

Random Numbers

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1 UNIFORM RANDOM NUMBERS

Let U be a uniform random variable between 0 and 1.

- 1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .

Solution:

Download the following files and execute the C program.

```
wget https://github.com/kamujuaakash/
Assignment1/blob/main/Exercise/Exercise
-1/1.1/exrand.c
wget https://github.com/kamujuaakash/
Assignment1/blob/main/Exercise/Exercise
-1/1.1/coeffs.h
```

Download the above files and execute the following commands

```
gcc exrand.c -lm
./a.out
```

- 1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \quad (1.1)$$

Solution:

The following code plots Fig. 1.2

```
wget https://github.com/kamujuaakash/
Assignment1/blob/main/Exercise/Exercise
-1/1.2/cdf_plot.py
```

Download the above files and execute the following commands to produce Fig.1.2

```
python3 cdf_plot.py
```

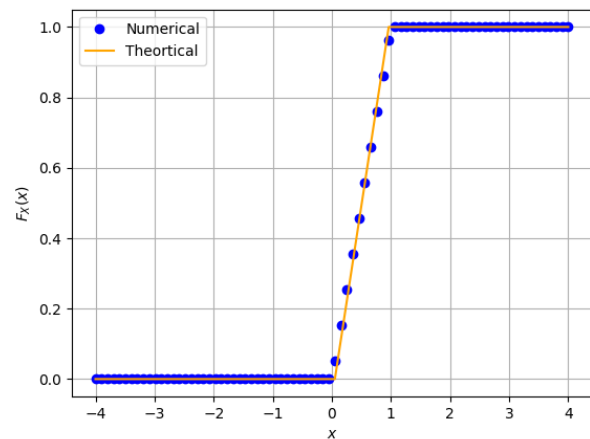


Fig. 1.2: The CDF of U

- 1.3 Find a theoretical expression for $F_U(x)$.

Solution:

Given U is a uniform Random Variable

$$p_U(x) = 1 \text{ for } 0 < x < 1$$

$$F_U(x) = \int_{-\infty}^x p_U(x) dx$$

$$\Rightarrow F_U(x) = \begin{cases} 0 & x \leq 0 \\ x & 0 < x < 1 \\ 1 & x \geq 1 \end{cases}$$

- 1.4 The mean of U is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^N U_i \quad (1.2)$$

and its variance as $\text{var}[U] = E[U - E[U]]^2$
Write a C program to find the mean and variance of U .

Solution:

Download the following files and execute the

C program.

```
wget https://github.com/kamujuaakash/
Assignment1/blob/main/Exercise/Exercise
-1/1.4/m.c
wget https://github.com/kamujuaakash/
Assignment1/blob/main/Exercise/Exercise
-1/1.4/coeffs.h
```

Download the above files and execute the following commands

```
gcc m.c -lm
./a.out
```

1.5 Verify your result theoretically given that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x)$$

Solution:

$$\begin{aligned} E[U] &= \int_{-\infty}^{\infty} x dF_U(x) \\ E[U] &= \int_0^1 x \\ \Rightarrow E[U] &= \frac{1}{2} \\ E[U^2] &= \int_{-\infty}^{\infty} x^2 dF_U(x) \\ E[U^2] &= \int_0^1 x^2 dF_U(x) \\ \Rightarrow E[U^2] &= \frac{1}{3} \\ \text{var}[U] &= E[U - E[U]]^2 \\ \Rightarrow \text{var}[U] &= E[U^2] - E[U]^2 \\ \boxed{\text{var}[U] = \frac{1}{12} = 0.0833} \end{aligned}$$

2 CENTRAL LIMIT THEOREM

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \quad (2.1)$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution: Download the following files and execute the C program.

```
wget https://github.com/kamujuaakash/
Assignment1/blob/main/Exercise/Exercise
-2/2.1/exrand.c
wget https://github.com/kamujuaakash/
Assignment1/blob/main/Exercise/Exercise
-2/2.1/coeffs.h
```

Download the above files and execute the following commands

```
gcc exrand.c -lm
./a.out
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Solution:

- $F_X(x) = P(X \leq x)$
- $Q_X(x) = P(X > x)$
- $Q_X(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-\frac{u^2}{2}} du$
- $F_X(x) = 1 - Q_X(x)$ This can be used to calculate $F(x)$.

The CDF of X is plotted in Fig. 2.2 using the code below

```
wget https://github.com/kamujuaakash/
Assignment1/blob/main/Exercise/Exercise
-2/2.2/2.2.py
```

Download the above files and execute the following commands to produce Fig.2.2

```
python3 2.2.py
```

Some of the properties of CDF

1)

$$\lim_{x \rightarrow \infty} F_X(x) = 1$$

2) $F_X(x)$ is non decreasing function.

3) Symmetric about one point.

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \quad (2.2)$$

What properties does the PDF have?

Solution: The PDF of X is plotted in Fig. 2.3 using the code below

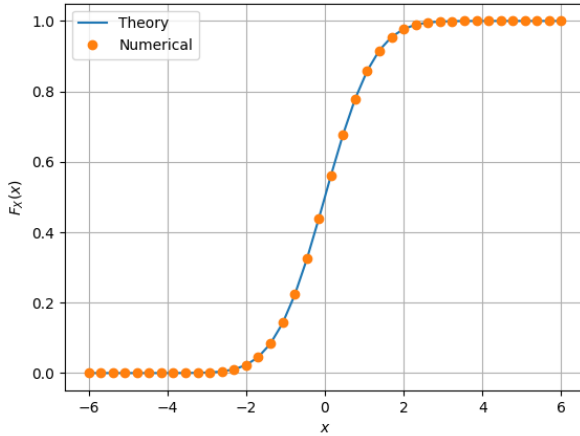


Fig. 2.2: The CDF of X

```
wget https://github.com/kamujuaakash/
Assignment1/blob/main/Exercise/Exercise
-2/2.3/2.3.py
```

Download the above files and execute the following commands to produce Fig.2.3

```
python3 2.3.py
```

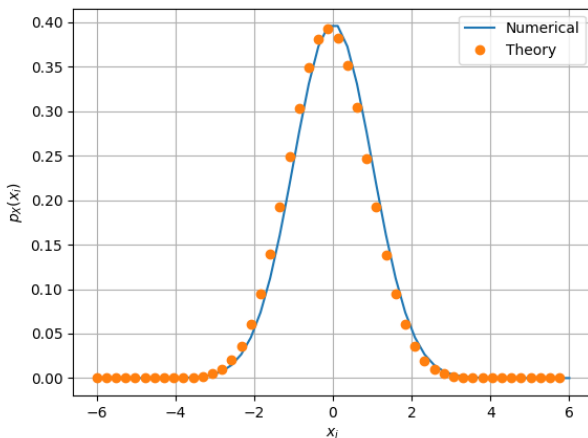


Fig. 2.3: The PDF of X

Let $\mu \approx 0$

Some of the properties of the PDF:

- Symmetric about $x = \mu$ in this case
- Decreasing function for $x > \mu$ and increasing for $x < \mu$ and attains maximum at $x = \mu$
- Area under the curve is unity.

2.4 Find the mean and variance of X by writing a C program.

Solution: Download the following files and execute the C program.

```
wget https://github.com/kamujuaakash/
Assignment1/blob/main/Exercise/Exercise
-2/2.4/m.c
wget https://github.com/kamujuaakash/
Assignment1/blob/main/Exercise/Exercise
-2/2.4/coeffs.h
```

Download the above files and execute the following commands

```
gcc m.c -lm
./a.out
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.3)$$

repeat the above exercise theoretically.

Solution:

1) CDF is given by

$$F_X(x) = \int_{-\infty}^{\infty} p_X(x) dx \quad (2.4)$$

$$F_X(x) = 1 \quad (2.5)$$

2) Mean is given by

$$E(x) = \int_{-\infty}^{\infty} x p_X(x) dx$$

$$\int_{-\infty}^{\infty} \left(x \exp\left(-\frac{x^2}{2}\right) \right) dx$$

Since the above function is odd

$$\Rightarrow E(x) = 0$$

3) Variance is given by

$$\text{var}[U] = E(U^2) - (E(U))^2$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 p_X(x) dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x^2 \exp\left(-\frac{x^2}{2}\right) dx$$

$$= \frac{1}{\sqrt{2\pi}} \left(x \int_{-\infty}^{\infty} x \exp\left(-\frac{x^2}{2}\right) dx \right)$$

$$- \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\int \left(x \exp\left(-\frac{x^2}{2}\right) dx \right) . dx \right)$$

$$\begin{aligned}
&= \left[-x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \right]_{-\infty}^{\infty} + \\
&\quad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx \\
&= \frac{\sqrt{2\pi}}{\sqrt{2\pi}} \\
&= 1 \implies \boxed{\text{var}[U] = 1}
\end{aligned}$$

3 FROM UNIFORM TO OTHER

3.1 Generate samples of

$$V = -2 \ln(1 - U) \quad (3.1)$$

and plot its CDF.

Solution: Running the below code generates samples of V from file uni.dat(U).

```
https://github.com/kamujuaakash/Assignment1/blob/main/Exercise/Exercise-3/3.1/3.1.py
```

Use the below command in the terminal to run the code:

```
python3 3.1.py
```

Now these samples are used to plot by running the below code,

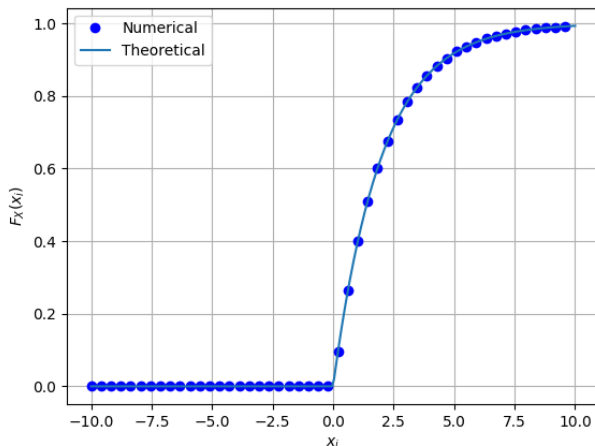


Fig. 3.1: CDF for (3)

```
wge https://github.com/kamujuaakash/Assignment1/blob/main/Exercise/Exercise-3/3.1/cdf.py
```

Use the below command to run the code:

```
python3 cdf.py
```

3.2 Theoretical expression for $F_V(x)$

$$\begin{aligned}
F_V(x) &= P\{V \leq x\} \\
&= P\{-2 \times \ln(1 - U) \leq x\} \\
&= P\{U \leq 1 - e^{(-\frac{x}{2})}\} \\
&= F_U\{1 - e^{(-\frac{x}{2})}\} \\
&= \begin{cases} 1 - e^{(-\frac{x}{2})} & 0 \leq x < \infty \\ 0 & x < 0 \end{cases}
\end{aligned}$$

4 TRIANGULAR DISTRIBUTION

4.1 Generate

$$T = U_1 + U_2 \quad (4.1)$$

Solution: Download the following files and execute the C program.

```
wget https://github.com/kamujuaakash/Assignment1/blob/main/Exercise/Exercise-4/4.1/4.1.c
wget https://github.com/kamujuaakash/Assignment1/blob/main/Exercise/Exercise-4/4.1/coeffs.h
```

Download the above files and execute the following commands

```
gcc 4.1.c
./a.out
```

4.2 Find the CDF of T .

Solution:

The CDF of T is plotted in figure using the code below

```
wget https://github.com/kamujuaakash/Assignment1/blob/main/Exercise/Exercise-4/4.2/4.2.py
```

Download the above files and execute the following commands to produce Fig.4.2

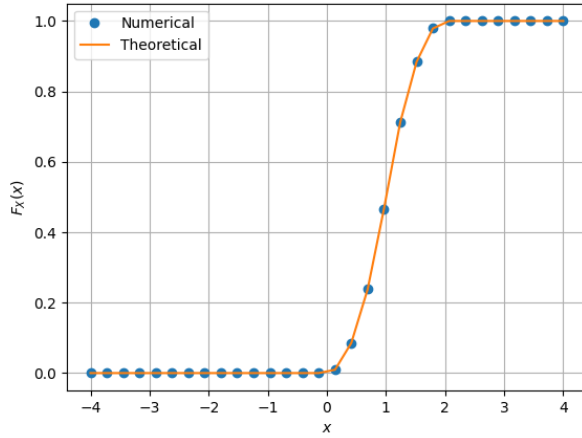
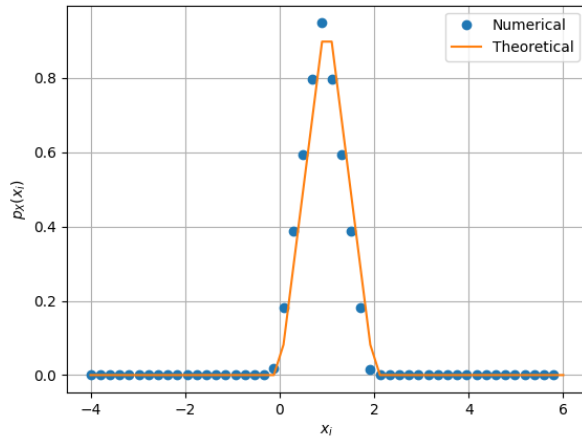
```
python3 4.2.py
```

4.3 Find the PDF of T .

Solution:

The PDF of T is plotted in Fig. 4.2 using the code below

```
wget https://github.com/kamujuaakash/Assignment1/blob/main/Exercise/Exercise-4/4.3/4.3.py
```

Fig. 4.2: The CDF of T Fig. 4.3: The PDF of T

Download the above files and execute the following commands to produce Fig.4.2

```
python3 4.3.py
```

4.4 Find the Theoretical Expression for the PDF and CDF of T

Solution:

$$T = U_1 + U_2 \quad (4.2)$$

$$\Rightarrow p_T(t) = \int_{-\infty}^t p_{U_1}(x)p_{U_2}(y)dx \quad (4.3)$$

$$\text{As, } p_{U_1}(x) = p_{U_1}(y) = p_U(u) \quad (4.4)$$

$$\Rightarrow p_T(t) = \int_{-\infty}^t p_U(u)p_U(t-u)du \quad (4.5)$$

a) Theoretical PDF

i) $t \leq 1$

$$p_T(t) = \int_0^t p_U(t-u)du \quad (4.6)$$

$$\Rightarrow p_T(t) = \int_0^t du = t \quad (4.7)$$

ii) $t > 1$

$$p_T(t) = \int_0^1 p_U(t-u)du \quad (4.8)$$

$$\Rightarrow p_T(t) = \int_{t-1}^1 du = 2-t \quad (4.9)$$

$$\Rightarrow P_T(t) = \begin{cases} t & 0 \leq t \leq 1 \\ 2-t & 1 < t \leq 2 \\ 0 & t < 0 \text{ or } t > 2 \end{cases}$$

b) Theoretical CDF

$$F_T(t) = \int_{-\infty}^t p_T(u)du \quad (4.10)$$

$$\Rightarrow F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \leq t \leq 1 \\ 2t-1-\frac{t^2}{2} & 1 < t \leq 2 \\ 1 & t > 2 \end{cases}$$

4.5 Verify your results through a plot

Solution: The Results are verified in the plots in Fig.4.2 and Fig.4.3