09.05.24

Степенни редове

$\sum_{n=0}^{\infty} a_n(x-a)^n$

разх. — абс сх — разх.

$a-R$ — a — $a+R$

D - област на сходимост

Развиване на функция в степенен ред

$$S(x) = \sum_{n=0}^{\infty} a_n (x-a)^n \quad x \in D$$

Th. Нека $S(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$. Тогава $S'(x) = \left(\sum_{n=0}^{\infty} a_n (x-a)^n\right)' =$

$$= \sum_{n=1}^{\infty} a_n \cdot n (x-a)^{n-1}$$

$$\int S(x)\,dx = \int \sum_{n=0}^{\infty} a_n (x-a)^n\,dx = \sum_{n=0}^{\infty} \int a_n (x-a)^n = \sum_{n=0}^{\infty} a_n \frac{(x-a)^{n+1}}{n+1}$$

и $S(x)$, $S'(x)$, $\int S(x)$ имат един и същ радиус на сход.

$\underline{S(a) = a_0}$ $\qquad S'(x) = \sum_{n=1}^{\infty} n \cdot a_n (x-a)^{n-1}$

$\underline{S'(a) = 1 a_1}$ $\qquad S''(x) = \sum_{n=2}^{\infty} n \cdot (n-1)\, a_n (x-a)^{n-2}$

$\underline{S''(a) = 2 \cdot 1 \cdot a_2}$ $\qquad$ - - - ——

- - - - - -

$S^{(n)}(a) = n!\, a_n \qquad a_n = \frac{S^{(n)}(a)}{n!}$

ред на Тейлър за функ. f ок. т. a

f - n пъти диференцируема $\forall n \in \mathbb{N}$ $\qquad \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$

$a=0$ - ред на Маклорен

Основни развития

1. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad x \in \mathbb{R}$

2. $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad x \in \mathbb{R}$

3. $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad x \in \mathbb{R}$

4. $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \quad x \in (-1, 1]$

5. $(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n \quad x \in (-1, 1) \quad \binom{\alpha}{n} = \frac{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-n+1)}{n!}$

5.1 $\frac{1}{1+x} = (1+x)^{-1} = \sum_{n=0}^{\infty} \binom{-1}{n} x^n$

$$\binom{-1}{n} = \frac{(-1)(-1-1)(-1-2)\dots(-1-n+1)}{n!} = \frac{(-1)(-2)(-3)\dots(-n)}{n!} = (-1)^n$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

5.2. $\sqrt{1+x} = (1+x)^{1/2} = \sum_{n=0}^{\infty} \binom{1/2}{n} x^n$

$$\binom{1/2}{n} = \frac{1/2\,(1/2-1)(1/2-2)\dots(1/2-n+1)}{n!} = \frac{1/2\cdot(-1/2)(-3/2)\dots\left(-\frac{2n-3}{2}\right)}{n!} =$$

$$= \frac{(-1)^{n-1}1.3.5\dots(2n-3)}{2^n\; n!} = \frac{(-1)^{n-1}(2n-3)!!}{(2n)!!}$$

$$\sqrt{1+x} = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}(2n-3)!!}{(2n)!!} x^n$$

Развийте в ред на Маклорен и намерете област на сходимост

① $f(x) = \cos^2 5x = \frac{1+\cos 10x}{2} = \frac{1}{2} + \frac{1}{2}\cdot\sum_{n=0}^{\infty}\frac{(-1)^n (10x)^{2n}}{(2n)!} = \underbrace{\frac{1}{2}+\frac{1}{2}\cdot 1}_{=1}\cdot\sum_{n=1}^{\infty}\frac{(-1)^n 10^{2n}}{(2n)!}x^{2n} =$

$$= 1 + \sum_{n=1}^{\infty}\frac{(-1)^n 10^{2n}}{2.(2n)!} x^{2n} \qquad 10x\in\mathbb{R} \quad x\in\mathbb{R}$$

② $f(x) = \frac{1}{x^2+x-20} = \frac{1}{(x+5)(x-4)} = \frac{1}{9}\cdot\left(\frac{1}{x-4} - \frac{1}{x+5}\right) = \frac{1}{9}\left(\frac{1}{-4(1+(-\frac{x}{4}))} - \frac{1}{5.(1+\frac{x}{5})}\right) =$

$$= \frac{1}{9}\left(-\frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\left(-\frac{x}{4}\right)^n - \frac{1}{5}\sum_{n=0}^{\infty}(-1)^n\left(\frac{x}{5}\right)^n\right) =$$

$$= \frac{1}{9}\left(-\frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n.(-1)^n}{4^n}.x^n - \frac{1}{5}\sum_{n=0}^{\infty}\frac{(-1)^n}{5^n}x^n\right) =$$

$$= \sum_{n=0}^{\infty} -\frac{1}{9}\left(\frac{1}{4^{n+1}} + \frac{(-1)^n}{5^{n+1}}\right)x^n \qquad \left|\begin{array}{l} -1 < -\frac{x}{4} < 1 \\ -1 < \frac{x}{5} < 1 \end{array}\right. \quad \left|\begin{array}{l} -4 < x < 4 \\ -5 < x < 5 \end{array}\right. \Rightarrow x\in(-4;4)$$

③ $f(x) = (4x+3)e^{2x} = (4x+3)\sum_{n=0}^{\infty}\frac{(2x)^n}{n!} = 4x.\sum_{n=0}^{\infty}\frac{2^n x^n}{n!} + 3\sum_{n=0}^{\infty}\frac{2^n x^n}{n!} =$

$$= \sum_{n=0}^{\infty}\frac{4.2^n x^{n+1}}{n!} + \sum_{n=0}^{\infty}\frac{3.2^n.x^n}{n!} = \sum_{k=1}^{\infty}\frac{4\,2^{k-1}x^k}{(k-1)!} + \sum_{n=1}^{\infty}\frac{3.2^n.x^n}{n!} + 3 =$$

$$= 3 + \sum_{n=1}^{\infty}\left(\frac{4.2^{n-1}}{(n-1)!} + \frac{3.2^n}{n!}\right)x^n = 3 + \sum_{n=1}^{\infty}\frac{2^n}{n!}.(2n+3)x^n \qquad x\in\mathbb{R}$$

④ $f(x) = \ln\sqrt{\frac{1+x}{1-x}} = \ln\sqrt{|1+x|} - \ln\sqrt{|1-x|} = \frac{1}{2}\ln(1+x) - \frac{1}{2}\ln(1-x) =$ $\underset{1+(-x)}{}$

$\frac{1+x}{1-x} > 0$ $\quad - \; -1 \; + \; 1 \; -$ $\quad = \frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} - \frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(-x)^n}{n} =$

$= \sum_{n=1}^{\infty}\frac{1}{2n}\left((-1)^{n-1}+1\right)x^n$

$\begin{cases} -1 < x \leq 1 \\ -1 < -x \leq 1 \to -1 \leq x < 1 \end{cases} \Rightarrow x \in (-1, 1)$

⑤ $f(x) = \frac{1}{x}$ развийте в ред на Тейлър около 3

начин 1 $\sum_{n=0}^{\infty} a_n (x-3)^n$ $\qquad y = x-3$, $x = y+3$

$\frac{1}{x} = \frac{1}{y+3} = \frac{1}{3(1+\frac{y}{3})} = \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{y}{3}\right)^n =$

$= \sum_{n=0}^{\infty}(-1)^n\frac{1}{3^{n+1}}y^n = \sum_{n=0}^{\infty}(-1)^n\frac{1}{3^{n+1}}(x-3)^n$

$-1 < \frac{y}{3} < 1 \qquad -3 < y < 3 \qquad -3 < x-3 < 3 \qquad \boxed{0 < x < 6}$

начин 2 $\quad a_n = \frac{f^{(n)}(3)}{n!}$

$f'(x) = -1 \cdot x^{-2}$

$f''(x) = -1 \cdot (-2) \cdot x^{-3}$

$f^{(n)}(x) = (-1)^n n! \frac{1}{x^{n+1}}$

$a_n = (-1)^n \cancel{n!}\frac{1}{3^{n+1}} \cdot \frac{1}{\cancel{n!}} = (-1)^n \cdot \frac{1}{3^{n+1}}$

⑥ $f(x) = \ln(x^2 + 7x + 12)$ ред на Тейлър около -2

$\sum_{n=0}^{\infty} a_n (x+2)^n$

$f(x) = \ln((x+3)(x+4)) = \ln(x+3) + \ln(x+4) = \ln(y+1) + \ln(y+2) =$

$y = x+2 \qquad x = y-2$

$= \ln(y+1) + \ln(2(1+\frac{y}{2})) = \ln(y+1) + \ln 2 + \ln(1+\frac{y}{2}) =$

$= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}y^n + \ln 2 + \sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{y}{2}\right)^n \cdot \frac{1}{n} =$

$= \ln 2 + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(1+\frac{1}{2^n}\right)y^n = \ln 2 + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(1+\frac{1}{2^n}\right)(x+2)^n$

$\left.\begin{array}{l} -1 < y \leq 1 \\ -1 < \frac{y}{2} \leq 1 \to -2 < y \leq 2 \end{array}\right\} \Rightarrow -1 < y < 1 \qquad -1 < x+2 \leq 1 \qquad \boxed{-3 < x \leq -1}$

⑦ $f(x) = \operatorname{arctg} x$ (Маклорен)

$$f'(x) = \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n . (x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$\int f'(x)\, dx = \int \sum_{n=0}^{\infty} (-1)^n x^{2n}\, dx = \sum_{n=0}^{\infty} \int (-1)^n x^{2n} dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + c$$

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + c$$

$x=0$ $\quad f(0) = 0 + c \qquad f(0) = \operatorname{arctg} 0 = 0 + c \Rightarrow c = 0$

$$\operatorname{arctg} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + c$$

$f'(x) \rightarrow \quad -1 < x^2 < 1 \quad 0 \le x^2 < 1 \quad -1 < x < 1$

крайни точки $\quad x=1 \quad \sum_{n=0}^{\infty} (-1)^n . \frac{1^{2n+1}}{2n+1}$ сходящ

$x=-1 \quad \sum_{n=0}^{\infty} (-1)^n \frac{(-1)^{2n+1}}{2n+1}$ сходящ $\quad \Rightarrow D_f : -1 \le x \le 1$

⑧ $f(x) = \int_0^x \sin t^2\, dt$

$$f'(x) = \sin x^2 = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!} \qquad x \in \mathbb{R}$$

$$\int f'(x)\, dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!} dx = \sum_{n=0}^{\infty} \int \frac{(-1)^n x^{4n+2}}{(2n+1)!} dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(2n+1)!\,(4n+3)} + c$$

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(2n+1)!\,(4n+3)} + c \qquad f(0) = 0 + c \quad \Rightarrow c = 0$$

$$f(0) = \int_0^0 \sin t^2 dt = 0$$

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(2n+1)!\,(4n+3)} \qquad x \in \mathbb{R}$$

⑨ $f(x) = \int_0^x \operatorname{arctg} \frac{t+\sqrt{3}}{1-t\sqrt{3}}\, dt$

$$f'(x) = \operatorname{arctg} \frac{x+\sqrt{3}}{1-x\sqrt{3}} \qquad f''(x) = \frac{1}{1+\left(\frac{x+\sqrt{3}}{1-x\sqrt{3}}\right)^2} \cdot \frac{1(1-x\sqrt{3})-(x+\sqrt{3})(-\sqrt{3})}{(1-x\sqrt{3})^2} =$$

$$= \frac{1}{\frac{(1-x\sqrt{3})^2+(x+\sqrt{3})^2}{\cancel{(1-x\sqrt{3})^2}}} \cdot \frac{1-\cancel{x\sqrt{3}}+\cancel{\sqrt{3}x}+3}{\cancel{(1-x\sqrt{3})^2}} = \frac{4}{1-2\cancel{\sqrt{3}x}+3x^2+x^2+2\cancel{\sqrt{3}x}+3} = \frac{4}{4x^2+4} = \frac{1}{x^2+1}$$

$$f''(x) = \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n (x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$\int f''(x)\,dx = \int \sum_{n=0}^{\infty} (-1)^n x^{2n}\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C$$

$$f'(0) = 0 + C \qquad \operatorname{arctg} \frac{\sqrt{3}}{1} = C \Rightarrow C = \frac{\pi}{3}$$

$$f'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} + \frac{\pi}{3}$$

$$\int f'(x)\,dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} + \frac{\pi}{3}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+1)(2n+2)} + \frac{\pi}{3}x + C_1$$

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+1)(2n+2)} + \frac{\pi}{3}x + C_1$$

$$f(0) = 0 + C_1$$

$$\int_0^0 \operatorname{arctg} \frac{t+\sqrt{3}}{1+t\sqrt{3}}\,dt = 0 + C_1 \Rightarrow C_1 = 0$$

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+1)(2n+2)} + \frac{\pi}{3}x \qquad x \in [-1;1]$$

$$f''(x) \longrightarrow \quad -1 < x^2 < 1 \Rightarrow -1 < x < 1$$

крайни точки $x=1$ $\quad \frac{\pi}{3} + \sum_{n=0}^{\infty} \frac{(-1)^n\, 1^{2n+2}}{(2n+1)(2n+2)}$ сходящ

$x=-1$ $\quad -\pi/3 + \sum_{n=0}^{\infty} \frac{(-1)^n\, (-1)^{2n+2}}{(2n+1)(2n+2)}$ сходящ

(10) $f(x) = x\ln(x+\sqrt{1+x^2}) + x\arcsin x + \sqrt{1-x^2} - \sqrt{1+x^2}$

$(1+(-x^2))^{1/2} \quad (1+x^2)^{1/2}$

$g(x) = x\ln(x+\sqrt{1+x^2}) + x\arcsin x$

$$f'(x) = \ln(x+\sqrt{1+x^2}) + x \cdot \frac{1}{x+\sqrt{1+x^2}} \cdot \left(1 + \frac{1}{2\sqrt{1+x^2}} \cdot 2x\right) + \arcsin x + \cancel{x \cdot \frac{1}{\sqrt{1-x^2}} + \frac{1}{2\sqrt{1-x^2}} \cdot -2x} - \frac{1}{2\sqrt{1+x^2}}2x$$

$$= \ln(x+\sqrt{1+x^2}) + \cancel{\frac{x}{x+\sqrt{1+x^2}} \cdot \frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}} + \arcsin x - \cancel{\frac{x}{\sqrt{1+x^2}}} = \ln(x+\sqrt{1+x^2}) + \arcsin x$$

$$f''(x) = \frac{1}{x+\sqrt{1+x^2}} \cdot \left(1 + \frac{1}{\cancel{2}\sqrt{1+x^2}} \cdot \cancel{2}x\right) + \frac{1}{\sqrt{1-x^2}} = \frac{1}{\cancel{x+\sqrt{1+x^2}}} \cdot \frac{\cancel{\sqrt{1+x^2}+x}}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1-x^2}}$$

$$\frac{1}{\sqrt{1+x^2}} = (1+x^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{-1/2}{n} (x^2)^n$$

$$\binom{-\frac{1}{2}}{n} = \frac{-\frac{1}{2}\cdot\left(-\frac{1}{2}-1\right)\cdots\left(-\frac{1}{2}-n+1\right)}{n!} = \frac{-\frac{1}{2}\cdot\left(-\frac{3}{2}\right)\cdots\left(-\frac{2n-1}{2}\right)}{n!} = \frac{(-1)^n\,(2n-1)!!}{(2n)!!}$$

$$\frac{1}{\sqrt{1+x^2}} = \sum_{n=0}^{\infty} \frac{(-1)^n\,(2n-1)!!}{(2n)!!}\, x^{2n}$$

$$\frac{1}{\sqrt{1-x^2}} = \sum_{n=0}^{\infty} \frac{(-1)^n\,(2n-1)!!}{(2n)!!}\,(-1)^n\, x^{2n} = \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!}\, x^{2n}$$

$$f''(x) = \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!}\,\left((-1)^n+1\right) x^{2n} \quad \Bigg| \quad \begin{matrix} -1 < x^2 < 1 \\ -1 < -x^2 < 1 \end{matrix} \Rightarrow -1 < x < 1$$

$$f'(x) = \int f''(x)\,dx = \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!}\,\left((-1)^n+1\right)\cdot\frac{x^{2n+1}}{2n+1} + C$$

$$f'(0) = 0 + C \qquad 0 = 0 + C \Rightarrow C = 0$$

$$f'(x) = \sum_{n=0}^{\infty} \frac{(2n-1)!!\,\left((-1)^n+1\right)}{(2n)!!\,(2n+1)}\cdot x^{2n+1}$$

$$f(x) = \int f'(x)\,dx = \sum_{n=0}^{\infty} \frac{(2n-1)!!\,\left((-1)^n+1\right)}{(2n)!!\,(2n+1)}\,\frac{x^{2n+2}}{2n+2} + C_1$$

$$f(x) = \sum_{n=0}^{\infty} \frac{(2n-1)!!\,\left((-1)^n+1\right)}{(2n+2)!!\,(2n+1)}\, x^{2n+2} + \underset{0}{\underset{\shortparallel}{C_1}}$$

$$f(0) = 0 + C_1 \qquad 0 = C_1$$

крайни точки $x = 1$, $x = -1$ еднакви редове $\rightarrow$ $\displaystyle\sum_{n=0}^{\infty} \frac{(2n-1)!!\,\left((-1)^n+1\right)}{(2n+2)!!\,(2n+1)}$

$$\frac{(2n-1)!!\,\left((-1)^n+1\right)}{(2n+2)!!\,(2n+1)} \le \frac{2\,(2n-1)!!}{(2n+2)!!\,(2n+1)}$$ с тякой критерий

⑨ $\lim\limits_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

$\lim\limits_{x\to 0}\left(\lim\limits_{y\to 0} \frac{xy}{x^2+y^2}\right) = \lim\limits_{x\to 0} \frac{0}{x^2} = \lim\limits_{x\to 0} 0 = 0$

$\lim\limits_{y\to 0}\left(\lim\limits_{x\to 0} \frac{xy}{x^2+y^2}\right) = \lim\limits_{y\to 0} \frac{0}{y^2} = \lim\limits_{y\to 0} 0 = 0$

$\lim\limits_{\substack{(x,y)\to(0,0)\\ y=kx}} f(x,y) = \lim\limits_{\substack{(x,y)\to(0,0)\\ y=kx}} \frac{xy}{x^2+y^2} = \lim\limits_{x\to 0} \frac{xkx}{x^2+k^2x^2} = \lim\limits_{x\to 0} \frac{\cancel{x^2} k}{\cancel{x^2}(k^2+1)} = \lim\limits_{x\to 0} \frac{k}{k^2+1}$

по всяка права $y=kx$ има разл. граници т.к. $g(k)$ не е константна ф-я

$\Rightarrow \lim f(x,y)$ не същ.

⑩ $\lim\limits_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$

$\lim\limits_{x\to 0}\left(\lim\limits_{y\to 0} \frac{xy^2}{x^2+y^4}\right) = \lim\limits_{x\to 0} \frac{0}{x^2} = \lim\limits_{x\to 0} 0 = 0$

$\lim\limits_{y\to 0}\left(\lim\limits_{x\to 0} \frac{xy^2}{x^2+y^4}\right) = \lim\limits_{y\to 0} \frac{0}{y^4} = \lim\limits_{y\to 0} 0 = 0$

$\lim\limits_{\substack{(x,y)\to(0,0)\\ y=kx}} \frac{xy^2}{x^2+y^4} = \lim\limits_{x\to 0} \frac{x k^2 x^2}{x^2+k^4x^4} = \lim\limits_{x\to 0} \frac{\overset{x}{\cancel{x^3}} k^2}{\cancel{x^2}(1+k^4x^2)} = \lim\limits_{x\to 0} \frac{x.k^2}{1+k^4x^2} = 0$

$\lim\limits_{\substack{(x,y)\to(0,0)\\ y=\sqrt{x}}} \frac{xy^2}{x^2+y^4} = \lim\limits_{x\to 0} \frac{x.x}{x^2+x^2} = \lim\limits_{x\to 0} \frac{\cancel{x^2}}{2\cancel{x^2}} = \frac{1}{2} \neq 0 \Rightarrow \lim f(x,y)$ не същ.