Fundamental Algorithms
Courant, Fall 2017

Homework 12

Instructor: Amir Shpilka
Due Date: Wed Dec 13, 11:55pm

1. (5 points) In a directed graph $G = (V, E)$, each edge $(u, v) \in E$ has an associated independent value $r(u, v) \in \mathbb{R}$, $0 \leq r(u, v) \leq 1$ which represents how secure the channel from vertex u to vertex v is, i.e. the probability that the channel will not fail. Give the most efficient algorithm you can to find the most reliable path between two given vertices s and t, and state its running time.

24.3-6

   Hint: Notice, here we need to *maximize the product*, while we know how to *minimize the sum*. To "turn" maximum into minimum, use $\max_{(a_1,\ldots a_k)\in S}(a_1 \cdot \ldots \cdot a_k) = \min_{(a_1,\ldots a_k)\in S}(\frac{1}{a_1} \cdot \ldots \cdot \frac{1}{a_k})$. To "turn" product into sum, use $\log(ab) = \log(a) + \log(b)$. Make sure you carefully explain how you follow these hints.

2. Tucker, who lives in a node s of a weighted graph G (with non-negative weights), is invited to an exciting party located at a node h, where he will meet a girl of his dreams, Sharona. Naturally, Tucker wants to get from t to h as soon as possible, but he is told to buy some beer on the way over. He can get beer at any supermarket, and the supermarkets form a subset of the vertices $B \subset V$. Thus, starting at s, he must go to some node $b \in B$ of his choice, and then head from b to h using the shortest total route possible (assume he wastes no time in the supermarket). Help Tucker to meet Sharona as soon as possible, by solving the following sub-problems. Note that the graph can be either directed or undirected, so either give solutions for both cases separately, or justify why your solution works for both cases.

   (a) (2 points) Compute the shortest distance from s to all supermarkets $b \in B$.

   (b) (3 points) Compute the shortest distance from every supermarket $b \in B$ to h. Can one simply add a new "fake" source s' connected to all supermarkets with zero-weight edges and run Dijkstra from s'?

   (c) (4 points) Combine parts (a) and (b) to solve the full problem.

   (d) (4 points) Your solution in part (c) used two calls to the Dijkstra's algorithm (one in part (a) and one in part (b)). Define a new graph G' on $2n$ vertices and at most $2m + n$ edges (and "appropriate" weights on these edges), so that the original problem can be solved using a *single* Dijkstra call on G'.

   (e) (5 points) Assume now that, in addition to beer, Tucker also needs to buy flowers for the beautiful Sharona, and the set of flower shops is $F \subset V$. As with supermarkets, Tucker can choose any flower shop $f \in F$, and it does not matter if he buys first beer, then flowers or vice versa, so he will naturally choose the best among all these options (which supermarket, which flower shop, and in what order). Show how to solve this problem for the poor Tucker.

3. Let $G = (V, E)$ be a directed graph with weighted edges; edge weights could be positive, negative or zero.

   (a) (4 points) Describe an $O(n^2)$ algorithm that takes as input $v \in V$ and returns an edge weighted graph $G' = (V', E')$ such that $V' = V \setminus \{v\}$ and the shortest path distance from u to w in G' for any $u, w \in V'$ is equal to the shortest path distance from u to w in G.

   (b) (4 points) Now assume that you have already computed the shortest distance for all pairs of vertices in G'. Give an $O(n^2)$ algorithm that computes the shortest distance in G from v to all nodes in V' and from all nodes in V' to v.

   (c) (4 points) Use part (a) and (b) to give a recursive $O(n^3)$ algorithm to compute the shortest distance between all pairs of vertices $u, v \in V$.

24-3

4. (5 points) Arbitrage is the use of discrepancies in currency exchange rates to transform one unit of a currency into more than one unit of the same currency. For example, suppose that 1 US dollar buys 0.5 British pounds, 1 British pound buys 10.0 Hong Kong dollars, and 1 Hong Kong dollars buys 0.21 US dollars. Then, by converting currencies, a clever trader can start with 1 US dollar and buy 0.5 * 10.0 * 0.21 = 1.05 US dollars, making a profit of 5 percent.

   Your bank gives you a list of m tuples, each tuple of the form (x, y, r) denoting that they will exchange 1 unit of currency x for r units of currency y. Assume there are n currencies in total. Give the fastest algorithm you can to determine if arbitrage is possible, and prove the running time and correctness.