# Homework 10



22.2-8 1. The diameter of an undirected tree $T = (V, E)$ on $n$ vertices $V$ (and $(n-1)$ edges $E$) is the largest of all shortest paths distances in the tree: $D = \max_{x,y \in V} \delta(x, y)$. You will design an $O(n)$ algorithm to compute $D$ and will prove its correctness as follows.

   (a) (7 points) Let $r$ be the root of $T$. Let $b$ is the furthest node from $r$ in $T$. Show that the diameter path in $T$ either ends or starts at $b$.

   (b) (5 points) Assuming part (a), irrespective of whether or not you solved it, design an $O(n)$ algorithm to compute $D$. For partial credit, give a slower algorithm.

2. (6 points) An undirected graph is said to be connected if there is a path between any two vertices in the graph. Given a connected undirected graph $G = (V, E)$, where $V = \{1, \ldots, n\}$, give an algorithm that runs in time $O(|V| + |E|)$ and finds a permutation $\pi : [n] \mapsto [n]$ such that the subgraph of $G$ induced by the vertices $\{\pi(1), \ldots, \pi(i)\}$ is connected for any $i \leq n$. Justify briefly the correctness and running time of your algorithm.

22.3-11 3. (a) (4 points) Explain how a vertex $u$ of a directed graph can end up in a depth-first tree containing only $u$, although $u$ has both incoming and outgoing edges.

   (b) (4 points) Assume $u$ is part of some directed cycle in $G$. Can $u$ still end up all by itself in the depth-first forest of $G$? Justify your answer.

   **Hint**: Recall the White Path Theorem.

22.3-9 4. (4 points) Give a counterexample to the conjecture that if a directed graph $G$ contains a path from $u$ to $v$, then any depth-first search must result in $v.d \leq u.f$.

22.3-12 5. (8 points) Show that we can use a depth-first search of an undirected graph $G$ to identify the connected components of $G$, and that the depth-first forest contains as many trees as $G$ has connected components. More precisely, show how to modify depth-first search so that it assigns to each vertex $v$ an integer label $v.cc$ between 1 and $k$, where $k$ is the number of connected components of $G$, such that $u.cc = v.cc$ if and only if $u$ and $v$ are in the same connected component. Your solution should also run in time $O(|V| + |E|)$.

   Briefly justify correctness and running time.

23.1-2 6. (4 points) Professor Jeff conjectures the following: let $G = (V, E)$ be a connected, undirected graph with a real-valued weight function $w$ defined on $E$. Let $A$ be a subset of $E$ that is included in some minimum spanning tree for $G$, let $(S, V \setminus S)$ be any cut of $G$ that respects $A$, and let $(u, v)$ be a safe edge for $A$ crossing $(S, V \setminus S)$. Then, $(u, v)$ is a light edge for the cut. Show that the professors conjecture is incorrect by giving a counterexample.

   Recall that we say a cut $(S, V \setminus S)$ respects a set of edges $A$ if no edge $(u, v) \in A$ has one node in $S$ and one node in $V \setminus S$.

23.1-3 7. (4 points) Show that if an edge $(u, v)$ is contained in some minimum spanning tree, then it is a light edge crossing some cut of the graph.

23.1-4 8. (4 points) Give a simple example of a connected graph such that the set of edges $\{(u, v) \mid$ there exists a cut $(S, V \setminus S)$ such that $(u, v)$ is a light edge crossing $(S, V \setminus S)\}$ does not form a minimum spanning tree.