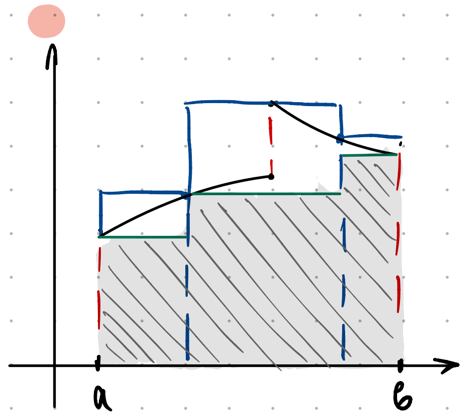
Определени интеграли

23.02.23



? Лице на специфична равнинна фигура

$f: [a,b] \to \mathbb{R}$ ограничена

$\sup\{f(x) : x \in [a,b]\} . (b-a)$ горна оценка

$\inf\{f(x) : x \in [a,b]\} . (b-a)$ долна оценка

$\tau: a = x_0 < x_1 < \dots < x_n = b_n$ (подразбиване на интервала $[a,b]$)

крайна редица

$[x_{i-1}, x_i]$: $M_i := \sup\{f(x) : x \in [x_{i-1}, x_i]\} . (x_i - x_{i-1})$ горна оценка за i-ти интервал

Голяма сума на Дарбу $S_f(\tau) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$ за f при подразбиване τ

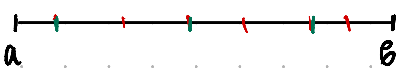
$m_i := \inf\{f(x) : x \in [x_{i-1}, x_i]\} . (x_i - x_{i-1})$ долна оценка за i-ти интервал

Малка сума на Дарбу $s_f(\tau) = \sum_{i=1}^{n} m_i (x_i - x_{i-1})$ за f при подразбиване τ

$$s_f(\tau) \leq S_f(\tau)$$

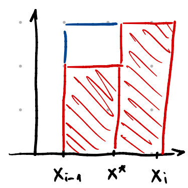
Лема 1 Ако $\tau^* \geq \tau$, то $S_f(\tau^*) \leq S_f(\tau)$ и $s_f(\tau^*) \leq s_f(\tau)$

τ^*, τ - 2 подразбивания; τ^* е по-фино от τ $(\tau^* \geq \tau)$ ако τ^* съдържа всички делящи точки от τ



Док: БОО. τ^* се получава от τ с прибавяне на точка, т.е $\tau^* = \tau \cup \{x^*\}$

$\tau := a = x_0 < x_1 < \dots < x_n = b$



$x^* \in [x_{i-1}, x_i]$

$\tau^* := a = x_0 < x_1 < \dots < x_{i-1} < x^* < x_i < \dots < x_n = b$

$i \in \{1, 2, \dots, n\}$

$S_f(\tau) - S_f(\tau^*) = \sum_{j=1}^{n} \sup_{[x_{j-1},x_j]} f . (x_j - x_{j-1}) - \Big(\sum_{j=1}^{i-1} \sup_{[x_{j-1},x_j]} f . (x_j - x_{j-1}) + \sup_{[x_{i-1},x^*]} f . (x^* - x_{i-1}) + \sup_{[x^*, x_i]} f . (x_i - x^*) +$

$$+\sum_{j=i+1}^{n} \sup_{[x_{j-1},x_j]} f\,(x_j-x_{j-1})\Bigg) = \sup_{[x_{i-1},x_i]} f\,(x_i-x_{i-1}) - \sup_{[x_{i-1},x^*]} f\,(x^*-x_{i-1}) - \sup_{[x^*,x_i]} f\,(x_i-x^*) =$$

$$= \sup_{[x_{i-1},x_i]} f\,(x_i-x_{i-1}) - \sup_{[x_{i-1},x_i]} f\,(x^*-x_{i-1}) - \sup_{[x_{i-1},x_i]} f\,(x_i-x^*) = 0$$

$$\sup_{[x_{i-1},x^*]} f \leq \sup_{[x_{i-1},x_i]} f \leq \sup_{[x^*,x_i]} f \leq \sup_{[x_{i-1},x_i]} f$$

<u>Лема 2</u> τ_1, τ_2 произволни подр. на $[a,b]$ Тогава $s_f(\tau_1) \leq S_f(\tau_2)$

Своц. $\tau^* \supseteq \tau_1,\ \tau^* \supseteq \tau_2$

$$s_f(\tau_1) \underset{\uparrow}{\leq} s_f(\tau^*) \leq S_f(\tau^*) \underset{\uparrow}{\leq} S_f(\tau_2)$$

$Л_1: \tau^* \supseteq \tau_1$ $Л_1: \tau^* \supseteq \tau_2$

$$[s_f(\tau_1), S_f(\tau_1)] \cap [s_f(\tau_2), S_f(\tau_2)] \neq \emptyset \quad \forall\, \tau_1, \tau_2 \text{ - подразб.}$$

$f: [a,b] \longrightarrow \mathbb{R}$ ограничена

$\overline{\int_a^b} f := \inf\{S_f(\tau) : \tau$ подразбиване на $[a,b]\}$ – горен интеграл на f в $[a,b]$

$\underline{\int_a^b} f := \sup\{s_f(\tau) : \tau$ подразбиване на $[a,b]\}$ – долен интеграл на f в $[a,b]$

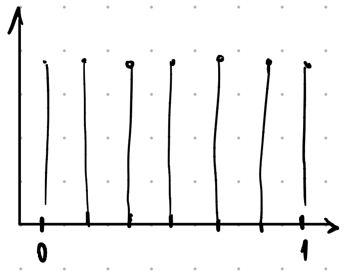
от Лема 2 $s_f(\tau_1) \leq \underbrace{S_f(\tau_2)}_{\tau_2 \text{ фикс.}}$ $\forall\, \tau_1, \tau_2$ подразбивания на $[a,b]$

$\Rightarrow \underline{\int_a^b} f \leq S_f(\tau_2)$ $\forall\, \tau_2$ подразбиване на $[a,b]$ $\Rightarrow \underline{\int_a^b} f \leq \overline{\int_a^b} f$

Дефиниция: $f: [a,b] \longrightarrow \mathbb{R}$ се нарича **интегруема по Риман** ако е ограничена и $\underline{\int_a^b} f = \overline{\int_a^b} f$. В този случай числото $\underline{\int_a^b} f = \overline{\int_a^b} f$ се нарича Риманов интеграл по f в $[a,b]$ и се бележи $\int_a^b f$ или $\int_a^b f(x)dx$

Пример за неинтегруема ф-я: функция на Дирихле

$$f(x)=\begin{cases}0, & \text{ако } x\in \mathbb{I}\cap[0,1]\\ 1, & \text{ако } x\in\mathbb{Q}\cap[0,1]\end{cases} \qquad f:[0,1]\to\mathbb{R}$$



$$s_f(\tau)=\sum_{i=1}^{n} 0\cdot(x_i-x_{i-1})=0 \qquad \underline{\int_0^1} f=0$$

$$S_f(\tau)=\sum_{i=1}^{n} 1\cdot(x_i-x_{i-1})=1 \qquad \overline{\int_0^1} f=1$$

Критерий за интегруемост

$f:[a,b]\to\mathbb{R}$ ограничена. Твърдим, че:

~ f е интегруема по Риман $\iff \forall\varepsilon>0\ \exists\tau_1,\tau_2$ подр. на $[a,b]$: $S_f(\tau_1)-s_f(\tau_2)<\varepsilon \iff$

$\iff \forall\varepsilon>0\ \exists\tau$ подр. на $[a,b]$: $S_f(\tau)-s_f(\tau)<\varepsilon$

Док: $(\Rightarrow)$ $\varepsilon>0$

$$\int_a^b f+\frac{\varepsilon}{2}=\overline{\int_a^b} f+\frac{\varepsilon}{2}>\overline{\int_a^b} f \Rightarrow \exists\,\tau_1 \text{ подр. на } [a,b],\ S_f(\tau_1)<\int_a^b f+\frac{\varepsilon}{2}$$

$$\int_a^b f-\frac{\varepsilon}{2}=\underline{\int_a^b} f-\frac{\varepsilon}{2}<\underline{\int_a^b} f \Rightarrow \exists\,\tau_2 \text{ подр. на } [a,b],\ s_f(\tau_2)>\int_a^b f-\frac{\varepsilon}{2}$$

$$S_f(\tau_1)-s_f(\tau_2)<\int_a^b f+\frac{\varepsilon}{2}-\int_a^b f+\frac{\varepsilon}{2}$$

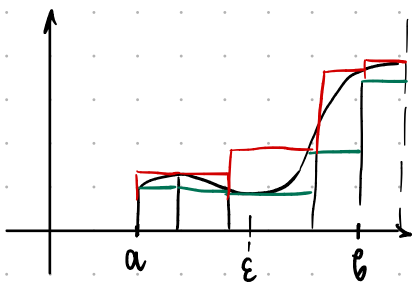
$$S_f(\tau_1)-s_f(\tau_2)<\varepsilon.$$

$(\Leftarrow)$ Допускаме противното $\Rightarrow \forall\tau_1,\tau_2$: $S_f(\tau_1)-s_f(\tau_2)\geq\overline{\int_a^b} f-\underline{\int_a^b} f>0$

$\Uparrow$ $\tau_1:=\tau,\ \tau_2:=\tau$

$\Downarrow$ $\varepsilon>0$ $\quad\tau_1,\tau_2\to S_f(\tau_1)-s_f(\tau_2)<\varepsilon$

$\tau\geq\tau_1,\ \tau\geq\tau_2 \Rightarrow S_f(\tau)-s_f(\tau)\leq S_f(\tau_1)-s_f(\tau_2)<\varepsilon$



τ: $a=x_0<x_1<\ldots<x_n=b$

$$S_f(\tau)-s_f(\tau)=\sum_{i=1}^{n}\Big(\sup_{[x_{i-1},x_i]} f-\inf_{[x_{i-1},x_i]} f\Big)(x_i-x_{i-1})$$

$f: [a,b] \to \mathbb{R}$ $\qquad \omega(f;[a,b]) = \sup\{|f(x)-f(y)| : x,y \in [a,b]\}$

ограничена

осцилация?

<u>Лема</u> $\omega(f;[a,b]) = \sup\limits_{[a,b]} f - \inf\limits_{[a,b]} f$ $\qquad x,y \in [a,b] \to f(x) \leq \sup\limits_{[a,b]} f,\ f(y) \geq \inf\limits_{[a,b]} f$

$|f(x)-f(y)| \to \left| \begin{array}{l} f(x)-f(y) \leq \sup f - \inf f \\ f(y)-f(x) \leq \sup f - \inf f \end{array} \right. \Rightarrow \omega(f,[a,b]) \leq \sup f - \inf f$

$\varepsilon > 0$

$\sup f - \inf f - \varepsilon$

$x_0 \in [a,b],\ f(x_0) > \sup f - \frac{\varepsilon}{2}$

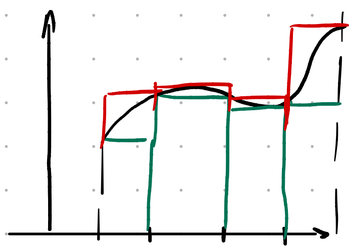
$y_0 \in [a,b],\ f(y_0) < \inf f + \frac{\varepsilon}{2}$ $\to |f(x_0)-f(y_0)| > \sup f - \inf f - \varepsilon$

$\forall \varepsilon > 0\ \exists \tau$ подразбиване на $[a,b]$: $\sum\limits_{i=1}^{n} \omega(f,[x_{i-1},x_i])$

<u>Твърдение:</u> Непрекъснатите функции са интегруеми

<u>Док:</u> $f: [a,b] \to \mathbb{R}$ от Вайерщрас $\to$ ограничена

непрекъсната



$\varepsilon > 0$

Кантор $\Rightarrow \forall \varepsilon > 0\ \exists \delta > 0\ \forall x', x'' \in [a,b],\ |x'-x''| < \delta : |f(x')-f(x'')| < \frac{\varepsilon}{2(b-a)}$

Взимаме τ: $a = x_0 < x_1 < \dots < x_n = b$

такива, че $\underbrace{\max\{x_i - x_{i-1} : i \in \{1,\dots,n\}\}}_{d(\tau)} < \delta$

Тогава

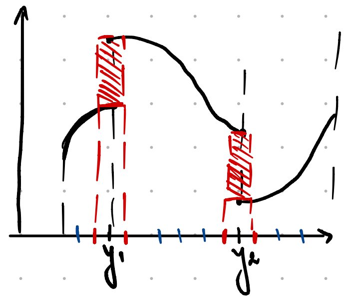
$S_f(\tau) - s_f(\tau) = \sum\limits_{i=1}^{n} \omega(f;[x_{i-1},x_i])(x_i - x_{i-1}) \leq \sum\limits_{i=1}^{n} \frac{\varepsilon}{2(b-a)}(x_i - x_{i-1}) = \frac{\varepsilon}{2\cancel{(b-a)}} \underbrace{\cancel{\sum\limits_{i=1}^{n}(x_i - x_{i-1})}}_{b-a}$

$= \frac{\varepsilon}{2} < \varepsilon$

$[x_i, x_{i-1}]$ $\qquad x', x'' \in [x_{i-1}, x_i]$

$x_i - x_{i-1} < \delta$ $\qquad \Rightarrow |x'-x''| < \delta \Rightarrow |f(x')-f(x'')| < \frac{\varepsilon}{2(b-a)} \Rightarrow \omega(f,[x_{i-1},x_i]) \leq \frac{\varepsilon}{2(b-a)}$

Твърдение 2 Нека $f:[a,b]\to\mathbb{R}$ е ограничена и има краен брой точки на прекъсване. Тогава f е интегруема



Доказателство: y_1, y_2 - точки на прекъсване на f

$\eta>0 \qquad C=[a,b]\setminus\bigcup_{i=1}^{k}(y_i-\eta, y_i+\eta)$

обединение на краен брой затворени интервали и f е непрекъсната в/у C

Кантор $\Rightarrow \exists\delta>0\ \forall x', x''\in C, |x'-x''|<\delta : |f(x')-f(x'')|<\frac{\varepsilon}{2(b-a)}$

τ: $a=x_0<x_1<\ldots<x_n=b$ такова, че $\begin{cases}[x_{i-1},x_i]\subset C, \text{ тогава } x_i-x_{i-1}<\delta \\ [x_{i-1},x_i]=[y_j-\eta, y_j+\eta]\cap[a,b] \text{ за някое } j\in\{1,2,\ldots n\}\end{cases}$

застраховка от излизане извън $[a,b]$

$$S_f(\tau)-s_f(\tau)=\sum_{i=1}^{n}\omega(f;[x_{i-1},x_i])(x_i-x_{i-1})\le\sum_{[x_{i-1},x_i]\subset C}^{n}\omega(f;[x_{i-1},x_i])(x_i-x_{i-1})+$$

$$+\sum_{j=1}^{k}\omega(f;[y_j-\eta,y_j+\eta]\cap[a,b])\cdot 2\eta \le \ldots \qquad \omega\le M-m \qquad M:=\sup_{[a,b]}f \ ;\ m:=\inf_{[a,b]}f$$

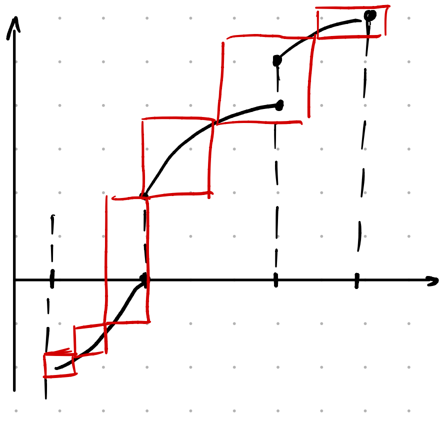
$$\le\frac{\varepsilon}{4(b-a)}\cdot\underbrace{\sum_{[x_{i-1},x_i]\subset C}(x_i-x_{i-1})}_{\le b-a}+(M-m)\cdot 2\eta\cdot k$$

$$S_f(\tau)-s_f(\tau)\le\frac{\varepsilon}{4\cancel{(b-a)}}\cdot\cancel{(b-a)}+(M-m)\cdot2\eta\cdot k<\varepsilon$$

Трябваше $0<\eta<\frac{\varepsilon}{4k(M-m)}$

Твърдение Монотонните функции са интегруеми

$f:[a,b]\to\mathbb{R}$ БОО f е растяща



$f(a)\le f(x)\le f(b)\ \forall x\in[a,b]\Rightarrow f$ е ограничена

τ: $a=x_0<x_1<\ldots<x_n=b$

$x\in[x_{i-1},x_i]\to f(x_{i-1})\le f(x)\le f(x_i)$

$$\inf_{[x_{i-1},x_i]}f(x)=f(x_{i-1}),\ \sup_{[x_{i-1},x_i]}f(x)=f(x_i)$$