

Schmidt Decomposition

```
In[*]:= Quit[]
```

```
In[*]:= Let[Qubit, S]
```

Two Qubits

```
In[*]:= bs = Basis[S@{1, 2}]
```

```
Out[*]=  
 $\{ |0_{S_1}0_{S_2}\rangle, |0_{S_1}1_{S_2}\rangle, |1_{S_1}0_{S_2}\rangle, |1_{S_1}1_{S_2}\rangle \}$ 
```

```
In[*]:= v = Total@bs
```

```
Out[*]=  
 $|0_{S_1}0_{S_2}\rangle + |0_{S_1}1_{S_2}\rangle + |1_{S_1}0_{S_2}\rangle + |1_{S_1}1_{S_2}\rangle$ 
```

```
In[*]:= KetFactor[v]
```

```
Out[*]=  
 $(|0_{S_1}\rangle + |1_{S_1}\rangle) \otimes (|0_{S_2}\rangle + |1_{S_2}\rangle)$ 
```

```
In[*]:= {val, α, β} = SchmidtDecomposition[v, S[1], S[2]]
```

```
Out[*]=  
 $\{ \{2, 0\}, \left\{ \frac{|0_{S_1}\rangle}{\sqrt{2}} + \frac{|1_{S_1}\rangle}{\sqrt{2}}, -\frac{|0_{S_1}\rangle}{\sqrt{2}} + \frac{|1_{S_1}\rangle}{\sqrt{2}} \right\}, \left\{ \frac{|0_{S_2}\rangle}{\sqrt{2}} + \frac{|1_{S_2}\rangle}{\sqrt{2}}, -\frac{|0_{S_2}\rangle}{\sqrt{2}} + \frac{|1_{S_2}\rangle}{\sqrt{2}} \right\} \}$ 
```

```
In[*]:= val
```

```
Out[*]=  
 $\{2, 0\}$ 
```

```
In[*]:= α
```

β

```
Out[*]=  
 $\left\{ \frac{|0_{S_1}\rangle}{\sqrt{2}} + \frac{|1_{S_1}\rangle}{\sqrt{2}}, -\frac{|0_{S_1}\rangle}{\sqrt{2}} + \frac{|1_{S_1}\rangle}{\sqrt{2}} \right\}$ 
```

```
Out[*]=  
 $\left\{ \frac{|0_{S_2}\rangle}{\sqrt{2}} + \frac{|1_{S_2}\rangle}{\sqrt{2}}, -\frac{|0_{S_2}\rangle}{\sqrt{2}} + \frac{|1_{S_2}\rangle}{\sqrt{2}} \right\}$ 
```

```
In[*]:= new = α[[1]] ** β[[1]] * 2 // Elaborate
```

```
Out[*]=  
 $|0_{S_1}0_{S_2}\rangle + |0_{S_1}1_{S_2}\rangle + |1_{S_1}0_{S_2}\rangle + |1_{S_1}1_{S_2}\rangle$ 
```

```
In[*]:= v - new
```

```
Out[*]=  
0
```

```

In[*]:= w = Total@Rest[bs]
Out[*]=

$$|0_{S_1}1_{S_2}\rangle + |1_{S_1}0_{S_2}\rangle + |1_{S_1}1_{S_2}\rangle$$


In[*]:= KetFactor[w]
Out[*]=

$$|0_{S_1}1_{S_2}\rangle + |1_{S_1}0_{S_2}\rangle + |1_{S_1}1_{S_2}\rangle$$


In[*]:= {val, α, β} = SchmidtDecomposition[w, S[1], S[2]];
In[*]:= val
Out[*]=

$$\left\{ \sqrt{\frac{1}{2}} (3 + \sqrt{5}), \sqrt{\frac{1}{2}} (3 - \sqrt{5}) \right\}$$


In[*]:= α // N
β // N
Out[*]=

$$\{0.525731 |0_{S_1}\rangle + 0.850651 |1_{S_1}\rangle, 0.850651 |0_{S_1}\rangle - 0.525731 |1_{S_1}\rangle\}$$

Out[*]=

$$\{0.525731 |0_{S_2}\rangle + 0.850651 |1_{S_2}\rangle, -0.850651 |0_{S_2}\rangle + 0.525731 |1_{S_2}\rangle\}$$


In[*]:= new = Total@MapThread[#1 * CircleTimes[#2, #3] &, {val, α, β}]
Out[*]=

$$\sqrt{\frac{1}{2}} (3 - \sqrt{5}) \left( \frac{1}{10} (-5 - \sqrt{5}) |0_{S_1}0_{S_2}\rangle + \frac{|0_{S_1}1_{S_2}\rangle}{\sqrt{5}} + \frac{|1_{S_1}0_{S_2}\rangle}{\sqrt{5}} + \frac{1}{10} (-5 + \sqrt{5}) |1_{S_1}1_{S_2}\rangle \right) +$$


$$\sqrt{\frac{1}{2}} (3 + \sqrt{5}) \left( \frac{1}{10} (5 - \sqrt{5}) |0_{S_1}0_{S_2}\rangle + \frac{|0_{S_1}1_{S_2}\rangle}{\sqrt{5}} + \frac{|1_{S_1}0_{S_2}\rangle}{\sqrt{5}} + \frac{1}{10} (5 + \sqrt{5}) |1_{S_1}1_{S_2}\rangle \right)$$


In[*]:= new - w // FullSimplify
Out[*]=
0

```

```
In[*]:= more = SchmidtForm[w, S[1], S[2]]
```

```
Out[*]=
```

$$\sqrt{\frac{1}{2} (3 - \sqrt{5})} \left(\frac{|0_{S_1}\rangle}{\sqrt{1 + \left(1 + \frac{1}{2} (-1 - \sqrt{5})\right)^2}} + \frac{\left(1 + \frac{1}{2} (-1 - \sqrt{5})\right) |1_{S_1}\rangle}{\sqrt{1 + \left(1 + \frac{1}{2} (-1 - \sqrt{5})\right)^2}} \right) \otimes$$

$$\left(\frac{(-1 - \sqrt{5}) |0_{S_2}\rangle}{2 \sqrt{1 + \frac{1}{4} (-1 - \sqrt{5})^2}} + \frac{|1_{S_2}\rangle}{\sqrt{1 + \frac{1}{4} (-1 - \sqrt{5})^2}} \right) +$$

$$\sqrt{\frac{1}{2} (3 + \sqrt{5})} \left(\frac{|0_{S_1}\rangle}{\sqrt{1 + \left(1 + \frac{1}{2} (-1 + \sqrt{5})\right)^2}} + \frac{\left(1 + \frac{1}{2} (-1 + \sqrt{5})\right) |1_{S_1}\rangle}{\sqrt{1 + \left(1 + \frac{1}{2} (-1 + \sqrt{5})\right)^2}} \right) \otimes$$

$$\left(\frac{(-1 + \sqrt{5}) |0_{S_2}\rangle}{2 \sqrt{1 + \frac{1}{4} (-1 + \sqrt{5})^2}} + \frac{|1_{S_2}\rangle}{\sqrt{1 + \frac{1}{4} (-1 + \sqrt{5})^2}} \right)$$

```
In[*]:= more // N // KetRegulate
```

```
Out[*]=
```

$$0.618034 \left(0.850651 |0_{S_1}\rangle - 0.525731 |1_{S_1}\rangle \right) \otimes \left(-0.850651 |0_{S_2}\rangle + 0.525731 |1_{S_2}\rangle \right) +$$

$$1.61803 \left(0.525731 |0_{S_1}\rangle + 0.850651 |1_{S_1}\rangle \right) \otimes \left(0.525731 |0_{S_2}\rangle + 0.850651 |1_{S_2}\rangle \right)$$

```
In[*]:= ReleaseTimes[more]
```

```
Out[*]=
```

$$\sqrt{\frac{1}{2} (3 - \sqrt{5})} \left(\frac{1}{10} (-5 - \sqrt{5}) |0_{S_1}0_{S_2}\rangle + \frac{|0_{S_1}1_{S_2}\rangle}{\sqrt{5}} + \frac{|1_{S_1}0_{S_2}\rangle}{\sqrt{5}} + \frac{1}{10} (-5 + \sqrt{5}) |1_{S_1}1_{S_2}\rangle \right) +$$

$$\sqrt{\frac{1}{2} (3 + \sqrt{5})} \left(\frac{1}{10} (5 - \sqrt{5}) |0_{S_1}0_{S_2}\rangle + \frac{|0_{S_1}1_{S_2}\rangle}{\sqrt{5}} + \frac{|1_{S_1}0_{S_2}\rangle}{\sqrt{5}} + \frac{1}{10} (5 + \sqrt{5}) |1_{S_1}1_{S_2}\rangle \right)$$

```
In[*]:= % - w // FullSimplify
```

```
Out[*]=
```

0

(1+2) Qubits

```
In[*]:= bs = Basis[S@{1, 2, 3}]
```

```
Out[*]=
```

$$\{ |0_{S_1}0_{S_2}0_{S_3}\rangle, |0_{S_1}0_{S_2}1_{S_3}\rangle, |0_{S_1}1_{S_2}0_{S_3}\rangle,$$

$$|0_{S_1}1_{S_2}1_{S_3}\rangle, |1_{S_1}0_{S_2}0_{S_3}\rangle, |1_{S_1}0_{S_2}1_{S_3}\rangle, |1_{S_1}1_{S_2}0_{S_3}\rangle, |1_{S_1}1_{S_2}1_{S_3}\rangle \}$$

```
In[*]:= v = Total[bs]
```

```
Out[*]=
```

$$|0_{S_1}0_{S_2}0_{S_3}\rangle + |0_{S_1}0_{S_2}1_{S_3}\rangle + |0_{S_1}1_{S_2}0_{S_3}\rangle +$$

$$|0_{S_1}1_{S_2}1_{S_3}\rangle + |1_{S_1}0_{S_2}0_{S_3}\rangle + |1_{S_1}0_{S_2}1_{S_3}\rangle + |1_{S_1}1_{S_2}0_{S_3}\rangle + |1_{S_1}1_{S_2}1_{S_3}\rangle$$

```
In[*]:= KetFactor[v]
```

```
Out[*]=
```

$$\left(\left| 0_{S_1} \right\rangle + \left| 1_{S_1} \right\rangle \right) \otimes \left(\left| 0_{S_2} \right\rangle + \left| 1_{S_2} \right\rangle \right) \otimes \left(\left| 0_{S_3} \right\rangle + \left| 1_{S_3} \right\rangle \right)$$

```
In[*]:= {val, α, β} = SchmidtDecomposition[v, S[1], S@{2, 3}]
```

```
Out[*]=
```

$$\left\{ \left\{ 2\sqrt{2}, 0 \right\}, \left\{ \frac{\left| 0_{S_1} \right\rangle}{\sqrt{2}} + \frac{\left| 1_{S_1} \right\rangle}{\sqrt{2}}, -\frac{\left| 0_{S_1} \right\rangle}{\sqrt{2}} + \frac{\left| 1_{S_1} \right\rangle}{\sqrt{2}} \right\}, \right. \\ \left. \left\{ \frac{1}{2} \left| 0_{S_2} 0_{S_3} \right\rangle + \frac{1}{2} \left| 0_{S_2} 1_{S_3} \right\rangle + \frac{1}{2} \left| 1_{S_2} 0_{S_3} \right\rangle + \frac{1}{2} \left| 1_{S_2} 1_{S_3} \right\rangle, -\frac{\left| 0_{S_2} 0_{S_3} \right\rangle}{\sqrt{2}} + \frac{\left| 1_{S_2} 1_{S_3} \right\rangle}{\sqrt{2}} \right\} \right\}$$

```
In[*]:= w = Total@Most@bs
```

```
Out[*]=
```

$$\left| 0_{S_1} 0_{S_2} 0_{S_3} \right\rangle + \left| 0_{S_1} 0_{S_2} 1_{S_3} \right\rangle + \left| 0_{S_1} 1_{S_2} 0_{S_3} \right\rangle + \left| 0_{S_1} 1_{S_2} 1_{S_3} \right\rangle + \left| 1_{S_1} 0_{S_2} 0_{S_3} \right\rangle + \left| 1_{S_1} 0_{S_2} 1_{S_3} \right\rangle + \left| 1_{S_1} 1_{S_2} 0_{S_3} \right\rangle$$

```
In[*]:= KetFactor[w]
```

```
Out[*]=
```

$$\left| 0_{S_1} 0_{S_2} 0_{S_3} \right\rangle + \left| 0_{S_1} 0_{S_2} 1_{S_3} \right\rangle + \left| 0_{S_1} 1_{S_2} 0_{S_3} \right\rangle + \left| 0_{S_1} 1_{S_2} 1_{S_3} \right\rangle + \left| 1_{S_1} 0_{S_2} 0_{S_3} \right\rangle + \left| 1_{S_1} 0_{S_2} 1_{S_3} \right\rangle + \left| 1_{S_1} 1_{S_2} 0_{S_3} \right\rangle$$

```
In[*]:= {val, α, β} = SchmidtDecomposition[w, S[1], S@{2, 3}]
```

```
Out[*]=
```

$$\left\{ \left\{ \sqrt{\frac{1}{2}} (7 + \sqrt{37}), \sqrt{\frac{1}{2}} (7 - \sqrt{37}) \right\}, \left\{ \frac{(1 + \sqrt{37}) \left| 0_{S_1} \right\rangle}{6 \sqrt{1 + \frac{1}{36} (1 + \sqrt{37})^2}} + \frac{\left| 1_{S_1} \right\rangle}{\sqrt{1 + \frac{1}{36} (1 + \sqrt{37})^2}}, \right. \right. \\ \frac{(1 - \sqrt{37}) \left| 0_{S_1} \right\rangle}{6 \sqrt{1 + \frac{1}{36} (1 - \sqrt{37})^2}} + \frac{\left| 1_{S_1} \right\rangle}{\sqrt{1 + \frac{1}{36} (1 - \sqrt{37})^2}} \left. \right\}, \\ \left\{ \frac{(1 + \frac{1}{6} (1 + \sqrt{37})) \left| 0_{S_2} 0_{S_3} \right\rangle}{\sqrt{\frac{1}{36} (1 + \sqrt{37})^2 + 3 (1 + \frac{1}{6} (1 + \sqrt{37}))^2}} + \frac{(1 + \frac{1}{6} (1 + \sqrt{37})) \left| 0_{S_2} 1_{S_3} \right\rangle}{\sqrt{\frac{1}{36} (1 + \sqrt{37})^2 + 3 (1 + \frac{1}{6} (1 + \sqrt{37}))^2}} + \right. \\ \frac{(1 + \frac{1}{6} (1 + \sqrt{37})) \left| 1_{S_2} 0_{S_3} \right\rangle}{\sqrt{\frac{1}{36} (1 + \sqrt{37})^2 + 3 (1 + \frac{1}{6} (1 + \sqrt{37}))^2}} + \frac{(1 + \sqrt{37}) \left| 1_{S_2} 1_{S_3} \right\rangle}{6 \sqrt{\frac{1}{36} (1 + \sqrt{37})^2 + 3 (1 + \frac{1}{6} (1 + \sqrt{37}))^2}}, \\ \frac{(1 + \frac{1}{6} (1 - \sqrt{37})) \left| 0_{S_2} 0_{S_3} \right\rangle}{\sqrt{\frac{1}{36} (1 - \sqrt{37})^2 + 3 (1 + \frac{1}{6} (1 - \sqrt{37}))^2}} + \frac{(1 + \frac{1}{6} (1 - \sqrt{37})) \left| 0_{S_2} 1_{S_3} \right\rangle}{\sqrt{\frac{1}{36} (1 - \sqrt{37})^2 + 3 (1 + \frac{1}{6} (1 - \sqrt{37}))^2}} + \\ \left. \frac{(1 + \frac{1}{6} (1 - \sqrt{37})) \left| 1_{S_2} 0_{S_3} \right\rangle}{\sqrt{\frac{1}{36} (1 - \sqrt{37})^2 + 3 (1 + \frac{1}{6} (1 - \sqrt{37}))^2}} + \frac{(1 - \sqrt{37}) \left| 1_{S_2} 1_{S_3} \right\rangle}{6 \sqrt{\frac{1}{36} (1 - \sqrt{37})^2 + 3 (1 + \frac{1}{6} (1 - \sqrt{37}))^2}} \right\} \right\}$$

```
In[*]:= val
```

```
Out[*]=
```

$$\left\{ \sqrt{\frac{1}{2}} (7 + \sqrt{37}), \sqrt{\frac{1}{2}} (7 - \sqrt{37}) \right\}$$

```

In[*]:= more = SchmidtForm[w, S[1], S@{2, 3}] // N
Out[*]=
2.55761 (0.76302 |0S1⟩ + 0.646375 |1S1⟩) ⊗
(0.551059 |0S20S3⟩ + 0.551059 |0S21S3⟩ + 0.551059 |1S20S3⟩ + 0.298333 |1S21S3⟩) +
0.677214 (-0.646375 |0S1⟩ + 0.76302 |1S1⟩) ⊗
(0.172243 |0S20S3⟩ + 0.172243 |0S21S3⟩ + 0.172243 |1S20S3⟩ - 0.954462 |1S21S3⟩)

In[*]:= new = ReleaseTimes[more]
Out[*]=
0.677214 (-0.111333 |0S10S20S3⟩ - 0.111333 |0S10S21S3⟩ -
0.111333 |0S11S20S3⟩ + 0.61694 |0S11S21S3⟩ + 0.131425 |1S10S20S3⟩ +
0.131425 |1S10S21S3⟩ + 0.131425 |1S11S20S3⟩ - 0.728273 |1S11S21S3⟩) +
2.55761 (0.420469 |0S10S20S3⟩ + 0.420469 |0S10S21S3⟩ + 0.420469 |0S11S20S3⟩ +
0.227634 |0S11S21S3⟩ + 0.356191 |1S10S20S3⟩ +
0.356191 |1S10S21S3⟩ + 0.356191 |1S11S20S3⟩ + 0.192835 |1S11S21S3⟩)

In[*]:= new - w // Garner // Chop
Out[*]=
0

```

Entanglement, and so what?

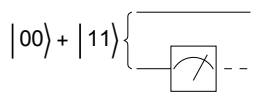
```

In[*]:= Let[Qubit, S]

In[*]:= v = Ket[] + Ket[S@{1, 2} → 1] // KetRegulate
Out[*]=
|0S10S2⟩ + |1S11S2⟩

In[*]:= qc1 = QuantumCircuit[v,
  Measurement[S[2, 3]],
  "PortSize" → {1.65, 0.5}]
Out[*]=

```



```

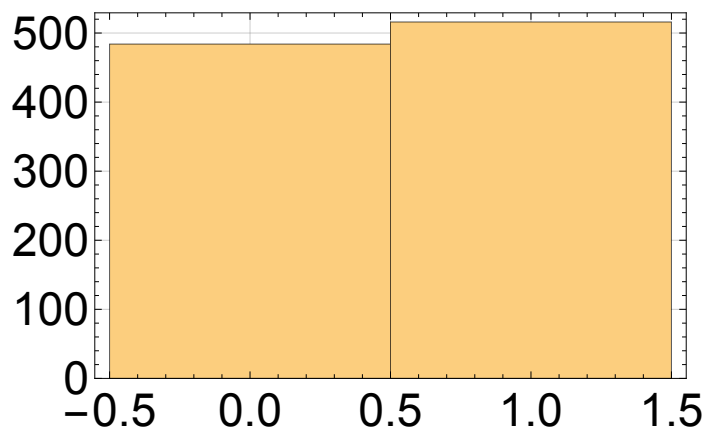
In[*]:= data = Table[Elaborate[qc1]; Readout[S[2, 3]], 20]
Out[*]=
{1, 0, 1, 1, 0, 1, 1, 1, 0, 0, 1, 0, 1, 1, 1, 0, 0, 1, 0, 1}

```

```
In[ ]:= EchoTiming[data = Table[Elaborate[qc1]; Readout[S[2, 3]], 1000];]
Histogram[data, ImageSize → Medium]
```

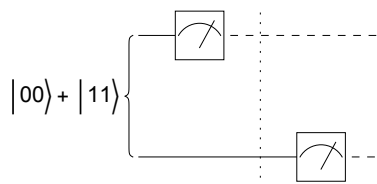
2.33801

Out[]:=



```
In[ ]:= qc2 = QuantumCircuit[v,
  Measurement[S[1, 3]], "Separator",
  Measurement[S[2, 3]],
  "Invisible" → S@{1.5},
  "PortSize" → {1.65, 0.5}]
```

Out[]:=



```
In[ ]:= data = Table[Elaborate[qc2]; Readout[S[{1, 2}, 3]], 20]
```

Out[]:=

```
{ {0, 0}, {1, 1}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {1, 1}, {1, 1}, {0, 0},
  {0, 0}, {0, 0}, {0, 0}, {1, 1}, {0, 0}, {1, 1}, {0, 0}, {0, 0}, {1, 1}, {1, 1} }
```

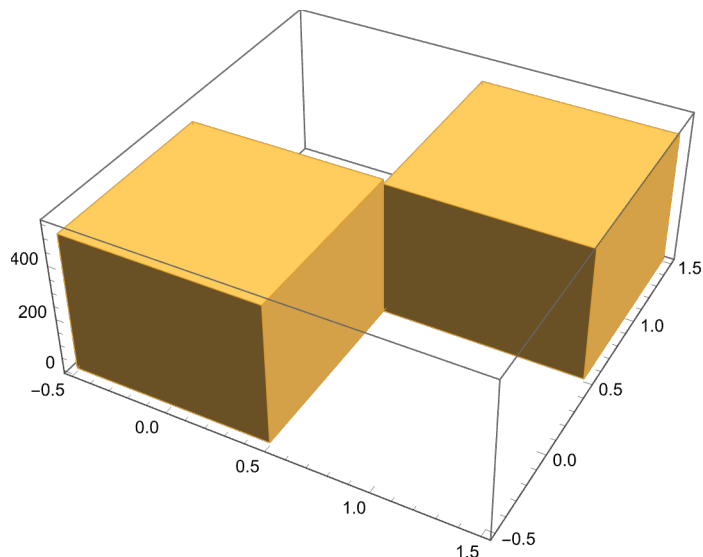
```

In[ ]:= EchoTiming[data = Table[Elaborate[qc2];
      Readout[S[{1, 2}, 3]], 1000];]
Histogram3D[data, ImageSize → Medium]

```

3.56709

Out[]:=



Summary

Functions

- `SchmidtDecomposition`
- `SchmidtForm`

Related Links

- Chapters 1 and 2 of the Quantum Workbook (2022, 2023).
- Tutorial: “Quantum States”
- Tutorial: “Quantum Computation: Overview”