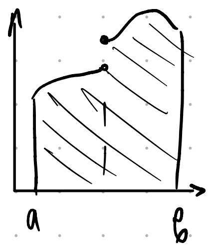
02.03.23

$f: [a,b] \to \mathbb{R}$



f - ограничена

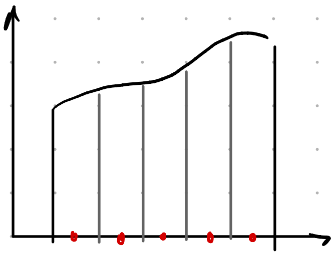
$\tau: x_0 = a < x_1 < \dots < x_n = b$

$$s_f(\tau) = \sum_{i=1}^{n} \inf_{[x_{i-1}, x_i]} f \,(x_i - x_{i-1}) \qquad S_f(\tau) = \sum_{i=1}^{n} \sup_{[x_{i-1}, x_i]} f \,(x_i - x_{i-1})$$

$$\underline{\int_a^b} f := \sup \{ s_f(\tau), \tau \text{ подр.} \} \qquad \overline{\int_a^b} f := \inf \{ S_f(\tau), \tau \text{ подр.} \}$$

долен интеграл горен интеграл

$$\int_a^b f(x)dx$$



Свойства на интеграли - суми на Риман

$f: [a,b] \to \mathbb{R}$

$\tau: a = x_0 < x_1 < \dots < x_n = b$

$\xi: \{ \xi_1, \dots, \xi_n \}$ представителни точки

$\xi_i \in [x_{i-1}, x_i] \ \forall 1 \le i \le n$

Сума на Риман за f за подр. τ

$$\sigma_f(\tau, \xi) = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})$$

с представителни точки ξ

Твърдение: $f: [a,b] \to \mathbb{R}$ ограничена $\tau: a = x_0 < x_1 < \dots < x_n = b$

Тогава

$$s_f(\tau) = \inf \{ \sigma_f(\tau, \xi) : \xi \text{ предст. т. за } \tau \}$$

$$S_f(\tau) = \sup \{ \sigma_f(\tau, \xi) : \xi \text{ предст. т. за } \tau \}$$

$\tau: a = x_0 < x_1 < \dots < x_n = b$

$\xi_i \in [x_{i-1}, x_i] \ \forall 1 \le i \le n$

$$\inf_{[x_{i-1}, x_i]} f \le f(\xi_i) \le \sup_{[x_{i-1}, x_i]} f \qquad \cdot (x_i - x_{i-1})$$

и сумиране

$$\sum_{i=1}^{n} \inf_{[x_{i-1}, x_i]} f \,(x_i - x_{i-1}) \le f(\xi_i)(x_i - x_{i-1}) \le \sum_{i=1}^{n} \sup_{[x_{i-1}, x_i]} f \,(x_i - x_{i-1})$$

$$s_f(\tau) \le \sigma(\tau, \xi) \le S_f(\tau)$$

$\varepsilon>0,$
$sf(\tau)+\varepsilon$

$\eta>0 \longrightarrow \xi_i^0\in[x_{i-1},x_i],\ f(\xi_i^0)<\inf\limits_{[x_{i-1},x_i]} f+\eta \quad | \text{ умн. с } (x_i-x_{i-1})$

ξ_i^0 - специално подбр. представителни точки

$$\sum_{i=1}^{n} f(\xi_i^0)(x_i-x_{i-1})<\sum_{i=1}^{n}\left(\inf_{[x_{i-1},x_i]} f+\eta\right)(x_i-x_{i-1}) \equiv$$

$$\equiv\ \sigma_f(\tau;\xi^0)<sf(\tau)+\eta\underbrace{\sum_{i=1}^{n}(x_i-x_{i-1})}_{b-a}\ \equiv$$

$$\equiv\ \sigma_f(\tau,\xi^0)<sf(\tau)+\frac{\varepsilon}{\cancel{b-a}}\cancel{(b-a)} \equiv \sigma_f(\tau,\xi^0)<sf(\tau)+\varepsilon$$

диаметър на подразб. $d(\tau):=\max\{x_i-x_{i-1},\ \forall\, 1\le i\le n\}$

$\tau:\ a=x_0<x_1<\dots<x_n=b$

Казваме, че сумите на Риман за $f[a,b]\to\mathbb{R}$ имат граница $I\in\mathbb{R}$ (и пишем $\lim\limits_{d(\tau)\to 0}\sigma_f(\tau,\xi)=I$) когато $d(\tau)$ клони към нула, ако $\forall\varepsilon>0\ \exists\delta>0$

такова, че $\forall$ избор на подразбиване τ за $[a,b]$ с $d(\tau)<\delta$ и $\forall$ избор на ξ - предст. точки за τ, е в сила $|\sigma_f(\tau,\xi)-I|<\varepsilon$

Теорема 1 $f:[a,b]\to\mathbb{R}$ и нека съществува $\lim\limits_{d(\tau)\to 0}\sigma_f(\tau,\xi)=I$. Тогава f е ограничена, f е интегруема по Риман и $\int_a^b f=I$.

Доказателство: $f:[a,b]\to\mathbb{R},\ \varepsilon=3>0 \Rightarrow$

$\Rightarrow \exists\delta>0\ \forall\tau, d(\tau)<\delta\ \forall\xi$ пр. т. за τ: $|\sigma_f(\tau,\xi)-I|<\varepsilon$

Нека $\tau:\ a=x_0<x_1<\dots<x_n=b$ е такова, че $d(\tau)<\delta$. Тогава за всеки избор на $\xi_i\in[x_{i-1},x_i],\ \forall 1\le i\le n$, е в сила $I-3<\sum\limits_{i=1}^{n} f(\xi_i)(x_i-x_{i-1})<I+3$

Достатъчно е да докажем, че f е ограничена в $[x_{i-1},x_i]$. Фиксираме $\xi_j\in[x_{j-1},x_j],\ j\neq i$. Получаваме

долна гр. за f в $[x_{i-1},x_i]$ горна гр. за f в $[x_{i-1},x_i]$

$$\frac{I-3-\sum\limits_{j\neq i}^{n} f(\xi_j)(x_j-x_{j-1})}{x_i-x_{i-1}}<f(\xi_i)<\frac{I+3-\sum\limits_{j\neq i}^{n} f(\xi_j)(x_j-x_{j-1})}{x_i-x_{i-1}}\quad \forall\xi_i\in[x_{i-1},x_i]$$

интегруемост: $\varepsilon>0 \to \exists\delta\ \forall\tau, d(\tau)<\delta, \forall\xi$ пр.т за τ: $|\sigma_f(\tau,\xi)-J|<\frac{\varepsilon}{3}$

Фиксираме τ, $d(\tau)<\delta$, тогава $J-\frac{\varepsilon}{3}<\sigma_f(\tau,\xi)<J+\frac{\varepsilon}{3} \to$

$$J-\frac{\varepsilon}{3}\le s_f(\tau)\le S_f(\tau)\le J+\frac{\varepsilon}{3}$$

$\Rightarrow S_f(\tau)-s_f(\tau)=(J+\frac{\varepsilon}{3})-(J-\frac{\varepsilon}{3})=\frac{2}{3}\varepsilon<\varepsilon \Rightarrow f$ е интегруема

разстоянието от интеграла до ε е

$$-\frac{\varepsilon}{3}\le\int_a^b f-J\le\frac{\varepsilon}{3} \Rightarrow \left|\int_a^b f-J\right|<\varepsilon\ \forall\varepsilon>0 \Rightarrow \boxed{J=\int_a^b f}$$

Теорема 2 Ако $f:[a,b]\to\mathbb{R}$ е интегруема по Риман (следователно f е и ограничена), то съществува $\lim\limits_{d(\tau)\to 0}\sigma_f(\tau,\xi)=\int_a^b f$

Доказателство Търсим колко дребно трябва да бъде τ.

Нека $\varepsilon>0$. $\int_a^b f+\frac{\varepsilon}{2}>\int_a^b f$. Съществува τ_0 подр. на $[a,b]$, $S_f(\tau)<\int_a^b f+\frac{\varepsilon}{2}$

τ_0: $a=y_0<y_1<\dots<y_n=b$. $\delta>0 = \frac{\varepsilon}{2k(M-m)}$; τ е произв. подр. за $[a,b]$, $d(\tau)<\delta$.

$\tau^*\to$ смесване на делящи точки от τ и τ_0 $\qquad \tau^*\ge\tau_0$

$$S_f(\tau)\le S_f(\tau^*)+k.(M-m)d(\tau)\le S_f(\tau_0)+k(M-m)d(\tau_0)<\int_a^b f+\frac{\varepsilon}{2}+k(M-m)d(\tau)$$

Трябва $\delta=\frac{\varepsilon}{2k(M-m)}$. БОО $\exists\delta>0\ \forall\tau, d(\tau)<\delta$ е в сила

$$\int_a^b f-\varepsilon<s_f(\tau)\le S_f(\tau)<\int_a^b f+\varepsilon \Rightarrow$$

$\forall\tau, d(\tau)<\delta\ \forall\xi$ пр.т. за τ: $\int_a^b f-\varepsilon<s_f(\tau)\le\sigma_f(\tau,\xi)\le S_f(\tau)<\int_a^b f+\varepsilon \Rightarrow$

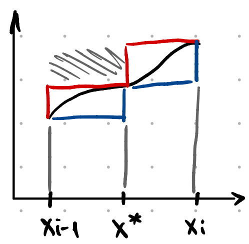
$$\left|\sigma_f(\tau,\xi)-\int_a^b f\right|<\varepsilon$$

Лема τ: $a=x_0<x_1<\dots<x_n=b$ подр. на $[a,b]$

$\tau^*\ge\tau$, τ^* се получава от τ чрез прибавяне на k точки

Тогава $\quad 0\le S_f(\tau)-S_f(\tau^*)\le k.(M-m)d(\tau) \qquad m=\inf\limits_{[a,b]} f \quad M=\sup\limits_{[a,b]} f$

$$0\le s_f(\tau^*)-s_f(\tau)\le k.(M-m)d(\tau)$$



БОО τ^* се получава от τ чрез прибавяне на точка x^*

τ^*: $a=x_0<x_1<\dots<x_{i-1}<x^*<x_i<\dots<x_n=b$

$$Sf(\tau) - Sf(\tau^*) = \sup_{[x_{i-1}, x_i]} f (x_i - x_{i-1}) - \sup_{[x_{i-1}, x^*]} f (x^* - x_{i-1}) - \sup_{[x^*, x_i]} f (x_i - x^*) \leq$$

$$\leq M(x_i - x_{i-1}) - m(x^* - x_{i-1}) - m(x_i - x^*) = M(x_i - x_{i-1}) - m(x_i - x_{i-1}) =$$

$$= (M - m)(x_i - x_{i-1}) \leq (M - m) d(\tau)$$

Основни свойства на интеграла

I Линейност

Ако $f, g : [a, b] \to \mathbb{R}$ са интегруеми и $\lambda \in \mathbb{R}$ =>

=> $f + g, \lambda f$ са интегруеми и $\int_a^b (f+g) = \int_a^b f + \int_a^b g$, $\int_a^b \lambda f = \lambda . \int_a^b f$

$$\sigma_{f+g}(\tau, \xi) = \sigma_f(\tau, \xi) + \sigma_g(\tau, \xi), \quad \sigma_{\lambda f}(\tau, \xi) = \lambda . \sigma_f(\tau, \xi)$$

Доказателство $\varepsilon > 0$. $\lim_{d(\tau) \to 0} \sigma_f(\tau, \xi) = \int_a^b f \Rightarrow \exists \delta_1 > 0 \, \forall \tau, d(\tau) < \delta_1, \forall \xi : |\sigma_f(\tau, \xi) - \int_a^b f| < \frac{\varepsilon}{2}$

$\lim_{d(\tau) \to 0} \sigma_g(\tau, \xi) = \int_a^b g \Rightarrow \exists \delta_2 > 0 \, \forall \tau, d(\tau) < \delta_2, \forall \xi : |\sigma_g(\tau, \xi) - \int_a^b g| < \frac{\varepsilon}{2}$

$\delta := \min\{\delta_1, \delta_2\} > 0$

τ - подразб., $d(\tau) < \delta$

ξ пр. т. за τ

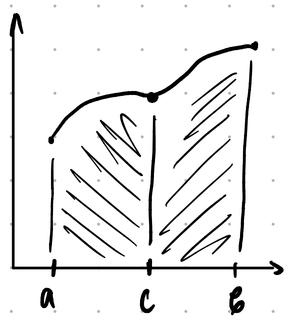
$$\Rightarrow |\sigma_{f+g}(\tau, \xi) - (\int_a^b f + \int_a^b g)| =$$

довърши!

$$= |(\sigma_f(\tau, \xi) - \int_a^b f) + (\sigma_g(\tau, \xi) - \int_a^b g)| \leq |\ldots| + |\ldots| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

II Адитивност

$f : [a, b] \to \mathbb{R}$, f интегруема $\quad a < c < b$

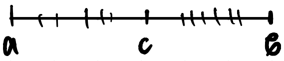


Тогава $f|_{[a,c]}$ + $f|_{[c,b]}$ са интегруеми и

$$\int_a^b f = \int_a^c f + \int_c^b f$$

Доказателство: f е интегруема => $\exists \tau$ подр. на $[a, b]$, $Sf(\tau) - sf(\tau) < \varepsilon$, $\varepsilon > 0$

$\tau^* = \tau \cup \{c\}$ => $Sf(\tau^*) - sf(\tau^*) < \varepsilon$



$\tau^* = \tau_1 \cup \tau_2$ τ_1 подр на $[a,c]$ τ_2 подр. на $[c,b]$

$$S_f(\tau^*) = S_f(\tau_1) + S_f(\tau_2) \quad \text{и} \quad s_f(\tau^*) = s_f(\tau_1) + s_f(\tau_2)$$

$$\varepsilon > S_f(\tau^*) - s_f(\tau^*) = [S_f(\tau_1) - s_f(\tau_1)] + [S_f(\tau_2) - s_f(\tau_2)]$$

τ_n - редица от подр. на $[a,b]$, съдържащи c като деляща точка, ξ_n пр т. за τ_n

$d(\tau_n) \to 0$

$\tau_n = \tau_n' \cup \tau_n''$

τ_n' - подр на $[a,c]$

τ_n'' - подр. на $[c,b]$

$$\begin{array}{ccccc} \sigma_f(\tau_n, \xi_n) & = & \sigma_f(\tau_n', \xi_n') & + & \sigma_f(\tau_n'', \xi_n'') \\ \downarrow{\scriptstyle n\to\infty} & & \downarrow{\scriptstyle n=\infty} & & \downarrow{\scriptstyle n=\infty} \\ \int_a^b f & = & \int_a^c f & + & \int_c^b f \end{array}$$

Забележка: $f: \Delta \to \mathbb{R}$, Δ - интервал, $a, b \in \Delta$

$$\int_a^a f = 0 \qquad \int_b^a f = -\int_a^b f$$

III Позитивност $\underset{\text{интегруема}}{f: [a,b] \to \mathbb{R}}$ и $f(x) \geq 0 \ \forall x \in [a,b]$ Тогава $\int_a^b f \geq 0$.

Следствие 1 $\underset{\text{интегруеми}}{f, g: [a,b] \to \mathbb{R}}$ $f(x) \leq g(x) \ \forall x \in [a,b]$ $\int_a^b f \leq \int_a^b g$

$$g - f \geq 0 : \quad 0 \leq \int_a^b (g-f) = \int_a^b g - \int_a^b f$$

Следствие 2 $\underset{\text{интегруема}}{f: [a,b] \to \mathbb{R}}$ Тогава $|f|$ е интегруема в $[a,b]$ и $\left| \int_a^b f \right| \leq \int_a^b |f|$

$$\big| |f(x)| - |f(y)| \big| \leq |f(x) - f(y)| \quad (|a| - |b| \leq |a-b|)$$

$\tau: a = x_0 < x_1 < \dots < x_n = b$ $\quad \omega(|f|, [x_{i-1}, x_i]) \leq \omega(f, [x_{i-1}, x_i]) \Rightarrow$

$$\Rightarrow S_{|f|}(\tau) - s_{|f|}(\tau) = \sum_{i=1}^{n} \omega(|f|, [x_{i-1}, x_i])(x_i - x_{i-1}) \leq$$

$$\leq \sum_{i=1}^{n} \omega(f, [x_{i-1}, x_i])(x_i - x_{i-1}) = S_f(\tau) - s_f(\tau)$$

$$-|f| \le f \le |f| \equiv -\int_a^b |f| \le \int_a^b f \le \int_a^b |f|$$

IV. Теорема за средните стойности $f, g : \underset{\text{интегруеми}}{[a,b]} \longrightarrow \mathbb{R}$ и

$g(x) \ge 0 \ \forall x \in [a,b]$. $M = \sup\limits_{[a,b]} f$, $m = \inf\limits_{[a,b]} f$ $\to$ $m \le f(x) \le M \ \forall x \in [a,b]$

$$m.g(x) \le g(x) f(x) \le M.g(x) \quad \forall x \in [a,b]$$

$$m \int_a^b g(x)dx \le \int_a^b g(x) f(x) \le M. \int_a^b g(x)dx$$