$$\int_0^x \operatorname{arctg}^2 t\, dt \ge \underbrace{\int_0^{x_0} \operatorname{arctg}^2 t\, dt}_{I_1} + \int_{x_0}^x \underset{f(x_0)>0}{f(x_0)}\, dt = I_1 + f(x_0)\int_{x_0}^x 1\, dt = I_1 + f(x_0)(x-x_0) \xrightarrow[x\to+\infty]{} +\infty \Rightarrow \left[\frac{\infty}{\infty}\right]$$

$f(t)=\operatorname{arctg}^2 t$ - непрекъсната в $(-\infty,+\infty) \Rightarrow$ интегруема

$$F(x)=\int_0^x f(t)\, dt \overset{\text{Н.Л.}}{\Longrightarrow} F'(x)=f(x)=\operatorname{arctg}^2 x$$

29.02.24

Теорема (Лайбниц-Нютон)

$f: D\to\mathbb{R}$, f интегруема във всеки огр. инт. в D

Ако f е непр. в т. $x\in D$, то $F(x)=\int_a^x f(t)dt$ е диференцируема в x и $F'(x)=f(x)$

$f:[a,b]\to\mathbb{R}$ непрекъсната $\quad F(x)=\int_a^x f(t)dt \quad x\in[a,b] \Rightarrow F'(x)=f(x) \ \forall x\in[a,b] \quad F(b)=\int_a^b f(t)dt=F(b)-\underbrace{F(a)}_{=0}$

Следствие (ф-ла на Лайбниц Нютон)

$f:[a,b]$ непр., $\Phi:[a,b]\to\mathbb{R}$ - примитивна на f в $[a,b] \Rightarrow$

$$\int_a^b f(x)\,dx = \Phi(b)-\Phi(a)=\Phi(x)\Big|_a^b$$

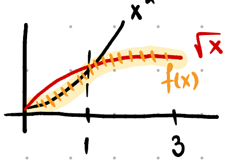
① $\displaystyle\int_1^3 2x+3\, dx = \left(\frac{\cancel{2}x^2}{\cancel{2}}+3x\right)\Big|_1^3 = 9+9-1-3=18-4=14$

② $\displaystyle\int_0^2 \frac{3}{1+x}\,dx = 3\int_0^2 \frac{1}{1+x}\,d(1+x) = 3\ln|1+x|\Big|_0^2 = 3\ln 3-3\ln 1=3\ln 3$

③ $\displaystyle\int_0^2 \frac{1}{x^2-2x+2}\,dx = \int_0^2\frac{1}{(x-1)^2+1}\,d(x-1) = \operatorname{arctg}(x-1)\Big|_0^2 = \operatorname{arctg}1-\operatorname{arctg}(-1)=\frac{\pi}{4}-\left(-\frac{\pi}{4}\right)=\frac{\pi}{2}$

④ $\displaystyle\int_0^\pi \frac{\sin x}{1+\cos^2 x}\,dx = \int_0^\pi \frac{-1}{1+\cos^2 x}\,d\cos x = -\operatorname{arctg}(\cos x)\Big|_0^\pi = -\left(\operatorname{arctg}(-1)-\operatorname{arctg}1\right) = -\left(-\frac{\pi}{4}-\frac{\pi}{4}\right)=\frac{\pi}{2}$

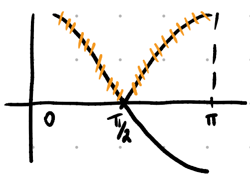
⑤ $\displaystyle\int_0^3 f(x)\,dx = \qquad f(x)=\begin{cases} x^2 & x\in[0;1] \\ \sqrt{x} & x\in(1,3]\end{cases}$



$$=\int_0^1 x^2\,dx+\int_1^3\sqrt{x}\,dx = \frac{x^3}{3}\Big|_0^1 + x^{3/2}\cdot\frac{2}{3}\Big|_1^3 = \left(\frac{1}{3}-0\right)+\left(\cancel{3}\sqrt{3}\cdot\frac{2}{\cancel{3}}-\frac{2}{3}\right) = -\frac{1}{3}+2\sqrt{3}$$

⑥ $\displaystyle\int_0^\pi\sqrt{\frac{1+\cos 2x}{2}}\,dx = \int_0^\pi\sqrt{\frac{\cos^2x+\cancel{\sin^2x}+\cos^2x-\cancel{\sin^2x}}{2}}\,dx = \int_0^\pi\sqrt{\frac{\cancel{2}\cos^2x}{\cancel{2}}}\,dx = \int_0^\pi|\cos x|\,dx = \int_0^{\pi/2}\cos x\,dx+\int_{\pi/2}^\pi -\cos x\,dx =$

$|\cos x| = \begin{cases}\cos x & 0\le x\le\frac{\pi}{2} \\ -\cos x & \frac{\pi}{2}\le x\le\pi\end{cases}$



$$=\sin x\Big|_0^{\pi/2}-\sin x\Big|_{\pi/2}^{\pi} = (1-0)-(0-1)=2$$

интегриране по части

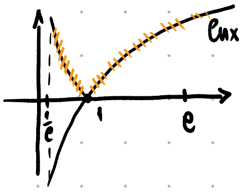
$$\int_a^b f(x)\,d\,g(x) = f(x)g(x)\Big|_a^b - \int_a^b g(x)\,d\,f(x)$$

$$P(x).\begin{cases}\sin x\\ \cos x\\ e^x\end{cases} dx \qquad \begin{cases}\text{arctg}\,x\\ \arcsin x\\ \ln x\end{cases}\cdot P(x)\,dx$$

(7) $$\int_0^{\pi/4} x\sin x\,dx = \int_0^{\pi/4} x\,d(-\cos x) = (-\cos x).x\Big|_0^{\pi/4} + \int_0^{\pi/4} -\cos x\,dx = -\left(\frac{\sqrt2}{2}.\frac{\pi}{4} - 1.0\right) + \sin x\Big|_0^{\pi/4} =$$

$$= -\frac{\sqrt2}{2}.\frac{\pi}{4} + \frac{\sqrt2}{2} - 0 = \frac{\sqrt2}{2}\left(1-\frac{\pi}{4}\right) > 0$$

(8) $$\int_{1/e}^e |\ln x|\,dx = \int_{1/e}^1 -\ln x\,dx + \int_1^e \ln x\,dx = -\ln x.x\Big|_{1/e}^1 + \int_{1/e}^1 x\,d\ln x + x.\ln x\Big|_1^e - \int_1^e x\,d\ln x =$$



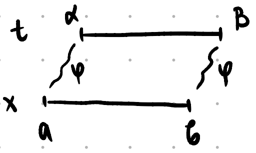
$$|\ln x| = \begin{cases}\ln x & 1\le x\le e\\ -\ln x & \frac1e\le x\le 1\end{cases}$$

$$= \cancel{-\underbrace{\ln 1}_{0}.1} + \underbrace{\ln\frac1e}_{-1}.\frac1e + \int_{1/e}^1 x.\frac1x\,dx + e.\underbrace{\ln e}_{1} - \cancel{1.\underbrace{\ln 1}_{0}} - \int_1^e x.\frac1x\,dx =$$

$$= -\frac1e + x\Big|_{1/e}^1 + e - x\Big|_1^e = -\frac1e + 1 - \frac1e + \cancel{e} - \cancel{e} + 1 = 2 - \frac2e$$

смяна на променливите

$\int_a^b f(x)\,dx$ $\quad f:[a,b]\to\mathbb{R}$ непр. $\qquad x=\varphi(t) \quad t\in[\alpha;\beta]$



$\varphi(\alpha)=a \qquad \varphi(t)\in[a,b]\ \forall t\in[\alpha,\beta]$

$\varphi(\beta)=b \qquad \varphi(t)$ непр. диф. ($\varphi'(t)$ непр.)

$$\Rightarrow \int_a^b f(x)\,dx = \int_\alpha^\beta f(\varphi(t))\underbrace{\varphi'(t)\,dt}_{dx}$$

(8) $$\int_1^4 \frac{2\sqrt x - 1}{1+\sqrt x}\,dx = \int_2^3 \frac{2(t-1)-1}{t}.2(t-1)\,dt = \int_2^3\frac{(2t-3).(2t-2)}{t}\,dt =$$

$t = 1+\sqrt x$ (II н. $t=\sqrt x$)

$t-1=\sqrt x \qquad x=1 \quad t=1+\sqrt1=2$

$(t-1)^2 = x \qquad x=4 \quad t=1+\sqrt4=3$

$dx = 2(t-1)\,dt$

$$=\int_2^3\frac{4t^2-4t-6t+6}{t}\,dt = \int_2^3 4t-4-6+\frac6t\,dt =$$

$$= \left(\frac{4t^2}{2}-10t+6\ln|t|\right)\Big|_2^3 = (\overbrace{2.9-30}^{-12}+6\ln 3)-(\overbrace{2.4-20}^{-12}+6\ln 2) =$$

$$= 6\ln 3 - 6\ln 2$$

$x=(t-1)^2 \iff t-1=\pm\sqrt{x}$

$t-1=\sqrt{x}$ $\qquad$ $t-1=-\sqrt{x}$

$x=1 \quad t-1=-1 \quad t=0$

$x=4 \quad t-1=-2 \quad t=-1$

$\int_0^1 \dots 2(t-1)\,dt =$ общия резултат

$1 \ \text{———} \ 4$

$\varphi \{ \qquad \}\varphi$

$0 \ \text{———} \ -1$

(9) $\int_1^2 x e^{x^2} dx = \frac{1}{2}\int_1^2 e^{x^2} dx^2 = \frac{1}{2}\int_1^4 e^t dt = \frac{1}{2} e^t\Big|_1^4 = \frac{1}{2}(e^4-e^1)$

$x^2=t \qquad x=1 \quad t=1$

$\qquad\quad x=2 \quad t=4$

(10) $\cos^2x=1-\sin^2x$

$\int_{\pi/6}^{\pi/3}\frac{\cos^3x}{\sqrt[3]{\sin x}}dx = \int_{\pi/6}^{\pi/3}\frac{\cos^2x}{\sqrt[3]{\sin x}}\,d\sin x = \int_{1/2}^{\sqrt3/2}\frac{1-t^2}{\sqrt[3]{t}}dt = \int_{1/2}^{\sqrt3/2}\frac{1}{\sqrt[3]{t}}dt - \int_{1/2}^{\sqrt3/2}\frac{t^2}{\sqrt[3]{t}}dt = \int_{1/2}^{\sqrt3/2} t^{-1/3}dt - \int_{1/2}^{\sqrt3/2} t^{5/3}dt =$

$t=\sin x$

$x=\frac{\pi}{6} \quad t=\sin\frac{\pi}{6}=\frac{1}{2}$

$x=\frac{\pi}{3} \quad t=\sin\frac{\pi}{3}=\frac{\sqrt3}{2}$

$=\left(t^{2/3}\cdot\frac{3}{2}-t^{8/3}\frac{3}{8}\right)\Big|_{1/2}^{\sqrt3/2} = \left(t^{2/3}\left(\frac{3}{2}-t^2\cdot\frac{3}{8}\right)\right)\Big|_{1/2}^{\sqrt3/2} =$

$=\sqrt[3]{\frac{3}{4}}\cdot\left(\frac{3}{2}-\frac{3}{4}\cdot\frac{3}{8}\right)-\sqrt[3]{\frac{1}{4}}\cdot\left(\frac{3}{2}-\frac{1}{4}\cdot\frac{3}{8}\right)=\sqrt[3]{\frac{3}{4}}\cdot\frac{39}{32}-\sqrt[3]{\frac{1}{4}}\cdot\frac{45}{32}$

(11) $\int_0^{\ln3}\frac{e^x\sqrt{e^x+1}}{e^x+5}dx = \int_1^3\frac{\cancel{t}\sqrt{t+1}}{t+5}\cdot\frac{1}{\cancel{t}}dt = \int_1^3\frac{\sqrt{t+1}}{t+5}dt = \int_{\sqrt2}^2\frac{y}{y^2+4}\,2y\,dy = \int_{\sqrt2}^2\frac{2y^2+8-8}{y^2+4}dy = \int_{\sqrt2}^2\frac{2\cancel{(y^2+4)}}{\cancel{y^2+4}}-\frac{8}{y^2+4}dy =$

$t=e^x \qquad x=0 \quad t=e^0=1$

$x=\ln t \qquad x=\ln3 \quad t=e^{\ln3}=3$

$dx=\frac{1}{t}dt$

$y=\sqrt{t+1} \qquad t=1 \quad y=\sqrt2$

$y^2=t+1 \qquad t=3 \quad y=\sqrt{1+3}=2$

$t=y^2-1$

$2y\,dy=dt$

$=\int_{\sqrt2}^2 2\,dy-\int_{\sqrt2}^2\frac{8}{y^2+4}dy=$

$=2y\Big|_{\sqrt2}^2-\int_{\sqrt2}^2\frac{\cancel{8}^{2}\cdot2}{\cancel{4}\left(\frac{y^2}{4}+1\right)}\cdot d\frac{y}{\cancel{2}}=$

$=2\cdot2-2\sqrt2-4\int_{\sqrt2}^2\frac{1}{\left(\frac{y}{2}\right)^2+1}\,d\frac{y}{2}=4-2\sqrt2-4\cdot\mathrm{arctg}\frac{y}{2}\Big|_{\sqrt2}^2=4-2\sqrt2-4\left(\underbrace{\mathrm{arctg}\,1}_{\pi/4}-\mathrm{arctg}\frac{\sqrt2}{2}\right)=4-2\sqrt2-\pi+4\,\mathrm{arctg}\frac{\sqrt2}{2}$

(12) $\int_0^2\sqrt{4-x^2}\,dx=\int_0^{\pi/2}\sqrt{4-4\sin^2t}\,2\cos t\,dt=\int_0^{\pi/2}\sqrt{4\cos^2t}\,2\cos t\,dt=\int_0^{\pi/2}2\,|\cos t|\,2\cos t\,dt=\int_0^{\pi/2}4\cos^2t\,dt=\int_0^{\pi/2}\cancel{4}^{2}\cdot\frac{1+\cos2t}{\cancel{2}}dt=$

$\cos t>0$

$\sqrt{a^2-x^2}\to x=a\sin t$

$\qquad\qquad x=a\cos t$

$x=2\sin t \qquad x=0 \quad 2\sin t=0\Rightarrow\sin t=0 \quad t=0$

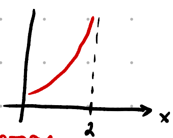
$dx=2\cos t\,dt \qquad x=2 \quad 2\sin t=2\Rightarrow\sin t=1 \quad t=\frac{\pi}{2}$

$=\int_0^{\pi/2}2\,dt+\int_0^{\pi/2}2\cos2t\,dt=2t\Big|_0^{\pi/2}+\int_0^{\pi/2}\cos2t\,d2t=$

$=\cancel{2}\frac{\pi}{\cancel{2}}-2\cdot0+\sin2t\Big|_0^{\pi/2}=\pi+\underbrace{\sin\pi}_{0}-\underbrace{\sin0}_{0}=\pi$

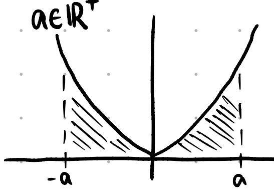
II начин $\int_0^2\sqrt{4-x^2}\,dx=x\cdot\sqrt{4-x^2}\Big|_0^2-\int_0^2x\cdot d\sqrt{4-x^2}=2\cdot0-0\cdot2-\int_0^2x\cdot\frac{1}{\cancel{2}\sqrt{4-x^2}}\cdot-\cancel{2}x\,dx=+\int_0^2\frac{x^2}{\sqrt{4-x^2}}dx$

несобствен интеграл



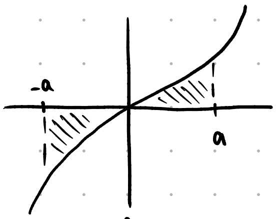
(13) $f:[-a,a]\to\mathbb{R}$ - непр.

$a\in\mathbb{R}^+$



1) $f(x)$ - четна $\qquad \int_{-a}^{a}f(x)\,dx=2\int_0^a f(x)\,dx$

$f(-x)=f(x)$



2) $f(x)$ - нечетна $\qquad \int_{-a}^{a} f(x)\,dx = 0$

$f(-x) = -f(x)$

Док: $$\int_{-a}^{a} f(x)\,dt = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx = \int_{a}^{0} f(-t)\,dt + \int_{0}^{a} f(x)\,dx = \circledast$$

$t=-x \quad x=-t \qquad x=-a \quad t=a$

$\qquad dx=-dt \qquad x=0 \quad t=0$

1) $f(t)$ - четна, $f(-t)=f(t)$ $$\circledast_1 = -\int_{a}^{0} f(t)dt + \int_{0}^{a} f(x)dx = \int_{0}^{a} f(t)dt + \int_{0}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

2) $f(t)$ - нечетна, $f(-t)=-f(t)$ $$\circledast_2 \; -\int_{a}^{0} -f(t)dt + \int_{0}^{a} f(x)\,dx = \int_{a}^{0} f(t)\,dt + \int_{0}^{a} f(x)\,dx = \int_{a}^{a} f(x)\,dx = 0$$

(14) $$\int_{-\pi/2}^{\pi/2} \sqrt{\cos x - \cos^3 x}\,dx = 2\int_{0}^{\pi/2} \sqrt{\cos x - \cos^3 x}\,dx = 2\int_{0}^{\pi/2} \sqrt{\cos x}\,\underbrace{\sqrt{1-\cos^2 x}}_{\sqrt{\sin^2 x}}\,dx = 2\int_{0}^{\pi/2} \sqrt{\cos x}\,\sin x\,dx = -2\int_{0}^{\pi/2} \sqrt{\cos x}\,d\cos x$$

$|\sin x|$

$\sin x \in [0, \frac{\pi}{2}] \Rightarrow > 0$

$f(x) = \sqrt{\cos x - \cos^3 x}$ $\qquad t = \cos x \qquad x=0 \quad \cos 0 = 1 = t$

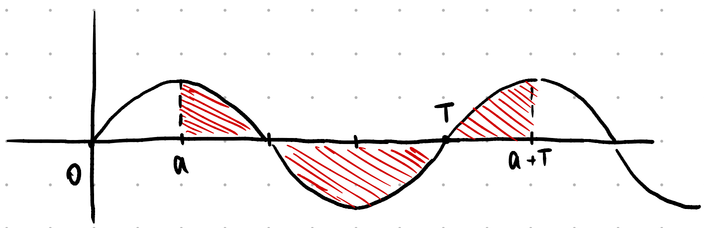
$\qquad x=\pi/2 \quad \cos \pi/2 = 0 = t$

$f(-x) = \sqrt{\cos(-x) - \cos^3(-x)} = \sqrt{\cos x - \cos^3 x} = f(x)$

$f(-x) = f(x) \Rightarrow f$ е четна

$$= -2\int_{1}^{0} \sqrt{t}\,dt = -2\,t^{3/2}\cdot\frac{2}{3}\Big|_{1}^{0} = -0 + \frac{4}{3} = \frac{4}{3}$$

(15) $f: D \to \mathbb{R}$ периодична с период T т.е $f(x+T) = f(x)$



$$\int_{a}^{a+T} f(x)\,dx = \int_{0}^{T} f(x)\,dx$$

$$\int_{a}^{a+T} f(x)\,dx = \int_{a}^{T} f(x)dx + \int_{T}^{a+T} f(x)\,dx = \int_{a}^{T} f(x)\,dx + \int_{0}^{a} \underset{f(t)}{f(t+T)}\,dt = \int_{a}^{T} f(x)dx + \int_{0}^{a} f(t)\,dt = \int_{0}^{T} f(x)\,dx$$

$x=t+T \qquad x=T \quad t=0$

$t=x-T \quad dt=dx \quad x=a+T \quad t=a$

(16) $$\int_{0}^{100\pi} \sqrt{1-\cos^2 x}\,dx = \int_{0}^{100\pi} \sqrt{\sin^2 x}\,dx = \int_{0}^{100\pi} |\sin x|\,dx = 100\int_{0}^{\pi} \sin x\,dx = 100\,(-\cos x)\Big|_{0}^{\pi} = 100\,(-(-1)+1) = 2\cdot 100 = 200$$

$$\int_{0}^{\pi/2} \sin x\,dx = \int_{\pi/2}^{\pi} \sin x\,dx$$

$|\sin x|$ е периодична с период π т.е. $|\sin(x+\pi)| = |-\sin x| = |\sin x|$