Digital Signal Processing

EE3900: Linear Systems and Signal Processing

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Contents

***Abstract*—This manual provides a simple introduction to digital signal processing.**

1. Software Installation

Run the following commands

```
sudo apt−get update
sudo apt−get install libffi−dev libsndfile1 python3
    −scipy python3−numpy python3−matplotlib
sudo pip install cffi pysoundfile
```

2. Digital Filter

2.1 Download the sound file from

```
wget https://github.com/kamujuaakash/
    EE3900/blob/main/codes/Sound_Noise.
    wav
```

2.2 You will find a spectrogram at https://academo.org/demos/spectrum-analyzer. Upload the sound file that you downloaded in Problem 2.1 in the spectrogram and play. Observe the spectrogram. What do you find?
Solution: There are a lot of yellow lines between 440 Hz to 5.1 KHz. These represent the synthesizer key tones. Also, the key strokes are audible along with background noise.

2.3 Write the python code for removal of out of band noise and execute the code.
Solution:

```
import soundfile as sf
from scipy import signal

#read .wav file
input_signal,fs = sf.read('codes/
    Sound_Noise.wav')

#sampling frequency of Input signal
sampl_freq=fs

#order of the filter
order=4

#cutoff frquency 4kHz
cutoff_freq=4000.0

#digital frequency
Wn=2*cutoff_freq/sampl_freq

# b and a are numerator and denominator
    polynomials respectively
b, a = signal.butter(order,Wn, 'low')

#filter the input signal with butterworth filter
```

```
output_signal = signal.filtfilt(b, a,
    input_signal)
#output_signal = signal.lfilter(b, a,
    input_signal)

#write the output signal into .wav file
sf.write('codes/Sound_With_ReducedNoise.
    wav', output_signal, fs)
```

2.4 The output of the python script in Problem 2.3 is the audio file Sound_With_ReducedNoise.wav. Play the file in the spectrogram in Problem 2.2. What do you observe?
Solution: The key strokes as well as background noise is subdued in the audio. Also, the signal is blank for frequencies above 5.1 kHz.

3. Difference Equation

3.1 Let

$$x(n) = \left\{ \underset{\uparrow}{1}, 2, 3, 4, 2, 1 \right\} \tag{3.1}$$

Sketch $x(n)$.
Solution: The following code yields Fig. 3.1

```
wget https://github.com/kamujuaakash/
    EE3900/blob/main/codes/3.1.py
```

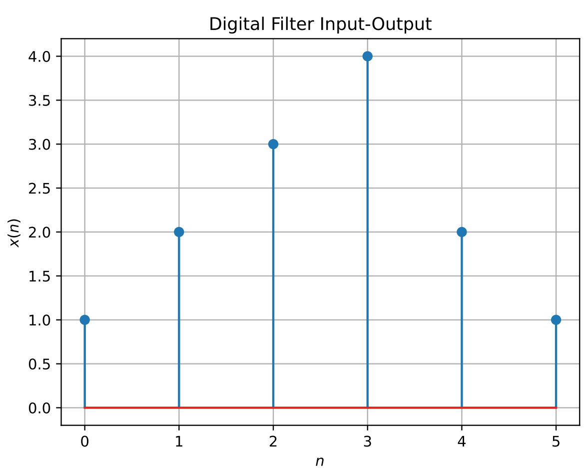

Fig. 3.1.

3.2 Let

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2),$$
$$y(n) = 0, n < 0 \tag{3.2}$$

Sketch $y(n)$.
Solution: The following code yields Fig. 5.3.

```
wget https://github.com/kamujuaakash/
    EE3900/blob/main/codes/xnyn.py
```

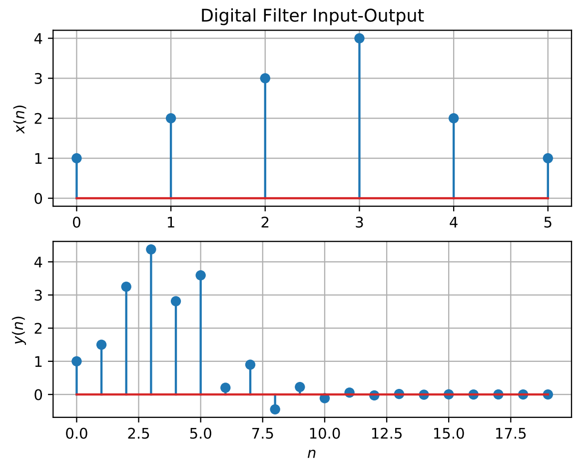

Fig. 3.2.

4. Z-transform

4.1 The Z-transform of $x(n)$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.1}$$

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \tag{4.2}$$

and find

$$\mathcal{Z}\{x(n-k)\} \tag{4.3}$$

Solution: From (4.1),

$$\mathcal{Z}\{x(n-k)\} = \sum_{n=-\infty}^{\infty} x(n-1)z^{-n} \tag{4.4}$$

$$= \sum_{n=-\infty}^{\infty} x(n)z^{-n-1} = z^{-1}\sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.5}$$

resulting in (4.2). Similarly, it can be shown that

$$\mathcal{Z}\{x(n-k)\} = z^{-k}X(z) \tag{4.6}$$

4.2 Obtain $X(z)$ for x(n) define in problem 3.1. **Solution:** From (4.1),

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.7}$$

$$= \sum_{n=0}^{\infty} x(n)z^{-n} \tag{4.8}$$

$$= 1 \cdot z^{-1} + 2 \cdot z^{-2} + 3 \cdot z^{-3} + 4 \cdot z^{-4} + 2 \cdot z^{-5} + 1 \cdot z^{-6} \tag{4.9}$$

$$= \frac{z^5 + 2z^4 + 3z^3 + +4z^2 + 2z + 1}{z^6} \tag{4.10}$$

4.3 Find

$$H(z) = \frac{Y(z)}{X(z)} \tag{4.11}$$

from (3.2) assuming that the Z-transform is a linear operation.
Solution: Applying (4.6) in (3.2),

$$Y(z) + \frac{1}{2}z^{-1}Y(z) = X(z) + z^{-2}X(z) \tag{4.12}$$

$$\implies H(z) = \frac{Y(z)}{X(z)} = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{4.13}$$

4.4 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.14}$$

and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.15}$$

is

$$U(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.16}$$

Solution: It is easy to show that

$$\delta(n) \overset{\mathcal{Z}}{\rightleftharpoons} 1 \tag{4.17}$$

and from (4.15),

$$U(z) = \sum_{n=0}^{\infty} z^{-n} \tag{4.18}$$

$$= \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.19}$$

using the fomula for the sum of an infinite geometric progression.

4.5 Show that

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - az^{-1}} \quad |z| > |a| \tag{4.20}$$

Solution: From (4.1),

$$\mathcal{Z}\{a^n u(n)\} = \sum_{n=-\infty}^{\infty} u(n) \left(\frac{a}{z}\right)^{-n} \tag{4.21}$$

From (4.15),

$$\mathcal{Z}\{a^n u(n)\} = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^{-n} \tag{4.22}$$

$$= \frac{1}{1 - \frac{a}{z}} \tag{4.23}$$

$$= \frac{1}{1 - az^{-1}} \quad |z| > |a| \tag{4.24}$$

using the formula for the sum of an infinite geometric progression.

4.6 Let

$$H\left(e^{\jmath\omega}\right) = H\left(z = e^{\jmath\omega}\right). \tag{4.25}$$

Plot $|H\left(e^{\jmath\omega}\right)|$. Is it periodic? If so,find the period. $H(e^{\jmath\omega})$ is known as the *Discret Time Fourier Transform* (DTFT) of $x(n)$.
Solution:

$$H\left(e^{\jmath\omega}\right) = H\left(z = e^{\jmath\omega}\right) \tag{4.26}$$

$$= \frac{1 + e^{-2\jmath\omega}}{1 + \frac{1}{2}e^{-\jmath\omega}} \tag{4.27}$$

$$= \frac{2\left(1 + e^{-2\jmath\omega}\right)}{2 + e^{-\jmath\omega}} \tag{4.28}$$

$$= \frac{2\left(1 + \cos 2\omega - \jmath \sin 2\omega\right)}{2 + \cos\omega - \jmath \sin\omega} \tag{4.29}$$

$$= \frac{2\left|\left(2\cos^2\omega - \jmath\sin 2\omega\right)\right|}{\left|2 + \cos\omega - \jmath\sin\omega\right|} \tag{4.30}$$

$$= \frac{2\sqrt{4\cos^4\omega + 4\sin^2\omega\cos^2\omega}}{\sqrt{(2 + \cos\omega)^2 + \sin^2\omega}} \tag{4.31}$$

$$= \frac{4\cos\omega}{\sqrt{5 + 4\cos\omega}} \tag{4.32}$$

$$|H\left(e^{\jmath\omega}\right)| = \frac{|4\cos\omega|}{\sqrt{5 + 4\cos\omega}} \tag{4.33}$$

Since $\left|H(e^{j\omega})\right|$ is function of cosine we can say it is periodic.And from the plot 4.6 we can say that it is symmetric about $\omega = 0$ (even function) and it is periodic with period 2π.You can find the same from the theoritical expression $\left|H\left(e^{j\omega}\right)\right|$,

$$H(e^{\jmath\omega}) = H(e^{j(-\omega)}) \,(\text{cos is an even function}) \tag{4.34}$$

And to find period, the period of $|\cos(\omega)|$ is π and the period of $\sqrt{5 + 4\cos(\omega)}$ is 2π. So the period of division of both will be 2π. This gives us the period of $\left|H(e^{j\omega})\right|$ as 2π The following code plots Fig. 4.6.

```
wget https://github.com/kamujuaakash/
    EE3900/blob/main/codes/hn.py
```

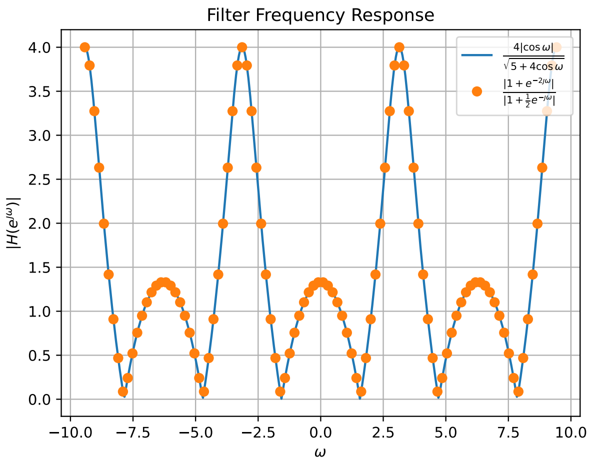

Fig. 4.6. $|H(e^{\jmath\omega})|$

4.7 Express $h(n)$ in terms of $H(e^{\jmath\omega})$
Solution:

$$\int_{-\pi}^{\pi} H(e^{\jmath\omega})e^{\jmath\omega n}d\omega \tag{4.35}$$

$$= \int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} h(k)e^{-\jmath\omega k}e^{\jmath\omega n}d\omega \tag{4.36}$$

$$= \sum_{k=-\infty}^{\infty} h(k) \int_{-\pi}^{\pi} e^{\jmath\omega(n-k)}d\omega \tag{4.37}$$

Now,

$$\int_{-\pi}^{\pi} e^{\jmath\omega(n-k)}d\omega = \begin{cases} \int_{-\pi}^{\pi} d\omega & n-k=0 \\ \left.\frac{\exp(\jmath\omega(n-k))}{\jmath(n-k)}\right|_{-\pi}^{\pi} & n-k\neq 0 \end{cases} \tag{4.38}$$

$$= \begin{cases} 2\pi & n-k=0 \\ 0 & n-k\neq 0 \end{cases} \tag{4.39}$$

$$= 2\pi\delta(n-k) \tag{4.40}$$

Thus,

$$\int_{-\pi}^{\pi} H(e^{\jmath\omega})e^{\jmath\omega n}d\omega = 2\pi \sum_{k=-\infty}^{\infty} h(k)\delta(n-k) \tag{4.41}$$

$$= 2\pi h(n) * \delta(n) \tag{4.42}$$

$$= 2\pi h(n) \tag{4.43}$$

Therefore, $h(n)$ is given by the inverse DTFT (IDTFT) of $H(e^{\jmath\omega})$

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{\jmath\omega})e^{\jmath\omega n}d\omega \tag{4.44}$$

5. Impulse Response

5.1 Using long division, find

$$h(n), \quad n < 5 \tag{5.1}$$

for H(z) in (4.13).
Solution:

$$H(z) = \frac{1+z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{5.2}$$

Substitute $z^{-1} = x$

$$\begin{array}{r l} & \quad 2x - 4 \\ \frac{1}{2}x + 1) & \overline{\;x^2 \qquad + 1} \\ & \underline{-\,x^2 - 2x} \\ & \qquad -\,2x + 1 \\ & \qquad \underline{\;\;2x + 4} \\ & \qquad\qquad 5 \end{array}$$

$$\implies 1 + z^{-2} = \left(1 + \frac{1}{2}z^{-1}\right)\left(-4 + 2z^{-1}\right) + 5 \tag{5.3}$$

$$\implies H(z) = -4 + 2z^{-1} + \frac{5}{1+\frac{1}{2}z^{-1}} \tag{5.4}$$

On applying the inverse Z-transform on both sides of the equation

$$H(z) \overset{\mathcal{Z}}{\rightleftharpoons} h(n) \tag{5.5}$$

$$-4 \overset{\mathcal{Z}}{\rightleftharpoons} -4\delta(n) \tag{5.6}$$

$$2z^{-1} \overset{\mathcal{Z}}{\rightleftharpoons} 2\delta(n-1) \tag{5.7}$$

$$\frac{5}{1+\frac{1}{2}z^{-1}} \overset{\mathcal{Z}}{\rightleftharpoons} 5\left(-\frac{1}{2}\right)^n u(n) \tag{5.8}$$

$$\tag{5.9}$$

Therefore,

$$h(n) = -4\delta(n) + 2\delta(n-1) + 5\left(-\frac{1}{2}\right)^n u(n) \tag{5.10}$$

5.2 Find an expression for $h(n)$ using $H(z)$, given that in (4.20), given that

$$h(n) \overset{\mathcal{Z}}{\rightleftharpoons} H(z) \tag{5.11}$$

and there is a one to one relationship between $h(n)$ and $H(z)$. $h(n)$ is known as the *impulse response* of the system defined by (3.2).
Solution: From (4.13),

$$H(z) = \frac{1}{1+\frac{1}{2}z^{-1}} + \frac{z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{5.12}$$

$$\implies h(n) = \left(-\frac{1}{2}\right)^{n} u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.13}$$

using (4.20) and (4.6).

5.3 Sketch $h(n)$. Is it bounded? Justify theoretically.
Solution:
The following code plots

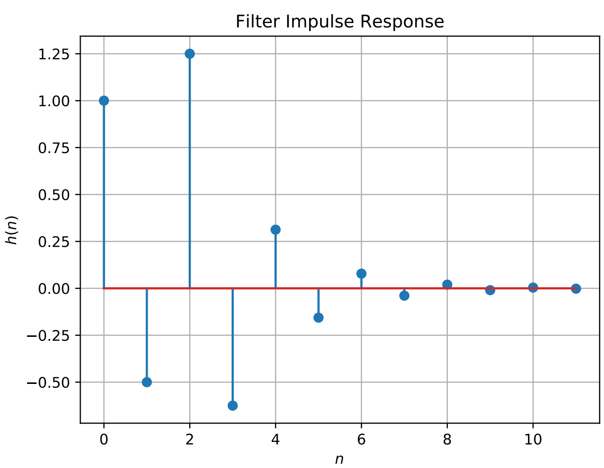

Fig. 5.3.

```
wget https://github.com/kamujuaakash/
    EE3900/blob/main/Simulation/codes/5.3.
    py
```

$$\lim_{n\to\infty} h(n) = \lim_{n\to\infty} \left(-\frac{1}{2}\right)^{n} u(n) + \lim_{n\to\infty} \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.14}$$

$$h(n) = \begin{cases} 5\left(\frac{-1}{2}\right)^{n} & n \geq 2 \\ \left(\frac{-1}{2}\right)^{n} & 0 \leq n < 2 \\ 0 & n < 0 \end{cases} \tag{5.15}$$

Maximum value and minimum value are always bounded in this case. $\therefore h(n)$ is bounded

5.4 Is it convergent? Justify using the ratio test.
Solution:

$$h(n+1) = \left(-\frac{1}{2}\right)^{n+1} u(n+1) + \left(-\frac{1}{2}\right)^{n-1} u(n-1) \tag{5.16}$$

According to ratio test, L is given by $\lim_{n\to\infty} \left|\frac{h(n+1)}{h(n)}\right|$, if $L < 1$ then h(n) is convergent.

$$L = \lim_{n\to\infty} \left|\frac{h(n+1)}{h(n)}\right| \tag{5.17}$$

$$= \lim_{n\to\infty} \left|\frac{\left(-\frac{1}{2}\right)^{n+1} u(n+1) + \left(-\frac{1}{2}\right)^{n-1} u(n-1)}{\left(-\frac{1}{2}\right)^{n} u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2)}\right| \tag{5.18}$$

$$= \left|\frac{-\frac{1}{2} + -\frac{1}{2}^{-1}}{1 + -\frac{1}{2}^{-}2}\right| \tag{5.19}$$

$$= \left|\frac{-\frac{1}{2} - 2}{1+4}\right| \tag{5.20}$$

$$= \left|\frac{\frac{-5}{2}}{5}\right| \tag{5.21}$$

$$= \frac{1}{2} \tag{5.22}$$

As $L < 1$, $h(n)$ is convergent.

5.5 The system with $h(n)$ is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \tag{5.23}$$

Is the system defined by (3.2) stable for the impulse response in (5.11)?
Solution:

$$\sum_{n=-\infty}^{\infty} h(n) = \sum_{n=0}^{\infty} h(n) \tag{5.24}$$

$$= \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^{n} u(n) + \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.25}$$

$$= 2\left(\frac{1}{1+\frac{1}{2}}\right) \tag{5.26}$$

$$= \frac{4}{3} \tag{5.27}$$

As $\sum_{n=-\infty}^{\infty} h(n) = \frac{4}{3}$ is less than ∞, the system defined by (3.2) is stable for the impulse response in (5.1).

5.6 Verify the above result using a python code. **Solution:** The following code determines if it is convergent or not:

```
wget https://github.com/kamujuaakash/
    EE3900/blob/main/Simulation/codes/5.5.
    py
```

5.7 Compute and sketch $h(n)$ using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2), \quad (5.28)$$

This is the definition of $h(n)$.
Solution: The following code plots Fig. 5.7. Note that this is the same as Fig. **??**.

```
wget https://github.com/kamujuaakash/
    EE3900/blob/main/Simulation/codes/5.7.
    py
```

Computing,
$h(0) = 1$
$h(1) = -\frac{1}{2}h(0)$
$h(2) = -\frac{1}{2}h(1) + 1$
Parallely, $h(n) = -\frac{1}{2}h(n-1)$

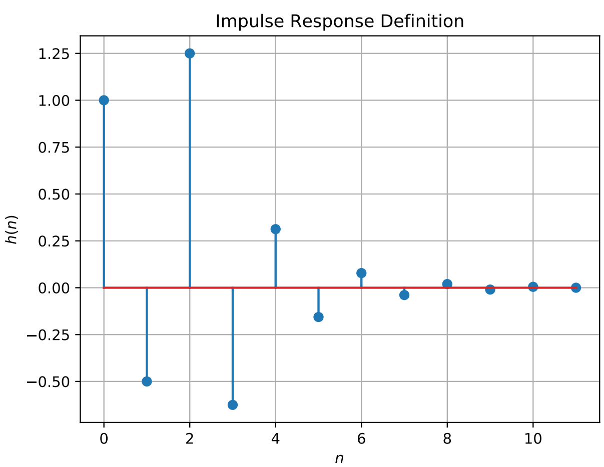

Fig. 5.7. $h(n)$ from the definition

5.8 Compute

$$y(n) = x(n) * h(n) = \sum_{n=-\infty}^{\infty} x(k)h(n-k) \quad (5.29)$$

Comment. The operation in (5.29) is known as *convolution*.
Solution: The following code plots Fig. 5.8. Note that this is the same as $y(n)$ in Fig. 5.3.

```
wget https://github.com/kamujuaakash/
    EE3900/blob/main/Simulation/codes/5.8.
    py
```

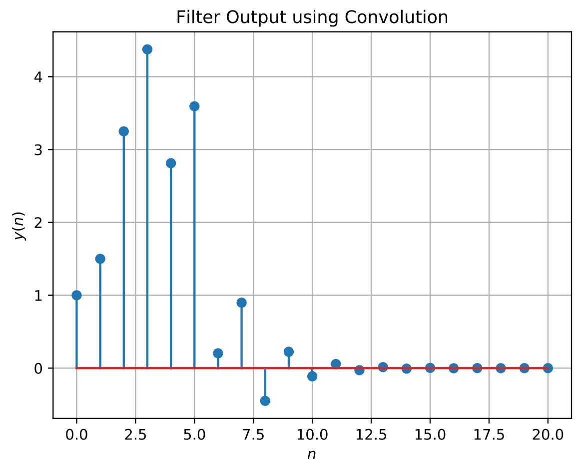

Fig. 5.8. $y(n)$ from the definition of convolution

5.9 Express the above convolution using a toeplitz matrix. **Solution:**

$$\mathbf{y} = \mathbf{x} \circledast \mathbf{h} \quad (5.30)$$

$$\mathbf{y} = \begin{pmatrix} h_1 & 0 & . & . & . & 0 \\ h_2 & h_1 & . & . & . & 0 \\ h_3 & h_2 & h_1 & . & . & 0 \\ h_{m-1} & . & . & . & h_2 & h_1 \\ h_m & h_{m-1} & . & . & . & h_2 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & h_m \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ . \\ x_n \end{pmatrix} \quad (5.31)$$

$$\mathbf{y} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{-1}{2} & 1 & 0 & 0 & 0 & 0 \\ \frac{5}{4} & \frac{-1}{2} & 1 & 0 & 0 & 0 \\ \frac{-5}{8} & \frac{5}{4} & \frac{-1}{2} & 1 & 0 & 0 \\ \frac{5}{16} & \frac{-5}{8} & \frac{5}{4} & \frac{-1}{2} & 1 & 0 \\ \frac{-5}{32} & \frac{5}{16} & \frac{-5}{8} & \frac{5}{4} & \frac{-1}{2} & 1 \\ \frac{5}{64} & \frac{-5}{32} & \frac{5}{16} & \frac{-5}{8} & \frac{5}{4} & \frac{-1}{2} \\ 0 & \frac{5}{64} & \frac{-5}{32} & \frac{5}{16} & \frac{-5}{8} & \frac{5}{4} \\ 0 & 0 & \frac{5}{64} & \frac{-5}{32} & \frac{5}{16} & \frac{-5}{8} \\ 0 & 0 & 0 & \frac{5}{64} & \frac{-5}{32} & \frac{5}{16} \\ 0 & 0 & 0 & 0 & \frac{5}{64} & \frac{-5}{32} \\ & & & & & \frac{5}{64} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1. \\ 1.5 \\ 3.25 \\ 4.375 \\ 2.8125 \\ 3.59375 \\ 0.203125 \\ 0.9375 \\ -0.390625 \\ 0.3125 \\ 0. \\ 0.078125 \end{pmatrix} \quad (5.32)$$

And this is what we got in (5.29)

5.10 Show that

$$y(n) = \sum_{n=-\infty}^{\infty} x(n-k)h(k) \tag{5.33}$$

Solution: Substitute $k \to n - k$ then

$$y(n) = x(n) * h(n) \tag{5.34}$$

$$= \sum_{n=-\infty}^{\infty} x(k)h(n-k) = \sum_{n=-\infty}^{\infty} x(n-k)h(k) \tag{5.35}$$

6. DFT and FFT

6.1 Compute

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-\jmath 2\pi kn/N}, \quad k = 0, 1, \ldots, N-1 \tag{6.1}$$

and $H(k)$ using $h(n)$. **Solution:** The following code plots Fig. 6.1.

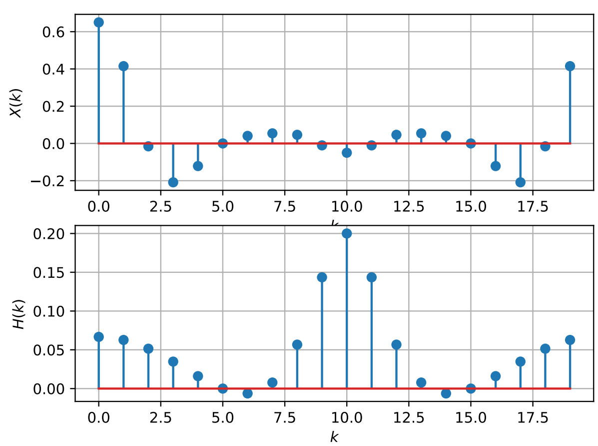

Fig. 6.1. $X(k), H(k)$ from the DFT

```
wget https://github.com/kamujuaakash/
    EE3900/blob/main/codes/xkhkdft.py
```

6.2 Compute

$$Y(k) = X(k)H(k) \tag{6.2}$$

Solution: The following code plots Fig. 6.2.

```
wget https://github.com/kamujuaakash/
    EE3900/blob/main/codes/ykdft.py
```

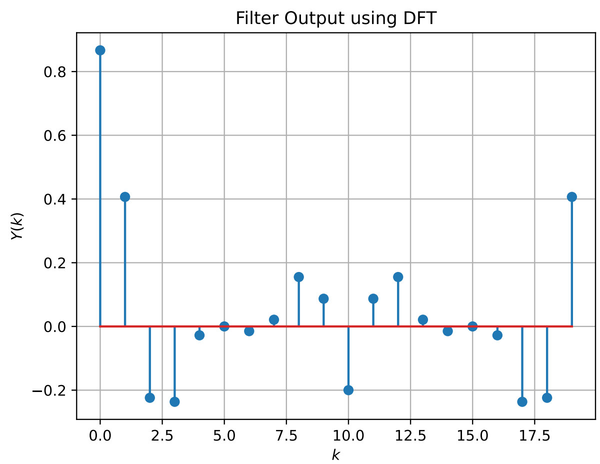

Fig. 6.2. $Y(k)$ from the DFT

6.3 Compute

$$y(n) = \frac{1}{N}\sum_{k=0}^{N-1} Y(k) \cdot e^{\jmath 2\pi kn/N}, \quad n = 0, 1, \ldots, N-1 \tag{6.3}$$

Solution: The following code plots Fig. 5.8. Note that this is the same as $y(n)$ in Fig. 5.3.

```
wget https://github.com/kamujuaakash/
    EE3900/blob/main/codes/yndft.py
```

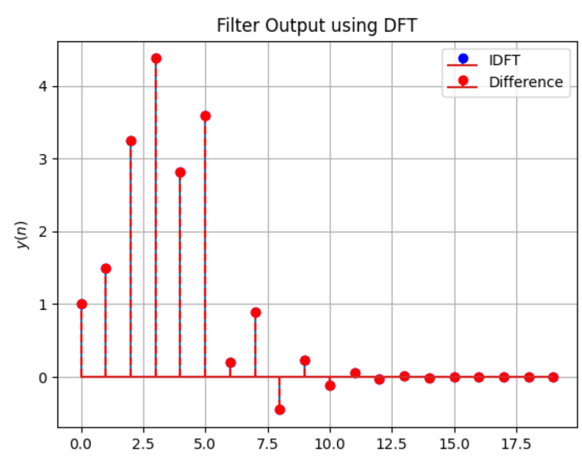

Fig. 6.3. $y(n)$ from the DFT

6.4 Repeat the previous exercise by computing $X(k), H(k)$ and $y(n)$ through FFT and IFFT. **Solution:** Download the below python code for the plot 6.4,

```
wget https://github.com/kamujuaakash/
    EE3900/blob/main/Simulation/codes/6.4.
    py
```

Then run the following command,

```
python3 6.4.py
```

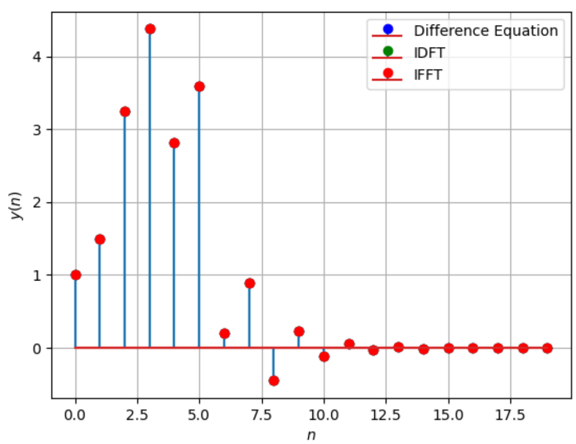

Fig. 6.4. The plot of $y(n)$ using IFFT

7. FFT

1. The DFT of $x(n)$ is given by
$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}, \quad k = 0, 1, \ldots, N-1 \tag{7.1}$$
2. Let
$$W_N = e^{-j2\pi/N} \tag{7.2}$$
Then the N-point $DFTmatrix$ is defined as
$$\mathbf{F}_N = [W_N^{mn}], \quad 0 \le m, n \le N-1 \tag{7.3}$$
where W_N^{mn} are the elements of $\mathbf{F}_N$.
3. Let
$$\mathbf{I}_4 = \begin{pmatrix} \mathbf{e}_4^1 & \mathbf{e}_4^2 & \mathbf{e}_4^3 & \mathbf{e}_4^4 \end{pmatrix} \tag{7.4}$$
be the 4×4 identity matrix. Then the 4 point *DFT permutation matrix* is defined as
$$\mathbf{P}_4 = \begin{pmatrix} \mathbf{e}_4^1 & \mathbf{e}_4^3 & \mathbf{e}_4^2 & \mathbf{e}_4^4 \end{pmatrix} \tag{7.5}$$
4. The 4 point $DFTdiagonalmatrix$ is defined as
$$\mathbf{D}_4 = diag\begin{pmatrix} W_8^0 & W_8^1 & W_8^2 & W_8^3 \end{pmatrix} \tag{7.6}$$
5. Show that
$$W_N^2 = W_{N/2} \tag{7.7}$$
Solution: From (7.2),
$$W_N = e^{-j2\pi/N} \tag{7.8}$$
Consider,
$$W_N^2 = \left(e^{-j2\pi/N}\right)^2 \tag{7.9}$$
$$= e^{-j2\pi/(N/2)} \tag{7.10}$$
$$= W_{N/2} \tag{7.11}$$
Hence proved.
6. Show that
$$\mathbf{F}_4 = \begin{bmatrix} \mathbf{I}_2 & \mathbf{D}_2 \\ \mathbf{I}_2 & -\mathbf{D}_2 \end{bmatrix} \begin{bmatrix} \mathbf{F}_2 & 0 \\ 0 & \mathbf{F}_2 \end{bmatrix} \mathbf{P}_4 \tag{7.12}$$
Solution: From the eq (7.5),
$$\mathbf{P}_4 = \begin{pmatrix} \mathbf{e}_4^1 & \mathbf{e}_4^3 & \mathbf{e}_4^2 & \mathbf{e}_4^4 \end{pmatrix} \tag{7.13}$$
Clearly $\mathbf{P}_4$ is an elementary matrix of $\mathbf{I}_4$, so on multiplication with a matrix it will interchange the rows/columns of matrix depending on positions of unit vectors.
From that it follows ,
$$\mathbf{P}_4^2 = \mathbf{I}_4 \tag{7.14}$$
So it is similar to prove that,
$$\mathbf{F}_4\mathbf{P}_4 = \begin{bmatrix} \mathbf{I}_2 & \mathbf{D}_2 \\ \mathbf{I}_2 & -\mathbf{D}_2 \end{bmatrix} \begin{bmatrix} \mathbf{F}_2 & 0 \\ 0 & \mathbf{F}_2 \end{bmatrix} \tag{7.15}$$
Now from (7.3),
$$\mathbf{F}_2 = \begin{bmatrix} W_2^{0.0} & W_2^{0.1} \\ W_2^{1.0} & W_2^{1.1} \end{bmatrix} \tag{7.16}$$
$$= \begin{bmatrix} W_2^0 & W_2^0 \\ W_2^0 & W_2^1 \end{bmatrix} \tag{7.17}$$
Using the result (7.11), we can write
$$\mathbf{F}_2 = \begin{bmatrix} W_4^0 & W_4^0 \\ W_4^0 & W_4^2 \end{bmatrix} \tag{7.18}$$
And $\mathbf{D}_2$ is a diagonal matrix,
$$\mathbf{D}_2 = diag\left(W_4^0, W_4^1\right) \tag{7.19}$$
$$= diag\,(1, W_4) \tag{7.20}$$

Then,

$$\mathbf{D}_2\mathbf{F}_2 = \begin{bmatrix}1 & 0\\0 & W_4^1\end{bmatrix}\begin{bmatrix}W_4^0 & W_4^0\\W_4^0 & W_4^2\end{bmatrix} \tag{7.21}$$

$$= \begin{bmatrix}W_4^0 & W_4^0\\W_4^1 & W_4^3\end{bmatrix} \tag{7.22}$$

And for $k \in \mathcal{N}$ and N be a even integer we know that,

$$W_N^{Nk} = 1 \tag{7.23}$$

$$W_N^{Nk+N/2} = -1 \tag{7.24}$$

Using that we can write,

$$-\mathbf{D}_2\mathbf{F}_2 = \begin{bmatrix}W_4^2 & W_4^6\\W_4^3 & W_4^9\end{bmatrix} \tag{7.25}$$

And from (7.3),

$$\mathbf{F}_4 = \begin{bmatrix}W_4^0 & W_4^0 & W_4^0 & W_4^0\\W_4^0 & W_4^1 & W_4^2 & W_4^3\\W_4^0 & W_4^2 & W_4^4 & W_4^6\\W_4^0 & W_4^3 & W_4^6 & W_4^9\end{bmatrix} \tag{7.26}$$

And

$$\mathbf{F}_4\mathbf{P}_4 = \begin{bmatrix}W_4^0 & W_4^0 & W_4^0 & W_4^0\\W_4^0 & W_4^2 & W_4^1 & W_4^3\\W_4^0 & W_4^4 & W_4^2 & W_4^6\\W_4^0 & W_4^6 & W_4^3 & W_4^9\end{bmatrix} \tag{7.27}$$

This is same as,

$$\begin{bmatrix}\mathbf{F}_2 & \mathbf{D}_2\mathbf{F}_2\\\mathbf{F}_2 & -\mathbf{D}_2\mathbf{F}_2\end{bmatrix} \tag{7.28}$$

$$\implies \begin{bmatrix}\mathbf{I}_2 & \mathbf{D}_2\\\mathbf{I}_2 & -\mathbf{D}_2\end{bmatrix}\begin{bmatrix}\mathbf{F}_2 & 0\\0 & \mathbf{F}_2\end{bmatrix} \tag{7.29}$$

Hence proved.

7. Show that

$$\mathbf{F}_N = \begin{bmatrix}\mathbf{I}_{N/2} & \mathbf{D}_{N/2}\\\mathbf{I}_{N/2} & -\mathbf{D}_{N/2}\end{bmatrix}\begin{bmatrix}\mathbf{F}_{N/2} & 0\\0 & \mathbf{F}_{N/2}\end{bmatrix}\mathbf{P}_N \tag{7.30}$$

Solution: As we saw earlier, it is similar to prove that

$$\mathbf{F}_N\mathbf{P}_N = \begin{bmatrix}\mathbf{I}_{N/2} & \mathbf{D}_{N/2}\\\mathbf{I}_{N/2} & -\mathbf{D}_{N/2}\end{bmatrix}\begin{bmatrix}\mathbf{F}_{N/2} & 0\\0 & \mathbf{F}_{N/2}\end{bmatrix} \tag{7.31}$$

Assuming that N is even, consider LHS

$$\mathbf{F}_N\mathbf{P}_N = \begin{bmatrix}W_N^{0\times0} & W_N^{0\times2} & .. & W_N^{0\times1} & W_N^{0\times3}..\\W_N^{1\times0} & W_N^{1\times2} & .. & W_N^{1\times1} & W_N^{1\times3}..\\ & & .. & & \\ & & .. & & \\W_N^{N/2\times0} & W_N^{N/2\times2} & .. & W_N^{N/2\times1} & W_N^{N/2\times3}..\\ & & .. & & \\ & & .. & & \\W_N^{N-1\times0} & W_N^{N-1\times2} & .. & W_N^{N-1\times1} & W_N^{N-1\times3}..\end{bmatrix} \tag{7.32}$$

On multiplying with $\mathbf{P}_N$ (permutation matrix), the odd-numbered columns of $\mathbf{F}_N$ shifted towards left.

Now we can divide the above matrix (7.32), into four sub-matrices as,

$$= \begin{bmatrix}\left[W_N^{n\times2m}\right] & \left[W_N^{n\times(2m+1)}\right]\\\left[W_N^{(n+\frac{N}{2})\times(2m)}\right] & \left[W_N^{(n+\frac{N}{2})\times(2m+1)}\right]\end{bmatrix} \tag{7.33}$$

where, $0 \le n, m \le \dfrac{N}{2} - 1$

$$= \begin{bmatrix}\left[\left(W_N^{n\times m}\right)^2\right] & \left[W_N^n\left(W_N^{n\times m}\right)^2\right]\\\left[W_N^{Nm}\left(W_N^{n\times m}\right)^2\right] & \left[W_N^{Nm+N/2}W_N^n\left(W_N^{n\times m}\right)^2\right]\end{bmatrix} \tag{7.34}$$

Using (7.23), (7.24) and (7.11)

$$= \begin{bmatrix}\left[W_{\frac{N}{2}}^{n\times m}\right] & \left[W_N^n W_{\frac{N}{2}}^{n\times m}\right]\\\left[W_{\frac{N}{2}}^{n\times m}\right] & \left[-W_N^n W_{\frac{N}{2}}^{n\times m}\right]\end{bmatrix} \tag{7.35}$$

Now from def (7.3) and (7.6), we can write,

$$= \begin{bmatrix}\mathbf{F}_{\frac{N}{2}} & \mathbf{D}_{\frac{N}{2}}\mathbf{F}_{\frac{N}{2}}\\\mathbf{F}_{\frac{N}{2}} & -\mathbf{D}_{\frac{N}{2}}\mathbf{F}_{\frac{N}{2}}\end{bmatrix} \tag{7.36}$$

$$\implies \mathbf{F}_N\mathbf{P}_N = \begin{bmatrix}\mathbf{I}_{N/2} & \mathbf{D}_{N/2}\\\mathbf{I}_{N/2} & -\mathbf{D}_{N/2}\end{bmatrix}\begin{bmatrix}\mathbf{F}_{N/2} & 0\\0 & \mathbf{F}_{N/2}\end{bmatrix} \tag{7.37}$$

Hence proved.

Note : If we want to do the above matrix decomposition recursively the value of N should in the form of 2^k.

8. Find

$$\mathbf{P}_4\mathbf{x} \tag{7.38}$$

Solution: Let $\mathbf{x}$,

$$\mathbf{x} = \begin{bmatrix}x(0)\\x(1)\\x(2)\\x(3)\end{bmatrix} \tag{7.39}$$

and $\mathbf{P}_4$ is 4 - point permutation matrix.
So,

$$\mathbf{P}_4\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} \tag{7.40}$$

$$= \begin{bmatrix} x(0) \\ x(2) \\ x(1) \\ x(3) \end{bmatrix} \tag{7.41}$$

9. Show that

$$\mathbf{X} = \mathbf{F}_N\mathbf{x} \tag{7.42}$$

where $\mathbf{x}, \mathbf{X}$ are the vector representations of $x(n), X(k)$ respectively.
Solution: From (7.1),

$$X(k) = \sum_{n=0}^{N-1} x(n)W^{kn} \tag{7.43}$$

Now we will try to convert the above expression into matrix equations,

$$X(0) = \sum_{n=0}^{N-1} x(n)W^{0.n} \tag{7.44}$$

$$= \begin{pmatrix} W^{0.0} \\ W^{0.1} \\ W^{0.2} \\ W^{0.(N-1)} \end{pmatrix}^T \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(N-1) \end{pmatrix} \tag{7.45}$$

$$X(1) = \begin{pmatrix} W^{1.0} \\ W^{1.1} \\ W^{1.2} \\ W^{1.(N-1)} \end{pmatrix}^T \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(N-1) \end{pmatrix} \tag{7.46}$$

.
.

$$X(N-1) = \begin{pmatrix} W^{(N-1)\times 0} \\ W^{(N-1)\times 1} \\ W^{(N-1)\times 2} \\ W^{(N-1)\times(N-1)} \end{pmatrix}^T \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(N-1) \end{pmatrix} \tag{7.47}$$

$\mathbf{X} =$

$$\begin{bmatrix} W_N^{0\times 0} & W_N^{0\times 1} & .. & W_N^{0\times N-1} \\ .. & .. & .. & .. \\ W_N^{N-1\times 0} & W_N^{N-1\times 1} & .. & W_N^{N-1\times N-1} \end{bmatrix} \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(N-1) \end{pmatrix} \tag{7.48}$$

From def (7.3),

$$\mathbf{X} = \mathbf{F}_N\mathbf{x} \tag{7.49}$$

Hence proved.

10. Derive the following Step-by-step visualisation of 8-point FFTs into 4-point FFTs and so on

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} + \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.50}$$

$$\begin{bmatrix} X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} - \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.51}$$

4-point FFTs into 2-point FFTs

$$\begin{bmatrix} X_1(0) \\ X_1(1) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.52}$$

$$\begin{bmatrix} X_1(2) \\ X_1(3) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.53}$$

$$\begin{bmatrix} X_2(0) \\ X_2(1) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.54}$$

$$\begin{bmatrix} X_2(2) \\ X_2(3) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.55}$$

$$P_8 \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \\ x(5) \\ x(6) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \\ x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} \tag{7.56}$$

$$P_4 \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(4) \\ x(2) \\ x(6) \end{bmatrix} \tag{7.57}$$

$$P_4 \begin{bmatrix} x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(1) \\ x(5) \\ x(3) \\ x(7) \end{bmatrix} \tag{7.58}$$

Therefore,

$$\begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} = F_2 \begin{bmatrix} x(0) \\ x(4) \end{bmatrix} \tag{7.59}$$

$$\begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} = F_2 \begin{bmatrix} x(2) \\ x(6) \end{bmatrix} \tag{7.60}$$

$$\begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} = F_2 \begin{bmatrix} x(1) \\ x(5) \end{bmatrix} \tag{7.61}$$

$$\begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} = F_2 \begin{bmatrix} x(3) \\ x(7) \end{bmatrix} \tag{7.62}$$

Solution: The 8-point FFT can be expressed as,

$$X(k) = \sum_{0}^{7} x(n) e^{\frac{-2\pi kn}{8}} \tag{7.63}$$

$$= \sum_{0}^{3} x(2n) e^{\frac{-2\pi kn}{4}} + \sum_{1}^{3} e^{\frac{-2\pi k(2n+1)}{8}} \tag{7.64}$$

$$= \sum_{0}^{3} x(2n) e^{\frac{-2\pi kn}{4}} + e^{\frac{-2\pi k}{8}} \sum_{1}^{3} x(2n) e^{\frac{-2\pi kn}{4}} \tag{7.65}$$

Call these 4 - point FFTs as X_1 and X_2,

$$X(k) = X_1(k) + W_8^k X_2(k) \tag{7.66}$$

Now consider,

$$X(k+4) = X_1(k+4) + W_8^{k+4} X_2(k+4) \tag{7.67}$$

$$= X_1(k) - W_8^k X_2(k) \tag{7.68}$$

Since the twiddle factors along with X_1 and X_2 are of 4-point $X_1(k+4) = X_1(k)$ and $X_2(k+4) = X_2(k)$.
With that (7.68) we can see how (7.50) and (7.51) are derived.
Now consider these 4-point FFTs,

$$X_1(k) = \sum_{0}^{1} x(4n) e^{\frac{-j2\pi nk}{2}} + e^{\frac{-j2\pi k}{4}} \sum_{0}^{1} x(4n+2) e^{\frac{-j2\pi nk}{2}} \tag{7.69}$$

$$= X_3(k) + W_4^k X_4(k) \tag{7.70}$$

where, $X_3(k)$ and $X_4(k)$ are 2-point FFTs of $x_1(n) = x_1(4n)$ and $x_2(n) = x(4n+2)$.
And you can see that,

$$X_1(k+2) = X_3(k) - W_4^k X_4(k) \tag{7.71}$$

With that we can see how we got (7.10) and (7.53).
And similarly we can write the 2-point FFTs from $X_2(k)$ as $X_5(k)$ and $X_6(k)$ of subsequences $x(4n+1)$ and $x(4n+3)$.
With that we can get (7.54) and (7.55).
Mathematically we can write these 2-point FFTs as,

$$\begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} = F_2 \begin{bmatrix} x(0) \\ x(4) \end{bmatrix} \tag{7.72}$$

$$\begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} = F_2 \begin{bmatrix} x(2) \\ x(6) \end{bmatrix} \tag{7.73}$$

$$\begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} = F_2 \begin{bmatrix} x(1) \\ x(5) \end{bmatrix} \tag{7.74}$$

$$\begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} = F_2 \begin{bmatrix} x(3) \\ x(7) \end{bmatrix} \tag{7.75}$$

where, the subsequences required for each 2-point FFT can be obtained from (7.56) , (7.57) and (7.58).

11. For

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \tag{7.76}$$

compute the DFT using (7.42)
Solution: Download the below python code,

```
wget https://github.com/kamujuaakash/
    EE3900/blob/main/Simulation/codes/
    X_k_dft.py
```

Then run the following command on terminal,

```
python3 X_k_dft.py
```

The plot of DFT can be seen in Fig 7.11

12. Repeat the above exercise using the FFT after zero padding **x**.
Solution: Download the below python code,

```
wget https://github.com/kamujuaakash/
    EE3900/blob/main/Simulation/codes/
    X_k_fft.py
```

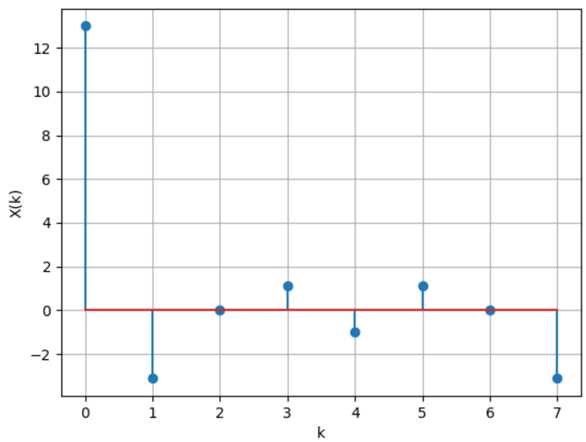

Fig. 7.11. DFT using DFT matrix

Then run the following command on terminal,

```
python3 X_k_fft.py
```

The plot of DFT can be seen in Fig 7.13

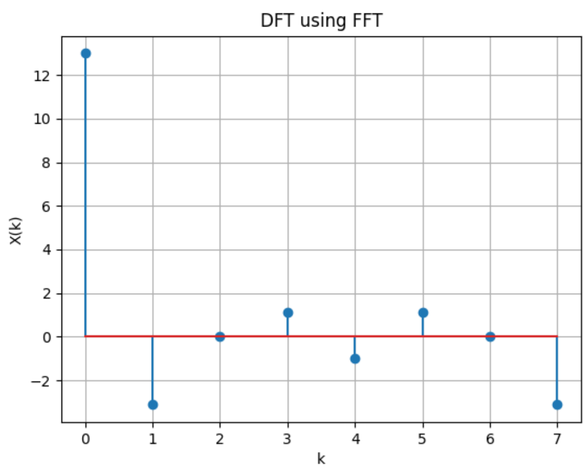

Fig. 7.12. FFT using Matrix decompostion

13. Write a C program to compute the 8-point FFT.
Solution: Download the C code from the following link

```
wget https://github.com/kamujuaakash/
    EE3900/blob/main/Simulation/codes/
    X_k_fft.c
```

Then run the following command,

```
cc X_k_fft.c −lm
./a.out
```

Download the below python code which uses fft.dat file from the C code.

```
wget https://github.com/kamujuaakash/
    EE3900/blob/main/Simulation/codes/
    X_k_8point.py
```

Then run the following command for the plot,

```
python3 X_k_8point.py
```

You will get output of DFT of $x(n)$.

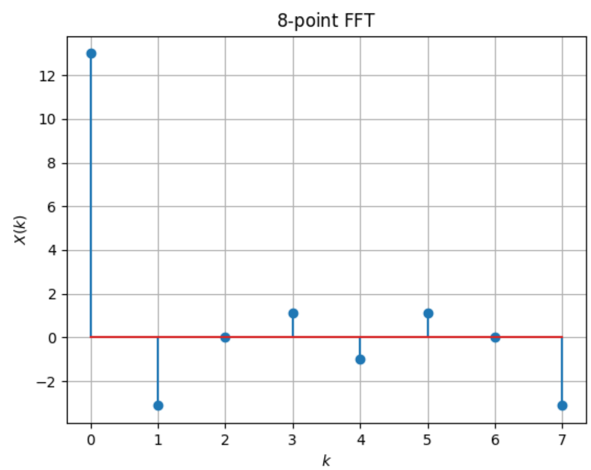

Fig. 7.13. FFT using C code

14. Compare FFT and Convolution for Input Signal.

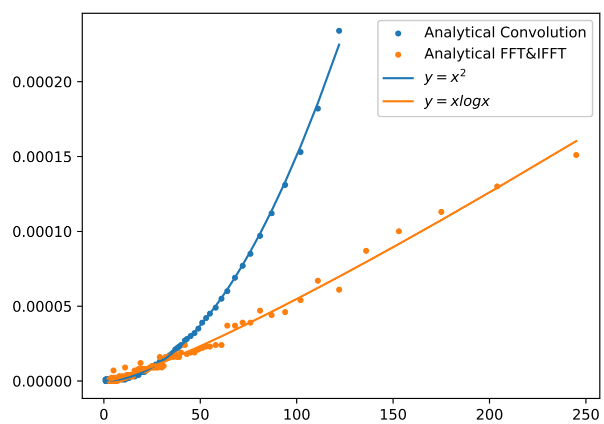

Fig. 7.14. Time Comparison

8. Exercises

Answer the following questions by looking at the python code in Problem 2.3

8.1 The command

```
output_signal = signal.lfilter(b, a,
    input_signal)
```

in Problem 2.3 is executed through the following difference equation

$$\sum_{m=0}^{M} a(m)\, y(n-m) = \sum_{k=0}^{N} b(k)\, x(n-k) \quad (8.1)$$

where the input signal is $x(n)$ and the output signal is $y(n)$ with initial values all 0. Replace **signal.filtfilt** with your own routine and verify.
Solution: On taking the Z-transform on both sides of the difference equation

$$\sum_{m=0}^{M} a(m)\, z^{-m} Y(z) = \sum_{k=0}^{N} b(k)\, z^{-k} X(z) \quad (8.2)$$

$$\implies H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{N} b(k)\, z^{-k}}{\sum_{m=0}^{M} a(m)\, z^{-m}} \quad (8.3)$$

For obtaining the discrete Fourier transform, put $z = \jmath\frac{2\pi i}{I}$ where I is the length of the input signal and $i = 0, 1, \dots, I-1$
Download the following Python code that does the above

```
wget https://github.com/kamujuaakash/
    EE3900/blob/main/Simulation/codes/8.1.
    py
```

8.2 Repeat all the exercises in the previous sections for the above a and b
Solution: The polynomial coefficients obtained are

$$\mathbf{a} = \begin{pmatrix} 1.000 \\ -2.519 \\ 2.561 \\ -1.206 \\ 0.220 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 0.003 \\ 0.014 \\ 0.021 \\ 0.014 \\ 0.003 \end{pmatrix} \quad (8.4)$$

The difference equation is then given by

$$\mathbf{a}^\top \mathbf{y} = \mathbf{b}^\top \mathbf{x} \quad (8.5)$$

where

$$\mathbf{y} = \begin{pmatrix} y(n) \\ y(n-1) \\ y(n-2) \\ y(n-3) \\ y(n-4) \end{pmatrix} \qquad \mathbf{x} = \begin{pmatrix} x(n) \\ x(n-1) \\ x(n-2) \\ x(n-3) \\ x(n-4) \end{pmatrix} \quad (8.6)$$

Download the following Python code

```
wget https://github.com/kamujuaakash/
    EE3900/blob/main/Simulation/codes/8.2.
    py
```

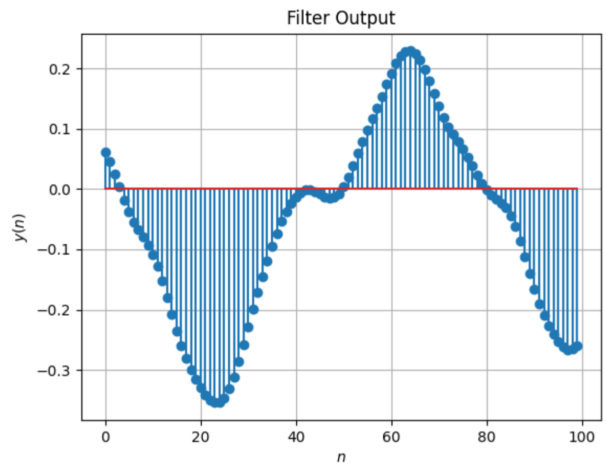

Fig. 8.2. Plot of $y(n)$

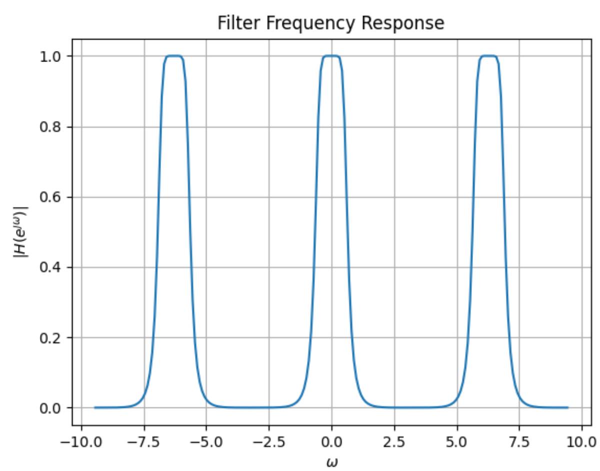

Fig. 8.2. Plot of $|H(e^{\jmath\omega})|$

8.3 What is the sampling frequency of the input signal?
Solution: The sampling frequency of the input signal is 44 100 Hz = 44.1 kHz

8.4 What is the type, order and cutoff frequency of the above Butterworth filter?
Solution:
Type: low-pass
Order: 4
Cutoff frequency: 4000 Hz = 4 kHz

8.5 Modify the code with different input parameters to get the best possible output.

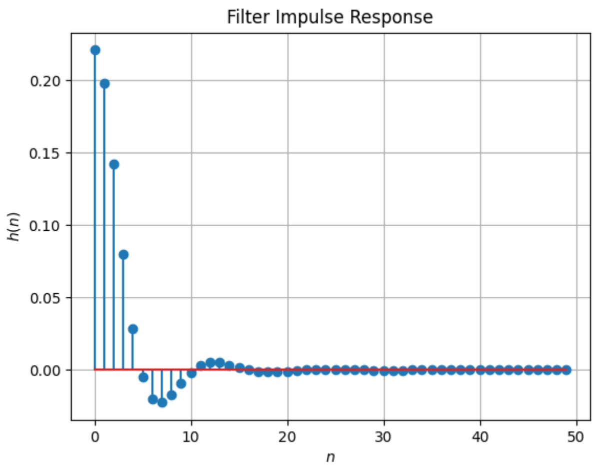

Fig. 8.2. Plot of $h(n)$

Solution:
Order: 10
Cutoff frequency: 3000 Hz = 3 kHz