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A Quantum Computation Workbook

September 1, 2021

Preface

This book is intended as an introductory text for a university course on quantum computation and as a self-learning guide. It is an attempt to collect some fundamental principles and elementary methods in the field of quantum computation and quantum information and then reorganize them in a compact and integrated form. A proper introduction to (and hence learning of) quantum computation and quantum information turns out to be demanding, especially to undergraduate students, because it needs to cover various subjects from different fields of science including the physics of quantum mechanics, the mathematics of advanced linear algebra, and the computer science of information theory, to list the least. Each field of science usually favors its specific language, so another layer of difficulties is to organize the subjects consistently in a unified language.

Two digital materials accompany the book. Both are freely available through the respective GitHub repositories at the links below.

- Q3, a Mathematica application (Appendix D):
<https://github.com/quantum-mob/Q3App>
- QuantumWorkbook, a compilation of the demonstrations in this book written in the Wolfram Language (Appendix E):
<https://github.com/quantum-mob/QuantumWorkbook>

Q3 consists of tools and utilities that perform symbolic calculations and numerical simulations useful in the study of quantum information processing, quantum many-body systems, and quantum spin systems. With *Q3*, one can avoid many of the tedious calculations involved in various principles and theorems of quantum theory. Furthermore, numerous visualization and simulation tools can help deepen the understanding of core concepts. *QuantumWorkbook* is a compilation of Mathematica Notebook files that contain the code used to generate the demonstrations in the book. Readers themselves can run and modify the code to build their own examples from the demonstrations and to experiment with fresh ideas. Both materials can be helpful companions, particularly in a course of self-learning, of the various subjects of quantum physics present in the book and for the general study of the overall subject as well.

This book is a result of several years of experience in teaching quantum computation and quantum information. It is a helpful resource to select and present particular topics of the subject at the undergraduate level. Although

the book aims to be self-contained, it nonetheless assumes some basic knowledge of quantum mechanics. Chapter 1 summarizes the fundamental postulates of quantum mechanics and effectively provides a brief review of basic concepts and fundamental principles of quantum mechanics. Chapter 2 presents and describes the properties of elementary quantum gates for universal quantum computation. These are the building blocks of quantum algorithms and quantum communication protocols. Chapter 3 discusses physical methods and principles to implement elementary quantum gates and also introduces different quantum computation schemes. Chapter 4 introduces some widely known quantum algorithms to help grasp the idea of the so-called ‘quantum supremacy’ of quantum algorithms over their classical counterparts. Chapter 5 is dedicated to decoherence effects and introduces mathematical methods including the Kraus representation and the Lindblad equation to describe the phenomena. Chapter 6 is devoted to quantum error-correction codes through a discussion of the basic principles, procedures, and examples. Chapter 7 introduces quantum information theory. It discusses distance measures for quantum information, measures for quantum entanglement degree, and entropies of information content. To maintain a coherent structure and to focus on the primary topics, we collect the corresponding mathematical theories in the appendices and refer to them from the main text as necessary.

The author is indebted to students in his classes for pointing out numerous mistakes and typographical errors in the manuscript. Many people have contributed to the development of Q3 by testing and actively using it. The author gives particular thanks to Ha-Eum Kim, Myeongwon Lee, and Su-Ho Choi for their energetic discussions and constructive feedback in the early stages of the development of Q3. He also appreciates bug reports and valuable comments by Boris Laurent, Mi Jung So, Yeong-ho Je, and Dongni Chen.

Seoul, Korea
August 2021

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Contents

1	The Postulates of Quantum Mechanics	11
1.1	Quantum States	12
1.1.1	Pure States	12
1.1.2	Mixed States	17
1.2	Time Evolution of Quantum States	26
1.2.1	Unitary Dynamics	26
1.2.2	Quantum Noisy Dynamics	30
1.3	Measurements on Quantum States	30
1.3.1	Projection Measurements	31
1.3.2	Generalized Measurements	35
2	Quantum Computation: Overview	43
2.1	Single-Qubit Gates	44
2.1.1	Pauli Gates	44
2.1.2	Hadamard Gate	48
2.1.3	Rotations	51
2.2	Two-Qubit Gates	55
2.2.1	CNOT, CZ, and SWAP	55
2.2.2	Controlled-Unitary Gate	65
2.2.3	General Unitary Gate	70
2.3	Multi-Qubit Controlled Gates	78
2.3.1	Gray Code	78
2.3.2	Multi-Qubit Controlled-NOT	80
2.4	Universal Quantum Computation	86
2.5	Measurements	88
3	Realizations of Quantum Computers	95
3.1	Quantum Bits	96
3.2	Dynamical Scheme	98
3.2.1	Implementation of Single-Qubit Gates	99
3.2.2	Implementation of CNOT	105
3.3	Geometric/Topological Scheme	108
3.3.1	A Toy Model	109

3.3.2	Geometric Phase	114
3.4	Measurement-Based Scheme	117
3.4.1	Single-Qubit Rotations	121
3.4.2	CNOT Gate	124
3.4.3	Graph States	127
4	Quantum Algorithms	133
4.1	Quantum Teleportation	134
4.1.1	Nonlocality in Entanglement	135
4.1.2	Implementation of Quantum Teleportation	138
4.2	Deutsch-Jozsa Algorithm & Variants	142
4.2.1	Quantum Oracle	142
4.2.2	Deutsch-Jozsa Algorithm	148
4.2.3	Bernstein-Vazirani Algorithm	150
4.2.4	Simon's Algorithm	151
4.3	Quantum Fourier Transform (QFT)	155
4.3.1	Definition and Physical Meaning	156
4.3.2	Quantum Implementation	157
4.3.3	Semiclassical Implementation	161
4.4	Quantum Phase Estimation (QPE)	165
4.4.1	Definition	165
4.4.2	Implementation	167
4.4.3	Accuracy	169
4.4.4	Simulation of the von Neumann Measurement	170
4.5	Applications	172
4.5.1	The Period-Finding Algorithm	172
4.5.2	The Order-Finding Algorithm	179
4.5.3	Quantum Factorization Algorithm	181
4.5.4	Quantum Search Algorithm	182
5	Decoherence	193
5.1	How Does Decoherence Occur?	194
5.2	Quantum Operations	200
5.2.1	Kraus Representation	202
5.2.2	Choi Isomorphism	210
5.2.3	Unitary Representation	213
5.2.4	Examples	214
5.3	Measurements as Quantum Operations	219
5.4	Quantum Master Equation	220
5.4.1	Derivation	222
5.4.2	Examples	223
5.4.3	Solution Methods	226
5.4.4	Examples Revisited	232
5.5	Distance between Quantum States	237

5.5.1	Norms and Distances	237
5.5.2	Hilbert-Schmidt and Trace Norms	238
5.5.3	Hilbert-Schmidt and Trace Distances	244
5.5.4	Fidelity	248
6	Quantum Error-Correction Codes	259
6.1	Elementary Examples: Nine-Qubit Code	260
6.1.1	Bit-Flip Errors	260
6.1.2	Phase-Flip Error	263
6.1.3	Shor's Nine-Qubit Code	265
6.2	Quantum Error Correction	269
6.2.1	Quantum Error-Correction Conditions	269
6.2.2	Discretization of errors	271
6.3	Stabilizer Formalism	272
6.3.1	Pauli Group	275
6.3.2	Properties of Stabilizers	280
6.3.3	Unitary Gates in Stabilizer Formalism	283
6.3.4	Clifford Group	284
6.3.5	Measurements in Stabilizer Formalism	290
6.3.6	Examples	292
6.4	Stabilizer Codes	298
6.4.1	Bit-Flip Code	300
6.4.2	Phase-Flip Code	301
6.4.3	Nine-Qubit Code	303
6.4.4	Five-Qubit Code	305
6.5	Surface Codes	309
6.5.1	Toric Codes	309
6.5.2	Planar Codes	315
6.5.3	Recovery Procedure	317
7	Quantum Information Theory	323
7.1	Shannon Entropy	323
7.1.1	Definition	323
7.1.2	Relative Entropy	325
7.1.3	Mutual Information	327
7.1.4	Data Compression	329
7.2	Von Neumann Entropy	330
7.2.1	Definition	330
7.2.2	Relative Entropy	333
7.2.3	Quantum Mutual Information	335
7.3	Entanglement and Entropy	337
7.3.1	What is Entanglement?	337
7.3.2	Separability Tests	338
7.3.3	Entanglement Distillation	341

7.3.4	Entanglement Measures	343
A	Linear Algebra	347
A.1	Vectors	347
A.1.1	Vector Space	347
A.1.2	Hermitian Product	349
A.1.3	Basis	350
A.1.4	Representations	351
A.2	Linear Operators	353
A.2.1	Linear Maps	353
A.2.2	Representations	354
A.2.3	Hermitian Conjugate of Operators	355
A.3	Dirac's Bra-Ket Notation	357
A.4	Spectral Theorems	360
A.4.1	Spectral Decomposition	360
A.4.2	Functions of Operators	362
A.5	Factorization of Operators	364
A.6	Tensor-Product Spaces	366
A.6.1	Vectors in a Product Space	366
A.6.2	Operators on a Product Space	368
B	Superoperators	371
B.1	Operators as Vectors	371
B.2	Superoperators	377
B.2.1	Matrix Representation	377
B.2.2	Operator-Sum Representation	379
B.2.3	Choi Isomorphism	384
B.3	Partial Trace	386
B.4	Partial Transposition	388
C	Group Theory	393
C.1	The Concept	393
C.2	Classes	397
C.3	Invariant Subgroups	398
C.4	Cosets and Quotient Groups	399
C.5	Product Groups	401
D	Mathematica Application Q3	403
D.1	Installation	403
D.2	Quick Start	404
E	Integrated Compilation of Demonstrations	405
E.1	Installation	405
E.2	Quick Start	406

F	Solutions to Select Problems	407
F.1	The Postulates of Quantum Mechanics	407
F.2	Quantum Computation: Overview	407
F.3	Quantum Computers	410
F.4	Quantum Algorithms	412
F.5	Decoherence	413
F.6	Quantum Error-Correction Codes	417
F.7	Quantum Information Theory	421
	Bibliography	423
	Index	432

