1) Линейност

2) Адитивност

3) Позитивност

4) Теорема за средните стойности

$f, g : [a, b] \to \mathbb{R}$ интегруеми

$g(x) \geq 0 \quad \forall x \in [a, b]$

$m \leq f(x) \leq M \quad \forall x \in [a, b]$

Тогава $m . \int_a^b g(x) \leq \int_a^b f(x) g(x)\, dx \leq M . \int_a^b g(x) dx$

Лема $f, g : [a, b] \to \mathbb{R}$ интегруеми $\Rightarrow f.g$ е интегруема в $[a, b]$

$\tau : a = x_0 < x_1 < \ldots < x_n = b$

$\varepsilon > 0 \qquad S_{fg}(\tau) - s_{fg}(\tau) =$

$= \sum_{i=1}^{n} \omega(f.g ; [x_{i-1}, x_i]) (x_i - x_{i-1}) \leq$

$\leq \sum_{i=1}^{n} (M_g . \omega(f ; [x_{i-1}, x_i]) + M_f . \omega(g ; [x_{i-1}, x_i]) . (x_i - x_{i-1})$

$|f.g(x) - f.g(y)| = \quad x, y \in [x_{i-1}, x_i]$

$= |f(x)g(x) - f(y)g(y)| =$

$= |f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)| \leq$

$\leq |f(x)| . |g(x) - g(y)| + |g(y)| . |f(x) - f(y)| \leq$

$f(x) \leq M_f \qquad g(x) \leq M_g \quad \forall x \in [a, b]$

$\leq M_f . \omega(g ; [x_{i-1}, x_i]) + M_g . \omega(f ; [x_{i-1}, x_i]) \Rightarrow$

$\Rightarrow \omega(f.g ; [x_{i-1}, x_i] \leq M_g . \omega(f ; [x_{i-1}, x_i] + M_f . \omega(g ; [x_{i-1}, x_i]$

$= M_g \sum_{i=1}^{n} \omega(f, [x_{i-1}, x_i]) (x_i - x_{i-1}) + M_f . \sum_{i=1}^{n} \omega(g ; [x_{i-1}, x_i]) (x_i - x_{i-1})$

$S_{fg}(\tau) - s_{fg}(\tau) \leq M_g (S_f(\tau) - s_f(\tau)) + M_f . (S_g(\tau) - s_g(\tau))$

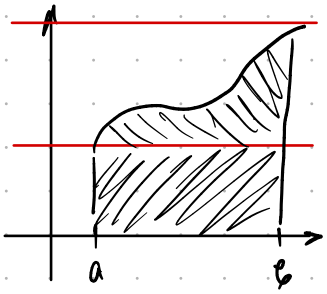
$\varepsilon > 0 \quad f$ интегруема $\Rightarrow \exists\, \tau_1$ подр. $S_f(\tau_1) - s_f(\tau_1) < \frac{\varepsilon}{(M_g + 1) . 2}$

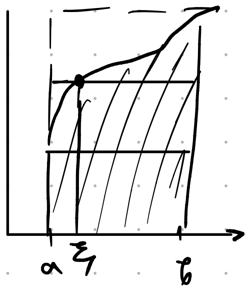
g интегруема $\Rightarrow \exists\, \tau_2$ подр. $S_f(\tau_2) - s_f(\tau_2) < \frac{\varepsilon}{(M_f + 1) . 2}$

$\tau \geq \tau_1, \tau \geq \tau_2$

$$\Rightarrow S_{fg}(\tau) - s_{fg}(\tau) < M_g \cdot \frac{\varepsilon}{2.(M_g+1)} + M_f \cdot \frac{\varepsilon}{2.(M_f+1)} < \varepsilon.$$

$$\boxed{\begin{aligned} S_f(\tau) - s_f(\tau) &\leq S_f(\tau_1) - s_f(\tau_1) \\ S_g(\tau) - s_g(\tau) &\leq S_g(\tau_2) - s_g(\tau_2) \end{aligned}}$$





Следствие: $f: [a,b] \to \mathbb{R}$ непр.

$g: [a,b] \to \mathbb{R}$ интегр., $g(x) \geq 0 \ \forall x \in [a,b]$

$\Rightarrow$ Съществува т. $\xi \in [a,b]$, $\int_a^b f(x) g(x) dx = f(\xi) . \int_a^b g(x) dx$

Док. Случай 1: $\int_a^b g(x) dx = 0$: $0 = m . \underbrace{\int_a^b g(x) dx}_{0} \leq \int_a^b f(x) g(x) dx \leq M . \underbrace{\int_a^b g(x) dx}_{0} = 0$

$\Rightarrow \int_a^b f(x) g(x) dx = 0 \Rightarrow \xi \in [a,b]$ произв.

Случай 2 $\int_a^b g(x) dx > 0$:

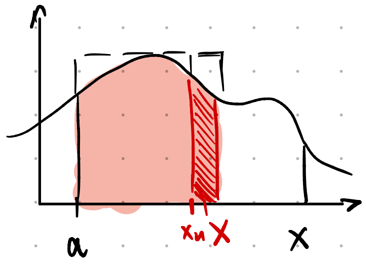
$$m = \inf_{[a,b]} f \leq \underbrace{\boxed{\frac{\int_a^b f(x) g(x) dx}{\int_a^b g(x) dx}}}_{C} \leq \sup_{[a,b]} f = M$$

f непр: Болцано + Вайерщрас $\Rightarrow \exists \ \xi \in [a,b], f(\xi) = c$

Определеният интеграл като ф-я на горната си граница

$f: \Delta \to \mathbb{R}$ $\quad$ f е интегруема в $[a,b] \ \forall x \in \Delta$

Δ интервал, $a \in \Delta$



$$F(x) = \int_a^x f(t)\,dt \qquad F: \Delta \to R$$

Твърдение $F: \Delta \to \mathbb{R}$ е непрекъсната

$x \in \Delta \qquad [x-\varepsilon, x+\varepsilon] \subset \Delta \to f$ интегруема в $[x-\varepsilon, x+\varepsilon] \to f$ ограничена в $[x-\varepsilon, x+\varepsilon]$ от M

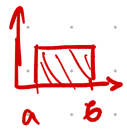
$$F(y) - F(x) = \int_a^y f(t)\,dt - \int_a^x f(t)\,dt = \int_x^y f(t)\,dt$$

$y \in [x-\varepsilon, x+\varepsilon]$

$$|F(y)-F(x)| = \left|\int_x^y f(t)\,dt\right| \leq \int_{\min\{x,y\}}^{\max\{x,y\}} |f(t)|\,dt \leq M.|x-y|$$

Забележка $a \leq b$

$$\int_a^b 1\,dt = b-a$$



$$\Rightarrow \lim_{y \to x} F(y) = F(x)$$

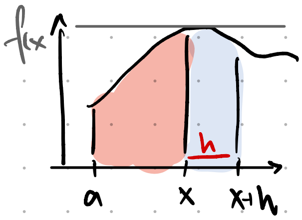
Ако x е: десен край на Δ $\quad [x-\varepsilon, x] \subset \Delta$ и ....

ляв край на Δ $\quad [x, x+\varepsilon] \subset \Delta$ и ....

Теорема на Нютон и Лайбниц

Нека допълнително f е непрекъсната в $x \in \Delta$. Тогава F е диференцируема в x и $F'(x) = f(x)$. $\qquad F(x) = \int_a^x f(t)\,dt$

$$\left|\frac{F(x+h)-F(x)}{h} - f(x)\right| = \left|\frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h} - f(x)\right| =$$



$$= \left|\frac{1}{h}\int_x^{x+h} f(t)\,dt - \frac{1}{h}\int_x^{x+h} f(x)\,dt\right| = \left|\frac{1}{h}\int_x^{x+h} [f(t)-f(x)]\,dt\right|$$

$$\int_x^{x+h} 1\,dt = h$$

$$h>0 \to \left|\frac{1}{h}\int_x^{x+h}(f(t)-f(x))dt\right| \le \frac{1}{|h|}\int_x^{x+h}|f(t)-f(x)|dt \le \frac{1}{|h|}\int_x^{x+h}\varepsilon dt = \varepsilon$$

$$h<0 \to \left|\frac{1}{h}\int_x^{x+h}(f(t)-f(x))dt\right| \le \frac{1}{|h|}\int_{x+h}^{x}|f(t)-f(x)|dt \le \frac{1}{|h|}\int_{x+h}^{x}\varepsilon dt = \varepsilon \cdot \frac{|h|}{|h|} = \varepsilon$$

$\varepsilon>0$ от f непр. в $x \Rightarrow \exists\, \delta>0$ такова че $\forall t\in\Delta, |t-x|<\delta : |f(t)-f(x)|<\varepsilon$

$|h|<\delta$, $x+h\in\Delta$

$$\left|\frac{F(x+h)-F(x)}{h} - f(x)\right| \le \varepsilon \Rightarrow \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} = f(x)$$

<u>Следствие 1</u> $f:\Delta\to\mathbb{R}$ Δ интервал, f непрекъсната, $a\in\Delta$

Тогава $F(x)=\int_a^x f(t)dt$ е диференцируема в Δ и $F'(x)=f(x)$ $\forall x\in\Delta$.

(Непр. функции в интервал имат примитивна)

Следствие 2 $f:[a,b]\to\mathbb{R}$ непрекъсната $G(x)$ - произволна примитивна ф-я за f в $[a,b]$

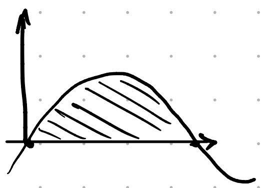
Тогава $\int_a^b f(x)dx = G(b)-G(a) =: G(x)\Big|_a^b$

G - примитивна и $F(x)=\int_a^x f(t)dt$ примит. за f в $[a,b]$

$F=G+C$, C - константа

$0=F(a)=G(a)+C \Rightarrow C=-G(a) \Rightarrow \int_a^b f(t)dt = F(b) = G(b)+C = G(b)-G(a)$

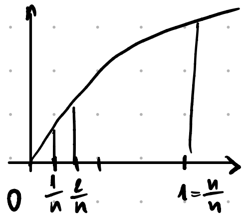
Пример 1.



$$\int_0^{\pi}\sin x\,dx = -\cos x\Big|_0^{\pi} = -\cos\pi+\cos 0 = 1+1=2$$

Пример 2 $a_n = \frac{1}{n\sqrt{n}} \sum_{i=1}^{n} \sqrt{i}$

$$a_n = \sum_{i=1}^{n} \sqrt{\frac{i}{n}} \cdot \frac{1}{n} = \sum_{i=1}^{n} \sqrt{\xi_i}\,(x_i - x_{i-1})$$



$\frac{i}{n},\ i \in [1,n]$

$$T_n: 0 = 0 < \frac{1}{n} < \frac{2}{n} < \dots < \frac{n}{n} = 1 \qquad \xi_i = \frac{1}{n} = \sigma_{\sqrt{i}}(T_n; \xi_n) \longrightarrow \int_0^1 \sqrt{x}\,dx \Rightarrow$$

$$\operatorname{diam} T_n = \frac{1}{n} \qquad \lim_{n\to\infty} a_n = \int_0^1 \sqrt{x}\,dx = \frac{x^{3/2}}{3/2}\Big|_0^1 = \frac{2}{3}$$

$$\left(\int_2^x \frac{dt}{\ln t}\right)' = \frac{1}{\ln x} \qquad \left(\int_2^{x^3} \frac{dt}{\ln t}\right)' = \frac{1}{\ln x^3} \cdot 3.x^2$$

$x > 1$

$$F(y) = \int_2^y \frac{dt}{\ln t} \qquad G(x) = \int_2^{x^3} \frac{dt}{\ln t}$$

$$G(x) = F(x^3) \qquad G'(x) = F'(x^3) \cdot 3x^2$$

$$\left(\int_{\sin x}^{x^3} \frac{dt}{\ln t}\right)' = \left(\int_{\sin x}^{1/2} \frac{dt}{\ln t} + \int_{1/2}^{x^3} \frac{dt}{\ln t}\right)' = \left(-\int_{1/2}^{\sin x} \frac{dt}{\ln t} + \int_{1/2}^{x^3} \frac{dt}{\ln t}\right) =$$

$x \in (0,1)$

$$= \frac{-1}{\ln(\sin x)} \cdot \cos x + \frac{3x^2}{\ln x^3}$$

$$f: \Delta \to \mathbb{R} \qquad a \in \Delta, b \in \Delta \qquad \int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

Δ инт., f непр.

$$\int_a^b f(x)\,dg(x) = \int_a^b f(x).g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b g(x)\,df(x) = f(x)g(x)\Big|_a^b - \int_a^b g(x).f'(x)\,dx$$

Пример $I_n = \int_0^{\pi/2} \cos^n x\,dx = \frac{(n-1)!!}{n!!} \begin{cases} \frac{\pi}{2}, & \text{ако } n \text{ четно} \\ 1, & \text{ако } n \text{ нечетно} \end{cases}$

$n > 1$

$$I_n = \int_0^{\pi/2} \cos^{n-1} x\,d\sin x = \sin x.\cos^{n-1} x\Big|_0^{\pi/2} - \int_0^{\pi/2} \sin x\,d\cos^{n-1} x = -\int_0^{\pi/2} \sin x\,(n-1)\cos^{n-2} x\,.-\sin x\,dx$$

$$= (n-1)\int_0^{\pi/2} \sin^2 x \cos^{n-2} x = (n-1)\int_0^{\pi/2} \cos^{n-2} - \cos^n dx$$

$$I_n = (n-1).I_{n-2} - (n-1) I_n = \frac{n-1}{n} . I_{n-2} = \frac{n-1}{n} . \frac{n-3}{n-2} . I_{n-4} \dots = \frac{(n-1)!!}{n!!} \begin{cases} I_0, \text{ ако } n \text{ четно} \\ I_1, \text{ ако } n \text{ нечетно} \end{cases}$$

$$I_0 = \int_0^{\pi/2} 1\, dx = \frac{\pi}{2} \qquad I_1 = \int_0^{\pi/2} \cos x\, dx = \sin x \Big|_0^{\pi/2} = 1$$

Теорема за смяна на променливите в определен интеграл

$f: \Delta_x \to \mathbb{R}$ непрекъсната, Δ_x - инт, $\Delta_x \supset [a,b]$ $\quad \int_a^b f(x)dx$

$\varphi: [c,d] \to \Delta_x$ диференцируема, $\quad \varphi(c)=a, \varphi(d)=b \quad \varphi'$ непрекъсната

$$\Rightarrow \int_a^b f(x)dx = \int_c^d f(\varphi(t))\, d\varphi(t) = \int_c^d f(\varphi(t))\, \varphi'(t)\, dt$$

F - примитивна за f в Δ

$\Phi(t) = F(\varphi(t)) \qquad \Phi'(t) = F'(\varphi(t)).\varphi'(t) = f(\varphi(t)).\varphi'(t)$

$$\int_c^d \underbrace{f(\varphi(t))\, \varphi'(t)}_{\text{непр.}} dt = \Phi(d) - \Phi(c) = F(\varphi(d)) - F(\varphi(c)) = F(b) - F(a) = \int_a^b f(x)dx$$

$$\int_0^{\pi/2} \sin^n x\, dx = \int_{\pi/2}^{0} \sin^n\left(\frac{\pi}{2} - t\right) d\left(\frac{\pi}{2} - t\right) = -\int_0^{\pi/2} \cos^n t\, d(-1)dt = \int_0^{\pi/2} \cos^n t\, dt$$

$x = \frac{\pi}{2} - t$

Формула на Уолис

$$a_n = \frac{1}{2n+1}\left(\frac{(2n)!!}{(2n-1)!!}\right)^2 \qquad 0 \le x \le \frac{\pi}{2}$$

$$\sin^{2n+1} x \le \sin^{2n} x \le \sin^{2n-1} x \quad \forall x \in \left[0, \frac{\pi}{2}\right]$$

$$\int_0^{\pi/2} \sin^{2n+1} x\, dx \le \int_0^{\pi/2} \sin^{2x} dx \le \int_0^{\pi/2} \sin^{2n-1} dx$$

$$\frac{(2n)!!}{(2n+1)!!} \le \frac{(2n-1)!!}{(2n)!!} \frac{\pi}{2} \le \frac{(2n-2)!!}{(2n-1)!!}$$

ляво $((2n)!!)^2 \le (2n-1)!!\,(2n+1)!!\cdot\frac{\pi}{2}$

$$a_n \le \frac{\pi}{2}$$

дясно $((2n-1)!!)^2\cdot\frac{\pi}{2} \le (2n-2)!!\,(2n)!!$

$$\frac{((2n)!!)^2}{2n} \ge \frac{\pi}{2}\left((2n-1)!!\right)^2$$

$$\frac{((2n)!!)^2}{((2n-1)!!)^2} \ge 2n\cdot\frac{\pi}{2}$$

$$\left(1-\frac{1}{2n+1}\right)\frac{\pi}{2} \le a_n \le \frac{\pi}{2} \Rightarrow a_n \xrightarrow[n\to\infty]{} \frac{\pi}{2}$$