Controlled-Unitary Gates: Special Examples

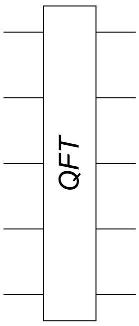
```
In[]:= Let[Qubit, S]
```

Controlled-Phase Gates

This is the quantum Fourier transform circuit.

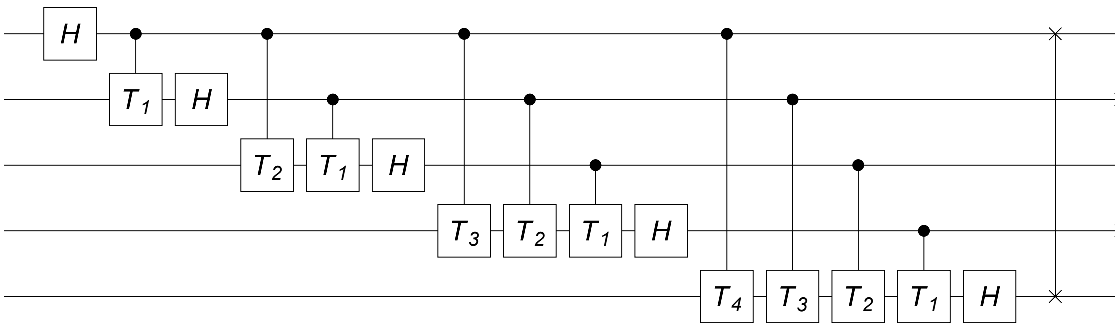
```
In[]:= qft = QuantumCircuit[QFT[S@{1, 2, 3, 4, 5}]]
```

Out[]=




```
In[]:= qft = QuantumCircuit[Expand@QFT[S@{1, 2, 3, 4, 5}]]
```

Out[]=




```
In[]:= phases = 2 Pi / HoldForm /@ {1, 2, 4, 8, 16};
ops = S@{0, 3, 7, 8, 9};
MatrixForm[{phases, ops}]
```

Out[]//MatrixForm=

$$\begin{pmatrix} \frac{2\pi}{1} & \frac{2\pi}{2} & \frac{2\pi}{4} & \frac{2\pi}{8} & \frac{2\pi}{16} \\ S^0 & S^z & S^S & S^T & S^F \end{pmatrix}$$

In[ ]:=
```
S[1, C[-1]]
S[1, C[-2]]
S[1, C[-3]]
S[1, C[-4]]
```

Out[ ]=
S_1^Z

Out[ ]=
S_1^S

Out[ ]=
S_1^T

Out[ ]=
S_1^F

In[ ]:=
```
S[1, C[-5]]
S[1, C[-6]]
S[1, C[-7]]
S[1, C[-8]]
```

Out[ ]=
$S_1^{\frac{2\pi}{2^5}}$

Out[ ]=
$S_1^{\frac{2\pi}{2^6}}$

Out[ ]=
$S_1^{\frac{2\pi}{2^7}}$

Out[ ]=
$S_1^{\frac{2\pi}{2^8}}$

- For about the QFT algorithm, see the Q3 tutorial “Quantum Fourier Transform”.

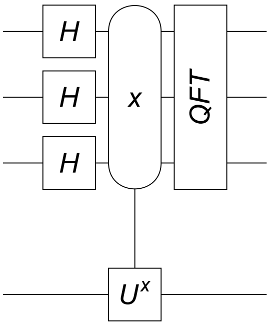
Controlled-Exponentiation Gates

In[ ]:=
```
Let[Qubit, S, T]
$n = 3;
cc = S[Range@$n, $]
```

Out[ ]=
$\{S_1, S_2, S_3\}$

In[ ]:=
```
QuantumCircuit[
 Through[cc[6]],
 cexp = ControlledExp[cc, T[C[-5]]],
 QFT[cc],
 "Invisible" → S@{$n + 1}]
```

Out[ ]=



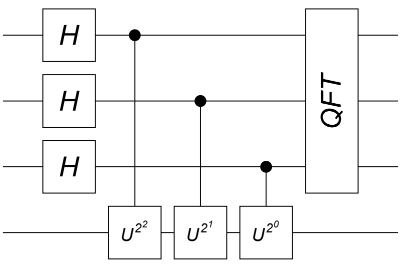
In[ ]:=
```
Expand@cexp
```

Out[ ]=

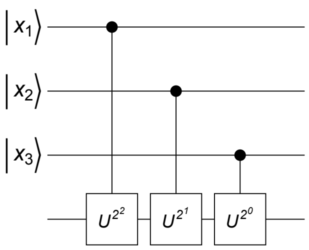
$$\mathrm{Sequence}\Big[\mathrm{ControlledGate}\Big[\{S_1\} \to \{1\}, \frac{1}{8}\left(3 + 2\,(-1)^{1/8} + 3\,(-1)^{1/4}\right) + \frac{1}{8}\left(3 + 2\,(-1)^{1/16} - 2\,(-1)^{3/16} - 3\,(-1)^{1/4}\right) T^z + \frac{1}{4}\left(-1 + (-1)^{1/16}\right)\left(-1 + (-1)^{1/8}\right) T^{\frac{2\pi}{2^5}}, \mathrm{Label} \to U^{2^2}, \mathrm{LabelSize} \to 0.65\Big], \mathrm{ControlledGate}\Big[\{S_2\} \to \{1\}, \frac{1}{2}\left(1 + (-1)^{1/8}\right) + \frac{1}{2}\left(1 - (-1)^{1/8}\right) T^z, \mathrm{Label} \to U^{2^1}, \mathrm{LabelSize} \to 0.65\Big], \mathrm{ControlledGate}\Big[\{S_3\} \to \{1\}, T^{\frac{2\pi}{2^5}}, \mathrm{Label} \to U^{2^0}, \mathrm{LabelSize} \to 0.65\Big]\Big]$$

In[ ]:=
```
qpe = QuantumCircuit[Through[cc[6]],
  Expand[cexp], QFT[cc]]
```

Out[ ]=



In[ ]:=
```
Let[Binary, x]
qc = QuantumCircuit[Ket[cc → x@Range[$n]], Expand[cexp]]
```
Out[ ]=



Note that $(U^{2^2})^{x_1} (U^{2^1})^{x_2} (U^{2^0})^{x_3} = U^{x_1 2^2} U^{x_2 2^1} U^{x_3 2^0} = U^{x_1 2^2 + x_2 2^1 + x_3 2^0}$.
Now, we see that $x_1 2^2 + x_2 2^1 + x_3 2^0 = (x_1 x_2 x_3)_2 = x$.

- For the QPE algorithm, see the Q3 tutorial “Quantum Phase Estimation”.

Summary

Functions

- `ControlledExp`
- `ControlledGate`
- `S[…,C[-k]]=Phase[` $\frac{2\pi}{2^k}$ `,S[…,3]]`
- `Hadamard`

Related Links

- Chapters 2 and 4 of the Quantum Workbook (2022, 2023).
- Tutorial: “Quantum Computation: Overview”
- Tutorial: “Quantum Phase Estimation”.
- Tutorial: “Quantum Fourier Transform”.