$$1 = A(1-z)(1+2z+z^2) + B(1+2z+z^2) + C(1+z)(1-2z+z^2) + D(1-2z+z^2)$$

$z^3: \quad 0 = -A + C \quad \Rightarrow A = C$

$z^2: \quad 0 = A - 2A + B - 2C + C + D = -A - C + B + D \Rightarrow A = B = C = D$

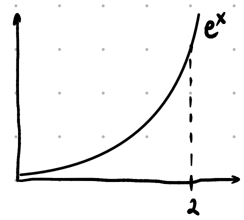
$z^1: \quad 0 = 2A - A + 2B + C - 2C - 2D = A - C + 2(B-D) \Rightarrow B = D$

$z^0: \quad 1 = A + B + C + D \Rightarrow A = B = C = D = \frac{1}{4}$

$$(*) \quad \frac{1}{8}\int_0^{\sin(\operatorname{arctg} 4)} \frac{1}{1-z} + \frac{1}{1+z} + \frac{1}{(1-z)^2} + \frac{1}{(1+z)^2}\, dz = \frac{1}{8}\left(-\ln|1-z| + \ln|1+z| + \frac{1}{1-z} - \frac{1}{1+z}\right)\Bigg|_0^{\sin(\operatorname{arctg} 4)}$$

21.03.24

зад 1



$f(x) = e^x, \; x \in [0,2]$

$$l = \int_0^2 \sqrt{1 + (f'(x))^2}\, dx = \int_0^2 \sqrt{1+e^{2x}}\, dx = \int_{\sqrt{2}}^{\sqrt{1+e^4}} t \cdot \frac{t}{t^2-1}\, dt = \int_{\sqrt{2}}^{\sqrt{1+e^4}} \frac{t^2-1+1}{t^2-1}\, dt = \int_{\sqrt{2}}^{\sqrt{1+e^4}} 1 + \frac{1}{t^2-1}\, dt =$$

$t = \sqrt{1+e^{2x}}$ $\quad x = 0 \quad t = \sqrt{1+1} = \sqrt{2}$ $\quad \sqrt{1+e^{2x}} = \sqrt{1+e^{\cancel{2}\cdot\frac{\ln(t^2-1)}{\cancel{2}}}} =$

$t^2 = 1 + e^{2x}$ $\quad x = 2 \quad t = \sqrt{1+e^4}$

$e^{2x} = t^2 - 1$ $\quad = \sqrt{1 + t^2 - 1} = \sqrt{t^2} = t$

$x = \frac{\ln(t^2-1)}{2}$ $\quad dx = \frac{1}{\cancel{2}} \cdot \frac{1}{t^2-1} \cdot \cancel{2} t\, dt$

$$= \int_{\sqrt{2}}^{\sqrt{1+e^4}} 1 + \frac{1}{(t-1)(t+1)}\, dt = \int_{\sqrt{2}}^{\sqrt{1+e^4}} 1 + \frac{1}{2}\cdot\frac{1}{t-1} - \frac{1}{2}\frac{1}{t+1}\, dt = \left(t + \frac{1}{2}\ln|t-1| - \frac{1}{2}\ln|t+1|\right)\Bigg|_{\sqrt{2}}^{\sqrt{1+e^4}} =$$

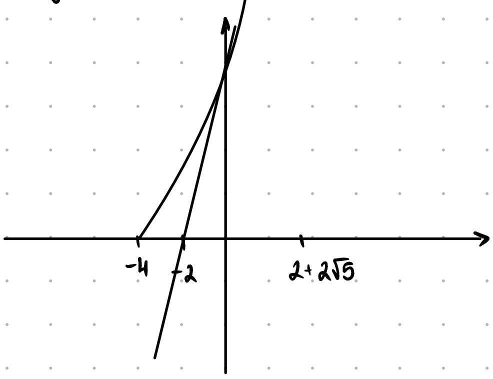
$\frac{A}{t-1} + \frac{B}{t+1} = \frac{1}{(t-1)(t+1)}$ $\quad A(t+1) + B(t-1) = 1$ $\quad t = -1 \quad -\cancel{A} - B + \cancel{A} - B - 1 = 0 \quad -2B = 1 \quad B = -\frac{1}{2}$

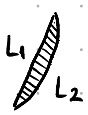
$At + A + Bt - B - 1 = 0$

$(A+B)t + A - B - 1 = 0$ $\quad t = 1 \quad A + \cancel{B} + A - \cancel{B} - 1 = 0 \quad 2A = 1 \quad A = \frac{1}{2}$

$$= \sqrt{1+e^4} + \frac{1}{2}\ln(\sqrt{1+e^4} - 1) - \frac{1}{2}\ln(\sqrt{1+e^4} + 1) - \sqrt{2} - \frac{1}{2}\ln(\sqrt{2} - 1) + \frac{1}{2}\ln(\sqrt{2} + 1)$$

зад 2 Намерете дълж. на контура на фиг., зададена от $y = 4x + 8$ и $y = \sqrt{(x+4)^3}$





пресечни точки: $\left| \begin{array}{l} y = 4x+8 \\ y = \sqrt{(x+4)^3} \end{array} \right.$ $\quad 4x + 8 = \sqrt{(x+4)^3} \Rightarrow (4x+8)^2 = (x+4)^3, \; 4x+8 \geq 0$

$x \geq -2$

$16x^2 + 64x + \cancel{64} = x^3 + 12x^2 + 48x + \cancel{64}$

$x^3 - 4x^2 - 16x = 0$ $\quad \left| \begin{array}{l} x_1 = 0 \\ y_1 = 8 \end{array} \right.$ $\quad \left| \begin{array}{l} x_2 = 2 + 2\sqrt{5} \\ y_2 = 16 + 8\sqrt{5} \end{array} \right.$

$x(x^2 - 4x - 16) = 0$

$x_1 = 0$ $\quad D = 16 + 64 = 80$

$x_{2,3} = \frac{4 \pm 4\sqrt{5}}{2}$ $\quad x_2 = 2 + 2\sqrt{5}$

$x_3 = 2 - 2\sqrt{5} < -2$ не е реш.

$l = l_1 + l_2$

$l_1: \; y_1 = 4x + 8, \; x \in [0; 2+2\sqrt{5}]$

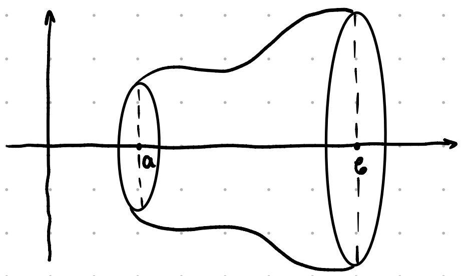
$$\int_0^{2+2\sqrt{5}} \sqrt{1 + (y_1')^2}\, dx = \int_0^{2+2\sqrt{5}} \sqrt{1+16}\, dx = \sqrt{17}\, x \Big|_0^{2+2\sqrt{5}} = \sqrt{17}(2 + 2\sqrt{5})$$

$l_2: \; y_2 = \sqrt{(x+4)^3} \;, \; x \in [0, 2+2\sqrt{5}]$

$$\int_0^{2+2\sqrt{5}} \sqrt{1+(y_2')^2}\, dx = \int_0^{2+2\sqrt{5}} \sqrt{1+\frac{9}{4}(x+4)}\, dx = \int_0^{2+2\sqrt{5}} \frac{\sqrt{10+\frac{9}{4}x}}{\frac{9}{4}}\, dx \cdot \frac{9}{4} = \frac{4}{9}\left(10+\frac{9}{4}x\right)^{\frac{3}{2}} \cdot \frac{2}{3}\Bigg|_0^{2+2\sqrt{5}}$$

$$y_2' = \frac{3}{2}(x+4)^{\frac{1}{2}} \cdot 1 = \frac{3}{2}\sqrt{x+4}$$

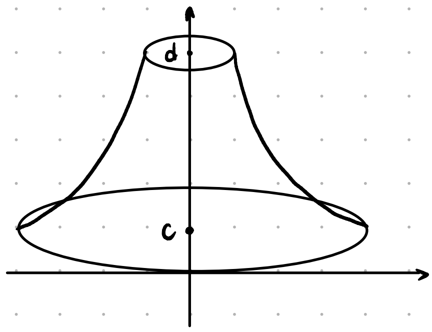
④ обем на ротационно тяло



$f(x)$ – непрекъсната в $[a,в]$

завъртаме $f(x)$ около Ox, $f(x) \geq 0$

$$V = \pi \cdot \int_a^в f^2(x)\, dx$$

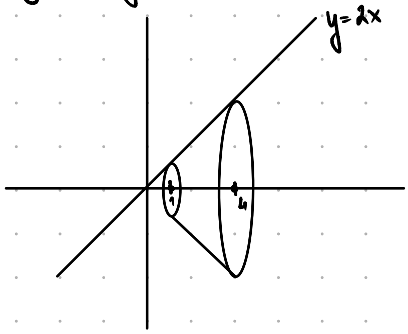


$g(y)$ – непрекъсната в $[c,d]$

завъртаме $g(y)$ около Oy, $g(y) \geq 0$

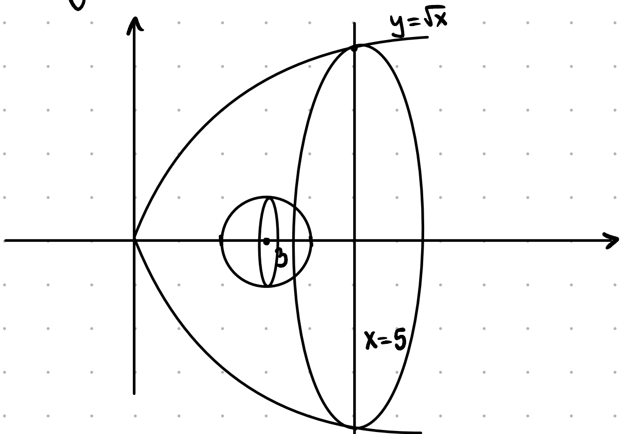
$$V = \pi \cdot \int_a^в g^2(y)\, dy$$

3 зад $\quad y=2x$, $x \in [1,4]$, завъртаме около $Ox \to V=?$



$$V = \pi \cdot \int_1^4 4x^2\, dx = \pi \cdot 4 \cdot \frac{x^3}{3}\Bigg|_1^4 = \pi \cdot 4 \cdot \frac{4^3}{3} - \pi \cdot 4\, \frac{1^3}{3} = \pi \cdot 4 \cdot \frac{\cancel{63}^{21}}{\cancel{3}} = 84\pi$$

зад 4 Нам. обема на тялото, заградено от $y=\sqrt{x}$, $y=\sqrt{6x-x^2-8}$, $x=5$ при зав. около Ox



$y = \sqrt{6x-x^2-8}$

$y^2 = 6x - x^2 - 8$

$x^2 - 6x + y^2 + 9 = -8 + 9$

$(x-3)^2 + y^2 = 1 \to$ окр с център $(3,0)$ и рад 1

пресечни точки

$$\begin{cases} y=\sqrt{x} & x\geq 0 \\ y=\sqrt{6x-x^2-8}, & 6x-x^2-8\geq 0 \end{cases}$$

$x=6x-x^2-8$

$x^2-5x+8=0$

$D=25-32<0 \Rightarrow$ няма

$V=V_1-V_2$

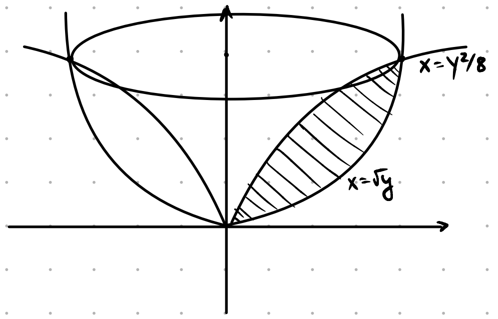
$$V_1=\pi.\int_0^5 y_1^2(x)\,dx=\pi\int_0^5 x\,dx=\pi\,\frac{x^2}{2}\Big|_0^5=\frac{25}{2}\pi$$

$$V_2=\pi\int_2^4 y_2^2(x)\,dx=\pi\int_2^4 6x-x^2-8\,dx=\pi\left(6\frac{x^2}{2}-\frac{x^3}{3}-8x\right)\Big|_2^4=\pi\left(3.16-\frac{64}{3}-32-12+\frac{8}{3}+16\right)=$$

$=4\pi$

$$V=\frac{25}{2}\pi-\frac{8}{2}\pi=\frac{17}{2}\pi$$

зад 5 Намерете обема на тялото, получено при завъртането на $D:\begin{cases} y\geq x^2 \\ 8x\geq y^2\end{cases}$ около Oy



пресечни точки: $\begin{cases} y=x^2 \\ 8x=y^2\end{cases}$

$8x=x^4$

$x(x^3-8)=0$

$x_1=0 \quad x_2=2$

$y_1=0 \quad y_2=4$

$\begin{cases} x_1(y)=\sqrt{y} \\ y\in[0;4]\end{cases}$ $\quad\begin{cases} x_2(y)=y^2/8 \\ y\in[0;4]\end{cases}$

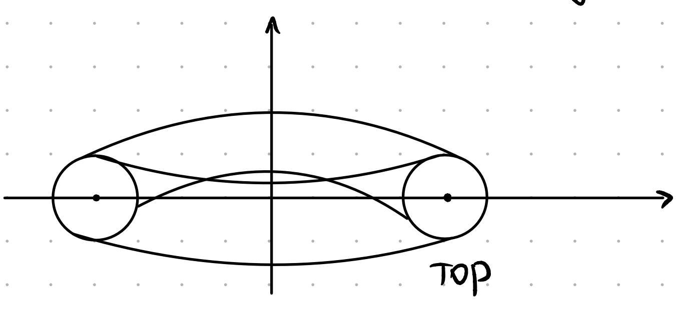
$$V=V_1-V_2=\pi\int_0^4 x_1^2(y)\,d(y)-\pi\int_0^4 x_2^2(y)\,dy=\pi\int_0^4 y\,dy-\pi\int_0^4\frac{1}{64}\cdot y^4\,dy=\pi\,\frac{y^2}{2}\Big|_0^4-\pi\,\frac{y^5}{64.5}\Big|_0^4=\pi\left(8-\frac{16}{5}\right)=\frac{24}{5}\pi$$

зад 6 $(x-a)^2+y^2=r^2$

$a>r>0$

завъртане около Oy

$(x-a)^2+y^2=r^2$ - окр. с център $(a,0)$ и радиус r.



$V=V_1-V_2, \quad y\in[-r,r]$

$$V=\pi\int_{-r}^{r}x_1^2(y)\,dy-\pi\int_{-r}^{r}x_2^2(y)\,dy=$$

$$=\pi\int_{-r}^{r}x_1^2(y)-x_2^2(y)\,dy=\circledast$$

$(x-a)^2+y^2=r^2$

$(x-a)^2=r^2-y^2$

$x-a=\pm\sqrt{r^2-y^2}$

$x=a\pm\sqrt{r^2-y^2}$

$-\ x_1(y)=a+\sqrt{r^2-y^2}\Rightarrow x_1^2(y)=a^2+2a\sqrt{r^2-y^2}+r^2-y^2$

$\ \ \ x_2(y)=a-\sqrt{r^2-y^2}\Rightarrow x_2^2(y)=a^2-2a\sqrt{r^2-y^2}+r^2-y^2$

$x_1^2(y)-x_2^2(y)=4a\sqrt{r^2-y^2}$

$$(*) = \pi \int_{-r}^{r} 4a\sqrt{r^2-y^2}\, dy = 4a\pi . 2\int_0^r \sqrt{r^2-y^2}\, dy = 8a\pi \int_0^{\pi/2} \sqrt{r^2-r^2\sin^2 t}\,.\, r\cos t\, dt =$$

четна ф-я
в сим. интервал.

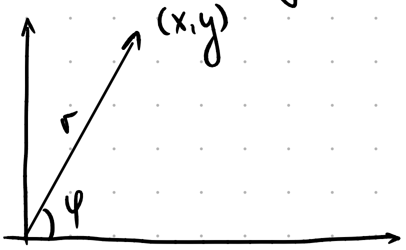
$y = r.\sin y \qquad y=0 \quad \sin t = 0,\ t=0$

$dy = r\cos t\, dt \qquad y = r \quad \sin t = 1,\ t = \pi/2$

$$= 8a\pi \int_0^{\pi/2} r.\cos t . r\cos t\, dt = 8a\pi . r^2 \int_0^{\pi/2} \cos^2 t\, dt = 8a\pi . r^2 \int_0^{\pi/2} \frac{1+\cos 2t}{2}\cdot\frac{1}{2}\, d2t =$$

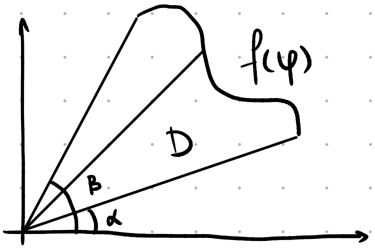
$$= 8a\pi . r^2 \left(\frac{1}{\cancel{4}_2}\cdot \cancel{2}t + \frac{1}{4}\sin 2t\right)\Big|_0^{\pi/2} = {}^2\cancel{8}a\pi r^2 . \frac{\pi}{\cancel{4}} = 2a\pi^2 r^2$$

Полярни координати



$$\left| \begin{array}{ll} x = r.\cos\varphi, & r \geq 0 \\ y = r.\sin\varphi, & a \leq \varphi < a + 2\pi \end{array} \right.$$

$$x^2+y^2 = r^2\cos^2\varphi + r^2\sin^2\varphi = r^2$$



$0 \leq \alpha \leq \beta \leq 2\pi$

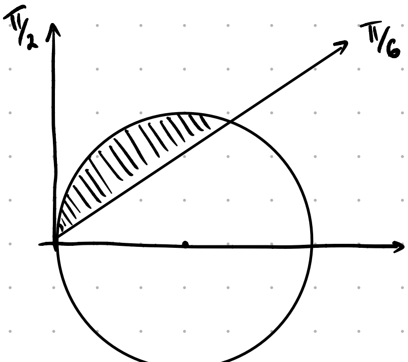
$$D: \left| \begin{array}{l} \alpha \leq \varphi \leq \beta \\ 0 \leq r \leq f(\varphi) \end{array} \right. \qquad f(\varphi) \text{ непрекъсната}$$

$$S_D = \frac{1}{2}\int_\alpha^\beta f^2(\varphi)\, d\varphi$$

зад 7 $D: \left| \begin{array}{l} \pi/6 \leq \varphi \leq \pi/2 \\ 0 \leq r \leq 2\cos\varphi \end{array} \right. \qquad S_D = ?$

$$S_D = \frac{1}{2}\int_{\pi/6}^{\pi/2} (2\cos\varphi)^2\, d\varphi = \frac{1}{2}\int_{\pi/6}^{\pi/2} 4\cos^2\varphi\, d\varphi = 2\int_{\pi/6}^{\pi/2} \frac{1+\cos 2\varphi}{2}\cdot\frac{1}{2}\, d2\varphi =$$

$$= \left(\frac{1}{\cancel{2}}\cancel{2}\varphi + \frac{1}{2}\cdot\sin 2\varphi\right)\Big|_{\pi/6}^{\pi/2} = \pi/2 - \left(\pi/6 + \frac{\sqrt{3}}{2}\cdot\frac{1}{2}\right) = \frac{1}{3}\pi - \frac{\sqrt{3}}{4}$$



$r \leq 2\cos\varphi \quad |.r \qquad x^2 - 2x + 1 + y^2 \leq 1$

$r^2 \leq 2r\cos\varphi \qquad (x-1)^2 + y^2 \leq 1$

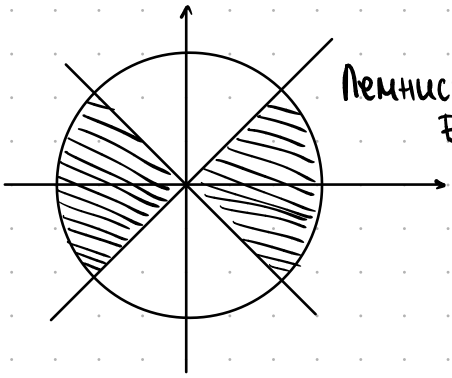
$x^2+y^2 \leq 2x \qquad$ кръг с център (1,0) и рад 1 зад 8 $D: (x^2+y^2)^2 \le x^2-y^2$ $\qquad S_D = ?$

$x^2-y^2 \ge (x^2+y^2)^2 \ge 0$

$x^2-y^2 \ge 0 \qquad |x| \ge |y| \qquad (x-y)(x+y) \ge 0$

$(x,y) \in D \Rightarrow (-x,y) \in D$ // сим. относно Oy

$(x,-y) \in D \Rightarrow (-x,-y) \in D$ // сим. относно Ox



Лемниската на Бернули

$S_D = 4 S_{D_1} = 4 S_1$ (от симетрии)

$D_1: \left| \begin{array}{l} (x^2+y^2)^2 \le x^2-y^2 \\ x \ge 0 \\ y \ge 0 \end{array} \right.$

полярни координати: $\begin{array}{ll} x = r\cos\varphi & r \ge 0 \\ y = r\sin\varphi & 0 \le \varphi \le 2\pi \end{array}$

$D_1: \left| \begin{array}{ll} (r^2)^2 \le r^2\cos^2\varphi - r^2\sin^2\varphi & | : r^2 \\ r\cos\varphi \ge 0 & | : r \\ r\sin\varphi \ge 0 & | : r \end{array} \right. \qquad D_1: \left| \begin{array}{l} r^2 \le \cos 2\varphi \\ \cos\varphi \ge 0 \\ \sin\varphi \ge 0 \end{array} \right. \left.\begin{array}{l} \\ \\ \end{array}\right\} \Rightarrow 0 \le \varphi \le \frac{\pi}{2}$

$D_1: \left| \begin{array}{l} 0 \le r \le \sqrt{\cos 2\varphi} \\ \cos 2\varphi \ge 0 \\ 0 \le \varphi \le \pi/2 \end{array} \right. \qquad \left.\begin{array}{l} 0 \le 2\varphi \le \pi \\ \cos 2\varphi \ge 0 \end{array}\right\} \Rightarrow 0 \le 2\varphi \le \frac{\pi}{2}, \quad 0 \le \varphi \le \frac{\pi}{4}$

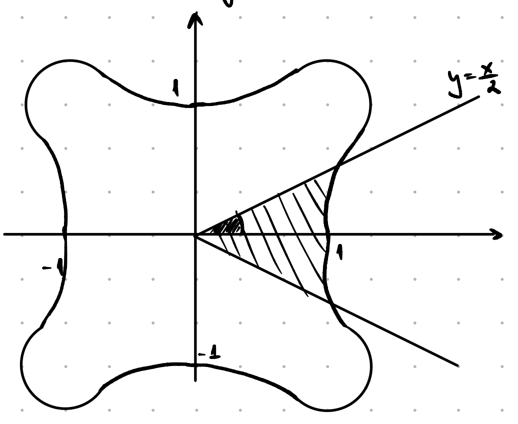
$D_1: \left| \begin{array}{l} 0 \le \varphi \le \pi/4 \\ 0 \le r \le \sqrt{\cos 2\varphi} \end{array} \right.$

$S_1 = \frac{1}{2}\int_0^{\pi/4} \sqrt{\cos 2\varphi}^{\,2}\, d\varphi = \frac{1}{2}\int_0^{\pi/4} \cos 2\varphi \cdot \frac{1}{2}\, d2\varphi = \frac{1}{4}\sin 2\varphi \Big|_0^{\pi/4} = \frac{1}{4}$

$S_D = 4 \cdot S_1 = 1$

зад 9 $D: \left| \begin{array}{l} x^4+y^4 \le x^2+y^2 \\ x \ge 2y \\ x \ge -2y \end{array} \right. \qquad S_D = ?$

$x^4+y^4 \le x^2+y^2 \longrightarrow (x,y) \Rightarrow (-x,y), (x,-y), (-x,-y)$



$\begin{array}{ll} x \ge 2y & x \ge -2y \\ y \le \frac{x}{2} & y \ge -\frac{x}{2} \end{array}$

$(x,y): \left.\begin{array}{l} x \ge 2y \\ x \ge -2y \end{array}\right\} \Rightarrow x = 0$

$(x,y) \Rightarrow (x,-y)$ сим. относно Ox

$S_D = 2S_{D_1} = 2S_1$

$D_1: \begin{cases} x^4+y^4 \le x^2+y^2 \\ x \ge 2y \\ x \ge -2y \\ x \ge 0 \\ y \ge 0 \end{cases}$

Полярни координати: $\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$ $\quad r \ge 0$, $\quad 0 \le \varphi \le 2\pi$

$x^4+y^4 = (x^2+y^2)^2 - 2x^2y^2$

$D_1: \begin{cases} (r^2)^2 - 2r^2\cos^2\varphi\, r^2\sin^2\varphi \le r^2 & |:r^2 \\ r\cos\varphi \ge 2\sin\varphi & |:r \\ r\cos\varphi \ge -2\sin\varphi & |:r \\ r\cos\varphi \ge 0 \\ r\sin\varphi \ge 0 \end{cases}$ $\left.\begin{matrix} r\cos\varphi \ge 0 \\ r\sin\varphi \ge 0 \end{matrix}\right\} \Rightarrow 0 \le \varphi \le \frac{\pi}{2}$

$D_1: \begin{cases} r^2 - 2r^2\cos^2\varphi\sin^2\varphi \le 1 \\ \cos\varphi \ge 2\sin\varphi \\ \cos\varphi \ge -2\sin\varphi \\ 0 \le \varphi \le \pi/2 \end{cases}$ $\Bigg| \cdot \frac{1}{2\cos\varphi}$

$D_1: \begin{cases} r^2(1-2\cos^2\varphi\sin^2\varphi) \le 1 \\ \frac{1}{2} \ge \operatorname{tg}\varphi \\ 0 \le \varphi \le \pi/2 \end{cases}$ $\Rightarrow 0 \le \varphi \le \operatorname{arctg}\frac{1}{2}$ $\quad$ ($\operatorname{tg}\varphi$ е растяща в $[0; \pi/2]$)

$D_1: \begin{cases} 0 \le \varphi \le \operatorname{arctg}\frac{1}{2} \\ r^2\underbrace{(1-\frac{1}{2}\sin^2 2\varphi)}_{\ge 0} \le 1 \end{cases}$ $\quad D_1: \begin{cases} 0 \le \varphi \le \operatorname{arctg}\frac{1}{2} \\ 0 \le r \le \sqrt{\dfrac{1}{1-\frac{1}{2}\sin^2 2\varphi}} \end{cases}$ $\quad \begin{cases} \alpha \le \varphi \le \beta \\ 0 \le r \le f(\varphi) \end{cases}$

$$S_1 = \frac{1}{2}\int_0^{\operatorname{arctg}\frac{1}{2}} \frac{1}{1-\frac{1}{2}\sin^2 2\varphi}\, d\varphi = \frac{1}{2}\int_0^{\operatorname{arctg}\frac{1}{2}} \frac{1}{2-\sin^2 2\varphi}\, d\varphi$$