Lecture 6 — Discrete RVs, Expectation and Problem Solving (CSE400)

Course: CSE400 — Fundamentals of Probability in Computing
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1. Outline (as listed)

- Previous Lecture Recap: Random Variables (RVs)
- Independent Events (Examples)
- Types of Discrete Random Variables: Bernoulli RV, Binomial RV, Geometric RV, Poisson RV
- Expectation of RVs: definition and example; expectation of a function of RV; linear operation with expectation; n^{th} moments and central moments (variance, skewness, kurtosis)
- The Cumulative Density Function (CDF): definition, properties, examples
- The Probability Density Function (PDF): definition, properties, examples

  Note (scope of this scribe): The attached PDF excerpt provides detailed slide content for random variables, PMF, and Bayes' theorem examples; other outline items above are listed but not developed in the shown slides.

2. Random Variables (Motivation and Concept)

2.1 Definition (Random Variable as a Function)

A random variable (X) on a sample space (Ω) is a function

$$X : \Omega \to \mathbb{R}$$

that assigns to each sample point ($\omega \in \Omega$) a real number ($X(\omega)$).

2.2 Discreteness Restriction (as stated)

Until further notice, attention is restricted to random variables that are **discrete**, i.e., they take values in a range that is **finite or countably infinite**. Even though (X) is defined as mapping (Ω) to $\mathbb{R}$, the actual set of values $\{X(\omega) : \omega \in \Omega\}$ that (X) takes is a **discrete subset** of $\mathbb{R}$.

3. Visualizing a Distribution (Bar Diagram Interpretation)

The distribution of a random variable can be visualized as a bar diagram:

- The **x-axis** represents the values the random variable can take on.
- The **height** of the bar at value a is the probability $\Pr[X = a]$.
- These probabilities can be computed by looking at the probability of the **corresponding event** in the sample space.

4. Discrete vs Continuous Variables (Guide / Checklist Items)

4.1 Discrete Variable (listed properties)

- Countable support
- Probability mass function
- Probabilities assigned to single values
- Each possible value has strictly positive probability

4.2 Continuous Variable (listed properties)

- Uncountable support
- Probability density function (PDF)
- Probabilities assigned to intervals of values
- Each possible value has zero probability

5. Worked Example 1 — Tossing 3 Fair Coins

5.1 Experiment and Random Variable

Experiment: toss **3 fair coins**.
Let (Y) denote the **number of heads** that appear. Then Y takes one of the values $0, 1, 2, 3$.

5.2 Probabilities (with listed outcome groupings)

$$\Pr(Y = 0) = \Pr((t, t, t)) = \frac{1}{8}$$

$$\Pr(Y = 1) = \Pr((t, t, h), (t, h, t), (h, t, t)) = \frac{3}{8}$$

$$\Pr(Y = 2) = \Pr((t, h, h), (h, t, h), (h, h, t)) = \frac{3}{8}$$

$$\Pr(Y = 3) = \Pr((h, h, h)) = \frac{1}{8}$$

5.3 Total Probability Check (as shown)

Since Y must take one of the values 0 through 3,

$$1 = \Pr\left(\bigcup_{i=0}^{3}\{Y = i\}\right) = \sum_{i=0}^{3} \Pr(Y = i).$$

6. Probability Mass Function (PMF)

6.1 Discrete Random Variable (concept statement)

A random variable that can take on at most a **countable number** of possible values is said to be **discrete**.

6.2 Definition and Notation

Let (X) be a discrete random variable with range (possible values)

$$R_X = x_1, x_2, x_3, \ldots \quad \text{(finite or countably infinite).}$$

The function

$$P_X(x_k) = P(X = x_k), \quad \text{for } k = 1, 2, 3, \ldots$$

is called the **Probability Mass Function (PMF)** of (X).

6.3 Required Property (Normalization)

Since (X) must take one of the values x_k, the slides state:

$$\sum_{k=1}^{\infty} P_X(x_k) = 1.$$

7. Worked Example 2 — PMF with $p(i) = c\frac{\lambda^i}{i!}$

7.1 Given

The probability mass function of a random variable (X) is given by

$$p(i) = c\frac{\lambda^i}{i!}, \quad i = 0, 1, 2, \ldots$$

where λ is some positive value.
Find $P[X = 0]$ and $P[X > 2]$.

7.2 Step 1 — Use $\sum_{i=0}^{\infty} p(i) = 1$ to solve for c

Since $\sum_{i=0}^{\infty} p(i) = 1$,

$$c\sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = 1.$$

Using the given series identity on the slide,

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!},$$

we have:

$$ce^{\lambda} = 1 \quad \text{or} \quad c = e^{-\lambda}.$$

7.3 Step 2 — Compute $P[X = 0]$

$$P[X = 0] = e^{-\lambda} \frac{\lambda^0}{0!} = e^{-\lambda}.$$

7.4 Step 3 — Compute $P[X > 2]$

$$P[X > 2] = 1 - P[X \le 2] = 1 - P[X = 0] - P[X = 1] - P[X = 2].$$

With $c = e^{-\lambda}$,

$$P[X > 2] = 1 - e^{-\lambda} - \lambda e^{-\lambda} - \frac{\lambda^2 e^{-\lambda}}{2}.$$

8. Bayes' Theorem (Recap on Slides)

8.1 Stated Relationship

Using:

$$\Pr(AB_i) = \Pr(B_i \mid A)\Pr(A),$$

we get:

$$\Pr(B_i \mid A) = \frac{\Pr(A \mid B_i)\Pr(B_i)}{\sum_{j=1}^{n} \Pr(A \mid B_j)\Pr(B_j)}.$$

This is known as the **Bayes Formula** [**Proposition 3.1**].

8.2 Terminology (as given)

- $\Pr(B_i)$ is the **a priori probability** (probabilities formed from self-evident or presupposed models).
- $\Pr(B_i \mid A)$ is the **posteriori probability** (probabilities derived or calculated after observing certain events of event B_i given A).

9. Worked Example 3 — Auditorium with 30 Rows (Bayes Application)

9.1 Problem Setup

A certain auditorium has **30 rows** of seats.

- Row 1 has **11** seats, Row 2 has **12** seats, Row 3 has **13** seats, and so on to Row 30 which has **40** seats.

A door prize is to be given away by:

1. randomly selecting a **row** (with equal probability of selecting any of the 30 rows), and
2. then randomly selecting a **seat within that row** (each seat in the row equally likely).

Task (as written): compute the probability that **Seat 15** was selected given that **Row 20** was selected, and also find the probability that **Row 20** was selected given that **Seat 15** was selected.

9.2 Notation Used on Slide

- S_{15}: event "Seat 15 was selected"
- R_{20}: event "Row 20 was selected"
- $\Pr_a(\cdot)$ notation appears on the slide for these probabilities.

9.3 Part A — Compute $\Pr_a(S_{15} \mid R_{20})$

From the slide:

$$\Pr_a(S_{15} \mid R_{20}) = \frac{1}{30}.$$

(Reason encoded in the slide's row-size pattern: Row 20 has $(20 + 10 = 30)$ seats, and seats are equally likely within the chosen row.)

9.4 Part B — Compute $\Pr_a(R_{20} \mid S_{15})$

Step 1 — Compute $\Pr_a(S_{15})$ via total probability

The slide expands:

$$\Pr_a(S_{15}) = \sum_{k=5}^{30} \Pr_a(S_{15} \mid R_k) \Pr_a(R_k).$$

(The lower limit $k = 5$ reflects that Seat 15 first exists when a row has at least 15 seats; by the stated pattern, Row 5 has $(5 + 10 = 15)$ seats.)

Using the row-size rule (Row k has $k + 10$ seats), the slide substitutes:

$$\Pr_a(S_{15}) = \sum_{k=5}^{30} \left(\frac{1}{k+10} \cdot \frac{1}{30} \right) = 0.0342.$$

Step 2 — Apply Bayes' formula

$$\Pr_a(R_{20} \mid S_{15}) = \frac{\Pr_a(S_{15} \mid R_{20}) \Pr_a(R_{20})}{\Pr_a(S_{15})}.$$

With $\Pr_a(S_{15} \mid R_{20}) = \frac{1}{30}$, $\Pr_a(R_{20}) = \frac{1}{30}$, and $\Pr_a(S_{15}) = 0.0342$, the slide computes:

$$\Pr_a(R_{20} \mid S_{15}) = \frac{\frac{1}{30} \cdot \frac{1}{30}}{0.0342} = 0.0325.$$