25.05.23

$f: U \to \mathbb{R}$, $\underset{\text{отв.}}{U \subset \mathbb{R}^n}$, $x_0 \in U$

$\underset{\text{линеен оператор}}{df(x_0): \mathbb{R}^n \to \mathbb{R}}$ такъв, че $f(x) = f(x_0) + df(x_0)(x - x_0) + \varphi(x, x_0)$ и $\dfrac{\varphi(x, x_0)}{\|x - x_0\|} \xrightarrow[x \to x_0]{} 0$

f диференцируема, то $\dfrac{df}{dx_i}(x_0) = \lim\limits_{\lambda \to 0} \dfrac{f(x_0 + \lambda e_i) - f(x_0)}{\lambda}$ същ. $\forall i \in \{1, \dots, n\}$ и

$$\frac{df}{dx_i}(x_0) = df(x_0)(e_i)$$

$$\operatorname{grad} f(x_0) = \left(\frac{df}{dx_1}(x_0), \dots, \frac{df}{dx_n}(x_0)\right) \qquad d f(x_0)(h) = \langle \operatorname{grad} f(x_0), h \rangle$$

Ако $\dfrac{df}{dx_i}$ същ в U и са непр. в x_0, то f е диф. в x_0.

$f: U \to \mathbb{R}^m$, U отв в $\mathbb{R}^n$, $x_0 \in U$ $\qquad f(x) = \begin{pmatrix} f_1(x) \\ \dots \\ f_m(x) \end{pmatrix}$

f диф, ако $f(x) = f(x_0) + df(x_0)(x - x_0) + o(\|x - x_0\|)$ $\qquad \underset{\text{линеен}}{df(x_0): \mathbb{R}^n \to \mathbb{R}^m}$

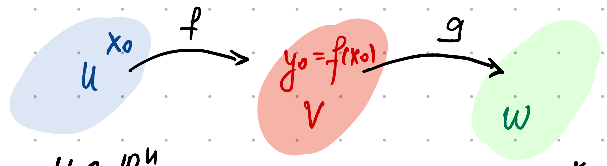
$\Updownarrow$

f_i диф в x_0 $\forall i \in \{1, \dots, n\}$ и

$$d f(x_0)(h) = \begin{pmatrix} \operatorname{grad} f_1(x_0) \\ \dots \\ \operatorname{grad} f_m(x_0) \end{pmatrix} . h = f'(x_0) . h$$

$$f'(x_0) = \underbrace{\begin{pmatrix} \frac{df_1}{dx_1} & \dots & \frac{df_1}{dx_m} \\ \frac{df_m}{dx_1} & \dots & \frac{df_m}{dx_m} \end{pmatrix}}_{\text{матрица}} (x_0)$$

Th.



$x_0 \in U \subset \mathbb{R}^n$ отв. $\qquad W \subset \mathbb{R}^k$

$y_0 = f(x_0) \in V \subset \mathbb{R}^n$ отв.

$f: U \to V$

$g: V \to \mathbb{R}^k$

$g \circ f: U \to \mathbb{R}^k$

f е диф. в x_0

g е диф. в y_0

Тогава $g \circ f$ е диференцируема в x_0 и

$$d(\underset{\substack{\uparrow \\ \text{композиция}}}{g \circ f})(x_0) = dg(f(x_0)) \circ df(x_0)$$

$df(x_0): \mathbb{R}^n \to \mathbb{R}^m$

$dg(y_0): \mathbb{R}^m \to \mathbb{R}^k$

$f'(x_0) \cdot h = df(x_0)(h)$ $\quad d[g\circ f)(x_0)](h) = [g'(f(x_0)) \cdot f'(x_0)] \cdot h$ $\quad$ (произв. на матр)

$g'(y_0) \cdot q = dg(y_0) \cdot h$ $\quad (g\circ f)'(x_0) = g'(f(x_0)) \cdot f'(x_0)$

$$\frac{d(g\circ f)}{dx_i}(x_0) = \sum_{j=1}^{m} \frac{dg_s}{dg_j}(f(x_0)) \frac{dg_j}{dx_i}(x_0) \quad i \in \{1, \dots, m\}$$

$s \in \{1, \dots, k\}$

док: БОО, $k=1$

f_j диф в x_0 $\longrightarrow$ $f_j(x) = f_j(x_0) + df_j(x_0)(x - x_0) + \varphi_j(x, x_0)$ $\quad \frac{\varphi_j(x, x_0)}{\|x - x_0\|} \xrightarrow[x \to x_0]{} 0$

$\forall j \in \{1, \dots, m\}$

g диф в $y_0 = f(x_0)$ $\quad g(y) = g(y_0) + dg(y_0)(y - y_0) + \varphi(y, y_0)$ $\quad \frac{\varphi(y, y_0)}{\|y - y_0\|} \xrightarrow[y \to y_0]{} 0$

$F(x) = g(f(x))$ $\quad F(x_1, \dots, x_n) = g[f_1(x_1, \dots, x_n), f_2(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_m)]$

пример: $f(x,y) = x^y$ $\quad [0, +\infty) \times \mathbb{R}$ $\quad F(t) = f(t,t) = t^t$

$\frac{df}{dx}(x,y) = y x^{y-1}$ $\quad t \longrightarrow (t,t) \xrightarrow{f} t$

$$F'(t) = \underbrace{\frac{df}{dx}}_{\text{произв. по 1ви арг.}}(t,t) \cdot 1 + \underbrace{\frac{df}{dy}}_{\text{произв. по IIри арг.}}(t,t) \cdot 1 =$$

$\frac{df}{dy}(x,y) = x^y \cdot \ln x$

$$= t \cdot t^{-1} + t^t \cdot \ln t = t^t(1 + \ln t)$$

отместване

$$F(x) - F(x_0) = g(f(x)) - g(\underset{y_0}{\underset{\shortparallel}{f(x_0)}}) = dg(y_0)(f(x) - f(x_0)) + \varphi(f(x), f(x_0)) =$$

$$= \sum_{j=1}^{m} \frac{dg}{dy_j}(y_0)(f_j(x) - f_j(x_0)) + \varphi(f(x), f(x_0)) = \sum_{j=1}^{\infty} \frac{dg}{dy_j}(y_0)(df_j(x_0)(x - x_0) + \varphi_j(x, x_0)) + \varphi(f(x), f(x_0))$$

$$= \sum_{j=1}^{m} \frac{dg}{dy_j}(y_0)\left(\sum_{i=1}^{n} \frac{df_i}{dx_i}(x_0)(x - x_0)_i + \varphi_j(x, x_0)\right) + \varphi(f(x), f(x_0)) =$$

$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{m} \frac{dg}{dy_j}(y_0) \cdot \frac{df_i}{dx_i}(x_0)\right) \cdot (x - x_0) + \sum_{j=1}^{m} \frac{dg}{dy_j}(y_0) \cdot \varphi_j(x, x_0) + \varphi(f(x), f(x_0)) =$$

$$F(x) = F(x_0) + \sum_{i=1}^{n} \left(\sum_{j=1}^{m} \frac{dg}{dy_j}(f(x_0)) \frac{df_i}{dx_i}(x_0)\right)(x - x_0) + \sum_{j=1}^{m} \frac{dg}{dy_j}(f(x_0)) \varphi_j(x, x_0) + \varphi(f(x), f(x_0))$$

$$\frac{dF}{dx_i}(x_0)=\sum_{j=1}^{m}\frac{dg}{dy_j}(f(x_0))\cdot\frac{df_j}{dx_i}(x_0)$$

$$\frac{1}{\|x-x_0\|}\cdot\sum_{j=1}^{m}\frac{dg}{dy_j}(f(x_0)).\ \varphi_j(x,x_0)=\sum_{j=1}^{m}\frac{dg}{dy_j}(f(x_0)).\boxed{\frac{\varphi_j(x,x_0)}{\|x-x_0\|}}\xrightarrow[x\to x_0]{}0$$

$$\frac{\psi(f(x),f(x_0))}{\|x-x_0\|}=\frac{\|f(x)-f(x_0)\|}{\|x-x_0\|}\cdot\alpha(f(x))$$

$f(x)\xrightarrow[x\to x_0]{}f(x_0)$ (f диф. в x_0 $\Rightarrow$ f непр. в x_0)

$$\alpha(y)=\begin{cases}\dfrac{\psi(y,y_0)}{\|y-y_0\|}, & \text{ако } y\neq y_0\\ 0, & \text{ако } y=y_0\end{cases}$$

$\alpha: V\to\mathbb{R}$, α е непрекъсната в y_0

α непр. в $y_0=f(x_0)$

$$\Rightarrow\ \alpha(f(x))\xrightarrow[x\to x_0]{}\alpha(f(x_0))=\alpha(y_0)=0$$

$$\frac{\psi(f(x),f(x_0))}{\|x-x_0\|}=\boxed{\frac{\|f(x)-f(x_0)\|}{\|x-x_0\|}}\cdot\alpha(f(x))$$

док., че е огр.

$\alpha(f(x))\xrightarrow[x\to x_0]{}0$

$$f(x)-f(x_0)=\begin{pmatrix}\operatorname{grad}f_1(x_0)\\ \cdots\\ \operatorname{grad}f_m(x_0)\end{pmatrix}(x-x_0)+\begin{pmatrix}\varphi_1(x,x_0)\\ \cdots\\ \varphi_m(x,x_0)\end{pmatrix}$$

$$0\leq\frac{\|f(x)-f(x_0)\|}{\|x-x_0\|}\leq\frac{1}{\|x-x_0\|}\sqrt{\sum_{j=1}^{m}(\langle\operatorname{grad}f_j(x_0),x-x_0\rangle)^2}+\frac{1}{\|x-x_0\|}\sqrt{\sum_{j=1}^{m}[\varphi_j(x,x_0)]^2}\leq$$

$$\leq\frac{1}{\cancel{\|x-x_0\|}}\sqrt{\sum_{j=1}^{m}\underbrace{\|\operatorname{grad}f_j(x_0)\|^2}_{const}\cdot\cancel{\|x-x_0\|^2}}+\sqrt{\sum_{j=1}^{m}\left(\frac{\varphi_j(x,x_0)}{\|x-x_0\|}\right)^2}=$$

$$=\sqrt{\sum_{j=1}^{m}\|\operatorname{grad}f_j(x_0)\|^2}+\boxed{\sqrt{\sum_{j=1}^{m}\left(\frac{\varphi_j(x,x_0)}{\|x-x_0\|}\right)^2}}\xrightarrow[x\to x_0]{}0$$

$\Rightarrow\ \dfrac{\|f(x)-f(x_0)\|}{\|x-x_0\|}$ е ограничен, което завършва доказателството

Инвариантност на формата на диференциала

$$df(x_0)(h)=\sum_{i=1}^{n}\frac{df}{dx_i}(x_0).h_i=\sum_{i=1}^{n}\frac{df}{dx_i}(x_0).dx_i(x_0).h$$

$(h_1,\dots,h_n)=h\xmapsto{x_i}h_i$

$$df(x_0)=\sum_{i=1}^{\infty}\frac{df}{dx_i}(x_0).dx_i$$

$$\boxed{df=\sum_{i=1}^{\infty}\frac{df}{dx_i}dx_i}$$

$x_i:\mathbb{R}^n\to\mathbb{R}$

$x_i(h)=h_i$

$dx_i(x_0)(h)=h_i$

$$g\longrightarrow dg=\sum_{i=1}^{n}\frac{dg}{dy_i}dy_i$$

$$F(x_1,\dots,x_n)=g(f_1(x_1,\dots,x_n),\dots,f_m(x_1,\dots,x_n))$$

$$dF=\sum_{i=1}^{n}\frac{dF}{dx_i}dx_i=\sum_{i=1}^{n}\left(\sum_{j=1}^{m}\frac{dg}{dy_j}\cdot\frac{df_i}{dx_i}\right)dx_i=\sum_{j=1}^{m}\frac{dg}{dy_j}\underbrace{\left(\sum_{i=1}^{n}\frac{df_i}{dx_i}dx_i\right)}_{dy_j=df_j}=$$

$$\begin{cases} y_1=f_1(x_1,\dots,x_n) \\ \dots \\ y_n=f_n(x_1,\dots,x_n)\end{cases}\qquad =\sum_{j=1}^{m}\frac{dg}{dy_j}dy_j$$

заменяме y_j с функ.на x

$g(u,v)=u+v$

по инвариантността на място на u и v можем да впишем функции

$$d(u+v)=\frac{d(u+v)}{du}\cdot du+\frac{d(u+v)}{dv}\cdot dv=du+dv$$

$$d(u\cdot v)=\frac{d(u\cdot v)}{du}\cdot du+\frac{d(u\cdot v)}{dv}\cdot dv=v\cdot du+u\cdot dv$$

$$d\left(\frac{u}{v}\right)=\frac{d\left(\frac{u}{v}\right)}{du}du+\frac{d\left(\frac{u}{v}\right)}{dv}dv=\frac{1}{v}du+u\cdot\left(-\frac{1}{v^2}\right)du=\frac{vdu-udv}{v^2}$$

$d(u+v)=du+dv$

$d(u\cdot v)=v\cdot du+u\cdot dv$

$d\left(\frac{u}{v}\right)=\frac{vdu+u\cdot dv}{v^2}$

пример:

1) $Z(x,y)$, $x,y\in\mathbb{R}^2$

$Z'_x=Z'_y$

$$\begin{cases}u=x+y\\ v=x-y\end{cases}\longleftrightarrow\begin{cases}x=x(u,v)\\ y=y(u,v)\end{cases}$$

$w(u,v)=z(u(x,y),v(x,y))$

$z(x,y)=w(u(x,y),v(x,y))$

$$\boxed{w'_u(x+y,x-y)}$$

$$z'_x(x,y)=w'_u(u(x,y),v(x,y))\cdot u'_x(x,y)+w'_v(u(x,y),v(x,y))\cdot v'_x(x,y)=$$

$$=w'_u(u(x,y),v(x,y))\cdot 1+w'_v(u(x,y),v(x,y))\cdot 1$$

$z'_x=w'_u+w'_v$

$z'_y=w'_u\cdot u_y+w'_v\cdot v_y=w'_u\cdot 1+w'_v\cdot(-1)=w'_u-w'_v$

$z'_x=z'_y$ $\qquad \cancel{w'_u}+w'_v-\cancel{w'_u}+w'_v=0$

$w'_v=0$

$w(u,v)=\varphi(u)$, φ диференцируема

$z(x,y)=\varphi(x+y)$

$\Rightarrow \varphi$ е гладка

$x \cdot z'_x + y z'_y = 0$

$$\begin{cases} u = x \\ v = \frac{y}{x} \end{cases} \qquad w(u,v)$$

$z(x,y) = w\left(x, \frac{y}{x}\right)$

$z'_x(x,y) = w'_u \cdot 1 - w'_v \cdot \frac{y}{x^2}$

$z'_y(x,y) = w'_u \cdot 0 + \frac{1}{x} \cdot w'_v$

$x w'_u - \cancel{\frac{y}{x} w'_v} + \cancel{\frac{y}{x} w'_v} = 0$

$x w'_u = 0 \sim u \cdot w'_u = 0$

$\Rightarrow w'_u = 0$

$w(u,v) = \varphi(v) \quad \varphi$ е диф.

$y z'_x - x z'_y = 0$

$$\begin{cases} u = x \\ v = x^2 + y^2 \end{cases} \qquad w(u,v)$$

$z(x,y) = w(x, x^2+y^2)$

$z'_x(x,y) = w'_u \cdot 1 + w'_v \cdot 2x + \cancel{w'_v \cdot 0}$

$z'_y(x,y) = \cancel{w'_u \cdot 0} + \cancel{w'_v \cdot 0} + w'_v \cdot 2y$

$y \cdot w'_u + \cancel{2xy w'_v} - \cancel{2xy w'_v} = 0$

$y \cdot w'_u = 0 \Rightarrow w'_u = 0$

$w(u,v) = \varphi(v) \quad \varphi$ е диф.

$f : U \to \mathbb{R}, \quad U \subset \mathbb{R}^n$

$\underbrace{\frac{df}{dx_i}(x)}_{f'_{x_i}(x)}, \; x \in U \to \frac{df}{dx_i} : U \to \mathbb{R}$

$\frac{d}{dx_j}\left(\frac{df}{dx_i}\right) = \frac{d^2 f}{dx_i \, dx_j}$

$\underline{f''_{x_i x_j} := (f'_{x_i})'_{x_j}}$ частна производна от II ред

Теорема на Шварц

$f : U \to \mathbb{R}$, U отв. в $\mathbb{R}^n$

f двукратно гладка $(f \in C^2(U))$

$\hookrightarrow \frac{d^2 f}{dx_i x_j}$ съществуват и са непрекъснати в U $\Rightarrow \frac{d^2 f}{dx_i dx_j} = \frac{d^2 f}{dx_j dx_i}$

$i, j \in \{1, \dots, n\}$

пример: $f(x,y) = (x^2 + \operatorname{arctg}(e^{\sqrt{y}})) \ln y$

f''_{xxy} - лесно

f''_{yxx} - гадно