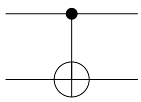
Controlled-Unitary Gates

```
In[ ]:= Let[Qubit, S]
```

CNOT vs Controlled-X

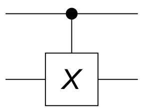
```
In[ ]:= cnot = QuantumCircuit[CNOT[S[1], S[2]]]
```

Out[ ]=



```
In[ ]:= cx = QuantumCircuit[ControlledGate[S[1], S[2, 1]]]
```

Out[ ]=




```
In[ ]:= Elaborate[cnot - cx]
```

Out[ ]=

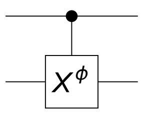
```
0
```

Controlled-Unitary Gates

Let us now consider a controlled-unitary gate of the following form.

```
In[ ]:= QuantumCircuit[ControlledGate[S[1], Phase[ϕ, S[3, 1]]]]
```

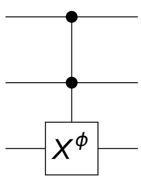
Out[ ]=



One also think of multi-control unitary gate of the following form.

```
In[ ]:= QuantumCircuit[ControlledGate[S@{1, 2}, Phase[ϕ, S[3, 1]]]]
```

Out[ ]=

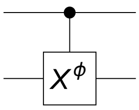


Here is the full form of **ControlledGate**.

In[∘]:=
```
ControlledGate[{S[1, $], S[2, $]} → {1, 1}, Phase[ϕ, S[3, 3]]]
```
Out[∘]=

$\text{ControlledGate}[\{S_1, S_2\} \to \{1, 1\}, S_3^z(\phi)]$

In[∘]:=
```
qc = QuantumCircuit[ControlledGate[S[1], Phase[ϕ, S[3, 1]]]]
```
Out[∘]=



In[∘]:=
```
mat = Matrix[qc];
mat // MatrixForm
```
Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2}+\frac{e^{i\phi}}{2} & \frac{1}{2}-\frac{e^{i\phi}}{2} \\ 0 & 0 & \frac{1}{2}-\frac{e^{i\phi}}{2} & \frac{1}{2}+\frac{e^{i\phi}}{2} \end{pmatrix}$$

In[∘]:=
```
in = Basis[S@{1, 2}];
out = qc ** in;
Thread[in → out] // TableForm
```
Out[∘]//TableForm=

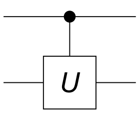
$$\left|0_{S_1}0_{S_2}\right\rangle \to \left|0_{S_1}0_{S_2}\right\rangle$$
$$\left|0_{S_1}1_{S_2}\right\rangle \to \left|0_{S_1}1_{S_2}\right\rangle$$
$$\left|1_{S_1}0_{S_2}\right\rangle \to \frac{1}{2}\left(1+e^{i\phi}\right)\left|1_{S_1}0_{S_2}\right\rangle + \frac{1}{2}\left(1-e^{i\phi}\right)\left|1_{S_1}0_{S_2}1_{S_3}\right\rangle$$
$$\left|1_{S_1}1_{S_2}\right\rangle \to \frac{1}{2}\left(1+e^{i\phi}\right)\left|1_{S_1}1_{S_2}\right\rangle + \frac{1}{2}\left(1-e^{i\phi}\right)\left|1_{S_1}1_{S_2}1_{S_3}\right\rangle$$

CNOT vs ControlledGate

In[∘]:=
```
op = I * EulerRotation[{Pi / 3, Pi / 2, Pi / 6}, S[2]];
Matrix[op] // SimplifyThrough // MatrixForm
```
Out[∘]//MatrixForm=

$$\begin{pmatrix} \frac{1}{2}+\frac{i}{2} & \left(-\frac{1}{4}-\frac{i}{4}\right)\left(i+\sqrt{3}\right) \\ \left(\frac{1}{4}+\frac{i}{4}\right)\left(1+i\sqrt{3}\right) & -\frac{1}{2}+\frac{i}{2} \end{pmatrix}$$

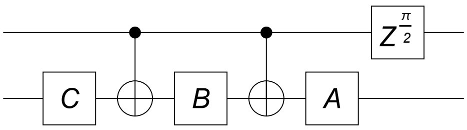
In[∘]:=
```
cu = ControlledGate[S[1], op, "Label" → "U"];
QuantumCircuit[cu]
```
Out[∘]=



In[∘]:=
```
Expand[cu]
```
Out[∘]=

$$\text{Sequence}\left[\text{Rotation}\left[-\text{ArcTan}\left[\frac{-\frac{1}{4}+\frac{\sqrt{3}}{4}}{\frac{1}{4}+\frac{\sqrt{3}}{4}}\right], S_2^z, \text{Label}\to C\right], \text{CNOT}[\{S_1\}\to\{1\}, \{S_2\}],\right.$$
$$\left\{\text{Cos}\left[\frac{\pi}{8}\right]^2+\frac{i\, S_2^y}{2\sqrt{2}}+\frac{i\, S_2^z}{2\sqrt{2}}-i\, S_2^x\, \text{Sin}\left[\frac{\pi}{8}\right]^2, \text{Label}\to B\right\}, \text{CNOT}[\{S_1\}\to\{1\}, \{S_2\}],$$
$$\left\{\frac{1}{2}\sqrt{3}\,\text{Cos}\left[\frac{\pi}{8}\right]-\frac{1}{2}\, i\,\text{Cos}\left[\frac{\pi}{8}\right] S_2^z+\frac{1}{2}\, i\, S_2^x\,\text{Sin}\left[\frac{\pi}{8}\right]-\frac{1}{2}\, i\sqrt{3}\, S_2^y\,\text{Sin}\left[\frac{\pi}{8}\right], \text{Label}\to A\right\},$$
$$\left.S_1^z\left(\frac{\pi}{2}\right)\right]$$

In[∘]:=
```
qc = QuantumCircuit[Expand[cu]]
```
Out[∘]=



In[∘]:=
```
Elaborate[cu] - Elaborate[cu]
```
Out[∘]=

0

- For a proof of the decomposition, see the Q3 tutorial titled “Controlled-Unitary Gates”.

Summary

Functions

- **ControlledGate**
- **Hadamard**

Related Links

- Chapters 2 of the Quantum Workbook (2022, 2023).
- Tutorial: “Quantum Computation: Overview”