Quantum Oracle: Properties

Episode 26. Classical Oracle

Episode 27. Quantum Oracle: Definition

Episode 28. Quantum Oracle: Properties

Preliminary

Here, let us set up classical and quantum oracles to be used in the examples below.

```
In[ ]:= $m = 3;
        $n = 2;
```

```
In[ ]:= f[1] = f[2] = 3;
        f[7] = 2;
        f[_Integer] = 0;
```

```
In[ ]:= ff = Oracle[f, $m, $n];
        xx = Tuples[{0, 1}, $m];
        yy = ff /@ xx;
```

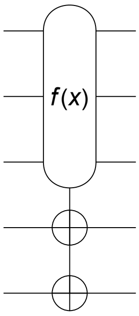
```
In[ ]:= Thread[xx → yy] // TableForm
Out[ ]//TableForm=
        {0, 0, 0} → {0, 0}
        {0, 0, 1} → {1, 1}
        {0, 1, 0} → {1, 1}
        {0, 1, 1} → {0, 0}
        {1, 0, 0} → {0, 0}
        {1, 0, 1} → {0, 0}
        {1, 1, 0} → {0, 0}
        {1, 1, 1} → {1, 0}
```

```
In[ ]:= Let[Qubit, S, T]
```

```
In[ ]:= SS = S[Range[$m], $];
        TT = T[Range[$n], $];
```

```
In[ ]:= op = Oracle[f, SS, TT]
Out[ ]=
        Oracle[f, {S₁, S₂, S₃}, {T₁, T₂}]
```

In[∘]:=
```
qc = QuantumCircuit[op]
```
Out[∘]=



Making Copies

The CNOT gate makes a copy of the computational basis state of the control register to the target register when the latter has been prepared initially in the state $|0\rangle$,

$$|x\rangle \otimes |0\rangle \mapsto |x\rangle \otimes |x\rangle.$$

The quantum oracle has a similar property, but it makes a copy of the image $|f(x)\rangle$ rather than the input state $|x\rangle$ itself of the native register to the ancillary register,

$$|x\rangle \otimes |0\rangle \mapsto |x\rangle \otimes |f(x)\rangle.$$

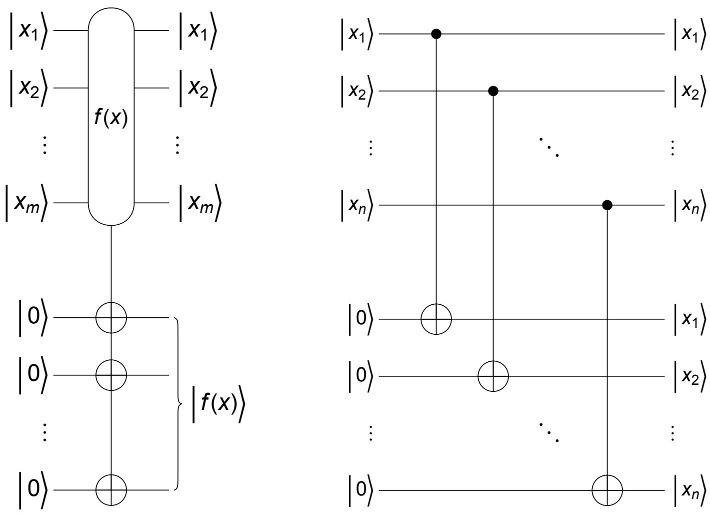
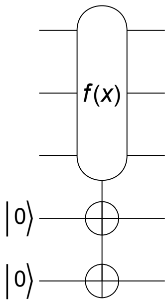


Figure 4. **(left)** A quantum circuit demonstrating the feature $|x\rangle \otimes |0\rangle \mapsto |x\rangle \otimes |f(x)\rangle$ of the quantum oracle corresponding to classical oracle $f : \{0, 1\}^m \to \{0, 1\}^n$. **(right)** A quantum circuit demonstrating the feature $|x\rangle \otimes |0\rangle \mapsto |x\rangle \otimes |x\rangle$ of the CNOT gates.

Example

In[ ]:=
```
QuantumCircuit[Ket[TT], op]
```

Out[ ]=



In[ ]:=
```
in = Basis[SS];
ProductForm[in, {SS, TT}]
```

Out[ ]=

{|000⟩ ⊗ |00⟩, |001⟩ ⊗ |00⟩, |010⟩ ⊗ |00⟩, |011⟩ ⊗ |00⟩,
|100⟩ ⊗ |00⟩, |101⟩ ⊗ |00⟩, |110⟩ ⊗ |00⟩, |111⟩ ⊗ |00⟩}

In[ ]:=
```
out = op ** in;
ProductForm[out, {SS, TT}]
```

Out[ ]=

{|000⟩ ⊗ |00⟩, |001⟩ ⊗ |11⟩, |010⟩ ⊗ |11⟩, |011⟩ ⊗ |00⟩,
|100⟩ ⊗ |00⟩, |101⟩ ⊗ |00⟩, |110⟩ ⊗ |00⟩, |111⟩ ⊗ |10⟩}

In[ ]:=
```
ProductForm[Thread[in → out], {SS, TT}] // TableForm
```

Out[ ]//TableForm=

|000⟩ ⊗ |00⟩ → |000⟩ ⊗ |00⟩
|001⟩ ⊗ |00⟩ → |001⟩ ⊗ |11⟩
|010⟩ ⊗ |00⟩ → |010⟩ ⊗ |11⟩
|011⟩ ⊗ |00⟩ → |011⟩ ⊗ |00⟩
|100⟩ ⊗ |00⟩ → |100⟩ ⊗ |00⟩
|101⟩ ⊗ |00⟩ → |101⟩ ⊗ |00⟩
|110⟩ ⊗ |00⟩ → |110⟩ ⊗ |00⟩
|111⟩ ⊗ |00⟩ → |111⟩ ⊗ |10⟩

In[ ]:=
```
Thread[xx → yy] // TableForm
```

Out[ ]//TableForm=

{0, 0, 0} → {0, 0}
{0, 0, 1} → {1, 1}
{0, 1, 0} → {1, 1}
{0, 1, 1} → {0, 0}
{1, 0, 0} → {0, 0}
{1, 0, 1} → {0, 0}
{1, 1, 0} → {0, 0}
{1, 1, 1} → {1, 0}

Superposition State

Furthermore, suppose that a native quantum register is in the superposition

$$\frac{1}{2^{m/2}} \sum_{x=0}^{2^m-1} \left| x \right\rangle$$

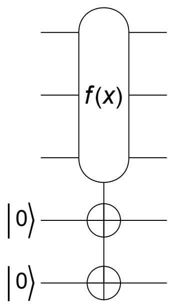
and the ancillary quantum register in in the state $\left|0\right\rangle \equiv \left|0\right\rangle^{\otimes n}$. The quantum oracle transforms the state as

$$\frac{1}{2^{m/2}} \sum_{x=0}^{2^m-1} \left| x \right\rangle \otimes \left| 0 \right\rangle \mapsto \frac{1}{2^{m/2}} \sum_{x=0}^{2^m-1} \left| x \right\rangle \otimes \left| f(x) \right\rangle$$

Just like the CNOT gate, the above state from the quantum oracle is also entangled in general unless f is a constant or very special function. In this case, the entanglement is controlled by the (classical) oracle f.

In[ ]:=
```
QuantumCircuit[Ket[TT], op]
```

Out[ ]=



In[ ]:=
```
in = Total@Basis[SS];
ProductForm[in, {SS, TT}]
```

Out[ ]=

$\left|000\right\rangle \otimes \left|00\right\rangle + \left|001\right\rangle \otimes \left|00\right\rangle + \left|010\right\rangle \otimes \left|00\right\rangle + \left|011\right\rangle \otimes \left|00\right\rangle + \left|100\right\rangle \otimes \left|00\right\rangle + \left|101\right\rangle \otimes \left|00\right\rangle + \left|110\right\rangle \otimes \left|00\right\rangle + \left|111\right\rangle \otimes \left|00\right\rangle$

In[ ]:=
```
out = op ** in;
ProductForm[out, {SS, TT}]
```

Out[ ]=

$\left|000\right\rangle \otimes \left|00\right\rangle + \left|001\right\rangle \otimes \left|11\right\rangle + \left|010\right\rangle \otimes \left|11\right\rangle + \left|011\right\rangle \otimes \left|00\right\rangle + \left|100\right\rangle \otimes \left|00\right\rangle + \left|101\right\rangle \otimes \left|00\right\rangle + \left|110\right\rangle \otimes \left|00\right\rangle + \left|111\right\rangle \otimes \left|10\right\rangle$

In[∘]:=
```
Thread[xx → yy] // TableForm
```
Out[∘]//TableForm=
```
{0, 0, 0} → {0, 0}
{0, 0, 1} → {1, 1}
{0, 1, 0} → {1, 1}
{0, 1, 1} → {0, 0}
{1, 0, 0} → {0, 0}
{1, 0, 1} → {0, 0}
{1, 1, 0} → {0, 0}
{1, 1, 1} → {1, 0}
```

Marking

The controlled-unitary gate induces a phase shift on the control register (rather than on the target register) when the target register is in an eigenstate of the unitary operator.

A similar method can be used to induce a phase shift conditionally on every term that satisfies a certain condition:

$$\left|x\right\rangle \mapsto \left|x\right\rangle(-1)^{f(x)}$$

More generally,

$$\sum_{x=0}^{2^n-1}\left|x\right\rangle \mapsto \sum_{x=0}^{2^n-1}\left|x\right\rangle(-1)^{f(x)}$$

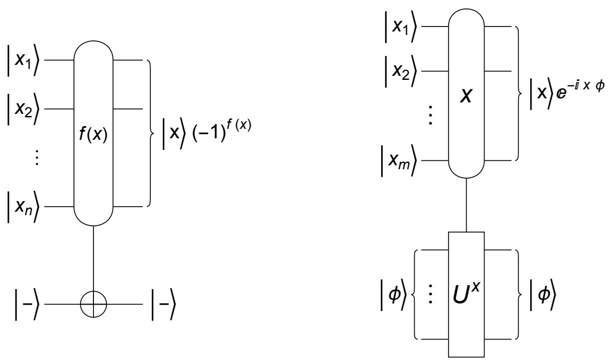
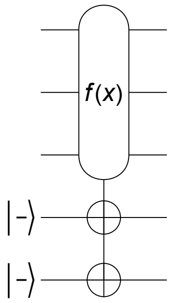


Figure 5. **(left)** A quantum circuit to induces conditional phase shifts depending on function value $f(x)$ at input value $x := (x_1, x_2, \ldots, x_n)$. **(right)** For comparison, the controlled exponentiation of a unitary gate induces an x-dependent phase shift, where $x := (x_1\, x_2 \ldots x_m)_2$.

Example

In[ ]:=
```
QuantumCircuit[ProductState[T[{1, 2}] → {1, -1}, "Label" → Ket[{"-"}]], op]
```

Out[ ]=



In[ ]:=
```
in = Basis[SS] ** ProductState[T[1] → {1, 1}, T[2] → {1, -1}];
KetFactor[in]
```

Out[ ]=

$$\{ |0_{S_1}0_{S_2}0_{S_3}\rangle \otimes (|0_{T_1}\rangle + |1_{T_1}\rangle) \otimes (|0_{T_2}\rangle - |1_{T_2}\rangle),\ |0_{S_1}0_{S_2}1_{S_3}\rangle \otimes (|0_{T_1}\rangle + |1_{T_1}\rangle) \otimes (|0_{T_2}\rangle - |1_{T_2}\rangle),$$
$$|0_{S_1}1_{S_2}0_{S_3}\rangle \otimes (|0_{T_1}\rangle + |1_{T_1}\rangle) \otimes (|0_{T_2}\rangle - |1_{T_2}\rangle),$$
$$|0_{S_1}1_{S_2}1_{S_3}\rangle \otimes (|0_{T_1}\rangle + |1_{T_1}\rangle) \otimes (|0_{T_2}\rangle - |1_{T_2}\rangle),$$
$$|1_{S_1}0_{S_2}0_{S_3}\rangle \otimes (|0_{T_1}\rangle + |1_{T_1}\rangle) \otimes (|0_{T_2}\rangle - |1_{T_2}\rangle),$$
$$|1_{S_1}0_{S_2}1_{S_3}\rangle \otimes (|0_{T_1}\rangle + |1_{T_1}\rangle) \otimes (|0_{T_2}\rangle - |1_{T_2}\rangle),$$
$$|1_{S_1}1_{S_2}0_{S_3}\rangle \otimes (|0_{T_1}\rangle + |1_{T_1}\rangle) \otimes (|0_{T_2}\rangle - |1_{T_2}\rangle),$$
$$|1_{S_1}1_{S_2}1_{S_3}\rangle \otimes (|0_{T_1}\rangle + |1_{T_1}\rangle) \otimes (|0_{T_2}\rangle - |1_{T_2}\rangle) \}$$

In[ ]:=
```
out = op ** in;
KetFactor[out]
```

Out[ ]=

$$\{ |0_{S_1}0_{S_2}0_{S_3}\rangle \otimes (|0_{T_1}\rangle + |1_{T_1}\rangle) \otimes (|0_{T_2}\rangle - |1_{T_2}\rangle),$$
$$-(|0_{S_1}0_{S_2}1_{S_3}\rangle \otimes (|0_{T_1}\rangle + |1_{T_1}\rangle) \otimes (|0_{T_2}\rangle - |1_{T_2}\rangle)),$$
$$-(|0_{S_1}1_{S_2}0_{S_3}\rangle \otimes (|0_{T_1}\rangle + |1_{T_1}\rangle) \otimes (|0_{T_2}\rangle - |1_{T_2}\rangle)),$$
$$|0_{S_1}1_{S_2}1_{S_3}\rangle \otimes (|0_{T_1}\rangle + |1_{T_1}\rangle) \otimes (|0_{T_2}\rangle - |1_{T_2}\rangle),$$
$$|1_{S_1}0_{S_2}0_{S_3}\rangle \otimes (|0_{T_1}\rangle + |1_{T_1}\rangle) \otimes (|0_{T_2}\rangle - |1_{T_2}\rangle),$$
$$|1_{S_1}0_{S_2}1_{S_3}\rangle \otimes (|0_{T_1}\rangle + |1_{T_1}\rangle) \otimes (|0_{T_2}\rangle - |1_{T_2}\rangle),$$
$$|1_{S_1}1_{S_2}0_{S_3}\rangle \otimes (|0_{T_1}\rangle + |1_{T_1}\rangle) \otimes (|0_{T_2}\rangle - |1_{T_2}\rangle),$$
$$|1_{S_1}1_{S_2}1_{S_3}\rangle \otimes (|0_{T_1}\rangle + |1_{T_1}\rangle) \otimes (|0_{T_2}\rangle - |1_{T_2}\rangle) \}$$

In[∘]:= `Thread[KetFactor[in] → KetFactor[out]] // TableForm`

Out[∘]//TableForm=

$\left|0_{S_1}0_{S_2}0_{S_3}\right\rangle \otimes \left(\left|0_{T_1}\right\rangle + \left|1_{T_1}\right\rangle\right) \otimes \left(\left|0_{T_2}\right\rangle - \left|1_{T_2}\right\rangle\right) \to \left|0_{S_1}0_{S_2}0_{S_3}\right\rangle \otimes \left(\left|0_{T_1}\right\rangle + \left|1_{T_1}\right\rangle\right) \otimes \left(\left|0_{T_2}\right\rangle - \left|1_{T_2}\right\rangle\right)$

$\left|0_{S_1}0_{S_2}1_{S_3}\right\rangle \otimes \left(\left|0_{T_1}\right\rangle + \left|1_{T_1}\right\rangle\right) \otimes \left(\left|0_{T_2}\right\rangle - \left|1_{T_2}\right\rangle\right) \to -\left(\left|0_{S_1}0_{S_2}1_{S_3}\right\rangle \otimes \left(\left|0_{T_1}\right\rangle + \left|1_{T_1}\right\rangle\right) \otimes \left(\left|0_{T_2}\right\rangle - \left|1_{T_2}\right\rangle\right)\right)$

$\left|0_{S_1}1_{S_2}0_{S_3}\right\rangle \otimes \left(\left|0_{T_1}\right\rangle + \left|1_{T_1}\right\rangle\right) \otimes \left(\left|0_{T_2}\right\rangle - \left|1_{T_2}\right\rangle\right) \to -\left(\left|0_{S_1}1_{S_2}0_{S_3}\right\rangle \otimes \left(\left|0_{T_1}\right\rangle + \left|1_{T_1}\right\rangle\right) \otimes \left(\left|0_{T_2}\right\rangle - \left|1_{T_2}\right\rangle\right)\right)$

$\left|0_{S_1}1_{S_2}1_{S_3}\right\rangle \otimes \left(\left|0_{T_1}\right\rangle + \left|1_{T_1}\right\rangle\right) \otimes \left(\left|0_{T_2}\right\rangle - \left|1_{T_2}\right\rangle\right) \to \left|0_{S_1}1_{S_2}1_{S_3}\right\rangle \otimes \left(\left|0_{T_1}\right\rangle + \left|1_{T_1}\right\rangle\right) \otimes \left(\left|0_{T_2}\right\rangle - \left|1_{T_2}\right\rangle\right)$

$\left|1_{S_1}0_{S_2}0_{S_3}\right\rangle \otimes \left(\left|0_{T_1}\right\rangle + \left|1_{T_1}\right\rangle\right) \otimes \left(\left|0_{T_2}\right\rangle - \left|1_{T_2}\right\rangle\right) \to \left|1_{S_1}0_{S_2}0_{S_3}\right\rangle \otimes \left(\left|0_{T_1}\right\rangle + \left|1_{T_1}\right\rangle\right) \otimes \left(\left|0_{T_2}\right\rangle - \left|1_{T_2}\right\rangle\right)$

$\left|1_{S_1}0_{S_2}1_{S_3}\right\rangle \otimes \left(\left|0_{T_1}\right\rangle + \left|1_{T_1}\right\rangle\right) \otimes \left(\left|0_{T_2}\right\rangle - \left|1_{T_2}\right\rangle\right) \to \left|1_{S_1}0_{S_2}1_{S_3}\right\rangle \otimes \left(\left|0_{T_1}\right\rangle + \left|1_{T_1}\right\rangle\right) \otimes \left(\left|0_{T_2}\right\rangle - \left|1_{T_2}\right\rangle\right)$

$\left|1_{S_1}1_{S_2}0_{S_3}\right\rangle \otimes \left(\left|0_{T_1}\right\rangle + \left|1_{T_1}\right\rangle\right) \otimes \left(\left|0_{T_2}\right\rangle - \left|1_{T_2}\right\rangle\right) \to \left|1_{S_1}1_{S_2}0_{S_3}\right\rangle \otimes \left(\left|0_{T_1}\right\rangle + \left|1_{T_1}\right\rangle\right) \otimes \left(\left|0_{T_2}\right\rangle - \left|1_{T_2}\right\rangle\right)$

$\left|1_{S_1}1_{S_2}1_{S_3}\right\rangle \otimes \left(\left|0_{T_1}\right\rangle + \left|1_{T_1}\right\rangle\right) \otimes \left(\left|0_{T_2}\right\rangle - \left|1_{T_2}\right\rangle\right) \to \left|1_{S_1}1_{S_2}1_{S_3}\right\rangle \otimes \left(\left|0_{T_1}\right\rangle + \left|1_{T_1}\right\rangle\right) \otimes \left(\left|0_{T_2}\right\rangle - \left|1_{T_2}\right\rangle\right)$

In[∘]:= `Thread[xx → yy] // TableForm`

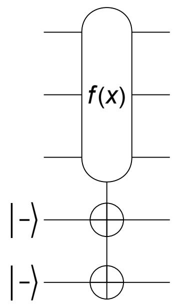
Out[∘]//TableForm=

```
{0, 0, 0} → {0, 0}
{0, 0, 1} → {1, 1}
{0, 1, 0} → {1, 1}
{0, 1, 1} → {0, 0}
{1, 0, 0} → {0, 0}
{1, 0, 1} → {0, 0}
{1, 1, 0} → {0, 0}
{1, 1, 1} → {1, 0}
```

Superposition

In[∘]:= `QuantumCircuit[ProductState[T[{1, 2}] → {1, -1}, "Label" → Ket[{"-"}]], op]`

Out[∘]=



In[∘]:=
```
in = Total@Basis[SS] ** ProductState[T[1] → {1, 1}, T[2] → {1, -1}];
KetFactor[in]
```

Out[∘]=

$\left(\left|0_{S_1}\right\rangle + \left|1_{S_1}\right\rangle\right) \otimes \left(\left|0_{S_2}\right\rangle + \left|1_{S_2}\right\rangle\right) \otimes \left(\left|0_{S_3}\right\rangle + \left|1_{S_3}\right\rangle\right) \otimes \left(\left|0_{T_1}\right\rangle + \left|1_{T_1}\right\rangle\right) \otimes \left(\left|0_{T_2}\right\rangle - \left|1_{T_2}\right\rangle\right)$

In[∘]:=
```
out = op ** in;
ProductForm[out, {SS, TT}]
```

Out[∘]=

$|000\rangle \otimes |00\rangle - |000\rangle \otimes |01\rangle + |000\rangle \otimes |10\rangle - |000\rangle \otimes |11\rangle - |001\rangle \otimes |00\rangle + |001\rangle \otimes |01\rangle - |001\rangle \otimes |10\rangle + |001\rangle \otimes |11\rangle - |010\rangle \otimes |00\rangle + |010\rangle \otimes |01\rangle - |010\rangle \otimes |10\rangle + |010\rangle \otimes |11\rangle + |011\rangle \otimes |00\rangle - |011\rangle \otimes |01\rangle + |011\rangle \otimes |10\rangle - |011\rangle \otimes |11\rangle + |100\rangle \otimes |00\rangle - |100\rangle \otimes |01\rangle + |100\rangle \otimes |10\rangle - |100\rangle \otimes |11\rangle + |101\rangle \otimes |00\rangle - |101\rangle \otimes |01\rangle + |101\rangle \otimes |10\rangle - |101\rangle \otimes |11\rangle + |110\rangle \otimes |00\rangle - |110\rangle \otimes |01\rangle + |110\rangle \otimes |10\rangle - |110\rangle \otimes |11\rangle + |111\rangle \otimes |00\rangle - |111\rangle \otimes |01\rangle + |111\rangle \otimes |10\rangle - |111\rangle \otimes |11\rangle$

In[∘]:=
```
KetFactor[out]
```

Out[∘]=

$\left(|0_{S_1} 0_{S_2} 0_{S_3}\rangle - |0_{S_1} 0_{S_2} 1_{S_3}\rangle - |0_{S_1} 1_{S_2} 0_{S_3}\rangle + |0_{S_1} 1_{S_2} 1_{S_3}\rangle + |1_{S_1} 0_{S_2} 0_{S_3}\rangle + |1_{S_1} 0_{S_2} 1_{S_3}\rangle + |1_{S_1} 1_{S_2} 0_{S_3}\rangle + |1_{S_1} 1_{S_2} 1_{S_3}\rangle\right) \otimes \left(|0_{T_1}\rangle + |1_{T_1}\rangle\right) \otimes \left(|0_{T_2}\rangle - |1_{T_2}\rangle\right)$

In[∘]:=
```
Thread[xx → yy] // TableForm
```

Out[∘]//TableForm=
```
{0, 0, 0} → {0, 0}
{0, 0, 1} → {1, 1}
{0, 1, 0} → {1, 1}
{0, 1, 1} → {0, 0}
{1, 0, 0} → {0, 0}
{1, 0, 1} → {0, 0}
{1, 1, 0} → {0, 0}
{1, 1, 1} → {1, 0}
```

Summary

Keywords

- Oracle
- Quantum oracle
- Quantum decision making

Functions

- **Oracle**
- **ControlledExp**

Related Links

- Section 4.2 of the Quantum Workbook (2022, 2023).
- Tutorial: Quantum Oracle
- Tutorial: Quantum Decision Algorithms