1. Bilinear Transform

1.1. In Fig. **??**, consider the case when S is switched to Q right in the beginning. Formulate the differential equation
Solution: The differential equation is the same as before

$$\frac{v_c(t)-0}{R_1}+\frac{v_c(t)-v_2(t)}{R_2}+\frac{\mathrm{d}q}{\mathrm{d}t}=0 \tag{1.1}$$

$$\text{i.e., } \frac{v_c(t)}{R_1}+\frac{v_c(t)-v_2(t)}{R_2}+C_0\frac{\mathrm{d}v_c}{\mathrm{d}t}=0 \tag{1.2}$$

but with a different initial condition

$$q(0^-)=q(0)=0 \tag{1.3}$$

1.2. Find $H(s)$ considering the ouput voltage at the capacitor
Solution: On taking the Laplace transform on both sides of this equation

$$\frac{V_c(s)}{R_1}+\frac{V_c(s)-V_2(s)}{R_2}+sQ(s)-0=0 \tag{1.4}$$

$$\implies V_c(s)\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+sC_0V_c(s)=\frac{V_2(s)}{R_2} \tag{1.5}$$

$$\implies \frac{V_c(s)}{V_2(s)}=\frac{\frac{1}{R_2}}{\frac{1}{R_1}+\frac{1}{R_2}+sC_0} \tag{1.6}$$

The transfer function is thus

$$H(s)=\frac{\frac{1}{R_2C_0}}{s+\frac{1}{R_1C_0}+\frac{1}{R_2C_0}} \tag{1.7}$$

On substituting the values, we get

$$H(s)=\frac{5\times10^5}{s+1.5\times10^6} \tag{1.8}$$

1.3. Plot $H(s)$. What kind of filter is it?
Solution: Download the following Python code that plots Fig. 1

```
wget https://github.com/Ankit−Saha−2003/
    EE3900/raw/main/Circuit/codes/4.3.py
```

Run the codes by executing

```
python 4.3.py
```

./figs/4.3.png

Fig. 1.3. Plot of $H(s)$

Consider the frequency-domain transfer function by putting $s=\jmath\omega$

$$H(\jmath\omega)=\frac{5\times10^5}{\jmath\omega+1.5\times10^6} \tag{1.9}$$

$$\implies \left|H(\jmath\omega)\right|=\frac{5\times10^5}{\sqrt{\omega^2+2.25\times10^{12}}} \tag{1.10}$$

As ω increases, $\left|H(\jmath\omega)\right|$ decreases
In other words, the amplitude of high-frequency signals gets diminished and they get filtered out
Therefore, this is a low-pass filter

1.4. Using trapezoidal rule for integration, formulate the difference equation by considering

$$y(n)=y(t)|_{t=n} \tag{1.11}$$

Solution:

$$\frac{v_c(t)}{R_1}+\frac{v_c(t)-v_2(t)}{R_2}+C_0\frac{\mathrm{d}v_c}{\mathrm{d}t}=0 \tag{1.12}$$

$$\implies C_0\frac{\mathrm{d}v_c}{\mathrm{d}t}=\frac{2u(t)-v_c(t)}{R_2}-\frac{v_c(t)}{R_1} \tag{1.13}$$

$$\implies v_c(t)|_{t=n}^{n+1}=\int_n^{n+1}\left(\frac{2u(t)-v_c(t)}{R_2C_0}-\frac{v_c(t)}{R_1C_0}\right)\mathrm{d}t \tag{1.14}$$

By the trapezoidal rule of integration

$$\int_a^b f(t)\mathrm{d}t\approx\frac{b-a}{2}(f(a)+f(b)) \tag{1.15}$$

Consider $y(t) = v_c(t)$

$$y(n+1) - y(n) = \frac{1}{R_2C_0}(u(n) + u(n+1)) - \frac{1}{2}(y(n+1) + y(n))\left(\frac{1}{R_1C_0} + \frac{1}{R_2C_0}\right) \quad (1.16)$$

Thus, the difference equation is

$$y(n+1)\left(1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0}\right) = y(n)\left(1 - \frac{1}{2R_1C_0} - \frac{1}{2R_2C_0}\right) + \frac{1}{R_2C_0}(u(n) + u(n+1)) \quad (1.17)$$

1.5. Find $H(z)$
Solution: Let $\mathcal{Z}\{y(n)\} = Y(z)$
On taking the Z-transform on both sides of the difference equation

$$zY(z)\left(1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0}\right) = Y(z)\left(1 - \frac{1}{2R_1C_0} - \frac{1}{2R_2C_0}\right) + \frac{1}{R_2C_0}\left(\frac{1}{1-z^{-1}} + \frac{z}{1-z^{-1}}\right) \quad (1.18)$$

$$Y(z)\left(z + \frac{z}{2R_1C_0} + \frac{z}{2R_2C_0} - 1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0}\right) = \frac{1}{R_2C_0}\frac{1+z}{1-z^{-1}} \quad (1.19)$$

Also

$$v_2(t) = 2 \qquad \forall t \geq 0 \quad (1.20)$$
$$\implies x(n) = 2u(n) \quad (1.21)$$
$$\implies X(z) = \frac{2}{1-z^{-1}} \qquad |z| > 1 \quad (1.22)$$

Thus, the transfer function in z-domain is

$$H(z) = \frac{Y(z)}{X(z)} \quad (1.23)$$
$$= \frac{\frac{1+z}{2R_2C_0}}{z + \frac{z}{2R_1C_0} + \frac{z}{2R_2C_0} - 1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0}} \quad (1.24)$$
$$= \frac{\frac{1+z^{-1}}{2R_2C_0}}{1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0} - z^{-1} + \frac{z^{-1}}{2R_1C_0} + \frac{z^{-1}}{2R_2C_0}} \quad (1.25)$$

On substituting the values

$$H(z) = \frac{2.5 \times 10^5(1 + z^{-1})}{7.5 \times 10^5 + 1 + (7.5 \times 10^5 - 1)z^{-1}} \quad (1.26)$$

with the ROC being

$$|z| > \max\left(1, \left|\frac{7.5 \times 10^5 - 1}{7.5 \times 10^5 + 1}\right|\right) \quad (1.27)$$
$$\implies |z| > 1 \quad (1.28)$$

1.6. How can you obtain $H(z)$ from $H(s)$?
Solution: The Z-transform can be obtained from the Laplace transform by the substitution

$$s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} \quad (1.29)$$

where T is the step size of the trapezoidal rule (1 in our case)
This is known as the bilinear transform
Thus

$$H(z) = \frac{\frac{1}{R_2C_0}}{2\frac{1-z^{-1}}{1+z^{-1}} + \frac{1}{R_1C_0} + \frac{1}{R_2C_0}} \quad (1.30)$$
$$= \frac{\frac{1+z^{-1}}{2R_2C_0}}{1 - z^{-1} + \left(\frac{1}{2R_1C_0} + \frac{1}{2R_2C_0}\right)(1 + z^{-1})} \quad (1.31)$$
$$= \frac{\frac{1+z^{-1}}{2R_2C_0}}{1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0} - z^{-1} + \frac{z^{-1}}{2R_1C_0} + \frac{z^{-1}}{2R_2C_0}} \quad (1.32)$$
$$= \frac{2.5 \times 10^5(1 + z^{-1})}{7.5 \times 10^5 + 1 + (7.5 \times 10^5 - 1)z^{-1}} \quad (1.33)$$

which is the same as what we obtained earlier