The MuSig Schnorr Signature Scheme

December 5, 2018

Contents

1 Introduction

This report investigates MuSig, which is provably secure in the *plain public-key model*. However, the case of interactive signature aggregation where each signer signs their own message must still be proven by a complete security analysis.

Multi-signatures are a form of technology used to add additional security for cryptocurrency transactions. A multi-signature protocol which allows a group of signers to produce a short, joint signature on a common message. [1]

2 Schnorr Signatures for Bitcoin

Schnorr signatures produce a smaller on-chain size, support faster validation and have better privacy. They natively allow for combining multiple signatures into one through aggregation. They permit more complex spending policies, including k-of-n and more to be represented as as single signature for a single key.

Signature aggregation also has its challenges. This included the rogue-key attack, where a participant steals funds using a specifically constructed key. This is easily solved for simple multi-signatures, however, through an enrollment procedure, where the keys sign themselves, supporting it across multiple inputs of a transaction requires plain public-key security, meaning there is no setup.

An additional attack, termed the Russel attacks, after Russel O'Connor, who was discovered for multi-party schemes where a party could claim ownership of someone else's key and so spend their other outputs.

Peter Wuille discussed the issues and their solutions, which refines the Bellare-Neven (BN) scheme. He also discussed the performance improvements that were implemented for the scaler multiplication fo the BN scheme and how they enable batch validation on the blockchain. A pair of BIPs are in process to make these advances a reality for Bitcoin.[2]

3 Key Aggregation for Schnorr Signatures

MuSig is a simple multi-signature scheme that is novel in combining:

1. Support for key aggregation;

2. Security in the plain public-key model.

There are two versions of MuSig, that are provably secure, which differ based on the number of communication rounds:

- Three-round MuSig only relies on the Discrete Logarithm (DL) assumption, on which ECDSA (Elliptic Curve Digital Signature Algorithm) also relies

- Two-round MuSig instead relies on the slightly stronger One-More Discrete Logarithm (OMDL) assumption

A multi-signature scheme is a combination of a signing and verification algorithm, where multiple signers (each with their own private/public key) jointly sign a single message, resulting in a single signature. This can then be verified by anyone knowing the message and the public keys of the signers.

Note: in the context of Bitcoin, the term 'multisig' refers to a k-of-n policy, where k can be different from n. While in the cryptographic literature, the term multi signature really only refers to n-of-n policies, however, k-of-n can be constructed on top of n-of-n.

The term *key aggregation* refers to multi-signatures that look like a single-key signature, but with respect to an aggregated public key that is a function of only the participants' public keys. Thus, verifiers do not require the knowledge of the original participants' public keys- they can just be given the aggregated key. In some use cases, this leads to better privacy and performance. MuSig is effectively a key aggregation scheme for Schnorr signatures.

There are other multi-signature schemes that already exist that provide key aggregation for Schnorr signatures, however they come with some limitations, such as needing to verify that participants actually have the private key corresponding to the pubic keys that they claim to have. *Security in the plain public-key model* means that no limitations exist. All that is needed from the participants is their public keys. [3]

3.1 Applications of mulit-signatures in Bitcoin

The most obvious use case for mulit-signatures with regards to Bitcoin is as a more efficient replacement of n-of-n multisig scripts and other policies that permit a number of possible combinations of keys (including k-of-n, using key trees, MAST, or traditional threshold schemes). For these, a native multi-signature scheme means that what is left is one signature per transaction input.

A key aggregation scheme also lets us reduce the number of public keys per input to one, as a user can send coins to the aggregate of all involved key, rather than including them all in the script. This leads to smaller on-chain footprint, faster validation, and better privacy. As a result, MuSig is a good choice here.

Instead of creating restrictions with one signature per input, one signature can be used for the entire transaction. Key aggregation cannot be used across multiple inputs, as the public keys are committed to by the outputs, and those can be spent independently. MuSig can be used here (with key aggregation done by the verifier).

On a technical standing, in order to combine all the transaction inputs' signatures, a multi-signature scheme is not necessary, instead an aggregate signature scheme can be used. The distinction is simply that in an aggregate signature, each signer has their own message, instead of one message shared by all.

Aggregate signatures can be categorized as being:

- Interactive: Interactive aggregate signatures (IAS) require the signers to cooperate, while non-interactive schemes all the aggregation to be done by anyone

- Non-interactive: These allow the aggregation to be done by anyone

No non-interactive aggregation schemes are known that only rely on the DL assumption, but interactive ones are trivial to construct: where a multi-signature scheme has every participant sign the concatenation of all messages. The paper by Blockstream, focusing on key aggregation for Schnorr Signatures shows that this is not always a desirable construction, and gives an IAS variant of BN with better properties instead. [3]

3.1.1 Details

Notation

- $x, x_1, x_{2...}$ are private keys with corresponding public keys $X, X_1, X_2... X_i = x_i G$, with G the generator

- The message being signed is m

- $H()$ is a cryptographic hash function

Schnorr Signatures

- Signatures are $(R, s) = (rG, r + H(X, R, m)x)$ where r is a random nonce chosen by the signer

- Verification requires $sG = R + H(X, R, m)X$

Naive Schnorr multi-signatures

- Call X the sum of the X_i points

- Each signer chooses a random nonce r_i and shares $R_i = r_i G$ with the other signers

- Call R the sum of the R_i points

- Each signer computes $s_i = r_i + H(X, R, m)x_i$

- The final signature is (R, s) where s is the sum of the s_i values

- Verification requires $sG = R + H(X, R, m)X$, where X is the sum of the individual public keys

It is interesting to note that this satisfies the definition of a *key aggregation scheme*, as multiple parties can jointly produce a signature that is a valid single-key signature for the sum of the keys.

The issue arises in that this scheme is not secure. Consider the following scenario:

- Alice and Bob want to produce a multi-signature together.

- Alice has a key pair (x_A, X_A) and Bob has (x_B, X_B). However, nothing prevents Bob from claiming that his public key is $X'_B = X_B - X_A$.

- If he does so, others will assume that $X_A + X'_B$ is the aggregated key that Alice and Bob need to cooperate in order to sign for

- Unfortunately, that is equal to X_B, thus Bob can clearly sign for this by himself

- This is called a rogue-key attack

- One way to avoid this is requiring that Alice and Bob prove first that they actually possess the private keys corresponding to their claimed public keys; however this is not always possible

- Ideally a scheme needs to be constructed whose security does not rely on out-of-band verification of the keys.

4 Simple Schnorr Multi-Signatures with Applications to Bitcoin

The paper describes a new Schnorr-based multi-signature scheme called MuSig, which is provably secure in the *plain public-model.* This means that signers are only required to have a public key, but they do not have to prove knowledge of the private key corresponding to their public key to some certification authority or to other signers prior to engaging the protocol.

This new scheme provides improvements to Bellare and Neven (ACM-CCS 2006) and its variants by Bagherzandi *et al.* (ACM-CCS 2008) and Ma *et al.* (Des. Codes Cryptogr., 2010) in two respects:

Write Sections on Bagerzandi and Ma

1. It is simple and efficient, as it has the same key and signature size as standard Schnorr signatures;

2. It allows *key aggregation*, where the joint signature can be verified just as a standard Schnorr signature with respect to a single "aggregated" public key which can be computed from the individual public keys of the signers. [1]

4.1 Multi-signatures

Introduced by Itakura and Nakamura [4], multi-signature protocols allow a group of signers (that individually possess their own private/public key pair) to produce a single signature σ on a message m. Verification of the given signature σ can be publicly performed given the message and the set of public keys of all signers.

A simple way to change a standard signature scheme into a multi-signature scheme is to have each signer produce a stand-alone signature for m with its private key and to then concatenate all individual signatures.

The transformation of a standard signature scheme to a multi-signature scheme needs to useful and practical, thus the newly calculated multi-signature scheme must produce signatures where the size is independent of the number of signers and similar to that of the original signature scheme. [1]

4.1.1 Rogue Attacks

Rogue attacks are a significant concern when implementing multi-signature schemes. Here a subset of corrupted singers, manipulate the public keys computed as functions of the public keys of honest users, allowing them to easily produce forgeries for the set of public keys (despite them not knowing the associated secret keys.

Proposals from [5], [6], [7], [8], [9], [10], [11] were thus undone before a formal model was put forward along with a provably secure scheme from Micali, Ohta, and Reyzin. [12]Unfortunately, despite being provably secure this scheme is costly and an impractical interactive key generation protocol. [1]

A means of generically preventing rogue-key attacks is to make it mandatory for users to prove knowledge (or possession [13]) of the secret key during public key registration with a certification authority. Certification authority is a setting known as the knowledge of secret key (KOSK) assumption. The pairing-based multi-signature schemes by Boldyreva [14] and Lu *et al.* [15] rely on the KOSK assumption in order to maintain security. However, this as can be seen from [16]and [13] this assumption is problematic.

As it stands, the Bellare and Neven [16] provides the most practical multi-signature scheme, based on the Schnorr signature scheme, which is provably secure that does not contain any assumption on the key setup. Since the only requirement of this scheme is that each potential signer has a public key, this setting is referred to as the *plain-key model*.

Should Bellare and Neven reference be included in this statement

4.1.2 Schnorr Signature Scheme

The Schnorr signature scheme uses:[17]

- A cyclic group G of prime order p

- A generator g of G

- A hash function H

- A private/public key pair is a pair $(x, X)\epsilon\{0, ..., p - 1\}\times G$ where $X = g^x$

- To sign a message m, the signer draws a random integer r in Z_p, computes $R = g^r$, $c = H(X, R, m)$, and $s = r + cx$

- The signature is the pair (R, s) , and its validity can be checked by verifying whether $g^s = RX^c$

The above described is referred to as the so-called "key-prefixed" variant of the scheme, which sees the public key hashed together with R and m[18]. This variant was thought to have a better multi-user security bound than the classic variant [19], however in [20] the key-prefixing was seen as unnecessary to enable good multi-user security for Schnorr signatures.

For the development of the new Schnorr-based multi-signature scheme [1], key-prefixing seemed a requirement for the security proof to go through, despite not knowing the form of an attack. The rationale also follows the process in reality, as messages signed in Bitcoin always indirectly commits to the public key.

4.1.3   Design of a Schnorr multi-signature scheme

The naive way to design a Schnorr multi-signature scheme would be as follows:

- A group of n signers want to cosign a message m

  - Let $L = \{X_1 = g^{x_1}, ..., X_n = g^{x_n}\}$ be the multi-set of all public keys [1]

  - Each cosigner randomly generates and communicates to others a share $R_{i=}g^{r_i}$

  - Each of the cosigners them computes $R = \Pi_{i=1}^{n} R_i$, $c = H(\tilde{X}, R, m)$

  - Where $\tilde{X} = \Pi_{i=1}^{n} X_i$ is the product of individual public keys, and a partial signature $s_i = r_i + cx_i$

  - Partial signatures are then combined into a single signature (R, s) where $s = \Sigma_{i=1}^{n} s_i \bmod p$

  - The validity of a signature (R, s) on message m for public keys $\{X_1, ...X_n\}$, is equivalent to $g^s = R\tilde{X}^c$ where $\tilde{X} = \Pi_{i=1}^{n} X_i$ and $c = H(\tilde{X}, R, m)$

  - Note that this is exactly the verification equation for a traditional key-prefixed Schnorr signature with respect to public key $\tilde{X}$, a property termed *key aggregation*

However, as mentioned above, [7], [9], [10] and[12] these protocols are vulnerable to a rogue-key attack where a corrupted signer sets its public key to $X_1 = g^{x_1}(\Pi_{i=2}^{n} X_i)^{-1}$, allowing the signer to produce signatures for public keys $\{X_1, ...X_n\}$ by themself.

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[1]No constraints are imposed on the key setup, the adversary thus can choose corrupted public keys at random, hence the same public key can appear more than once in L

4.1.4 Micali-Ohta-Reyzin Multi-signature scheme

The Micali-Ohta-Reyzin multi-signature scheme [12]solves the rogue-key attack using a sophisticated interactive key generation protocol.

4.1.5 Bellare and Neven

Bellare-Neven (BN) [16]proceeded differently in order to avoid any key setup. It is a more widely known plain public-key multi-signature scheme, that does not support key aggregation. It is possible to use BN multi-signatures where the individual keys are MuSig aggregates. BN multi-signature scheme is secure without such assumptions. Below are details:

- Call $L = H(X_1, X_{2...})$

- Each signer chooses a random nonce r_i and shares $R_i = r_i G$ with the other signers

- Call R the sum of the R_i points

- Each signer computes $s_i = r_i + H(L, X_i, R, m)x_i$

- The final signature is (R, s) where $s$is the sum of the s_i values

- Verification requires $sG = R + H(L, X_1, R, m)X_2 + ...$

Technically, BN has a pre-commit round, where the signers initially reveal $H(R_i)$to each other, prior to revealing the R_i points themselves. This step is a requirement in order to prove security under the DL assumption, but it can be dismisses if instead the OMDL assumption is accepted.

Furthermore, when an IAS is desired (where each signer has their own message), $L = H((X_1, m_{1)}, (X_2, m_2), ...)$ and $s_i = r_i + H(L, R, i)x_i$ is used for signing (and analogous for verification).

The resulting signature does not satisfy the normal Schnorr equation anymore, nor any other equation that can be written as a function of a combination of the public keys; the key aggregation property is lost in order to gain security in the plain public-key model.

This is where MuSig comes in. It recovers the *key aggregation property without losing security:*

- Call $L = H(X_1, X_2...)$

- Call X the sum of all $H(L, X_i)X_i$

- Each signer chooses a random nonce r_i, and shares $R_i = r_i G$ with the other signers

- Call R the sum of the R_i points

- Each signer computes $s_i = r_i + H(X, R, m)H(L, X_i)x_i$

- The final signature is (R, s) where s is the sum of the s_i values

- Verification again satisfies $sG = R + H(X, R, m)X$

So what was needed was to define X not as a simple sum of the individual public keys X_i, but as a sum of multiples of those keys, where the multiplication factor depends on a hash of all participating keys. [3]

Their main idea is to have each cosigner use a distinct "challenge" c_i when computing their partial signature $s_i = r_i + c_i x_i$, defined as $c_i = H(< L > X_i, R, m)$, where as before

Bellare and Neven showed that this yields a multi-signature scheme provably secure in the *plain public-key* model under the Discrete Logarithm assumptions, modeling H and H' as random oracles. However, this scheme does not allow key aggregation anymore since the entire list of public keys is required for verification.

4.1.6 The new scheme

The new proposed Schnorr-based multi-signature scheme can be seen as a variant of the BN scheme, allowing key aggregation in the *plain public-key model*. This scheme consists of three rounds, the first two being exactly the same as in BN. Challenges c_i are changed from $c_i = H(< L > X_i, R, m)$ to $c_i = H_{agg}(< L > X_i).H_{sig}(\tilde{X}, R, m)$, where $\tilde{X}$ is the so-called aggregated public key corresponding to the multi-set of public keys $L = \{X_1, ... X_n\}$, defined as $\tilde{X} = \prod_{i=1}^{n} X_i^{a_i c} = R\tilde{X}^c$ where $c = H_{sig}(\tilde{X}, R, m)$.

Basically , the key aggregation property has been recovered and can now be enjoyed by the naive scheme, which respect to a more complex aggregation key $\tilde{X} = \prod_{i=1}^{n} X_i^{a_i c}$. $c = H_{sig}(< L >, R, m)$ yields a secure scheme, however does not allow key aggregation since verification is impossible without knowing all the individual singer keys.

4.1.7 The benefits of key aggregation

- If a group of n signers want to authorize which all of them agree, but do not necessarily wish to reveal their individual public keys

- They can privately compute the aggregated key $\tilde{X}$ corresponding to their multi-set of public keys and publish it as an ordinary (non-aggregated) key.

- Signers are ensured that all of them will need to cooperate to produce a signature which is valid under $\tilde{X}$, whereas verifiers will not even learn that $\tilde{X}$ is in fact an aggregated key.

- Moreover, $\tilde{X}$ can be computed by a third party just from the list of public keys, without interacting with the signers.

- This property will rpove instrumental for obtaining a more compact and privacy-preserving variant of so-called n-of-n multi-signature transactions in Bitcoin.

Two variants of the BN multi-signature scheme have been previously proposed.

4.1.8 The Bagherzandi *et al.* Scheme

Bagherzandi *et al.*[21] reduced the number of rounds from three to two using an homomorphic commitment scheme. Unfortunatel, this increases the signature size and the computational cost of signing and verification.

4.1.9 The Ma *et al.* Scheme

Ma *et al.*[22]proposed a variation based on Okamoto's signature scheme[23], which involved the "double hashing" technique, which sees the reduction of the signature size compared to [21]while using only two rounds.

However, neither of these two variants allow for key aggregation.

Multi-signature schemes supporting key aggreagation are easier to come by in the KOSK model. In particular, Syta *et al.* proposed the CoSi scheme which can be seen as the naive Schnorr multi-signature scheme described earlier where the cosiners are organsied in a tree structure for fast signature generation.

5 Interactive Aggregate Signatures

In some situations, it may be useful to allow each participant to sign a different message rather than a single common one. An interactive aggregate signature (IAS) is one where each signer has its own message m_i to sign, and the joint signature proves that the i-th signer has signed m_i . These schemes are more general than multi-signature schemes, however they are less flexible than non-interactive aggregate signatures [24, 25] and sequential aggreagate signatures [LMRSO4].

According to Bellare and Neven [BN06], they suggested a generic way to turn any multi-signature scheme into an IAS scheme by the signer running the multi-signature protocol using as message the tuple of all public keys/message pairs involced in the IAS protocol.

For BN's scheme and the scheme of the Blockstream signature scheme, this does not increase the number of communication rounds as messages can be sent together with shares R_i.

However, the problem arise with the generic construction in the *plain public-key model.* We will now take a closer look at this transformation and show that is subtly fails to provide the strongest security guarantees in the *plain public key model.*

5.1 Syntax and Security Model

The syntax is adapted as follows. An IAS scheme II is a triple of algorithms (KeyGen, Sign, Ver). The randomization key generation algorithm takes no input and returns a private/public key pair $(\mathsf{sk}, \mathsf{pk}) \leftarrow_\$ \mathsf{KeyGen}()$. The signature algorithm Sign is run by each participant on input its key pair $(\mathsf{sk}, \mathsf{pk})$, a message m, and a set of public key/message pairs for other cosigners $S' = \left\{ (pk_1', m_1'), ... (pk_{n-1}', m_{n-1}') \right\}$ such that $(\mathsf{pk}, m) \notin S'$; it returns a signature σ for the set of public key/message pairs $S = S' \cup \{(\mathsf{pk}, m)\}$.[2] The determinisitic verification algorithm Ver takes as input a set of public key/message pairs $S = \{(\mathsf{pk_1}, m_1), ..., (\mathsf{pk_n}, m_n)\}$ and a signature σ, and returns 1 if the signature is valid for S and 0 otherwise.

As another means, one can specify the signature algorithm to take as input a key pair

5.2 Revisions

A previous version of this paper, dated January 15, 2918, proposed a 2-round vaiant of MuSig, where the intital commitment round is omitted claiming a security proof unsre the One More Discrete Logarithm (OMDL) assumptions [BP02][BNPS03]. However, a flaw in the secuirty proof was discovered by Drijvers *et al.* [DEFN18], hwo also showed through a meta-reduction that this scheme cannot be proved secure using an algebraic black box reduction inder the DL or OMDL assumption (assuming the OMDL problem is hard).

In more details, observe that in the 2-round variant of MuSig, an adversary (controlling public keys $X_2, ..., X_n$) can impose the value of $R = \Pi_{i=1}^n R_i$ used in signature protocols since he can choose $R_2, ..., R_n$ after having recived R_1 from the honest signer (contolling public key $X_1 = g^{x_1}$). This forbids to use the textbook way of simulating the honest signer in the Random Oracle model without knowing x_1 by randomly drawing s_1 and c, computing

There is no attach currently known against the 2-rounf variant of MuSig and that it might be secure although thais is not provable under standard assumptions from exisiting techniques.

6 The Discrete Logarithm Problem

Definition 1 (DL problem)

- Let $\mathbb{G}$, p, g be group parameters- it is fixed, but the bit length k and p can be regarded as a security parameter if neccesary.

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[2]S is a set, i.e., no public key/message pair repeats, even though the same public key might appear mulitple times. This is without loss of generality since any repeated public key/message pair in S can be deleted by the signature/verification algorithm.

- An algorithm $\mathcal{A}$ is said to $(t, \varepsilon$ $solve$ the DL problem with with respect to $\mathbb{G}, p, g$ if on input a random gorup element X, it runs in time at most t and returns $x \in \{0, ..., p-1\}$ such that

7 Elliptic Curve Digital Signature Algorithm

Currently in Bitcoin ECDSA is implemented. To sign a message m fwe hash it and treat this hash as a number: $z = hash(m)$. We also need a random or random-looking number k. We prefer not to trusut random number generators (too many failures and vulnerabilities are realted to bad random number generators) so we usually use RFC6979 to calulate deterministic k absed on our secret and teh message we are signing.

Using a private key pk we can generate a signature for message m consisting of two numbers: r($x$coordinate of the random point $R = kG$ and $s = (z+rpk)/k$. Then, using our public key $P = pkG$ anyone can verify our signature by checking that point $(\frac{z}{s})G + (\frac{r}{s})P$ has $x$coordinate equal to r.

Insert Image

This algorithm is very common, hoever it can be improved. Firstly, signature verification includes inversion $(1/s)$ snf two points multiplications an these operations are very computationally heavy. In Bitocin every node has to verify all the tansactions. This means that when you broadcast a transastion, thousands of computers will have to verify yoru signature. Making verification procress simpler will be very beneficial even if signing is more difficult.

Secondly, every node has to verify every signature individually. In the case of m-of-n multisig transaction node may even have to verify the same signautre several times. For example, transaction of 7-of-11 multisig input will contrain 7 signatues and require from 7 to 11 signaute verifications on every node in teh network. Also such transaction will take a huge amount of space in teh block and you will have to pay large fees for that.

8 Schnorr signatures

Schnorr signatures are generated slightly differently. Instead of two scalars (r, s)we use a point R and a scalar . Like ECDSA, R is considered a random point on the elliptic curve $(R = kG)$. Second part of the signaute is calculated slightly differently: $s = k + hash(P, R, m)pk$. Here pk is your private key, $P = pkG$ is yoru public key, m is the message. Then onc can verify this signatuare by checking that $sG = R + hash(P, R, m)P$

Insert image: Visualisation of the Schnorr signature verification

This equation is linear, so equations can be added and sutracted with each other and still stay valid. This leads to a nice feature of Schnorr signatures.

8.1 Batch Validation

To verify a block in Bitcoin blockchain we need to make sure that all signatures in the block are valid. If one of them is not valid we don't care which one- we just reject eh whole block and that's it.

With ECDSA every signature has to be verified separately. Meaning that if we have 1000 signatures in the block we will need to compute 1000 inversions and 2000 point multiplication. In total approximately 3000 heavy operations.

With Schnorr signatures we can add up all the signature verification equatiosn and save some computational power. In total for a block with 1000 transactions we need to verify that:

$(s1+s2+...+s1000)G=(R1+...+R1000)+(hash(P1,R1,m1)P1+hash(P2,R2,m2)P2+...+hash(P1000,R1000,$ $m1000)P1000$

Here we have a bunch of point additions (almost free in sense of computational power) and 1001 point mulitplication. This is already a factor of 3 improvement- we meed to compute roughyl one heavy operation per signature.

Insert image: Batch validation of two signatures. As verification equation is linear the sum of several equations is valid as soon as all signatures are valid. We save some computational power as scalare and point additions are much easier than point muliplication.

8.2 Key aggregation

There is a need to keep one's bitcoin safe, so we might want to use at least two different private keys to control bitcoins. Once we will use on a laptop or a phone and another one- on a hardware wallet/cold wallet. So when one of them is compromised we still have control over our bitcoins.

Currently it is implemented via 2-of-2 multisg script. This requires two separate signatures to be included in the transaction.

With Schnorr signatures we can use a pair of private keys $(pk1, pk2)$ and generate a shared signature corresponding to a shared public key $P = P1+P2 = pk1G + pk2G$. To generate this signature we need to choose a random number on every device $(k1, k2)$, generate a random point $Ri = kiG$, add them up to calcaulate a common $hash(P, R1 + R2, m)$ and then get $s1$ and $s2$ from every device $(si = ki + hash(P, R, m)pki)$. Then we can add up these signatures and use a pair $(R, s) = (R1 + R2, s1 + s2)$ as our signature for shared public key P. Everyone else won't be able to say if it is an aggregated signature or

8.3 Merkle Multi-signatures

MuSig and key aggregation require all signers to sign a transation. What happens if you want to make a 2-of-3 multisig? Can we use signature aggregation or will we have to use our usual $OP_C HECKMULTISIGandseparatesignatures$?

it is possible, but with a small change in the protocol. A new op-code similar to $OP_C HECKMULTISIGcanbedevelopedthatchecksifaggregatedsignaturecorrespondstoaparticularite$

For example, if we use a 2-of-3 multisig with public keys $P1$, $P2$ and $P3$, then we need to construct a Merkle tree of aggregated public keys for all combinations we can use: $(P1, P2)$, $(P2, P3)$, $(P1, P3)$ and put the root in the locking script. To spend bitcoins we provide a signature and a proof that our public keys is in the tree. For 2-of-3 multisig there are only 3 elements in the tree and the proof will consist fo two hashes-the one we want to use and its neighbour. For a 7-of-11 multisig there will be already $11!/7!/4!=300$ possible key combinations and the proof will require 8 elements. In general the number of elements in the proof scels almost linear with the number of keys in multisig (its $log2(n!m!/(n-m)!)$).

But with the Merkle tree of public keys we are not limited to m-of-n multisigs. We can make a tree with any public keys we want. For example , if we have a laptop, a phone, a hardware wallet and a recovery seed, we can construct a structure that would allow us to spend bitcoins with a laptop and a hardware

9 Applications to Bitcoin

Each input ontains a signature of a modified version of the transaction to be validated with its referened output's key. In fact some outputs even require multiple signatures to be spent. Transactions spending such an output are often referred to as m-of-n multi-signature transactions [And11], and the current implementation corresponding to the triviail way of building a multi-signature scheme by concatenating individual signatures. Additionally, a threshold policy can be enforced where only m valid signatues out of the n possible ones are needed to redeem the transaction (again this is the most straightforward way to turn a multi-signature scheme into some kind of basic threshold signature scheme).

Today Bitcoin uses ECDSA signatures [ANS05][NIS13] over the `secp256k1` curve [SEC10] to authenticate transactions. As Bitcoin nodes fully verify all transactions, signature size and verification time are important design considerationsm while signing time is much less so. Besiges, signatures account for a large part of the size of Bitcoin transactions. Because of this, using multi-signatures seems appealing. However, designing multiparty ECDSA signature schemes is notably cumbersome [MR01][GGN16][Lin17]due to the modular inversion involced in signing, and moving to Schnorr signatures would definitely help deploy compact multi-signatures. While several multi-signature schemes could off an improvement over the currently avaiable method, two properties increas the possible impact:

- The availability of key aggregatio nremoves the need for verifiers to see all the involved key, improving bandwidth, privacy, and validation cost

- Security under the *plain public-key model* enables multi-signatures across multiple inputs of a transaction, where the choice of signers cannot be committed to in advance. This greatly increases the number of situatiosn in which mulit-signatures are beneficial.

9.1 Introduction

Bitcoin contains every transaction since the system's inception, resulting in a final state, the set of unspent coins. Each unspent coin has an associated value (expressed as a mulitple of the currency unit, 10^{-8} bitcoin) and a programmable public key of the owner. Every transaction consumes one or more coins, proviging a signature for each to authorize its spending, and creates one ot more new coins, with a total value not larger than the value of the consumeed coins.

Bitcoins uses a programmable generalisation of a digitial signautre scheme. Instead of a public key, a predicate that determines spendability is included in every output (implemented in a concise programming language, called *Bitcoin Script)*. When spending, instead of a signautrem, a witness that satisfies the predicate is provided. In practice, most output predicates effectively corresond to a single ECDSA verification. This is also how Bitcoin supports a naive version of multi-signatures with a threshold policy: coins can be assigned a prdicated that requires valid signatures for multiple public keys. Several use cases for the exist, including low-trust escrow services [GBGN17] and split-device security. While using thOPe predicate language to implement multi-signautes is very flexible, it is inefficient in terms of size, computational cost, and privacy.

As a global consensus system, kept in check by the ability for everyt participant to validate all updates to the ledger, the size of signatures and predicates, and teh computational cost for verifying them are the primary limiting factors for its scalability. The computational requiremnets for signing, or the communication overhead between different signers are far less constrained. Bitcon does not have any central trusted party, so it is not generally possible to introduce new cryptographic shcemes that requrie a trrustes setup. Finally, to function as a currency, a hgih degree of fungibility and privacy is desirable. Among other thinfs, this means that ideally the predictes of coins fo not lead information about the owner. In particilat, if several styles of predicates are in use, the choise may reveal what software or service is being used to manage it.

9.2 Native Multi-Signature Support

An obvious improvement is to replace the need for implementating n-of-n multi-signautes in an ad-hoc fashion with a constant-size mulit-signature primitive like Bellare-Neven, While this is on itself an improvemtn in terms of size, it still requires the prediccate itself-whose size also matters- to contain all of the signers' public keys. Key aggregation improves upon this further, as a single-key predicate can be used instead which is both smaller and has lower computational cost for verification. It also improves provacy, as the participant keys and their count remain private to the signers.

When generalizing to the m-of-$n$scenario, several options exist. One is to forego key aggregation, and still include all potential signer keys in the predicates while still only producing a single singature for teh chosen combination of keys. Alternatvely, a Merkle tree [Mer87] where the leaves are permitted combinations of keys (in aggregated form) can be employed. The predicate in this case would

take as input an aggregated public key, a signature with it, and a proof. Its validity would depend on the signature being valid wiht hte provided key, and the proof estabishing that the key is in fact one the leaves of the Merkle tree, identified by its root hash. This approach is very generic, as it works for any subset of combinations of keys, and as a result has very good privacy as the exact licy is not visible from the proof. It is only feasible however when the total number of combinations is tractable.

Some key aggregation schemes that do not protect against rogue-key attacks can be used instead in the above cases, under the assumption that the sender is given a proof of knowledge/possession for the receivers' private keys. However, these schemes are difficult to prove secure except by using very large prrofs of knowledge [MOR01]. As those proofs of knowledge/possession for the receivers' private keys. However, these schemes are difficult to prove secure except by using very large proofs of knowledge/possession do not need to be seen by verfiiers- they are effectively cetified by the sender's signature on teh transaction whihc includes the predicate- they do not burden validation. However, passing them arounf to senders is inconveneient, and easy to get wrong .Using a scheme that is secure in the *plain public-key model* categorically avoids these concerns.

Another alternative is to use an algorithm whose key generation requires a trusted setup, for example in the KOSK model. While many of these schemes have been proven secure [Bol03, LOS+06], they rely on mechanisms that are usually not implemented by certification aythorities [BN06, RY07].

9.3 Cross-Input Multi-Signatures

The previous sections explained how the numbers of signautres per input can gnerally by reduced to one, but we would like to go further, and replace it with a signel signature per transaction. Doing so requires a fundamental change in validation semantics, as the validitiy of separate inputs is not loger independent. As a result , the outputs can no longer be modelled as predicates. Instead, we model them as functions that return a boolean plus a set of zero or more public keys.

Overall validity requires all returend booleans to be True and a multi-signature of the transaction with L the union of all returned keys.

More concretly, this can be implemented by providing an alternative to the signature checking opcode $OP_C HECKSIG and related opcodes in teh Script language. Instead of returning th widemultiset of keys. Finally, after all inputs are verified, a multi-signature presnet in the transaction is ver signture, or choose to only use the orginial opcodes. Adding these new opcodes is possible in backwa-compatibleway.$

9.4 Protection against Rogue-Key Attacks

When taking cross-inptu signatures input signatures, there is no published commitment to the set of signers,a s each transaction input can independetly spend an output that requires authorizartion from distinct participants. This functionality was not restrticited as it would then interfere with fungibility improvemtns

like CoinJoin [Max13]. Due to the lack of certitification, security against rogue-key attacks is of great importance.

If it is assumed that transactions used a single multi-signature that was vulnerable to rogue-attacks, like the simpler schemes previously descibed, an attacker could identify an arbitrarynumber of outputs he wants to steal, with the public keys $X_1, ..., X_{n-t}$ and then use the rogue-key $X_{n-t+1}, ..., X_n$ such that he can sign for the aggregated key $\tilde{X}$.He would then send a small amount of his own money to outputs with predicates corresponmding to the keys $X_{n-t+1}, ..., X_n$. Finally, he can create a transaction that spends all of the victim coins together with the ones he just dreated by forging a multi-signature for the whole transaction.

It can be seen that in the cse of mulit-signatures across inputs, theft can occur by merely being able to forge a signature over a set of keys that includes at least one key not controlled by the attacker, just what the *plain public-key model* considewrs a win for the attacker. This is in contrast to the single-inout multi-signature casem where theft is only possible by forging a signature for the exact (aggregated)keys contained in an exitisting output. As a result, it is no longer possible to rely on proofs of knowledge/possession that are private to the signers.

To anaylse the impact of multi-signatres, a simlation on Bitcoin's historical blockcahin was performed to determine the portential space saving. Figure 3 shows the cumlative blockchian size together with what the size would be if all transactions' signatures were replaced with just one per transtion, giving an idiciation of what could have been saved if MuSig had been used since the beginning. Note that this only encompasses the savings from using multi-signatures, and dones no tinduce the savings that are possible from key aggregation.

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