}\}$

$\{\lambda\in\mathbb{R} : x_0+\lambda\ell\in U\}$ – околност на 0

$B_\delta(x_0)\subset U$

$x_0+\lambda\ell\in B_\delta(x_0) \iff \|\lambda\ell\|<\delta \iff |\lambda|.\|\ell\|<\delta$

$$\lim \frac{f(x_0+\lambda\ell)-f(x_0)}{\lambda} =: \frac{\partial f}{\partial \ell}(x_0)$$

производна на f по направление ℓ в т. x_0

Твърдение. Ако f е диференцуема в x_0, то f има производна в x_0 по всяко направление. При това

$$\frac{\partial f}{\partial \ell}(x_0)=df(x_0)(\ell)=\langle \operatorname{grad} f(x_0),\ell\rangle$$

Доказателство:

$$\lim_{\lambda\to 0}\frac{f(x_0+\lambda\ell)-f(x_0)}{\lambda}=\lim_{\lambda\to 0}\frac{f(x_0+\lambda\ell)-f(x_0)+df(x_0)(\lambda\ell)-df(x_0)(\lambda\ell)}{\lambda}=$$

$$=\lim\left[\frac{f(x_0+\lambda\ell)-f(x_0)-\lambda df(x_0)(\ell)}{\lambda}+\frac{\cancel{\lambda} fd(x_0)(\ell)}{\cancel{\lambda}}\right]=$$

$$=df(x_0)(\ell)+\lim_{\lambda\to 0}\underbrace{\frac{f(x_0+\lambda\ell)-f(x_0)-df(x_0)(\lambda\ell)}{\|\lambda\ell\|}}_{0}.\|\ell\|.\operatorname{sign}\lambda=$$

$$=df(x_0)(\ell), \text{ ако } \|\ell\|\neq 0$$

Ако $\|\ell\|=0$, тривиално

Следствие.

$$\max\left\{\frac{\partial f}{\partial \ell}(x_0) : \|\ell\| = 1\right\} = \|grad\, f(x_0)\|$$ достига за $\ell = \frac{grad\, f(x_0)}{\|grad\, f(x_0)\|}$

$$\|grad\, f(x_0)\| \cdot \|\ell\|$$

$$\left|\frac{\partial f}{\partial \ell}(x_0)\right| = |\langle grad\, f(x_0), \ell\rangle| \le \|grad\, f(x_0)\| \cdot \|\ell\|$$

$-\|grad\, f(x_0)\|$ достига се за

$$\min\left\{\frac{\partial f}{\partial \ell}(x_0) : \|\ell\| = 1\right\} = -\|grad\, f(x_0)\|$$

$$\ell = -\frac{grad\, f(x_0)}{\|grad\, f(x_0)\|}$$

Теорема. U отворено в $\mathbb{R}^n$

$f : U \to \mathbb{R}$

$x_0 \in U$

$\frac{\partial f}{\partial x_i}$ съществуват в U

$i = \{1, \dots, n\}$

Нека $\frac{\partial f}{\partial x_i}$ е непрекъснат в x_0

$i \in \{1, \dots, n\}$

Тогава f е диференцируема в x_0.

Означение: U отворено в $\mathbb{R}^n$, $f : U \to \mathbb{R}$

f се нарича гладка ($f \in C^1(U, \mathbb{R})$), ако $\frac{\partial f}{\partial x_i}$ съществуват и са непрекъснати в U за всяко $i \in \{1, \dots, n\}$

Следствие. Гладките функции са диференцируеми

Доказателство на теоремата:

$\delta > 0,\ B_\delta(x_0) \subset U$

$h \in \mathbb{R}^n,\ \|h\| < \delta$

$$f(x_0+h) - f(x_0) = \left(f(x_0 + h_1e_1) - f(x_0)\right) + \left(f(x_0 + h_1e_1 + h_2e_2) - f(x_0 + h_1e_1)\right) +$$

$$\dots + \Big(f(\underbrace{x_0 + \sum_{i=1}^{n-1} h_ie_i + h_ne_n}_{x_0+h}) - f(x_0 + \sum_{i=0}^{n-1} h_ie_i)\Big) = \frac{\partial f}{\partial x_1}(\xi_1)h_1 + \dots$$

$$+ \frac{\partial f}{\partial x_n}(\xi_n)h_n =$$

$=$

$$f(x_0 + \sum_{i=1}^{k-1} h_ie_i + h_ke_k) - f(x_0 + \sum_{i=1}^{k-1} h_ie_i) = \varphi_k(h_k) - \varphi_k(0) =$$

$$= \varphi_k(h_k) - \varphi_k(0) = \underset{0<\theta_k<1}{\varphi_k'(\theta_k \cdot h_k) \cdot h_k} = \frac{\partial f}{\partial x_k}(\xi_k) \cdot h_k, \text{ където } \underset{\xi_k \in B_\delta(x_0)}{\xi_k} = x_0 + \sum_{i=1}^{k-1} h_ie_i + \theta_k h_k e_k$$

$$\|x_0 + \sum_{i=1}^{k-1} h_ie_i - x_0\| \leq \|\sum_{i=1}^{k-1} h_ie_i\| = \sqrt{\sum_{i=1}^{k-1} h_i^2} \leq \|h\| < \delta$$

$$\varphi_k(\lambda) = f(x_0 + \sum_{i=1}^{k-1} h_ie_i + \lambda e_k)$$

λ между 0 и h_k

$$\varphi_k'(\lambda) = \frac{\partial f}{\partial x_k}(x_0 + \sum_{i=1}^{k-1} h_ie_i + \lambda e_k)$$

$$f(x_0+h) - f(x_0) - df(x_0)(h) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\xi_i)h_i - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x_0)h_i =$$

$$= \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i}(\xi_i) - \frac{\partial f}{\partial x_i}(x_0)\right) \cdot h_i$$

$$\left|\frac{f(x_0+h)-f(x_0)-df(x_0)(h)}{\|h\|}\right| =$$

$$= \frac{1}{\|h\|}\left|\sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}(\xi_i)-\frac{\partial f}{\partial x_i}(x_0)\right)\cdot h_i\right| \leq$$

$$\leq \frac{1}{\cancel{\|h\|}}\sqrt{\sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}(\xi_i)-\frac{\partial f}{\partial x_i}(x_0)\right)^2}\cdot\cancel{\|h\|} \xrightarrow[h\to 0]{} 0$$

$$\|\xi_k-x_0\| = \left\|\sum_{i=1}^{k-1} h_i e_i + \theta_k h_k \cdot e_k\right\| = \sqrt{\sum_{i=1}^{k-1} h_i^2 + \theta_k^2 h_k^2} \leq \|h\| \xrightarrow[h\to 0]{} 0$$

$\frac{\partial f}{\partial x_i}$ непрекъсната в $x_0 \Rightarrow \frac{\partial f}{\partial x_i}(\xi_i) \xrightarrow[h\to 0]{} \frac{\partial f}{\partial x_k}(x_0)$