$\circledast$ $X = L.X \cup M$ L, M- езици

$L^* = \bigcup\limits_{n\geq 0} L^n$

$= L^0 \cup \bigcup\limits_{n\geq 1} L^n$

$= \{\varepsilon\} \cup \bigcup\limits_{n\geq 0} L^{n+1}$

$= \{\varepsilon\} \cup L.L^*$

- $X = L^*.M$ е решение: $\boxed{\text{вярно}}$

$L^*.M \overset{?}{=} L.(L^*.M) \cup M =$

$= (L.L^*).M \cup \{\varepsilon\}.M =$

$= (L.L^* \cup \{\varepsilon\}).M =$

$= L^*.M$

- Нека $\{\varepsilon\} \in L$, тогава $X = \Sigma^*$ е решение: $\boxed{\text{вярно}}$

$\Sigma^* = (\overbrace{\{\varepsilon\} \cup L}^{L}).\Sigma^* \cup M =$

$= \Sigma^* \cup L.\Sigma^* \cup M$

$= \Sigma^*$

$M \in \Sigma^*$ $\quad L.\Sigma^* \in \Sigma^*$

$\Sigma^* \cup M = \Sigma^*$ $\quad \Sigma^* \cup L.\Sigma^* = \Sigma^*$

$L^*.M$ е най-малко решение на $\circledast$:

Ако $A = L.A \cup M$, то $L^*.M \subseteq A$

т.е. $L^n.M \subseteq A$ за всяко n

$n=0$ $\quad L^0.M = M$

Щом $A = L.A \cup M$, то $M \subseteq A$

$n>0$ $\quad L^n.A = L.L^{n-1}.A$

От ИП $L^{n-1}.M \subseteq A$

$\Big\}$ $L^n.M \subseteq L.A \subseteq L.A \cup M \subseteq A$

$X = L.X \cup M$, $\varepsilon \notin L$

Тогава $L^*.M$ е единствено решение

Нека $A = L.A \cup M$

Знаем, че $L^*.M \subseteq A$

$P(n) \overset{def}{=} (\forall \alpha \in \Sigma^{\leq n})[\alpha \in A \Rightarrow \alpha \in L^*M]$

ще докажем, че $A \subseteq L^*.M$

$(\forall \alpha \in \Sigma^*)[\alpha \in A \Rightarrow \alpha \in L^*.M]$

• $P(0)$?

Нека $\varepsilon \in A$. Защо $\varepsilon \in L^*.M$?

$\varepsilon = L.A \cup M$

$\left.\begin{array}{l}\varepsilon = L.A \cup M \\ \varepsilon \notin L \Rightarrow \varepsilon \notin L.A\end{array}\right\}\ \varepsilon \in M \Rightarrow \varepsilon \in L^* M$

• $P(n)$?

Нека $|\alpha| \leq n$, $\alpha \in A$. Защо $\alpha \in L^*.M$?

сл. 1) Ако $|\alpha| \leq n-1$, то от $P(n-1)$ имаме $\alpha \in L^*.M$

сл. 2) Нека $|\alpha| = n$.

Щом $\alpha \in A \Rightarrow \begin{cases} \alpha \in M \Rightarrow \alpha \in L^*.M \\ \text{или} \\ \alpha \in L.A \end{cases}$

Нека $\alpha \in L.A$

$\varepsilon \in L \Rightarrow \alpha = \alpha_1.\alpha_2$, където

$\alpha_1 \in L$, $|\alpha_1| \geq 1$

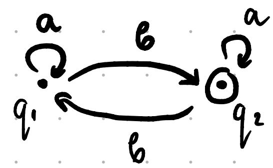
$\alpha_2 \in A$, $|\alpha_2| \leq n-1$ $\xrightarrow{P(n-1)}$

$\left.\begin{array}{l}\alpha_2 \in L^*.M \\ \alpha_1 \in L\end{array}\right\}\ \alpha = \alpha_1.\alpha_2 \in \overbrace{L.L^*.M}^{\subseteq L^* M}$

$\Downarrow$

$\alpha \in L^*.M$

$\mathcal{A}$:



$\mathcal{L}_A(q) = \{\alpha \in \Sigma^* \mid \delta^*(q,\alpha) \in F\}$

$x_1 = \{a\}.x_1 \cup \{b\}.x_2 \cup \emptyset$

$x_2 = \{a\}.x_2 \cup \{b\}.x_1 \cup \{\varepsilon\}$

$\mathcal{L}_A(q_1) = \{a\}.\mathcal{L}_A(q_1) \cup \{b\}.\mathcal{L}_A(q_2) \cup \{\varepsilon\}$

$\mathcal{L}_A(q_2) = \{a\}.\mathcal{L}_A(q_2) \cup \{b\}.\mathcal{L}_A(q_1) \cup \{\varepsilon\}$

$x_2 = \underbrace{\{a\}}_{L}.x_2 \cup \underbrace{(\{b\}.x_1 \cup \{\varepsilon\})}_{M}$

$x_2 = \{a\}^*.(\{b\}.x_1 \cup \{\varepsilon\}) = \{a\}^*\{b\}x_1 \cup \{a\}^*$

$x_1 = \{a\}.x_1 \cup \{b\}\{a^*\}\{b\}x_1 \cup \{b\}\{a\}^* \cup \{\varepsilon\}$

$x_1 = \underbrace{(\{a\} \cup \{b\}\{a^*\}\{b\})}_{L} x_1 \cup \underbrace{(\{b\}\{a\}^* \cup \{\varepsilon\})}_{M}$

$x_1 = (\{a\} \cup \{b\}\{a^*\}\{b\})^*.(\{b\}\{a\}^* \cup \{\varepsilon\})$

$$**$$

$$\begin{cases} x_1 = L_{1,1}.x_1 \cup L_{1,2}.x_2 \cup \dots \cup L_{1,n}.x_n \cup M_1 \\ \dots \\ x_n = L_{n,1}.x_1 \cup L_{n,2}.x_2 \cup \dots \cup L_{n,n}.x_n \cup M_n \end{cases} \qquad \varepsilon \notin L_{i,j} \quad \forall i,j = 1$$

(**) Има единствено решение и ако $L_{i,j}, M_i$ са регулярни, то решението е съставено от регулярни езици

Индукция по n

$n=1$: $x_1 = L_{1,1}.x_1 \cup M_1$. Знаем, че $L_{1,1}^*.M$ е единствено решение

$n>1$: $x_n = L_{n,1}.x_1 \cup \left(\bigcup_{i=1}^{n-1} L_{n,i}.x_1 \cup M_n\right)$

$x_n = L_{n,n}^*\left(\bigcup_{i=1}^{n} L_{n,i} x_i \cup M_n\right)$

$\underline{x_n = L_{n,n}^*.L_{n,1} x_1 \cup \dots \cup L_{n,n}^*.L_{n,n-1}.x_{n-1} \cup L_{n,n}^* M_n}$

$x_1 = L_{1,1} x_1 \cup \dots \cup L_{1,n-1}.x_{n-1} \cup L_{1,n}.L_{n,n}^*.L_{n,1}.x_1 \cup \dots \cup L_{1,n}.L_{n,n}^*.L_{n,n-1}.x_{n-1} \cup$
$\cup L_{1,n}.L_{n,n}^* M_n \cup M_1$

$x_1 = (L_{1,1} \cup L_{1,n}.L_{n,n}^* L_{n,1}) x_1 \cup (L_{1,2} \cup L_{1,n} L_{n,n}^* L_{n,2}) x_2 \cup \dots \cup$
$\cup (L_{1,n} \cup L_{1,n}.L_{n,n}^*.L_{n,n}) x_n \cup L_{1,n} L_{n,n}^* M_n \cup M_n$

$$\left.\begin{array}{l} x_1 = \bigcup_{i=1}^{n-1} (L_{1,i} \cup L_{1,n} L_{n,n}^* L_{n,i}).x_i \cup L_{1,n}.L_{n,n}^* M_n \cup M_1 \\ \vdots \\ x_n = \bigcup_{i=1}^{n-1} (L_{n-1,i} \cup L_{n-1,n} L_{n,n}^* L_{n,i}) x_i \cup L_{n-1,n} L_{n,n}^* M_n \cup M_{n-1} \end{array}\right\} \xRightarrow{\text{ИП}} \text{има единств. решение } (A_1 \dots A_{n-1})$$

$x_n = L_{n,n}^*\left(\bigcup_{i=1}^{n} L_{n,i} A_i \cup M_n\right)$ е единствено решение на уравнението за x_n

$\mathcal{A} = (\Sigma, Q, \delta, q_{start}, F)$

$Q = \{q_1, \dots q_n\}$ $\quad R_{i,j} = \{a \in \Sigma \mid \delta(q_i, a) = q_j\}$ $\quad L_i = \mathcal{L}_A(q_i)$ $\quad P_i = \begin{cases} \emptyset, & q_i \notin F \\ \{\varepsilon\}, & q_i \in F \end{cases}$

$$\left|\begin{array}{l} L_1 = \bigcup_{i=1}^{n} R_{1,i}.L_i \cup P_1 \\ \vdots \\ L_n = \bigcup_{i=1}^{n} R_{n,i}.L_i \cup P_n \end{array}\right.$$

$\varepsilon \notin R_{i,j}$, $\forall i,j = \{1,\dots,n\}$

$R_{i,j}, P_i$ са регулярни

$(L_1,\dots,L_n)$ е решение

$\begin{array}{ccc} \| & & \| \\ A_1 & \dots & A_n \end{array}$

регулярни

$$\left.\left|\begin{array}{l} X_1 = \bigcup_{i=1}^{n} R_{1,i}.X_i \cup P_1 \\ \vdots \\ X_n = \bigcup_{i=1}^{n} R_{n,i}.X_i \cup P_n \end{array}\right.\right\}$$ Има единствено решение $A_1 \dots A_n$ от регулярни езици