Scribe Information

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Lecture: 11: Transformation of Random Variables
Course: CSE400 - Fundamentals of Probability in Computing
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Date: February 10, 2026

Definitions

Transformation of Random Variables: The learning of transformation techniques for probability density functions (PDF) of random variables.

Function of Two Random Variables: The process involving joint transformations and their derived distributions.

Notation and Assumptions

X and Y represent random variables, with transformations such as $Z_1 = X + Y$, $Z_2 = X - Y$, and $Z_3 = X/Y$.

$f_X(x)$ represents the probability density function (PDF) of the random variable X.

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$F_X(x)$ represents the cumulative distribution function (CDF).

Assumption: The PDF of X, denoted as $f_X(x)$, is known a priori.

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$y = g(x)$ denotes a new transformed random variable.

$f_Y(y)$ and $F_Y(y)$ are used to denote the PDF and CDF of the new random variable.

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Theorems

Objective: To find the CDF and PDF of a new transformed random variable, $f_Y(y)$.

General PDF Transformation Formula: For a transformation, the PDF is defined by the following equation:

$$f_Y(y) = \left. \frac{f_X(x)}{\left|\frac{dy}{dx}\right|} \right|_{x=g^{-1}(y)}$$

Proofs

Derivation for a Monotonically Increasing Transformation:
Step 1 (S_1): Establish the CDF.

$$F_Y(y) = Pr(Y \leq y) = Pr(g(x) \leq y)$$

By using the inverse $x = g^{-1}(y)$, the expression becomes:
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$$F_{PDF} = Pr(x \leq g^{-1}(y)) = F_X(g^{-1}(y))$$

Step 2 (S_2): Differentiate with respect to y.

$$f_Y(y) = \frac{d}{dy}[F_X(g^{-1}(y))]$$

Applying the chain rule yields:

$$f_Y(y) = f_X(g^{-1}(y)) \cdot \frac{d}{dy}g^{-1}(y) = f_X(x)\frac{dx}{dy}\bigg|_{x=g^{-1}(y)}$$

Derivation for a Decreasing Transformation:
Step 1 (S_1): Establish the CDF considering the decreasing nature.

$$F_Y(y) = Pr(Y \leq y) = Pr(X \geq g^{-1}(y)) = 1 - F_X(g^{-1}(y))$$

Step 2 (S_2): Differentiate the CDF.
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$$f_Y(y) = 0 - f_X(x)\frac{dx}{dy}\bigg|_{x=g^{-1}(y)}$$

Step 3 (S_3): Change the limits for y.
Detailed Derivation for the Case $Z = X + Y$:
By isolating variables, we establish $Y = Z - X$ and $X = Z - Y$.
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The cumulative distribution function for Z is:

$$F_Z(z) = Pr(Z \leq z) = Pr(X + Y \leq z)$$

The region is integrated using a strip with respect to the joint distribution $f_{X,Y}(x, y)dydx$.

Worked Examples

Example 1: Uniformly Distributed Continuous Random Variable (CRV)
Given a uniformly distributed random variable over the interval $(-1, 1)$. The PDF is $f_X(x) = 1/2$ for $-1 < x < 1$, and 0 otherwise. * Transformation: $y = g(x) = \sin(\frac{\pi x}{2})$.
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Inverse: $x = \frac{2}{\pi}\sin^{-1}(y)$.
Limits calculation: * For $x = -1$, $y = \sin(\frac{-\pi}{2}) = -1$. For $x = 1$, $y = \sin(\frac{\pi}{2}) = 1$.
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Derivative: The derivative with respect to y is:

$$\frac{dx}{dy} = \frac{2}{\pi}\frac{1}{\sqrt{1-y^2}} = \frac{1}{\pi\sqrt{1-y^2}}$$

The goal is to find the resulting $f_Y(y)$.
Example 2: Derived Distributions for $Z = X + Y$
(i) Find the PDF $f_Z(z)$.
(ii) Find $f_Z(z)$, if X and Y are independent.
(iii) Let $X \sim N(0, 1)$ and $Y \sim N(0, 1)$, prove that $Z \sim N(0, 2)$.
(iv) If X and Y are exponential distributed random variables with a parameter, find $f_Z(z)$.