Lecture 6: Discrete Random Variables, Expectation and Problem Solving

(Based strictly on Lecture 6 slides and referenced textbook material)

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1 Random Variables

1.1 Definition

A **random variable** (X) on a sample space (Ω) is a function

$$X : \Omega \to \mathbb{R}$$

that assigns to each sample point $(\omega \in \Omega)$ a real number $(X(\omega))$.

1.2 Discrete Random Variables

Until further notice, attention is restricted to **discrete random variables**, i.e., random variables that take values in a range that is finite or countably infinite.

Although (X) maps (Ω) to $(\mathbb{R})$, the actual set of values

$$\{X(\omega) : \omega \in \Omega\}$$

is a discrete subset of $(\mathbb{R})$.

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2 Visualization of Distributions

The distribution of a discrete random variable can be visualized using a **bar diagram**.

- The **x-axis** represents the possible values that the random variable can take.

- The **height of the bar** at value (a) represents

$$\Pr[X = a].$$

Each probability corresponds to the probability of the associated event in the sample space.

3 Discrete vs Continuous Random Variables

3.1 Discrete Random Variable

- Countable support

- Probability Mass Function (PMF)

- Probabilities assigned to single values

- Each possible value has strictly positive probability

3.2 Continuous Random Variable

- Uncountable support

- Probability Density Function (PDF)

- Probabilities assigned to intervals of values

- Each single value has probability zero

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4 Example: Tossing Three Fair Coins

Let the experiment consist of tossing **three fair coins**. Let (Y) denote the number of heads that appear.

The random variable (Y) takes values in $(\{0, 1, 2, 3\})$ with probabilities:

$$P(Y = 0) = \frac{1}{8}$$
$$P(Y = 1) = \frac{3}{8}$$
$$P(Y = 2) = \frac{3}{8}$$
$$P(Y = 3) = \frac{1}{8}$$

Since (Y) must take one of the values (0) through (3),

$$1 = P\left(\bigcup_{i=0}^{3} \{Y = i\}\right) = \sum_{i=0}^{3} P(Y = i).$$

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5 Probability Mass Function (PMF)

5.1 Definition

A random variable that can take on at most a countable number of possible values is said to be **discrete**.

Let (X) be a discrete random variable with range

$$R_X = \{x_1, x_2, x_3, \ldots\}$$

(finite or countably infinite).

The function

$$P_X(x_k) = P(X = x_k), \quad k = 1, 2, 3, \ldots$$

is called the **Probability Mass Function (PMF)** of (X).

5.2 Property of PMF

Since (X) must take exactly one value from its range,

$$\sum_{k=1}^{\infty} P_X(x_k) = 1.$$

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6 Example: PMF with Exponential Series

The probability mass function of a random variable (X) is given by

$$p(i) = c\frac{\lambda^i}{i!}, \quad i = 0, 1, 2, \ldots$$

where $(\lambda > 0)$.

6.1 Finding the Constant (c)

Since

$$\sum_{i=0}^{\infty} p(i) = 1,$$

we have

$$c\sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = 1.$$

Using

$$e^{\lambda} = \sum_{i=0}^{\infty} \frac{\lambda^i}{i!},$$

it follows that

$$ce^{\lambda} = 1 \quad \Rightarrow \quad c = e^{-\lambda}.$$

6.2 Required Probabilities

$$P(X = 0) = e^{-\lambda}$$

$$P(X > 2) = 1 - P(X \leq 2)$$

$$P(X > 2) = 1 - [P(X = 0) + P(X = 1) + P(X = 2)]$$

7 Bayes' Theorem

7.1 Bayes' Formula (Proposition 3.1)

Using

$$\Pr(A \cap B) = \Pr(B \mid A) \Pr(A),$$

we obtain

$$\Pr(B_i \mid A) = \frac{\Pr(A \mid B_i) \Pr(B_i)}{\sum_{j=1}^{n} \Pr(A \mid B_j) \Pr(B_j)}.$$

7.2 Terminology

- $(\Pr(B_i))$: **a priori probability**

- $(\Pr(B_i \mid A))$: **posterior probability**

8 Bayes' Theorem Example: Auditorium with 30 Rows

An auditorium has **30 rows of seats**.

- Row 1 has 11 seats

- Row 2 has 12 seats

- Row 3 has 13 seats

- . . .

- Row 30 has 40 seats

A door prize is awarded by:

1. Randomly selecting a row (each row equally likely).

2. Randomly selecting a seat within that row (each seat equally likely).

Computations

- Probability that Seat 15 is selected given Row 20 is selected:

$$P(S_{15} \mid R_{20}) = \frac{1}{30}$$

- Probability that Row 20 was selected given Seat 15 was selected:

$$P(R_{20} \mid S_{15}) = \frac{P(S_{15} \mid R_{20})P(R_{20})}{P(S_{15})}$$