

# Controlled-Unitary Gates

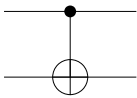
```
In[*]:= Let[Qubit, S]
```

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## CNOT vs Controlled-X

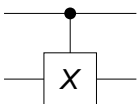
```
In[*]:= cnot = QuantumCircuit[CNOT[S[1], S[2]]]
```

Out[\*]=



```
In[*]:= cx = QuantumCircuit[ControlledGate[S[1], S[2, 1]]]
```

Out[\*]=



```
In[*]:= Elaborate[cnot - cx]
```

Out[\*]=

0

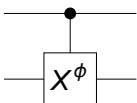
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## Controlled-Unitary Gates

Let us now consider a controlled-unitary gate of the following form.

```
In[*]:= QuantumCircuit[ControlledGate[S[1], Phase[ $\phi$ , S[3, 1]]]]
```

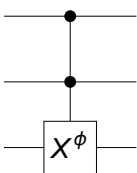
Out[\*]=



One also think of multi-control unitary gate of the following form.

```
In[*]:= QuantumCircuit[ControlledGate[S@{1, 2}, Phase[ $\phi$ , S[3, 1]]]]
```

Out[\*]=

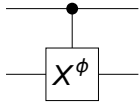


Here is the full form of ControlledGate.

```
In[*]:= ControlledGate[{S[1, $], S[2, $]} → {1, 1}, Phase[ϕ, S[3, 3]]]
Out[*]= ControlledGate[{S1, S2} → {1, 1}, S3Z(ϕ)]
```

---

```
In[*]:= qc = QuantumCircuit[ControlledGate[S[1], Phase[ϕ, S[3, 1]]]]
Out[*]=
```



```
In[*]:= mat = Matrix[qc];
mat // MatrixForm
```

```
Out[*]//MatrixForm=
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} + \frac{e^{i\phi}}{2} & \frac{1}{2} - \frac{e^{i\phi}}{2} \\ 0 & 0 & \frac{1}{2} - \frac{e^{i\phi}}{2} & \frac{1}{2} + \frac{e^{i\phi}}{2} \end{pmatrix}$$

```
In[*]:= in = Basis[S@{1, 2}];
out = qc ** in;
Thread[in → out] // TableForm
```

```
Out[*]//TableForm=
```

$$\begin{array}{l} |0_{S_1}0_{S_2}\rangle \rightarrow |0_{S_1}0_{S_2}\rangle \\ |0_{S_1}1_{S_2}\rangle \rightarrow |0_{S_1}1_{S_2}\rangle \\ |1_{S_1}0_{S_2}\rangle \rightarrow \frac{1}{2} (1 + e^{i\phi}) |1_{S_1}0_{S_2}\rangle + \frac{1}{2} (1 - e^{i\phi}) |1_{S_1}0_{S_2}1_{S_3}\rangle \\ |1_{S_1}1_{S_2}\rangle \rightarrow \frac{1}{2} (1 + e^{i\phi}) |1_{S_1}1_{S_2}\rangle + \frac{1}{2} (1 - e^{i\phi}) |1_{S_1}1_{S_2}1_{S_3}\rangle \end{array}$$


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## CNOT vs ControlledGate

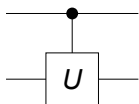
```
In[*]:= op = I * EulerRotation[{Pi / 3, Pi / 2, Pi / 6}, S[2]];
Matrix[op] // SimplifyThrough // MatrixForm
```

```
Out[*]//MatrixForm=
```

$$\begin{pmatrix} \frac{1}{2} + \frac{i}{2} & \left(-\frac{1}{4} - \frac{i}{4}\right) (i + \sqrt{3}) \\ \left(\frac{1}{4} + \frac{i}{4}\right) (1 + i\sqrt{3}) & -\frac{1}{2} + \frac{i}{2} \end{pmatrix}$$

```
In[*]:= cu = ControlledGate[S[1], op, "Label" → "U"];
QuantumCircuit[cu]
```

```
Out[*]=
```



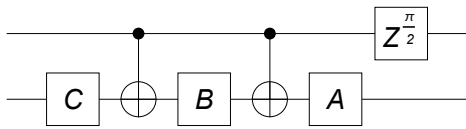
In[\*]:= Expand[cu]

Out[\*]=

$$\text{Sequence}\left[\text{Rotation}\left[-\text{ArcTan}\left[\frac{-\frac{1}{4} + \frac{\sqrt{3}}{4}}{\frac{1}{4} + \frac{\sqrt{3}}{4}}\right], S_2^z, \text{Label} \rightarrow C\right], \text{CNOT}[\{S_1\} \rightarrow \{1\}, \{S_2\}], \right. \\ \left. \left\{\text{Cos}\left[\frac{\pi}{8}\right]^2 + \frac{i S_2^y}{2\sqrt{2}} + \frac{i S_2^z}{2\sqrt{2}} - i S_2^x \text{Sin}\left[\frac{\pi}{8}\right]^2, \text{Label} \rightarrow B\right\}, \text{CNOT}[\{S_1\} \rightarrow \{1\}, \{S_2\}], \right. \\ \left. \left\{\frac{1}{2}\sqrt{3}\text{Cos}\left[\frac{\pi}{8}\right] - \frac{1}{2}i\text{Cos}\left[\frac{\pi}{8}\right]S_2^z + \frac{1}{2}iS_2^x\text{Sin}\left[\frac{\pi}{8}\right] - \frac{1}{2}i\sqrt{3}S_2^y\text{Sin}\left[\frac{\pi}{8}\right], \text{Label} \rightarrow A\right\}, \right. \\ \left. S_1^z\left(\frac{\pi}{2}\right)\right]$$

In[\*]:= qc = QuantumCircuit[Expand[cu]]

Out[\*]=



In[\*]:= Elaborate[cu] - Elaborate[cu]

Out[\*]=

0

- For a proof of the decomposition, see the Q3 tutorial titled “Controlled-Unitary Gates”.

## Summary

### Functions

- ControlledGate
- Hadamard

### Related Links

- Chapters 2 of the Quantum Workbook (2022, 2023).
- Tutorial: “Quantum Computation: Overview”