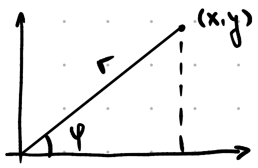
Полярни координати



$$\begin{cases} x = r\cos\varphi & r \geq 0 \\ y = r\sin\varphi & 0+\alpha \leq \varphi < 2\pi+\alpha \end{cases}$$

① $$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{r\to 0} \frac{\cancel{r^2}\sin\varphi\cos\varphi}{\cancel{r^2}} = \lim_{r\to 0} \sin\varphi\cos\varphi = \underset{\overset{\parallel}{g(\varphi)}}{\sin\varphi\cos\varphi} \neq const$$

$$\begin{cases} x = r\cos\varphi & r \geq 0 \\ y = r\sin\varphi & 0 \leq \varphi < 2\pi \end{cases}$$

$\varphi = 0 \quad 0$

$\varphi = \frac{\pi}{4} \quad \frac{1}{2}$

$$r = \sqrt{x^2+y^2} \xrightarrow[\substack{x\to 0 \\ y\to 0}]{} 0$$

$$\Rightarrow \nexists \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

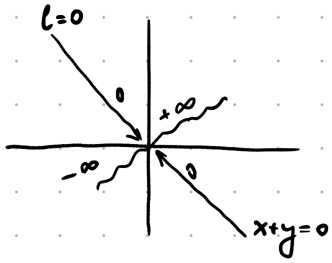
② $$\lim_{(x,y)\to(0,0)} \frac{x+y}{x^2+y^2} = \lim_{r\to 0} \frac{\cancel{r}\cos\varphi + \cancel{r}\sin\varphi}{r^{\cancel{2}}} = \lim_{r\to 0} \frac{1}{r}(\sin\varphi + \cos\varphi)$$

$$\begin{cases} x = r\cos\varphi & r \geq 0 \\ y = r\sin\varphi & 0 \leq \varphi < 2\pi \end{cases}$$

1) $\cos\varphi + \sin\varphi = 0 \quad \varphi = \frac{3\pi}{4}, \frac{7\pi}{4}$

2) $\cos\varphi + \sin\varphi > 0 \quad l = +\infty$

3) $\cos\varphi + \sin\varphi < 0 \quad l = -\infty$



$$\Rightarrow \nexists \lim_{(x,y)\to(0,0)} \frac{x+y}{x^2+y^2}$$

③ $$\lim_{(x,y)\to(0,0)} \frac{x^4-xy^3+y^4}{x^4+x^2y^2+y^4} = \lim_{r\to 0} \frac{r^4\cos^4\varphi - r\cos\varphi\, r^3\sin^3\varphi + r^4\sin^4\varphi}{r^4\cos^4\varphi + r^2\cos^2\varphi\, r^2\sin^2\varphi + r^4\sin^4\varphi} =$$

$$\begin{cases} x = r\cos\varphi & r \geq 0 \\ y = r\sin\varphi & 0 \leq \varphi < 2\pi \end{cases}$$

$$= \lim_{r\to 0} \frac{\cancel{r^4}(\cos^4\varphi - \cos\varphi\sin^3\varphi + \sin^4\varphi)}{\cancel{r^4}(\cos^4\varphi + \cos^2\varphi\sin^2\varphi + \sin^4\varphi)} =$$

$$= \frac{\cos^4\varphi - \cos\varphi\sin^3\varphi + \sin^4\varphi}{\cos^4\varphi + \cos^2\varphi\sin^2\varphi + \sin^4\varphi} = g(\varphi) \neq const$$

$\varphi = 0 \quad 1$

$\varphi = \frac{\pi}{4} \quad \frac{1}{3}$

$$\Rightarrow \nexists \lim_{(x,y)\to(0,0)} \frac{x^4-xy^3+y^4}{x^4+x^2y^2+y^4}$$

Ако (x,y) клони в друга точка примерно $(1,0)$ изместваме координатната система чрез полагане

$t = x-1 \xrightarrow[x\to 1]{} 0$

$s = y \xrightarrow[y\to 0]{} 0$

④ Изследвайте за непрекъснатост

$$f(x,y)=\begin{cases} \dfrac{x^3-y^3}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$$

$f(x,y)$ е непрекъсната в $\mathbb{R}\setminus\{0,0\}$ като комп. на непрекъснати ф-ии

$f(x,y)$ е непр. в $(0,0)$, ако $\exists \lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0,0)$

$$\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2} = \lim_{r\to 0} \frac{\cancel{r^3}\cos^3\varphi - \cancel{r^3}\sin^3\varphi}{\cancel{r^2}} = \lim_{r\to 0} r\underbrace{(\cos^3\varphi-\sin^3\varphi)}_{\text{ограничена}} = 0 = f(0,0) \Rightarrow$$

$\left| \begin{array}{ll} x=r\cos\varphi & r\geq 0 \\ y=r\sin\varphi & 0\leq\varphi<2\pi \end{array} \right.$

$\Rightarrow f(x,y)$ е непрекъсната в $(0,0)$

⑤ Додефинирайте (ако е възможно) функцията $f(x,y)=(1+xy^2)^{\frac{1}{x^2+y^2}}$ в $(0,0)$ т.че тя да е непрекъсната в $(0,0)$

$$f(0,0) = \lim_{(x,y)\to(0,0)} (1+xy^2)^{\frac{1}{x^2+y^2}} = [1^\infty] = \lim_{(x,y)\to(0,0)} \underbrace{\boxed{(1+xy^2)^{\frac{1}{xy^2}}}}_{\to e}{}^{xy^2\cdot\frac{1}{x^2+y^2}} =$$

$$= e^{\lim\limits_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2}} = e^{\ell} = e^0 = 1$$

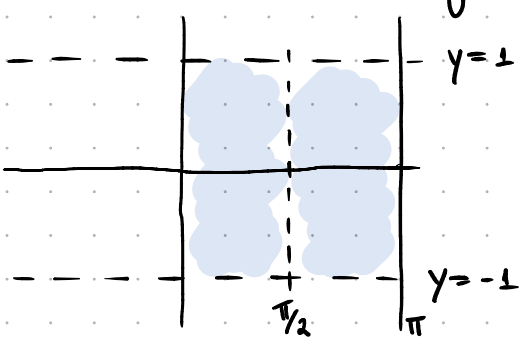
$$\ell = \lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2} = \lim_{r\to 0} \frac{r\cos\varphi\, \cancel{r^2}\sin^2\varphi}{\cancel{r^2}} = \lim_{r\to 0} r\underbrace{\cos\varphi\sin^2\varphi}_{\text{ограничена}} = 0$$

$\left| \begin{array}{ll} x=r\cos\varphi & r\geq 0 \\ y=r\sin\varphi & 0\leq\varphi<2\pi \end{array} \right.$

$\Rightarrow f(0,0)=1$

⑥ Изследвайте за непрекъснатост

$$f(x,y)=\begin{cases} \dfrac{\ln(1+y\cos x)}{\cos x}, & 0\leq x\leq\pi,\ x\neq\frac{\pi}{2},\ |y|<1 \\ y, & x=\frac{\pi}{2},\ |y|<1 \end{cases}$$



$f(x,y)$ е непрекъсната за $\left| \begin{array}{l} 0\leq x\leq\frac{\pi}{2},\ x\neq\frac{\pi}{2} \\ |y|<1 \end{array} \right.$

като композиция от непрекъснати ф-ии

Остава да проверим за т. $(\frac{\pi}{2}, y)$, $-1<y<1$

? $\lim\limits_{(x,y)\to(\pi/2,0)} f(x,y) = f(\frac{\pi}{2}, y_0)$, $-1<y_0<1$. Нека $-1<y_0<1$

. $\lim\limits_{(x,y)\to(\pi/2,y_0)} y = y_0 = f(\frac{\pi}{2}, y_0)$

• $\lim\limits_{(x,y)\to(\pi/2,y_0)} \frac{\ln(1+y\cos x)}{\cos x} = \lim\limits_{\substack{t\to0\\ y\to y_0}} \frac{\ln(1+yt)}{t} \overset{(\sim)}{=} \lim\limits_{\substack{t\to0\\ y\to y_0}} \frac{y\cancel{t}}{\cancel{t}} = y_0 = f(\frac{\pi}{2}, y_0)$

$t=\cos x \xrightarrow[x\to\frac{\pi}{2}]{} 0$ $\quad yt \xrightarrow[t\to0]{} 0$ $\quad \ln(1+yt)\sim yt \quad t\to0$

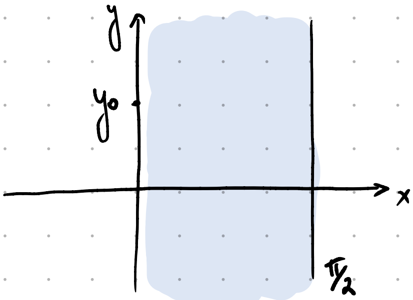
$\Rightarrow f(x,y)$ непрекъснатата в $(\frac{\pi}{2}, y_0)$

$\Rightarrow f(x,y)$ непрекъснатата в $(\frac{\pi}{2}, y)$, $-1<y<1$

⑦ Додефинирайте (ако е възможно) функцията

$$f(x,y) = \frac{y\sin x - \operatorname{arctg}(y\sin x)}{\sin^3 x} \quad 0<x\leq\frac{\pi}{2},\ y\in\mathbb{R}$$

за точките от вида $(0,y)$ т.че $f(x,y)$ да бъде непр. в тях.



$f(0,y_0) = \lim\limits_{(x,y)\to(0,y_0)} f(x,y) \quad y_0\in\mathbb{R}$

$\lim\limits_{(x,y)\to(0,y_0)} \frac{y\sin x - \operatorname{arctg}(y\sin x)}{\sin^3 x} = \lim\limits_{(t,y)\to(0,y_0)} \frac{yt - \operatorname{arctg}(yt)}{t^3} =$

$t=\sin x \xrightarrow[x\to0]{} 0 \quad yt \xrightarrow[\substack{t\to0\\ y\to y_0}]{} 0 \quad \operatorname{arctg} z = z - \frac{z^3}{3} + o(z^3) \quad z\to0$

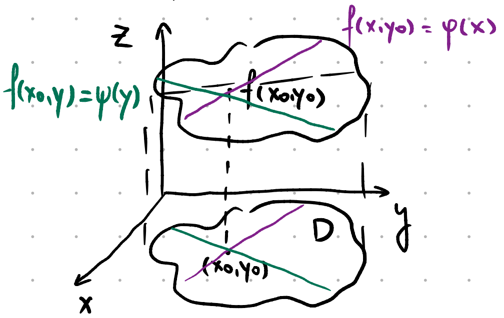
$= \lim\limits_{(t,y)\to(0,y_0)} \frac{yt - \left(yt - \frac{(yt)^3}{3} + o(yt)^3\right)}{t^3} = \lim\limits_{(t,y)\to(0,y_0)} \frac{y^3\cancel{t^3}}{3\cancel{t^3}} - \underbrace{\boxed{\frac{o(y^3t^3)}{y^3t^3}}}_{\to0} y^3 = \frac{y_0^3}{3} - 0 \qquad f(0,y) = \frac{y^3}{3} \quad \forall y\in\mathbb{R}$

$\lim\limits_{(x,y)\to(0,y_0)} \frac{y^3}{3} = \frac{y_0^3}{3} = f(0,y_0)$

$$\Rightarrow f(x,y) = \begin{cases} \frac{y\sin x - \operatorname{arctg}(y\sin x)}{\sin^3 x}, & 0<x\leq\frac{\pi}{2},\ y\in\mathbb{R} \\ \frac{y^3}{3}, & x=0,\ y\in\mathbb{R} \end{cases}$$

Частни производни и диференцируемост

$\mathbb{R}^2$: $\quad f: D \to \mathbb{R} \quad D \subseteq \mathbb{R}^2$ отворено



$$f'_x(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+h,y_0)-f(x_0,y_0)}{h} = \lim_{h\to 0} \frac{\varphi(x_0+h)-\varphi(x_0)}{h} = \varphi'(x_0)$$

$$\parallel$$

$\dfrac{df}{dx}(x_0,y_0) \longrightarrow$ частна производна на f по x в т. (x_0,y_0)

$$f'_y(x_0,y_0) := \frac{df}{dy}(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0,y_0+h)-f(x_0,y_0)}{h} =$$

$$= \lim_{h\to 0} \frac{\psi(y_0+h)-\psi(y_0)}{h} = \psi'(y_0)$$

① $f'_x,\ f'_y$

А) $f(x,y) = 3x^2y + 5y^4 - 2xy$

$f'_x(x,y) = 6xy + 0 - 2y$

$f'_y(x,y) = 3x^2 + 20y - 2x$

Б) $f(x,y) = \sin(1+x^2+2y^2)$

$f'_x(x,y) = \cos(1+x^2+2y^2)\cdot(1+x^2+2y^2)'_x =$

$= \cos(1+x^2+2y^2)\cdot 2x$

$f'_y(x,y) = \cos(1+x^2+2y^2)\cdot(1+x^2+2y^2)'_y =$

$= \cos(1+x^2+2y^2)\cdot 4y$

$$f''_{xx} = (f'_x)'_x = \frac{d^2f}{dx^2} = \frac{d}{dx}\left(\frac{d}{dx}f\right) \qquad f''_{yy} = (f'_y)'_y = \frac{d^2}{dy^2}f = \frac{d}{dy}\left(\frac{d}{dy}f\right)$$

$$f''_{xy} = (f'_x)'_y = \frac{d^2f}{dxdy} = \frac{d}{dy}\left(\frac{d}{dx}f\right) \qquad f''_{yx} = (f'_y)'_x = \frac{d^2}{dydx}f = \frac{d}{dx}\left(\frac{d}{dy}f\right)$$

② Частни производни до втори ред

А) $f(x,y) = 6x^3y + 2xy + y^3$

$f'_x(x,y) = 18x^2y + 2y + 0 \qquad f''_{xx} = (f'_x)'_x(x,y) = (18x^2y+2y)'_x = 36y + 0$

$f'_y(x,y) = 6x^3 + 2x + 3y^2 \qquad f''_{xy} = (f'_x)'_y(x,y) = (18x^2y+2y)'_y = 18x^2 + 2$

$f''_{yx} = (f'_y)'_x = (6x^3+2x+3y^2)'_x = 18x^2 + 2$

$f''_{yy} = (f'_y)'_y = (6x^3+2x+3y^2)'_y = 0 + 0 + 6y$

б) $f(x,y) = \operatorname{arctg}\frac{y}{x}$ $\qquad f''_{xx},\ f''_{xy},\ f''_{yy}$

$$f'_x(x,y) = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(\frac{y}{x}\right)'_x = \frac{1}{\frac{x^2+y^2}{\cancel{x^2}}}\cdot y\cdot\left(-\frac{1}{\cancel{x^2}}\right) = -\frac{y}{x^2+y^2}$$

$$f'_y(x,y) = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(\frac{y}{x}\right)'_y = \frac{1}{\frac{x^2+y^2}{x^2}}\cdot\frac{1}{x} = \frac{x}{x^2+y^2}$$

$$f''_{xx}(x,y) = \left(-\frac{y}{x^2+y^2}\right)'_x = -y\cdot\left(-\frac{1}{(x^2+y^2)^2}\right)\cdot 2x = \frac{2xy}{(x^2+y^2)^2}$$

$$f''_{xy}(x,y) = \left(-\frac{y}{x^2+y^2}\right)'_y = \frac{-1(x^2+y^2)+y2y}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$f''_{yy}(x,y) = \left(\frac{x}{x^2+y^2}\right)'_y = x\cdot\left(-\frac{1}{(x^2+y^2)^2}\right)\cdot 2y = \frac{-2xy}{(x^2+y^2)^2}$$

③ Докажете $f(x,y) = xe^{x+y}+yx+y^2$ удовлетворява диф. уравн.

$$f''_{xx}-2f''_{xy}+f''_{yy}=0$$

$f'_x(x,y) = 1.e^{x+y}+x.e^{x+y}+y+0$

$f''_{xx}(x,y) = (e^{x+y}+xe^{x+y})'_x = e^{x+y}+e^{x+y}+xe^{x+y} = (2+x)e^{x+y}$

$f''_{xy}(x,y) = (e^{x+y}+xe^{x+y})'_y = e^{x+y}+xe^{x+y}+\underline{1}$

$f'_y(x,y) = xe^{x+y}+x+2y$

$f''_{yy}(x,y) = (xe^{x+y}+x+2y)'_y = xe^{x+y}+2$

$$f''_{xx}-2f''_{xy}+f''_{yy} = (2+x)e^{x+y}-2.e^{x+y}-2xe^{x+y}-\cancel{2}+xe^{x+y}+\cancel{2} =$$

$$= \cancel{2e^{x+y}}+\cancel{xe^{x+y}}-\cancel{2e^{x+y}}-\cancel{2xe^{x+y}}+\cancel{xe^{x+y}} = 0$$

④ $f(x,y) = x(x-y)\cdot\sin\frac{1}{x^2+y^2}$, $(x,y)\neq(0,0)$

А) додеф. $f(x,y)$ в т. $(0,0)$ т.че $f(x,y)$ е непр. в $(0,0)$

б) $f'_x(0,0)=?$

$f'_y(0,0)=?$