Digital Signal Processing

Aakash Kamuju

Contents

***Abstract*—This manual provides a simple introduction to digital signal processing.**

1 Software Installation

Run the following commands

```
sudo apt−get update
sudo apt−get install libffi−dev libsndfile1 python3
    −scipy python3−numpy python3−matplotlib
sudo pip install cffi pysoundfile
```

2 Digital Filter

2.1 Download the sound file from

```
wget https://github.com/kamujuaakash/
    EE3900/blob/main/codes/Sound_Noise.
    wav
```

2.2 You will find a spectrogram at https://academo.org/demos/spectrum-analyzer. Upload the sound file that you downloaded in Problem 2.1 in the spectrogram and play. Observe the spectrogram. What do you find?
Solution: There are a lot of yellow lines between 440 Hz to 5.1 KHz. These represent the synthesizer key tones. Also, the key strokes are audible along with background noise.

2.3 Write the python code for removal of out of band noise and execute the code.
Solution:

```
import soundfile as sf
from scipy import signal

#read .wav file
input_signal,fs = sf.read('codes/
    Sound_Noise.wav')

#sampling frequency of Input signal
sampl_freq=fs

#order of the filter
order=4

#cutoff frquency 4kHz
cutoff_freq=4000.0

#digital frequency
Wn=2*cutoff_freq/sampl_freq

# b and a are numerator and denominator
    polynomials respectively
b, a = signal.butter(order,Wn, 'low')

#filter the input signal with butterworth filter
output_signal = signal.filtfilt(b, a,
    input_signal)
#output_signal = signal.lfilter(b, a,
    input_signal)

#write the output signal into .wav file
sf.write('codes/Sound_With_ReducedNoise.
    wav', output_signal, fs)
```

2.4 The output of the python script in Problem 2.3 is the audio file Sound_With_ReducedNoise.wav. Play the file in the spectrogram in Problem 2.2. What do you observe?
Solution: The key strokes as well as background noise is subdued in the audio.

Also, the signal is blank for frequencies above 5.1 kHz.

3 Difference Equation

3.1 Let

$$x(n) = \left\{\underset{\uparrow}{1}, 2, 3, 4, 2, 1\right\} \tag{3.1}$$

Sketch $x(n)$.
Solution: The following code yields Fig. 3.1

```
wget https://github.com/kamujuaakash/
    EE3900/blob/main/codes/3.1.py
```

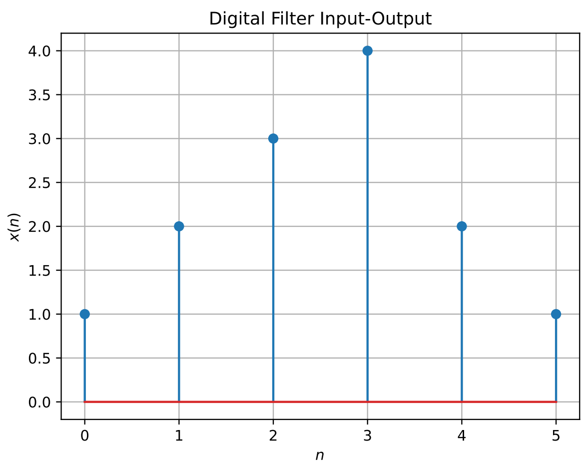

Fig. 3.1

3.2 Let

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2), \quad y(n) = 0, n < 0 \tag{3.2}$$

Sketch $y(n)$.
Solution: The following code yields Fig. 5.3.

```
wget https://github.com/kamujuaakash/
    EE3900/blob/main/codes/xnyn.py
```

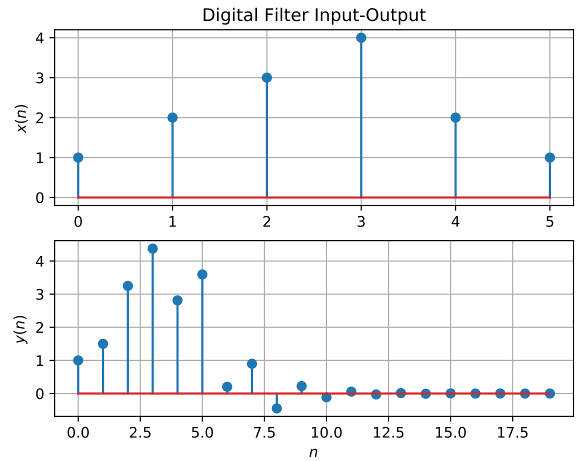

Fig. 3.2

4 Z-transform

4.1 The Z-transform of $x(n)$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.1}$$

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \tag{4.2}$$

and find

$$\mathcal{Z}\{x(n-k)\} \tag{4.3}$$

Solution: From (4.1),

$$\mathcal{Z}\{x(n-k)\} = \sum_{n=-\infty}^{\infty} x(n-1)z^{-n} \tag{4.4}$$

$$= \sum_{n=-\infty}^{\infty} x(n)z^{-n-1} = z^{-1}\sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.5}$$

resulting in (4.2). Similarly, it can be shown that

$$\mathcal{Z}\{x(n-k)\} = z^{-k}X(z) \tag{4.6}$$

4.2 Obtain $X(z)$ for x(n) define in problem 3.1. **Solution:** From (4.1),

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.7}$$

$$= \sum_{n=0}^{\infty} x(n)z^{-n} \tag{4.8}$$

$$= 1\cdot z^{-1} + 2\cdot z^{-2} + 3\cdot z^{-3} + 4\cdot z^{-4} + 2\cdot z^{-5} + 1\cdot z^{-6} \tag{4.9}$$

$$= \frac{z^5 + 2z^4 + 3z^3 + +4z^2 + 2z + 1}{z^6} \tag{4.10}$$

4.3 Find

$$H(z) = \frac{Y(z)}{X(z)} \tag{4.11}$$

from (3.2) assuming that the Z-transform is a linear operation.

Solution: Applying (4.6) in (3.2),

$$Y(z)+\frac{1}{2}z^{-1}Y(z)=X(z)+z^{-2}X(z) \quad (4.12)$$

$$\implies H(z)=\frac{Y(z)}{X(z)}=\frac{1+z^{-2}}{1+\frac{1}{2}z^{-1}} \quad (4.13)$$

4.4 Find the Z transform of

$$\delta(n)=\begin{cases}1 & n=0\\ 0 & \text{otherwise}\end{cases} \quad (4.14)$$

and show that the Z-transform of

$$u(n)=\begin{cases}1 & n\geq 0\\ 0 & \text{otherwise}\end{cases} \quad (4.15)$$

is

$$U(z)=\frac{1}{1-z^{-1}}, \quad |z|>1 \quad (4.16)$$

Solution: It is easy to show that

$$\delta(n)\overset{\mathcal{Z}}{\rightleftharpoons}1 \quad (4.17)$$

and from (4.15),

$$U(z)=\sum_{n=0}^{\infty}z^{-n} \quad (4.18)$$

$$=\frac{1}{1-z^{-1}}, \quad |z|>1 \quad (4.19)$$

using the fomula for the sum of an infinite geometric progression.

4.5 Show that

$$a^n u(n)\overset{\mathcal{Z}}{\rightleftharpoons}\frac{1}{1-az^{-1}} \quad |z|>|a| \quad (4.20)$$

Solution: From (4.1),

$$\mathcal{Z}\{a^n u(n)\}=\sum_{n=-\infty}^{\infty}u(n)\left(\frac{a}{z}\right)^{-n} \quad (4.21)$$

From (4.15),

$$\mathcal{Z}\{a^n u(n)\}=\sum_{n=0}^{\infty}\left(\frac{a}{z}\right)^{-n} \quad (4.22)$$

$$=\frac{1}{1-\frac{a}{z}} \quad (4.23)$$

$$=\frac{1}{1-az^{-1}} \quad |z|>|a| \quad (4.24)$$

using the formula for the sum of an infinite geometric progression.

4.6 Let

$$H\left(e^{\jmath\omega}\right)=H\left(z=e^{\jmath\omega}\right). \quad (4.25)$$

Plot $|H(e^{\jmath\omega})|$. Is it periodic? If so,find the period. $H(e^{\jmath\omega})$ is known as the *Discret Time Fourier Transform* (DTFT) of $x(n)$.

Solution:

$$H\left(e^{\jmath\omega}\right)=H\left(z=e^{\jmath\omega}\right) \quad (4.26)$$

$$=\frac{1+e^{-2\jmath\omega}}{1+\frac{1}{2}e^{-\jmath\omega}} \quad (4.27)$$

$$=\frac{2\left(1+e^{-2\jmath\omega}\right)}{2+e^{-\jmath\omega}} \quad (4.28)$$

$$=\frac{2\left(1+\cos 2\omega-\jmath\sin 2\omega\right)}{2+\cos\omega-\jmath\sin\omega} \quad (4.29)$$

$$=\frac{2\left|\left(2\cos^2\omega-\jmath\sin 2\omega\right)\right|}{\left|2+\cos\omega-\jmath\sin\omega\right|} \quad (4.30)$$

$$=\frac{2\sqrt{4\cos^4\omega+4\sin^2\omega\cos^2\omega}}{\sqrt{(2+\cos\omega)^2+\sin^2\omega}} \quad (4.31)$$

$$=\frac{4\cos\omega}{\sqrt{5+4\cos\omega}} \quad (4.32)$$

$$\left|H\left(e^{\jmath\omega}\right)\right|=\frac{|4\cos\omega|}{\sqrt{5+4\cos\omega}} \quad (4.33)$$

Since $\left|H(e^{j\omega})\right|$ is function of cosine we can say it is periodic.And from the plot 4.6 we can say that it is symmetric about $\omega = 0$ (even function) and it is periodic with period 2π.You can find the same from the theoritical expression $\left|H\left(e^{j\omega}\right)\right|$,

$$H(e^{\jmath\omega})=H(e^{j(-\omega)})\,(\text{cos is an even function}) \quad (4.34)$$

And to find period, the period of $|\cos(\omega)|$ is π and the period of $\sqrt{5+4\cos(\omega)}$ is 2π. So the period of division of both will be 2π. This gives us the period of $\left|H(e^{j\omega})\right|$ as 2π The following code plots Fig. 4.6.

```
wget https://github.com/kamujuaakash/
    EE3900/blob/main/codes/hn.py
```

4.7 Express $h(n)$ in terms of $H\left(e^{\jmath\omega}\right)$

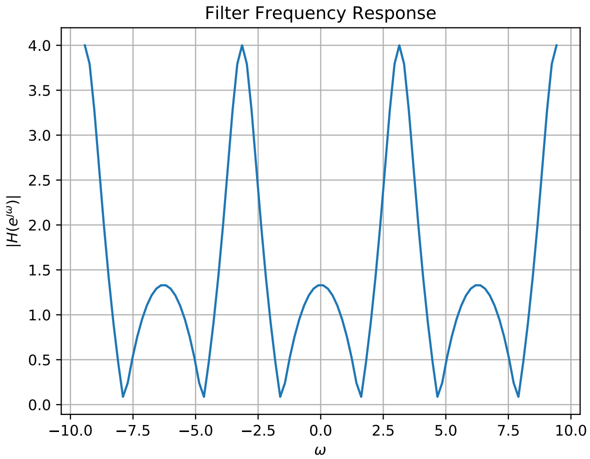

Fig. 4.6: $|H(e^{\jmath\omega})|$

Solution:

$$\int_{-\pi}^{\pi} H(e^{\jmath\omega})e^{\jmath\omega n}\omega \tag{4.35}$$

$$= \int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} h(k)e^{-\jmath\omega k}e^{\jmath\omega n}\omega \tag{4.36}$$

$$= \sum_{k=-\infty}^{\infty} h(k) \int_{-\pi}^{\pi} e^{\jmath\omega(n-k)}\omega \tag{4.37}$$

Now,

$$\int_{-\pi}^{\pi} e^{\jmath\omega(n-k)}\omega = \begin{cases} \int_{-\pi}^{\pi}\omega & n-k=0 \\ \left.\frac{\exp\left(\jmath\omega(n-k)\right)}{\jmath(n-k)}\right|_{-\pi}^{\pi} & n-k\neq 0 \end{cases} \tag{4.38}$$

$$= \begin{cases} 2\pi & n-k=0 \\ 0 & n-k\neq 0 \end{cases} \tag{4.39}$$

$$= 2\pi\delta(n-k) \tag{4.40}$$

Thus,

$$\int_{-\pi}^{\pi} H(e^{\jmath\omega})e^{\jmath\omega n}\omega = 2\pi\sum_{k=-\infty}^{\infty} h(k)\delta(n-k) \tag{4.41}$$

$$= 2\pi h(n) * \delta(n) \tag{4.42}$$

$$= 2\pi h(n) \tag{4.43}$$

Therefore, $h(n)$ is given by the inverse DTFT (IDTFT) of $H(e^{\jmath\omega})$

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{\jmath\omega})e^{\jmath\omega n}\omega \tag{4.44}$$

5 Impulse Response

5.1 Using long division, find

$$h(n), \quad n < 5 \tag{5.1}$$

for H(z) in (4.13). **Solution:** Using (4.13),

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1+z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{5.2}$$

$$= \frac{1}{1+\frac{1}{2}z^{-1}} + \frac{z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{5.3}$$

$$\tag{5.4}$$

Using the Infinite geometric progression,

$$1-\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}-\frac{1}{8}z^{-3}+\frac{1}{16}z^{-4}-\frac{1}{32}z^{-5}+\dots = \frac{1}{1+\frac{1}{2}z^{-1}} \tag{5.5}$$

Parallely,

$$\frac{z^{-2}}{1+\frac{1}{2}z^{-1}} = z^{-2}-\frac{1}{2}z^{-3}+\frac{1}{4}z^{-4}-\frac{1}{8}z^{-5}+\dots \tag{5.6}$$

Considering (5.5) and (5.6),

$$H(z) = 1-\frac{1}{2}z^{-1}+\frac{5}{4}z^{-2}-\frac{5}{8}z^{-3}+\frac{5}{16}z^{-4}-\frac{5}{32}z^{-5}+\dots \tag{5.7}$$

Comparing (5.7) with (4.8),

$$h(0) = 1 \tag{5.8}$$

$$h(1) = -\frac{1}{2} \tag{5.9}$$

$$h(2) = \frac{5}{4} \tag{5.10}$$

$$h(3) = -\frac{5}{8} \tag{5.11}$$

$$h(4) = \frac{5}{16} \tag{5.12}$$

$$h(5) = -\frac{5}{32} \tag{5.13}$$

5.2 Find an expression for $h(n)$ using $H(z)$, given that

$$h(n) \overset{\mathcal{Z}}{\rightleftharpoons} H(z) \tag{5.14}$$

and there is a one to one relationship between $h(n)$ and $H(z)$. $h(n)$ is known as the *impulse response* of the system defined by (3.2).

Solution: From (4.13),

$$H(z) = \frac{1}{1+\frac{1}{2}z^{-1}} + \frac{z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{5.15}$$

$$\implies h(n) = \left(-\frac{1}{2}\right)^{n} u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.16}$$

using (4.20) and (4.6).

5.3 Sketch $h(n)$. Is it bounded? Justify theoretically.

Solution:

The following code plots Fig. 5.3.

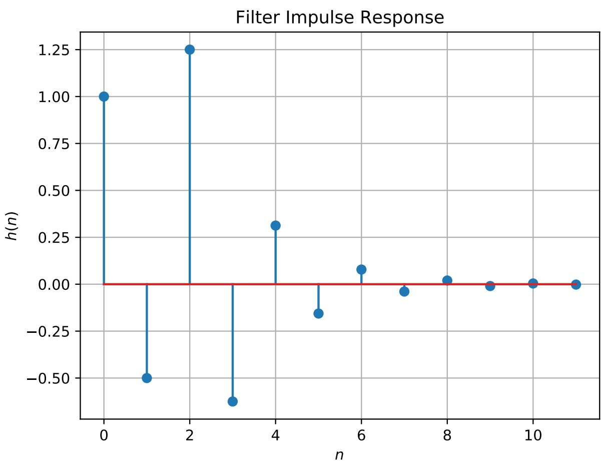

Fig. 5.3

```
wget https://github.com/kamujuaakash/
    EE3900/blob/main/codes/5.3.py
```

$$\lim_{n\to\infty} h(n) = \lim_{n\to\infty}\left(-\frac{1}{2}\right)^{n} u(n) + \lim_{n\to\infty}\left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.17}$$

$$h(n) = \begin{cases} 5\left(\frac{-1}{2}\right)^{n} & n \geq 2 \\ \left(\frac{-1}{2}\right)^{n} & 0 \leq n < 2 \\ 0 & n < 0 \end{cases} \tag{5.18}$$

Maximum value and minimum value are always bounded in this case. $\therefore$ $h(n)$ is bounded

5.4 Is it convergent? Justify using the ratio test.

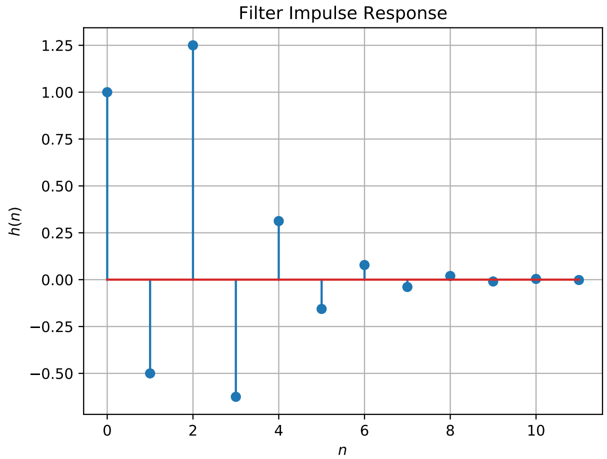

Fig. 5.3: $h(n)$ as the inverse of $H(z)$

Solution:

$$h(n+1) = \left(-\frac{1}{2}\right)^{n+1} u(n+1) + \left(-\frac{1}{2}\right)^{n-1} u(n-1) \tag{5.19}$$

According to ratio test, L is given by $\lim_{n\to\infty}\left|\frac{h(n+1)}{h(n)}\right|$, if $L < 1$ then h(n) is convergent.

$$L = \lim_{n\to\infty}\left|\frac{h(n+1)}{h(n)}\right| \tag{5.20}$$

$$= \lim_{n\to\infty}\left|\frac{\left(-\frac{1}{2}\right)^{n+1} u(n+1) + \left(-\frac{1}{2}\right)^{n-1} u(n-1)}{\left(-\frac{1}{2}\right)^{n} u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2)}\right| \tag{5.21}$$

$$= \left|\frac{-\frac{1}{2} + -\frac{1}{2}^{-1}}{1 + -\frac{1}{2}^{-}2}\right| \tag{5.22}$$

$$= \left|\frac{-\frac{1}{2} - 2}{1+4}\right| \tag{5.23}$$

$$= \left|\frac{\frac{-5}{2}}{5}\right| \tag{5.24}$$

$$= \frac{1}{2} \tag{5.25}$$

As $L < 1$, $h(n)$ is convergent.

5.5 Verify the above result using a python code.

Solution: The following code determines if it is convergent or not:

```
wget https://github.com/kamujuaakash/
    EE3900/blob/main/codes/5.5.py
```

5.6 The system with $h(n)$ is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \tag{5.26}$$

Is the system defined by (3.2) stable for the impulse response in (5.14)?
Solution:

$$\sum_{n=-\infty}^{\infty} h(n) = \sum_{n=0}^{\infty} h(n) \tag{5.27}$$

$$= \sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^{n} u(n) + \sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.28}$$

$$= 2\left(\frac{1}{1+\frac{1}{2}}\right) \tag{5.29}$$

$$= \frac{4}{3} \tag{5.30}$$

As $\sum_{n=-\infty}^{\infty} h(n) = \frac{4}{3}$ is less than ∞, the system defined by (3.2) is stable for the impulse response in (5.1).

5.7 Compute and sketch $h(n)$ using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2), \tag{5.31}$$

This is the definition of $h(n)$.
Solution: The following code plots Fig. 5.7. Note that this is the same as Fig. 5.3.

```
wget https://github.com/kamujuaakash/
    EE3900/blob/main/codes/hndef.py
```

Computing,
$h(0) = 1$
$h(1) = -\frac{1}{2}h(0)$
$h(2) = -\frac{1}{2}h(1) + 1$
Parallely, $h(n) = -\frac{1}{2}h(n-1)$

5.8 Compute

$$y(n) = x(n) * h(n) = \sum_{n=-\infty}^{\infty} x(k)h(n-k) \tag{5.32}$$

Comment. The operation in (5.32) is known as *convolution*.
Solution: The following code plots Fig. 5.8. Note that this is the same as $y(n)$ in Fig. 5.3.

```
wget https://github.com/kamujuaakash/
    EE3900/blob/main/codes/ynconv.py
```

5.9 Express the above convolution using a toeplitz matrix. **Solution:**

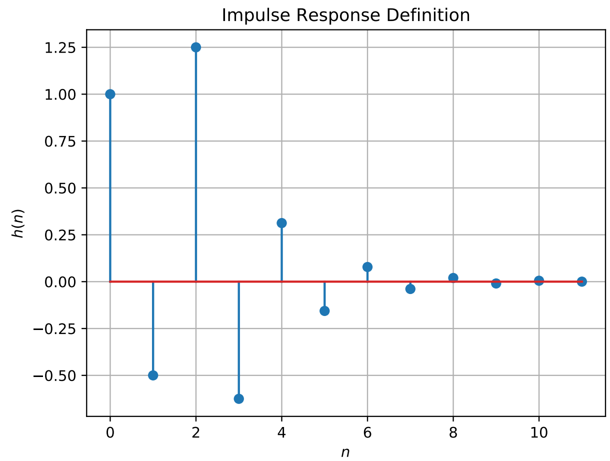

Fig. 5.7: $h(n)$ from the definition

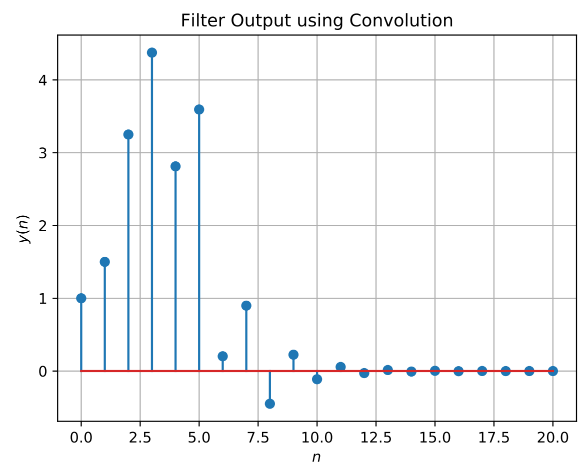

Fig. 5.8: $y(n)$ from the definition of convolution

5.10 Show that

$$y(n) = \sum_{n=-\infty}^{\infty} x(n-k)h(k) \tag{5.33}$$

Solution: Substitute $k \to n-k$ then

$$y(n) = x(n) * h(n) \tag{5.34}$$

$$= \sum_{n=-\infty}^{\infty} x(k)h(n-k) \quad = \sum_{n=-\infty}^{\infty} x(n-k)h(k) \tag{5.35}$$

6 DFT and FFT

6.1 Compute

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n) e^{-\jmath 2\pi kn/N}, \quad k = 0, 1, \ldots, N-1 \tag{6.1}$$

and $H(k)$ using $h(n)$. **Solution:** The following code plots Fig. 6.1.

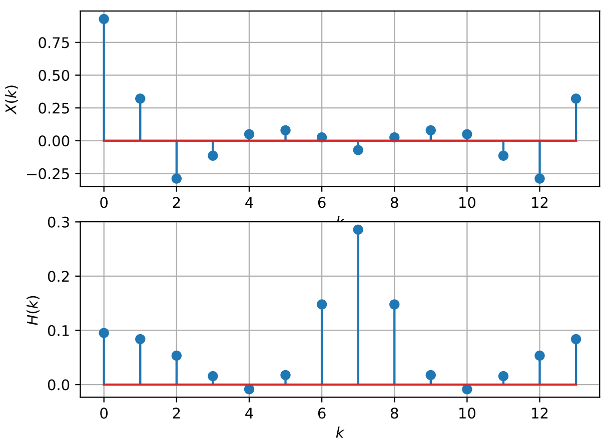

Fig. 6.1: $X(k), H(k)$ from the DFT

```
wget https://github.com/kamujuaakash/
    EE3900/blob/main/codes/xkhkdft.py
```

6.2 Compute

$$Y(k) = X(k)H(k) \tag{6.2}$$

Solution: The following code plots Fig. 6.2.

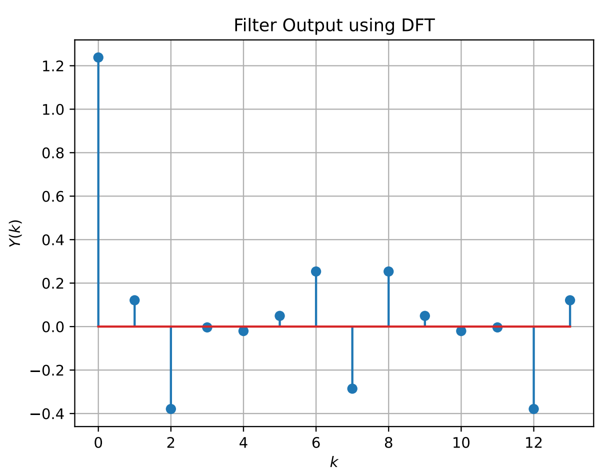

Fig. 6.2: $Y(k)$ from the DFT

```
wget https://github.com/kamujuaakash/
    EE3900/blob/main/codes/ykdft.py
```

6.3 Compute

$$y(n) = \frac{1}{N} \sum_{k=0}^{N-1} Y(k) \cdot e^{\jmath 2\pi kn/N}, \quad n = 0, 1, \ldots, N-1 \tag{6.3}$$

Solution: The following code plots Fig. 5.8. Note that this is the same as $y(n)$ in Fig. 5.3.

```
wget https://github.com/kamujuaakash/
    EE3900/blob/main/codes/yndft.py
```

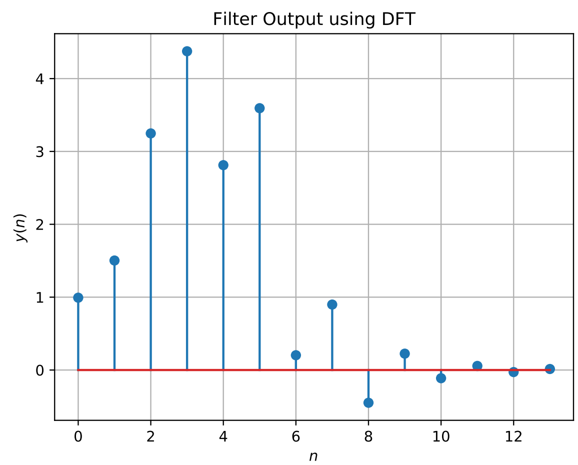

Fig. 6.3: $y(n)$ from the DFT

6.4 Repeat the previous exercise by computing $X(k), H(k)$ and $y(n)$ through FFT and IFFT.

6.5 Wherever possible, express all the above equations as matrix equations.

7 Exercises

Answer the following questions by looking at the python code in Problem 2.3.

7.1 The command

```
output_signal = signal.lfilter(b, a,
    input_signal)
```

in Problem 2.3 is executed through the following difference equation

$$\sum_{m=0}^{M} a(m)\, y(n-m) = \sum_{k=0}^{N} b(k)\, x(n-k) \tag{7.1}$$

where the input signal is $x(n)$ and the output signal is $y(n)$ with initial values all 0. Replace **signal.filtfilt** with your own routine and verify.

7.2 Repeat all the exercises in the previous sections for the above a and b.

7.3 What is the sampling frequency of the input signal?

Solution: Sampling frequency(fs)=44.1kHZ.

7.4 What is type, order and cutoff-frequency of the above butterworth filter

Solution: The given butterworth filter is low pass with order=2 and cutoff-frequency=4kHz.

7.5 Modifying the code with different input parameters and to get the best possible output.