Степенни редове

16.05.24

$S(x)=\sum_{n=0}^{\infty} a_n (x-a)^n \quad x\in D$

Намерете сумата:

① $\sum_{n=0}^{\infty} (n+1)x^n = S(x)$

$$\int S(x)\,dx = \int\sum_{n=0}^{\infty}(n+1)x^n\,dx = \sum_{n=0}^{\infty}\int (n+1)x^n\,dx = \sum_{n=0}^{\infty} \cancel{(n+1)}\frac{x^{n+1}}{\cancel{(n+1)}} = \sum_{n=0}^{\infty} x^{n+1} = x\cdot\frac{1}{1-x}+C \quad x\in(-1,1)$$

$$\int S(x)\,dx = \frac{x}{1-x}+C \qquad S(x)=\left(\frac{x}{1-x}\right)' = \frac{1\cdot(1-x)-x\cdot(-1)}{(1-x)^2} = \frac{(1-x)+x}{(1-x)^2} = \frac{1}{(1-x)^2}$$

крайни точки: $x=1 \quad \sum_{n=0}^{\infty}(n+1)$ разх.

$$\Rightarrow S(x)=\frac{1}{(1-x)^2} \quad x\in(-1,1)$$

$x=-1 \quad \sum_{n=0}^{\infty}(n+1)(-1)^n$ разх.

общия член не клони към 0

② $\sum_{n=0}^{\infty}(2n^2+3n+4)x^n = S(x)$

$$S(x)=\sum_{n=0}^{\infty}2n^2x^n+\sum_{n=0}^{\infty}3nx^n+\sum_{n=0}^{\infty}4x^n \qquad S(x)=2\cdot\frac{x(1+x)}{(1-x)^3}+\frac{3x}{(1-x)^2}+4\cdot\frac{1}{1-x} \quad x\in(-1,1)$$

$$4\sum_{n=0}^{\infty}x^n = 4\,\frac{1}{1-x}$$

$x=1$, $x=-1$ разх.

$$\sum_{n=0}^{\infty}x^n=\frac{1}{1-x}\ \Big|'$$

$$\sum_{n=0}^{\infty}n\cdot x^{n-1} = -\frac{1}{(1-x)^2}\cdot(-1)=\frac{1}{(1-x)^2}\ \Big|\cdot x$$

$$\sum_{n=0}^{\infty}n\cdot x^n=\frac{x}{(1-x)^2}\ \Big|'$$

$$\sum_{n=0}^{\infty}n^2x^{n-1}=\frac{1(1-x)^{\cancel{2}\,1}-x\cdot2\cancel{(1-x)}\cdot(-1)}{(1-x)^{\cancel{4}\,3}}=\frac{1-x+2x}{(1-x)^3}=\frac{1+x}{(1-x)^3}\ \Big|\cdot x$$

$$\sum_{n=0}^{\infty}n^2x^n=\frac{x(1+x)}{(1-x)^3}$$

③ $$\sum_{n=0}^{\infty}\frac{2n^2+3n+4}{3^n}=\sum_{n=0}^{\infty}(2n^2+3n+4)\left(\frac{1}{3}\right)^n \underset{\text{от ②}}{=} S\left(\frac{1}{3}\right) \qquad \frac{1}{3}\in(-1,1)$$

④ $\sum_{n=1}^{\infty} \frac{x^n}{n} = S(x)$

$S'(x) = \sum_{n=1}^{\infty} n \cdot \frac{x^{n-1}}{n} = \sum_{n=1}^{\infty} x^{n-1} = \frac{1}{1-x} \quad x \in (-1,1)$

$S(x) = \int \frac{1}{1-x} dx = -\int \frac{1}{1-x} d{-x} = -\ln|1-x| + C = -\ln|1-x| \qquad x \in [-1,1]$

$x=0 \quad \underset{0}{\underset{\shortparallel}{S(0)}} = -\ln|1| + C = 0 + C \Rightarrow C = 0$

крайни точки: $x=1 \quad \sum_{n=0}^{\infty} \frac{1}{n}$ разх.

$x=-1 \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{n}$ сходящ $= -\ln|1-(-1)| = -\ln 2$

⑤ $\sum_{n=2}^{\infty} \frac{1}{n^2-n} x^n = S(x)$

$S'(x) = \sum_{n=2}^{\infty} \frac{n}{n^2-n} x^{n-1} = \sum_{n=0}^{\infty} \frac{1}{n-1} x^{n-1}$

$S''(x) = \sum_{n=2}^{\infty} \frac{n-1}{n-1} x^{n-2} = \sum_{n=2}^{\infty} x^{n-2} = \frac{1}{1-x} \quad x \in (-1,1)$

$S'(x) = \int \frac{1}{1-x} dx = -\ln|1-x| + C = -\ln|1-x| \qquad \underset{0}{\underset{\shortparallel}{S'(0)}} = 0 + C \Rightarrow C = 0$

$S(x) = \int -\ln|1-x| dx = -x \cdot \ln|1-x| + \int x \, d\ln|1-x| = -x\ln|1-x| + \int x \cdot \frac{1}{1-x}(-1) dx =$

$= -x\ln|1-x| + \int \frac{1-x-1}{1-x} dx = -x\ln|1-x| + x + \ln|1-x| + C_1$

$S(x) = -x\ln|1-x| + x + \ln|1-x| + C_1 \qquad \underset{0}{\underset{\shortparallel}{S(0)}} = 0 + C_1 \Rightarrow C_1 = 0$

кр. точки $x=1 \quad \sum_{n=2}^{\infty} \frac{1}{n^2-n} \sim \sum_{n=2}^{\infty} \frac{1}{n^2}$ сходящ

$x=-1 \quad \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2-n}$ сходящ

⑥ $\sum_{n=2}^{\infty} \frac{2^n}{5^n(n^2-n)} = \sum_{n=2}^{\infty} \frac{1}{n^2-n} \left(\frac{2}{5}\right)^n \underset{⑤}{=} S\left(\frac{2}{5}\right)$

⑦ $\sum_{n=1}^{\infty} \frac{n}{(n-1)!} = \sum_{n=1}^{\infty} \frac{n}{(n-1)!} (1)^n = S(1) = e + e = 2e$

$S(x) = \sum_{n=1}^{\infty} \frac{n}{(n-1)!} x^n = x \cdot e^x + x^2 e^x$

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}\ \Big|'$$

$$e^x=\sum_{n=0}^{\infty}\frac{n\cdot x^{n-1}}{n!}=\sum_{n=0}^{\infty}\frac{x^{n-1}}{(n-1)!}\ \Big|\cdot x$$

$$xe^x=\sum_{n=0}^{\infty}\frac{x^n}{(n-1)!}\ \Big|'$$

$$1.e^x+xe^x=\sum_{n=0}^{\infty}\frac{n.x^{n-1}}{(n-1)!}\ \Big|\cdot x$$

$$xe^x+x^2e^x=\sum_{n=0}^{\infty}\frac{n.x^n}{(n-1)!}$$

⑧ $$\sum_{n=0}^{\infty}\frac{(-1)^n.n}{(2n+1)!}=\sum_{n=0}^{\infty}\frac{(-1)^n n}{(2n+1)!}1^{2n+1}=S(1)=\frac{1}{2}\cos 1-\frac{1}{2}\sin 1$$

$$S(x)=\sum\frac{(-1)^n n}{(2n+1)!}x^{2n+1}\qquad S(x)=x^2\sum_{n=0}^{\infty}\frac{(-1)^n n}{(2n+1)!}x^{2n+1}=x^2\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1-1)}{2(2n+1)!}x^{2n+1}$$

$$\sin x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$$

$$f(x)=x\int\sum_{n=0}^{\infty}\frac{(-1)^n n}{(2n+1)!}x^{2n+1}dx=x\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\cancel{n}\cdot\frac{x^{2n}}{2\cancel{n}}=$$

$$=x\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2(2n+1)!}=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2(2n+1)!}=\frac{\sin x}{2}$$

$$g(x)=\left(\frac{f(x)}{x}\right)'=\left(\frac{\sin x}{2x}\right)'=\frac{-\cos x.x-\sin x}{2x^2}$$

$$S(x)=x^2.g(x)=\frac{1}{2}\cos x-\frac{\sin x}{2}\qquad x\in\mathbb{R}$$

Функции на повече от една променлива

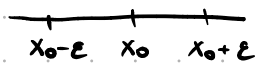
$X=(x_1,\dots,x_n)\in\mathbb{R}^n$ $\quad f:D\to\mathbb{R}\quad D\subseteq\mathbb{R}^n$ $\quad\varepsilon$ - околност на x_0

$$\rho(x,y)=\sqrt{\sum_{i=1}^{n}(x_i-y_i)^2}\qquad x,y\in\mathbb{R}^n$$

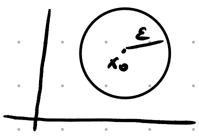
$$\|x-y\|=\rho(x,y)$$
$$\|x\|=\rho(0,x)=\sqrt{\sum_{i=1}^{n}x_i}$$

$$\{x\in\mathbb{R}^n:\ \|x-x_0\|<\varepsilon\}$$

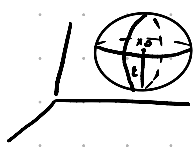
$\mathbb{R}^1$:



$\mathbb{R}^2$:



$\mathbb{R}^3$:



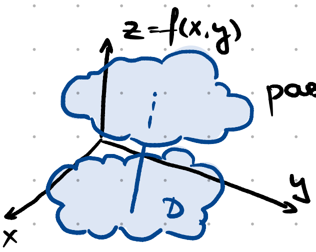
<u>Коши</u>: $\lim_{x\to x_0}f(x)=\ell \iff \forall\varepsilon>0\ \exists\delta>0:\ x\in D\setminus\{x_0\},\ \|x-x_0\|<\delta\quad |f(x)-\ell|<\varepsilon$

<u>Хайне</u>: $\lim_{x\to x_0}f(x)=\ell \iff \forall\{x_n\}_{n=1}^{\infty}:\ x_n\in D\setminus\{x_0\}$ и $x_n\xrightarrow[n\to\infty]{}x_0\quad f(x_n)\xrightarrow[n\to\infty]{}\ell$

Нека x_0 не е изолирана. $f(x)$ е непр. в т. x_0, ако $f(x) = f(x_0)$

$\mathbb{R}^2 \quad (x,y) \quad x,y \in \mathbb{R}$

$f: \underset{\subseteq \mathbb{R}^2}{D} \to \mathbb{R}$



равнина

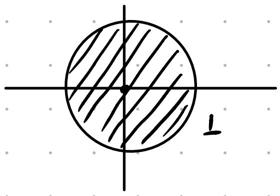
$\mathbb{R}^3: (x,y,z)$

$f: \underset{\subseteq \mathbb{R}^3}{D} \to \mathbb{R}$

① Определете дефиниционната област на:

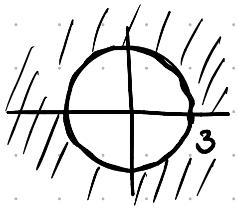
а) $f(x,y) = \sqrt{1-x^2-y^2}$ D: $1-x^2-y^2 \geq 0$

$x^2+y^2 \leq 1$

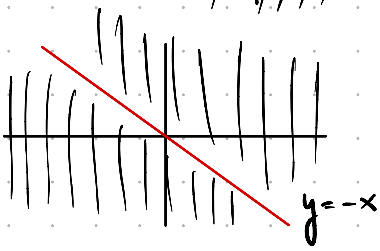


б) $f(x,y) = \sqrt{x^2+y^2-9}$ D: $x^2+y^2-9 \geq 0$

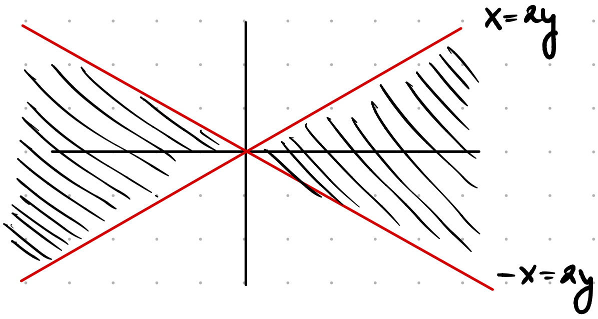
$x^2+y^2 \geq 3^2$



в) $f(x,y) = \dfrac{2x+3y+1}{x+y}$ D: $x \neq -y$



г) $f(x,y) = \arcsin \dfrac{2y}{x}$ D: $-1 \leq \dfrac{2y}{x} \leq 1$ $\begin{cases} x>0 \\ -x \leq 2y \leq x \end{cases} \cup \begin{cases} x<0 \\ -x \geq -2y \geq x \end{cases}$



② $\lim\limits_{(x,y)\to(0,0)} \dfrac{\ln(1+xy)}{3xy} = \lim\limits_{t\to 0} \dfrac{\ln(1+t)}{3t} = \dfrac{1}{3}$ $\to 1$

$t = xy \xrightarrow[\substack{x\to 0 \\ y \to 0}]{} 0$

③ $\lim\limits_{(x,y)\to(0,1)} \dfrac{\sin 2xy}{x} = \lim\limits_{(x,y)\to(0,1)} \dfrac{\sin 2xy}{x \cdot 2y} \cdot 2y = 2$ $\to 1$ $\to 1$

$t = x \cdot y \xrightarrow[\substack{x\to 0 \\ y \to 1}]{} 0$ $\qquad \dfrac{\sin 2xy}{2xy} = \dfrac{\sin 2t}{2t} \xrightarrow[t\to 0]{} 1$

④ $$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} = \lim_{t\to 0} \frac{t}{\sqrt{t+1}-1} \cdot \frac{\sqrt{t+1}+1}{\sqrt{t+1}+1} = \lim_{t\to 0} \frac{\cancel{t}\,(\sqrt{t+1}+1)}{\cancel{t}+\cancel{1}-\cancel{1}} = 2$$

$$t = x^2+y^2 \xrightarrow[\substack{x\to 0\\ y\to 0}]{} 0$$

⑤ $$\lim_{(x,y)\to(\frac{1}{2},\frac{1}{2})} \frac{\ln(x+y)-x-y+1}{(x+y-1)^2} = \lim_{t\to 1} \frac{\ln t - t + 1}{(t-1)^2} = \lim_{z\to 0} \frac{\ln(z+1)-z-\cancel{1}+\cancel{1}}{z^2} = \lim_{z\to 0} \frac{\ln(z+1)-z}{z^2} =$$

$$t = x+y \xrightarrow[\substack{x\to \frac{1}{2}\\ y\to 1/2}]{} 1 \qquad z = t-1 \xrightarrow[t\to 1]{} 0 \qquad = \lim_{z\to 0} \frac{-\frac{z^2}{2}+o(z^2)}{z^2} = -\frac{1}{2}$$

⑥ $$\lim_{(x,y)\to(0,0)} \frac{1-\cos(x^2+y^2)}{(x^2+y^2)\,x^2y^2} = \lim_{\substack{q\to 0\\ p\to 0}} \frac{1-\cos p}{p} \cdot \frac{1}{q} = \lim_{p\to 0} \frac{1-\cos p}{p} \cdot \lim_{q\to 0} \frac{1}{q}$$ × неопределеност

$$0 \cdot +\infty$$

$$p = x^2+y^2 \to 0$$
$$q = x^2y^2 \to 0$$

$$\lim_{p\to 0} \frac{1-\cos p}{p} = \lim_{p\to 0} \frac{\sin p}{p} = 0$$

$$\lim_{(x,y)\to(0,0)} \underbrace{\frac{1-\cos(x^2+y^2)}{(x^2+y^2)^2}}_{\to \frac{1}{2}} \frac{x^2+y^2}{x^2y^2} = \frac{1}{2} \lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x^2y^2} = \frac{1}{2} \lim_{(x,y)\to(0,0)} \left(\frac{1}{y^2}+\frac{1}{x^2}\right) = +\infty$$

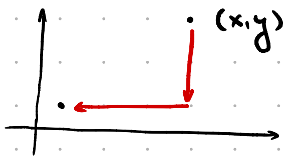
$$1-\cos p = 1 - 1 + \frac{p^2}{2} + o(p^2) \sim \frac{p^2}{2} \quad p\to 0$$

⑦ $$\lim_{(x,y)\to(0,0)} (1+x^2+y^2)^{\frac{3}{x^2+y^2}} = \lim_{t\to 0} (1+t)^{\frac{3}{t}} = \lim_{t\to 0} \Big(\underbrace{(1+t)^{1/t}}_{\to e}\Big)^3 = e^3$$

$$t = x^2+y^2 \xrightarrow[\substack{x\to 0\\ x\to 0}]{} 0$$

$$\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e$$
$$\lim_{y\to 0} (1+y)^{\frac{1}{y}} = e$$
$$y = \frac{1}{x}$$

повторни граници $$\lim_{y\to y_0}\left(\lim_{x\to x_0} f(x,y)\right) = \ell_1 \qquad \lim_{x\to x_0}\left(\lim_{y\to y_0} f(x,y)\right) = \ell_2$$



Ако съществува $\ell = \lim_{(x,y)\to(x_0,y_0)} f(x,y)$, то $\ell_1 = \ell_2 = \ell$. обратното не е в сила

⑧ $$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$$

$$\lim_{x\to 0}\left(\lim_{y\to 0} \frac{x^2-y^2}{x^2+y^2}\right) = \lim_{x\to 0} \frac{\cancel{x^2}}{\cancel{x^2}}\ 1 = 1$$

$$\neq \Rightarrow \lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} \text{ не съществува}$$

$$\lim_{y\to 0}\left(\lim_{x\to 0} \frac{x^2-y^2}{x^2+y^2}\right) = \lim_{y\to 0} \frac{-\cancel{y^2}}{\cancel{y^2}}\ -1 = -1$$

⑨ $\lim\limits_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

$$\lim_{x\to 0}\left(\lim_{y\to 0}\frac{xy}{x^2+y^2}\right) = \lim_{x\to 0}\frac{0}{x^2} = \lim_{x\to 0} 0 = 0$$

$$\lim_{y\to 0}\left(\lim_{x\to 0}\frac{xy}{x^2+y^2}\right) = \lim_{y\to 0}\frac{0}{y^2} = \lim_{y\to 0} 0 = 0$$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=kx}} f(x,y) = \lim_{\substack{(x,y)\to(0,0)\\ y=kx}}\frac{xy}{x^2+y^2} = \lim_{x\to 0}\frac{xkx}{x^2+k^2x^2} = \lim_{x\to 0}\frac{\cancel{x^2}k}{\cancel{x^2}(k^2+1)} = \lim_{x\to 0}\frac{k}{k^2+1}$$

по всяка права $y=kx$ има разл. граници т.к. $g(k)$ не е константна ф-я

$\Rightarrow \lim f(x,y)$ не същ.

⑩ $\lim\limits_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$

$$\lim_{x\to 0}\left(\lim_{y\to 0}\frac{xy^2}{x^2+y^4}\right) = \lim_{x\to 0}\frac{0}{x^2} = \lim_{x\to 0} 0 = 0$$

$$\lim_{y\to 0}\left(\lim_{x\to 0}\frac{xy^2}{x^2+y^4}\right) = \lim_{y\to 0}\frac{0}{y^4} = \lim_{y\to 0} 0 = 0$$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=kx}}\frac{xy^2}{x^2+y^4} = \lim_{x\to 0}\frac{x\,k^2x^2}{x^2+k^4x^4} = \lim_{x\to 0}\frac{\cancel{x^3}^{\,x}k^2}{\cancel{x^2}(1+k^4x^2)} = \lim_{x\to 0}\frac{x.\,k^2}{1+k^4x^2} = 0$$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=\sqrt{x}}}\frac{xy^2}{x^2+y^4} = \lim_{x\to 0}\frac{x.\,x}{x^2+x^2} = \lim_{x\to 0}\frac{\cancel{x^2}}{2\cancel{x^2}} = \frac{1}{2} \neq 0 \Rightarrow \lim f(x,y) \text{ не същ.}$$