# Lecture 11 Scribe: Transformation of Random Variables

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## 1 Outline and Learning Objective

### 1.1 Topics in this lecture

1. **Transformation of Random Variables:** learning transformation techniques for random variables.
2. **Function of Two Random Variables:** joint transformations and derived distributions.
3. **Illustrative Example:** detailed derivation for the case $Z = X + Y$.

### 1.2 Assumption / setup stated in the outline

**Assumption:** the PDF of the original RV (e.g., $f_X(x)$) is known *a priori*; the goal is how to find the PDF of the new transformed RV (e.g., $f_Y(y)$).

### 1.3 Noted example transformations (as written on the outline slide)

$$Z_1 = X + Y, \quad Z_2 = X - Y, \quad Z_3 = \frac{X}{Y}, \quad Z_4 = \sqrt{X^2 + Y^2}.$$

## 2 Transformation of One Random Variable: $Y = g(X)$

### 2.1 Definitions and notation (as used on the slides)

A transformation is defined by

$$Y = g(X).$$

CDF notation: $F_Y(y)$ and $F_X(x)$.
PDF notation: $f_Y(y)$ and $f_X(x)$.
The slides proceed via **Step S1 (CDF)** then **Step S2 (differentiate w.r.t. $y$)**.

### 2.2 Assumption / condition: monotonicity

The slide distinguishes monotonic cases:

- **Monotonically increasing** case
- **Monotonically decreasing** case

### 2.3 Case A: $g(\cdot)$ is monotonically increasing

**Step S1 (CDF method)**

$$F_Y(y) = \Pr(Y \le y) = \Pr(g(X) \le y) = \Pr\big(X \le g^{-1}(y)\big) = F_X\big(g^{-1}(y)\big).$$

**Step S2 (Differentiate w.r.t. $y$)**

$$f_Y(y) = \frac{d}{dy}\Big[F_X\big(g^{-1}(y)\big)\Big] = f_X\big(g^{-1}(y)\big) \cdot \frac{d}{dy}\big(g^{-1}(y)\big).$$

Let $x = g^{-1}(y)$:

$$f_Y(y) = f_X(x)\,\frac{dx}{dy}\Big|_{x=g^{-1}(y)}.$$

### 2.4 Case B: $g(\cdot)$ is monotonically decreasing

**Step S1 (CDF method)**

$$F_Y(y) = \Pr(Y \le y) = \Pr\big(X \ge g^{-1}(y)\big) = 1 - F_X\big(g^{-1}(y)\big).$$

**Step S2 (Differentiate w.r.t. $y$)**

$$f_Y(y) = \frac{f_X(x)}{\left|\dfrac{dy}{dx}\right|}\Bigg|_{x=g^{-1}(y)}.$$

**S3 note:** change the limits for $y$ (determine valid $y$-range from the mapping).

## 3 Worked Example: $X \sim \text{Uniform}(-1,1)$, $\quad Y = \sin\left(\frac{\pi x}{2}\right)$

### 3.1 Given (from the example slide)

- $X$ is uniform on $(-1, 1)$.
- Transformation:
$$Y = g(x) = \sin\left(\frac{\pi x}{2}\right).$$
- PDF of $X$:
$$f_X(x) = \begin{cases} \dfrac{1}{2}, & -1 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$
- Objective: find $f_Y(y)$.

### 3.2 Step-by-step solution (as shown)

**Step 1: Invert the transformation**

$$y = \sin\left(\frac{\pi x}{2}\right) \quad \Rightarrow \quad x = \frac{2}{\pi}\sin^{-1}(y).$$

**Step 2: Differentiate $x$ w.r.t. $y$**

$$\frac{dx}{dy} = \frac{2}{\pi} \cdot \frac{1}{\sqrt{1-y^2}}.$$

**Step 3: Apply the transformation formula**

$$f_Y(y) = f_X(x) \cdot \left|\frac{dx}{dy}\right|.$$

Substitute $f_X(x) = \frac{1}{2}$:

$$f_Y(y) = \frac{1}{2} \cdot \frac{2}{\pi} \cdot \frac{1}{\sqrt{1-y^2}} = \frac{1}{\pi\sqrt{1-y^2}}.$$

**Step 4: Determine the valid range of $y$ from endpoints**

- At $x = -1$: $y = \sin\left(-\frac{\pi}{2}\right) = -1$
- At $x = 1$: $y = \sin\left(\frac{\pi}{2}\right) = 1$

So the support is $-1 < y < 1$ (otherwise 0).

### 3.3 Final answer (as presented)

$$f_Y(y) = \begin{cases} \dfrac{1}{\pi\sqrt{1-y^2}}, & -1 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

## 4 Function of Two Random Variables: Example Setup ($Z = X + Y$)

### 4.1 Problem statements listed on the slide

For $Z = X + Y$, the slide lists:

1. Find the PDF of $Z$: $f_Z(z)$.
2. Find $f_Z(z)$ if $X$ and $Y$ are independent.
3. Let $X \sim N(0, 1)$ and $Y \sim N(0, 1)$; prove that $Z \sim N(0, 2)$.
4. If $X$ and $Y$ are exponential RVs with parameter $\lambda$, find $f_Z(z)$.

## 5 Detailed Derivation: CDF setup for $Z = X + Y$ via Region Integration

### 5.1 Definition of $Z$ and CDF start

$$Z = X + Y, \qquad F_Z(z) = \Pr(Z \le z) = \Pr(X + Y \le z).$$

### 5.2 Region description and the "strip" integrals

Using the joint pdf $f_{XY}(x, y)$ and the boundary line $x + y = z$:

**(A) Horizontal strip (H-strip) form**

$$F_Z(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{z-y} f_{XY}(x, y)\, dx\, dy.$$

**(B) Vertical strip form**

$$F_Z(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{z-x} f_{XY}(x, y)\, dy\, dx.$$

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*End of Lecture 11 scribe.*