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L6_S2_A.pdf
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Create a lecture scribe which is purely exam oriented reference work of CSE400.: Fundamentals of Probability in Computing - Lecture 6.

The only given context to use is the lecture slides and the textbook material. The scribe should be a close reproduction of the information learnt in the classroom, which can be used to write closed-notes exams.

Included definitions, notation, assumptions, formal statements of results, proofs or derivations, worked examples, only when they are present in the lecture materials, and in the form that they take there. Don't elaborate much with with explanations, intuition, alternative methods of approach, examples. Do not generalize, expand or paraphrase the material more than it is brought out.

Arrange the scribe in clear sections and headings in accordance with the structure and the logical flow of the lecture. Stick to a formal academic style of writing and style.

Never make up anything, never tell us how you came to your conclusion, and never add extra explanations anywhere.

Only the lecture scribe should be generated.

CSE400 – Fundamentals of Probability in Computing

Lecture 6: Discrete Random Variables, Expectation and Problem Solving

(Based strictly on Lecture 6 slides and referenced textbook material) L6_S2_A

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1. Random Variables

1.1 Definition

A **random variable** X on a sample space Ω is a function

$$X : \Omega \to \mathbb{R}$$

that assigns to each sample point $\omega \in \Omega$ a real number $X(\omega)$.

1.2 Discrete Random Variables

Until further notice, attention is restricted to **discrete random variables**, i.e., random variables that take values in a range that is finite or countably infinite.

Although X maps Ω to $\mathbb{R}$, the actual set of values

$$\{X(\omega) : \omega \in \Omega\}$$

is a discrete subset of $\mathbb{R}$.

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2. Visualization of Distributions

The distribution of a discrete random variable can be visualized using a **bar diagram**.

- The **x-axis** represents the possible values that the random variable can take.
- The **height of the bar** at value a represents

$$\Pr[X = a].$$

Each probability corresponds to the probability of the associated event in the sample space.

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3. Discrete vs Continuous Random Variables

3.1 Discrete Random Variable

- Countable support
- Probability Mass Function (PMF)
- Probabilities assigned to single values
- Each possible value has strictly positive probability

3.2 Continuous Random Variable

- Uncountable support
- Probability Density Function (PDF)
- Probabilities assigned to intervals of values
- Each single value has probability zero

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4. Example: Tossing Three Fair Coins

Let the experiment consist of tossing **three fair coins**.
Let Y denote the number of heads that appear.

The random variable Y takes values in $\{0, 1, 2, 3\}$ with probabilities:

$$
\begin{aligned}
P(Y=0) &= \frac{1}{8} \\
P(Y=1) &= \frac{3}{8} \\
P(Y=2) &= \frac{3}{8} \\
P(Y=3) &= \frac{1}{8}
\end{aligned}
$$

Since Y must take one of the values 0 through 3,

$$1 = P\left(\bigcup_{i=0}^{3}\{Y=i\}\right) = \sum_{i=0}^{3} P(Y=i).$$

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5. Probability Mass Function (PMF)

5.1 Definition

A random variable that can take on at most a countable number of possible values is said to be **discrete**.

Let X be a discrete random variable with range

$$R_X = \{x_1, x_2, x_3, \ldots\}$$

(finite or countably infinite).

The function

$$P_X(x_k) = P(X = x_k), \quad k = 1, 2, 3, \ldots$$

is called the **Probability Mass Function (PMF)** of X.

5.2 Property of PMF

Since X must take exactly one value from its range,

$$\sum_{k=1}^{\infty} P_X(x_k) = 1.$$

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6. Example: PMF with Exponential Series

The probability mass function of a random variable X is given by

$$p(i) = c\frac{\lambda^i}{i!}, \quad i = 0, 1, 2, \ldots$$

where $\lambda > 0$.

6.1 Finding the Constant c

Since

$$\sum_{i=0}^{\infty} p(i) = 1,$$

we have

$$c \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = 1.$$

Using

$$e^{\lambda} = \sum_{i=0}^{\infty} \frac{\lambda^i}{i!},$$

it follows that

$$ce^{\lambda} = 1 \quad \Rightarrow \quad c = e^{-\lambda}.$$

6.2 Required Probabilities

$$P(X = 0) = e^{-\lambda}$$

$$P(X > 2) = 1 - P(X \leq 2)$$

$$P(X > 2) = 1 - [P(X = 0) + P(X = 1) + P(X = 2)]$$

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7. Bayes' Theorem

7.1 Bayes' Formula (Proposition 3.1)

Using

$$\Pr(A \cap B) = \Pr(B \mid A) \Pr(A),$$

we obtain

$$\Pr(B_i \mid A) = \frac{\Pr(A \mid B_i) \Pr(B_i)}{\sum_{j=1}^{n} \Pr(A \mid B_j) \Pr(B_j)}.$$

7.2 Terminology

- $\Pr(B_i)$: **a priori probability**
- $\Pr(B_i \mid A)$: **posterior probability**

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8. Bayes' Theorem Example: Auditorium with 30 Rows

An auditorium has **30 rows of seats**.

- Row 1 has 11 seats
- Row 2 has 12 seats
- Row 3 has 13 seats
- ...
- Row 30 has 40 seats

A door prize is awarded by:

1. Randomly selecting a row (each row equally likely).
2. Randomly selecting a seat within that row (each seat equally likely).

Computations

- Probability that Seat 15 is selected given Row 20 is selected:

$$P(S_{15} \mid R_{20}) = \frac{1}{30}$$

- Probability that Row 20 was selected given Seat 15 was selected:

$$P(R_{20} \mid S_{15}) = \frac{P(S_{15} \mid R_{20})P(R_{20})}{P(S_{15})}$$

Sources