Random Numbers

Aakash Kamuju

Contents

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution:
Download the following files and execute the C program.

```
wget https://github.com/kamujuaakash/
    Assignment1/blob/main/Exercise/Exercise
    -1/1.1/exrand.c
wget https://github.com/kamujuaakash/
    Assignment1/blob/main/Exercise/Exercise
    -1/1.1/coeffs.h
```

Download the above files and execute the following commands

```
gcc exrand.c -lm
./a.out
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \le x) \tag{1.1}$$

Solution:
The following code plots Fig. 1.2

```
wget https://github.com/kamujuaakash/
    Assignment1/blob/main/Exercise/Exercise
    -1/1.2/cdf_plot.py
```

Download the above files and execute the following commands to produce Fig.1.2

```
python3 cdf_plot.py
```

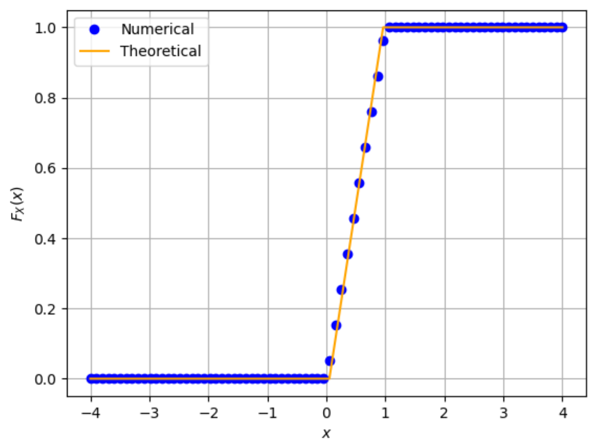

Fig. 1.2: The CDF of U

1.3 Find a theoretical expression for $F_U(x)$.
Solution:
Given U is a uniform Random Variable

$$p_U(x) = 1 \text{ for}$$

$$F_U(x) = \int_{-\infty}^{\infty} p_U(x)dx$$

$$\implies F_U(x) = \begin{cases} 0 & x \le 0 \\ x & 0 < x < 1 \\ 1 & x \ge 1 \end{cases}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.2}$$

and its variance as $\text{var}[U] = E[U - E[U]]^2$
Write a C program to find the mean and variance of U.
Solution:
Download the following files and execute the

C program.

```
wget https://github.com/kamujuaakash/
    Assignment1/blob/main/Exercise/Exercise
    −1/1.4/m.c
wget https://github.com/kamujuaakash/
    Assignment1/blob/main/Exercise/Exercise
    −1/1.4/coeffs.h
```

Download the above files and execute the following commands

```
gcc m.c −lm
./a.out
```

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x)$$

Solution:

$$E\left[U\right] = \int_{-\infty}^{\infty} x dF_U(x)$$

$$E\left[U\right] = \int_{0}^{1} x$$

$$\implies E\left[U\right] = \frac{1}{2}$$

$$E\left[U^2\right] = \int_{-\infty}^{\infty} x^2 dF_U(x)$$

$$E\left[U^2\right] = \int_{0}^{1} x^2 dF_U(x)$$

$$\implies E\left[U^2\right] = \frac{1}{3}$$

$$\mathrm{var}\left[U\right] = E\left[U - E\left[U\right]\right]^2$$

$$\implies \mathrm{var}\left[U\right] = E\left[U^2\right] - E\left[U\right]^2$$

$$\boxed{\mathrm{var}\left[U\right] = \frac{1}{12} = 0.0833}$$

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution: Download the following files and execute the C program.

```
wget https://github.com/kamujuaakash/
    Assignment1/blob/main/Exercise/Exercise
    −2/2.1/exrand.c
wget https://github.com/kamujuaakash/
    Assignment1/blob/main/Exercise/Exercise
    −2/2.1/coeffs.h
```

Download the above files and execute the following commands

```
gcc exrand.c −lm
./a.out
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution:

- $F_X(x) = P(X \le x)$
- $Q_X(x) = P(X > x)$
- $Q_X(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-\frac{u^2}{2}} du$
- $F_X(x) = 1 - Q_X(x)$ This can be used to calculate F (x).

The CDF of X is plotted in Fig. 2.2
using the code below

```
wget https://github.com/kamujuaakash/
    Assignment1/blob/main/Exercise/Exercise
    −2/2.2/2.2.py
```

Download the above files and execute the following commands to produce Fig.2.2

```
python3 2.2.py
```

Some of the properties of CDF
1)

$$\lim_{x\to\infty} F_X(x) = 1$$

2) $F_X(x)$ is non decreasing function.
3) Symmetric about one point.

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.2}$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 2.3 using the code below

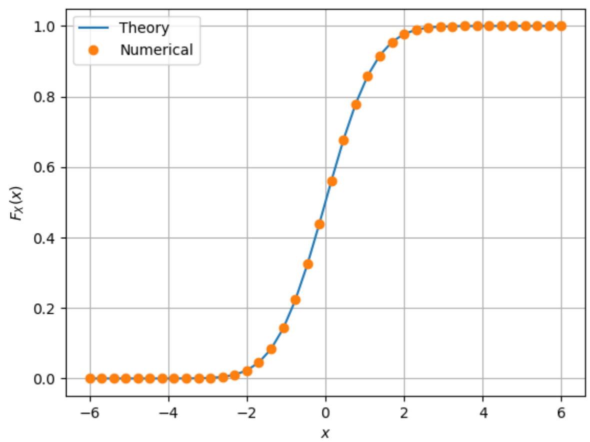

Fig. 2.2: The CDF of X

```
wget https://github.com/kamujuaakash/
    Assignment1/blob/main/Exercise/Exercise
    -2/2.3/2.3.py
```

Download the above files and execute the following commands to produce Fig.2.3

```
python3 2.3.py
```

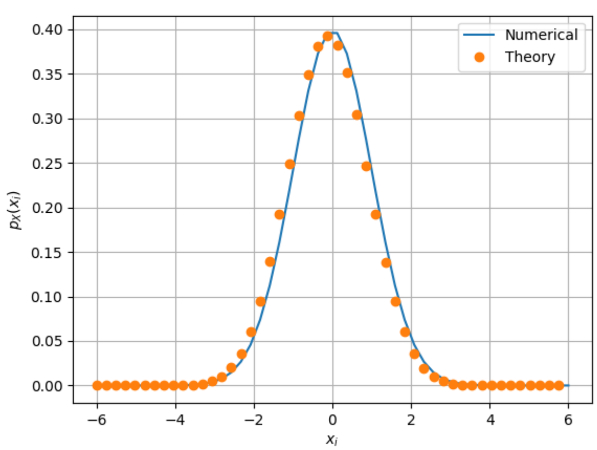

Fig. 2.3: The PDF of X

Let $\mu \approx 0$

Some of the properties of the PDF:

a) Symmetric about $x = \mu$ in this case
b) Decreasing function for $x > \mu$ and increasing for $x < \mu$ and attains maximum at $x = \mu$
c) Area under the curve is unity.

2.4 Find the mean and variance of X by writing a C program.
Solution: Download the following files and execute the C program.

```
wget https://github.com/kamujuaakash/
    Assignment1/blob/main/Exercise/Exercise
    -2/2.4/m.c
wget https://github.com/kamujuaakash/
    Assignment1/blob/main/Exercise/Exercise
    -2/2.4/coeffs.h
```

Download the above files and execute the following commands

```
gcc m.c -lm
./a.out
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.3)$$

repeat the above exercise theoretically.

Solution:

1) CDF is given by

$$F_X(x) = \int_{-\infty}^{\infty} p_X(x)dx \quad (2.4)$$

$$\boxed{F_X(x) = 1} \quad (2.5)$$

2) Mean is given by

$$E(x) = \int_{-\infty}^{\infty} xp_X(x)dx$$
$$\int_{-\infty}^{\infty} \left(xexp\left(-\frac{x^2}{2}\right)dx\right)$$
$$\implies \boxed{E(x) = 0}$$

3) Variance is given by

$$\text{var}\,[U] = E(U^2) - (E(U))^2$$
$$E\left(x^2\right) = \int_{-\infty}^{\infty} x^2 p_X(x)dx$$
$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x^2 exp\left(-\frac{x^2}{2}\right)dx$$
$$= \frac{1}{\sqrt{2\pi}}\left(x\int_{-\infty}^{\infty} xexp\left(-\frac{x^2}{2}\right)dx\right)$$
$$- \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\int \left(xexp\left(-\frac{x^2}{2}\right)dx\right).dx\right)$$

$$= \left[-x \frac{1}{\sqrt{2\pi}} exp \frac{-x^2}{2} \right]_{-\infty}^{\infty} +$$
$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} exp\left(-\frac{x^2}{2}\right) dx$$
$$= \frac{\sqrt{2\pi}}{\sqrt{2\pi}}$$
$$= 1 \implies \boxed{\text{var}[U] = 1}$$

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1-U) \tag{3.1}$$

and plot its CDF.
Solution: Running the below code generates samples of V from file uni.dat(U).

```
https://github.com/kamujuaakash/Assignment1
    /blob/main/Exercise/Exercise−3/3.1/3.1.py
https://github.com/kamujuaakash/Assignment1
    /blob/main/Exercise/Exercise−3/3.1/3.1.c
```

Use the below command in the terminal to run the code:

```
python3 3.1.py
gcc 3.1.c
./a.out
```

Now these samples are used to plot by running the below code,

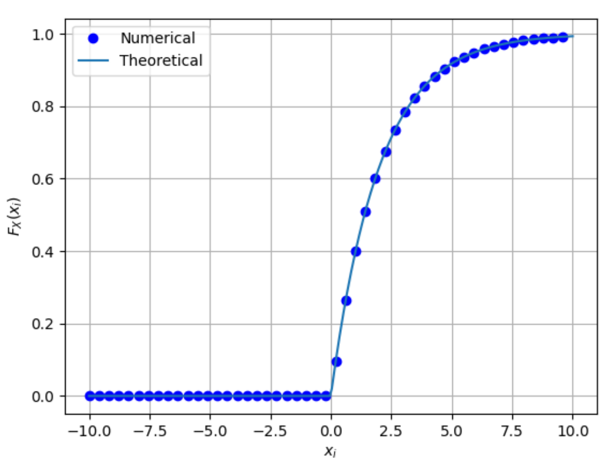

Fig. 3.1: CDF for (3)

```
wge https://github.com/kamujuaakash/
    Assignment1/blob/main/Exercise/Exercise
    −3/3.1/cdf.py
```

Use the below command to run the code:

```
python3 cdf.py
```

3.2 Theorotical expression for $F_V(x)$

$$\begin{aligned} F_V(x) &= P\{V \le x\} \\ &= P\{-2 \times \ln(1-U) \le x\} \\ &= P\{U \le 1 - e^{(-\frac{x}{2})}\} \\ &= F_U\{1 - e^{(-\frac{x}{2})}\} \\ &= \begin{cases} 1 - e^{(-\frac{x}{2})} & 0 \le x < \infty \\ 0 & x < 0 \end{cases} \end{aligned}$$

4 Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \tag{4.1}$$

Solution: Download the following files and execute the C program.

```
wget https://github.com/kamujuaakash/
    Assignment1/blob/main/Exercise/Exercise
    −4/4.1/4.1.c
wget https://github.com/kamujuaakash/
    Assignment1/blob/main/Exercise/Exercise
    −4/4.1/coeffs.h
```

Download the above files and execute the following commands

```
gcc 4.1.c
./a.out
```

4.2 Find the CDF of T.
Solution:
The CDF of T is plotted in figure using the code below

```
wget https://github.com/kamujuaakash/
    Assignment1/blob/main/Exercise/Exercise
    −4/4.2/4.2.py
```

Download the above files and execute the following commands to produce Fig.4.2

```
python3 4.2.py
```

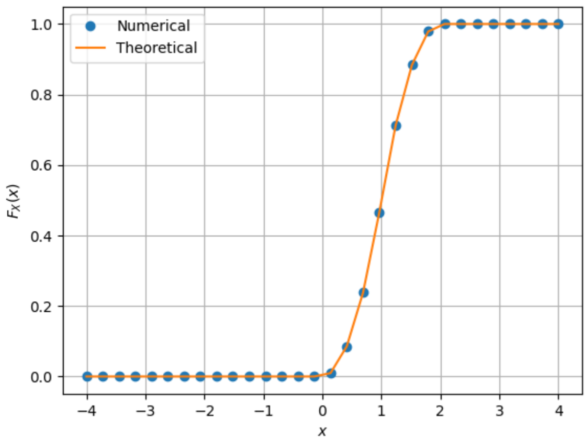

Fig. 4.2: The CDF of T

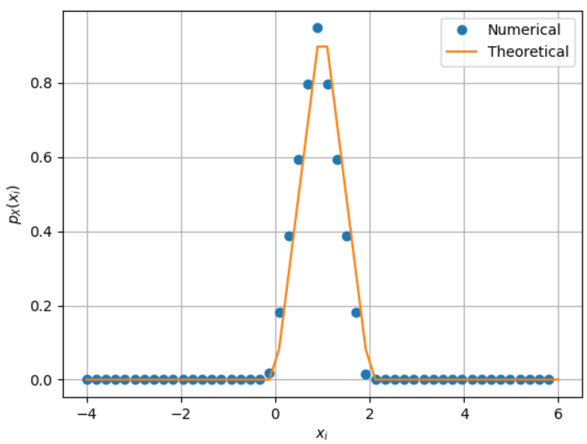

Fig. 4.3: The PDF of T

4.3 Find the PDF of T.
Solution:
The PDF of T is plotted in Fig. 4.2 using the code below

```
wget https://github.com/kamujuaakash/
    Assignment1/blob/main/Exercise/Exercise
    -4/4.3/4.3.py
```

Download the above files and execute the following commands to produce Fig.4.2

```
python3 4.3.py
```

4.4 Find the Theoretical Expression for the PDF and CDF of T

Solution:

$$T = U_1 + U_2 \tag{4.2}$$

$$\implies p_T(t) = \int_{-\infty}^{t} p_{U1}(x)p_{U2}(y)dx \tag{4.3}$$

$$\text{As,} p_{U1}(x) = p_{U1}(y) = p_U(u) \tag{4.4}$$

$$\implies p_T(t) = \int_{-\infty}^{t} p_U(u)p_U(t-u)du \tag{4.5}$$

a) Theoretical PDF
   i) $t \leq 1$

$$p_T(t) = \int_0^t p_U(t-u)du \tag{4.6}$$

$$\implies p_T(t) = \int_0^t du = t \tag{4.7}$$

   ii) $t > 1$

$$p_T(t) = \int_0^1 p_U(t-u)du \tag{4.8}$$

$$\implies p_T(t) = \int_{t-1}^{1} du = 2-t \tag{4.9}$$

$$\implies \boxed{P_T(t) = \begin{cases} t & 0 \leq t \leq 1 \\ 2-t & 1 < t \leq 2 \\ 0 & t < 0 \text{ or } t > 2 \end{cases}}$$

b) Theoretical CDF

$$F_T(t) = \int_{-\infty}^{t} p_T(u)du \tag{4.10}$$

$$\implies \boxed{F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \leq t \leq 1 \\ 2t - 1 - \frac{t^2}{2} & 1 < t \leq 2 \\ 1 & t > 2 \end{cases}}$$

4.5 Verify your results through a plot
Solution: The Results are verified in the plots in Fig.4.2and Fig.4.3