Quantum Operators

In[◦]:=
```
Quit[]
```

In[◦]:=
```
<< Q3`
```

Pauli Operators

In[◦]:=
```
Let[Qubit, S]
```

Elementary Pauli operators

In[◦]:=
```
S[2, 1]
S[2, 2]
S[2, 3]
```

Out[◦]=

S_2^X

Out[◦]=

S_2^Y

Out[◦]=

S_2^Z

In[◦]:=
```
S[2, All]
```

Out[◦]=

$\{S_2^X, S_2^Y, S_2^Z\}$

In[◦]:=
```
S[2, Full]
```

Out[◦]=

$\{S_2^0, S_2^X, S_2^Y, S_2^Z\}$

Extended Pauli Operators

In[◦]:=
```
S[2, 4]
S[2, 5]
S[2, 6]
```

Out[◦]=

S_2^+

Out[◦]=

S_2^-

Out[◦]=

S_2^H

Matrix Representation

In[ ]:=
```
mat = Matrix[S[2, 1]]
```
Out[ ]=

SparseArray[Specified elements: 2 Dimensions: {2, 2}]

In[ ]:=
```
MatrixForm[mat]
```
Out[ ]//MatrixForm=

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

In[ ]:=
```
mat = Matrix[S[2, 3]];
mat // MatrixForm
```
Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In[ ]:=
```
ExpressionFor[mat, S[2, $]]
```
Out[ ]=

S_2^Z

Action of Pauli operators on quantum states

Suppose that qubit `S[2,$]` is in the following quantum state.

In[ ]:=
```
in = 2 * Ket[S[1] → 0] - I * Ket[S[2] → 1]
```
Out[ ]=

$2 \left|0_{S_1}\right\rangle - i \left|1_{S_2}\right\rangle$

Operate Pauli X on the quantum state above.

In[ ]:=
```
out = S[2, 1] ** in
```
Out[ ]=

$-i \left|0_{S_1} 0_{S_2}\right\rangle + 2 \left|0_{S_1} 1_{S_2}\right\rangle$

In[ ]:=
```
Multiply[S[2, 1], in]
```
Out[ ]=

$-i \left|0_{S_1} 0_{S_2}\right\rangle + 2 \left|0_{S_1} 1_{S_2}\right\rangle$

An interesting operator is the Hadamard operator.

In[ ]:=
```
bs = Basis[S[2]]
```
Out[ ]=

$\left\{\left|0_{S_2}\right\rangle, \left|1_{S_2}\right\rangle\right\}$

In[ ]:=
```
out = S[2, 6] ** bs
```
Out[ ]=
$$\left\{\frac{|0_{S_2}\rangle}{\sqrt{2}}+\frac{|1_{S_2}\rangle}{\sqrt{2}},\ \frac{|0_{S_2}\rangle}{\sqrt{2}}-\frac{|1_{S_2}\rangle}{\sqrt{2}}\right\}$$

Multiplications of Two Operators

In[ ]:=
```
S[2, 1] ** S[2, 2]
```
Out[ ]=
$$i\ S_2^Z$$

In[ ]:=
```
S[2, 2] ** S[2, 3]
```
Out[ ]=
$$i\ S_2^X$$

In[ ]:=
```
S[2, 1] ** S[2, 3]
```
Out[ ]=
$$-i\ S_2^Y$$

Advanced Topic: Phase Operators

For quantum states, the relative phase difference is important leading to various interference effects.

In[ ]:=
```
in = S[2, 6] ** Ket[]
```
Out[ ]=
$$\frac{|0_{S_2}\rangle}{\sqrt{2}}+\frac{|1_{S_2}\rangle}{\sqrt{2}}$$

In[ ]:=
```
op = Phase[ϕ, S[2, 3]]
```
Out[ ]=
$$S_2^Z(\phi)$$

In[ ]:=
```
out = op ** in
```
Out[ ]=
$$\frac{|0_{S_2}\rangle}{\sqrt{2}}+\frac{e^{i\phi}\ |1_{S_2}\rangle}{\sqrt{2}}$$

In[ ]:=
```
op = Phase[ϕ, S[2, 1]]
```
Out[ ]=
$$S_2^X(\phi)$$

In[ ]:=
```
in = Ket[S[2] → 0]
```
Out[ ]=
$$|0_{S_2}\rangle$$

In[ ]:=
```
XBasisForm[in, S[2, $]]
```
Out[ ]=

$$\frac{\left|-_{S_2}\right\rangle}{\sqrt{2}}+\frac{\left|+_{S_2}\right\rangle}{\sqrt{2}}$$

In[ ]:=
```
out = op ** in
```
Out[ ]=

$$\frac{1}{2}\left(1+e^{i\phi}\right)\left|0_{S_2}\right\rangle+\frac{1}{2}\left(1-e^{i\phi}\right)\left|1_{S_2}\right\rangle$$

In[ ]:=
```
XBasisForm[out, S[2, $]]
```
Out[ ]=

$$\frac{e^{i\phi}\left|-_{S_2}\right\rangle}{\sqrt{2}}+\frac{\left|+_{S_2}\right\rangle}{\sqrt{2}}$$

Advanced Topic: Rotations

CNOT

In[ ]:=
```
op = CNOT[S[1, $], S[2, $]]
```
Out[ ]=

$\text{CNOT}[\{S_1\} \to \{1\}, \{S_2\}]$

In[ ]:=
```
in = Basis[S[{1, 2}]]
```
Out[ ]=

$\{\left|0_{S_1}0_{S_2}\right\rangle, \left|0_{S_1}1_{S_2}\right\rangle, \left|1_{S_1}0_{S_2}\right\rangle, \left|1_{S_1}1_{S_2}\right\rangle\}$

In[ ]:=
```
out = op ** in
```
Out[ ]=

$\{\left|0_{S_1}0_{S_2}\right\rangle, \left|0_{S_1}1_{S_2}\right\rangle, \left|1_{S_1}1_{S_2}\right\rangle, \left|1_{S_1}0_{S_2}\right\rangle\}$

In[ ]:=
```
Thread[in → out] // TableForm
```
Out[ ]//TableForm=

$\left|0_{S_1}0_{S_2}\right\rangle \to \left|0_{S_1}0_{S_2}\right\rangle$

$\left|0_{S_1}1_{S_2}\right\rangle \to \left|0_{S_1}1_{S_2}\right\rangle$

$\left|1_{S_1}0_{S_2}\right\rangle \to \left|1_{S_1}1_{S_2}\right\rangle$

$\left|1_{S_1}1_{S_2}\right\rangle \to \left|1_{S_1}0_{S_2}\right\rangle$

In[ ]:=
```
mat = Matrix[op];
mat // MatrixForm
```
Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

In[ ]:=
```
ExpressionFor[mat, S[{1, 2}, $]]
```
Out[ ]=

$$\frac{1}{2}-\frac{1}{2}S_1^Z S_2^+ - \frac{1}{2}S_1^Z S_2^- + \frac{S_1^Z}{2} + \frac{S_2^+}{2} + \frac{S_2^-}{2}$$

In[ ]:=
```
Elaborate[op]
```
Out[ ]=

$$\frac{1}{2}-\frac{1}{2}S_1^Z S_2^X + \frac{S_1^Z}{2} + \frac{S_2^X}{2}$$

In[ ]:=
```
ExpressionFor[mat, S[{1, 2}, $]] // Elaborate
```
Out[ ]=

$$\frac{1}{2}-\frac{1}{2}S_1^Z S_2^X + \frac{S_1^Z}{2} + \frac{S_2^X}{2}$$

Advanced Topic: Controlled-Unitary

Summary

Functions

- `Matrix`
- `ExpressionFor`
- `Multiply` (`**`)
- `S[k,1], S[k,2], S[k,3], …`
- `Phase`
- `CNOT`
- `Elaborate`

Related Links

- Chapters 1 and 2 of the Quantum Workbook (2022, 2023).
- Tutorial: “Quantum States”
- Tutorial: “Quantum Computation: Overview”