Lecture Scribe: CSE 400 - Fundamentals of Probability in Computing

Lecture 6: Discrete RVs, Expectation and Problem Solving

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1 Random Variables: Motivation and Concept

1.1 Definition

A random variable X on a sample space Ω is a function $X : \Omega \to \mathbb{R}$ that assigns a real number $X(\omega)$ to each sample point $\omega \in \Omega$.

1.2 Discrete Random Variables

- **Scope:** Attention is restricted to discrete random variables, which take values in a finite or countably infinite range.
- **Range:** Although X maps Ω to $\mathbb{R}$, the actual set of values $\{X(\omega) : \omega \in \Omega\}$ is a discrete subset of $\mathbb{R}$.
- **Characteristics**:
  - Possesses countable support.
  - Probabilities are assigned to single values.
  - Each possible value has a strictly positive probability.

2 Probability Mass Function (PMF)

2.1 Concept

A random variable that can take on at most a countable number of possible values is said to be discrete. The Probability Mass Function (PMF) gives the probability of each outcome.

2.2 Visualization and Computation

The distribution of a discrete random variable can be visualized as a bar diagram:

- **X-axis:** Represents the values the random variable can take on.
- **Height:** The height of the bar at value a is the probability $Pr[X = a]$.
- **Calculation:** Probabilities are computed by looking at the corresponding event in the sample space.

2.3 Legitimacy Condition

For a PMF to be legitimate, the sum of all probabilities must equal 1:

$$\sum PMF = 1$$

$$1 = P\left(\bigcup_i \{X = i\}\right) = \sum_i P\{X = i\}$$

3 Worked Example: Tossing Fair Coins

Experiment: Tossing 3 fair coins.
Random Variable (Y): Let Y denote the number of heads that appear.
Possible Values: $Y \in \{0, 1, 2, 3\}$.

Probability Calculations:

- $P\{Y = 0\} = P\{(t, t, t)\} = \frac{1}{8}$
- $P\{Y = 1\} = P\{(t, t, h), (t, h, t), (h, t, t)\} = \frac{3}{8}$
- $P\{Y = 2\} = P\{(t, h, h), (h, t, h), (h, h, t)\} = \frac{3}{8}$
- $P\{Y = 3\} = P\{(h, h, h)\} = \frac{1}{8}$

Legitimacy Verification:

$$\sum_{i=0}^{3} P\{Y = i\} = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = 1$$

4 Guide to Selecting a Discrete Probability Distribution

4.1 Binomial Distribution

Definition: $X = \#$ of successes (S) in n trials.
Conditions:

1. Identical trials.
2. Exactly 2 outcomes: Success (S) and Failure (F).
3. $P(S)$ and $P(F)$ remain constant across trials.
4. Trials are independent.

4.2 Poisson Distribution

Definition: $X = \#$ of times a rare event (S) occurs in a unit.
Conditions:

1. $P(S)$ remains constant across units.
2. Unit x values are independent.

4.3 Hypergeometric Distribution

Definition: $X = \#$ of successes (S) in trials.
Conditions:

1. n elements are drawn without replacement from N elements.
2. Exactly 2 outcomes: Success (S) and Failure (F).

5 Summary Comparison: Discrete vs. Continuous

Feature	**Discrete (Countable)**	**Continuous (Uncountable)**
Function Type	Prob. Mass Function (PMF)	Prob. Density Function (PDF)
Probability Assignment	To single values	To intervals of values
Point Probability	Strictly positive (> 0)	Zero (0)
Legitimacy Condition	$\sum PMF = 1$	$\int_{-\infty}^{\infty} f_x(x)dx = 1$