1) пример за $f(x)$ - диференцируема, но не е непрекъснато диференцируема т.е $f'(x)$ не е непрекъсната

2) $f: [-a;a] \to \mathbb{R}$ непрекъсната $f(x) = \underbrace{g(x)}_{\text{четна}} + \underbrace{r(x)}_{\text{нечетна}}$

07.03.24

① $$\int_1^2 \frac{\sin^2(\sqrt{x^2+1})}{1+\cos^2(\sqrt{x^2+1})}\cdot\frac{x}{\sqrt{x^2+1}}\,dx = \int_{\sqrt2}^{\sqrt5}\frac{\sin^2 t}{1+\cos^2 t}\cdot\frac{\cancel{\sqrt{t^2-1}}}{\cancel{t}}\cdot\frac{1}{\cancel{2}\cancel{\sqrt{t^2-1}}}\,\cancel{2t}\,dt = \int_{\sqrt2}^{\sqrt5}\frac{\sin^2 t}{1+\cos^2 t}\,dt =$$

$t = \sqrt{x^2+1}$ $x=1$ $t=\sqrt2$

$t^2 = x^2+1$ $x=2$ $t=\sqrt5$

$x^2 = t^2-1$

$x = \pm\sqrt{t^2-1}$ $dx = \frac{1}{2\sqrt{t^2-1}}\cdot 2t\,dt$

$y = tg\,t$ $t=\sqrt2$ $y = tg\sqrt2$

$t=\sqrt5$ $y = tg\sqrt5$

$t = arctg\,y$ $dt = \frac{1}{1+y^2}\,dy$

$\sin^2 t + \cos^2 t = 1$

$\frac{\sin^2 t}{\cos^2 t} = y^2 \Rightarrow \sin^2 t = y^2\cdot\cos^2 t$

$(1+y^2)\cos^2 t = 1$

$\cos^2 t = \frac{1}{1+y^2}$

$\sin^2 t = \frac{y^2}{1+y^2}$

$$= \int_{tg\sqrt2}^{tg\sqrt5}\frac{\frac{y^2}{1+y^2}}{1+\frac{1}{1+y^2}}\cdot\frac{1}{1+y^2}\,dy = \int_{tg\sqrt2}^{tg\sqrt5}\frac{\frac{y^2}{\cancel{1+y^2}}}{\frac{1+y^2+1}{\cancel{1+y^2}}}\cdot\frac{1}{1+y^2}\,dy = \int_{tg\sqrt2}^{tg\sqrt5}\frac{y^2}{(2+y^2)(1+y^2)}\,dy = \int_{tg\sqrt2}^{tg\sqrt5}\frac{2}{2+y^2} - \frac{1}{1+y^2}\,dy =$$

$$\frac{y^2}{(2+y^2)(1+y^2)} = \frac{Ay+B}{2+y^2} + \frac{Cy+D}{1+y^2}$$

$$= \int_{tg\sqrt2}^{tg\sqrt5}\frac{\cancel{2}\cdot\sqrt2}{\cancel{2}\left(1+\left(\frac{y}{\sqrt2}\right)^2\right)}\,d\frac{y}{\sqrt2} - \int_{tg\sqrt2}^{tg\sqrt5}\frac{1}{1+y^2}\,dy = \sqrt2\cdot arctg\frac{y}{\sqrt2}\Bigg|_{tg\sqrt2}^{tg\sqrt5} - arctg\,y\Bigg|_{tg\sqrt2}^{tg\sqrt5} = \sqrt2\,arctg\left(\frac{tg\sqrt5}{\sqrt2}\right) - \sqrt2\,arctg\left(\frac{tg\sqrt2}{\sqrt2}\right) -$$

$$arctg(tg\sqrt5) + arctg(tg\sqrt2)$$

② $$I_n = \int_0^{\pi/2}\sin^n x\,dx$$

$$I_0 = \int_0^{\pi/2}\sin^0 x\,dx = x\Big|_0^{\pi/2} = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$$I_1 = \int_0^{\pi/2}\sin x\,dx = -\cos x\Big|_0^{\pi/2} = -(0-1) = +1$$

$$I_n = \int_0^{\pi/2}\sin^n x\,dx = \int_0^{\pi/2}\sin x\cdot\sin^{n-1}x\,dx = -\int_0^{\pi/2}\sin^{n-1}x\,d\cos x = -\cos x\cdot\sin^{n-1}x\Big|_0^{\pi/2} + \int_0^{\pi/2}\cos x\,d\sin^{n-1}x =$$

$$= 0 - 0 + \int_0^{\pi/2}\cos x\,(n-1)\sin^{n-2}x\,\cos x\,dx = (n-1)\int_0^{\pi/2}\cos^2 x\,\sin^{n-2}x\,dx = (n-1)\int_0^{\pi/2}(1-\sin^2 x)\sin^{n-2}x\,dx =$$

$$= (n-1)\cdot\underbrace{\int_0^{\pi/2}\sin^{n-2}dx}_{I_{n-2}} - (n-1)\cdot\underbrace{\int_0^{\pi/2}\sin^n dx}_{I_n}$$

$I_n = (n-1)\cdot I_{n-2} - (n-1)\cdot I_n$

$I_n + (n-1)I_n = (n-1)I_{n-2}$

$nI_n = (n-1)I_{n-2}$

$$I_n = \frac{n-1}{n}I_{n-2} = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot I_{n-4} = \frac{(n-1)!!}{n!!}\begin{cases} I_1, & n-\text{нечетно} \\ I_0, & n-\text{четно}\end{cases}$$

$$\int_0^{\pi/2} \sin^n x\,dx = \frac{(n-1)!!}{n!!} \cdot \begin{cases} 1, & n\text{-нечетно} \\ \pi/2, & n\text{-четно} \end{cases} = \int_0^{\pi/2} \cos^n x\,dx$$

③ f - непр. в $[0,1] \Rightarrow \int_0^{\pi/2} f(\sin x)\,dx = \int_0^{\pi/2} f(\cos x)\,dx$

Д-во: $t = \frac{\pi}{2} - x$ $\quad x=0 \quad t=\frac{\pi}{2}$ $\quad x=\frac{\pi}{2} \quad t=0$

$$\int_0^{\pi/2} f(\sin x)dx = -\int_{\pi/2}^{0} f(\sin(\tfrac{\pi}{2}-t))dt = \int_0^{\pi/2} f(\cos t)\,dt$$

$x = \frac{\pi}{2} - t$

$dx = -dt$

④ f - непр. в $[0,1] \Rightarrow \int_0^{\pi} x f(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx$

Д-во: $t = \pi - x$ $\quad x=0 \quad t=\pi$

$x = \pi - t$ $\quad x=\pi \quad t=0$

$dx = -dt$

$$\int_0^{\pi} x f(\sin x)\,dx = -\int_{\pi}^{0} (\pi-t) f(\sin(\pi-t))\,dt = \int_0^{\pi} (\pi-t) f(\sin t)\,dt =$$

$$= \int_0^{\pi} \pi f(\sin t)dt - \int_0^{\pi} t f(\sin t)dt$$

$$I = \int_0^{\pi} \pi f(\sin t)dt - I$$

$$\frac{\cancel{2}I}{\cancel{2}} = \frac{\pi}{2}\int_0^{\pi} f(\sin t)\,dt$$

⑤ $$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = -\int_0^{\pi} \frac{x}{1+\cos^2 x}\,d\cos x = -\int_0^{\pi} x\,d\,\mathrm{arctg}(\cos x) = -x\,\mathrm{arctg}(\cos x)\Big|_0^{\pi} + \int_0^{\pi} \mathrm{arctg}(\cos x)\,dx =$$

$$= -\pi\,\mathrm{arctg}(-1) + 0 - \int_{-\pi/2}^{\pi/2} \mathrm{arctg}(\cos(\tfrac{\pi}{2}-t))\,dt = \frac{\pi^2}{4} + \int_{-\pi/2}^{\pi/2} \mathrm{arctg}(\sin t)\,dt = \frac{\pi^2}{4} + 0 = \frac{\pi^2}{4}$$

$t = \frac{\pi}{2} - x$ $\quad x=0 \quad t=\frac{\pi}{2}$

$\mathrm{arctg}(\sin(-t)) = \mathrm{arctg}(-\sin(t)) = -\mathrm{arctg}(\sin t) \Rightarrow$ нечетна $\Rightarrow = 0$

$x = \frac{\pi}{2} - t$ $\quad x=\pi \quad t=-\frac{\pi}{2}$

$dx = -dt$

⑥ f - непр в $[-a,a]$ $\quad f(x) = \underset{\text{четна}}{g(x)} + \underset{\text{нечетна}}{h(x)}$

$g(x) = \frac{f(x)+f(-x)}{2}$ $\qquad g(-x) = \frac{f(-x)+f(-(-x))}{2} = g(x)$ е четна

$h(x) = \frac{f(x)-f(-x)}{2}$ $\qquad h(-x) = \frac{f(-x)-f(-(-x))}{2} = \frac{-(f(x)-f(-x))}{2} = -h(x)$ е нечетна

③ $\int_{-3}^{3} \underbrace{\frac{\operatorname{arctg}\sqrt{|x|}}{1+(1+x^2)^x}}_{f(x)}\,dx = \int_{0}^{3} \frac{\operatorname{arctg}\sqrt{|x|}}{1+(1+x^2)^x} + \frac{\operatorname{arctg}\sqrt{|x|}}{1+(1+x^2)^{-x}}\,dx =$ ⊛

$f(-x) = \frac{\operatorname{arctg}\sqrt{|-x|}}{1+(1+(-x)^2)^{-x}} = \frac{\operatorname{arctg}\sqrt{|x|}}{1+(1+x^2)^{-x}} \neq \pm f(x) \Rightarrow$ нито четна, нито нечетна

$f(x) = \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2}$

$\int_{-3}^{3} f(x)\,dx = \int_{-3}^{3} \frac{f(x)+f(-x)}{2} + \underbrace{\int_{-3}^{3} \frac{f(x)-f(-x)}{2}}_{=0} = 2\int_{0}^{3} \frac{f(x)+f(-x)}{2}\,dx + 0$

⊛ $= \int_0^3 \operatorname{arctg}\sqrt{x}\left(\frac{1}{1+(1+x^2)^x} + \frac{1}{1+\frac{1}{(1+x^2)^{-x}}}\right)dx = \int_0^3 \operatorname{arctg}\sqrt{x}\left(\frac{1}{1+(1+x^2)^x} + \frac{1}{\frac{(1+x^2)^x+1}{(1+x^2)^x}}\right)dx =$

$= \int_0^3 \operatorname{arctg}\sqrt{x}\,dx = \int_0^{\sqrt{3}} \operatorname{arctg}t\; 2t\,dt = \int_0^{\sqrt{3}} \operatorname{arctg}t\,dt^2 = t^2\operatorname{arctg}t\Big|_0^{\sqrt{3}} - \int_0^{\sqrt{3}} t^2\,d\operatorname{arctg}t =$

$t=\sqrt{x}$ $\quad x=0 \;\; t=0$

$\quad x=3 \;\; t=\sqrt{3}$ $\quad = 3.\operatorname{arctg}\sqrt{3} - 0 - \int_0^{\sqrt{3}} \frac{t^2+1-1}{1+t^2}\,dt = 3.\frac{\pi}{3} - \int_0^{\sqrt{3}} 1 - \frac{1}{1+t^2}\,dt =$

$t^2 = x$

$dx = 2t\,dt$ $\quad = \pi - (t - \operatorname{arctg}t)\Big|_0^{\sqrt{3}} = \pi - \left(\sqrt{3} - \frac{\pi}{3} - 0\right) = \frac{4}{3}\pi - \sqrt{3}$

Пример $f(x)$ диференцируема, но не е непр. диференцируема

$f(x) = \begin{cases} x^2 \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

$f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{h^{\cancel{2}}\sin\frac{1}{h}}{\cancel{h}} = \lim_{h\to 0} h\sin\frac{1}{h} = 0$

$\Rightarrow f$ е диференцируема в 0

f е диф. в $\mathbb{R}\setminus\{0\}$ като комп. на диф. ф-ии

$\Big\} \Rightarrow f$ е диференцируема в $\mathbb{R}$

? $f'(x)$ е непрекъсната

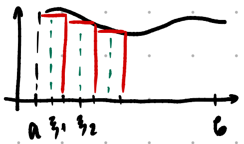
$f'(x) = 2x.\sin\frac{1}{x} + \cancel{x^2}\cos\frac{1}{x}\left(-\frac{1}{\cancel{x^2}}\right) = 2x.\sin\frac{1}{x} - \cos\frac{1}{x}$

$\lim_{x\to 0} f'(x) = \lim_{x\to 0} \underbrace{2x\sin\frac{1}{x}}_{\to 0} - \underbrace{\cos\frac{1}{x}}_{} \to$ не същ. $\Rightarrow f$ не е непр. диф. ($f'(x)$ не е непр.)

Приложение на определените интеграли

1) граници на редици

$f:[a,b] \to \mathbb{R}$ ограничена



$\tau: a = x_0 < x_1 < \dots < x_n = b$

$\xi = (\xi_1, \xi_2, \dots, \xi_n) \quad \xi_i \in [x_{i-1}, x_i]$

$\sigma_f(\tau, \xi) = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})$

$$\int_a^b f(x)\,dx = \lim_{diam(\tau)\to 0} \sigma_f(\tau, \xi)$$

① $S_n = \frac{1}{n^2} + \frac{2}{n^2} + \dots + \frac{n}{n^2} \quad \left(S_n = \frac{n(n+1)}{2.n^2} \xrightarrow[n\to\infty]{} \frac{1}{2} \right.$

a x b $\qquad t = \frac{x-a}{b-a}$

$0 \ \frac{1}{n} \ \frac{2}{n} \ \dots \ \frac{n-1}{n} \ 1 = \frac{n}{n}$

$\tau: x_i = \frac{i}{n} \quad i = 0, \dots, n \qquad x_i - x_{i-1} = \frac{1}{n}$

$\xi_i = \frac{i}{n} \quad i = 1, \dots, n \qquad f(\xi_i) = \xi_i \quad f(x) = x$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i}{n^2} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i}{n} \cdot \frac{1}{n} =$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \xi_i (x_i - x_{i-1}) = \lim_{n\to\infty} \underbrace{\sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})}_{\text{сума на Риман за } f(x)=x \text{ в } [0,1] \text{ и подр. } \tau} =$$

$$= \int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$$

② $S_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{n+n}$

0 1 $\qquad \tau: x_i = \frac{i}{n} \qquad x_i - x_{i-1} = \frac{1}{n}$

$f(\xi_i) = \frac{1}{1+\frac{i}{n}} \qquad \xi_i = \frac{i}{n} \qquad f(x) = \frac{1}{1+x}$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n+i} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} \cdot \frac{1}{1+\frac{i}{n}} =$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} (x_i - x_{i-1}) \cdot \frac{1}{1+\frac{i}{n}} = \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{f(\xi_i)(x_i - x_{i-1})}_{\text{сума на Риман за } f(x)=\frac{1}{1+x} \text{ в } [0,1] \text{ и подр. } \tau} =$$

$$= \int_0^1 \frac{1}{1+x}\,dx = \ln|1+x|\Big|_0^1 = \ln 2 - \ln 1 = \ln 2$$

③ $$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\cos^2(\frac{i\pi}{2})}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} \cos^2\left(\frac{i}{n} \cdot \pi\right) =$$

0 1 $\qquad \tau: x_i = \frac{i}{n} \qquad x_i - x_{i-1} = \frac{1}{n}$

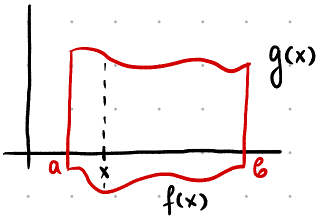
$\xi_i = \frac{i}{n}$

$$= \lim_{n\to\infty} (x_i - x_{i-1}) . \cos^2(\pi . \xi_i) = \lim_{n\to\infty} (x_i - x_{i-1}) f(\xi_i) =$$

$$= \int_0^1 \cos^2(\pi x)\,dx = \int_0^1 \frac{1 + \cos(2\pi x)}{2}\,dx = \int_0^1 \frac{1}{2}\,dx + \int_0^1 \frac{\cos(2\pi x)\,dx . 2\pi}{2 . 2\pi} = \frac{1}{2}x\Big|_0^1 + \frac{1}{4\pi}\sin(2\pi x)\Big|_0^1 = \frac{1}{2} - 0 + 0 = \frac{1}{2}$$

2) пресмятане на лице

D - криволинеен трапец



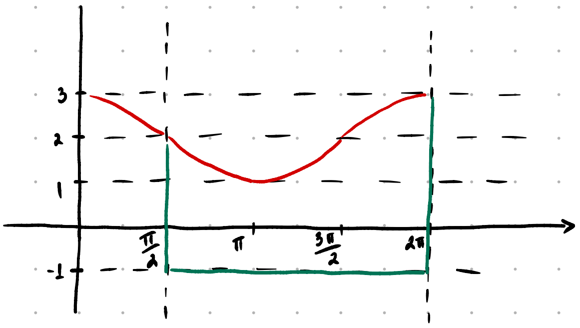
$D: \left| \begin{array}{l} a \le x \le b \\ f(x) \le y \le g(x) \end{array} \right.$

$f, g: [a,b] \to \mathbb{R}$ непр.

$f(x) \le g(x) \quad x \in [a,b]$

$$S_D = \int_a^b g(x) - f(x)\,dx$$

④ Намерете лицето на фигурата, ограничена от $y=\cos x+2$, $x=\frac{\pi}{2}$, $x=2\pi$, $y=-1$



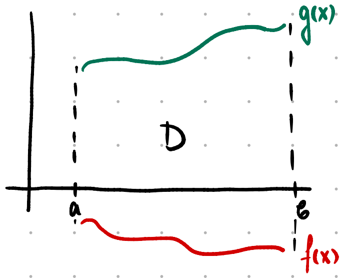
$$D: \left| \begin{array}{l} \frac{\pi}{2} \le x \le 2\pi \\ -1 \le y \le \cos x + 2 \end{array} \right.$$

$$S_D = \int_{\frac{\pi}{2}}^{2\pi} \cos x + 2 - (-1)\, dx = \int_{\pi/2}^{2\pi} \cos x + 3\, dx = (\sin x + 3x)\Big|_{\frac{\pi}{2}}^{2\pi} = (0+6\pi) - \left(1 + \frac{3\pi}{2}\right) = \frac{9}{2}\pi - 1$$

14.03.24

Приложение на определените интеграли

2) лица на равнинни фигури



D- криволинеен трапец

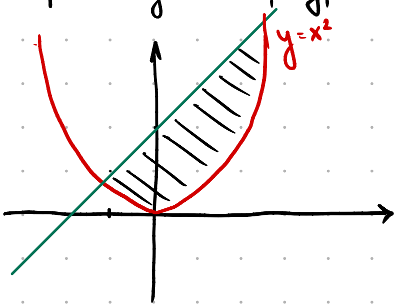
$$D: \left| \begin{array}{l} a \le x \le b \\ f(x) \le y \le g(x) \end{array} \right.$$

$f, g: [a,b] \to \mathbb{R}$ непр.

$f(x) \le g(x) \quad x \in [a,b]$

$$S_D = \int_a^b g(x) - f(x)\, dx$$

① Намерете лицето на фигурата, заградена от $y=x^2$ и $y-x=2$



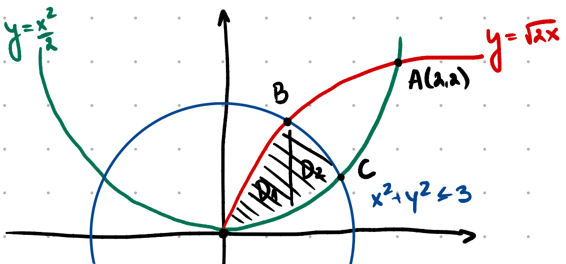
пресечни точки

$$\left| \begin{array}{l} y=x^2 \\ y-x=2 \end{array} \right. \qquad \begin{array}{ll} x^2-x-2=0 & \\ (x-2)(x+1)=0 & \\ x_1=2 & x_2=-1 \\ y_1=4 & y_2=1 \end{array}$$

$$D: \left| \begin{array}{l} -1 \le x \le 2 \\ x^2 \le y \le x+2 \end{array} \right.$$

$$S_D = \int_{-1}^{2} x+2-x^2\, dx = \left(\frac{x^2}{2} + 2x - \frac{x^3}{3}\right)\Big|_{-1}^{2} = \left(2+4-\frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right) = 4\frac{1}{2}$$

② $D \left| \begin{array}{l} y \ge \frac{x^2}{2} \\ y \le \sqrt{2x} \\ x^2+y^2 \le 3 \end{array} \right.$



$S_D = S_{D_1} + S_{D_2}$ т.к. $D = D_1 \cup D_2$

Окр. с ц. (a,b) и рад. r

$(x-a)^2 + (y-b)^2 = r^2$

$\le$ $\ge$

Пресечни точки A: $\left| \begin{array}{l} y=\frac{x^2}{2} \\ y=\sqrt{2x} \end{array} \right.$ $\quad \frac{x^2}{2} = \sqrt{2x}$ $\quad x^2 = 2\sqrt{2}\sqrt{x}$ $\quad x_1=0 \quad x_2=2$ $\quad A(2,2)$ $\quad 2^2+2^2=8>3 \Rightarrow$ A външ.