Digital Signal Processing
EE3900

Fourier Series

I Sai Pradeep
AI21BTECH11013

October 27, 2022

Contents

***Abstract*—This manual provides a simple introduction to Fourier Series**

1 Periodic Function

Let

$$x(t) = A_0 \left|\sin\left(2\pi f_0 t\right)\right| \tag{1.1}$$

1.1 Plot $x(t)$.
Solution:

```
wget https://github.com/Pradeep8802/EE3900
    −Digital−Signal−Processing/blob/main/
    charger/codes/1.1.py
```

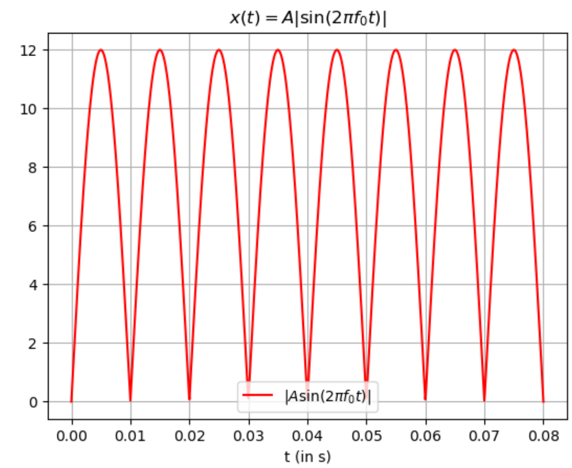

Fig. 0

1.2 Show that $x(t)$ is periodic and find its period.
Solution: A signal $x(t)$ is said to be periodic with fundamental period T if

$$x(t + nT) = x(t) \forall n \in \mathbb{Z} \tag{1.2}$$

Let T be fundamental period of $x(t)$. Comparing (1.2) and (1.1), we get

$$A_0 \left|\sin\left(2\pi f_0 t\right)\right| = A_0 \left|\sin\left(2\pi f_0 (t + T)\right)\right| \tag{1.3}$$
$$\left|\sin\left(2\pi f_0 t\right)\right| = \left|\sin\left(2\pi f_0 (t + T)\right)\right| \tag{1.4}$$
$$\left|\sin\left(2\pi f_0 t\right)\right| = \left|\sin\left(2\pi f_0 t + 2\pi f_0 T\right)\right| \tag{1.5}$$

As $|sin\theta|$ is periodic with fundamental period $F = \pi$, Hence,

$$\left|\sin\left(t\right)\right| = \left|\sin\left(t + F\right)\right| \tag{1.6}$$

Hence,$2\pi f_0 T = \pi$, therefore, fundamental period(T) is

$$T = \frac{\pi}{2\pi f_0} = \frac{1}{2f_0} \tag{1.7}$$

2 Fourier Series

Consider $A_0 = 12$ and $f_0 = 50$ for all numerical calculations.

2.1 If

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t} \tag{2.1}$$

show that

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-j2\pi k f_0 t} \, dt \tag{2.2}$$

Solution: From (2.1),

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t} \tag{2.3}$$

Mulitply $e^{-j2\pi l f_0 t}$ on both sides of (2.3), we get,

$$x(t)e^{-j2\pi l f_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t} e^{-j2\pi l f_0 t} \tag{2.4}$$

Integrating (2.4) w.r.t. t from $-T$ to T, and $T = \frac{1}{f_0}$, we get,

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t)e^{-j2\pi k f_0 t}\, dt = \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} \sum_{k=-\infty}^{\infty} c_k e^{j2\pi(k-l)f_0 t}\, dt \tag{2.5}$$

$$= \sum_{k=-\infty}^{\infty} c_k \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{j2\pi(k-l)f_0 t}\, dt \tag{2.6}$$

Consider the following cases.
case-1:$k = l$

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{j2\pi(k-l)f_0 t}\, dt = \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{0}\, dt \tag{2.7}$$

$$= \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} 1\, dt \tag{2.8}$$

case-2: $k \neq l$
Let $n = f_0(k-l)$, here $n \in \mathbb{Z}$

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{j2\pi(k-l)f_0 t}\, dt = \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{2n\pi}\, dt \tag{2.9}$$

Here, $2n\pi T = 2f_0(k-l)T\pi$, and $2n\pi T = (k-l)\pi$

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{j2n\pi}\, dt = \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} 1\, dt \tag{2.10}$$

$$= \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} \cos(2n\pi) + j\sin(2n\pi)\, dt \tag{2.11}$$

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{j2n\pi}\, dt = -\sin(2n\pi t)\Big|_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} \tag{2.12}$$

$$+ j\cos(2n\pi t)\Big|_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} \tag{2.13}$$

$$= -\sin(2n\pi t)\Big|_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} \tag{2.14}$$

$$+ j\cos(2n\pi t)\Big|_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} \tag{2.15}$$

$$= -\sin(2n\pi t)\Big|_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} \tag{2.16}$$

$$+ j\cos(2n\pi t)\Big|_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} \tag{2.17}$$

$$\tag{2.18}$$

$$= -\sin((k-l)\pi) + \sin(-(k-l)\pi) \tag{2.19}$$

$$+ j\cos((k-l)\pi) - j\cos(-(k-l)\pi) \tag{2.20}$$

$$\tag{2.21}$$

Since $k - l \in \mathbb{Z}$,$\sin((k-l)\pi) = 0$ and $\sin(-(k-l)\pi) = 0$, simillarly, as $\cos(\theta) = \cos(-\theta)$, we get $\cos((k-l)\pi) - \cos(-(k-l)\pi) = 0$
From (2.18),

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{j2n\pi}\, dt = \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} 1\, dt \tag{2.22}$$

$$= 0 + j0 = 0 \tag{2.23}$$

Hence, we have,

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{j2\pi(k-l)f_0 t}\, dt = \begin{cases} 0 & k \neq l \\ \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} 1\, dt & k = l \end{cases} \tag{2.24}$$

From (2.5),

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t)e^{-j2\pi k f_0 t}\, dt = \sum_{k=-\infty}^{\infty} c_k \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{j2\pi(k-l)f_0 t}\, dt \tag{2.25}$$

$$= c_k \times \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} 1\, dt \tag{2.26}$$

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-j2\pi k f_0 t}\, dt \tag{2.27}$$

$$\therefore c_k = \frac{2}{T} \int_{-\frac{1}{T}}^{\frac{1}{T}} x(t) e^{-j2\pi k f_0 t}\, dt \tag{2.28}$$

2.2 Find c_k for (1.1)
Solution: We know that,

$$c_k = 2f_0 \int_{0}^{\frac{1}{2f_0}} x(t) e^{-j2\pi k f_0 t}\, dt \tag{2.29}$$

when $t \in \left(0, \frac{1}{2f_0}\right)$, $x(t) = A_0 \sin(2\pi f_0 t)$

$$c_k = 2f_0 \int_{0}^{\frac{1}{2f_0}} A_0 \left(\frac{e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}}{2j}\right) e^{-j2\pi k f_0 t}\, dt \tag{2.30}$$

$$= A_0 f_0 \int_{0}^{\frac{1}{2f_0}} \left(\frac{e^{j2\pi(1-k)f_0 t} - e^{j2\pi(-1-k)f_0 t}}{j}\right) dt \tag{2.31}$$

$$= A_0 f_0 \Bigg(\frac{e^{j2\pi(1-k)f_0 t}}{-2\pi(1-k) f_0}\Bigg|_0^{\frac{1}{2f_0}} \tag{2.32}$$

$$- \frac{e^{j2\pi(-1-k)f_0 t}}{-2\pi(-1-k) f_0}\Bigg|_0^{\frac{1}{2f_0}} \Bigg) \tag{2.33}$$

$$= A_0 \left[\frac{e^{j\pi(1-k)} - 1}{2\pi(k-1)} - \frac{e^{-j\pi(1+k)} - 1}{2\pi(k+1)}\right] \tag{2.34}$$

Hence,

$$c_k = \begin{cases} \frac{2A_0}{\pi(1-k^2)} & k = even \\ 0 & k = odd \end{cases} \tag{2.35}$$

2.3 Verify (1.1) using python.
Solution:

```
wget https://github.com/Pradeep8802/EE3900
    −Digital−Signal−Processing/blob/main/
    charger/codes/2.3.py
        python3 2.3.py
```

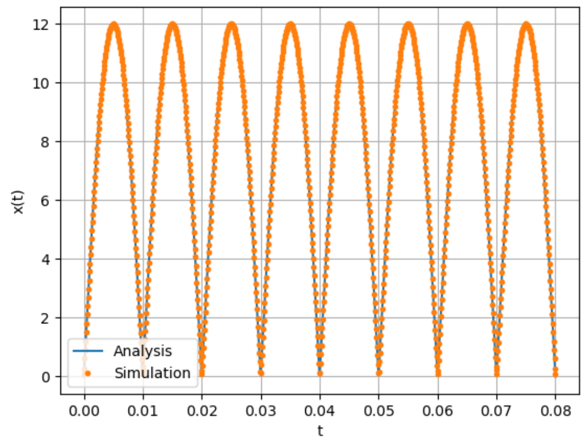

Fig. 0

2.4 Show that

$$x(t) = \sum_{k=0}^{\infty} \left(a_k \cos 2\pi k f_0 t + b_k \sin 2\pi k f_0 t\right) \tag{2.36}$$

and obtain the formulae for a_k and b_k.
Solution: Using (2.1),

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t} \tag{2.37}$$

As,

$$e^{j2\pi k f_0 t} = \cos(2\pi k f_0 t) + j \sin(2\pi k f_0 t) \tag{2.38}$$

From (2.1), we have,

$$x(t) = \sum_{k=-\infty}^{\infty} c_k \left[\cos(2\pi k f_0 t) + j \sin(2\pi k f_0 t)\right] \tag{2.39}$$

$$= \sum_{k=-\infty}^{\infty} c_k \cos(2\pi k f_0 t) + j c_k \sin(2\pi k f_0 t) \tag{2.40}$$

$$= \sum_{k=-\infty}^{-1} \left[c_k \cos(2\pi k f_0 t) + j c_k \sin(2\pi k f_0 t)\right] \tag{2.41}$$

$$+ c_0 + \sum_{k=1}^{\infty} \left[c_k \cos(2\pi k f_0 t) + j c_k \sin(2\pi k f_0 t)\right] \tag{2.42}$$

$$= \sum_{k=1}^{\infty} \left[c_{-k} \cos(2\pi k f_0 t) - j c_{-k} \sin(2\pi k f_0 t)\right] \tag{2.43}$$

$$+ c_0 + \sum_{k=1}^{\infty} \left[c_k \cos(2\pi k f_0 t) + j c_k \sin(2\pi k f_0 t)\right] \tag{2.44}$$

$$= c_0 + \sum_{k=1}^{\infty} \Bigg((c_k + c_{-k}) \cos(2\pi k f_0 t) \tag{2.45}$$

$$+ j(c_k - c_{-k}) \sin(2\pi k f_0 t) \Bigg) \tag{2.46}$$

Substituting $a_k = c_k + c_{-k}$ and $b_k = j(c_k - c_{-k})$,we get,

$$x(t) = c_0 + \sum_{k=1}^{\infty} (a_k \cos 2\pi k f_0 t + b_k \sin 2\pi k f_0 t) \tag{2.47}$$

$$= \sum_{k=0}^{\infty} (a_k \cos 2\pi k f_0 t + b_k \sin 2\pi k f_0 t) \tag{2.48}$$

$$\therefore a_k = \begin{cases} c_k + c_{-k} & k \neq 0 \\ c_0 & k = 0 \end{cases} \tag{2.49}$$

$$b_k = j(c_k - c_{-k}) \tag{2.50}$$

Using (2.2),

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-j2\pi k f_0 t}\, dt \tag{2.51}$$

$$c_{-k} = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{j2\pi k f_0 t}\, dt \tag{2.52}$$

$$a_k = c_k + c_{-k} = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) \left[e^{-j2\pi k f_0 t} + e^{j2\pi k f_0 t} \right] dt \tag{2.53}$$

$$= 2f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) \cos(2\pi k f_0 t)\, dt \tag{2.54}$$

Similarly, for b_k, we get,

$$b_k = -j \left\{ 2f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) \sin\{2\pi k f_0 t\}\, dt \right\} \tag{2.55}$$

2.5 Find a_k and b_k for (1.1)
Solution: Using (2.49) and (2.50) with (2.35),

$$a_k = c_k + c_{-k} = \begin{cases} \frac{4A_0}{\pi(1-k^2)} & k = even \\ \frac{2A_0}{\pi} & k = 0 \\ 0 & k = odd \end{cases} \tag{2.56}$$

$$b_k = j(c_k - c_{-k}) = 0 \tag{2.57}$$

2.6 Verify (2.36) using python.
Solution:

```
wget https://github.com/Pradeep8802/EE3900
    −Digital−Signal−Processing/blob/main/
    charger/codes/2.6.py
        python3 2.3.py
```

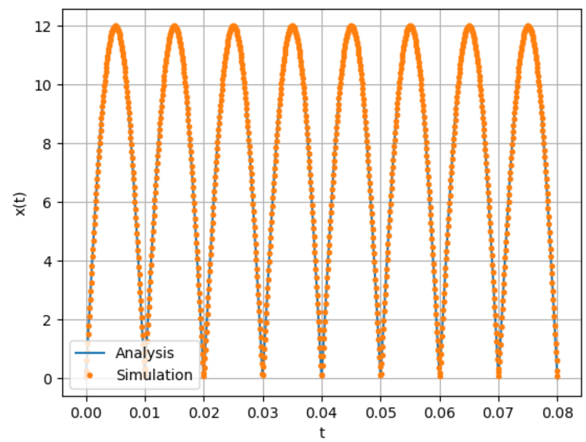

Fig. 0

3 Fourier Transform

3.1

$$\delta(t) = 0, \quad t \neq 0 \tag{3.1}$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \tag{3.2}$$

3.2 The Fourier Transform of $g(t)$ is

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi f t}\, dt \tag{3.3}$$

3.3 Show that

$$g(t - t_0) \xleftrightarrow{\mathcal{F}} G(f) e^{-j2\pi f t_0} \tag{3.4}$$

Solution: Let us consider $x = t - t_0$. Fourier transform of $g(t - t_0)$ is given as

$$g(t - t_0) \xleftrightarrow{\mathcal{F}} \int_{-\infty}^{\infty} g(t - t_0) e^{-j2\pi f t}\, dt \tag{3.5}$$

$$= \int_{-\infty}^{\infty} g(t - t_0) e^{-j2\pi f((t - t_0) + t_0)}\, du \tag{3.6}$$

$$= \int_{-\infty}^{\infty} g(x) e^{-j2\pi f(x + t_0)}\, dt \tag{3.7}$$

$$= \int_{-\infty}^{\infty} g(x) e^{-j2\pi f(x + t_0)}\, d(x - t_0) \tag{3.8}$$

$$= \int_{-\infty}^{\infty} e^{-j2\pi f t_0} g(x) e^{-j2\pi f x}\, d(x) \tag{3.9}$$

$$= e^{-j2\pi f t_0} \left\{ \int_{-\infty}^{\infty} g(x) e^{-j2\pi f x}\, d(x) \right\} \tag{3.10}$$

Using (3.3) in equation (3.10), we get,

$$g(t-t_0) \overset{\mathcal{F}}{\longleftrightarrow} G(f)e^{-\jmath 2\pi f t_0} \tag{3.11}$$

3.4 Show that

$$G(t) \overset{\mathcal{F}}{\longleftrightarrow} g(-f) \tag{3.12}$$

Solution: Let $g(t) \overset{\mathcal{F}}{\longleftrightarrow} G(f)$, then

$$g(t) = \int_{-\infty}^{\infty} G(f)e^{\jmath 2\pi f t}\,df \tag{3.13}$$

Consider $g(-k)$,

$$g(-k) = \int_{-\infty}^{\infty} G(f)e^{\jmath 2\pi f k}\,df \tag{3.14}$$

Let $f = t$,then,

$$g(-k) = \int_{-\infty}^{\infty} G(t)e^{\jmath 2\pi t k}\,dt \tag{3.15}$$

Substituting $k = f$ and in the (3.15), we get,

$$g(-f) = \int_{-\infty}^{\infty} G(t)e^{\jmath 2\pi f t}\,dt \tag{3.16}$$

Comparing (3.16) with (3.3), we can say that,

$$G(t) \overset{\mathcal{F}}{\longleftrightarrow} g(-f) \tag{3.17}$$

3.5 $\delta(t) \overset{\mathcal{F}}{\longleftrightarrow}?$

Solution: From (3.3), fourier transform of $\delta(t)$ is,

$$\delta(t) \overset{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\infty} \delta(t)e^{-\jmath 2\pi f t}\,dt \tag{3.18}$$

$$= \int_{-\infty}^{\infty} \delta(0)e^{-\jmath 2\pi f 0}\,dt \tag{3.19}$$

$$= \int_{-\infty}^{\infty} \delta(0)\,dt \tag{3.20}$$

$$= 1 \tag{3.21}$$

Hence, $\delta(t) \overset{\mathcal{F}}{\longleftrightarrow} 1$

3.6 $e^{-j2\pi f_0 t} \overset{\mathcal{F}}{\longleftrightarrow}?$

Solution: Suppose $g(t) \overset{\mathcal{F}}{\longleftrightarrow} G(f)$. Hence,

$$g(t) \overset{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\infty} g(t)e^{-\jmath 2\pi f t} \tag{3.22}$$

$$g(t)e^{-j2\pi f_0 t} \overset{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\infty} g(t)e^{-\jmath 2\pi f t}e^{-j2\pi f_0 t} \tag{3.23}$$

$$g(t)e^{-j2\pi f_0 t} \overset{\mathcal{F}}{\longleftrightarrow} G(f)e^{-j2\pi f_0 f} \tag{3.24}$$

$$\tag{3.25}$$

From (3.16),

$$g(t-f_0) \overset{\mathcal{F}}{\longleftrightarrow} G(f)e^{-\jmath 2\pi f t f_0} \tag{3.26}$$

$$g(t)e^{-j2\pi f_0 t} \overset{\mathcal{F}}{\longleftrightarrow} G(f)e^{-j2\pi f_0 f} \tag{3.27}$$

$$\tag{3.28}$$

From (3.13),

$$\delta(t) \overset{\mathcal{F}}{\longleftrightarrow} 1 \tag{3.29}$$

$$1 \overset{\mathcal{F}}{\longleftrightarrow} \delta(-f) = \delta(f) \tag{3.30}$$

Hence,

$$g(t-f_0) \overset{\mathcal{F}}{\longleftrightarrow} \delta((f+f_0)) \tag{3.31}$$

$$g(t)e^{\jmath 2\pi f_0 t} \overset{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\infty} g(t)e^{-\jmath 2\pi (f-f_0)t}\,dt \tag{3.32}$$

$$= G(f-f_0) \tag{3.33}$$

Hence,

$$e^{-\jmath 2\pi f_0 t} \overset{\mathcal{F}}{\longleftrightarrow} \delta(-(f+f_0)) = \delta(f+f_0) \tag{3.34}$$

3.7 $\cos(2\pi f_0 t) \overset{\mathcal{F}}{\longleftrightarrow}?$

Solution: We know that

$$\cos(2\pi f_0 t) = \frac{1}{2}\left(e^{\jmath 2\pi f_0 t} + e^{-\jmath 2\pi f_0 t}\right) \tag{3.35}$$

$$\tag{3.36}$$

Hence,

$$\mathcal{F}(\cos(2\pi f_0 t)) = \mathcal{F}(\frac{1}{2}\left(e^{\jmath 2\pi f_0 t} + e^{-\jmath 2\pi f_0 t}\right)) \tag{3.37}$$

$$\mathcal{F}(\cos(2\pi f_0 t)) = \frac{1}{2}\mathcal{F}(\left(e^{\jmath 2\pi f_0 t}\right)) + \frac{1}{2}\mathcal{F}(e^{-\jmath 2\pi f_0 t}) \tag{3.38}$$

$$= \frac{1}{2}\mathcal{F}(\left(e^{\jmath 2\pi f_0 t}\right)) + \frac{1}{2}\mathcal{F}(e^{-\jmath 2\pi f_0 t}) \tag{3.39}$$

$$= \frac{1}{2}\left(\delta(f+f_0) + \delta(f-f_0)\right) \tag{3.40}$$

3.8 Find the Fourier Transform of $x(t)$ and plot it. Verify using python. **Solution:** As obtained earlier,from equation (2.35),

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} \sum_{k=-\infty}^{\infty} c_k \tag{3.41}$$

$$\implies x(t) \overset{\mathcal{F}}{\longleftrightarrow} \sum_{k=-\infty}^{\infty} \frac{\delta f +}{} \tag{3.42}$$

$$\tag{3.43}$$

Substituting (**??**) in (2.1),

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} \sum_{k=-\infty}^{\infty} c_k \delta\left(f + k f_0\right) \tag{3.44}$$

$$= \frac{2A_0}{\pi} \sum_{k=-\infty}^{\infty} \frac{\delta\left(f + 2k f_0\right)}{1 - 4k^2} \tag{3.45}$$

The python code `codes/3_8.py` verifies (3.45).

3.9 Show that

$$\operatorname{rect} t \overset{\mathcal{F}}{\longleftrightarrow} \operatorname{sinc} t \tag{3.46}$$

Verify using python. **Solution:** We write

$$\operatorname{rect} t \overset{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\infty} \operatorname{rect} t e^{-\jmath 2\pi f t}\, dt \tag{3.47}$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-\jmath 2\pi f t}\, dt \tag{3.48}$$

$$= \frac{e^{\jmath \pi f} - e^{-\jmath \pi f}}{\jmath 2\pi f} = \frac{\sin \pi f}{\pi f} = \operatorname{sinc} f \tag{3.49}$$

The python code `codes/3_9.py` verifies (3.49). **Solution:** From (**??**), we have

$$\operatorname{sinc} t \overset{\mathcal{F}}{\longleftrightarrow} \operatorname{rect}(-f) = \operatorname{rect} f \tag{3.50}$$

Since $\operatorname{rect} f$ is an even function. The python code `codes/3_10.py` verifies (3.50).