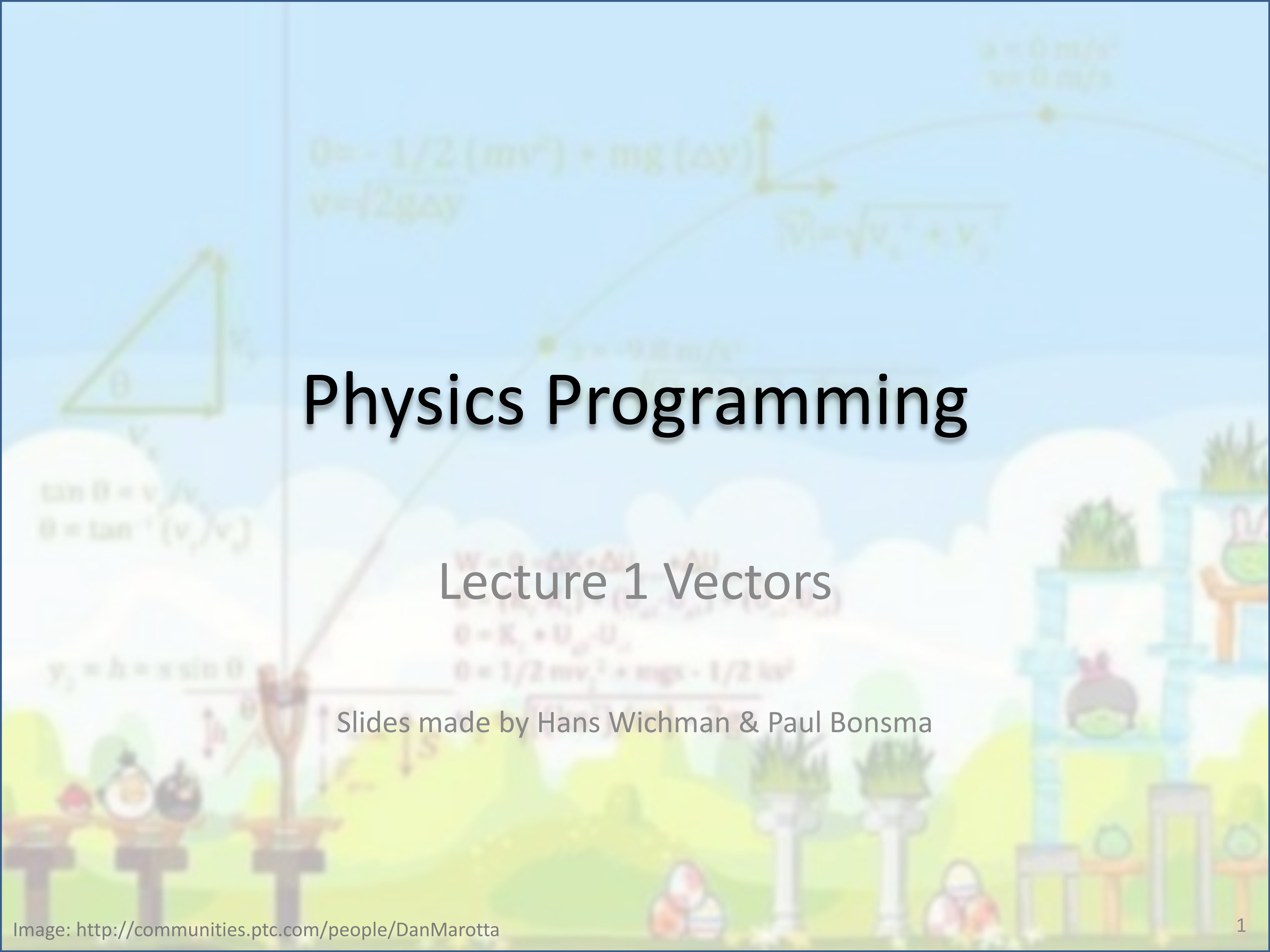


Physics Programming

Lecture 1 Vectors

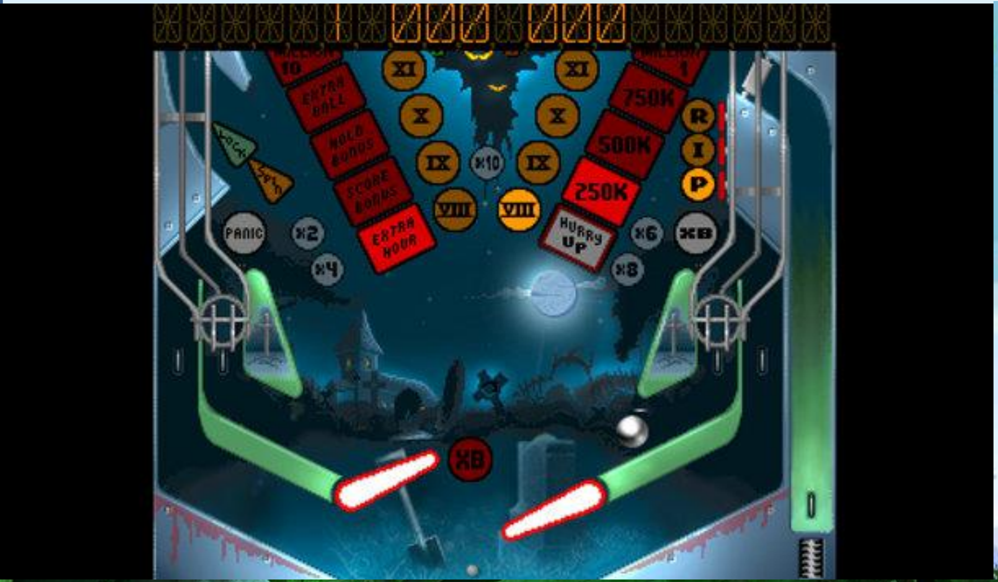
Slides made by Hans Wichman & Paul Bonsma

Image: http://communities.ptc.com/people/DanMarotta

What is physics programming about?

- In an interactive application: *making objects move realistically* → inspired by laws of physics
- Note: *Good interaction / Fun game mechanics* are more important than *super realistic physics simulation!*
- But: good physics simulation can help for:
  - Intuitive controls
  - Dynamic game play
  - Realism / immersion

Physics Game Examples



What kind of physics are typically used in games?

- Mainly: *Rigid body* physics.
- *Collisions* / objects bouncing into each other (=impulses)
- *Continuous forces*:
  - Gravity
  - Explosion force
  - Attracting / repelling (e.g. magnets)
  - Springs, hinges
  - Friction / viscosity

- During this course, we won’t be able to create a full featured physics engine – not even for 2D.
- For advanced physics: use a physics engine (Bullet, Farseer, Unity physics, …)

- So why do we learn about game physics?

- So why do we learn about game physics?
  - To create custom physics / dynamic movement for simple games (e.g. platformers, pinball, race game)
  - To get a better understanding of what physics engines can and cannot do
  - To learn about *vector math.* Also useful for:
    - Computer graphics: 3D rendering
    - Procedural generation
    - …
    - Basically: *the basis for many upcoming courses*

More reasons to pay attention

- You will create reusable code and learn more about Object Oriented Design
  (vectors, collisions, etc)
- Will be a source of reference for upcoming project
  (prepare you for next project)
- To allow you to study further on your own
  (more complicated subjects)
- In conclusion: effects ripple way beyond this course!

How difficult is Physics Programming?

- We start at the beginning, assuming only basic math and programming knowledge.
- We assume you passed 1.1 Programming, and can work with the GXPEngine + Visual Studio.
- We won't assume you have already seen vectors / trigonometry.

- Nevertheless:
  - If all of this is new to you and math is not your strength, it's a steep learning curve!
  - Staying on top of it every week is essential!

How difficult is Physics Programming?

It will **seem** hard, but practice makes perfect... So:

- *Go to the labs, ask feedback*
- Use your time well
- *Don't start too late with the assignments*
- If needed: study additional resources
- *Expected time investment: 80 hours!*

  (>13 hours per week!)

Grading

- Week 3.9: *Assessment:* based on a final programming assignment (grading criteria: see manual)

  → Main grade (between 1 – 10, where 5.5 = sufficient)

- In addition, there will be *five lab assignments*, that directly prepare for the final assignment.

  → 2.5 bonus points for the exam can be scored by *signing these off on time, during the labs. (added to the exam grade)*

- Discussing your solutions to the assignments with your lab teacher is perfect preparation for the assessment!
- To access the assignments, you need to complete a *short test of understanding* on Blackboard every week.

- Details on Blackboard

Outline Today

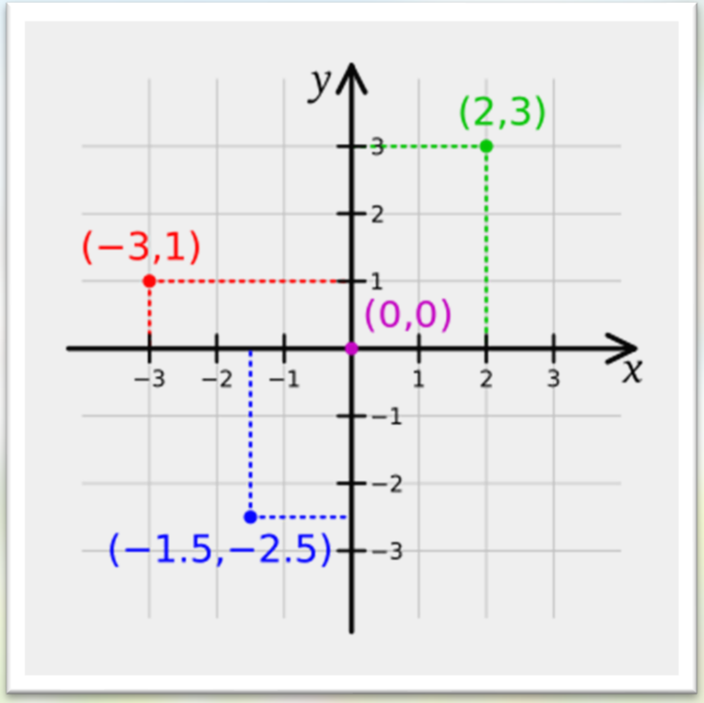
- Coordinate systems
- Vectors
- Vector addition and subtraction
- Vectors in code:
  - Struct vs class
  - Operator overloading
- Length (Pythagoras), scaling, normalizing
- Assignment 1 explanation

Coordinate systems

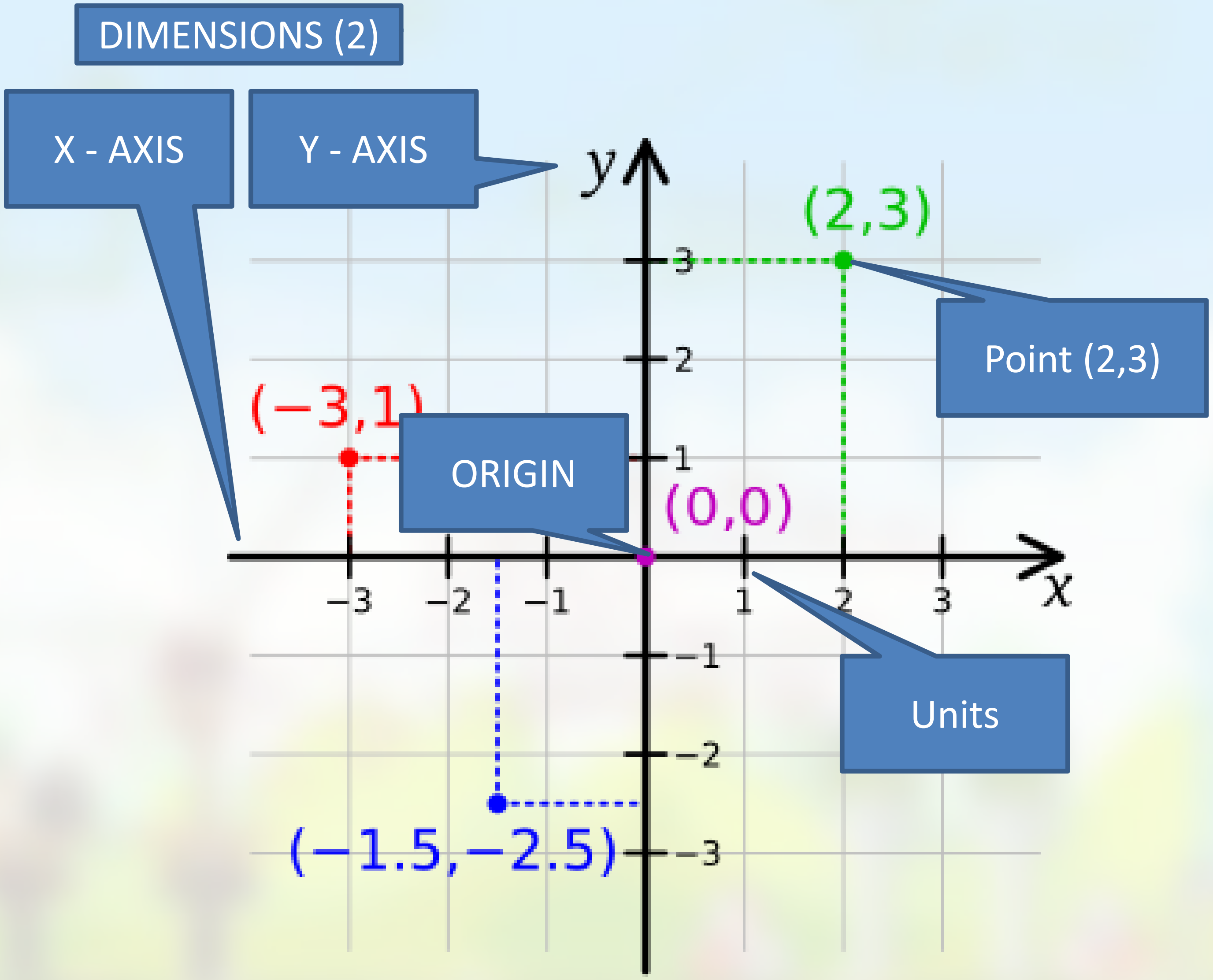
On axis, units, and your freedom to choose…

What is a Coordinate System?

- A framework to describe *positions in space* using *numbers* (a.k.a. *coordinates*).
- Most well known framework is the *Cartesian Coordinate System*:
  - *Perpendicular axes (=90°)*
  - Same *unit* on all
- There are other systems (e.g. *polar)* → not used for now.



Elements of Cartesian Coordinate System: Dimensions, Axis, Origin, Units & Points

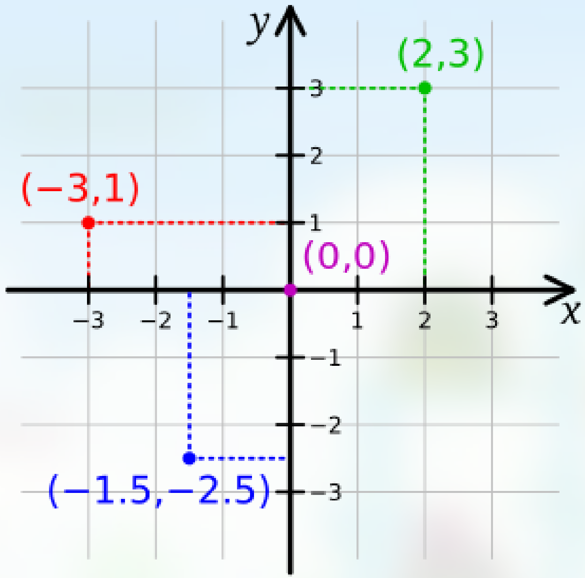


Freedom of choice

- Some things are fixed (because Cartesian):
  - Same unit on every axis
  - Axes are perpendicular.
- …we still have to choose:
  - The number of dimensions (1D/2D/3D/…)
  - The directions of the axis
  - The unit size

Example 1: High school math

- One customary approach in high school math is:
  - 2 dimensions
  - 1 unit = 1 centimeter
  - +x to the right
  - +y is up
- Lot of general math subjects will be discussed in this space (for example trigonometry)

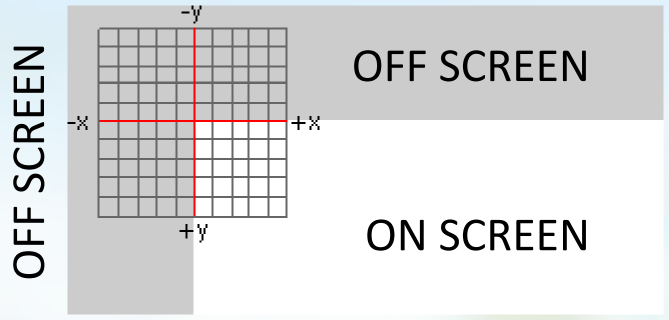


Example 2: Cocos2D

- Some game engines, such as Cocos2D use **almost** the same approach:
  - 2 dimensions
  - 1 unit = **1 pixel**
  - +x to the right
  - +y is up
  - (0,0) is bottom left
  - (screen width, screen height) is top right

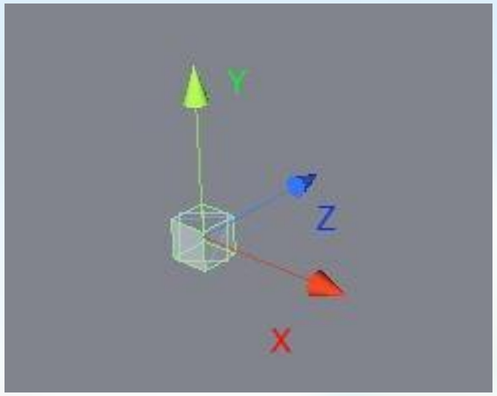
Example 3: GXPEngine

- The GXP Engine uses:
  - 2 dimensions
  - 1 unit = 1 pixel
  - +x to the right
  - +y is **DOWN**
  - **(0,0) is top left of screen**
  - The “floor” is at game.height



Example 4: Unity

- Unity3D uses:
  - 3 dimensions
  - 1 unit = 1 meter
  - +x to the right
  - +y is up
  - +z into the screen



Vectors

About general terminology often used in physics

Vectors

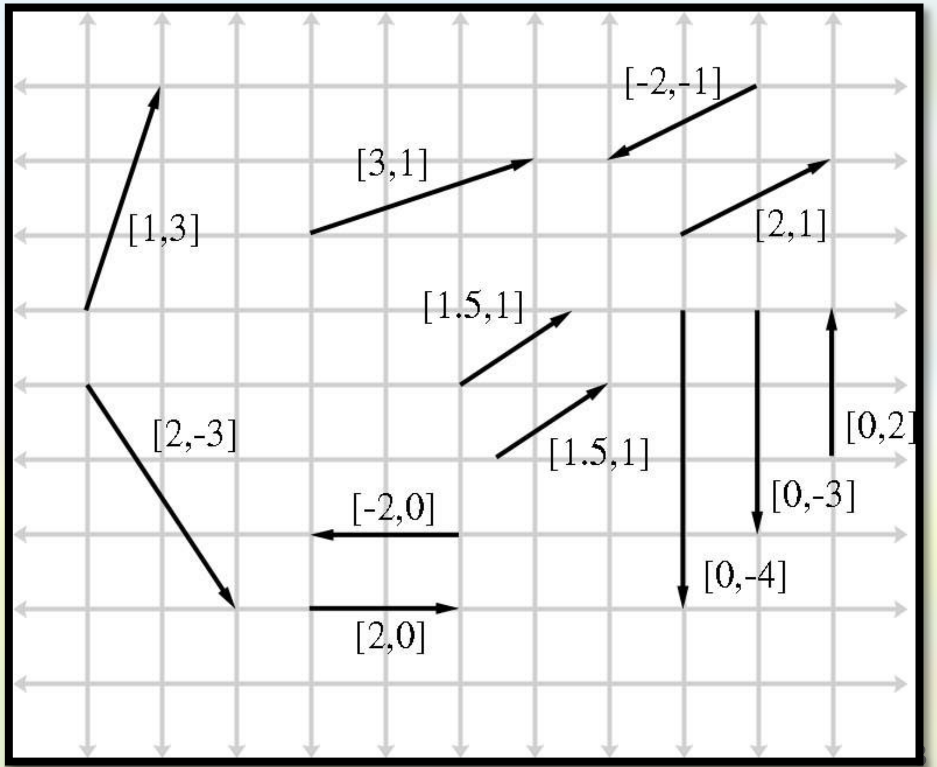
- In 2D games we use x & y properties to position and move objects around, e.g. an object is at position (x,y)
- In physics / 2d math we call this (x,y) pair …
  - a point
  - a position
  - a vector
- All are common, but for game math the name **Vector** is **most common.**

Vectors

- A 2d vector describes an (x,y) pair
- A 3d vector describes an (x,y,z) tuple
- The “textbook” notation for a vector v is $\vec{v}$
- When we refer to a Vector’s x and y components *(coordinates)*, it’s written like:
  - $\vec{v}.x$ or $\vec{v}_x$
  - $\vec{v}.y$ or $\vec{v}_y$
- Both notations are common!

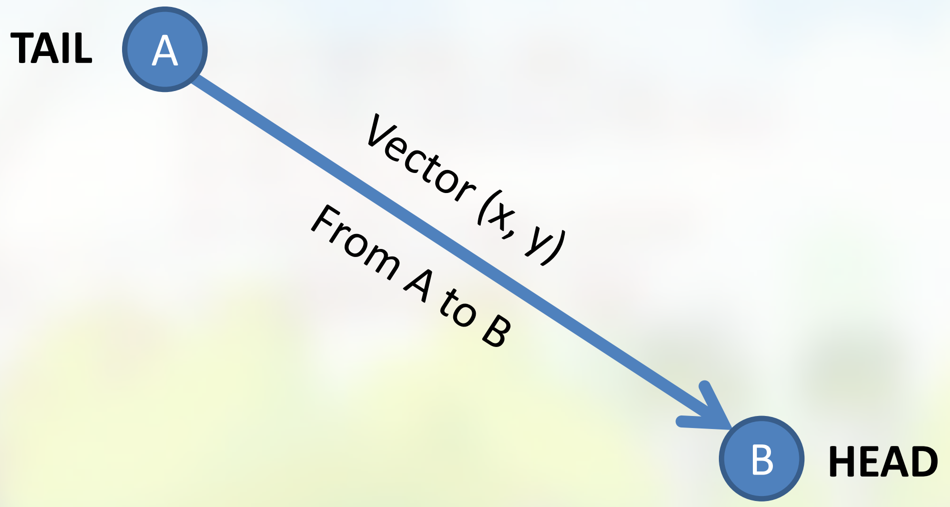
Drawing vectors / equality

- Vectors are often drawn as *arrows.*
- A vector can be *drawn anywhere.*
- Two vectors are *equal* if and only if all their coordinates are equal.



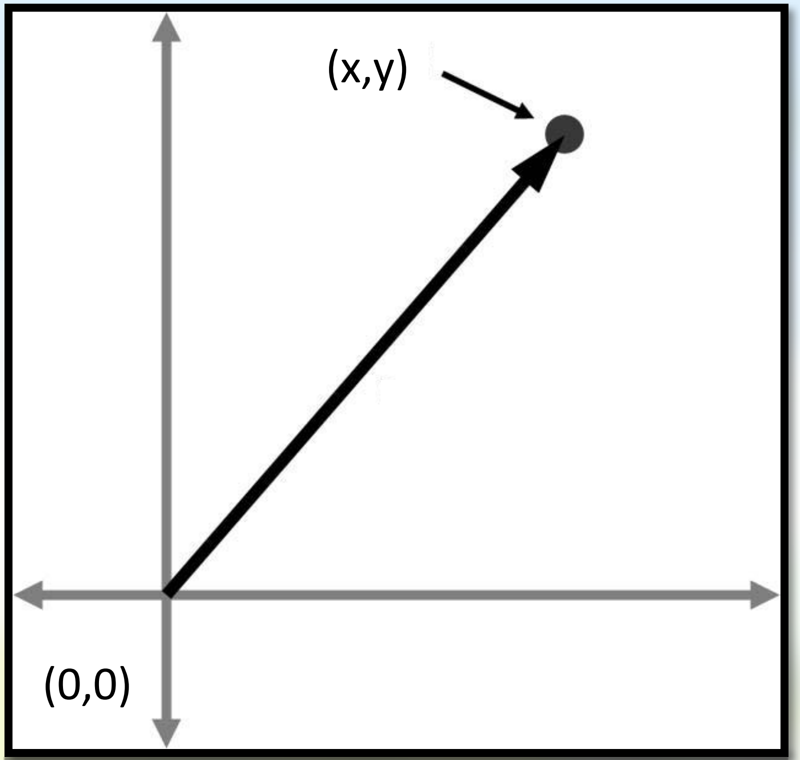
Vector terminology: tail & head / from .. to ..

- A *drawn vector* has a **tail** and a **head**:
- Below, vector (x,y) is drawn *from A to B:*



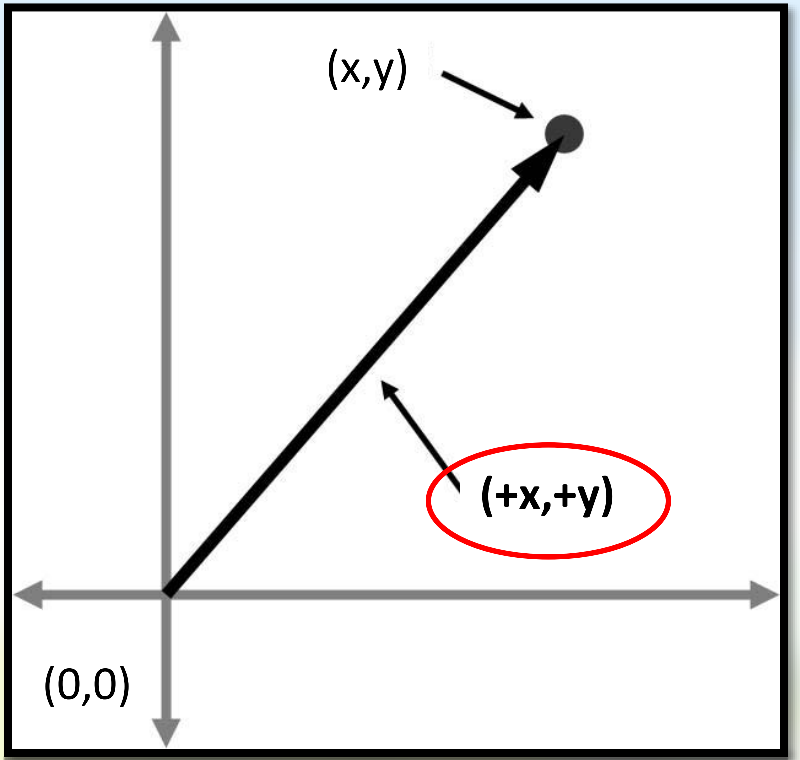
Vectors can describe a *position*

We are **at** (x,y) → absolute value



Vectors can describe a *displacement* / directional movement

Here **from** (0,0) **to** (x,y) → relative value



Modifying positions directly?

- We can make objects move by storing their position in a vector, and modifying this.
  - Quick demo: 002_ball_vector_position
- However, in real life, positions only change because we apply force, which causes acceleration, which causes velocity...

  (These are actually *Newton's* first two *laws* → later)
- Let's start with velocity...
  - what is velocity?
  - how does velocity change an object's position?

Velocity

- **Velocity is speed in a specific direction**

- Velocity, speed or direction?:
  - 80 miles an hour?
  - Going north?
  - Heading left?
  - Going west at 20 mph ?

Velocity

- **Velocity is speed in a specific direction**

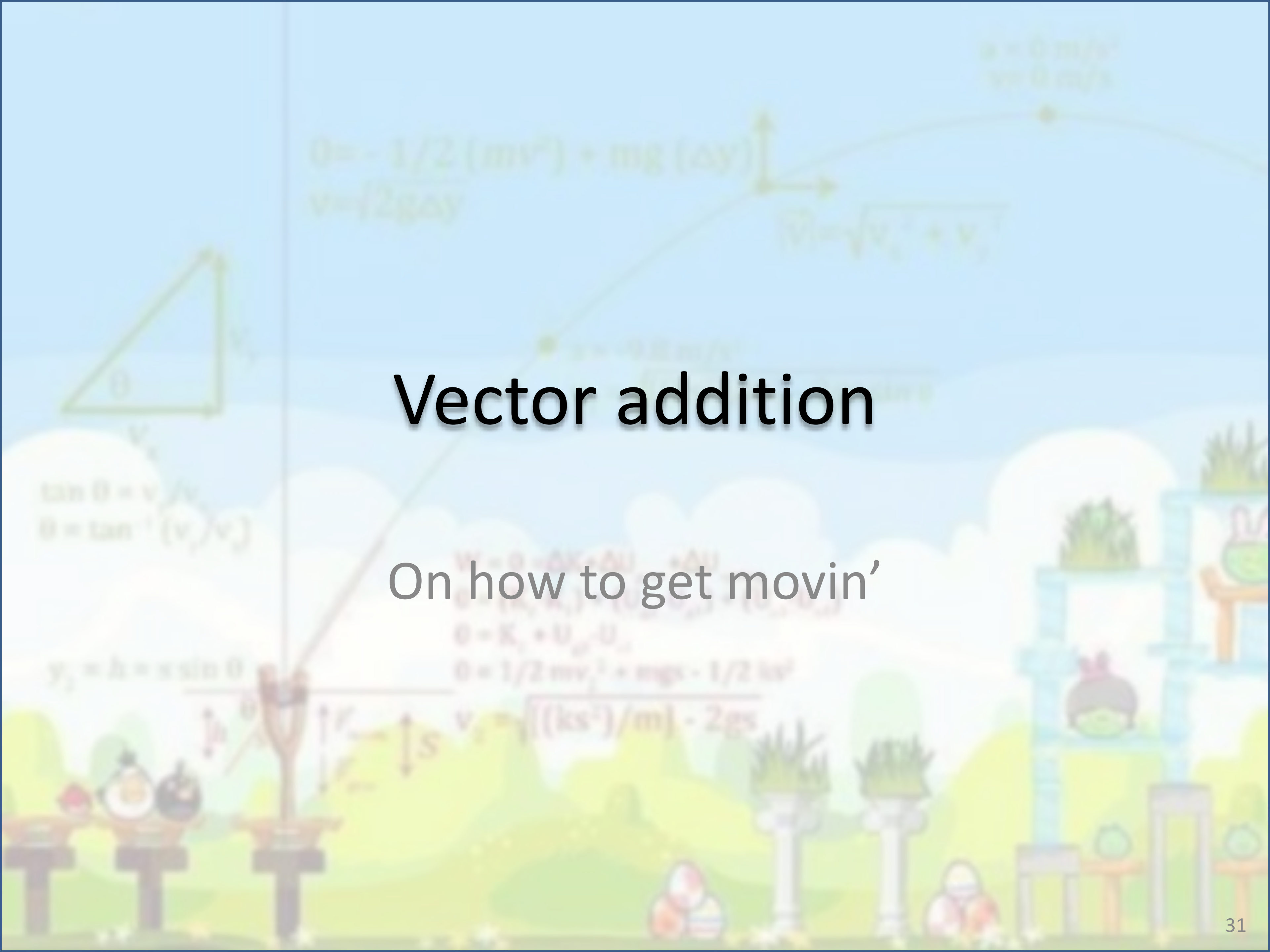
- Velocity, speed or direction?:
  - 80 miles an hour? *speed*
  - Going north? *direction*
  - Heading left? *direction*
  - Going west at 20 mph? *velocity*

Using Vec2 for velocity

- Velocity **is** directional movement
- Vectors **can describe** directional movement ("displacement")
- → we can store velocity as a vector too!

- For example:
  - If I say my velocity is (3,4) that means that I want to go 3 pixels to the right, and 4 down each frame.

- This requires vector **addition.**

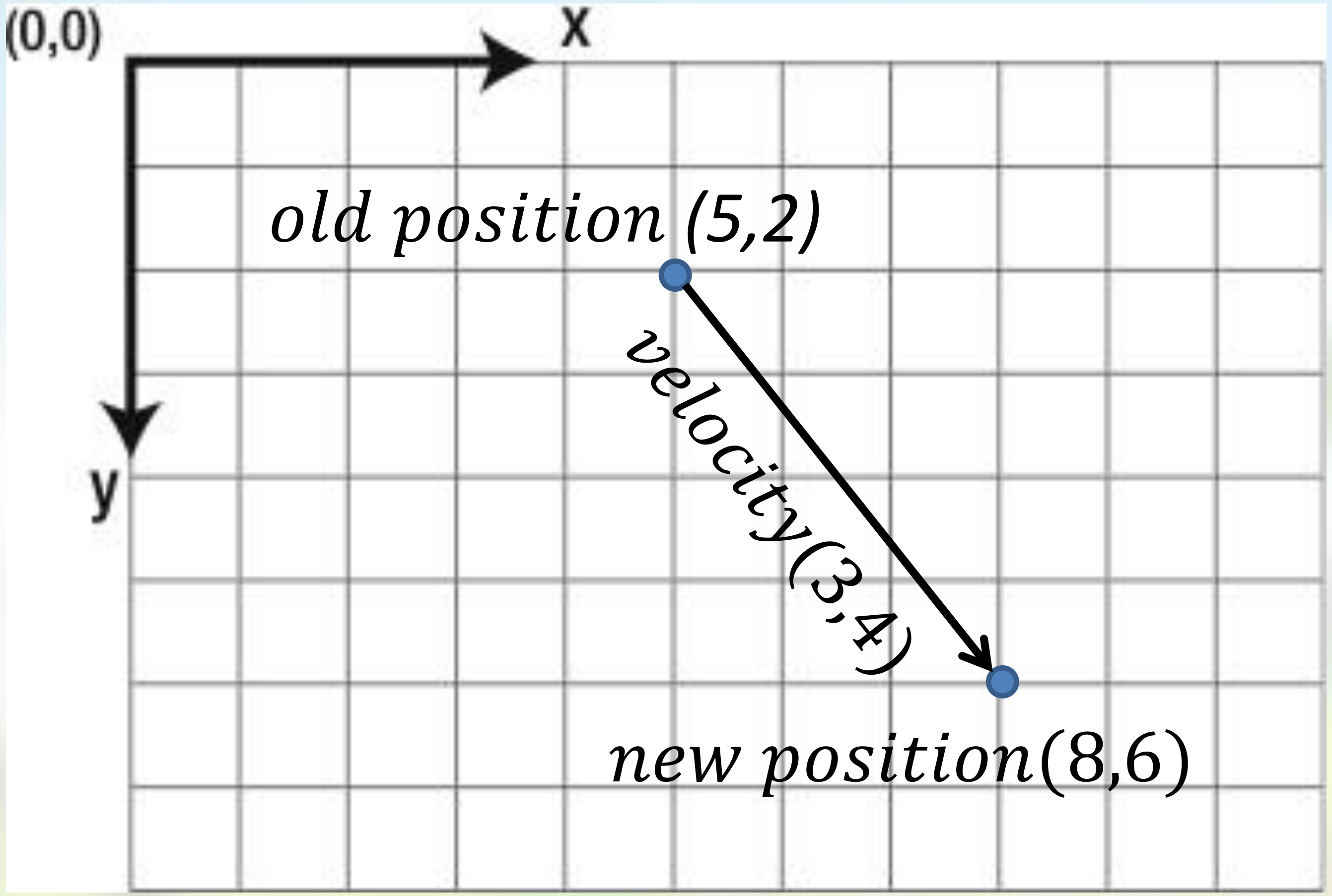


Vector addition

On how to get movin'

Vector addition in a picture

new position = old position + velocity

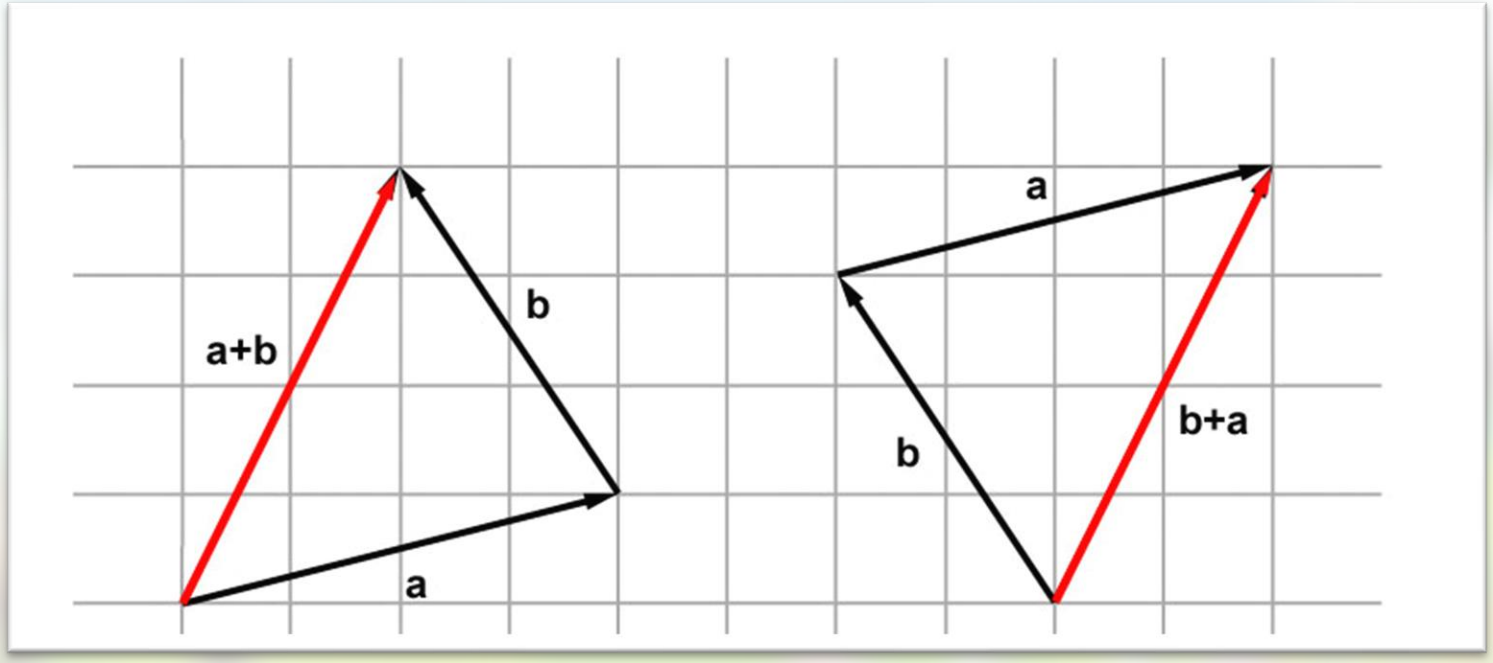


Vector addition: (mathematical) definition

- new position = old position + velocity
- Officially written as $\vec{\boldsymbol{p}}' = \vec{\boldsymbol{p}} + \vec{\boldsymbol{v}}$
- The $'$ in $\vec{\boldsymbol{p}}'$ means "**modified version of**"
- Is done through **component-wise** addition:
  - Given $\vec{p}$ = ($\vec{p}$.x, $\vec{p}$.y) and $\vec{v}$ = ($\vec{v}$.x, $\vec{v}$.y)
  - Then $\vec{p}'$ = $\vec{p}$ + $\vec{v}$ = ($\vec{p}$.x + $\vec{v}$.x, $\vec{p}$.y + $\vec{v}$.y)
  - So $\vec{p}'$ = $\vec{p}$ + $\vec{v}$ = (5,2) + (3,4) = $(5 + 3, 2 + 4)$ = (8,6)

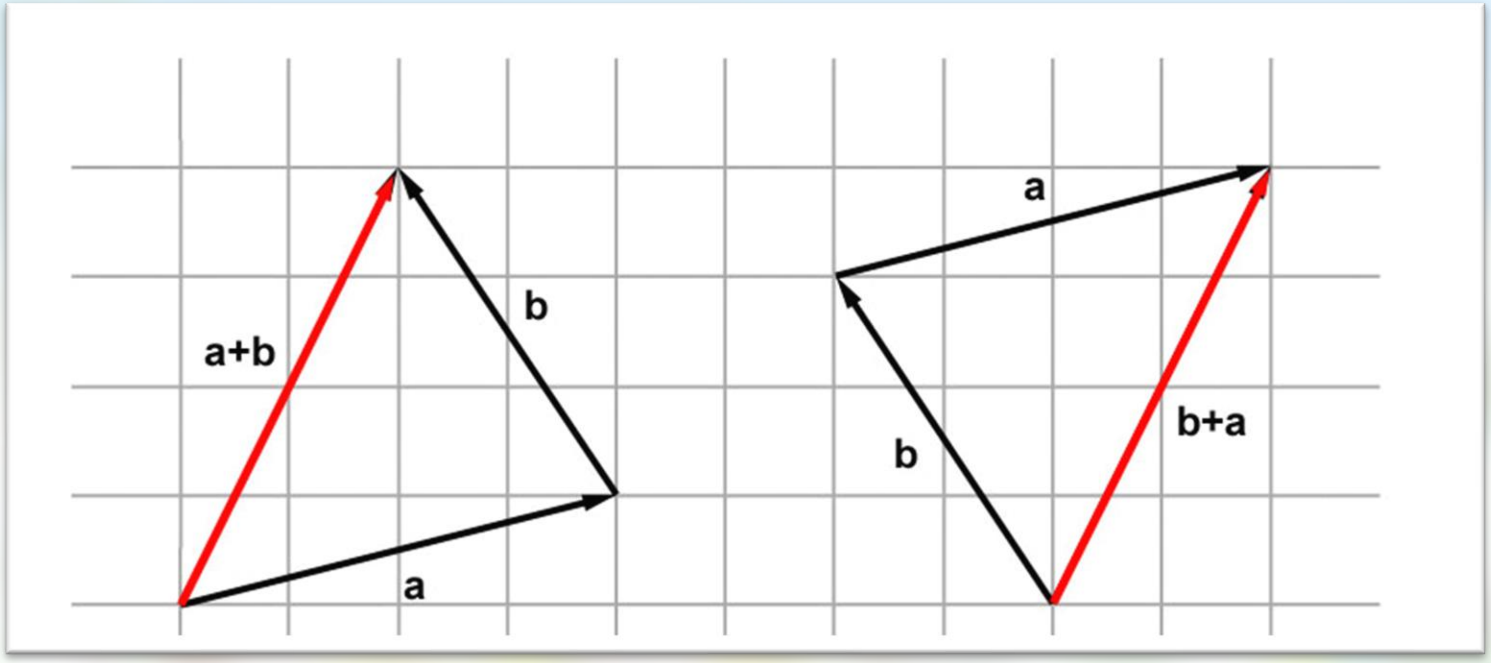
Visualizing vector addition

- The vector $\vec{a} + \vec{b}$ can be visualized by drawing $\overrightarrow{a}$ and $\vec{b}$ "head to tail" as follows:
- Order doesn't matter! → *"Commutative"*



Vector addition is *commutative*

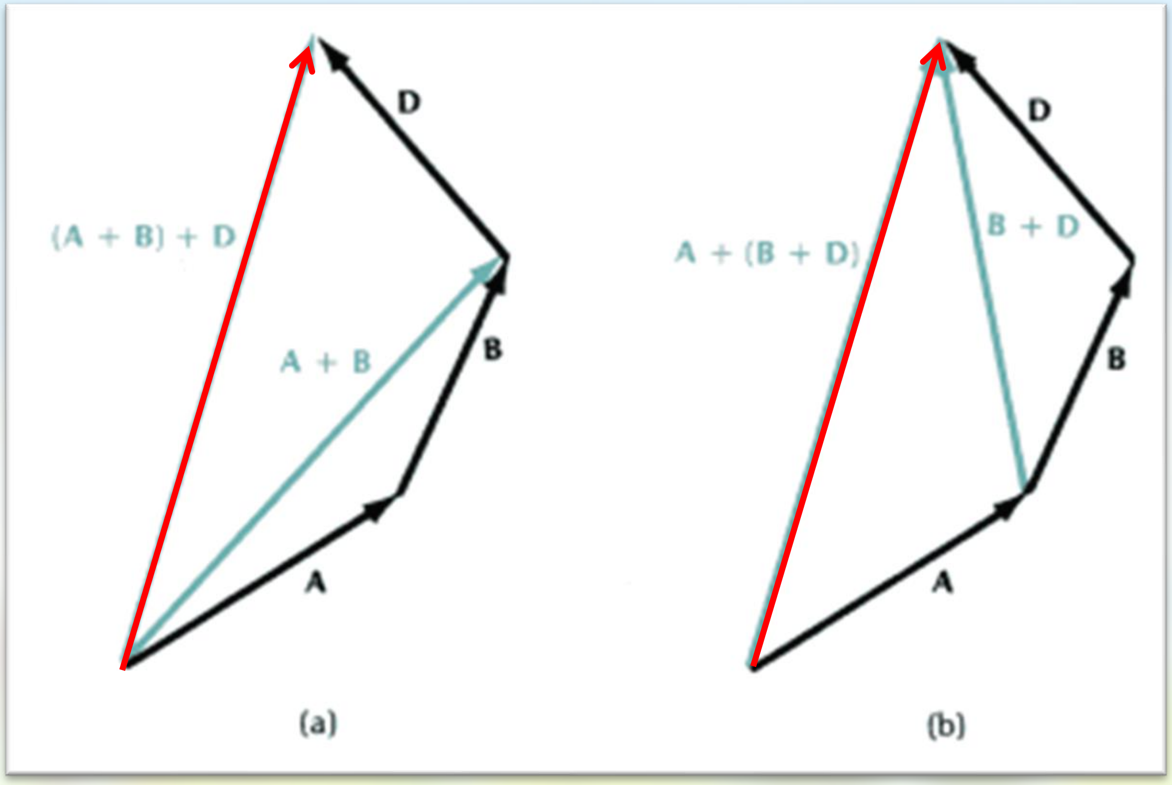
$$\vec{a} + \vec{b} = \vec{b} + \vec{a}$$



Note the end result (red) after adding

Vector addition is *associative*

$$(\vec{a} + \vec{b}) + \vec{d} = \vec{a} + (\vec{b} + \vec{d})$$



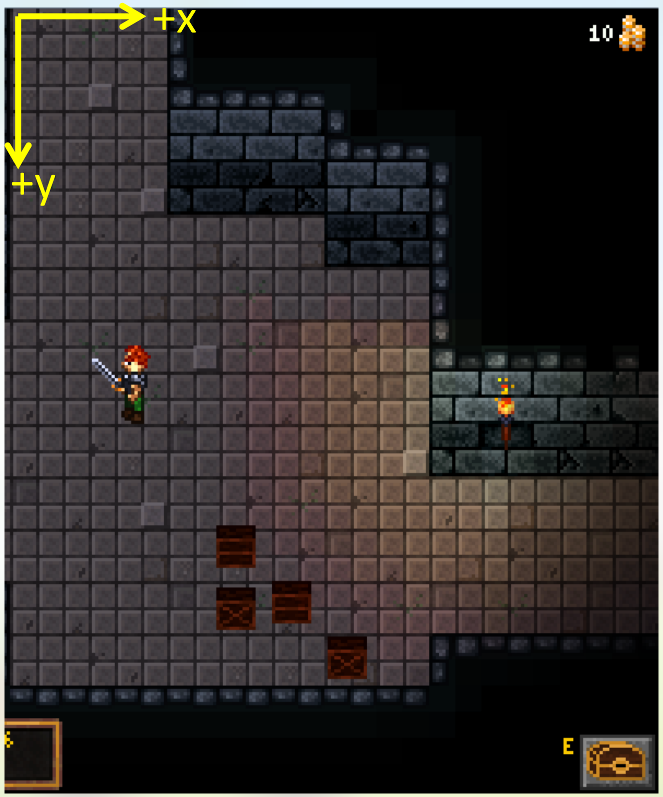
Note the end result (red) after adding

Vector addition

- **Summing up:**
  - the **order** in which we add vectors **doesn't matter**.

Treasure hunting

- You just found a map! Find the treasure with the least amount of steps possible, which way do you go?



Follow these steps to find the treasure!:

(-1,0)
(0,-2)
(1,-3)
(1, 3)
(0,2)

How many steps do you have to take?

Move to Target

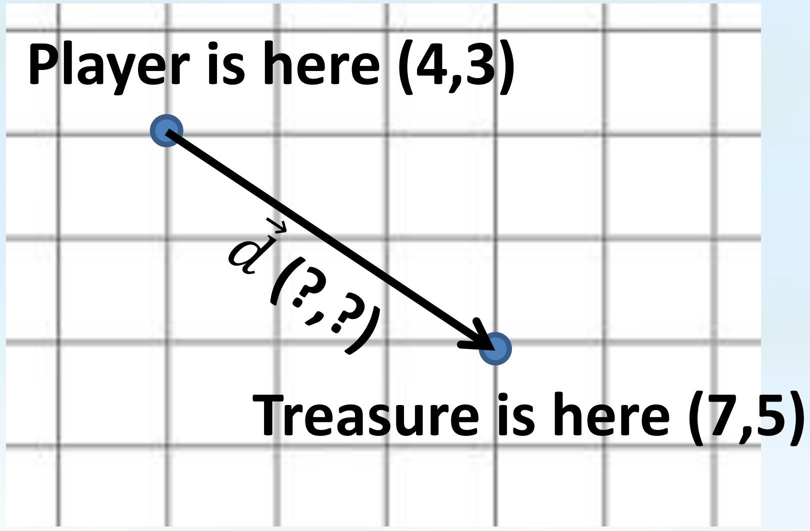
- What if instead, we know where we are (position $\vec{p}$), where the treasure is (target $\vec{t}$), and we want to know how to get there?
- $\vec{p} + ? = \vec{t}$
- So: $? = \vec{t} - \vec{p}$
- ➔ We need vector *subtraction*
- Definition:

$$t - \vec{p} = (\vec{t}.x - \vec{p}.x, \vec{t}.y - \vec{p}.y)$$

Subtracting vectors

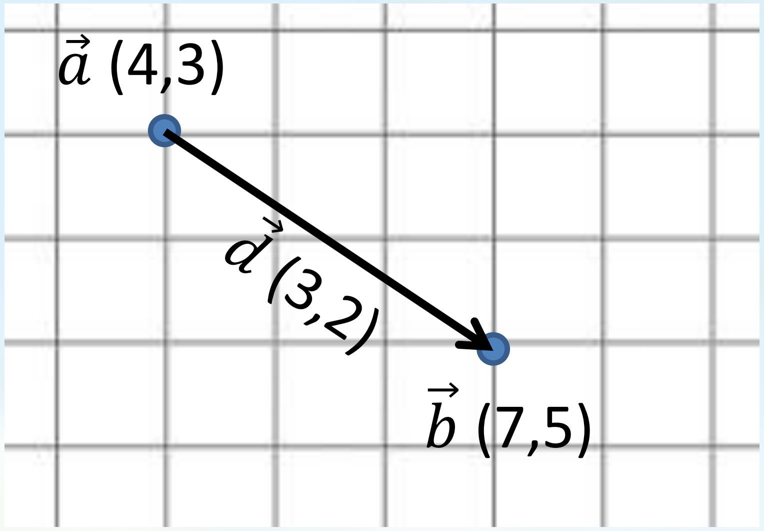
Where you learn how to go where you want

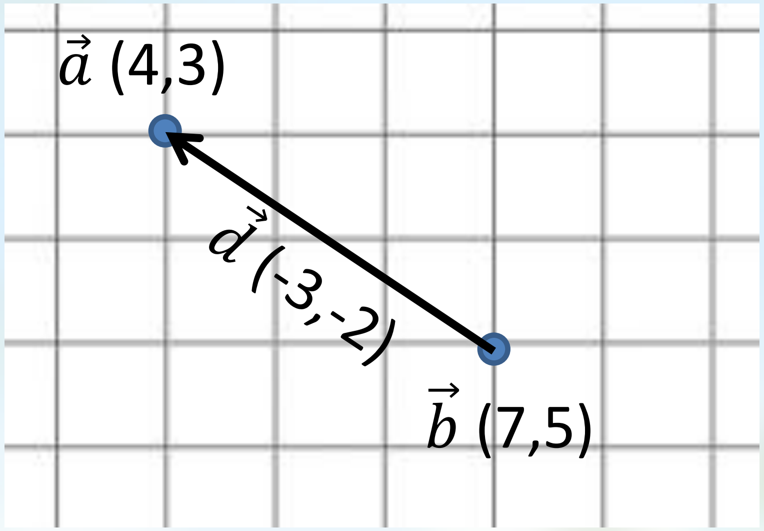
Imagine this scenario:



- Player is at (4,3), treasure is at (7,5)
- Player needs to move over vector $\vec{d}$
- $\vec{d}$ stands for **d**elta, delta means change/difference

Calculate delta using vector subtraction





- More general:
  - $\vec{b}$ - $\vec{a}$ describes the displacement from $\vec{a}$ to $\vec{b}$
  - $\vec{a}$ - $\vec{b}$ describes the displacement from $\vec{b}$ to $\vec{a}$
- Subtraction is not commutative: $\vec{b} - \vec{a} \neq \vec{a} - \vec{b}$!

  (…nor associative)

Vectors in code

On what vectors know and do, and how we can integrate them into the GXPEngine

Defining vectors in code

- Object oriented design:
  - What do vectors know?
  - What can vectors do?
  - What can we do with them?

Defining vectors in code

- Object oriented design:
  - What do vectors know?
    - they know about their x & y coordinates (and maybe a z in 3d space)
    - They know their length?
    - They know their direction?
  - What can vectors "do"?
    - they can be added
    - they can be subtracted
    - they can be ….

Defining vectors in code

- Object oriented design:
  - What do vectors know?
    - they know about their x & y coordinates (and maybe a z in 3d space)
    - ~~They know their length?~~ →can be deduced from *x & y!*
    - ~~They know their direction?~~ →can be deduced from *x & y!*
  - What can vectors “do”?
    - they can be added
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    - they can be ….

Defining vectors in code

- Object oriented design:
  - What do vectors know?
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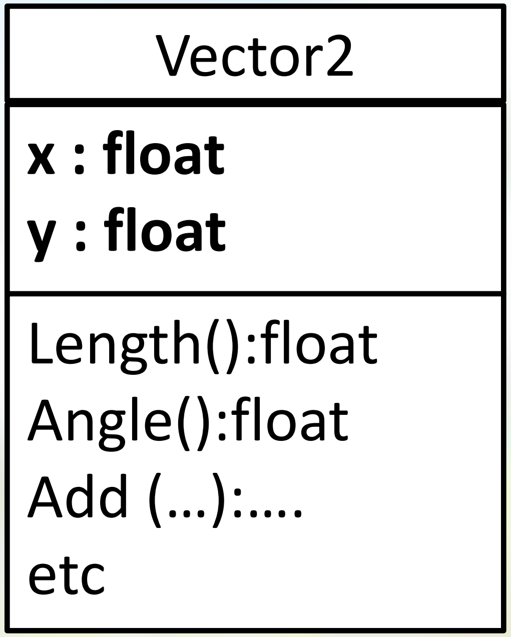
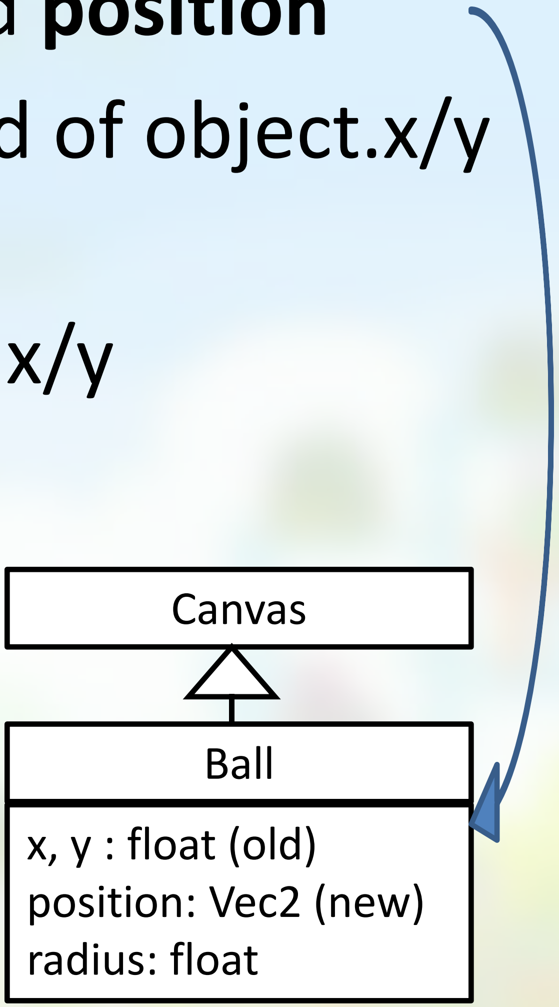


Defining vectors in code

```
class Vec2 {
    public float x = 0;
    public float y = 0;
}
```

How do we use our Vec2 for positions?

- Give objects a **Vec2** instance called **position**
- Change **object.position.**x/y instead of object.x/y
- On object Step() (or Update())
  copy object.position.x/y to object.x/y
- Demo +002_ball_vector_position



Vec2: Class or struct?

In C#, *classes* and *structs* are very similar. The only difference:

- A *class* is a *reference type* (it refers to an object in memory that has to be explicitly created using *new).*
- A *struct* is a *value type* (its value is stored directly where the variable is "defined").

Consider this code – what happens? What is the value of `v.x`?

```
Vec2 v = new Vec2(2,3);
Vec2 w = new Vec2(4,5);
v = w;
w.x = 6;
```

Class Case

```
Vec2 v = new Vec2(2,3);
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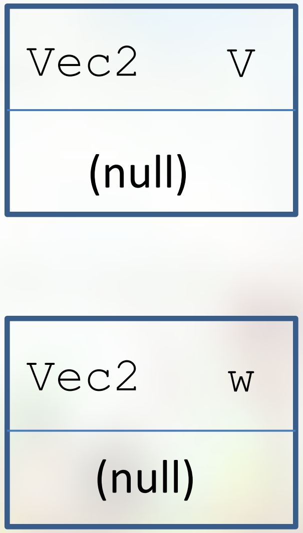
```

Variables:

Memory (heap):



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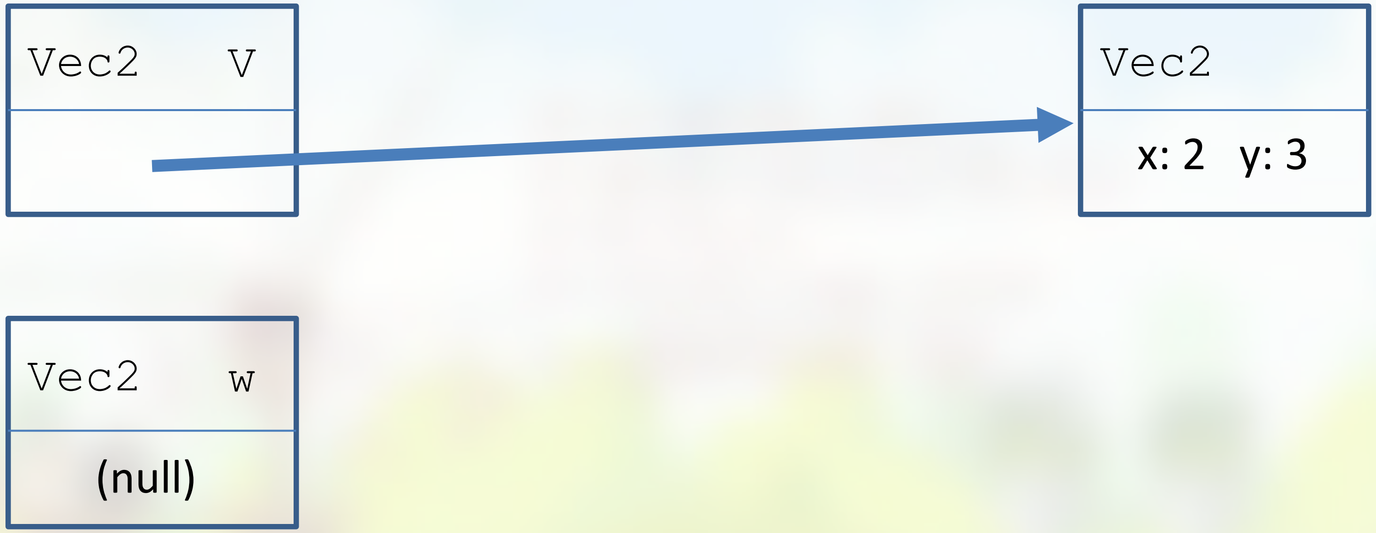

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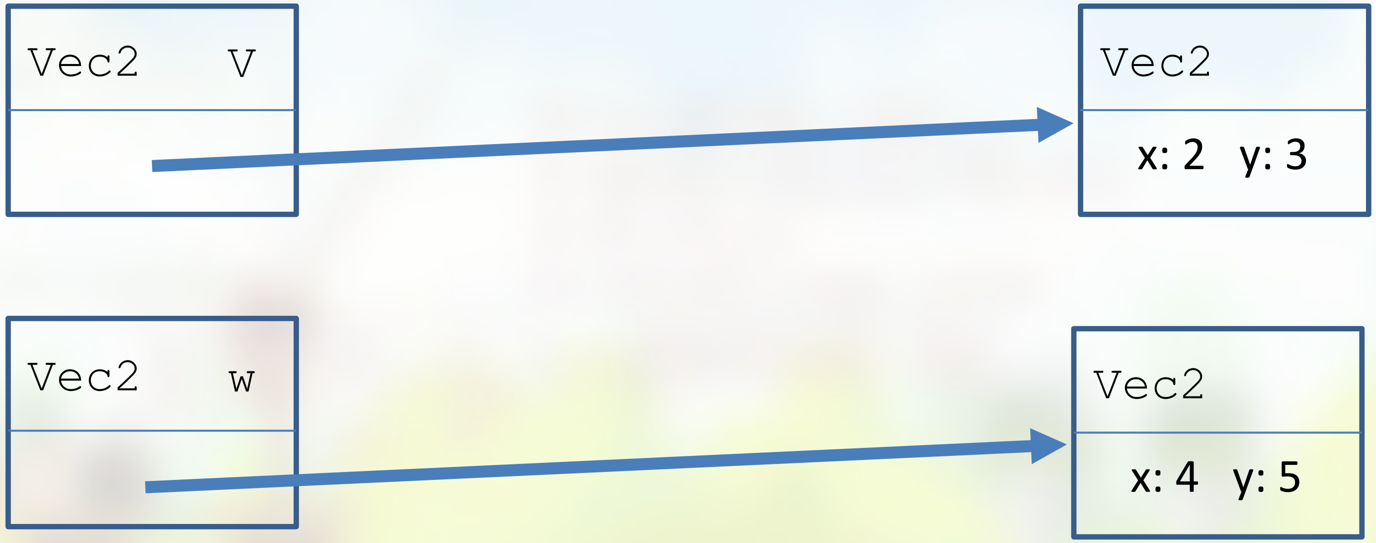

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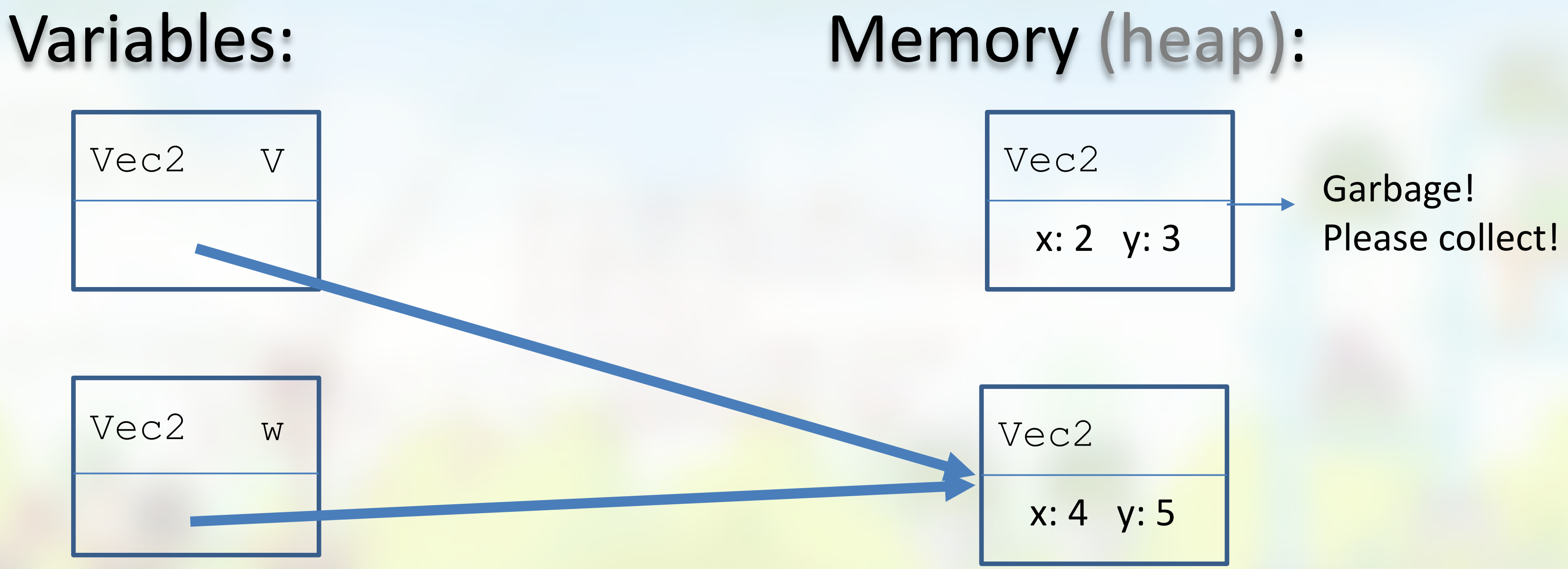

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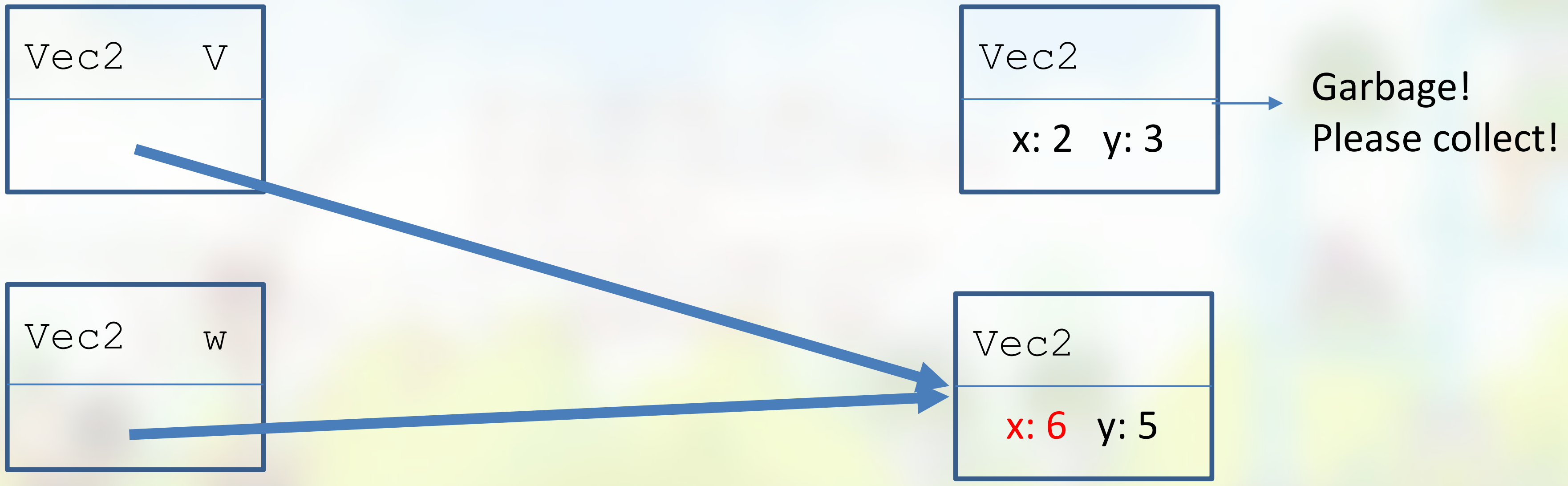

Struct Case

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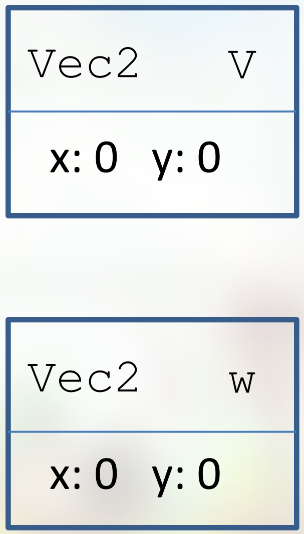
w.x = 6;
```

Variables:



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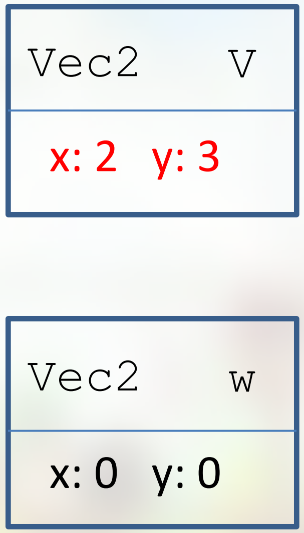

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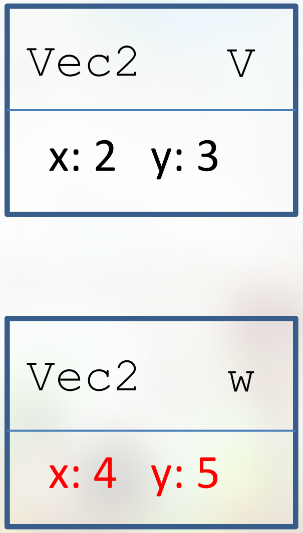

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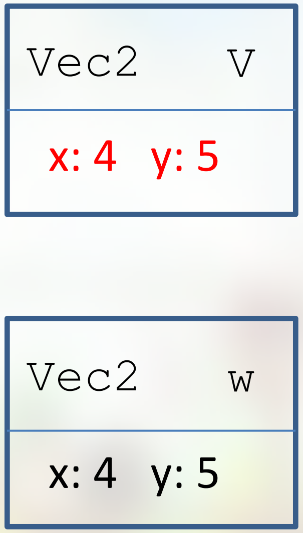

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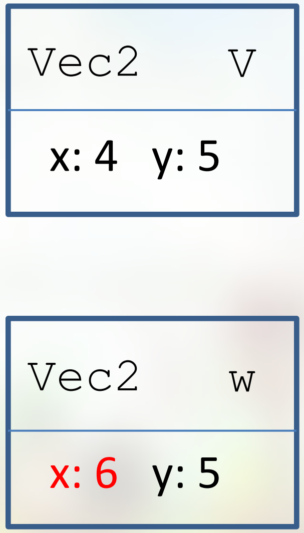


(Dis)advantages of using structs?

Advantages:

- No “shared reference” bugs – modifying a struct cannot have unexpected consequences

  (you’ll be *so* thankful for that!)

- Efficient: No *memory allocation, pointer dereferencing, garbage collection* needed for doing simple operations

  (…and by the end of this course, you’ll be doing a *lot* of operations with vectors!)

- Easy to understand

  (…unless you’re only used to classes, and don’t really know the difference!)

(Dis)advantages of using structs?

Disadvantages:

- For every assignment operation / method call with struct arguments, *all data in the struct is copied* – inefficient for large structs.

  (…but we only have two floats)

- Structs cannot be *null*.

  (…which is sometimes useful as a "special value")

- Sometimes you *want* to copy something, change it, and have the original changed too.
- You cannot use *inheritance* for structs.

Conclusion

- You should use classes for most things, especially for complex objects, but
- …it's best to make our Vec2 a struct.

    (See also Unity, and other C# libraries with vectors)

- …but you need to know the differences between structs and classes to avoid bugs!
- Study the example 001_class_vs_struct closely!

Vector addition in code

Vector addition in code:

- Here's one way to implement addition for Vec2 structs:

```
struct Vec2 {
    public float x = 0;
    public float y = 0;

    public void Add (Vec2 other) {
        x += other.x;
        y += other.y;
    }
}
```

Vector addition in code:

```
public void Add (Vec2 other) {
        x += other.x;
        y += other.y;
}
```

Some OO design decisions are already made here:

- Method parameter is Vec2, not two floats

  (Conceptually better, and we should start using Vec2 for everything anyway.)

- Calling the method *modifies* the struct

  (…which is somewhat controversial: https://docs.microsoft.com/en-us/dotnet/standard/design-guidelines/struct)

- We don’t return anything

  (Returning *this* is useful, but confusing combined with modification)

Operator overloading

Q: Can’t we just add a bunch a vectors together without modifying them? Preferably with very readable code, like this?:

```
   Vec2 sum = v1 + v2 + v3 + v4;
```

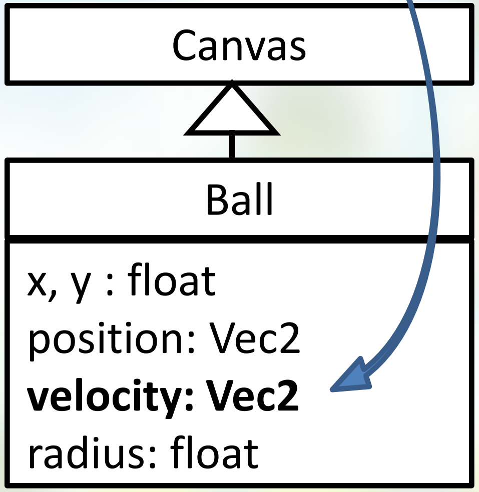
A: Yes, using *operator overloading*!

→ see 001_class_vs_struct

(and https://docs.microsoft.com/en-us/dotnet/csharp/language-reference/keywords/operator)

Using Vec2 for adding velocity

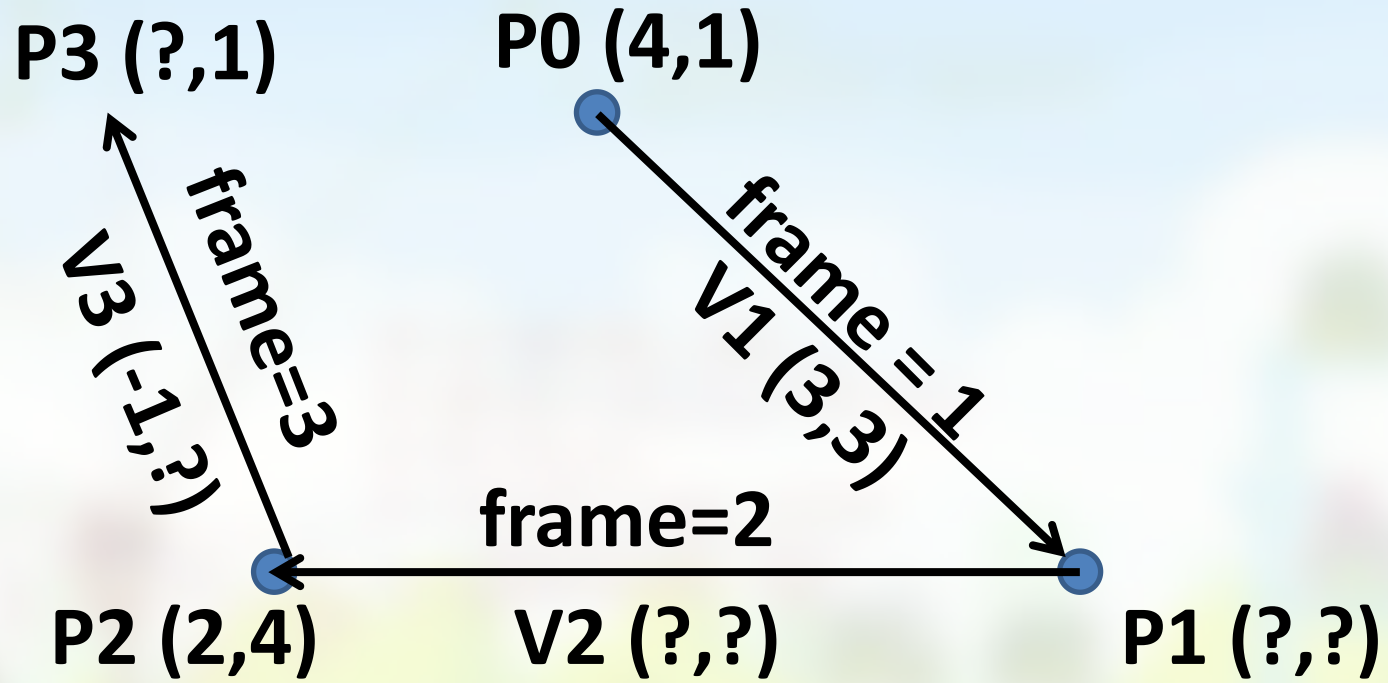
- How do we use Vec2 to implement velocity?
  - Give objects a **Vec2** instance called **velocity**
  - Stop updating **object.position** x,y directly
  - Change only **object.velocity** x,y directly
  - On object Step:
    - **Add** object velocity to object position
    - **Copy** object position to object x,y
  - This is called **Euler Integration** **(Oiler):**
    - We store explicit position
    - We store explicit velocity
    - We add velocity to position each frame
- Example +003_ball_vector_velocity
  - Note: we seem to move faster diagonally… Why?



Velocity, units?

- Velocity is movement over time
- This implies two different measurements:
  - one for distance
  - one for time
- For example:
  - kilometers/miles per hour
  - meters per second
- In our code, velocity is defined in pixels/frame

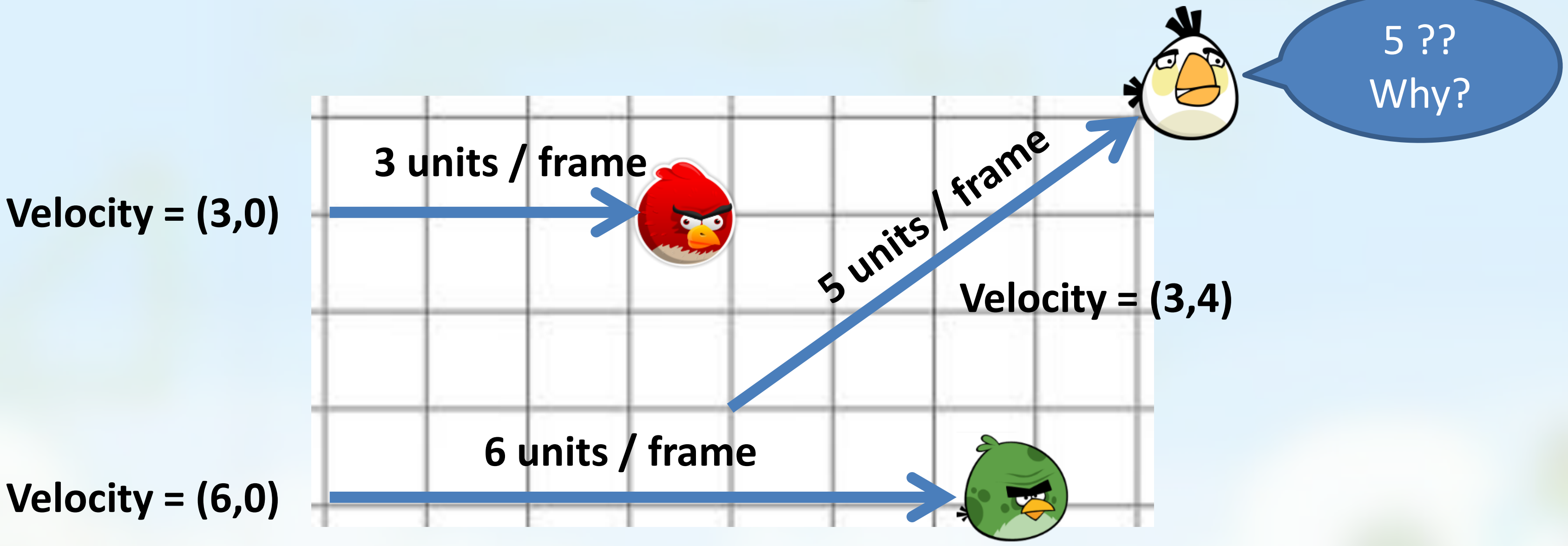
Velocity (and thus position) can change each frame!
Exercise: fill in the correct details



Vector length

On calculating the length of a vector to understand why we go faster diagonally…

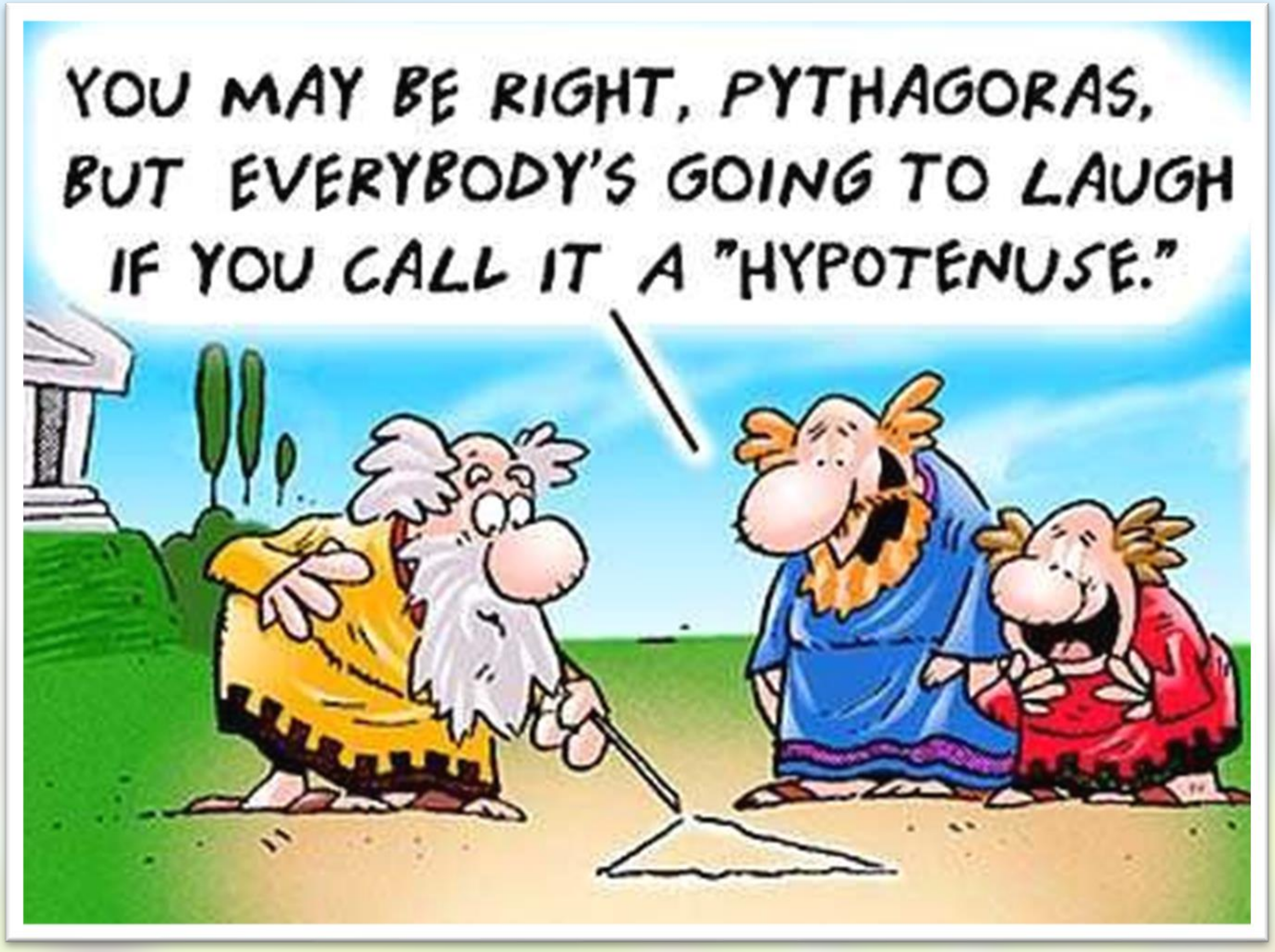
Vector length



- A vector defines both a direction and a length
- For velocity, vector length corresponds with speed
- The *length* of vector $\vec{v}$ is called *magnitude*, in textbook notation written as $|\vec{v}|$
- How can we calculate the length?

*direction == orientation / magnitude == size == length

Pythagoras, 572 BC – 500 BC (!)


YOU MAY BE RIGHT, PYTHAGORAS, BUT EVERYBODY'S GOING TO LAUGH IF YOU CALL IT A "HYPOTENUSE."

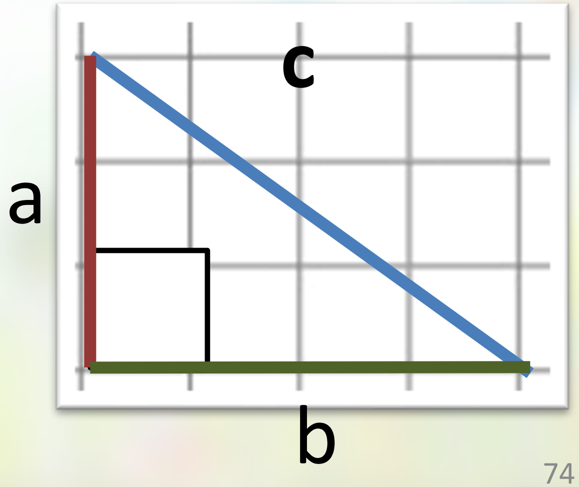
Pythagoras’ Theorem

- A *right-angled triangle* has one 90° angle.
- The side opposite to the 90° angle is called *hypotenuse*.

Pythagoras’s Theorem:

In a right-angled triangle, where *c* is the length of the hypotenuse and *a* and *b* are the lengths of the other sides, it holds that:

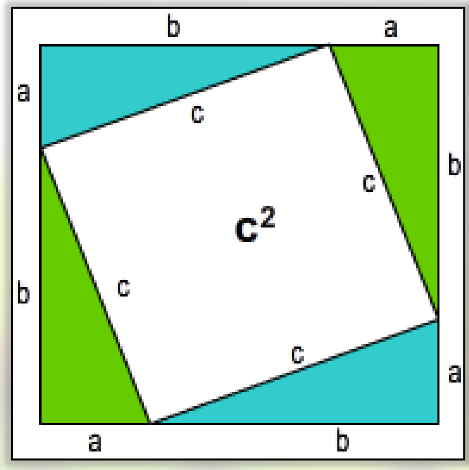
$$a^2 + b^2 = c^2$$



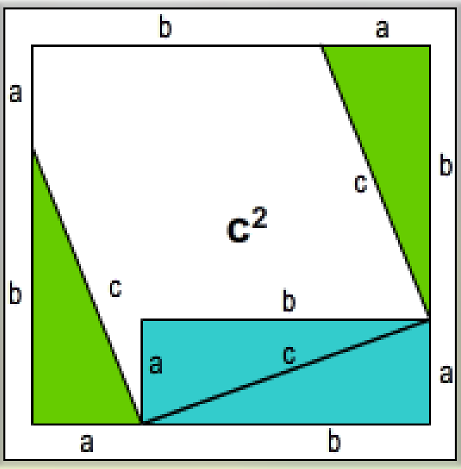
Pythagoras's Theorem – Proof sketch

- Draw an outer rectangle with area: (a+b)*(a+b)
- Draw an inner rectangle with area: c*c
- Rearrange the colored pieces to demonstrate that the empty area c*c == a*a + b*b

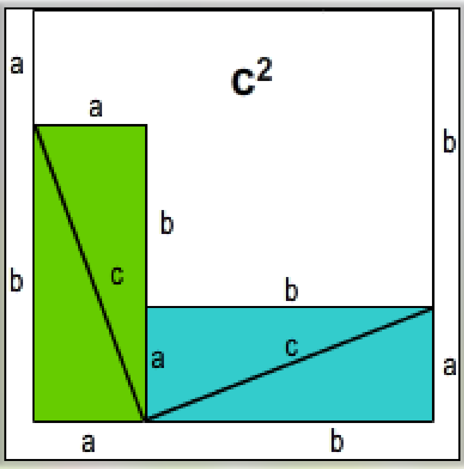
1



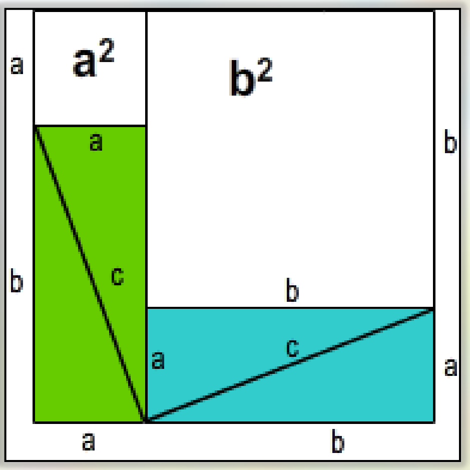
2



3.1

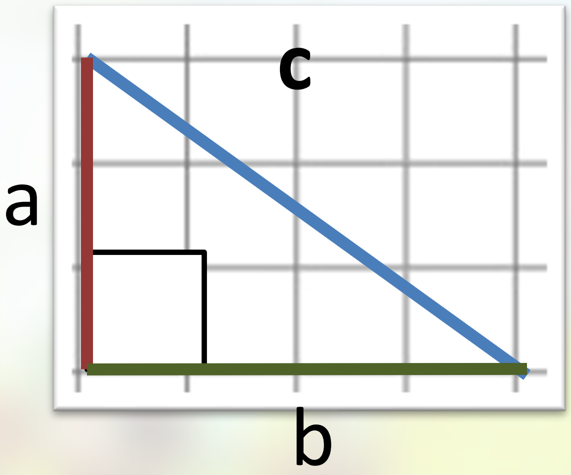


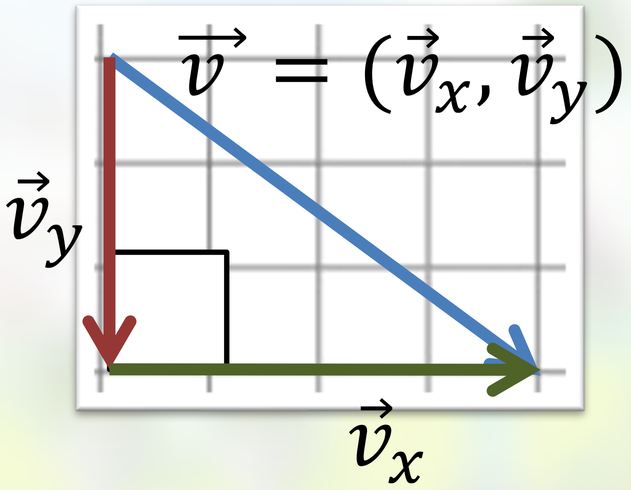
3.2



Pythagoras

- Pythagoras's Theorem: $c^2 = a^2 + b^2$
- Applied to vectors: $|\vec{v}|^2 = \vec{v}_x{}^2 + \vec{v}_y{}^2$





Resulting math for vector length

$$|\vec{v}|^2 = \vec{v}_x{}^2 + \vec{v}_y{}^2 \quad \rightarrow \quad |\vec{v}| = \sqrt{\vec{v}_x{}^2 + \vec{v}_y{}^2}$$

- Practice, calculate length of these vectors:
  - (3,0)
  - (0,4)
  - (3,4)
  - (4,5)
  - (5,0)
  - (5,5)

The result of these two prove we are indeed moving faster diagonally

Vector length in code

Implementing Vector length

- Add a method Length() to the Vec2 class:

```
public float Length() {
    return ...;
}
```

- This is part of assignment 1 !
- But this only **gets** the current length !

Setting vector length?

- How can we **change** the length of the velocity vector to enforce a **specific** speed?
  - For example: we want to go south east with a speed of 5 pixels per frame:
    - (5,5) ?
    - Direction is correct, but speed isn’t since it’s ≈ 7

Changing the vector length

On how we can change the length of a vector to enforce a specific speed

Vector scaling

- Scaling a vector $\vec{v}$ with a factor k *(=scalar)* is written as: $\vec{v}' = k * \vec{v}$
- Or shorter: $\vec{v}' = k\vec{v}$
- Scaling is done component-wise, like adding:

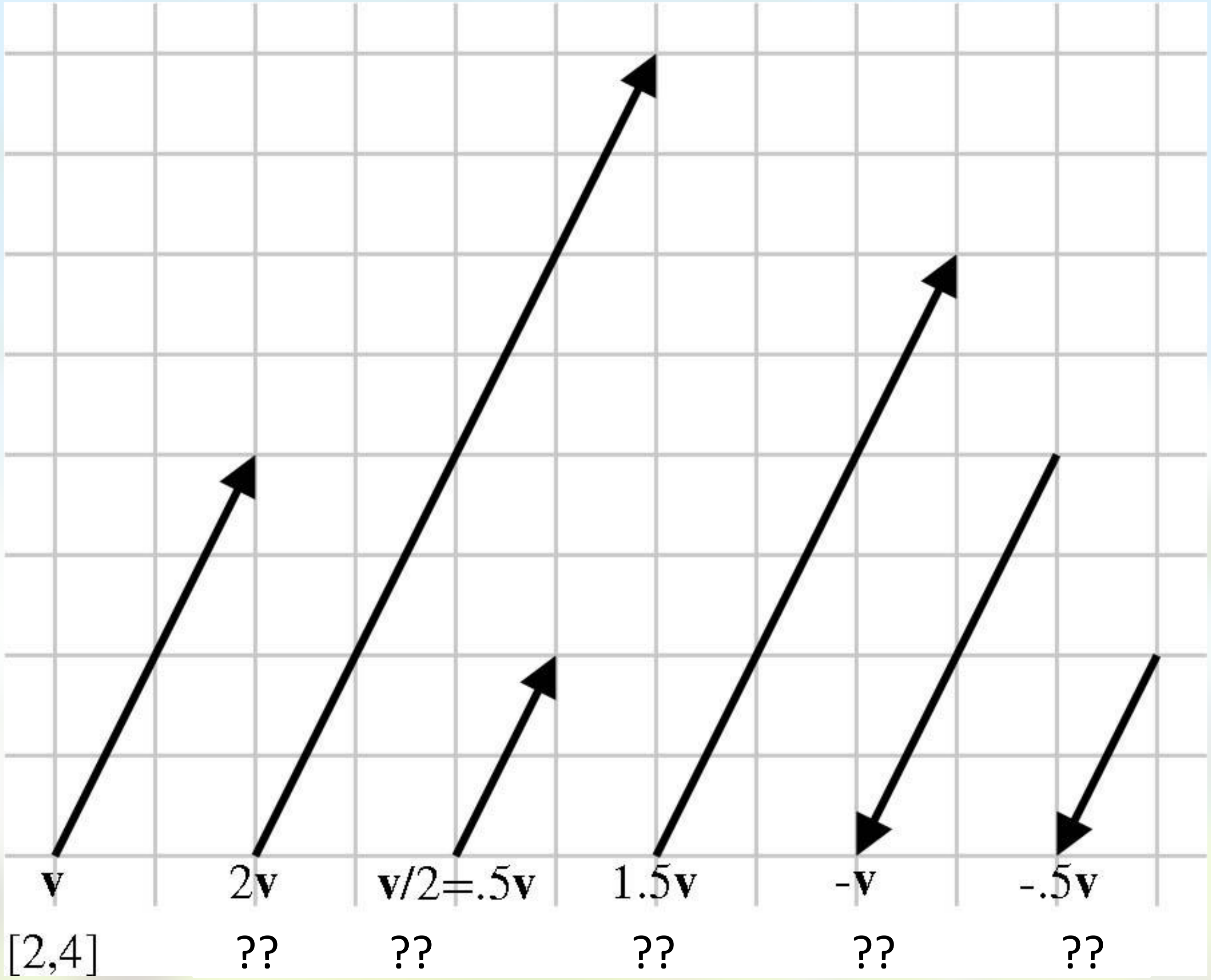
$$k * \vec{v} = (k * \vec{v}_x , k * \vec{v}_y)$$

- In code: implement a method that allows you to write:

```
Vec2 scaledVec = 3 * origVec;
```

(= *overload* the * operator.)

Vector scaling practice



Setting vector length

- Now we know how to change the length of a vector **without** changing its direction:
  - through **scaling**

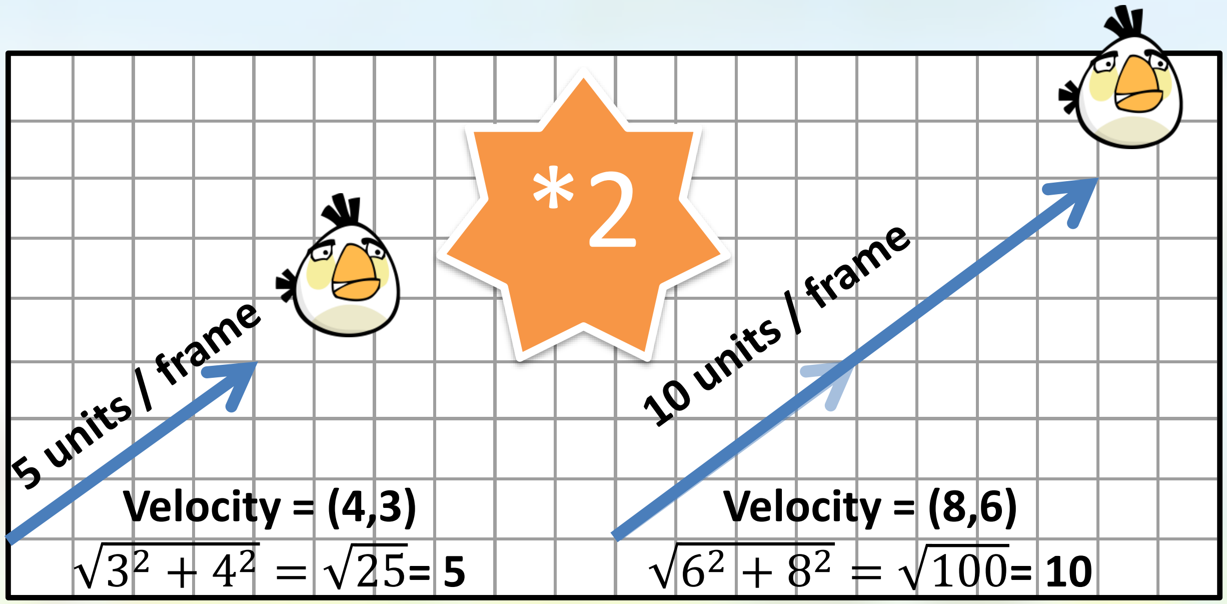
- Which brings us back to our previous question:
  - How can we change (scale) the length of the velocity vector to enforce a **specific** speed?
  - E.g. how much do I have to scale (8,6) to end up with a speed of 5?

Unit vectors

Where we learn to separate length from direction, so that we can travel at any desired speed

Effect of scaling on length

When we scale a vector,
the length change is proportional to the scale factor:



Velocity = (4,3)

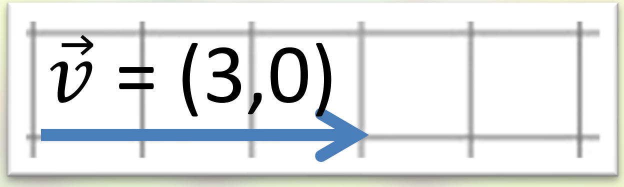
$\sqrt{3^2 + 4^2} = \sqrt{25}$= **5**

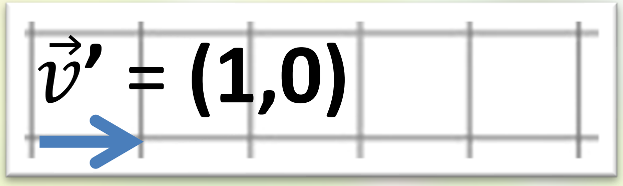
Velocity = (8,6)

$\sqrt{6^2 + 8^2} = \sqrt{100}$= **10**

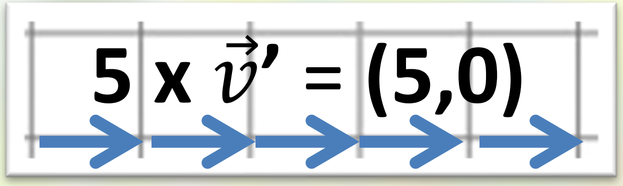
Scaling a vector to length L?

- So how do we scale the length of vector $\vec{v}$ to a specific length L?
- The easiest approach is:
  1. **First** scale $\vec{v}$ by $\frac{1}{|\vec{v}|}$ so that it has length 1
  2. **Then** scale this new $\vec{v}$ times L so that it has length L
- In pictures for $\vec{v}$(3,0) and **desired** speed = **5**:





convert to length 1



scale speed times

Normalizing & Unit vector

- Scaling a vector's length to **1** is called **normalizing:**
  - To normalize $\vec{v}$ you scale it by $\frac{1}{|\vec{v}|}$
  - The result is called a **unit vector**, written as $\hat{v}$:
  - $\hat{v} = \frac{1}{|\vec{v}|} * \vec{v}$ which equals $\hat{v} = (\frac{\vec{v}_x}{|\vec{v}|}, \frac{\vec{v}_y}{|\vec{v}|})$

Vector properties: Normalizing

- A little bit of practice, normalize this:
  - $\vec{v}(5,0)$
  - $\vec{v}(1,1)$
  - $\vec{v}(3,4)$

=??????

- How can we verify that a vector is normalized?

Vector normalizing

- Implement a method Normalize():

```
public void Normalize () {
            .....
}
```

- Watch out for zero length vectors…

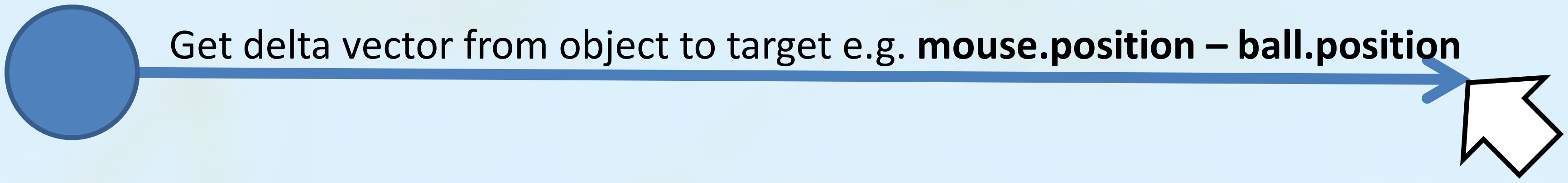
Updating example 2

- Now you can fix example 003:
  - set the velocity as before
  - normalize the velocity
  - scale the velocity with desired speed

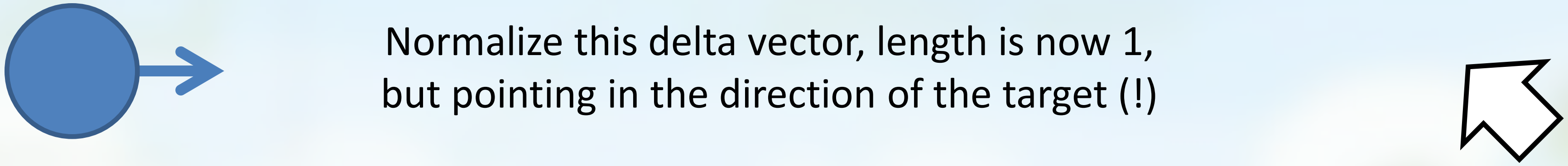
- Can we also use this knowledge to move **to** a specific **target** at a desired speed?

How can we use it to follow a target?

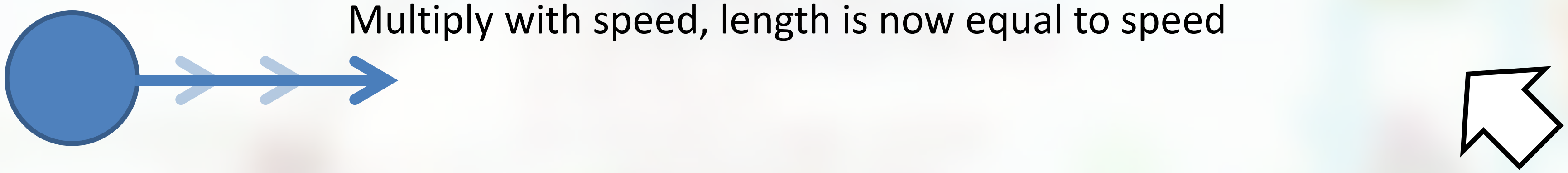
Get delta vector from object to target e.g. **mouse.position – ball.position**



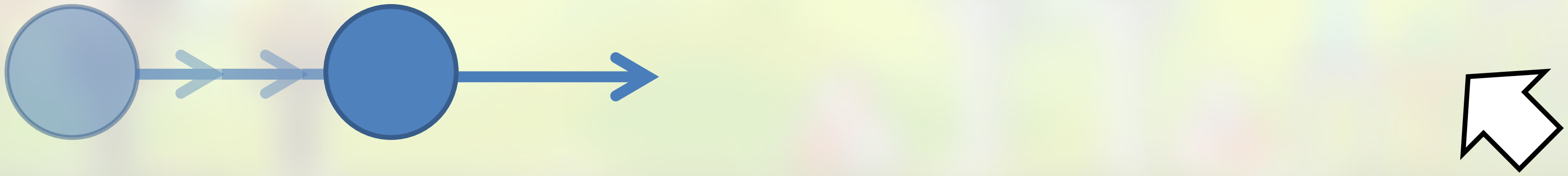
Normalize this delta vector, length is now 1,
but pointing in the direction of the target (!)



Multiply with speed, length is now equal to speed



Apply vector as velocity, moving ball with speed each frame towards target



Mouse Following

- Example 004_ball_follow_mouse:
  - Gets difference vector between ball and mouse
  - Normalizes difference vector (speed is now 1)
  - Multiplies the normalized vector with desired speed
  - Increases desired speed every frame
- Result:
  - Ball follows mouse at desired speed
- Part of assignment 1!

Lab work / Home work

- Complete the short test on Blackboard about vector basics.
- Doing this *unlocks* Assignment 1:

  Highly recommended:
  - Build the start of your Vec2 struct
  - Test whether you did this correctly → *unit tests*
  - Implementing small mouse following “game”
- Show this to your lab teacher for *feedback* (and bonus points) week 3.5 at the latest!

Look ahead: other things you can do

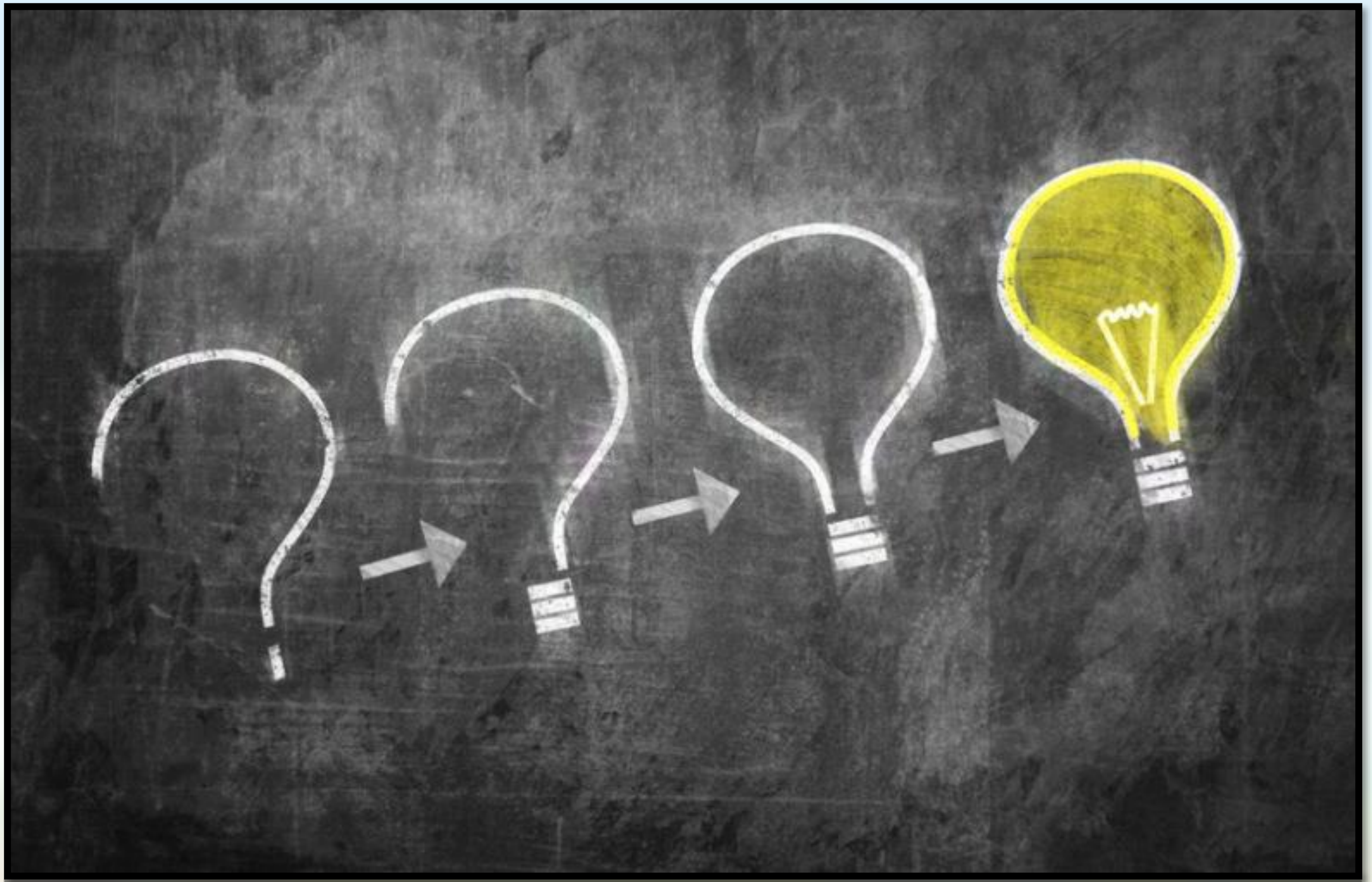
- Add *acceleration:* don't "overwrite" velocity every frame, but change it by *adding* to it (based on gravity, magnetism, spring constraints, …).
- Add *friction / viscosity:* scale the velocity every frame by a factor $0 < f < 1$, such that the object slows down gradually.
- *(Advanced:) Leading a target:* don't aim at the target's current position, but where it will be when the bullet arrives.

- <demo's>

Extra Resources Week 1

Basics:

- Introduction to vectors, by 3Blue1Brown:
  - https://www.youtube.com/watch?v=fNk_zzaMoSs
- See also the links in the manual, e.g.:
  - Math for Game Developers (check the first few video's): https://www.youtube.com/channel/UCEhBM2x5MG9-e_JSOzU068w
- Now that you know the right search terms, google can find a lot more info for you. ☺



Additional proofs

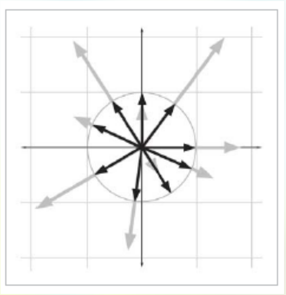
Scale proof

- Scaling $\vec{v}$ by k, changes the length of $\vec{v}$ by k:
  - If $\vec{v}' = k\vec{v}$, then $|\vec{v}'| = |k\vec{v}| = k|\vec{v}|$

$\vec{v}' = k\vec{v} = (k\vec{v}_x, k\vec{v}_y)$	1. According to definition of scaling
$\lvert k\vec{v}\rvert = \sqrt{(k\vec{v}_x)^2+(k\vec{v}_y)^2}$	2. According to pythagoras
$\lvert k\vec{v}\rvert = \sqrt{k^2\vec{v}_x{}^2 + k^2\vec{v}_y{}^2}$	3. According to $(k\vec{v}_x)^2 = k\vec{v}_x k\vec{v}_x = kk\vec{v}_x\vec{v}_x = k^2\vec{v}_x{}^2$
$\lvert k\vec{v}\rvert = \sqrt{k^2 * (\vec{v}_x{}^2 + \vec{v}_y{}^2)}$	4. Bring k^2 outside of the bracket
$\lvert k\vec{v}\rvert = \sqrt{k^2} * \sqrt{(\vec{v}_x{}^2 + \vec{v}_y{}^2)}$	5. Split roots
$\lvert k\vec{v}\rvert = k * \sqrt{\vec{v}_x{}^2 + \vec{v}_y{}^2} = k\lvert\vec{v}\rvert$	6. Simplify and use pythagoras definition

Normalizing & Unit vector

- To normalize $\vec{v}$ you scale it by $\frac{1}{|\vec{v}|}$
- The result is a **unit vector** written as $\hat{v}$,
  - $\hat{v} = \frac{1}{|v|} * v$ which equals $\hat{v} = (\frac{v_x}{|v|}, \frac{v_y}{|v|})$
- Proof:
  - Given $|\vec{v}'| = |k\vec{v}|$ =$k|\vec{v}|$ and k = $\frac{1}{|\vec{v}|}$
  - Then $|\hat{v}| = |k\vec{v}|$ = $k|\vec{v}|$ = $\frac{1}{|\vec{v}|}$ * $|\vec{v}|$ = $\frac{|\vec{v}|}{|\vec{v}|}$ = 1



Bonus

- For those brave souls who made it this far, who are hungry for more math… ☺

  Here's how you "lead a moving target":

  http://playtechs.blogspot.nl/2007/04/aiming-at-moving-target.html