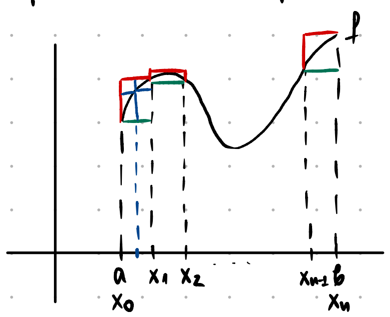
22.02.24

Определени интеграли

$f:[a,b]\to\mathbb{R}$, ограничена



Метод на Дарбу

$\tau: a=x_0<x_1<x_2<\dots<x_{n-1}<x_n=b$

подразбиване

$m_i=\inf\{f(x): x\in[x_{i-1},x_i]\}$ $\qquad s_f(\tau)=\sum_{i=1}^{n} m_i(x_i-x_{i-1})$ малка сума на Дарбу

$M_i=\sup\{f(x): x\in[x_{i-1},x_i]\}$ $\qquad S_f(\tau)=\sum_{i=1}^{n} M_i(x_i-x_{i-1})$ голяма сума на Дарбу

Ако τ_1 - по-дребно подразбиване на τ (съдържа точките от τ и допълнителни такива),

то $s_{\tau_1}\geq s_\tau$ и $S_{\tau_1}\leq S_\tau$

Нека τ_1,τ_2 - произволни и τ^* – по-дребно от τ_1 и τ_2

$s_{\tau_1}\leq s_{\tau^*}\leq S_{\tau^*}\leq S_{\tau_2}$ и $s_{\tau_1}\leq S_{\tau_1}$

долен интеграл $\underline{I}=\sup\{s_f(\tau): \tau\text{-подразбиване на }[a,b]\}$

горен интеграл $\overline{I}=\inf\{S_f(\tau): \tau\text{-подразбиване на }[a,b]\}$

$\underline{I}\leq\overline{I}$

Деф. (Дарбу) Нека $f:[a,b]\to\mathbb{R}$ е ограничена, f е интегруема по Риман, ако $\underline{I}=\overline{I}$ $\quad\int_a^b f(x)\,dx=\underline{I}=\overline{I}$

пример за неинтегруема функция

Функция на Дирихле

$$f(x)=\begin{cases}1, & x\in\mathbb{Q}\\ 0, & x\notin\mathbb{Q}\end{cases}$$

$\tau: 0=x_0<x_1<x_2<\dots<x_{n-1}<x_n=1$

$m_i=\inf\{f(x): x\in[x_{i-1},x_i]\}=0$

$M_i=\sup\{f(x): x\in[x_{i-1},x_i]\}=1$

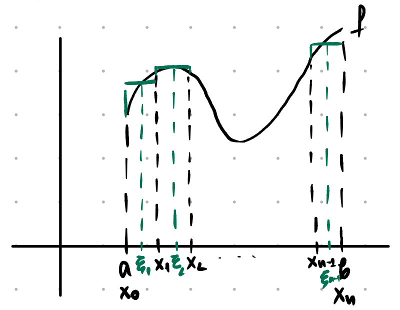
$s_f(\tau)=\sum_{i=1}^{n} m_i(x_i-x_{i-1})=0$ $\qquad\underline{I}=0$

$S_f(\tau)=\sum_{i=1}^{n} M_i(x_i-x_{i-1})=1$ $\qquad\overline{I}=1$

$\Rightarrow \underline{I}\neq\overline{I}\Rightarrow f$ не е интегруема в $[0,1]$

НДУ за интегруемост Нека $f:[a,b]\to\mathbb{R}$ е ограничена. f е интегруема в $[a,b]$ тстк.

$\forall\varepsilon>0\ \exists\tau$-подразбиване на $[a,b]$: $S_f(\tau)-s_f(\tau)<\varepsilon$



$f:[a,b]\to\mathbb{R}$, ограничена

Метод на Риман

$\tau: a=x_0<x_1<x_2<\ldots<x_{n-1}<x_n=b$

подразбиване

$\xi=(\xi_1,\xi_2,\ldots,\xi_n)$ - представителни точки

$\xi_i\in[x_{i-1},x_i] \quad i=1,2,\ldots,n$

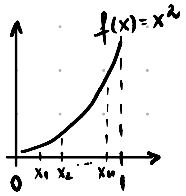
$\sigma_f(\tau,\xi)=\sum_{i=1}^{n} f(\xi_i)\cdot(x_i-x_{i-1})$ сума на Риман

$\operatorname{diam}(\tau)=\max\{x_i-x_{i-1},\ i=1,\ldots,n\}$

<u>Деф.</u> (Риман) Нека $f:[a,b]\to\mathbb{R}$ е ограничена. f е интегруема по Риман, ако $\exists\ I=\lim\limits_{\operatorname{diam}(\tau)\to 0}\sigma_f(\tau,\xi) \iff$

$\forall\varepsilon>0\ \exists\delta>0: \forall\tau$- подразбиване на $[a,b]$: $\operatorname{diam}(\tau)<\delta,\ \forall\xi \qquad |\sigma_f(\tau,\xi)-I|<\varepsilon$

① Докажете, че $f(x)=x^2$ е интегруема в $[0,1]$



$\tau: 0=x_0<x_1<x_2<\ldots<x_n=1$

$m_i=f(x_{i-1}) \qquad s_f(\tau)=\sum_{i=1}^{n} m_i(x_i-x_{i-1})=\sum_{i=1}^{n} f(x_{i-1})(x_i-x_{i-1})=\sum_{i=1}^{n} x_{i-1}^2(x_i-x_{i-1})$

$M_i=f(x_i) \qquad S_f(\tau)=\sum_{i=1}^{n} M_i(x_i-x_{i-1})=\sum_{i=1}^{n} f(x_i)(x_i-x_{i-1})=\sum_{i=1}^{n} x_i^2(x_i-x_{i-1})$

$S_f(\tau)-s_f(\tau)=\sum_{i=1}^{n} x_i^2(x_i-x_{i-1})-\sum_{i=1}^{n} x_{i-1}^2(x_i-x_{i-1})=\sum_{i=1}^{n}(x_i^2-x_{i-1}^2)(x_i-x_{i-1})$

Нека $\varepsilon>0$, $\tau_n: x_i=\frac{i}{n}$, $i=1,\ldots,n$ $\qquad 0\ \frac{1}{n}\ \frac{2}{n}\ \frac{3}{n}\ \ldots\ \frac{n-1}{n}\ 1=\frac{n}{n}$

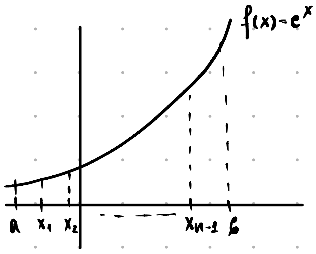
$S_f(\tau_n)-s_f(\tau_n)=\sum_{i=1}^{n}\underbrace{(x_i+x_{i-1})(x_i-x_{i-1})^2}_{\text{разлика от квадрати}}=\sum_{i=1}^{n}\frac{2i-1}{n}\cdot\frac{1}{n^2}=\frac{1}{n^3}\sum_{i=1}^{n}(2i-1)=\frac{1}{n^3}\cdot\frac{\cancel{1}+\cancel{2}n-\cancel{1}}{\cancel{2}}\cdot n=\frac{n^2}{n^3}=\frac{1}{n}\xrightarrow[n\to\infty]{}0 \Rightarrow$

за ε $\exists N: S_f(\tau_n)-s_f(\tau_n)<\varepsilon$ за $n>N \Rightarrow f(x)$ е интегруема в $[0,1]$

$\tau_n: x_i=\frac{i}{n},\ i=1,\ldots,n \quad \xi_i=\frac{i}{n} \qquad \sigma_f(\tau_n,\xi)=\sum_{i=1}^{n} f(\xi_i)(x_i-x_{i-1})=\sum_{i=1}^{n} f\left(\frac{i}{n}\right)\frac{1}{n}=\sum_{i=1}^{n}\frac{i^2}{n^2}\cdot\frac{1}{n}=\frac{1}{n^3}\sum_{i=1}^{n} i^2=\frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\xrightarrow[n\to\infty]{}\frac{2}{6}$

$\Rightarrow \int_0^1 x^2\,dx=\frac{1}{3}$

② Докажете, че $f(x)=e^x$ е интегруема в $[a,b]$, $a,b\in\mathbb{R}$ и пресметнете $\int_a^b e^x\,dx$



$\tau: a=x_0<x_1<x_2<\ldots<x_n=b$

$s_f(\tau)=\sum_{i=1}^{n} f(x_{i-1})(x_i-x_{i-1})=\sum_{i=1}^{n} e^{x_{i-1}}(x_i-x_{i-1})$

$S_f(\tau)=\sum_{i=1}^{n} f(x_i)(x_i-x_{i-1})=\sum_{i=1}^{n} e^{x_i}(x_i-x_{i-1})$

$S_f(\tau)-s_f(\tau)=\sum_{i=1}^{n}(e^{x_i}-e^{x_{i-1}})(x_i-x_{i-1})$

подразбиване на равни части

Нека $\varepsilon>0$ $\exists\, \tau_n: \ x_i = a + i.h_n \qquad h_n = \frac{b-a}{n}$

$$S_f(\tau) - s_f(\tau) = \sum_{i=1}^{n} (e^{x_i} - e^{x_{i-1}})(x_i - x_{i-1}) = \sum_{i=1}^{n} (e^{a+i.h_n} - e^{a+(i-1).h_n})(a+i.h_n - a+(i-1)h_n) = \sum_{i=1}^{n} (e^{a+ih_n} + e^{a+(i-1)h_n}).h_n =$$

$$= h_n.\sum_{i=1}^{n} (e^{a+i.h_n} - e^{a+(i-1)h_n}) = h_n.(e^{x_n} - e^{x_0}) = h_n(e^b - e^a) = \frac{b-a}{n}(e^b - e^a) \xrightarrow[n\to\infty]{} 0 \Rightarrow \text{за } \varepsilon>0\ \exists N: S_f(\tau_n) - s_f(\tau_n) < \varepsilon$$

за $n>N$

$$\sum_{i=1}^{n} e^{x_i} - e^{x_{i-1}} = \cancel{e^{x_1}} - e^{x_0} + \cancel{e^{x_2}} - \cancel{e^{x_1}} + e^{x_3} - \cancel{e^{x_2}} + \dots + e^{x_n} - \cancel{e^{x_{n-1}}} = -e^{x_0} + e^{x_n}$$

$\Rightarrow f(x) = e^x$ е интегруема в $[a,b]$

$\tau_n: \ x_i = a + i.h_n$

$\xi_i = a + ih_n$

$$\sigma_f(\tau_n, \xi) = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} e^{a+i.h_n}(a+i.h_n - (a+(i-1).h_n)) = \sum_{i=1}^{n} e^{a+i.h_n}.h_n = h_n.e^a \sum_{i=1}^{n} e^{ih_n} = h_n.e^a.\frac{e^{h_n}(e^{n.h_n}-1)}{e^{h_n}-1}$$

геом. пр. $q = e^{h_n}$

$$\lim_{n\to\infty} \sigma_f(\tau_n, \xi) = \lim_{n\to\infty} h_n.\frac{e^a.e^{h_n}(e^{n.h_n}-1)}{e^{h_n}-1} = \lim_{n\to\infty} \frac{b-a}{n}.\frac{e^a.e^{\frac{b-a}{n}}.(e^{n.\frac{b-a}{n}}-1)}{e^{\frac{b-a}{n}}-1} = (b-a)e^a.(e^{b-a}-1)\lim_{n\to\infty}\frac{e^{\frac{b-a}{n}}}{n(e^{\frac{b-a}{n}}-1)} \overset{*}{=}$$

$h_n = \frac{b-a}{n}$

$$= \cancel{(b-a)}e^a(e^{b-a}-1).\frac{1}{\cancel{b-a}} = e^b - e^a \Rightarrow \int_a^b e^x dx = e^b - e^a$$

$$* \lim_{y\to\infty}\frac{e^{\frac{b-a}{y}}}{y(e^{\frac{b-a}{y}}-1)} = 1.\lim_{y\to\infty}\frac{\frac{1}{y}}{e^{\frac{b-a}{y}}-1} = \left[\frac{0}{0}\right] \overset{\text{Лоп.}}{=} \lim_{y\to\infty}\frac{\cancel{-\frac{1}{y^2}}}{\underbrace{e^{\frac{b-a}{y}}}_{1}.\cancel{\left(-\frac{1}{y^2}\right)}(b-a)} = \frac{1}{b-a}$$

Свойства:

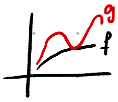
1) $\int_a^b f(x) \pm g(x)\, dx = \int_a^b f(x) dx \pm \int_a^b g(x)\, dx$

2) $\int_a^b c\, f(x)\, dx = c\int_a^b f(x)\, dx$

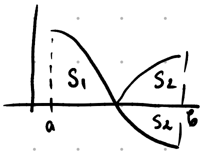
3) $\int_a^c f(x)\, dx + \int_c^b f(x)\, dx = \int_a^b f(x)\, dx$

4) $\int_a^a f(x)\, dx = 0$

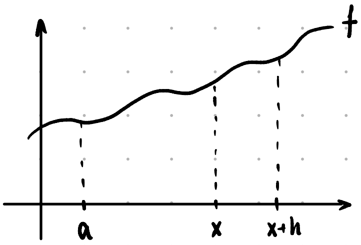
5) $\int_a^b f(x) dx \leq \int_a^b g(x)\, dx$, $f(x) \leq g(x)$, $x \in [a,b]$



6) $\int_a^b f(x) dx \leq \int_a^b |f(x)| dx$



7) $\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$



$f: D \to \mathbb{R}$, $a \in D$, f е интегруема във всеки ограничен и затворен интервал

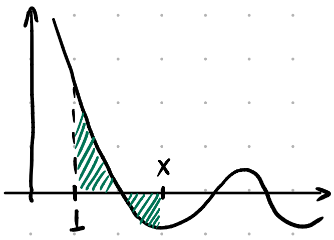
$$F(x) = \int_a^x f(t)\, dt$$

Теорема (Нютон-Лайбниц) $f: D \to \mathbb{R}$, D - интервал, f - интегруема във всеки ограничен и затворен интервал в D. Нека f е непрекъсната в т. $x \in D$, тогава

$F(x) = \int_a^b f(t)\,dt$ е диференцируема в т. x и $F'(x) = f(x)$

$$\lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \to 0} \frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h}$$

③ $F(x) = \int_1^x \frac{\cos t}{t}\,dt \qquad F'(x) = ? \qquad x > 1$



$f(t) = \frac{\cos t}{t}$ е непрекъсната за $t \neq 0 \Rightarrow f$ е интегруема

$F(x) = \int_1^x f(t)\,dt \xRightarrow{Н.-Л.} F'(x) = f(x) = \frac{\cos x}{x}$

④ $F(x) = \int_{x^2}^{e^x} \frac{1}{\ln t}\,dt \qquad F'(x) = ? \qquad x > 1$

$a > 1$: $\int_{x^2}^{e^x} \frac{1}{\ln t}\,dt = \int_{x^2}^{a} \frac{1}{\ln t}\,dt + \int_a^{e^x} \frac{1}{\ln t}\,dt = -\int_a^{x^2} \frac{1}{\ln t}\,dt + \int_a^{e^x} \frac{1}{\ln t}\,dt$

$f(t) = \frac{1}{\ln t}$ е непрекъсната в $(0;1) \cup (1;+\infty)$

$G(y) = \int_a^y f(t)\,dt \qquad F_1(x) = G(x^2) \qquad F_2(x) = G(e^x) \qquad \xRightarrow{Н.Л.} G'(y) = f(y) = \frac{1}{\ln y}$

$F_1'(x) = G'(x^2).(x^2)' = \frac{1}{\ln x^2} . 2x = \frac{1}{2\ln x} . 2x = \frac{1}{\ln x}$

$F_2'(x) = G'(e^x).(e^x)' = \frac{1}{\ln e^x} . e^x = \frac{e^x}{\ln e^x} = \frac{e^x}{x}$

⑤ $\lim_{x\to 0^+} \frac{\int_0^{x^2} \sin\sqrt{t}\,dt}{x^3} = \left[\frac{0}{0}\right] \overset{Лоп.}{=} \lim_{x \to 0^+} \frac{\sin|x| . 2\cancel{x}}{3x^{\cancel{2}}} = \lim_{x\to 0^+} \frac{\sin x . 2}{3x} = \frac{2}{3}$

$f(t) = \sin\sqrt{t}$ f е непрекъсната за $t \in [0, +\infty) \Rightarrow$ интегруема в $[a, b]$, $a \geq 0$

$F(x) = \int_0^{x^2} \sin\sqrt{t}\,dt = \int_0^{x^2} f(t)\,dt$

$G(y) = \int_0^y f(t)\,dt \xRightarrow{Н.Л.} G'(y) = f(y) = \sin\sqrt{y}$

$F(x) = G(x^2) \Rightarrow F'(x) = G'(x^2).(x^2)' = \sin\sqrt{x^2}.2x = \underline{\sin|x|\,2x}$

⑥ $\lim_{x\to+\infty} \frac{\int_0^x (\operatorname{arctg} t)^2\,dt}{\sqrt{x^2+1}} = \left[\frac{\infty}{\infty}\right] \overset{Лоп.}{=} \lim_{x\to+\infty} \frac{\operatorname{arctg}^2 x}{\frac{1}{\cancel{2}\sqrt{x^2+1}} . \cancel{2}x} = \frac{\frac{\pi^2}{4}}{1} = \frac{\pi^2}{4}$

$\operatorname{arctg}^2 t \xrightarrow[t\to+\infty]{} \frac{\pi^2}{4}$