Lecture Summary: Discrete Random Variables and Distributions

Scribe Notes

1 Random Variables: Concept and Definitions

Definition of Random Variable (RV): A random variable X on a sample space Ω is a function $X : \Omega \to R$ that assigns a real number $X(\omega)$ to each sample point $\omega \in \Omega$.

- **Discrete Random Variables:** These variables take values in a range that is either finite or countably infinite.
- **Discrete Subset:** While X maps to the set of real numbers (R), the actual set of values $\{X(\omega) : \omega \in \Omega\}$ that X takes is a discrete subset of R.

2 Probability Mass Function (PMF)

Definition: The PMF of a discrete random variable gives the probability of each outcome.

- **Visualization:** The distribution can be visualized as a bar diagram where the x-axis represents the possible values of the RV and the height of the bar at value a is the probability $Pr[X = a]$.
- **Legitimacy Condition:** For a discrete random variable, the sum of all probabilities in the PMF must equal 1:

$$\sum_i P\{X = i\} = 1$$

Worked Example: Tossing 3 Fair Coins

Experiment: Tossing 3 fair coins.
Random Variable (Y): Let Y denote the number of heads that appear.
Possible Values and Probabilities:

- $P\{Y = 0\} = P\{(t, t, t)\} = 1/8$
- $P\{Y = 1\} = P\{(t, t, h), (t, h, t), (h, t, t)\} = 3/8$

- $P\{Y = 2\} = P\{(t, h, h), (h, t, h), (h, h, t)\} = 3/8$
- $P\{Y = 3\} = P\{(h, h, h)\} = 1/8$

Verification: $\sum_{i=0}^{3} P\{Y = i\} = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = 1.$

3 Guide to Selecting a Discrete Probability Distribution

The following table provides conditions to identify the appropriate distribution:

Distribution	**Description of x**	**Necessary Conditions**
Binomial	Number of successes in n trials	1. Identical trials
		2. Two outcomes: Success (S) and Failure (F)
		3. $P(S)$ and $P(F)$ remain constant
		4. Trials are independent
Poisson	Successes for a rare event per unit	1. $P(S)$ remains constant across units
		2. Unit x values are independent
Hypergeometric	Number of successes in n trials	1. Drawn **without replacement** from N elements
		2. Two outcomes: Success (S) and Failure (F)

4 Expectation and Moments

- **Expectation (μ):** Defined as $E[X]$.
- **Function of RV:** The expectation of a function of a random variable, $E[g(X)]$.
- **Linearity:** Linear operations with expectation, e.g., $E[aX+b] = aE[X]+b$.
- **Moments:** n^{th} moments and central moments including **Variance**, **Skewness**, and **Kurtosis**.

5 CDF and PDF Properties

- **Cumulative Density Function (CDF):** Defined with specific properties and examples for discrete variables.
- **Probability Density Function (PDF):** Used for continuous variables to give the relative likelihood of outcomes in a range.
- **Legitimacy for Continuous RVs:** A PDF $f_x(x)$ is legitimate if:

$$\int_{-\infty}^{\infty} f_x(x)dx = 1$$