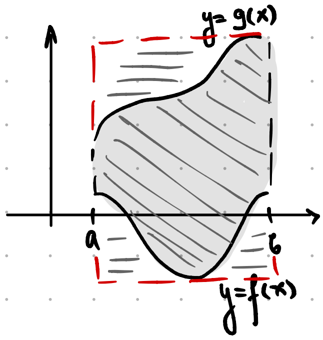
16.03.23

Геометрично приложение на определения интеграл

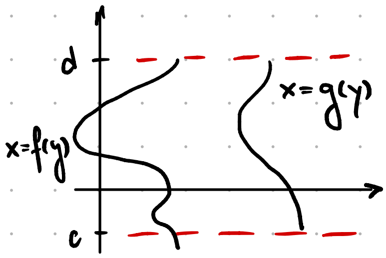


Криволинеен трапец

$$D=\{(x,y)\in\mathbb{R}^2:\ a\le x\le b,\ f(x)\le y\le g(x)\}$$

$f,g:[a,b]\longrightarrow\mathbb{R}$ $\quad f(x)\le g(x)\ \ \forall x\in[a,b]$

$$S(D)=\int_a^b (g(x)-c)\,dx-\int_a^b (f(x)-c)\,dx=\int_a^b (g(x)-f(x))\,dx$$

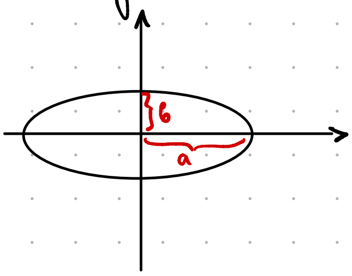


$$D'=\{(x,y)\in\mathbb{R}^2:\ c\le y\le d,\ f(y)\le x\le g(y)\}$$

непр. $f,g:[c,d]\to\mathbb{R}$, $f(x)\le g(x)\ \ \forall x\in D$

$$S(D')=\int_c^d (g(y)-f(y))\,dx$$

Лице на елипса



$$\{(x,y)\in\mathbb{R}^2:\ \frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1\}=\{(x,y)\in\mathbb{R}^2,\ -a\le x\le a$$

$a>0,\ b>0$ $\qquad -b\sqrt{1-\frac{x^2}{a^2}}\le y\le b\sqrt{1-\frac{x^2}{a^2}}$

$\frac{y^2}{b^2}\le 1-\frac{x^2}{a^2}$ $\qquad y^2\le b^2\left(1-\frac{x^2}{a^2}\right)$ $\qquad 1-\frac{x^2}{a^2}\ge 0$ $\qquad \frac{x^2}{a^2}\le 1$

$y\le b\sqrt{1-\frac{x^2}{a^2}}$ $\qquad y\ge -b\sqrt{1-\frac{x^2}{a^2}}$ $\qquad -\frac{x^2}{a^2}\ge -1$ $\qquad \frac{x}{a}\le\pm 1$ $\qquad x\le a$, $x\ge -a$

$$S=\int_a^b\left(b\sqrt{1-\frac{x^2}{a^2}}+b\sqrt{1-\frac{x^2}{a^2}}\right)dx=\int_a^b 2b\sqrt{1-\frac{x^2}{a^2}}\,dx=2b\int_a^b\sqrt{1-\frac{x^2}{a^2}}\,dx=$$

$$=2b\int_{-\pi/2}^{\pi/2}\sqrt{1-\sin^2 t}\,\cos t\,dt=2b\int_{-\pi/2}^{\pi/2}\cos t\cdot\cos t\,dt=$$

$x=a\sin t$ $\qquad x^2=a^2\sin^2 t$

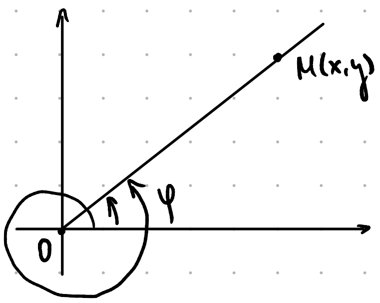
$t\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$dx=\cos t$

$$=2b\int_{-\pi/2}^{\pi/2}\frac{1+\cos 2t}{2}\,dt=2b\int_{-\pi/2}^{\pi/2}\frac{1}{2}\,dt+2b\int_{-\pi/2}^{\pi/2}\cos 2t\,dt=$$

$$= 2b\left(\frac{t}{2} + \frac{\sin 2t}{4}\right)\Bigg|_{-\pi/2}^{\pi/2} = 2b\left(\frac{2t+\sin 2t}{4}\right)\Bigg|_{-\pi/2}^{\pi/2} = 2b\left(\frac{\pi+\sin\pi}{4} - \frac{-\pi+\sin(-\pi)}{4}\right) =$$

$$= 2b\left(\frac{\pi}{4}+\frac{\pi}{4}\right) = 2b\frac{\pi}{2} = b\pi$$



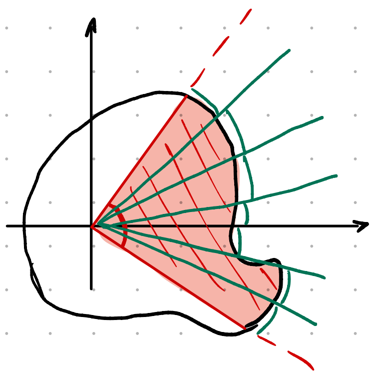
$|OM| = \sqrt{x^2+y^2} =: \rho$ полярен радиус $\qquad \rho \in [0;+\infty)$

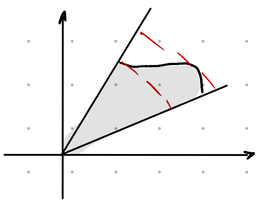
$\sphericalangle(\vec{Ox}, \vec{OM})$ = полярен ъгъл $\qquad \varphi \in [0, 2\pi)$

$$\left[\begin{array}{l} x = \rho\cos\varphi \\ y = \rho\sin\varphi \end{array}\right.$$

$\rho = f(x$

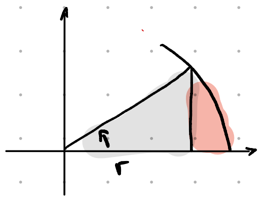
$$\{(\rho,\varphi): \ \varphi_1 \le \varphi \le \varphi_2 \quad 0 \le \rho \le f(\varphi)\}$$





$2\pi \longrightarrow \pi r^2$

$\alpha \longrightarrow \frac{\alpha}{2} r^2$



$\varphi_1 = \varphi_0 < \varphi_1 < \ldots < \varphi_n = \varphi_2 \qquad \varphi \in [\varphi_{i-1}, \varphi_i]$

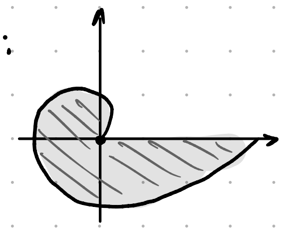
$$\frac{\varphi_i - \varphi_{i-1}}{2}\left(\inf_{\varphi\in[\varphi_{i-1},\varphi_i]} f(\varphi)\right)^2 \le S_i \le \frac{\varphi_i-\varphi_{i-1}}{2}\left(\sup_{\varphi\in[\varphi_{i-1},\varphi_i]} f(\varphi)\right)^2$$

$$\frac{1}{2}\sum_{i=1}^{n}\left(\inf_{[\varphi_{i-1},\varphi_i]} f^2\right)(\varphi_i-\varphi_{i-1}) \le S \le \frac{1}{2}\sum_{i=1}^{n}\left(\sup_{[\varphi_{i-1},\varphi_i]} f^2\right)(\varphi_i-\varphi_{i-1})$$

Малки и големи суми на Дарбу за $\frac{1}{2}f^2$ в интервала $[\varphi_1, \varphi_2]$

$\frac{1}{2}\int_{\varphi_1}^{\varphi_2} f^2(\varphi)d\varphi$ - Лице на фигура, зададена с полярни координати

пример:

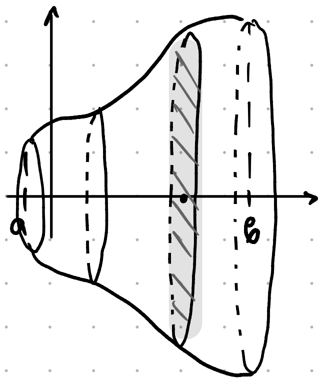


Архимедова спирала

$\rho = \varphi.k \ , \ k>0$

$$S = \frac{1}{2}\int_0^{2\pi}(k.\varphi)^2 d\varphi = k^2.\frac{1}{2}\,\frac{\varphi^3}{3}\Bigg|_0^{2\pi} = \frac{k^2}{6}.8\pi^3 = \frac{4}{3}k^2\pi^3$$

Ротационно тяло

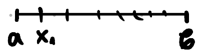


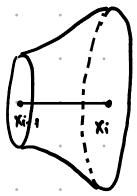
$f:[a,b]\longrightarrow\mathbb{R},\ f(x)\ge 0\quad \forall x\in[a,b]$

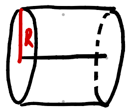
$K=\{(x,y,z)\in\mathbb{R}^3:\ x\in[a,b],\ \sqrt{y^2+z^2}\le f(x)\}$

рот. тяло

$a=x_0<x_1<\dots<x_n=b$







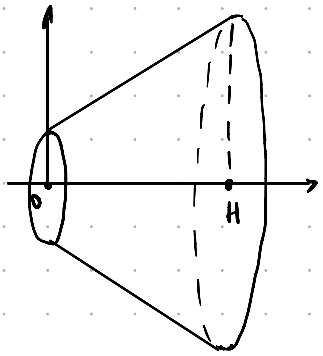
Обем

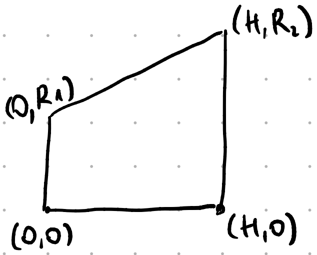
$$\pi\Big(\inf_{x\in[x_{i-1},x_i]} f(x)\Big)^2(x_i-x_{i-1})\le V_i\le\pi\Big(\sup_{x\in[x_{i-1},x_i]} f(x)\Big)^2(x_i-x_{i-1})$$

$$\sum_{i=1}^{n}\Big(\inf_{[x_{i-1},x_i]}\pi f^2(x)\Big)(x_i-x_{i-1})\le V\le\sum_{i=1}^{n}\Big(\sup_{[x_{i-1},x_i]}\pi f^2(x)\Big)(x_i-x_{i-1})$$

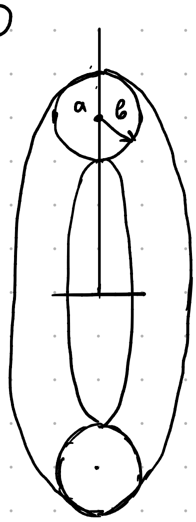
$$V=\pi\int_a^b f^2(x)\,dx$$

①





②



$a>b>0$

$x^2+(y-a)^2\le b^2$

$(y-a)^2\le b^2-x^2$

$y-a\le\pm\sqrt{b^2-x^2}$

$y\le a+\sqrt{b^2-x^2}$

$y\ge a-\sqrt{b^2-x^2}$

$$V=V_1-V_2=\pi\int_a^b\big(a+\sqrt{b^2-x^2}\big)^2dx-\pi\int_a^b\big(a-\sqrt{b^2-x^2}\big)^2dx=$$

$$=\pi\Big(\int_a^b\cancel{a^2}+2a\sqrt{b^2-x^2}+\cancel{b^2}-\cancel{x^2}-\cancel{a^2}+2a\sqrt{b^2-x^2}-\cancel{b^2}+\cancel{x^2}\,dx\Big)$$

$$=\pi\int_a^b 4a\sqrt{b^2-x^2}\,dx=4a\pi\int_a^b\sqrt{b^2-x^2}\,dx=$$

$$=4a\pi\int_{-\pi/2}^{\pi/2}\sqrt{b^2-b^2\sin^2t}\,\cos t\,dt=$$

$x=b\sin t$

$dx=\cos t$

$$= 4a\pi \int_{-\pi/2}^{\pi/2} b\sqrt{1-\sin^2 t}\,\cos t\,dt = 4ab\pi\int_{-\pi/2}^{\pi/2}\cos^2 t\,dt =$$

$$= 4ab\pi\int_{-\pi/2}^{\pi/2} \frac{1+\cos 2t}{2}\,dt = 4ab\pi\left(\frac{t}{2}+\frac{\sin 2t}{4}\right)\Big|_{-\pi/2}^{\pi/2} =$$

$$= 4ab\pi\left(\frac{2t+\sin 2t}{4}\right)\Big|_{-\pi/2}^{\pi/2} = 4ab\pi\left(\frac{\pi+\sin\pi}{4} - \frac{-\pi+\sin -\pi}{4}\right)$$

$$= 4ab\pi\left(\frac{\pi}{4}+\frac{\pi}{4}\right) = 4ab\pi\cdot\frac{\pi}{2} = 2ab\pi^2$$

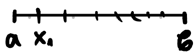
Лице на повърхнината

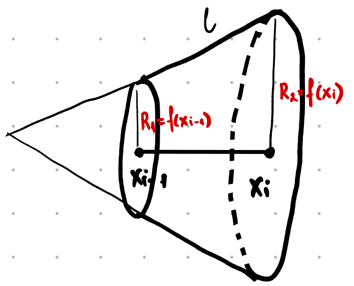
$f:[a,b]\longrightarrow\mathbb{R},\ f(x)\geq 0\quad \forall x\in[a,b]$ → диференцируема и f' непрекъсната

$K=\{(x,y,z)\in\mathbb{R}^3:\ x\in[a,b],\ \sqrt{y^2+z^2}\leq f(x)\}$

рот. тяло

$a=x_0<x_1<\dots<x_n=b$





Лице на ок. повърхнина конус

$$l\cdot\frac{R_1+R_2}{2}\cdot 2\pi$$

$a=x_0<x_1<x_2<\dots<x_n=b$

$$l=\sqrt{h^2+(R_1-R_2)^2}$$

$$S_{ok}\sim\sum_{i=1}^{n}2\pi\cdot\frac{f(x_{i-1})+f(x_i)}{2}\sqrt{(x_i-x_{i-1})^2+(f(x_{i-1})-f(x_i))^2}=$$

$$=2\pi\cdot\sum_{i=1}^{n}\frac{f(x_{i-1})+f(x_i)}{2}\sqrt{1+\left(\frac{f(x_i)-f(x_{i-1})}{x_i-x_{i-1}}\right)^2}\cdot(x_i-x_{i-1})=$$

от Th. Лагранж за ср. стойности =>
$\exists\,\xi_i\in[x_{i-1},x_i],\ \frac{f(x_i)-f(x_{i-1})}{x_i-x_{i-1}}=f'(\xi_i)$

$$=2\pi\sum_{i=1}^{n}\frac{f(x_{i-1})+f(x_i)}{2}\sqrt{1+(f'(\xi_i))^2}\,(x_i-x_{i-1})=$$

$$=2\pi\sum_{i=1}^{n}f(\xi_i)\sqrt{1+f'(\xi_i)^2}\,(x_i-x_{i-1})-2\pi\sum_{i=1}^{n}\left(\frac{f(x_i)-f(x_{i-1})}{2}-f(\xi_i)\right)$$

$$2\pi\sum_{i=1}^{n}f(\xi_i)\sqrt{1+f'(\xi_i)^2}\,(x_i-x_{i-1})=\sigma_g(\tau,\xi)\xrightarrow[d(\tau)\to 0]{}\int_a^b 2\pi f(x)\sqrt{1+f'(x)^2}\,dx$$

$g=2\pi f\sqrt{1+f'^2}$ непрекъсната

$$\left|2\pi\sum_{i=1}^{n}\left(\frac{f(x_i)-f(x_{i-1})}{2}-f(\xi_i)\right)\cdot\sqrt{1+f'(\xi_i)^2}\cdot(x_i-x_{i-1})\right|\le$$

$|f'(x)|\le M \quad \forall x\in[a,b]$

Вайерщрас

$$\le 2\pi\sum_{i=1}^{n}\left|\frac{f(x_i)-f(\xi_i)}{2}+\frac{f(x_{i-1})-f(\xi_i)}{2}\right|\sqrt{1+f'(\xi_i)^2}\cdot(x_i-x_{i-1})\le$$

$x,y\in[a,b]$

$|f(x)-f(y)|\le M\cdot|x-y|$

$\parallel$

$|f'(\eta)\cdot(x-y)| \qquad \eta\in[x,y]$

$$\le 2\pi\sum_{i=1}^{n}\left(\frac{1}{2}|x_i-\xi_i|+\frac{1}{2}|x_{i-1}-\xi_i|\right)\sqrt{1+M^2}\cdot(x_i-x_{i-1})\le$$

$$\le 2\pi\sqrt{1+M}\cdot\sum_{i=1}^{n}d(\tau)(x_i-x_{i-1})=d(\tau)\cdot 2\pi\sqrt{1.M^2}\cdot(b-a)\xrightarrow[d(\tau)=0]{}0$$

$$S_{ok}=2\pi\int_a^b f(x)\sqrt{1+(f'(x))^2}\,dx$$

Пример: Лице на тора

$a>b>0 \quad x^2+(y-a)^2=b^2 \qquad y_1(x)=\sqrt{b^2-x^2}+a \quad y_2(x)=-\sqrt{b^2-x^2}+a$

$x\in[-b,b]$ център $(0,0)$

$$S_{ок тора}=2\pi\int_{-b}^{b}\left[y_1(x)\cdot\sqrt{1+(y_1'(x))^2}+y_2(x)\cdot\sqrt{1+(y_2'(x))^2}\right]dx=$$

$$y_1'(x)=\frac{-2x}{2\sqrt{b^2-x^2}}=\frac{-x}{\sqrt{b^2-x^2}} \qquad y_2'(x)=\frac{x}{\sqrt{b^2-x^2}}$$

$$\sqrt{1+y_i'^2(x)}=\sqrt{1+\frac{x^2}{b^2-x^2}}=\sqrt{\frac{b^2-x^2+x^2}{b^2-x^2}}=\frac{b}{\sqrt{b^2-x^2}}$$

$$=2\pi\int_{-b}^{b}\left[(\sqrt{b^2-x^2}+a)\cdot\frac{b}{\sqrt{b^2-x^2}}+(-\sqrt{b^2-x^2}+a)\cdot\frac{b}{\sqrt{b^2-x^2}}\right]dx=$$

$$=2\pi\int_{-b}^{b}\frac{2ab}{\sqrt{b^2-x^2}}dx=4\pi ab\int_{-b}^{b}\frac{dx}{\sqrt{b^2-x^2}}=$$

$$=4\pi ab\int_{-b}^{b}\frac{d(\frac{x}{b})}{\sqrt{1-\frac{x^2}{b^2}}}=4\pi ab\arcsin\frac{x}{b}\Big|_{-b}^{b}=4\pi ab\left(\frac{\pi}{2}+\frac{\pi}{2}\right)=4\pi^2ab$$