

The Controlled-Z (CZ) Gate

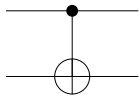
```
In[*]:= Let[Qubit, S]
```

What is it?

CNOT: Revisited

```
In[*]:= cnot = QuantumCircuit[CNOT[S[1], S[2]]]
```

Out[*]=



```
In[*]:= in = Basis[S@{1, 2}];
```

```
out = cnot ** in;
```

```
ProductForm[Thread[in → out], S@{1, 2}] // TableForm
```

Out[*]//TableForm=

$$\begin{array}{l} |0\rangle \otimes |0\rangle \rightarrow |0\rangle \otimes |0\rangle \\ |0\rangle \otimes |1\rangle \rightarrow |0\rangle \otimes |1\rangle \\ |1\rangle \otimes |0\rangle \rightarrow |1\rangle \otimes |1\rangle \\ |1\rangle \otimes |1\rangle \rightarrow |1\rangle \otimes |0\rangle \end{array}$$

```
In[*]:= Matrix[cnot] // MatrixForm
```

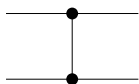
Out[*]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

CZ

```
In[*]:= cz = QuantumCircuit[CZ[S[1], S[2]]]
```

Out[*]=



```

In[*]:= in = Basis[S@{1, 2}];
out = cz ** in;
ProductForm[Thread[in → out], S@{1, 2}] // TableForm

Out[*]//TableForm=

$$\begin{array}{l}
|0\rangle \otimes |0\rangle \rightarrow |0\rangle \otimes |0\rangle \\
|0\rangle \otimes |1\rangle \rightarrow |0\rangle \otimes |1\rangle \\
|1\rangle \otimes |0\rangle \rightarrow |1\rangle \otimes |0\rangle \\
|1\rangle \otimes |1\rangle \rightarrow -|1\rangle \otimes |1\rangle
\end{array}$$


In[*]:= Matrix[cz] // MatrixForm

Out[*]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$


```


Advantages of CZ

- Easier to physically realize in many cases.

Application: Graph State

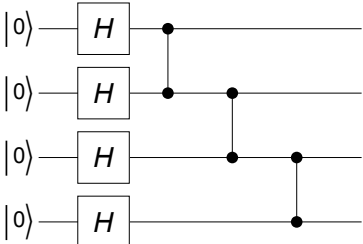
```

In[*]:= links = Chain[FlavorNone@S@{1, 2, 3, 4}]
Out[*]=
{S1 → S2, S2 → S3, S3 → S4}

In[*]:= g = Graph[links]
Out[*]=


```

```

In[*]:= qc = QuantumCircuit[Ket[S@{1, 2, 3, 4}], g]
Out[*]=


```

```
In[*]:= GraphState[g]
```

```
Out[*]:=
```

$$\begin{aligned} & \frac{1}{4} \left| 0_{S_1} 0_{S_2} 0_{S_3} 0_{S_4} \right\rangle + \frac{1}{4} \left| 0_{S_1} 0_{S_2} 0_{S_3} 1_{S_4} \right\rangle + \frac{1}{4} \left| 0_{S_1} 0_{S_2} 1_{S_3} 0_{S_4} \right\rangle - \frac{1}{4} \left| 0_{S_1} 0_{S_2} 1_{S_3} 1_{S_4} \right\rangle + \\ & \frac{1}{4} \left| 0_{S_1} 1_{S_2} 0_{S_3} 0_{S_4} \right\rangle + \frac{1}{4} \left| 0_{S_1} 1_{S_2} 0_{S_3} 1_{S_4} \right\rangle - \frac{1}{4} \left| 0_{S_1} 1_{S_2} 1_{S_3} 0_{S_4} \right\rangle + \frac{1}{4} \left| 0_{S_1} 1_{S_2} 1_{S_3} 1_{S_4} \right\rangle + \\ & \frac{1}{4} \left| 1_{S_1} 0_{S_2} 0_{S_3} 0_{S_4} \right\rangle + \frac{1}{4} \left| 1_{S_1} 0_{S_2} 0_{S_3} 1_{S_4} \right\rangle + \frac{1}{4} \left| 1_{S_1} 0_{S_2} 1_{S_3} 0_{S_4} \right\rangle - \frac{1}{4} \left| 1_{S_1} 0_{S_2} 1_{S_3} 1_{S_4} \right\rangle - \\ & \frac{1}{4} \left| 1_{S_1} 1_{S_2} 0_{S_3} 0_{S_4} \right\rangle - \frac{1}{4} \left| 1_{S_1} 1_{S_2} 0_{S_3} 1_{S_4} \right\rangle + \frac{1}{4} \left| 1_{S_1} 1_{S_2} 1_{S_3} 0_{S_4} \right\rangle - \frac{1}{4} \left| 1_{S_1} 1_{S_2} 1_{S_3} 1_{S_4} \right\rangle \end{aligned}$$

CZ vs CNOT

Recall the identity.

```
In[*]:= S[6] ** S[1] ** S[6] // Elaborate
```

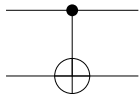
```
Out[*]:=
```

S^Z

Keeping the identity in mind, let us start with the CNOT gate.

```
In[*]:= cnot = QuantumCircuit[CNOT[S[1], S[2]]]
```

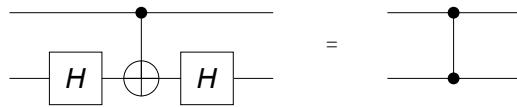
```
Out[*]:=
```



Using the identity above, we can see that the following two quantum circuits are equivalent.

```
In[*]:= Row@{
  new = QuantumCircuit[S[2, 6], cnot, S[2, 6]], "=",
  cz = QuantumCircuit[CZ[S[1], S[2]]]
}
```

```
Out[*]:=
```



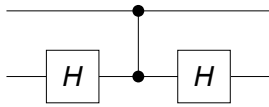
Indeed, check the matrix representation of the second quantum circuit.

```
In[*]:= Matrix[new] // MatrixForm
```

```
Out[*] // MatrixForm =
```

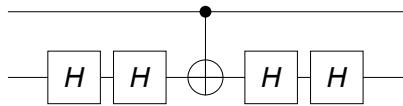
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

```
In[*]:= more = QuantumCircuit[S[2, 6], CZ[S[1], S[2]], S[2, 6]]
Out[*]=
```

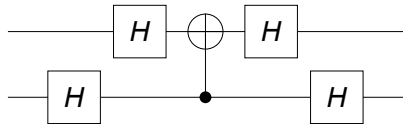


```
In[*]:= more - cnot // Elaborate
Out[*]=
0
```

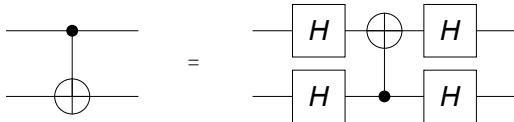
```
In[*]:= QuantumCircuit[S[2, 6], S[2, 6], cnot, S[2, 6], S[2, 6]]
Out[*]=
```



```
In[*]:= back = QuantumCircuit[S[2, 6], S[1, 6], CNOT[S[2], S[1]], S[1, 6], S[2, 6]]
Out[*]=
```

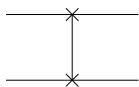


```
In[*]:= Row@{cnot, "=",
  QuantumCircuit[S[{1, 2}, 6], CNOT[S[2], S[1]], S[{1, 2}, 6]]
}
Out[*]=
```



SWAP Gate

```
In[*]:= swap = QuantumCircuit[SWAP[S[1], S[2]]]
Out[*]=
```



```
In[*]:= in = Basis[S@{1, 2}];
out = swap ** in;
ProductForm[Thread[in → out], S@{1, 2}] // TableForm
```

```
Out[*]//TableForm=
```

$ 0\rangle \otimes 0\rangle$	\rightarrow	$ 0\rangle \otimes 0\rangle$
$ 0\rangle \otimes 1\rangle$	\rightarrow	$ 1\rangle \otimes 0\rangle$
$ 1\rangle \otimes 0\rangle$	\rightarrow	$ 0\rangle \otimes 1\rangle$
$ 1\rangle \otimes 1\rangle$	\rightarrow	$ 1\rangle \otimes 1\rangle$

```
In[*]:= Matrix[swap] // MatrixForm
```

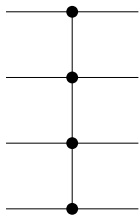
```
Out[*]//MatrixForm=
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Multi-Control Z Gate

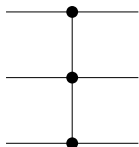
```
In[*]:= QuantumCircuit[CZ[S[{1, 2, 3}, $], S[4]]]
```

```
Out[*]=
```



```
In[*]:= $n = 2;
cc = S[Range[$n], $];
tt = S[$n + 1, $];
cnot = QuantumCircuit[CZ[cc, tt]]
```

```
Out[*]=
```



```
In[*]:= Matrix[cnot] // MatrixForm
```

```
Out[*]//MatrixForm=
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

```
In[*]:= in = Basis[cc, tt];
out = cnot ** in;
```

```
In[*]:= ProductForm[Thread[in → out], {cc, tt}] // TableForm
```

```
Out[*]//TableForm=
```

$$\begin{array}{l} |00\rangle \otimes |0\rangle \rightarrow |00\rangle \otimes |0\rangle \\ |00\rangle \otimes |1\rangle \rightarrow |00\rangle \otimes |1\rangle \\ |01\rangle \otimes |0\rangle \rightarrow |01\rangle \otimes |0\rangle \\ |01\rangle \otimes |1\rangle \rightarrow |01\rangle \otimes |1\rangle \\ |10\rangle \otimes |0\rangle \rightarrow |10\rangle \otimes |0\rangle \\ |10\rangle \otimes |1\rangle \rightarrow |10\rangle \otimes |1\rangle \\ |11\rangle \otimes |0\rangle \rightarrow |11\rangle \otimes |0\rangle \\ |11\rangle \otimes |1\rangle \rightarrow -|11\rangle \otimes |1\rangle \end{array}$$

Summary

Functions

- CZ vs CNOT
- Chain
- SWAP
- GraphState

Related Links

- Chapters 2 of the Quantum Workbook (2022, 2023).
- Tutorial: “Quantum Computation: Overview”