CSE400 - Fundamentals of Probability in Computing
Lecture 11: Transformation of Random Variables

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1. Transformation of Random Variables

Let

$$Y = g(X)$$

We consider two cases:

Case 1: $g(x)$ Monotonically Increasing (+ve case)

Step 1: CDF Method

$$\begin{aligned}
F_Y(y) &= P(Y \le y) \\
&= P(g(X) \le y) \\
&= P(X \le g^{-1}(y)) \\
&= F_X(g^{-1}(y))
\end{aligned}$$

Step 2: Differentiate to get PDF

$$\begin{aligned}
f_Y(y) &= \frac{d}{dy} F_X(g^{-1}(y)) \\
&= f_X(g^{-1}(y)) \cdot \frac{d}{dy} g^{-1}(y)
\end{aligned}$$

If $x = g^{-1}(y)$,

$$f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|$$

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Case 2: $g(x)$ Monotonically Decreasing (-ve case)

Step 1:

$$
\begin{aligned}
F_Y(y) &= P(Y \leq y) \\
&= P(X \geq g^{-1}(y)) \\
&= 1 - F_X(g^{-1}(y))
\end{aligned}
$$

Step 2: PDF

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| \quad \text{where } x = g^{-1}(y)$$

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Example

Let

$$X \sim \text{Uniform}(-1, 1)$$

$$f_X(x) = \begin{cases} \frac{1}{2}, & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Define

$$Y = \sin\left(\frac{\pi X}{2}\right)$$

Step 1: Inverse transformation

$$y = \sin\left(\frac{\pi x}{2}\right)$$

$$x = \frac{2}{\pi}\sin^{-1}(y)$$

Step 2: Derivative

$$\frac{dx}{dy} = \frac{2}{\pi} \cdot \frac{1}{\sqrt{1 - y^2}}$$

Step 3: Apply formula

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$$

$$= \frac{1}{2} \cdot \frac{2}{\pi} \cdot \frac{1}{\sqrt{1 - y^2}}$$

$$= \frac{1}{\pi\sqrt{1 - y^2}}, \quad -1 < y < 1$$

$$f_Y(y) = 0 \quad \text{otherwise}$$

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2. Function of Two Random Variables

Let

$$Z = X + Y$$

We want to find:

(i) $f_Z(z)$

(ii) $f_Z(z)$ if X and Y are independent

(iii) If $X \sim N(0,1)$ and $Y \sim N(0,1)$, prove $Z \sim N(0,2)$

(iv) If X and Y are exponential with parameter λ, find $f_Z(z)$

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Detailed Derivation for $Z = X + Y$

$$F_Z(z) = P(Z \leq z)$$

$$= P(X + Y \leq z)$$

$$= \iint_{x+y \leq z} f_{X,Y}(x,y)\,dx\,dy$$

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Horizontal Strip Method

$$F_Z(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{z-y} f_{X,Y}(x,y)\,dx\,dy$$

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Vertical Strip Method

$$F_Z(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{z-x} f_{X,Y}(x,y)\,dy\,dx$$

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If X and Y are Independent

$$f_{X,Y}(x,y) = f_X(x) f_Y(y)$$

Therefore,

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\,dx$$

This is called **Convolution**.

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Normal Case

If

$$X \sim N(0,1), \quad Y \sim N(0,1)$$

Then

$$Z = X + Y \sim N(0, 1+1) = N(0,2)$$

Because:

$$E(Z) = E(X) + E(Y) = 0$$

$$\mathrm{Var}(Z) = \mathrm{Var}(X) + \mathrm{Var}(Y) = 2$$

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Exponential Case

If

$$f_X(x) = \lambda e^{-\lambda x}, \quad x \geq 0$$

$$f_Y(y) = \lambda e^{-\lambda y}, \quad y \geq 0$$

Then

$$f_Z(z) = \int_0^z \lambda e^{-\lambda x} \lambda e^{-\lambda(z-x)} dx$$

$$= \lambda^2 e^{-\lambda z} \int_0^z dx$$

$$= \lambda^2 z e^{-\lambda z}, \quad z \geq 0$$

This is Gamma distribution with parameters $(2, \lambda)$.