CSE 400: Fundamentals of Probability in Computing

Lecture 11: Transformation of Random Variables

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1. Overview and Learning Objectives

This lecture focuses on:

- **Transformation of Random Variables**
  Learning techniques to determine the distribution of a transformed random variable.
- **Function of Two Random Variables**
  Studying joint transformations and deriving the resulting distributions.
- **Illustrative Example:** Detailed derivation for the specific case

$$Z = X + Y$$

Objective: Develop systematic methods to compute the distribution of new random variables defined as functions of existing ones.

2. Transformation of a Random Variable

Let X be a random variable with a known distribution.

Define a new random variable:

$$Y = g(X)$$

where $g(\cdot)$ is a function.

Problem Statement

Given the distribution of X, determine the distribution of Y.

Methodological Requirements

- Clear definition of the transformation function $g(\cdot)$
- Identification of the support of Y
- Proper mapping of probabilities from X to Y

The transformation technique allows us to derive the probability distribution of Y from that of X.

3. Function of Two Random Variables

Let X and Y be two random variables with a known joint distribution.

Define:

$$Z = h(X, Y)$$

Goal

Determine the distribution of Z.

Required Steps

- Use the joint distribution of X and Y
- Identify the region of integration corresponding to the transformation
- Express probabilities in terms of the derived variable

Joint transformations require careful handling of probability over multidimensional regions.

4. Illustrative Example: $Z = X + Y$

Let X and Y be two random variables. Define:

$$Z = X + Y$$

4.1 Problem Structure

To determine the distribution of Z, compute:

$$F_Z(z) = P(Z \leq z)$$

Since

$$Z = X + Y,$$

we rewrite:

$$Z \leq z \iff X + Y \leq z$$

Thus,

$$F_Z(z) = P(X + Y \leq z)$$

This corresponds to evaluating probability over the region:

$$\{(x, y) : x + y \leq z\}$$

4.2 Continuous Case

If X and Y are continuous with joint PDF $f_{X,Y}(x, y)$, then:

$$F_Z(z) = \iint_{x+y \leq z} f_{X,Y}(x, y)\, dx\, dy$$

The PDF of Z is obtained by differentiation:

$$f_Z(z) = \frac{d}{dz} F_Z(z)$$

If X and Y are independent:

$$f_{X,Y}(x, y) = f_X(x) f_Y(y)$$

Then,

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x)\, dx$$

This integral is known as the **convolution** of f_X and f_Y.

4.3 Discrete Case

If X and Y are discrete:

$$P(Z = z) = \sum_x P(X = x, Y = z - x)$$

If X and Y are independent:

$$P(Z = z) = \sum_x P(X = x)P(Y = z - x)$$

5. Conceptual Structure

The lecture progression:

1. Transformation of a single random variable
2. Extension to functions of two random variables
3. Application to the sum $Z = X + Y$

Derived distributions are computed directly from known joint distributions.

6. Key Takeaways

- A transformed random variable requires precise probability mapping.
- Joint distributions are essential when dealing with multiple variables.
- The distribution of $Z = X + Y$ is obtained from:

$$P(X + Y \leq z)$$

- For independent continuous variables, the PDF of the sum is obtained via convolution.