Deutsch-Jozsa Algorithm

Episode 29. Deutsch-Jozsa Algorithm

Episode 30. Bernstein-Vazirani Algorithm

Episode 31. Simon’s Algorithm

Statement of the Problem

1. Consider a classical function $f:\{0,\ 1\}^n \to \{0,\ 1\}$. It is a classical oracle taking n-bit inputs and returning 0 or 1.
2. The function is promised to be either constant (either 0 or 1 for all inputs) or balanced (0 for one half of the possible inputs and 1 for the other half).
3. The task is to determine whether f is constant or balanced by using the oracle the least number of times.

In classical algorithms, one has to evaluate the oracle $\left(2^{n-1}+1\right) \approx 2^n/2$ times in the worst case.

Quantum Implementation

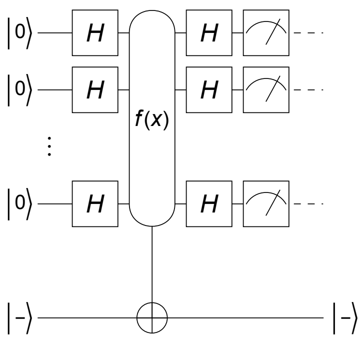


Figure 1. A quantum circuit for the Deutsch-Jozsa algorithm.

The first set of Hadamard gates:

$$\left|0\right\rangle \mapsto \frac{1}{2^{n/2}} \sum_{x=0}^{2^n-1} \left|x\right\rangle$$

The quantum oracle:

$$\frac{1}{2^{n/2}} \sum_{x=0}^{2^n-1} \left|x\right\rangle \mapsto \frac{1}{2^{n/2}} \sum_{x=0}^{2^n-1} \left|x\right\rangle \left(-1\right)^{f(x)}$$

The second set of Hadamard gates:

$$\frac{1}{2^{n/2}} \sum_{x=0}^{2^n-1} \left|x\right\rangle \left(-1\right)^{f(x)} \mapsto \frac{1}{2^n} \sum_{y=0}^{2^n-1} \left|y\right\rangle \sum_{x=0}^{2^n-1} (-1)^{f(x)+x\cdot y}$$

To see the effect of the function f on the final state, suppose that f is a constant function. Then, the output state is given by

$$\frac{(-1)^{f(0)}}{2^n} \sum_{y=0}^{2^n-1} \sum_{x=0}^{2^n-1} (-1)^{x\cdot y} = (-1)^{f(0)} \left|0\right\rangle.$$

Every measurement on each of the n qubits should yield zero with unit probability.

To make the analysis more explicit, consider the probability to find the n-qubit register in the state $\left|0\right\rangle \equiv \left|0\right\rangle \otimes \left|0\right\rangle \otimes \cdots \otimes \left|0\right\rangle$

$$P_0 = \frac{1}{2^n} \left| \sum_{x=0}^{2^n-1} (-1)^{f(x)} \right|^2.$$

1. If f is constant, then $P_0 = 1$.
2. When the function f is balanced, there are as many terms with -1 as 1, and the sum is always zero. There must be at least one qubit in the state $\left|1\right\rangle$ if the function f is balanced.

Example

In[◦]:=
```
Let[Qubit, S]
Let[Complex, c]
```

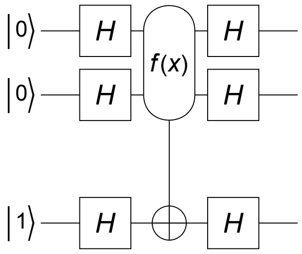
Consider a balanced function as an example.

In[◦]:=
```
Clear[f];
f[{0, 0}] = f[{1, 0}] = {0};
f[{0, 1}] = f[{1, 1}] = {1};
```

Here is a quantum circuit for the Deutsch-Jozsa algorithm. The final Hadamard gate on the third qubit is not necessary, but we put it here to make the output state more readable.

In[ ]:=
```
cc = {1, 2};
tt = {3};
all = Join[cc, tt];
qc = QuantumCircuit[Ket[S@tt → 1, S@all],
  S[all, 6], Oracle[f, S@cc, S@tt], S[all, 6],
  "Invisible" → S@{2.5}]
```

Out[ ]=



In[ ]:=
```
out = ExpressionFor[qc]
```

Out[ ]=

$\left|0_{S_1} 1_{S_2} 1_{S_3}\right\rangle$

Summary

Keywords

- Oracle
- Decision making
- Detusch-Jozsa problem

Functions

- `Oracle`

Related Links

- Section 4.2 of the Quantum Workbook (2022, 2023).
- Tutorial: Deutsch-Jozsa Algorithm