ii) $\int_0^{+\infty} \frac{x^\lambda}{(x^3+x^6)\ln^\lambda(1+\sqrt[4]{x})}dx = \underbrace{\int_0^1 \frac{x^\lambda}{(x^3+x^6)\ln^\lambda(1+\sqrt[4]{x})}dx}_{I_2} + \underbrace{\int_1^{+\infty} \frac{x^\lambda}{(x^3+x^6)\ln^\lambda(1+\sqrt[4]{x})}dx}_{I_1}$

I_1: $g_1(x) = \frac{1}{x^\alpha (\ln x)^\beta}$

$$\lim_{x\to\infty} \frac{f(x)}{g_1(x)} = \lim_{x\to\infty} \frac{x^\lambda}{(x^3+x^6)\ln^\lambda(1+\sqrt[4]{x})} \cdot x^\alpha \cdot (\ln x)^\beta = \lim_{x\to\infty} \frac{x^{\lambda+\alpha}}{x^3+x^6} \cdot \frac{\ln x^\beta}{\ln^\lambda(1+\sqrt[4]{x})} =$$

$$\lim_{x\to\infty} \frac{x^{\lambda+\alpha}}{x^6\underbrace{(\frac{1}{x^3}+1)}_{\to 1}} \cdot \frac{\ln^\beta x}{\ln^\lambda x \cdot \underbrace{\frac{\ln^\lambda(1+\sqrt[4]{x})}{\ln^\lambda x}}_{\to c}} = \lim_{x\to\infty} c\cdot x^{\lambda+\alpha-6}\cdot(\ln x)^{\beta-\lambda} = c \quad \text{при } \alpha = 6-\lambda$$

$\beta = \lambda$

$$\lim_{x\to\infty} \frac{\ln x}{\ln(1+\sqrt[4]{x})} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{1+\sqrt[4]{x}}\cdot\frac{1}{4}\cdot\frac{1}{\sqrt[4]{x^3}}} = 4$$

$\Rightarrow \int_1^{+\infty} \frac{1}{x^{6-\lambda}(\ln x)^\lambda}dx$

сх. $6-\lambda>1$ и $\beta\in\mathbb{R}$
$\lambda<5$
или $6-\lambda=1$ и $\beta>1$
$\lambda=5$
разх. иначе

$$I_2: \lim_{x\to0^+} \frac{x^\lambda}{(x^3+x^6)\ln^\lambda(1+\sqrt[4]{x})}\cdot x^\alpha \overset{(\sim)}{=} \lim_{x\to0^+} \frac{x^\lambda}{x^3(1+x^3)(\sqrt[4]{x})^\lambda}\cdot x^\alpha = \lim_{x\to0^+} x^{\alpha+\lambda-3-\frac{\lambda}{4}} = 1 \quad \text{при}$$

$\alpha = 3-\frac{3}{4}\lambda$

$I_2 \sim \int_0^1 \frac{1}{x^{3-\frac{3}{4}\lambda}}dx$ сх. $3-\frac{3}{4}\lambda<1$ $\lambda>\frac{8}{3}$

$I = I_1+I_2$ сх. $\lambda\in(\frac{8}{3};5]$

11.04.2024

Числови редове

$a_1,\dots,a_n,\dots$ $a_n\in\mathbb{R}$

$\sum_{n=1}^\infty a_n = a_1+a_2+\dots+a_n+a_{n+1}+\dots$

$\sum_{n=1}^\infty a_n$ е сходящ, ако $\{S_n\}_{n=1}^\infty$ е сходяща

$S_n = a_1,\dots a_n$ - частична сума

пример 1) $a_n=n$

$\sum_{n=1}^\infty n = 1+2+\dots+n+\dots$

$S_n = 1+2+\dots+n = \frac{n(n+1)}{2}$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\frac{n(n+1)}{2} = +\infty \Rightarrow$ разходящ

2) $a_n = (-1)^n$ $\qquad S_n = \begin{cases} 0 & , n\text{-четно} \\ -1 & , n\text{-нечетно} \end{cases}$ $\qquad \lim\limits_{n\to\infty} S_n$ не същ

$$\sum_{n=1}^{\infty} (-1)^n = -1+1-1+1-1+\ldots$$

3) $a_n = \left(\frac{1}{2}\right)^n$ $\qquad S_n = \frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2^n} = \frac{\cancel{1}}{2}\cdot\frac{1-\left(\frac{1}{2}\right)^n}{\cancel{1-\frac{1}{2}}} = 1-\left(\frac{1}{2}\right)^n$

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = \lim_{n\to\infty} S_n = 1 \qquad \lim_{n\to\infty} S_n = \lim_{n\to\infty} 1-\left(\frac{1}{2}\right)^n = 1$$

4) $a_n = q^n$

$$\sum_{n=1}^{\infty} q^n \qquad S_n = q+q^2+\ldots+q^n = \begin{cases} \frac{q}{1-q}\cdot(1-q)^n & , q\neq 1 \\ n & , q=1 \end{cases}$$

$$\lim_{n\to\infty} S_n = \begin{cases} \frac{q}{1-q} & , |q|<1 \quad \text{сходящ} \\ +\infty & , q>1 \\ \infty & , q<-1 \\ +\infty & , q=1 \\ \text{не същ} & , q=-1 \end{cases} \quad \text{разходящ}$$

5) $\sum\limits_{n=1}^{\infty} \frac{1}{n!} = e$ $\qquad$ сходящ

Необходимо условие $\quad$ Ако $\sum\limits_{n=1}^{\infty} a_n$ е сходящ, то $\lim\limits_{n\to\infty} a_n = 0$

НДУ за сходимост $\quad$ $\sum\limits_{n=1}^{\infty} a_n$ е сходящ $\iff \forall \varepsilon>0\, \exists N : |S_{n+m}-S_n|<\varepsilon \;\; \forall n>N, \forall m>0$

$$|S_{n+m}-S_n| = |a_{n+1}+a_{n+2}+\ldots+a_{n+m}|$$

$$\overbrace{\underbrace{a_1+a_2+\ldots+a_n}_{S_n}+a_{n+1}+\ldots+a_{n+m}}^{S_{n+m}}$$

6) $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ $\qquad a_1+a_2+\ldots+a_n+a_{n+1}+\ldots+a_{2n}$

$$|S_{2n}-S_n| = |a_{n+1}+a_{n+2}+\ldots+a_{2n}| = \left|\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}\right| \geq n\cdot\frac{1}{2n} = \frac{1}{2} \implies \text{разходящ}$$

$$\exists \varepsilon>0\; \forall N : |S_{n+m}-S_n|>\varepsilon$$

Принципи за сравнение

$$\sum_{n=1}^{\infty} a_n, \quad \sum_{n=1}^{\infty} b_n \qquad 0\leq a_n\leq b_n \quad \forall n\geq n_0$$

1) Ако $\sum_{n=1}^{\infty} a_n$ е разходящ $\Rightarrow \sum_{n=1}^{\infty} b_n$ е разходящ

2) Ако $\sum_{n=1}^{\infty} b_n$ е сходящ $\Rightarrow \sum_{n=1}^{\infty} a_n$ е сходящ

Следствие (гранична форма) $\sum_{n=1}^{\infty} a_n, \sum_{n=1}^{\infty} b_n \quad a_n \geq 0 \quad b_n \geq 0 \quad l = \lim_{n\to\infty} \frac{a_n}{b_n}$

1) $l \in \mathbb{R}, l \neq 0 \Rightarrow \sum_{n=1}^{\infty} a_n$ и $\sum_{n=1}^{\infty} b_n$ са едновременно сходящи/разходящи

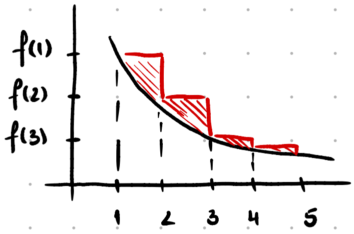
2) $l = 0$ и $\sum_{n=1}^{\infty} b_n$ е сходящ $\Rightarrow \sum_{n=1}^{\infty} a_n$ е сходящ

3) $l = \infty$ и $\sum_{n=1}^{\infty} a_n$ е разходящ $\Rightarrow \sum_{n=1}^{\infty} b_n$ е разходящ

Интегрален критерий

$f: [1, +\infty) \to [0, +\infty)$ и f е монотонно намаляваща

$\int_1^{+\infty} f(x)\,dx$ е сходящ $\Leftrightarrow \sum_{n=1}^{\infty} f(n)$ е сходящ



$\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ сходящ $\alpha > 1$, разходящ $\alpha \leq 1$

$\sum_{n=1}^{\infty} q^n$ сходящ $|q| < 1$, разходящ $|q| \geq 1$

Сходящи ли са редовете?

① $\sum_{n=1}^{\infty} \sin\frac{1}{n} \qquad \frac{1}{n} \xrightarrow[n\to\infty]{} 0 \quad \sin\frac{1}{n} \sim \frac{1}{n} \quad n\to\infty \quad \sum_{n=1}^{\infty} \sin\frac{1}{n} \sim \sum_{n=1}^{\infty} \frac{1}{n}$ разходящ $\Rightarrow \sum_{n=1}^{\infty} \sin\frac{1}{n}$ разходящ

② $\sum_{n=2}^{\infty} \frac{1}{n\ln n} \qquad f(x) = \frac{1}{x\ln x}$ – намаляваща, ≥ 0

$\int_2^{+\infty} \frac{1}{x\ln x}\,dx$ разходящ $\Rightarrow \sum_{n=2}^{\infty} \frac{1}{n\ln n}$ е разходящ

$\int_2^{+\infty} \frac{1}{x^\alpha (\ln x)^\beta}\,dx \quad$ $\alpha > 1$ сх. $\alpha < 1$ разх. $\alpha = 1, \beta > 1$ сх. $\alpha = 1, \beta \leq 1$ разх.

③ $\sum_{n=1}^{\infty} \frac{e^{-\sqrt{n}}}{\sqrt{n}} \qquad f(x) = \frac{e^{-\sqrt{x}}}{\sqrt{x}} = \frac{1}{e^{\sqrt{x}}\sqrt{x}} \to$ намаляваща в $[1, +\infty)$

$$\int_1^{+\infty} \frac{e^{-\sqrt{x}} \cdot 1}{\sqrt{x}}\,dx = \int_1^{+\infty} e^{-\sqrt{x}}\,d\,2\sqrt{x} = -2\int_1^{+\infty} e^{-\sqrt{x}}\,d(-\sqrt{x}) = -2\lim_{p\to+\infty} \int_1^{p} e^{-\sqrt{x}}\,d{-\sqrt{x}} =$$

$$= -2\lim_{p\to\infty} e^{-\sqrt{x}}\Big|_1^p = -2\lim_{p\to\infty} (\underset{\downarrow 0}{e^{-\sqrt{p}}} - e^{-1}) = \frac{2}{e} \Rightarrow \int_0^{+\infty} f(x)\,dx \text{ сходящ} \Rightarrow$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{e^{-\sqrt{n}}}{\sqrt{n}} \text{ е сходящ}$$

④ $\sum\limits_{n=1}^{\infty} \ln\left(\frac{n^2+2}{n^2+1}\right)$ $\qquad a_n = \ln\left(\frac{n^2+2}{n^2+1}\right) = \ln\left(1+\underset{\downarrow\, 0}{\frac{1}{n^2+1}}\right)$

$\sum\limits_{n=1}^{\infty} \frac{1}{n^\alpha}$ сх. $\alpha>1$
разх $\alpha \le 1$

$\sum\limits_{n=1}^{\infty} q^n$ сх. $|q|<1$
разх. $|q| \ge 1$

$\ln(1+y) \sim y$ при $y\to 0$ $\qquad a_n \sim \frac{1}{n^2+1}$ при $n\to\infty$

$\sum\limits_{n=1}^{\infty} \ln\left(\frac{n^2+2}{n^2+1}\right) \sim \sum\limits_{n=1}^{\infty}\frac{1}{n^2+1} \sim \sum\limits_{n=1}^{\infty}\frac{1}{n^2} \to$ сходящ $\Rightarrow \sum\limits_{n\to\infty} a_n$ сходящ

⑤ $\sum\limits_{n=2}^{\infty} \frac{3\sqrt{n}+2}{(n+1)^2\sqrt{n-1}}$ $\qquad \sum\limits_{n=1}^{\infty}\frac{1}{n^\alpha}$

$$\lim_{n\to\infty}\frac{\frac{3\sqrt{n}+2}{(n+1)^2\sqrt{n-1}}}{\frac{1}{n^\alpha}} = \lim_{n\to\infty}\frac{3\sqrt{n}+2}{(n+1)^2\sqrt{n-1}}\cdot n^\alpha = \lim_{n\to\infty}\frac{\cancel{\sqrt{n}}\overset{\to 3}{\left(3+\frac{2}{\sqrt{n}}\right)}}{n^2\underbrace{\left(1+\frac{1}{n}\right)^2\cancel{\sqrt{n}}\sqrt{1-\frac{1}{n}}}_{\to 1}}\cdot n^\alpha = \lim_{n\to\infty} 3\cdot n^{\alpha-2} = 3 \text{ при } \alpha=2$$

$\sum\limits_{n=1}^{\infty}\frac{3\sqrt{n}+2}{(n+1)^2\sqrt{n-1}} \sim \sum\limits_{n=1}^{\infty}\frac{1}{n^2} \to$ сходящ

⑥ $\sum\limits_{n=2}^{\infty} n\cdot tg\left(\frac{n^2+2}{n^3-2}\right)$ $\qquad tg\, y \sim y \quad y\to 0$

$\frac{n^2+2}{n^3-2}\to 0 \quad n\to\infty \Rightarrow tg\left(\frac{n^2+2}{n^3-2}\right)\sim\frac{n^2+2}{n^3-2}$ при $n\to\infty$

$\sum\limits_{n=2}^{\infty} n\cdot tg\left(\frac{n^2+2}{n^3-2}\right) \sim \sum\limits_{n=2}^{\infty} n\cdot\frac{n^2+2}{n^3-2} \sim \sum\limits_{n=2}^{\infty} 1$ разх.

$$\lim_{n\to\infty}\frac{n\cdot\frac{n^2+2}{n^3-2}}{1} = 1$$

⑦ $\sum\limits_{n=1}^{\infty}\frac{e^n+3n^3}{4^n+2\ln^2(n+1)}$

$$\lim_{n\to\infty}\frac{\frac{e^n+3n^3}{4^n+2\ln^2(n+1)}}{\frac{e^n}{4^n}} = \lim_{n\to\infty}\frac{\cancel{e^n}\left(1+\overset{\to 0}{\frac{3n^3}{e^n}}\right)}{\cancel{4^n}\left(1+\underset{\to 0}{\frac{2\ln^2(n+1)}{4^n}}\right)}\cdot\frac{\cancel{4^n}}{\cancel{e^n}} = 1$$

$\sum\limits_{n=1}^{\infty}\frac{e^n+3n^3}{4^n+2\ln^2(n+1)} \sim \sum\limits_{n=1}^{\infty}\frac{e^n}{4^n} = \sum\limits_{n=1}^{\infty}\left(\frac{e}{4}\right)^n \to$ сходящ

$0<\frac{e}{4}<1$ (геом. прогр. с $q<1$)

⑧ $\sum\limits_{n=2}^{\infty}\frac{\sqrt{n+1}\,\ln^3 n\cdot|\cos n|}{n^2}$ $\qquad a_n = \frac{\sqrt{n+1}\,\ln^3 n\,|\cos n|}{n^2} \le \frac{\sqrt{n+1}\,\ln^3 n}{n^2} \sim \frac{\sqrt{n}\,\ln^3 n}{n^2} = \frac{1}{n^{3/2}\ln^{-3} n}$

$\int\limits_2^{+\infty}\frac{1}{x^{3/2}\lg^{-3}x}dx$ $\quad \frac{3}{2}>1 \Rightarrow$ сходящ $\quad \lim\limits_{n\to\infty}\frac{\lg^3 x}{x^{3/2}} = 0$ $\quad \frac{\ln^3 x}{x^{3/2}}>0$ $\quad \exists x_0: \frac{\ln^3 x}{x^{3/2}}$ нам. за $x>x_0$

$\Rightarrow \sum\limits_{n=2}^{\infty}\frac{\sqrt{n+1}\,\ln^3 n}{n^2}$ е сходящ $\Rightarrow \sum\limits_{n=2}^{\infty}\frac{\sqrt{n+1}\,\ln^3 n\,|\cos n|}{n^2}$ е сходящ

⑨ $\sum\limits_{n=1}^{\infty}\frac{(n!)^2}{(2n)!}$ $\qquad a_n = \frac{n!\,\cancel{n!}}{(2n)\ldots(n+1)\cdot\cancel{n\ldots1}} = \frac{n!}{2n(2n-1)\ldots(n+1)} = \frac{n(n-1)(n-2)\ldots2\cdot1}{2n(2n-1)\ldots\ldots2\cdot1} =$

$$= \frac{\cancel{n}(n-1)(n-2)\dots 2.1}{2^n \cdot \cancel{n}\left(n-\frac{1}{2}\right)(n-1)\left(n-\frac{3}{2}\right)\dots\frac{n+1}{2}} \le \frac{1}{2^n}$$

$\sum\limits_{n=1}^{\infty} \frac{1}{2^n}$ сходящ $\Rightarrow \sum\limits_{n=1}^{\infty} a_n$ е сходящ

(геом. пр с $q<1$)

(10) $\sum\limits_{n=1}^{\infty} \frac{5+(-1)^n 3}{2^{n+3}}$ $\qquad a_n = \frac{5+(-1)^n 3}{2^{n+3}} = \begin{cases} \frac{5+3}{2^{n+3}} \sim \frac{1}{2^n} & n\text{-четно} \\ \frac{5-3}{2^{n+3}} = \frac{1}{2^{n+2}} & n\text{-нечетно} \end{cases}$

- $\sum\limits_{k=1}^{\infty} a_{2k} = \sum\limits_{k=1}^{\infty} \frac{1}{2^{2k}}$ сходящ
- $\sum\limits_{k=1}^{\infty} a_{2k+1} = \sum\limits_{k=1}^{\infty} \frac{1}{2^{2k+1+2}}$ сходящ

нечетни

$a_1, a_2, a_3, a_4, \dots, a_n, a_{n+1}, \dots$ $\qquad a_n \ge 0$

четни

$\Rightarrow \sum\limits_{n=1}^{\infty} a_n = \sum\limits_{k=1}^{\infty} a_{2k} + \sum\limits_{k=1}^{\infty} a_{2k+1} \Rightarrow$ сходящ

(11) $\sum\limits_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}} = \sum\limits_{n=2}^{\infty} \frac{1}{n^{\ln\ln n}}$

$(\ln n)^{\ln n} = e^{\ln((\ln n)^{\ln n})} = e^{\ln n \ln\ln n} = n^{\ln\ln n}$

$\frac{1}{n^{\ln\ln n}} < \frac{1}{n^2}$ за $n>n_0$ $\qquad \ln\ln n > 2$ за $n > e^{e^2} = n_0$

$\sum\limits_{n=2}^{\infty} \frac{1}{n^2}$ е сх. $\Rightarrow \sum\limits_{n=2}^{\infty} \frac{1}{n^{\ln\ln n}}$ е сходящ $\Rightarrow \sum\limits_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$ е сходящ

Критерий на Даламбер (гранична форма)

$\sum\limits_{n=1}^{\infty} a_n$, $a_n > 0$
1) $l>1$ разходящ
2) $l<1$ сходящ
3) $l=1$ не дава резултат (ако $\to 1^+ \Rightarrow$ разходящ)

$l = \lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n}$

Критерий на Коши (гранична форма)

$\sum\limits_{n=1}^{\infty} a_n$, $a_n > 0$
1) $l>1$ разходящ
2) $l<1$ сходящ
3) $l=1$ не дава резултат (ако $\to 1^+ \Rightarrow$ разходящ)

$l = \lim\limits_{n\to\infty} \sqrt[n]{a_n}$

(12) $\sum\limits_{n=1}^{\infty} \frac{n(n+1)}{3^n}$ $\quad a_n = \frac{n(n+1)}{3^n}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\cancel{(n+1)}(n+2)}{3^{n+1}} \cdot \frac{\cancel{3^n}}{n\cancel{(n+1)}} = \lim_{n\to\infty} \frac{n+2}{3n} = \frac{1}{3} < 1 \Rightarrow \text{сходящ}$$

(13) $\sum\limits_{n=1}^{\infty} \frac{n!}{n^n}$ $\quad a_n = \frac{n!}{n^n}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\overset{n!(n+1)}{\cancel{(n+1)!}}}{(n+1)^{(n+1)}} \cdot \frac{n^n}{\cancel{n!}} = \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \lim_{n\to\infty} \frac{1}{\left(\frac{n+1}{n}\right)^n} = \frac{1}{e} < 1 \Rightarrow \text{сходящ}$$

(14) $\sum\limits_{n=1}^{\infty} \frac{1}{3^n}\left(\frac{n+1}{n}\right)^{n^2}$ $\quad a_n = \frac{1}{3^n}\left(\frac{n+1}{n}\right)^{n^2}$

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{3^n}\cdot\left(\frac{n+1}{n}\right)^{n^2}} = \lim_{n\to\infty} \frac{1}{3}\left(1+\frac{1}{n}\right)^n = \frac{e}{3} < 1 \Rightarrow \text{сходящ}$$

(15) $\sum\limits_{n=1}^{\infty} \frac{1}{7^n}\left(\frac{n^2+2n+5}{n^2+2n+3}\right)^{n^3}$ $\quad a_n = \frac{1}{7^n}\left(\frac{n^2+2n+5}{n^2+2n+3}\right)^{n^3}$

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{7^n}\left(\frac{n^2+2n+5}{n^2+2n+3}\right)^{n^3}} = \lim_{n\to\infty} \frac{1}{7}\left(\frac{n^2+2n+5}{n^2+2n+3}\right)^{n^2} =$$

$$= \lim_{n\to\infty} \frac{1}{7}\cdot \underbrace{\left(1+\frac{2}{n^2+2n+3}\right)^{\frac{n^2+2n+3}{2}}}_{e}{}^{\cdot\frac{2}{n^2+2n+3}\cdot n^2} = \frac{1}{7}\, e^{\lim\limits_{n\to\infty}\frac{2n^2}{n^2+2n+3}} = \frac{1}{7}e^2 > 1 \text{ расх.}$$

$$\lim_{n\to\infty}(1+y_n)^{\frac{1}{y_n}} = e \qquad y_n \xrightarrow[n\to\infty]{} 0$$