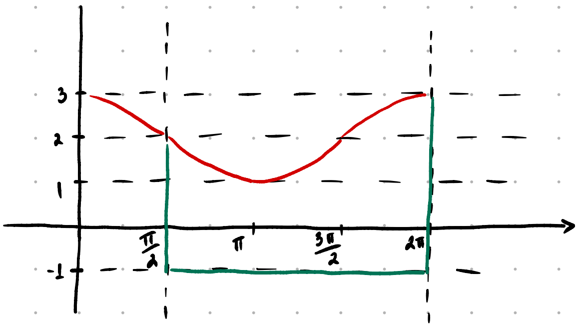
④ Намерете лицето на фигурата, ограничена от $y=\cos x+2$, $x=\frac{\pi}{2}$, $x=2\pi$, $y=-1$



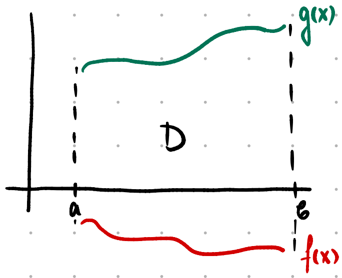
$$D: \left| \begin{array}{l} \frac{\pi}{2} \le x \le 2\pi \\ -1 \le y \le \cos x+2 \end{array} \right.$$

$$S_D = \int_{\frac{\pi}{2}}^{2\pi} \cos x+2-(-1)\,dx = \int_{\pi/2}^{2\pi} \cos x+3\,dx = (\sin x+3x)\Big|_{\frac{\pi}{2}}^{2\pi} = (0+6\pi)-\left(1+\frac{3\pi}{2}\right) = \frac{9}{2}\pi - 1$$

14.03.24

Приложение на определените интеграли

2) лица на равнинни фигури



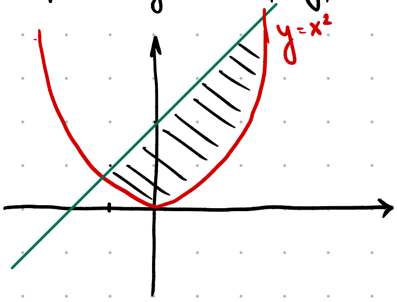
D- криволинеен трапец

$$D: \left| \begin{array}{l} a \le x \le b \\ f(x) \le y \le g(x) \end{array} \right.$$

$f, g: [a,b] \to \mathbb{R}$ непр.
$f(x) \le g(x) \quad x \in [a,b]$

$$S_D = \int_a^b g(x) - f(x)\,dx$$

① Намерете лицето на фигурата, заградена от $y=x^2$ и $y-x=2$



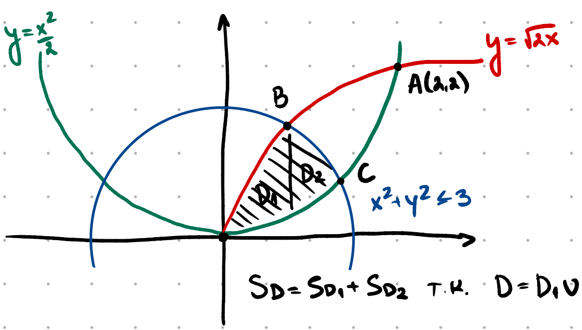
пресечни точки

$$\left| \begin{array}{l} y=x^2 \\ y-x=2 \end{array} \right. \qquad \begin{array}{ll} x^2-x-2=0 & \\ (x-2)(x+1)=0 & \\ x_1=2 & x_2=-1 \\ y_1=4 & y_2=1 \end{array}$$

$$D: \left| \begin{array}{l} -1 \le x \le 2 \\ x^2 \le y \le x+2 \end{array} \right.$$

$$S_D = \int_{-1}^{2} x+2-x^2\,dx = \left(\frac{x^2}{2}+2x-\frac{x^3}{3}\right)\Big|_{-1}^{2} = \left(2+4-\frac{8}{3}\right) - \left(\frac{1}{2}-2+\frac{1}{3}\right) = 4\frac{1}{2}$$

② $D \left| \begin{array}{l} y \ge \frac{x^2}{2} \\ y \le \sqrt{2x} \\ x^2+y^2 \le 3 \end{array} \right.$



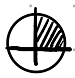
$S_D = S_{D_1} + S_{D_2}$ т.к. $D = D_1 \cup D_2$

Окр. с ц. (a,b) и рад. r
$(x-a)^2+(y-b)^2=r^2$
$\le$ $\ge$

Пресечни точки A: $\left| \begin{array}{l} y=\frac{x^2}{2} \\ y=\sqrt{2x} \end{array} \right.$ $\quad \frac{x^2}{2}=\sqrt{2x} \quad x^2 = 2\sqrt{2}\sqrt{x}$ $\quad x_1=0 \quad x_2=2$ $\quad A(2,2) \quad 2^2+2^2=8>3 \Rightarrow A$ външ.

B: $\begin{cases} y=\sqrt{2x} \\ x^2+y^2=3 \end{cases}$ $\quad x^2+2x-3=0$ $\quad x_1=-3<0$ $\quad x_2=1 \quad y_2=\sqrt{2}$ $\quad B(1,\sqrt{2})$

C: $\begin{cases} y=\frac{x^2}{2} \\ x^2+y^2=3 \end{cases}$ $\quad 2y=x^2$ $\quad y^2+2y-3=0$ $\quad y_1=-3<0$ $\quad C(\sqrt{2},1)$

$y_2=1 \quad x=\pm\sqrt{2}$ взимаме + тк. ни трябва 

$D_1 \begin{cases} 0\le x\le 1 \\ \frac{x^2}{2}\le y\le \sqrt{2x} \end{cases}$

$D_2 \begin{cases} 1\le x\le \sqrt{2} \\ \frac{x^2}{2}\le y\le \sqrt{3-x^2} \end{cases}$ $\quad y^2=3-x^2$

$y=\pm\sqrt{3-x^2}$ взимаме + тк. ни трябва 

$$S_{D_1}=\int_0^1 \sqrt{2x}-\frac{x^2}{2}\,dx=\left(\sqrt{2}\cdot x^{3/2}\cdot\frac{2}{3}-\frac{x^3}{2.3}\right)\Big|_0^1=\frac{2\sqrt{2}}{3}-\frac{1}{6}$$

$$S_{D_2}=\int_1^{\sqrt{2}} \sqrt{3-x^2}-\frac{x^2}{2}\,dx=\underbrace{\int_1^{\sqrt{2}}\sqrt{3-x^2}\,dx}_{I}-\frac{x^3}{2.3}\Big|_1^{\sqrt{2}}=\frac{3}{2}\left(\arcsin\sqrt{\frac{2}{3}}-\arcsin\frac{1}{\sqrt{3}}\right)-\frac{\sqrt{2}}{3}+\frac{1}{6}$$

$$I=\int_1^{\sqrt{2}}\sqrt{3-x^2}\,dx=x.\sqrt{3-x^2}\Big|_1^{\sqrt{2}}-\int_1^{\sqrt{2}} x\,d\sqrt{3-x^2}=\underbrace{\sqrt{2}-1.\sqrt{2}}_{0}-\int_1^{\sqrt{2}} x\,\frac{1}{2\sqrt{3-x^2}}(-2x)\,dx=-\int_1^{\sqrt{2}}\frac{3-x^2-3}{\sqrt{3-x^2}}\,dx=-\int_1^{\sqrt{2}}\sqrt{3-x^2}+3\int_1^{\sqrt{2}}\frac{1}{\sqrt{3-x^2}}\,dx$$

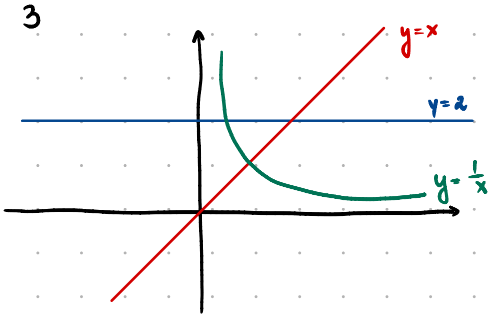
$$I=-I+3\int_1^{\sqrt{2}}\frac{1\cdot\sqrt{3}}{\sqrt{3}\sqrt{1-\left(\frac{x}{\sqrt{3}}\right)^2}}\,d\frac{x}{\sqrt{3}}$$

$$2I=3\arcsin\left(\frac{x}{\sqrt{3}}\right)\Big|_1^{\sqrt{2}}=3.\left(\arcsin\frac{\sqrt{2}}{\sqrt{3}}-\arcsin\frac{1}{\sqrt{3}}\right)$$

$$I=\frac{3}{2}\left(\arcsin\sqrt{\frac{2}{3}}-\arcsin\frac{1}{\sqrt{3}}\right)$$

$$S_D=\frac{3}{2}\left(\arcsin\sqrt{\frac{2}{3}}-\arcsin\frac{1}{\sqrt{3}}\right)+\frac{\sqrt{2}}{3}$$

(3) D: $\begin{cases} 0\le x\le y\le 2 \\ xy\ge 1 \end{cases}$ $\quad$ D: $\begin{cases} 0\le x \\ x\le y \\ y\le 2 \\ xy\ge 1 \end{cases}$

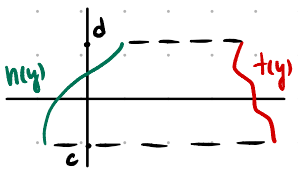


$xy\ge 1 \;|:x>0 \;(x=0 \;\; 0.y\ne 1)$

$y\ge\frac{1}{x}$

1 начин: $D=D_1\cup D_2$ 

2 начин: криволинеен трапец с хоризонтални основи



D: $\begin{cases} c\le y\le d \\ h(y)\le x\le f(y) \end{cases}$ $\quad h,f:[c,d]\to\mathbb{R}$ непр.

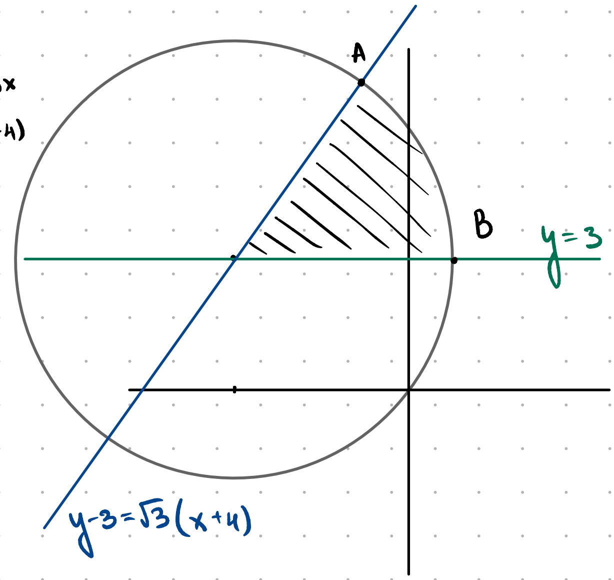
$h(y)\le f(y) \;\; \forall y\in[c,d]$

пресечни точки: $A \left| \begin{array}{l} y=x \\ y=\frac{1}{x}, x>0 \end{array} \right.$ $\quad x^2=1 \quad \begin{array}{l} x=\pm 1 \\ x>0 \end{array} \Rightarrow x=y=1$

$D: \left| \begin{array}{l} 1 \le y \le 2 \\ \frac{1}{y} \le x \le y \end{array} \right.$ $\quad S_D = \int_1^2 y - \frac{1}{y}\, dy = \left(\frac{y^2}{2} - \ln|y| \right) \Big|_1^2 = (2-\ln 2) - \left(\frac{1}{2} - \ln 1\right) = \frac{3}{2} - \ln 2$

④ $D \left| \begin{array}{l} x^2+y^2 \le 6y-8x \\ y-3 \le \sqrt{3}(x+4) \\ y \ge 3 \end{array} \right.$

кръг



$x^2+y^2 \le 6y-8x$
$x^2+8x+16+y^2-6y+9 \le 25$
$(x+4)^2+(y-3)^2 \le 25$
център $(-4,3)$ и радиус 5

$A \left| \begin{array}{l} y-3=\sqrt{3}(x+4) \\ (x+4)^2+(y-3)^2=25 \end{array} \right.$

$(x+4)^2+3(x+4)^2=25$
$4(x+4)^2=25$
$(x+4)^2=\left(\frac{5}{2}\right)^2$

$x+4=\pm\frac{5}{2}$
$x=-4\pm\frac{5}{2}$

$A\left(-4+\frac{5}{2},\ 3+\frac{5\sqrt{3}}{2}\right)$

$D \left| \begin{array}{l} 3 \le y \le 3+\frac{5\sqrt{3}}{2} \\ \frac{y-3}{\sqrt{3}} - 4 \le x \le -4+\sqrt{25-(y-3)^2} \end{array} \right.$

$y-3=\sqrt{3}(x+4)$
$\frac{y-3}{\sqrt{3}} = x+4$
$x = \frac{y-3}{\sqrt{3}} - 4$

$(x+4)^2+(y-3)^2=25$
$(x+4)^2 = 25-(y-3)^2$
$x+4 = \pm\sqrt{25-(y-3)^2}$
$x = -4 \pm \sqrt{25-(y-3)^2}$

$$S_D = \int_3^{3+\frac{5\sqrt{3}}{2}} -4+\sqrt{25-(y-3)^2} - \left(\frac{y-3}{\sqrt{3}} - 4\right) dy =$$

$$= \int_3^{3+\frac{5\sqrt{3}}{2}} \cancel{-4}+\sqrt{25-(y-3)^2} - \frac{y}{\sqrt{3}} + \sqrt{3} \cancel{+4}\, dy$$

$$= \underbrace{\int_3^{3+\frac{5\sqrt{3}}{2}} \sqrt{25-(y-3)^2}\, dy}_{I} - \left(\frac{y^2}{2\sqrt{3}} + \sqrt{3}y \Big|_3^{3+\frac{5\sqrt{3}}{2}} \right)$$

$$I = \int_3^{3+\frac{5\sqrt{3}}{2}} \sqrt{25-(y-3)^2}\, dy = \int_0^{\pi/3} \sqrt{25-25\sin^2 t}\ 5\cos t\, dt = \int_0^{\pi/3} 25\sqrt{\cos^2 t}\cos t\, dt = 25\int_0^{\pi/3} \cos^2 t\, dt =$$

$y-3 = 5\sin t \qquad y=3 \qquad 0=5\sin t \qquad t=0$

$dy = 5\cos t\, dt \qquad y=3+\frac{5\sqrt{3}}{2} \qquad \frac{5\sqrt{3}}{2} = 5\sin t \qquad t=\frac{\pi}{3}$

$$= 25\int_0^{\pi/3} \frac{1+\cos 2t}{2.2}\, d2t = 25\left(\frac{1}{\cancel{4}}.\cancel{2}t + \frac{\sin 2t}{4}\right)\Big|_0^{\pi/3} = 25\left(\frac{\pi}{6} + \frac{\sqrt{3}}{2.4}\right)$$

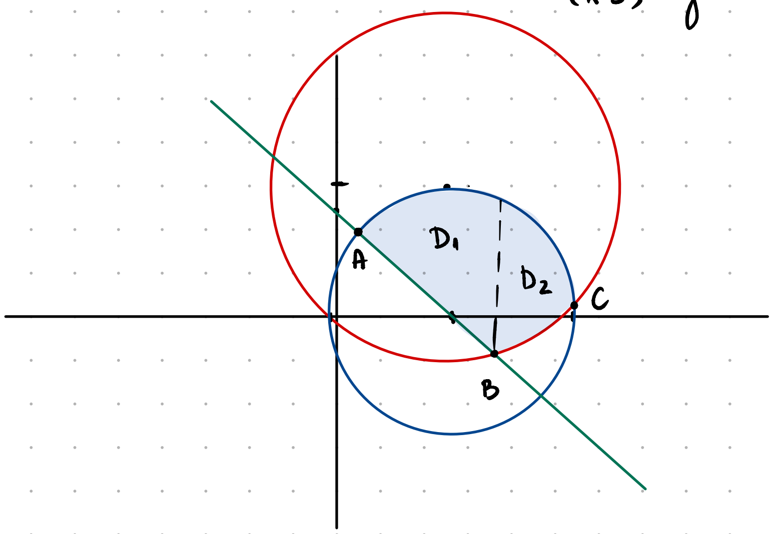
$$S_D = 25\left(\frac{\pi}{6} + \frac{\sqrt{3}}{8}\right) - \left(\frac{y^2}{2\sqrt{3}} + \sqrt{3}y \Big|_3^{3+\frac{5\sqrt{3}}{2}}\right)$$

⑤ $D \left| \begin{array}{l} x^2+y^2 \le 10x+12y \\ (x-5)^2+y^2 \le 30 \\ x+y \ge 5 \end{array} \right.$

$x^2-10x+25+y^2-12y+36 \le 61$

$(x-5)^2+(y-6)^2 \le 61$ център $(5,6)$ радиус $\sqrt{61}$

$(x-5)^2+y^2 \le 30$ център $(5,0)$ радиус $\sqrt{30}$



$A \left| \begin{array}{l} (x-5)^2+y^2=30 \\ x+y=5 \\ x<5 \end{array} \right.$ $5-x=y$ $2(x-5)^2=30$ $(x-5)^2=15$

$x-5=\pm\sqrt{15}$

$x=5-\sqrt{15}$

$y=\sqrt{15}$

$A(5-\sqrt{15}, \sqrt{15})$

$B \left| \begin{array}{l} x+y=5 \\ (x-5)^2+(y-6)^2=61 \\ x>5 \end{array} \right.$ $x-5=-y$

$y^2+y^2-12y+36=61$

$2y^2-12y-25=0$

$y_{1,2}=\frac{6\pm\sqrt{36+50}}{2}=$

$=\frac{6\pm\sqrt{86}}{2}$

$B\left(5-\frac{6-\sqrt{86}}{2}, \frac{6-\sqrt{86}}{2}\right)$

$C \left| \begin{array}{l} (x-5)^2+y^2=30 \\ (x-5)^2+(y-6)^2=61 \\ x>5 \Rightarrow x-5>0 \end{array} \right.$ $-$

$\cancel{y^2}-12y+36-\cancel{y^2}=31$

$12y=5$

$y=\frac{5}{12}$

$(x-5)^2+\frac{25}{144}=30$

$(x-5)^2=30-\frac{25}{144}$

$x-5=\pm\sqrt{\frac{4320-25}{144}}$

$x-5=\pm\frac{\sqrt{4295}}{12}$

$C\left(5+\frac{\sqrt{4295}}{12}, \frac{5}{12}\right)$

$D_1 \left| \begin{array}{l} x_A \le x \le x_B \\ 5-x \le y \le \sqrt{30-(x-5)^2} \end{array} \right.$

$x+y=5$

$y=5-x$

$(x-5)^2+y^2=30$

$y^2=30-(x-5)^2$

$y=\pm\sqrt{30-(x-5)^2}$

$D_2 \left| \begin{array}{l} x_B \le x \le x_C \\ 6-\sqrt{61-(x-5)^2} \le y \le \sqrt{30-(x-5)^2} \end{array} \right.$

$(x-5)^2+(y-6)^2=61$

$(y-6)^2=61-(x-5)^2$

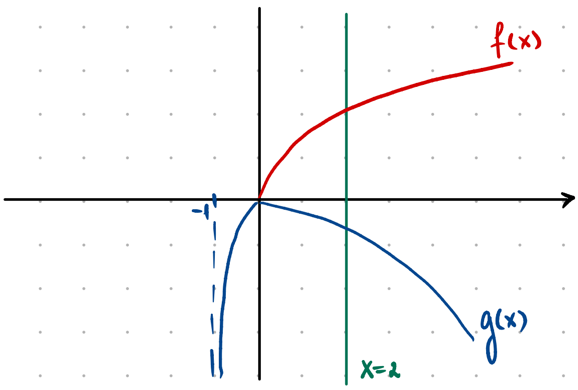
$y-6=\pm\sqrt{61-(x-5)^2}$

$y=6-\sqrt{61-(x-5)^2}$

$S_{D_1}=\int_{x_A}^{x_B} \sqrt{30-(x-5)^2}-(5-x)\,dx$ $S_{D_2}=\int_{x_B}^{x_C} \sqrt{30-(x-5)^2}-\left(6-\sqrt{61-(x-5)^2}\right)dx$

⑥ Намерете лицето на фигурата определена от $y=\frac{\sqrt{x}}{1+\sqrt{x}}$, $y=-x\ln(1+x)$, $x=2$

$x \ge 0$



$f(x)=\frac{\sqrt{x}}{1+\sqrt{x}}$ $g(x)=-x\ln(1+x)$

$x\in[0;2]$ $\left| \begin{array}{l} f(x)>0, \ x>0 \\ f(0)=0 \end{array} \right.$

$\left| \begin{array}{l} g(x)<0, \ x>0 \\ g(0)=0 \end{array} \right.$

$$D: \left| \begin{array}{l} 0 \le x \le 2 \\ -x\ln(1+x) \le y \le \frac{\sqrt{x}}{1+\sqrt{x}} \end{array} \right.$$

$$S_D = \int_0^2 \frac{\sqrt{x}}{1+\sqrt{x}} - \left(-x\ln(1+x)\right) dx = 2 - 2\sqrt{2} + 2\ln(1+\sqrt{2}) + \frac{3}{2}\ln 3$$

$$\int_0^2 \frac{\sqrt{x}+1-1}{1+\sqrt{x}} dx = \int_0^2 1 - \frac{1}{1+\sqrt{x}} dx = \int_0^{\sqrt{2}} \left(1 - \frac{1}{1+t}\right) 2t\, dt = \int_0^{\sqrt{2}} 2t - \frac{2t+2-2}{1+t} = \int_0^{\sqrt{2}} 2t - 2 + \frac{2}{1+t}\, dt =$$

$t = \sqrt{x}$ $\quad x=0 \;\; t=0$

$x = t^2$ $\quad x=2 \;\; t=\sqrt{2}$

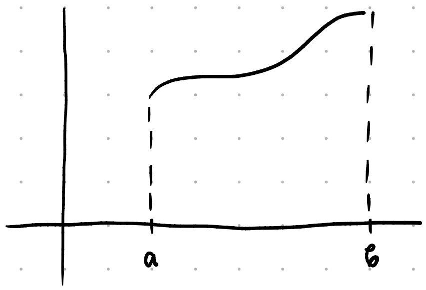
$dx = 2t\,dt$

$$= \left(t^2 - 2t + 2\ln|1+t|\right)\Big|_0^{\sqrt{2}} = 2 - 2\sqrt{2} + 2\ln|1+\sqrt{2}|$$

$$\int_0^2 x\ln(1+x)\,dx = \int_0^2 \ln(1+x)\, d\frac{x^2}{2} = \frac{x^2}{2}\ln(1+x)\Big|_0^2 - \int \frac{x^2}{2}\, d\ln(1+x) = 2\ln 3 - \int_0^2 \frac{x^2}{2}\frac{1}{1+x}dx = 2\ln 3 - \frac{1}{2}\int_0^2 \frac{\overbrace{x^2-1}^{(x-1)(x+1)}+1}{1+x}dx$$

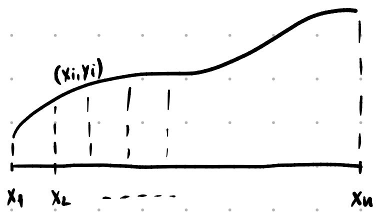
$$= 2\ln 3 - \frac{1}{2}\int_0^2 x - 1 + \frac{1}{1+x}dx = 2\ln 3 - \frac{1}{2}\left(\frac{x^2}{2} - x + \ln|1+x|\right)\Big|_0^2 = 2\ln 3 - \frac{1}{2}(2 - 2 + \ln 3) = \frac{3}{2}\ln 3$$

③ Дължина на крива



$f(x)$ – диференцируема в $(a-\varepsilon, b+\varepsilon)$

$$l = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$



$$\sum_{i=1}^n \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} = \sum_{i=1}^n \sqrt{\frac{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}{(x_i - x_{i-1})^2}}\,(x_i - x_{i-1}) =$$

$$= \sum_{i=1}^n \sqrt{1 + \underbrace{\left(\frac{y_i - y_{i-1}}{x_i - x_{i-1}}\right)}_{f'(x_{i-1})}^2}\,(x_i - x_{i-1})$$

① l на $\left| \begin{array}{l} y = x^2 \\ x \in [0;2] \end{array} \right.$

$f(x) = x^2$

$f'(x) = 2x$

$$l = \int_0^2 \sqrt{1 + (2x)^2}\,dx = \int_0^2 \sqrt{1 + 4x^2}\,dx = \int_0^{\operatorname{arctg} 4} \sqrt{1 + tg^2 t} \cdot \frac{1}{2} \cdot \frac{1}{\cos^2 t}\,dt = \int_0^{\operatorname{arctg} 4} \underbrace{\sqrt{\frac{\cos^2 t + \sin^2 t}{\cos^2 t}}}_{= \frac{1}{|\cos t|}} \cdot \frac{1}{2} \cdot \frac{1}{\cos^2 t}\,dt =$$

$2x = tg\,t$ $\quad x=0 \quad tg\,t = 0 \quad t=0$

$2dx = \frac{1}{\cos^2 t}dt$ $\quad x=2 \quad tg\,t = 4 \quad t = \operatorname{arctg} 4$

$$= \frac{1}{2}\int_0^{\operatorname{arctg} 4} \frac{1}{\cos^3 t} \cdot \frac{\cos t}{\cos t}\,dt = \frac{1}{2}\int_0^{\operatorname{arctg} 4} \frac{1}{\cos^4 t}\,d\sin t = \frac{1}{2}\int_0^{\sin(\operatorname{arctg} 4)} \frac{1}{(1-z^2)^2}\,dz = \frac{1}{2}\int_0^{\sin(\operatorname{arctg} 4)} \frac{1}{(1-z)^2(1-z)^2}\,dz = ⊛$$

$z = \sin t$ $\quad t=0 \quad z=0$

$t = \operatorname{arctg} 4 \quad z = \sin(\operatorname{arctg} 4)$

$\cos^4 t = (\cos^2 t)^2 = (1 - \sin^2 t)^2$

$$\frac{1}{(1-z)^2(1-z)^2} = \frac{A}{1-z} + \frac{B}{(1-z)^2} + \frac{C}{1+z} + \frac{D}{(1+z)^2}$$

$$1 = (1-z)(1+z)^2 + B(1+z)^2 + C(1+z)(1-z)^2 + D(1-z)^2$$