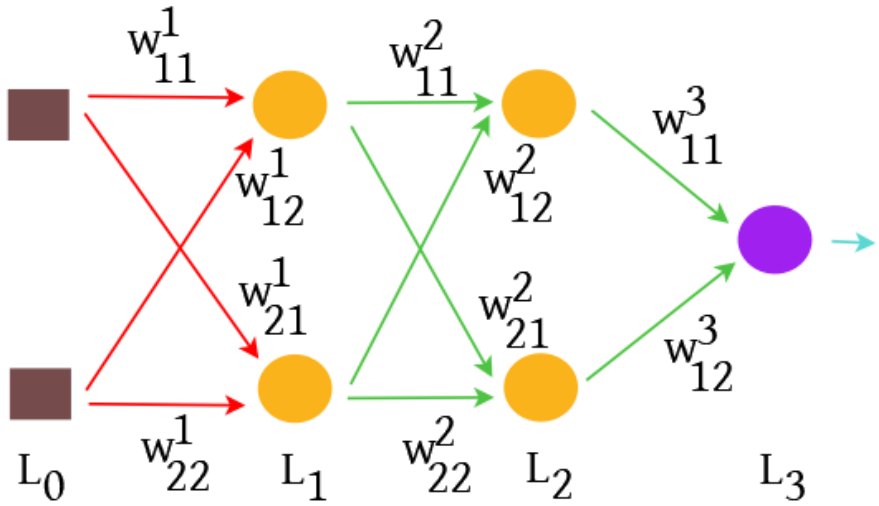
Deep Learning - Theory and Practice

IE 643
Lecture 8

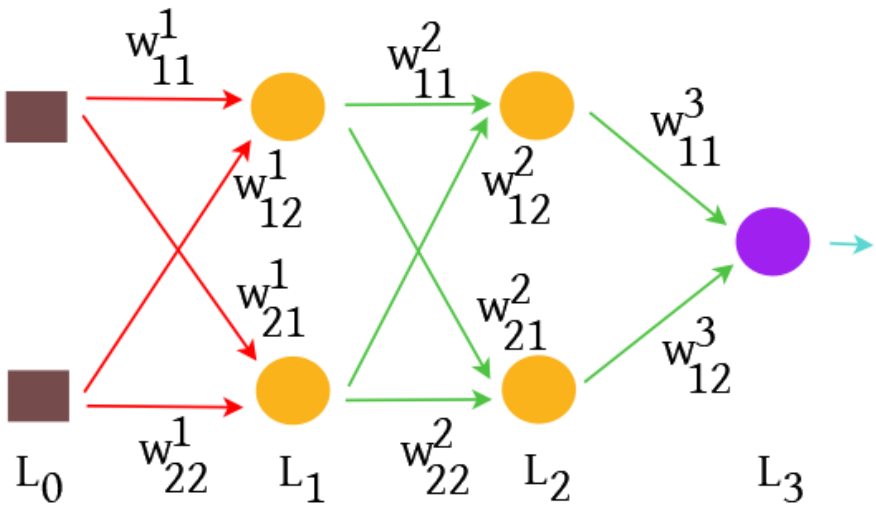
August 27, 2022.

Multi Layer Perceptron - Data Perspective



- **Input:** Training Data $D = \{(x^s, y^s)\}_{s=1}^{S}$.
- For each sample x^s the prediction $\hat{y}^s = \text{MLP}(x^s)$.
- **Error:** $e^s = E(y^s, \hat{y}^s)$.
- **Aim:** To minimize $\sum_{s=1}^{S} e^s$.

Multi Layer Perceptron - Data Perspective

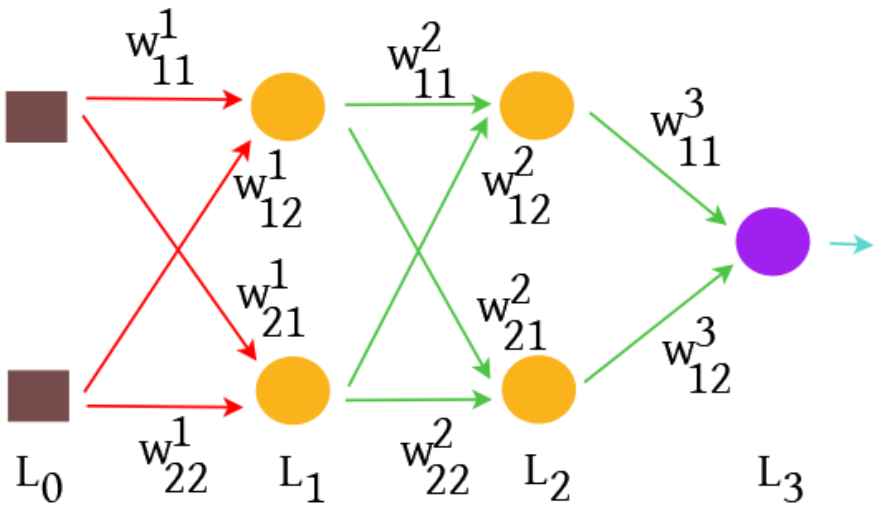


Optimization perspective

- Given training data $D = \{(x^s, y^s)\}_{s=1}^{S}$,

$$\min \sum_{s=1}^{S} e^s$$

Multi Layer Perceptron - Data Perspective



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- Given training data $D = \{(x^s, y^s)\}_{s=1}^{S}$,

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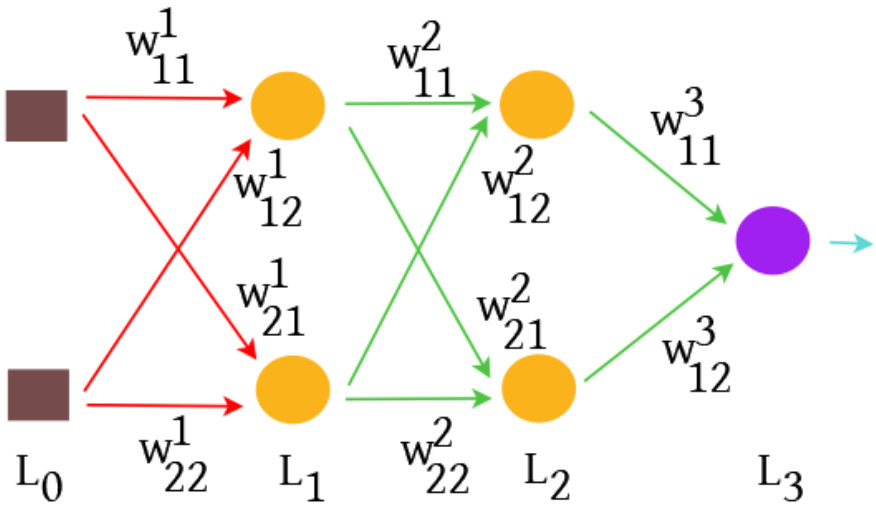
Multi Layer Perceptron - Data Perspective

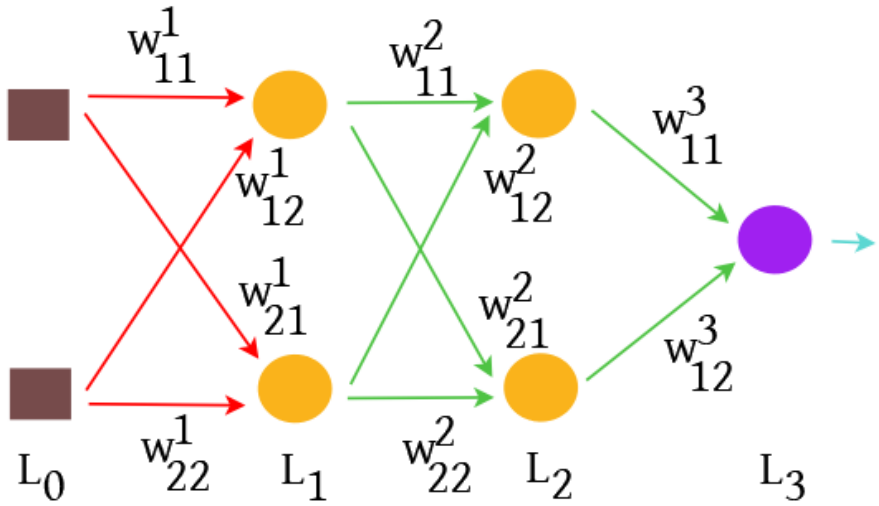


Optimization perspective

- Given training data $D = \{(x^s, y^s)\}_{s=1}^{S}$,

$$\min \sum_{s=1}^{S} e^s = \sum_{s=1}^{S} E(y^s, \hat{y}^s) = \sum_{s=1}^{S} E(y^s, \text{MLP}(x^s))$$

Multi Layer Perceptron - Data Perspective



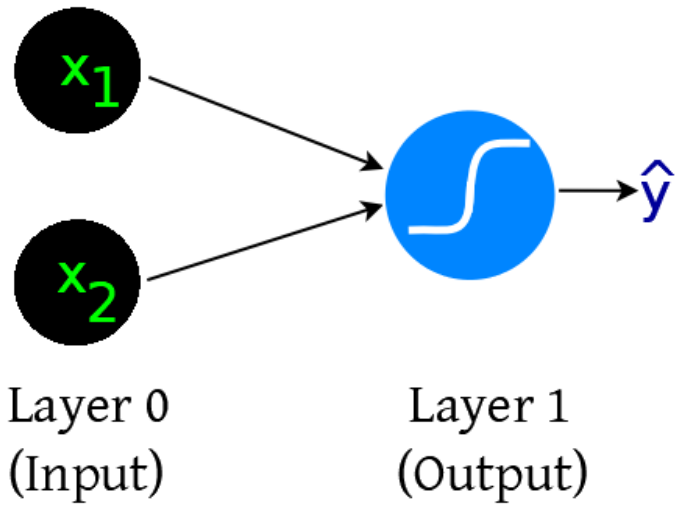
Optimization perspective

- Given training data $D = \{(x^s, y^s)\}_{s=1}^{S}$,

$$\min \sum_{s=1}^{S} e^s = \sum_{s=1}^{S} E(y^s, \hat{y}^s) = \sum_{s=1}^{S} E(y^s, \text{MLP}(x^s))$$

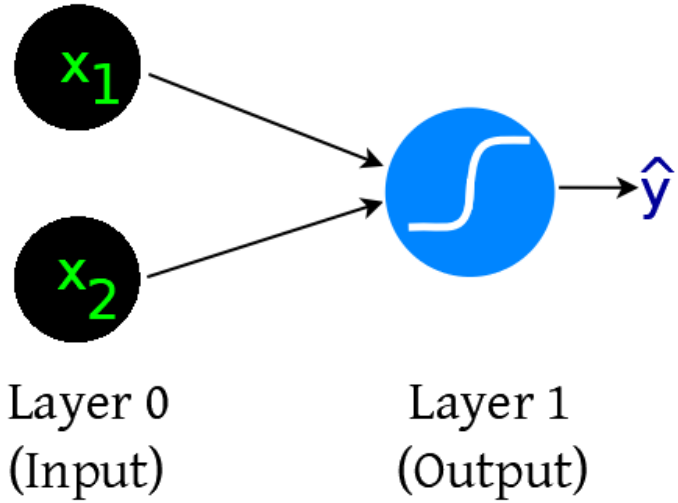
- Note: The minimization is over the weights of the MLP $W^1, \ldots, W^L$, where L denotes number of layers in MLP.

MLP - Data Perspective: A Simple Example



$$\hat{y} = \sigma(w_{11}^1 x_1 + w_{12}^1 x_2) = \frac{1}{1 + \exp\left(-[w_{11}^1 x_1 + w_{12}^1 x_2]\right)}$$

MLP - Data Perspective: A Simple Example



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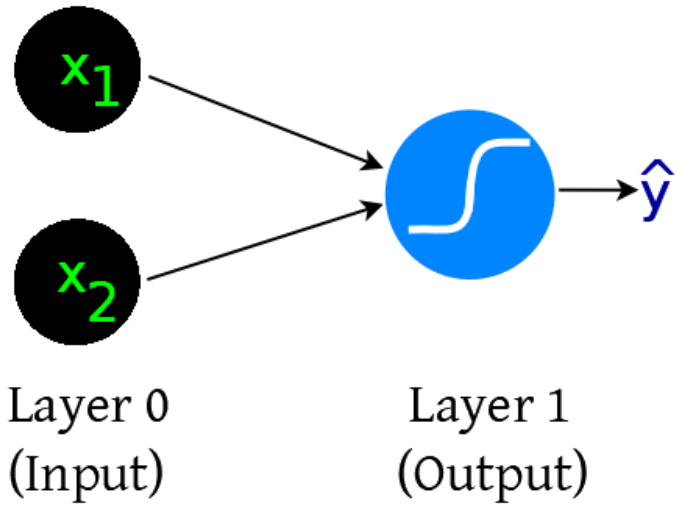
Property of 0-1 sigmoid $\sigma : \mathbb{R} \to [0, 1]$

- σ is continuous
- σ is monotonic
- $\sigma(z) \to \begin{cases} 0 & \text{if } z \to -\infty \\ 1 & \text{if } z \to +\infty \end{cases}$

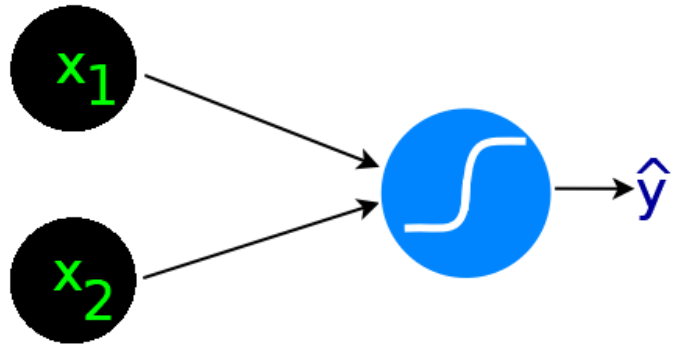
MLP - Data Perspective: A Simple Example



- Let

$$D = \{(x^1 = (-3, -3), y^1 = 1), \\ (x^2 = (-2, -2), y^2 = 1), \\ (x^3 = (4, 4), y^3 = 0), \\ (x^4 = (2, -5), y^4 = 0)\}.$$

MLP - Data Perspective: A Simple Example



x_1	x_2	y	$\hat{y} = \sigma(w_{11}^1 x_1 + w_{12}^1 x_2)$
-3	-3	1	$\sigma(-3w_{11}^1 - 3w_{12}^1)$
-2	-2	1	$\sigma(-2w_{11}^1 - 2w_{12}^1)$
4	4	0	$\sigma(4w_{11}^1 + 4w_{12}^1)$
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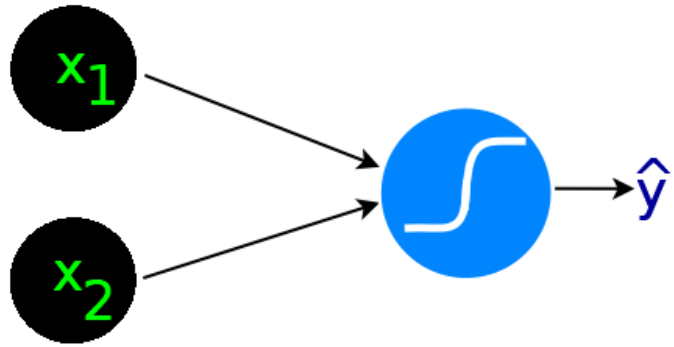
MLP - Data Perspective: A Simple Example



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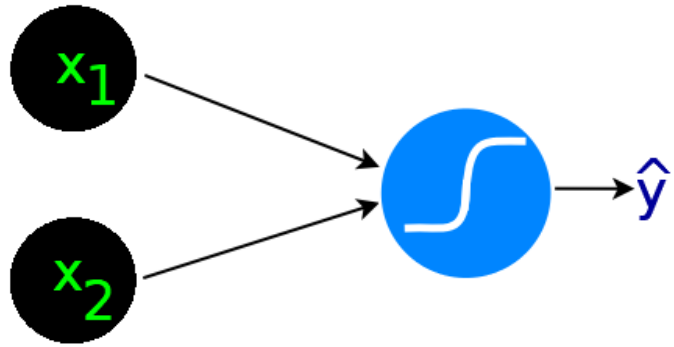
- **Assume:** $\text{Err}(y, \hat{y}) = (y - \hat{y})^2$.

MLP - Data Perspective: A Simple Example



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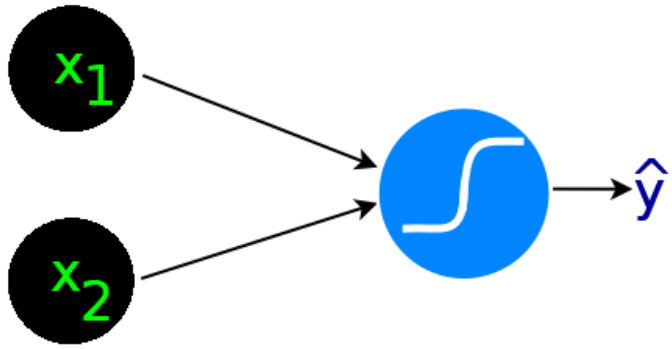
- **Assume:** $\text{Err}(y, \hat{y}) = (y - \hat{y})^2$.
- Popularly called the squared error.

MLP - Data Perspective: A Simple Example



Layer 0 (Input) Layer 1 (Output)

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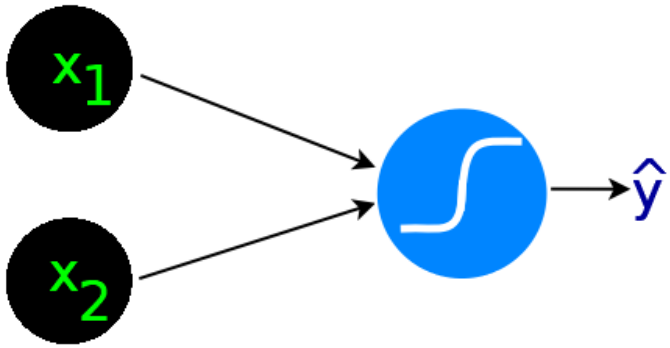
- Total error (or loss):

$$E = \sum_{i=1}^{4} e^i = \sum_{i=1}^{4} \text{Err}(y^i, \hat{y}^i)$$

MLP - Data Perspective: A Simple Example



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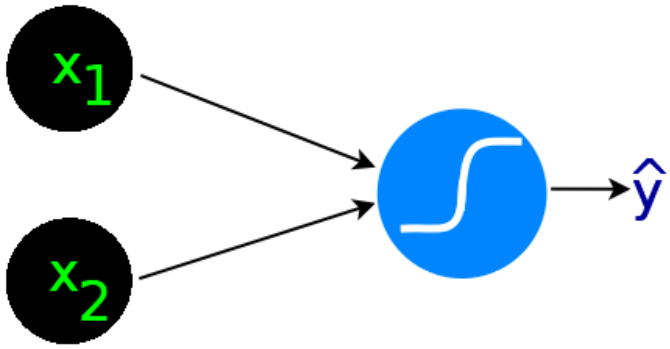
- Total error (or loss):

$$E = \sum_{i=1}^{4} \left(y^i - \frac{1}{1 + \exp\left(-\left[w_{11}^1 x_1^i + w_{12}^1 x_2^i\right]\right)} \right)^2$$

MLP - Data Perspective: A Simple Example



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- Aim: To minimize the total error (or loss), which is

$$\min_{w_{11}^1, w_{12}^1} E = \sum_{i=1}^{4} \left(y^i - \frac{1}{1 + \exp\left(-\left[w_{11}^1 x_1^i + w_{12}^1 x_2^i\right]\right)} \right)^2$$

MLP - Data Perspective: A Simple Example

Visualizing the loss surface:

x_1	x_2	y	$\hat{y} = \sigma(w_{11}^1 x_1 + w_{12}^1 x_2)$
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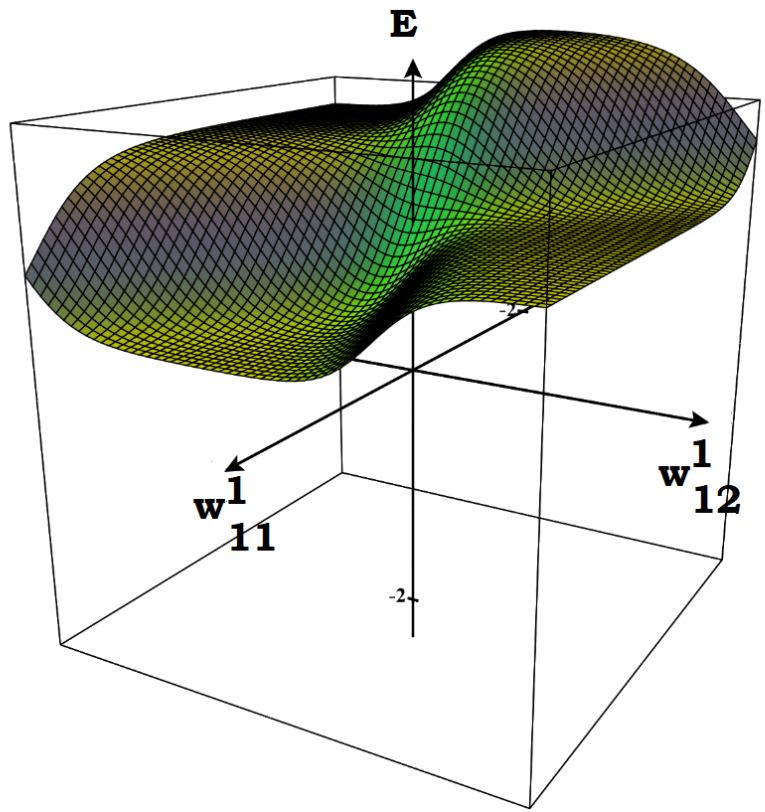


$$E = \sum_{i=1}^{4} \left(y^i - \frac{1}{1 + \exp\left(-\left[w_{11}^1 x_1^i + w_{12}^1 x_2^i\right]\right)} \right)^2$$

Optimization Concepts

General Optimization Problem

$$\min_{x \in \mathcal{C}} f(x)$$

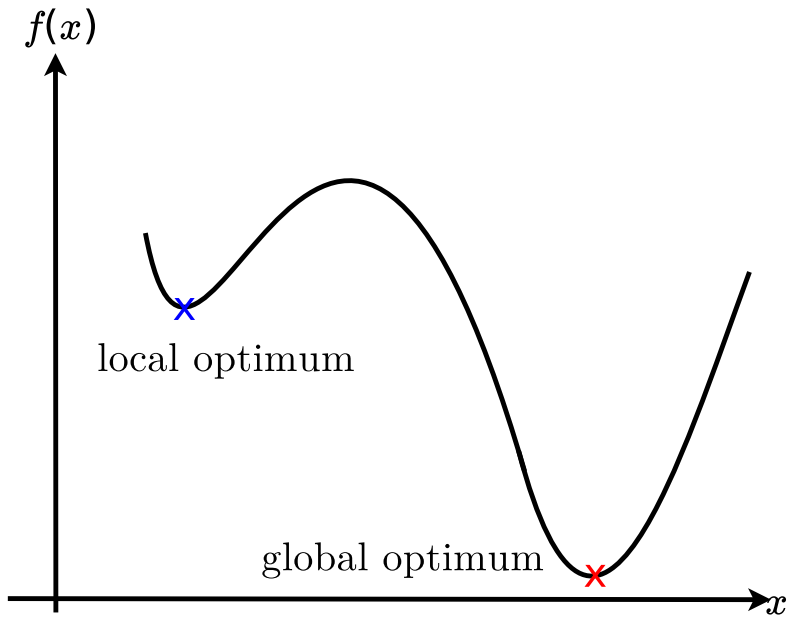
General Optimization Problem

$$\min_{x \in \mathcal{C}} f(x)$$

- f is called objective function and $\mathcal{C}$ is called feasible set.
- Let $f^* = \min_{x \in \mathcal{C}} f(x)$ denote the **optimal objective function value**.
- **Optimal Solution Set** $S^* = \{x \in \mathcal{C} : f(x) = f^*\}$.
- Let us denote by x^* an optimal solution in S^*.

General Optimization Problem

$$\min_{x \in \mathcal{C}} f(x) \qquad \text{(OP)}$$



General Optimization Problem

$$\min_{x \in \mathcal{C}} f(x) \qquad \text{(OP)}$$

Local Optimal Solution

A solution z to (OP) is called local optimal solution if $f(z) \leq f(\hat{z})$, $\forall \hat{z} \in \mathcal{N}(z, \epsilon)$ for some $\epsilon > 0$.

Note: $\mathcal{N}(z, \epsilon)$ denotes suitable $\epsilon-$neighborhood of z.

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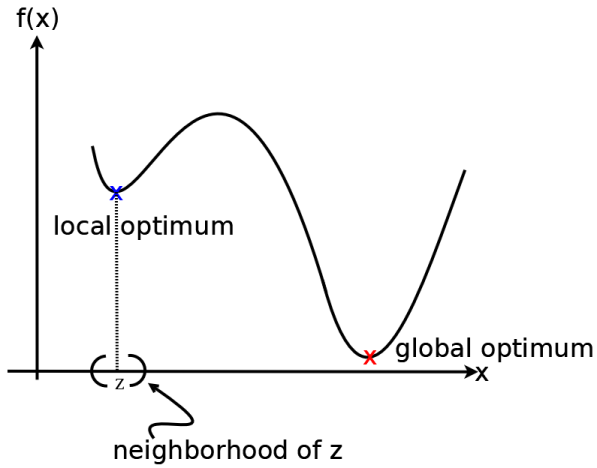

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$\epsilon-$ Neighborhood of $z \in \mathcal{C}$

$$\mathcal{N}(z, \epsilon) = \{u \in \mathcal{C} : \text{dist}(z, u) \leq \epsilon\}.$$

General Optimization Problem

$$\min_{x \in \mathcal{C}} f(x) \qquad \text{(OP)}$$

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Global Optimal Solution

A solution z to (OP) is called global optimal solution if $f(z) \leq f(\hat{z})$, $\forall \hat{z} \in \mathcal{C}$.

General Optimization Problem

$$\min_{x \in \mathcal{C}} f(x)$$

- **General Assumption:** $\mathcal{C} \subseteq \mathbb{R}^d$.

High Dimensional Representation - Notations

- Gradient of a function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ at a point x

$$\nabla f(x) = \begin{pmatrix} \frac{\partial f(x)}{\partial x_1} \\ \\ \frac{\partial f(x)}{\partial x_2} \\ \vdots \\ \vdots \\ \vdots \\ \frac{\partial f(x)}{\partial x_d} \end{pmatrix}$$

General Optimization Problem

$$\min_{x \in \mathcal{C}} f(x)$$

- $\mathcal{C} \subseteq \mathbb{R}^d$.
- $f : \mathcal{C} \longrightarrow \mathbb{R}$.

Directional derivative

Let $f : \mathcal{C} \longrightarrow \mathbb{R}$ be a function defined over $\mathcal{C} \subseteq \mathbb{R}^d$. Let $x \in int(\mathcal{C})$. Let $\mathbf{0} \neq d \in \mathbb{R}^d$. If the limit

$$\lim_{\alpha \downarrow 0} \frac{f(x + \alpha d) - f(x)}{\alpha}$$

exists, then it is called the directional derivative of f at x along the direction d, and is denoted by $f'(x; d)$.

Directional derivative

Interior of a set $\mathcal{C}$

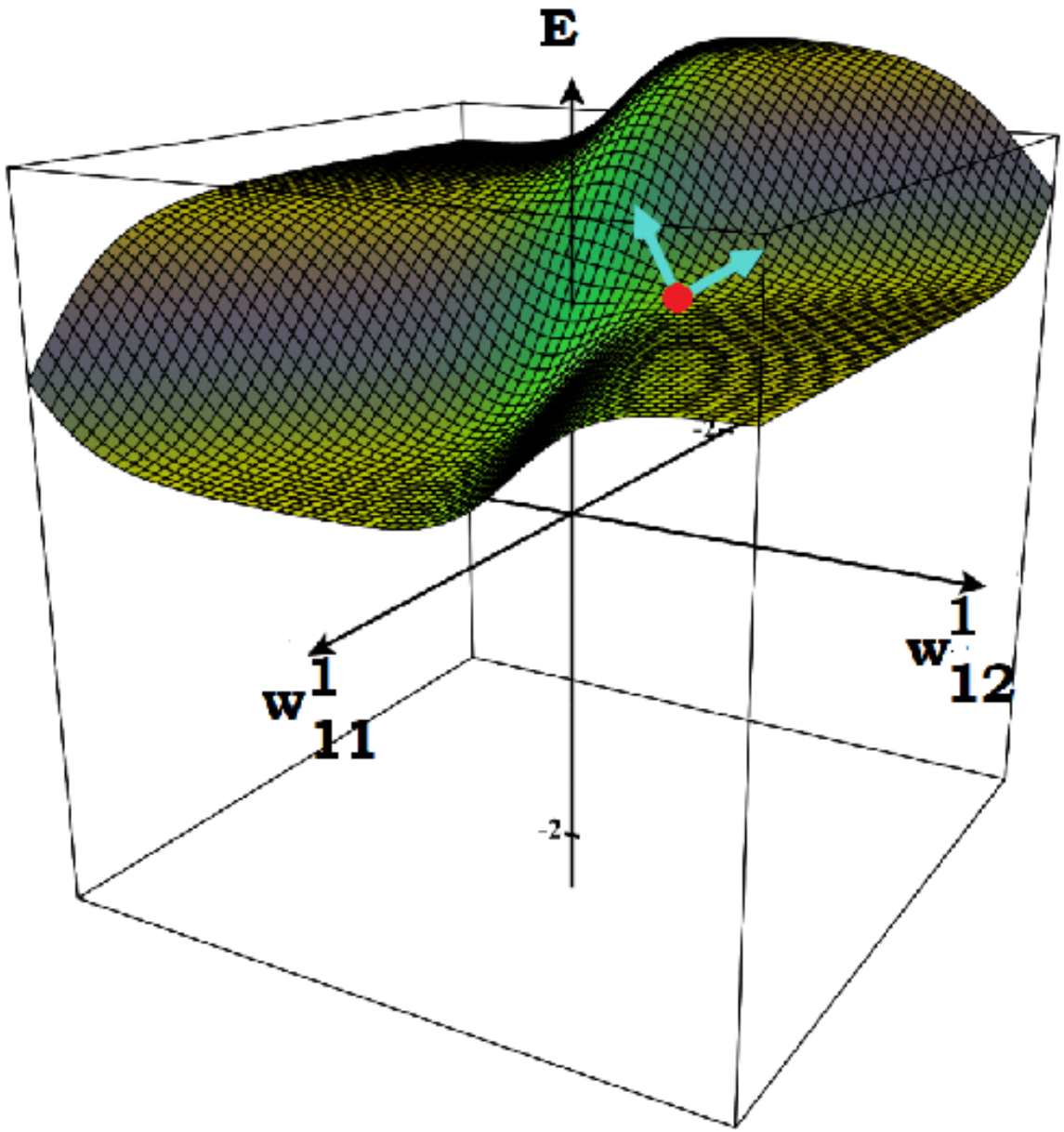
Let $\mathcal{C} \subseteq \mathbb{R}^d$. Then $int(\mathcal{C})$ is defined by:

$$int(\mathcal{C}) = \{x \in \mathcal{C} : B(x, \epsilon) \subseteq \mathcal{C}, \text{ for some } \epsilon > 0\},$$

where $B(x, \epsilon)$ is the open ball centered at x with radius ϵ given by

$$B(x, \epsilon) = \{y \in \mathcal{C} : \|x - y\| < \epsilon\}.$$

Directional derivative



Directional derivative

Let $f : \mathcal{C} \longrightarrow \mathbb{R}$ be a function defined over $\mathcal{C} \subseteq \mathbb{R}^d$. Let $x \in int(\mathcal{C})$. Let $d \neq \mathbf{0} \in \mathbb{R}^d$. If the limit

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exists, then it is called the directional derivative of f at x along the direction d, and is denoted by $f'(x; d)$.

Note: If all partial derivatives of f exist at x, then $f'(x; d) = \langle \nabla f(x), d \rangle$, where $\nabla f(x) = \left[\frac{\partial f(x)}{\partial x_1} \quad \ldots \frac{\partial f(x)}{\partial x_d} \right]^\top$.

Descent Direction

Let $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a continuously differentiable function over $\mathbb{R}^d$. Then a vector $\mathbf{0} \neq d \in \mathbb{R}^d$ is called a descent direction of f at x if the directional derivative of f at x is negative; that is,

$$f'(x; d) = \langle \nabla f(x), d \rangle < 0.$$

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Note: A natural candidate for a descent direction is $d = -\nabla f(x)$.

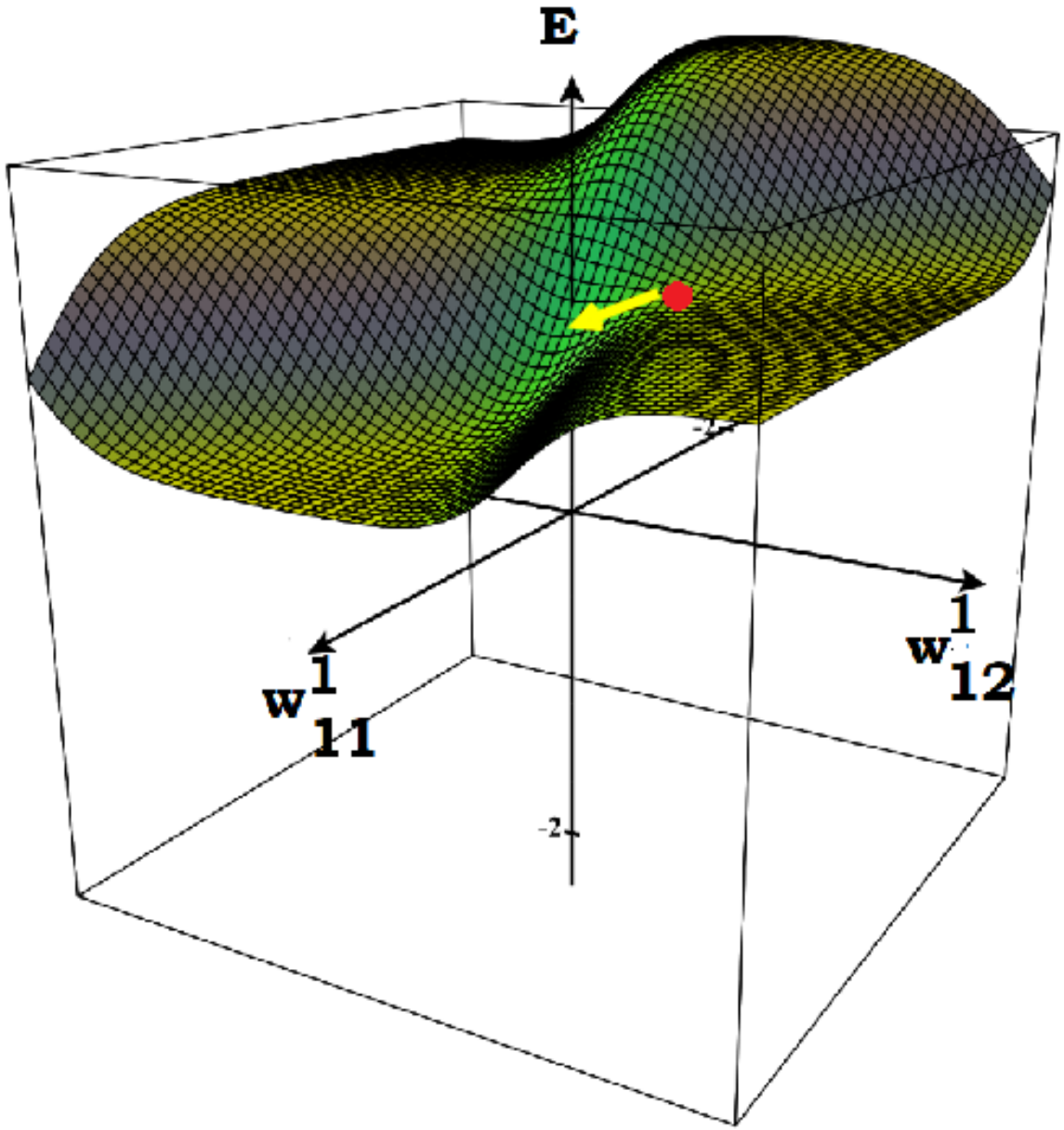
Descent Direction

Proposition

Let $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a continuously differentiable function over $\mathbb{R}^d$. Let $\mathbf{0} \neq d \in \mathbb{R}^d$ be a descent direction of f at x. Then there exists $\epsilon > 0$ such that $\forall \alpha \in (0, \epsilon]$ we have

$$f(x + \alpha d) < f(x).$$

Descent Direction



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Proof idea:

Descent Direction

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Proof idea: Since $\mathbf{0} \neq d \in \mathbb{R}^d$ is a descent direction, by definition of the directional derivative we have

$$f'(x; d) = \lim_{\alpha \downarrow 0} \frac{f(x + \alpha d) - f(x)}{\alpha} < 0$$

$\implies \exists \epsilon > 0$ such that $\forall \alpha \in (0, \epsilon]$, $f(x + \alpha d) < f(x)$.

Descent Direction

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$\implies \exists \epsilon > 0$ such that $\forall \alpha \in (0, \epsilon]$, $f(x + \alpha d) < f(x)$.

Note: If we cannot find such ϵ, d is no longer a descent direction. **Why?**

Algorithm Development using Descent Direction

Consider the general optimization problem:

$$\min_{x \in \mathbb{R}^d} f(x) \qquad \text{(GEN-OPT)}$$

where $f : \mathbb{R}^d \longrightarrow \mathbb{R}$

Algorithm to solve (GEN-OPT)

- Start with $x^0 \in \mathbb{R}^d$.
- For $k = 0, 1, 2, \ldots$
  - Find a descent direction d^k of f at x^k and $\alpha^k > 0$ such that $f(x^k + \alpha^k d^k) < f(x^k)$.
  - $x^{k+1} = x^k + \alpha^k d^k$.
  - Check for some stopping criterion and break from loop.

Characterization Of Local Optimum

Proposition

Let $f : \mathcal{C} \longrightarrow \mathbb{R}$ be a function over the set $\mathcal{C} \subseteq \mathbb{R}^d$. Let $x^\star \in int(\mathcal{C})$ be a local optimum point of f. Let all partial derivatives of f exist at $x^\star$. Then $\nabla f(x^\star) = \mathbf{0}$.

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Characterization Of Local Optimum

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Let $f : \mathcal{C} \longrightarrow \mathbb{R}$ be a function over the set $\mathcal{C} \subseteq \mathbb{R}^d$. Let $x^\star \in int(\mathcal{C})$ be a local optimum point of f. Let all partial derivatives of f exist at $x^\star$. Then $\nabla f(x^\star) = \mathbf{0}$.

Proof idea:

- Consider $e_i = [0 \ldots 0 \underbrace{1}_{i\text{-th coordinate}} 0 \ldots 0]^\top$ containing 1 at the i-th coordinate.

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- However $g'(0) = \langle \nabla f(x^\star), e_i \rangle = \frac{\partial f(x^\star)}{\partial x_i} = 0$. (How?)

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Algorithm Development using Descent Direction

Consider the general optimization problem:

$$\min_{x \in \mathbb{R}^d} f(x) \qquad \text{(GEN-OPT)}$$

where $f : \mathbb{R}^d \longrightarrow \mathbb{R}$.

Algorithm to solve (GEN-OPT)

- Start with $x^0 \in \mathbb{R}^d$.
- For $k = 0, 1, 2, \ldots$
  - Find a descent direction d^k of f at x^k and $\alpha^k > 0$ such that $f(x^k + \alpha^k d^k) < f(x^k)$.
  - $x^{k+1} = x^k + \alpha^k d^k$.
  - If $\|\nabla f(x^{k+1})\|_2 = 0$, set $x^* = x^{k+1}$, break from loop.
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Homework: Compare the structure of this algorithm with the Perceptron training algorithm and try to understand the perceptron update rule from an optimization perspective.

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Gradient Descent Algorithm to solve (GEN-OPT)

- Start with $x^0 \in \mathbb{R}^d$.
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Gradient Descent for our MLP Problem

Recall: For MLP, the loss minimization problem is:

$$\min_{w=(w_{11}^1, w_{12}^1)} E(w) = \sum_{i=1}^{4} e^i(w) = \sum_{i=1}^{4} \left(y^i - \frac{1}{1 + \exp\left(-\left[w_{11}^1 x_1^i + w_{12}^1 x_2^i\right]\right)} \right)^2$$

where $E : \mathbb{R}^2 \longrightarrow \mathbb{R}$.

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Gradient Descent:

- Function values $E(w^t)$ exhibit $O(1/\sqrt{k})$ convergence under minor assumptions and the assumption of existence of a local optimum.
- $O(1/k^2)$ convergence possible.
- Linear convergence also possible for strongly convex and smooth function $E(w)$.
- Arbitrary accuracy possible $|W(w^{gd}) - E(w^*)| \approx O(10^{-15})$.

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Gradient Descent:

- Blind to structure of $E(w)$.
- Finding proper α^k at each k is computationally intensive - takes at least $O(Sd)$ time.
- Storage complexity: $O(d)$

Stochastic Gradient Descent for our MLP Problem

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Stochastic Gradient Descent Algorithm to solve MLP Loss Minimization Problem

- Start with $w^0 \in \mathbb{R}^d$.
- For $k = 0, 1, 2, \ldots$
  - Choose a sample $j_k \in \{1, \ldots, 4\}$.
  - $w^{k+1} \leftarrow w^k - \gamma_k \nabla_w e^{j_k}(w^k)$.

Regularized Empirical Loss Minimization - Optimization Methods

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$\nabla_w e^{j_k}(w^k)$: Gradient at point w^k, of e^{i_k} with respect to w. Takes only $O(d)$ time.

Under suitable conditions on γ_k ($\sum_k \gamma_k^2 < \infty$, $\sum_k \gamma_k \to \infty$), this procedure converges **asymptotically**.
For smooth functions, $O(1/k)$ convergence possible (in theory!).
Typical choice: $\gamma_k = \frac{1}{k+1}$.

Mini-Batch Stochastic Gradient Descent for our MLP Problem

Mini-batch SGD Algorithm to solve MLP Loss Minimization Problem

- Start with $w^0 \in \mathbb{R}^d$.
- For $k = 0, 1, 2, \ldots$
  - Choose a block of samples $B_k \subseteq \{1, \ldots, 4\}$.
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- Restrictions on γ_k similar to that in SGD.
- **Asymptotic convergence** !

GD/SGD: Crucial Step

Recall: For MLP, the loss minimization problem is:

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Crucial step in Gradient Descent Algorithm

$w^{k+1} = w^k - \alpha^k \sum_{i=1}^{4} \nabla e^i(w^k)$

Crucial step in Stochastic Gradient Descent Algorithm

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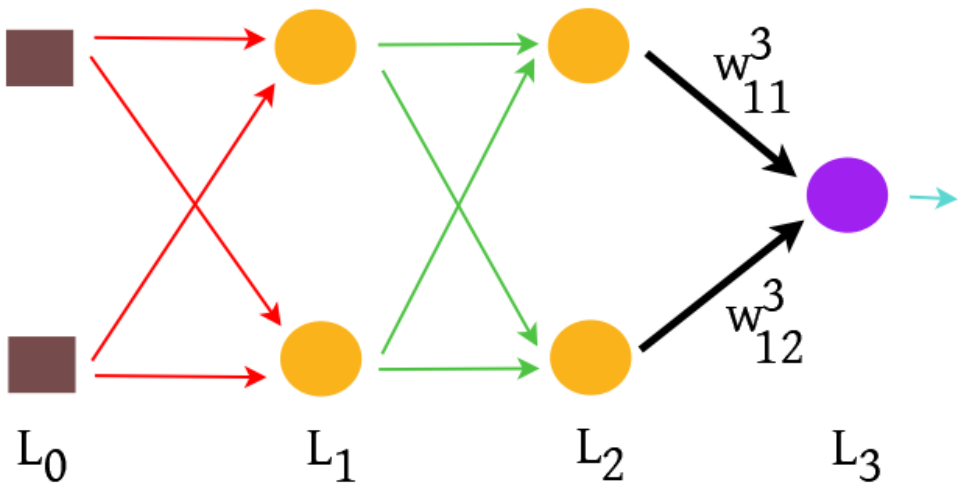
Note: $\nabla e^i(w^k)$, $\nabla_w e^{j_k}(w^k)$, $\nabla e^j(w^k)$ denote sample-wise gradient computation.

GD/SGD for MLP: Sample-wise Gradient Computation



- Consider an arbitrary training sample $(x, y) \in D$.

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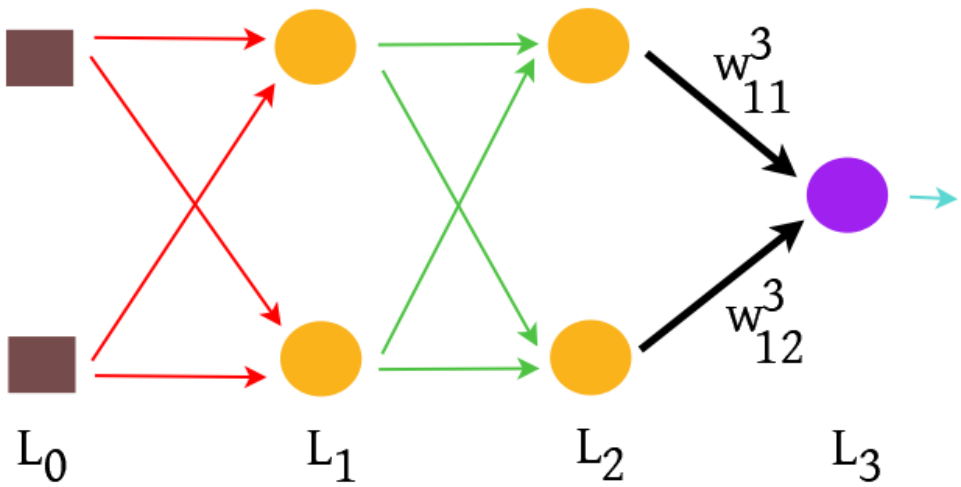

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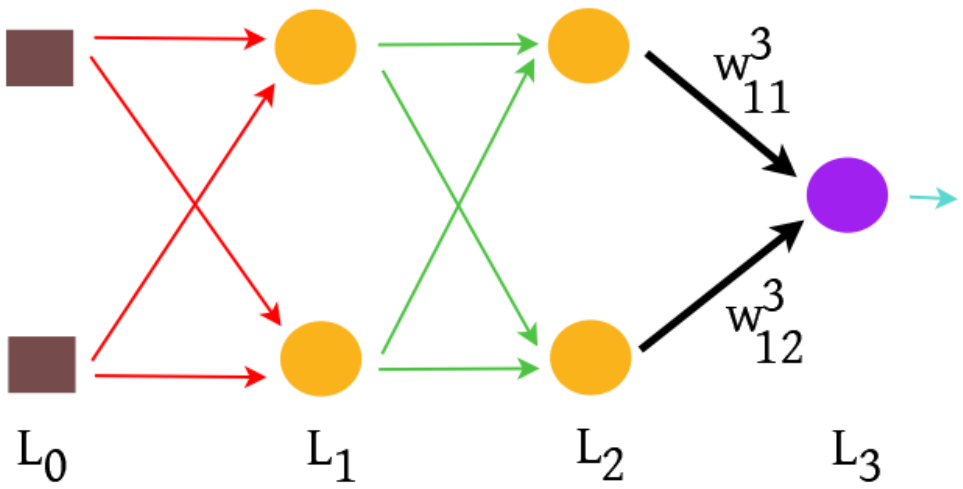

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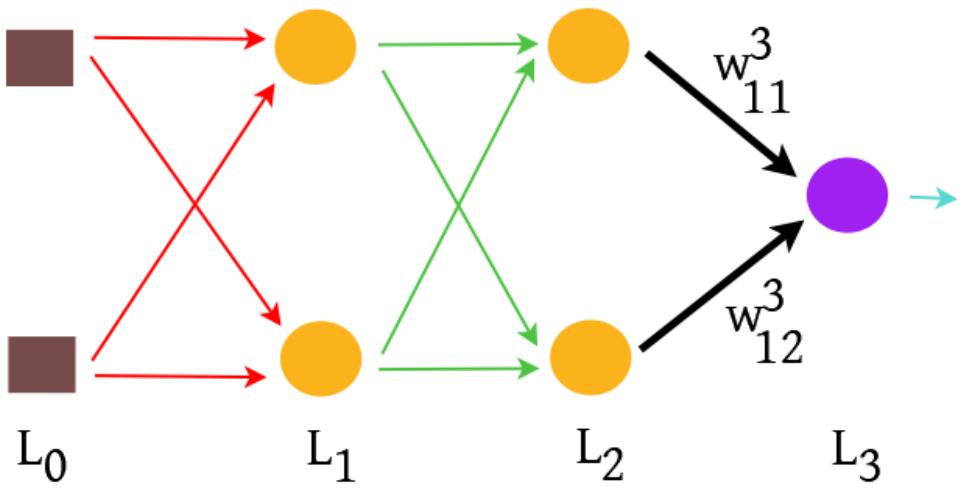

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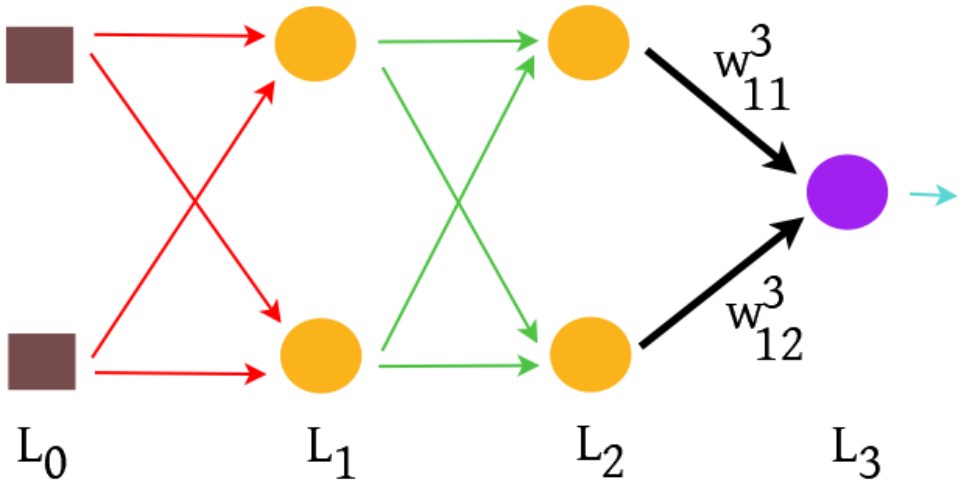
- **Aim:** To find $\nabla_w e = [\nabla_{w_{11}^1} e \ \nabla_{w_{12}^1} e \ \ldots \ \nabla_{w_{12}^3} e]^\top$.

GD/SGD for MLP: Sample-wise Gradient Computation



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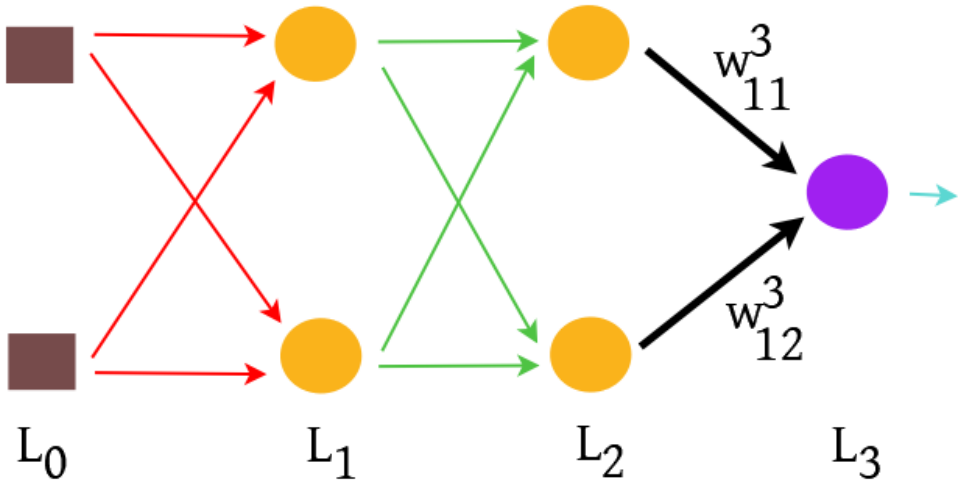

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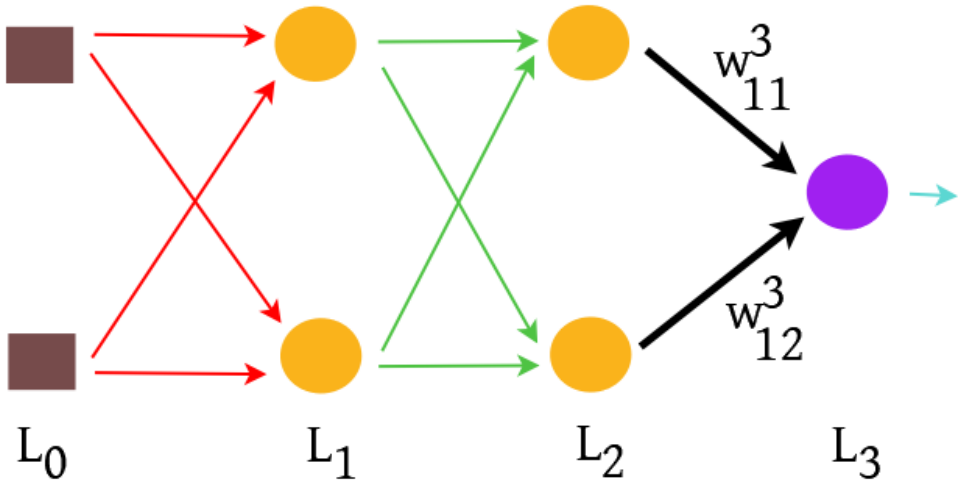

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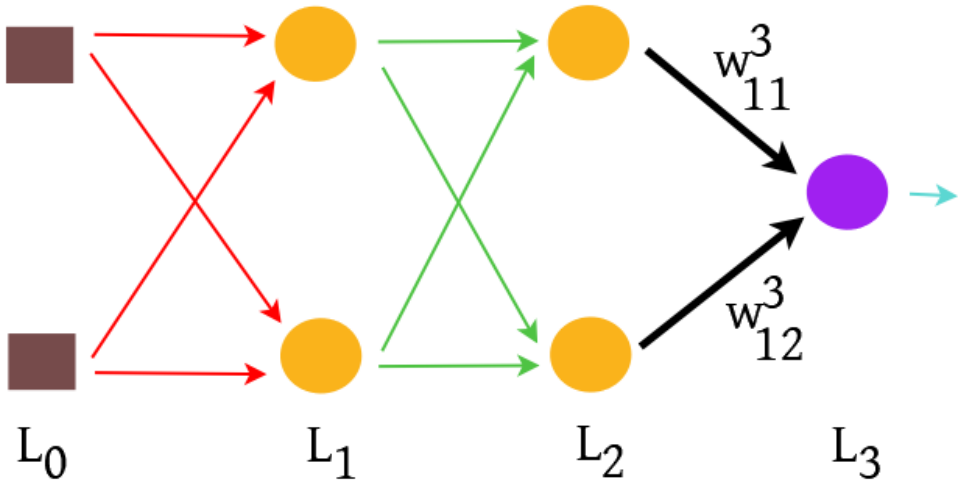

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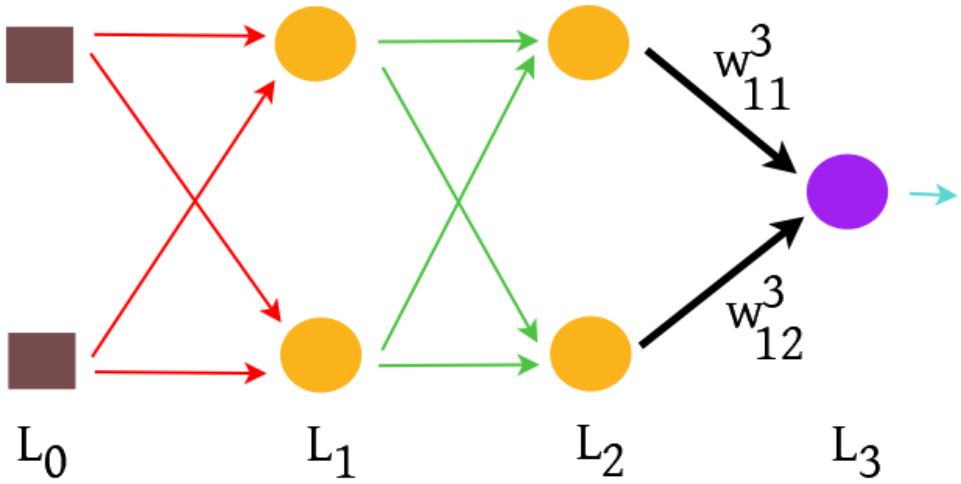

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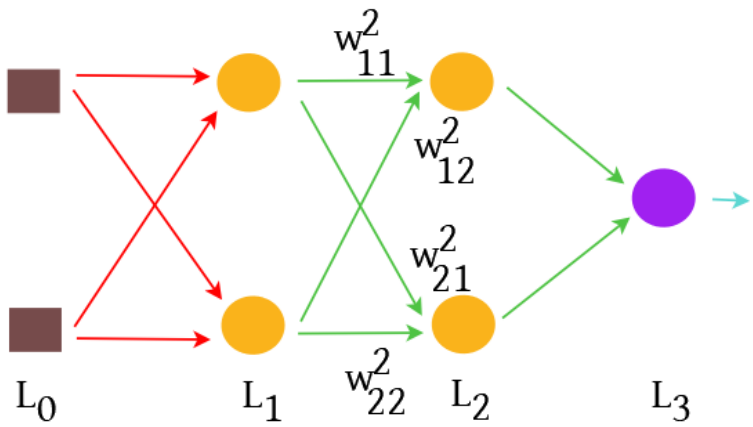
- Similarly, $\nabla_{w_{12}^3} e = \frac{\partial e}{\partial \hat{y}}\phi'(z_1^3)a_2^2$.

GD/SGD for MLP: Sample-wise Gradient Computation



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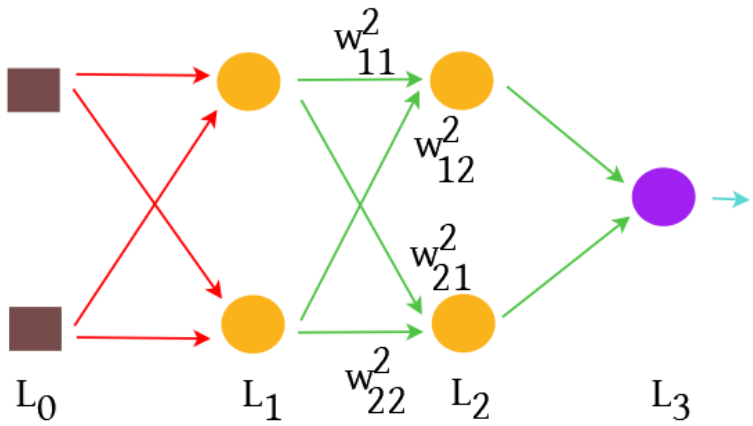

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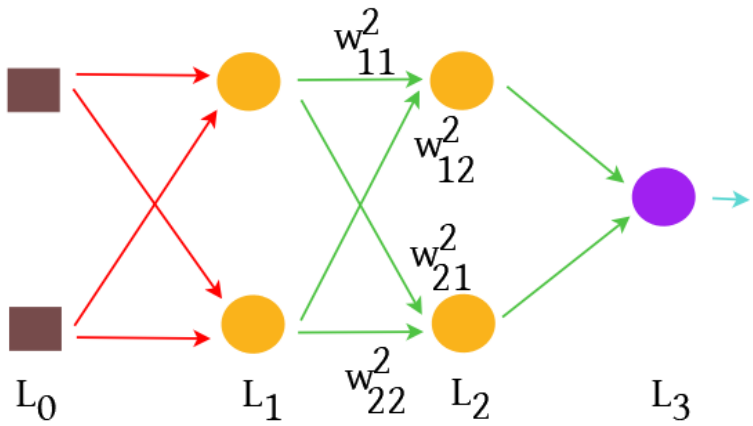
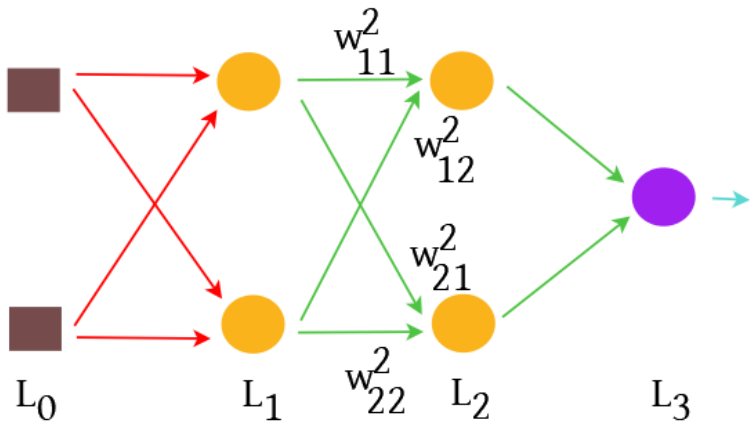
- Hence, $\nabla_{w_{11}^2} e = \frac{\partial e}{\partial z_1^2}\frac{\partial z_1^2}{\partial w_{11}^2} = \frac{\partial e}{\partial z_1^2} a_1^1$.

GD/SGD for MLP: Sample-wise Gradient Computation



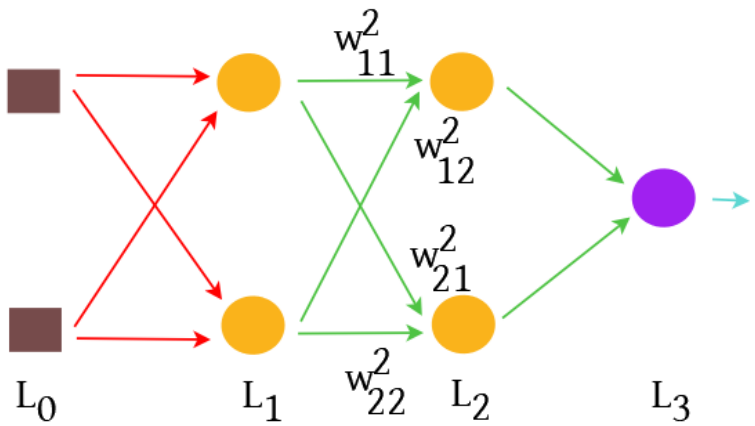
- We have at layer L_2: $a_1^2 = \phi\left(z_1^2\right) = \phi\left(w_{11}^2 a_1^1 + w_{12}^2 a_2^1\right)$.
- Hence, $\nabla_{w_{11}^2} e = \frac{\partial e}{\partial z_1^2}\frac{\partial z_1^2}{\partial w_{11}^2} = \frac{\partial e}{\partial z_1^2} a_1^1$.

GD/SGD for MLP: Sample-wise Gradient Computation



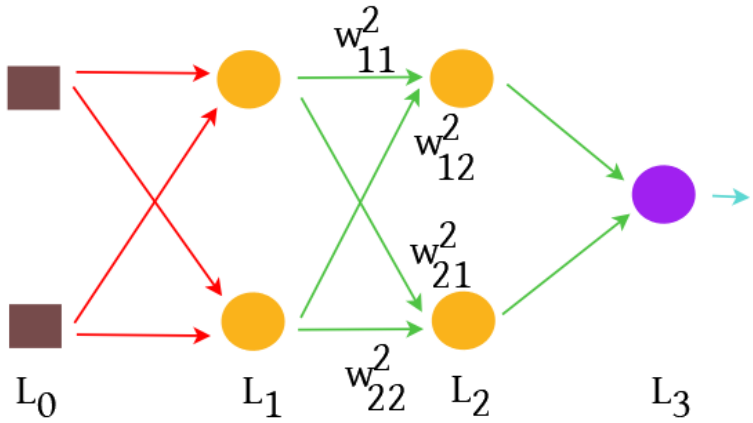
- We have at layer L_2: $a_1^2 = \phi\left(z_1^2\right) = \phi\left(w_{11}^2 a_1^1 + w_{12}^2 a_2^1\right)$.
- Hence, $\nabla_{w_{11}^2} e = \frac{\partial e}{\partial z_1^2}\frac{\partial z_1^2}{\partial w_{11}^2} = \frac{\partial e}{\partial z_1^2} a_1^1 = \frac{\partial e}{\partial a_1^2}\frac{\partial a_1^2}{\partial z_1^2} a_1^1 = \frac{\partial e}{\partial a_1^2}\phi'(z_1^2) a_1^1$.

GD/SGD for MLP: Sample-wise Gradient Computation



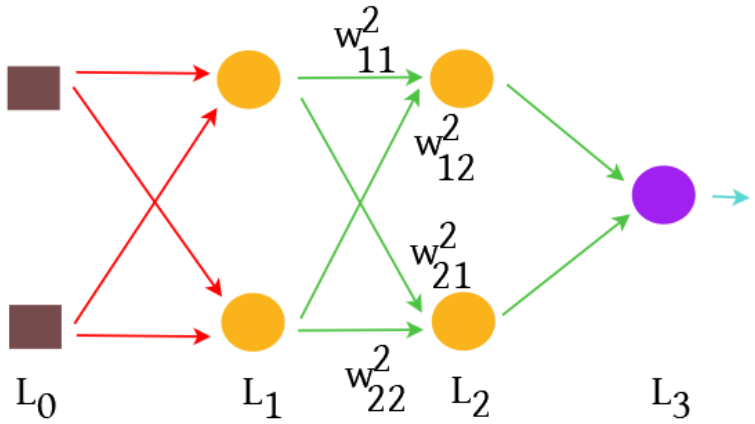
- We have at layer L_2: $a_1^2 = \phi\left(z_1^2\right) = \phi\left(w_{11}^2 a_1^1 + w_{12}^2 a_2^1\right)$.
- Hence, $\nabla_{w_{11}^2} e = \frac{\partial e}{\partial z_1^2}\frac{\partial z_1^2}{\partial w_{11}^2} = \frac{\partial e}{\partial z_1^2} a_1^1 = \frac{\partial e}{\partial a_1^2}\frac{\partial a_1^2}{\partial z_1^2} a_1^1 = \frac{\partial e}{\partial a_1^2}\phi'(z_1^2) a_1^1$.
- Now recall that $z_1^3 = \left(w_{11}^3 a_1^2 + w_{12}^3 a_2^2\right)$.

GD/SGD for MLP: Sample-wise Gradient Computation



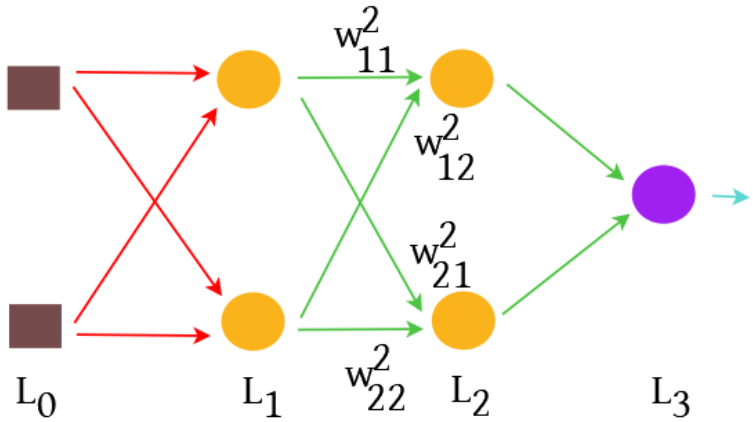
- We have at layer L_2: $a_1^2 = \phi\left(z_1^2\right) = \phi\left(w_{11}^2 a_1^1 + w_{12}^2 a_2^1\right)$.
- Hence, $\nabla_{w_{11}^2} e = \frac{\partial e}{\partial z_1^2}\frac{\partial z_1^2}{\partial w_{11}^2} = \frac{\partial e}{\partial z_1^2} a_1^1 = \frac{\partial e}{\partial a_1^2}\frac{\partial a_1^2}{\partial z_1^2} a_1^1 = \frac{\partial e}{\partial a_1^2}\phi'(z_1^2) a_1^1$.
- Now recall that $z_1^3 = \left(w_{11}^3 a_1^2 + w_{12}^3 a_2^2\right)$.
- Hence $\frac{\partial e}{\partial a_1^2} = \frac{\partial e}{\partial z_1^3}\frac{\partial z_1^3}{\partial a_1^2} = \frac{\partial e}{\partial z_1^3} w_{11}^3$.

GD/SGD for MLP: Sample-wise Gradient Computation



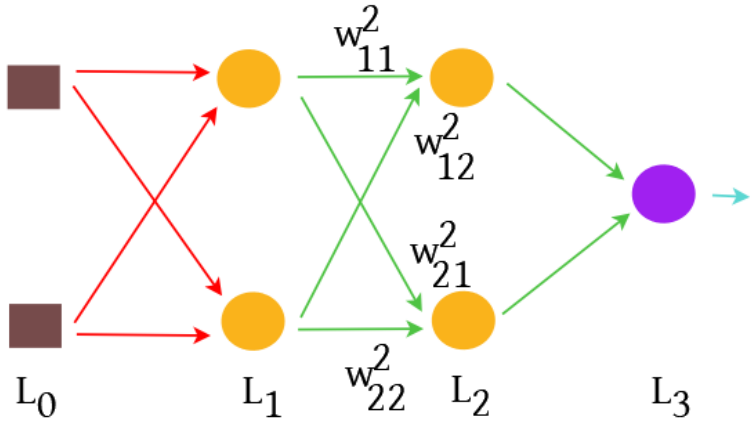
- We have at layer L_2: $a_1^2 = \phi\left(z_1^2\right) = \phi\left(w_{11}^2 a_1^1 + w_{12}^2 a_2^1\right)$.
- Hence, $\nabla_{w_{11}^2} e = \frac{\partial e}{\partial z_1^2}\frac{\partial z_1^2}{\partial w_{11}^2} = \frac{\partial e}{\partial z_1^2} a_1^1 = \frac{\partial e}{\partial a_1^2}\frac{\partial a_1^2}{\partial z_1^2} a_1^1 = \frac{\partial e}{\partial a_1^2}\phi'(z_1^2) a_1^1$.
- Now recall that $z_1^3 = \left(w_{11}^3 a_1^2 + w_{12}^3 a_2^2\right)$.
- Hence $\frac{\partial e}{\partial a_1^2} = \frac{\partial e}{\partial z_1^3}\frac{\partial z_1^3}{\partial a_1^2} = \frac{\partial e}{\partial z_1^3} w_{11}^3$.
- Recall: We have already computed $\frac{\partial e}{\partial z_1^3} = \frac{\partial e}{\partial \hat{y}}\phi'(z_1^3)$.

GD/SGD for MLP: Sample-wise Gradient Computation



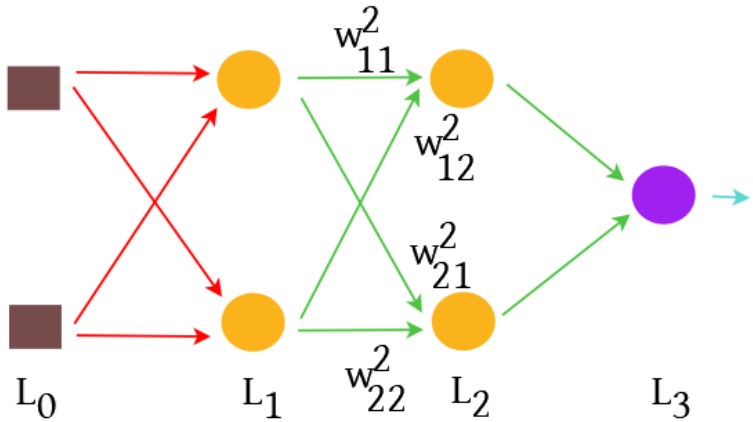
- We have at layer L_2: $a_1^2 = \phi\left(z_1^2\right) = \phi\left(w_{11}^2 a_1^1 + w_{12}^2 a_2^1\right)$.
- Hence, $\nabla_{w_{11}^2} e = \frac{\partial e}{\partial z_1^2}\frac{\partial z_1^2}{\partial w_{11}^2} = \frac{\partial e}{\partial z_1^2} a_1^1 = \frac{\partial e}{\partial a_1^2}\frac{\partial a_1^2}{\partial z_1^2} a_1^1 = \frac{\partial e}{\partial a_1^2}\phi'(z_1^2) a_1^1$.
- Now recall that $z_1^3 = \left(w_{11}^3 a_1^2 + w_{12}^3 a_2^2\right)$.
- Hence $\frac{\partial e}{\partial a_1^2} = \frac{\partial e}{\partial z_1^3}\frac{\partial z_1^3}{\partial a_1^2} = \frac{\partial e}{\partial z_1^3} w_{11}^3 = \frac{\partial e}{\partial \hat{y}}\phi'(z_1^3) w_{11}^3$.

GD/SGD for MLP: Sample-wise Gradient Computation



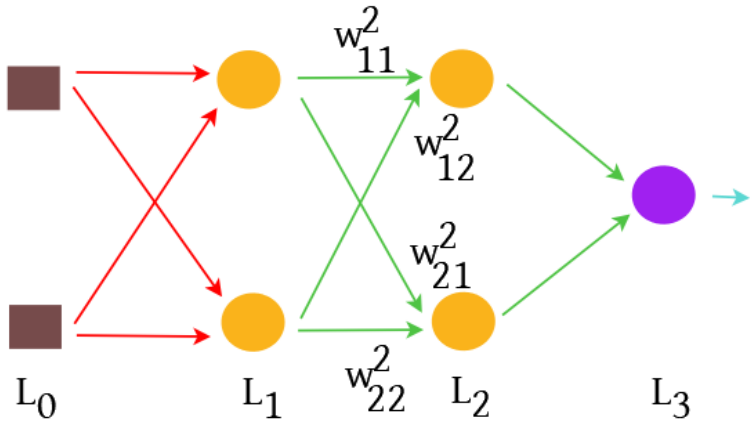
- We have at layer L_2: $a_1^2 = \phi\left(z_1^2\right) = \phi\left(w_{11}^2 a_1^1 + w_{12}^2 a_2^1\right)$.
- Hence, $\nabla_{w_{11}^2} e = \frac{\partial e}{\partial z_1^2}\frac{\partial z_1^2}{\partial w_{11}^2} = \frac{\partial e}{\partial z_1^2} a_1^1 = \frac{\partial e}{\partial a_1^2}\frac{\partial a_1^2}{\partial z_1^2} a_1^1 = \frac{\partial e}{\partial a_1^2}\phi'(z_1^2) a_1^1$.
- Now recall that $z_1^3 = \left(w_{11}^3 a_1^2 + w_{12}^3 a_2^2\right)$.
- Hence $\frac{\partial e}{\partial a_1^2} = \frac{\partial e}{\partial z_1^3}\frac{\partial z_1^3}{\partial a_1^2} = \frac{\partial e}{\partial z_1^3} w_{11}^3 = \frac{\partial e}{\partial \hat{y}}\phi'(z_1^3) w_{11}^3$.
- Combining, we have $\nabla_{w_{11}^2} e = \frac{\partial e}{\partial \hat{y}}\phi'(z_1^3) w_{11}^3 \phi'(z_1^2) a_1^1$.

GD/SGD for MLP: Sample-wise Gradient Computation



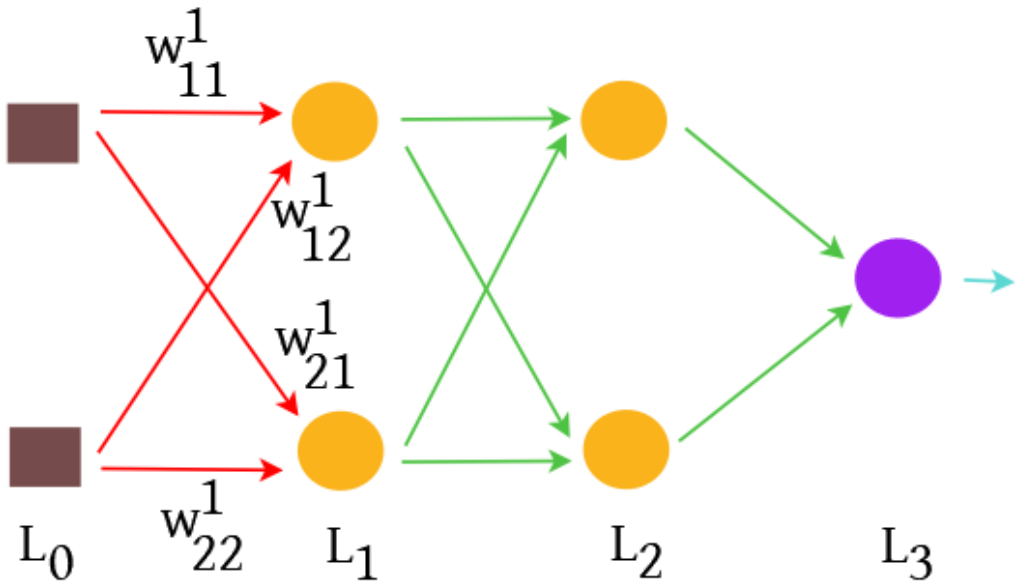
- Thus, $\nabla_{w^2_{11}} e = \frac{\partial e}{\partial \hat{y}} \phi'(z^3_1) w^3_{11} \phi'(z^2_1) a^1_1$.
- Similarly, $\nabla_{w^2_{12}} e = \frac{\partial e}{\partial \hat{y}} \phi'(z^3_1) w^3_{11} \phi'(z^2_1) a^1_2$.

GD/SGD for MLP: Sample-wise Gradient Computation



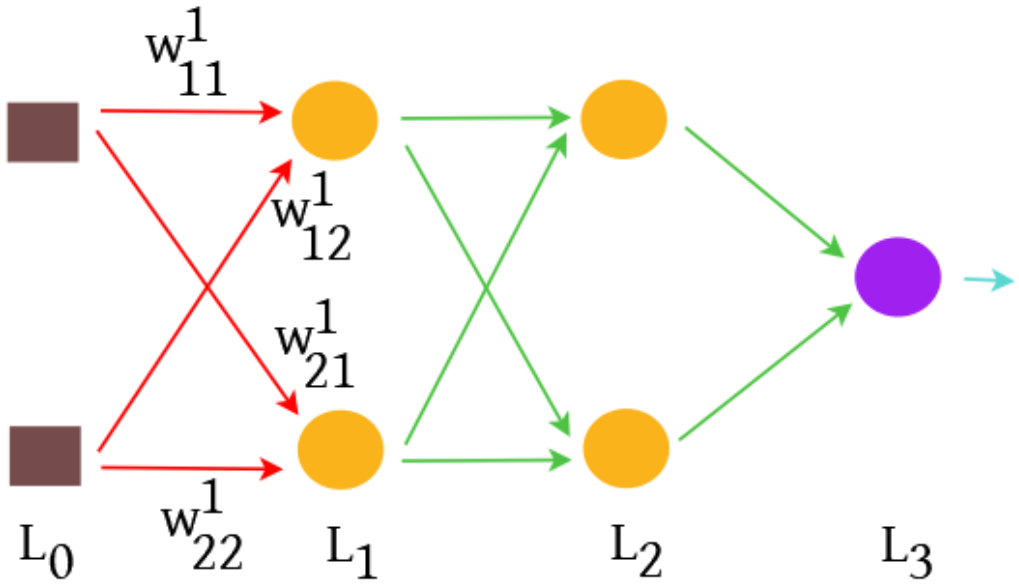
- Also, we have at layer L_2: $a_2^2 = \phi\left(z_2^2\right) = \phi\left(w_{21}^2 a_1^1 + w_{22}^2 a_2^1\right)$.
- Hence, $\nabla_{w_{21}^2} e = ?$, $\nabla_{w_{22}^2} e = ?$

GD/SGD for MLP: Sample-wise Gradient Computation



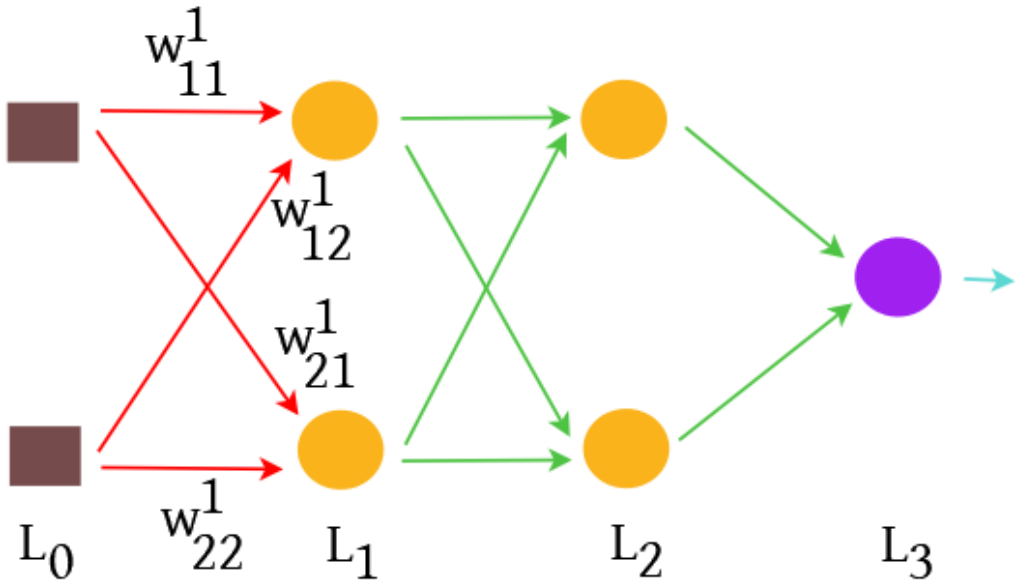
- We have at layer L_1: $a_1^1 = \phi\left(z_1^1\right) = \phi\left(w_{11}^1 x_1 + w_{12}^1 x_2\right)$.

GD/SGD for MLP: Sample-wise Gradient Computation



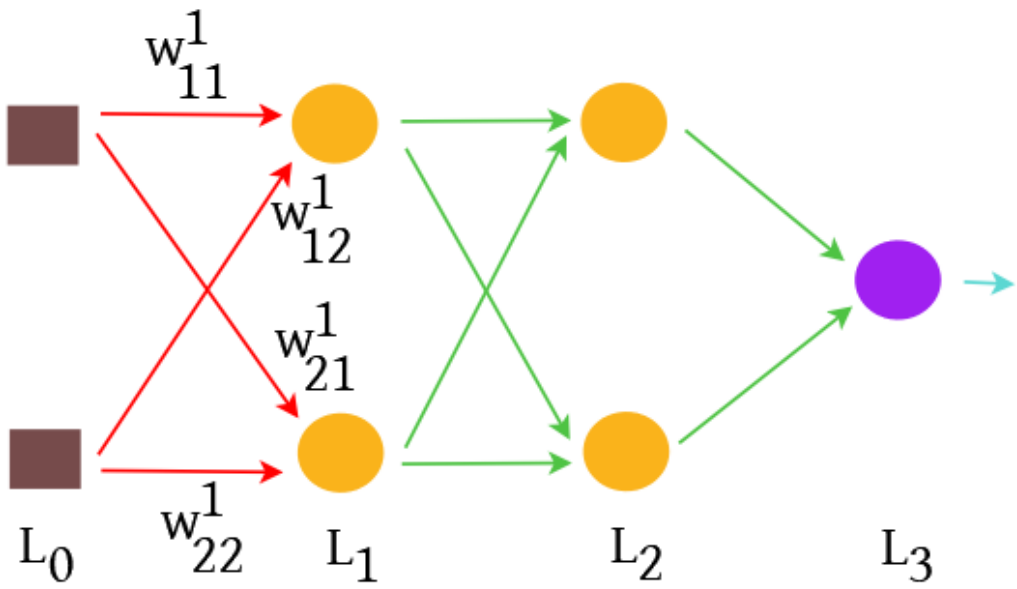
- We have at layer L_1: $a_1^1 = \phi\left(z_1^1\right) = \phi\left(w_{11}^1 x_1 + w_{12}^1 x_2\right)$.
- **Note:** $\nabla_{w_{11}^1} e = \frac{\partial e}{\partial z_1^1}\frac{\partial z_1^1}{\partial w_{11}^1} = \frac{\partial e}{\partial z_1^1} x_1$.

GD/SGD for MLP: Sample-wise Gradient Computation



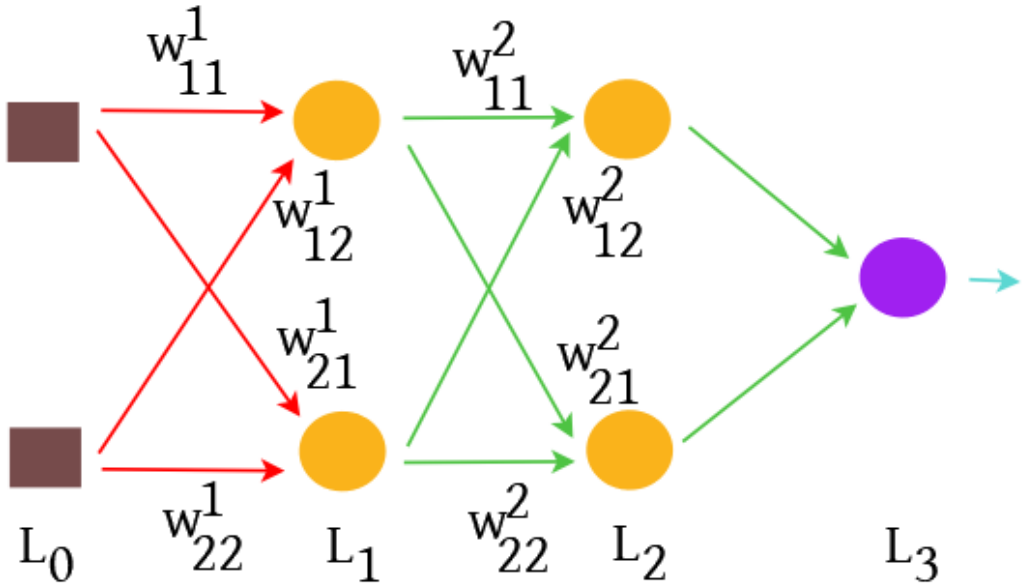
- We have at layer L_1: $a_1^1 = \phi\left(z_1^1\right) = \phi\left(w_{11}^1 x_1 + w_{12}^1 x_2\right)$.
- **Note:** $\nabla_{w_{11}^1} e = \frac{\partial e}{\partial z_1^1} x_1 = \frac{\partial e}{\partial a_1^1} \phi'(z_1^1) x_1$.

GD/SGD for MLP: Sample-wise Gradient Computation



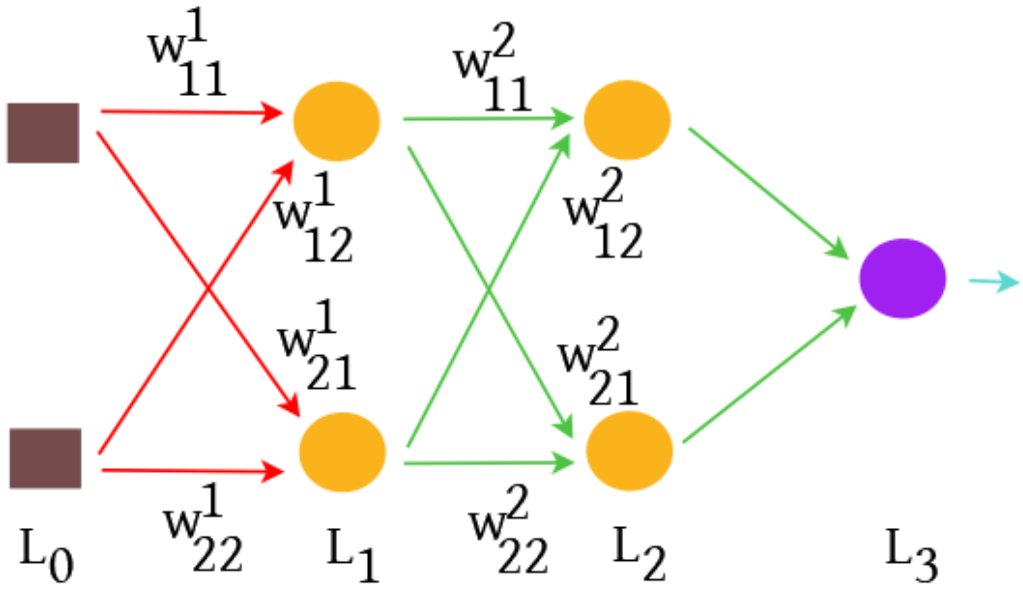
- We have at layer L_1: $a_1^1 = \phi\left(z_1^1\right) = \phi\left(w_{11}^1 x_1 + w_{12}^1 x_2\right)$.
- **Note:** $\nabla_{w_{11}^1} e = \frac{\partial e}{\partial z_1^1} x_1 = \frac{\partial e}{\partial a_1^1} \phi'(z_1^1) x_1$.
- Now we see that a_1^1 contributes to both z_1^2 and z_2^2.

GD/SGD for MLP: Sample-wise Gradient Computation



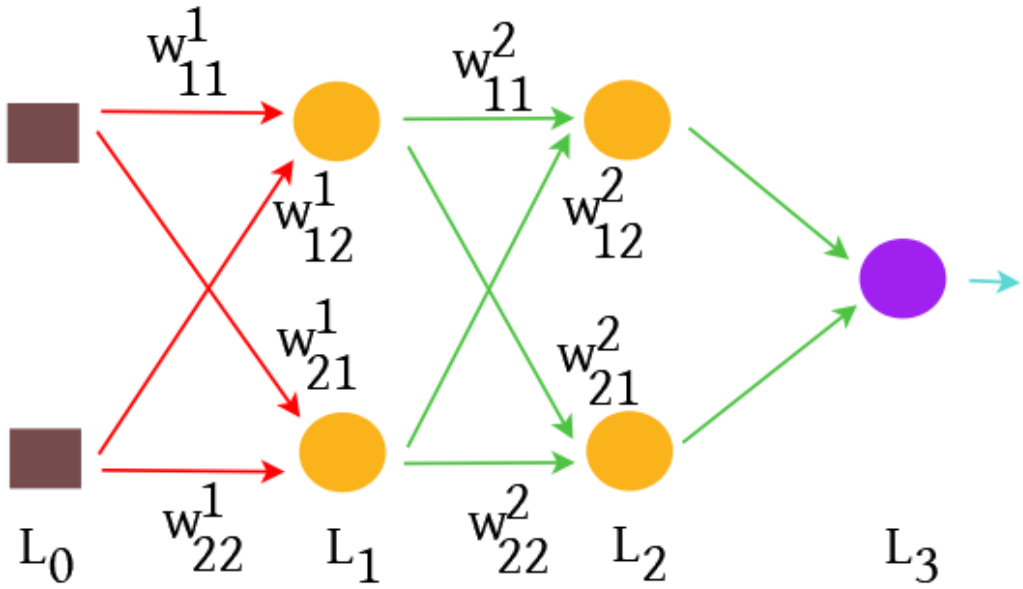
- We have at layer L_1: $a_1^1 = \phi\left(z_1^1\right) = \phi\left(w_{11}^1 x_1 + w_{12}^1 x_2\right)$.
- **Note:** $\nabla_{w_{11}^1} e = \frac{\partial e}{\partial z_1^1} x_1 = \frac{\partial e}{\partial a_1^1} \phi'(z_1^1) x_1$.
- Now we see that a_1^1 contributes to both z_1^2 and z_2^2.
- **Recall:** $z_1^2 = w_{11}^2 a_1^1 + w_{12}^2 a_2^1$ and $z_2^2 = w_{21}^2 a_1^1 + w_{22}^2 a_2^1$.

GD/SGD for MLP: Sample-wise Gradient Computation



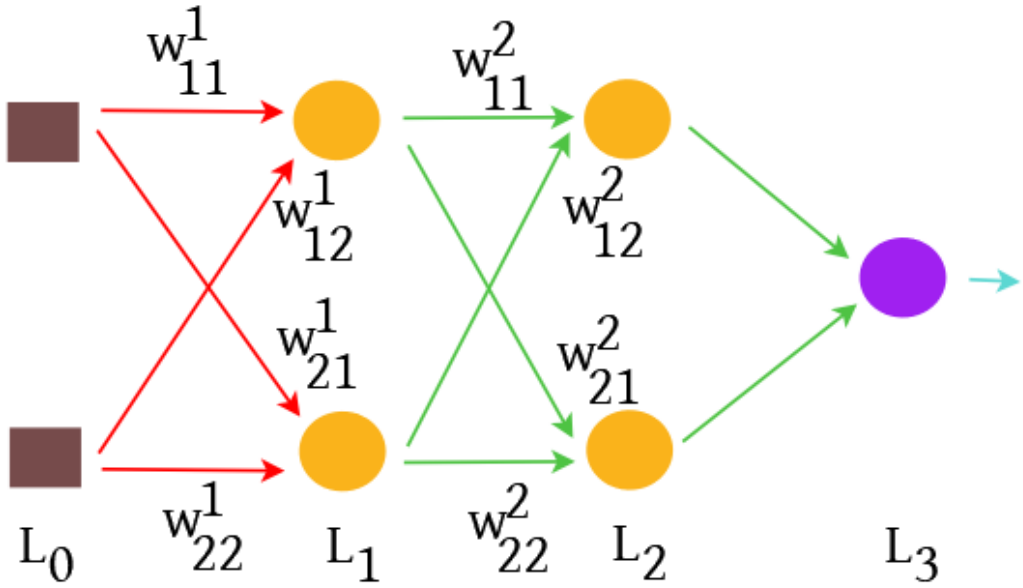
- We have at layer L_1: $a^1_1 = \phi\left(z^1_1\right) = \phi\left(w^1_{11}x_1 + w^1_{12}x_2\right)$.
- **Note:** $\nabla_{w^1_{11}} e = \frac{\partial e}{\partial z^1_1} x_1 = \frac{\partial e}{\partial a^1_1}\phi'(z^1_1)x_1$.
- Now we see that a^1_1 contributes to both z^2_1 and z^2_2.
- **Recall:** $z^2_1 = w^2_{11}a^1_1 + w^2_{12}a^1_2$ and $z^2_2 = w^2_{21}a^1_1 + w^2_{22}a^1_2$.
- Hence $\frac{\partial e}{\partial a^1_1} = \sum_{i=1}^{2} \frac{\partial e}{\partial z^2_i}\frac{\partial z^2_i}{\partial a^1_1}$.

GD/SGD for MLP: Sample-wise Gradient Computation



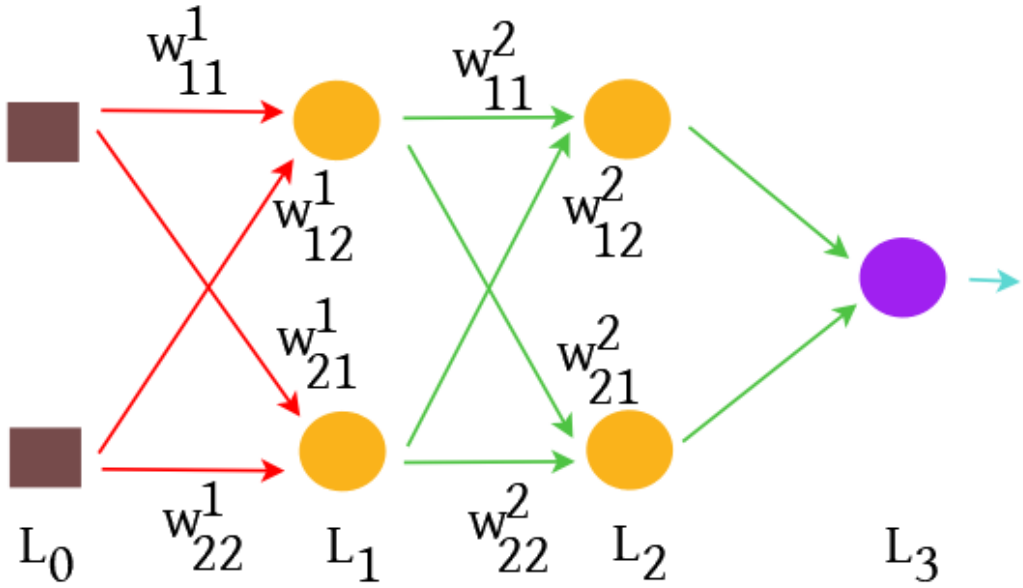
- We have at layer L_1: $a_1^1 = \phi\left(z_1^1\right) = \phi\left(w_{11}^1 x_1 + w_{12}^1 x_2\right)$.
- **Note:** $\nabla_{w_{11}^1} e = \frac{\partial e}{\partial z_1^1} x_1 = \frac{\partial e}{\partial a_1^1} \phi'(z_1^1) x_1$.
- Now we see that a_1^1 contributes to both z_1^2 and z_2^2.
- **Recall:** $z_1^2 = w_{11}^2 a_1^1 + w_{12}^2 a_2^1$ and $z_2^2 = w_{21}^2 a_1^1 + w_{22}^2 a_2^1$.
- Hence $\frac{\partial e}{\partial a_1^1} = \sum_{i=1}^2 \frac{\partial e}{\partial z_i^2} \frac{\partial z_i^2}{\partial a_1^1} = \sum_{i=1}^2 \frac{\partial e}{\partial z_i^2} w_{i1}^2$.

GD/SGD for MLP: Sample-wise Gradient Computation



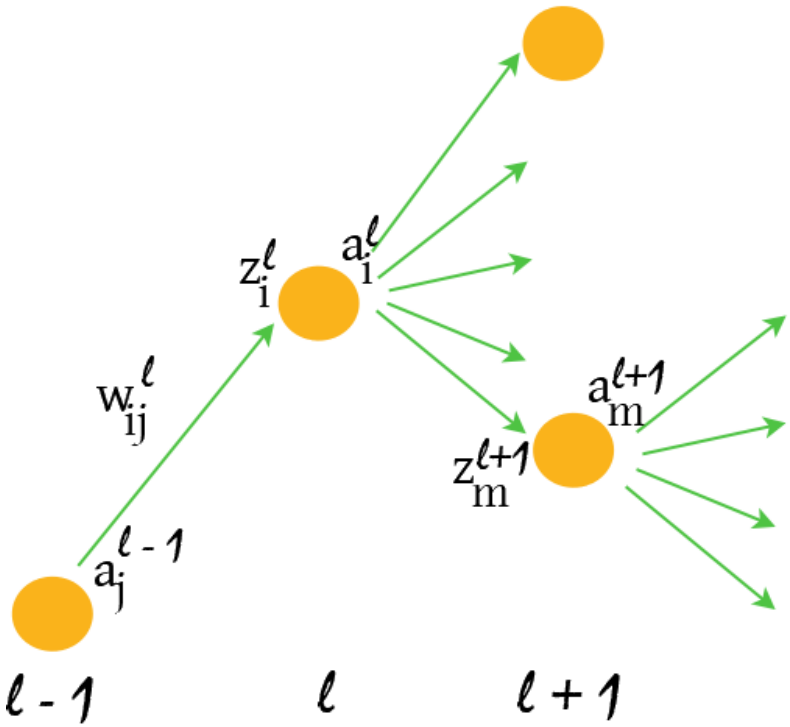
- We have at layer L_1: $a^1_1 = \phi\left(z^1_1\right) = \phi\left(w^1_{11}x_1 + w^1_{12}x_2\right)$.
- **Note:** $\nabla_{w^1_{11}} e = \frac{\partial e}{\partial z^1_1}x_1 = \frac{\partial e}{\partial a^1_1}\phi'(z^1_1)x_1$.
- Now we see that a^1_1 contributes to both z^2_1 and z^2_2.
- **Recall:** $z^2_1 = w^2_{11}a^1_1 + w^2_{12}a^1_2$ and $z^2_2 = w^2_{21}a^1_1 + w^2_{22}a^1_2$.
- Hence $\frac{\partial e}{\partial a^1_1} = \sum_{i=1}^{2} \frac{\partial e}{\partial z^2_i}\frac{\partial z^2_i}{\partial a^1_1} = \sum_{i=1}^{2} \frac{\partial e}{\partial z^2_i}w^2_{i1}$.
- Recall: We have already computed $\frac{\partial e}{\partial z^2_i}, i = 1, 2$.

GD/SGD for MLP: Sample-wise Gradient Computation



- We have at layer L_1: $a^1_1 = \phi\left(z^1_1\right) = \phi\left(w^1_{11}x_1 + w^1_{12}x_2\right)$.
- **Note:** $\nabla_{w^1_{11}} e = \frac{\partial e}{\partial z^1_1} x_1 = \frac{\partial e}{\partial a^1_1}\phi'(z^1_1)x_1$.
- Now we see that a^1_1 contributes to both z^2_1 and z^2_2.
- **Recall:** $z^2_1 = w^2_{11}a^1_1 + w^2_{12}a^1_2$ and $z^2_2 = w^2_{21}a^1_1 + w^2_{22}a^1_2$.
- Hence $\frac{\partial e}{\partial a^1_1} = \sum_{i=1}^{2} \frac{\partial e}{\partial z^2_i}\frac{\partial z^2_i}{\partial a^1_1} = \sum_{i=1}^{2} \frac{\partial e}{\partial z^2_i} w^2_{i1}$.
- Recall: We have already computed $\frac{\partial e}{\partial z^2_i} = \frac{\partial e}{\partial a^2_i}\phi'(z^2_i), i = 1, 2$.

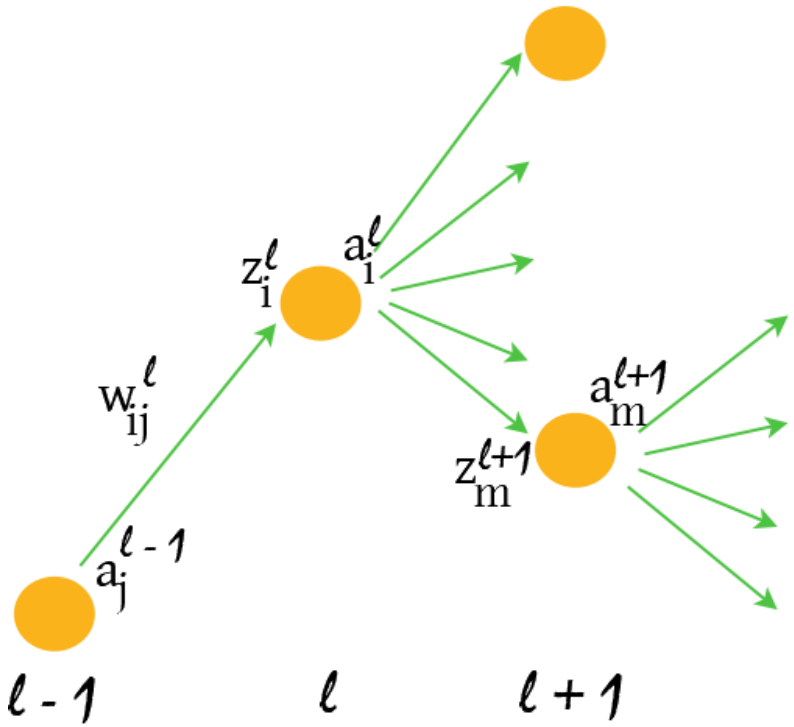
GD/SGD for MLP: Sample-wise Gradient Computation



Generalized setting:

$$\frac{\partial e}{\partial w_{ij}^\ell} = \frac{\partial e}{\partial z_i^\ell} a_j^{\ell-1}$$

GD/SGD for MLP: Sample-wise Gradient Computation

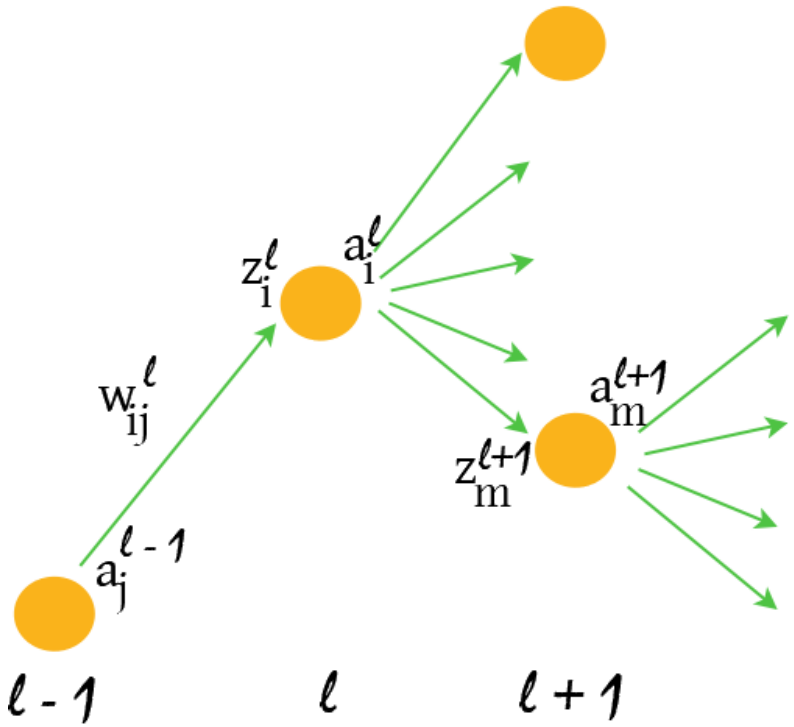


Generalized setting:

$$\frac{\partial e}{\partial w_{ij}^\ell} = \frac{\partial e}{\partial z_i^\ell} a_j^{\ell-1}$$

$$\frac{\partial e}{\partial z_i^\ell} = \frac{\partial e}{\partial a_i^\ell} \phi'(z_i^\ell)$$

GD/SGD for MLP: Sample-wise Gradient Computation



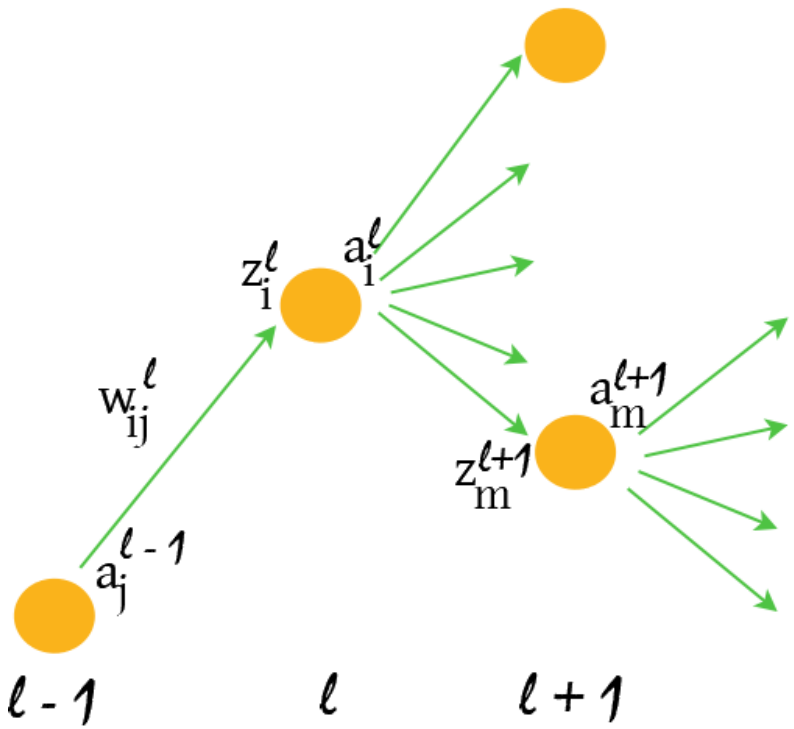
Generalized setting:

$$\frac{\partial e}{\partial w_{ij}^\ell} = \frac{\partial e}{\partial z_i^\ell} a_j^{\ell-1}$$

$$\frac{\partial e}{\partial z_i^\ell} = \frac{\partial e}{\partial a_i^\ell} \phi'(z_i^\ell)$$

$$\frac{\partial e}{\partial a_i^\ell} = \sum_{m=1}^{N_{\ell+1}} \frac{\partial e}{\partial z_m^{\ell+1}} w_{mi}^{\ell+1}$$

GD/SGD for MLP: Sample-wise Gradient Computation



Generalized setting:

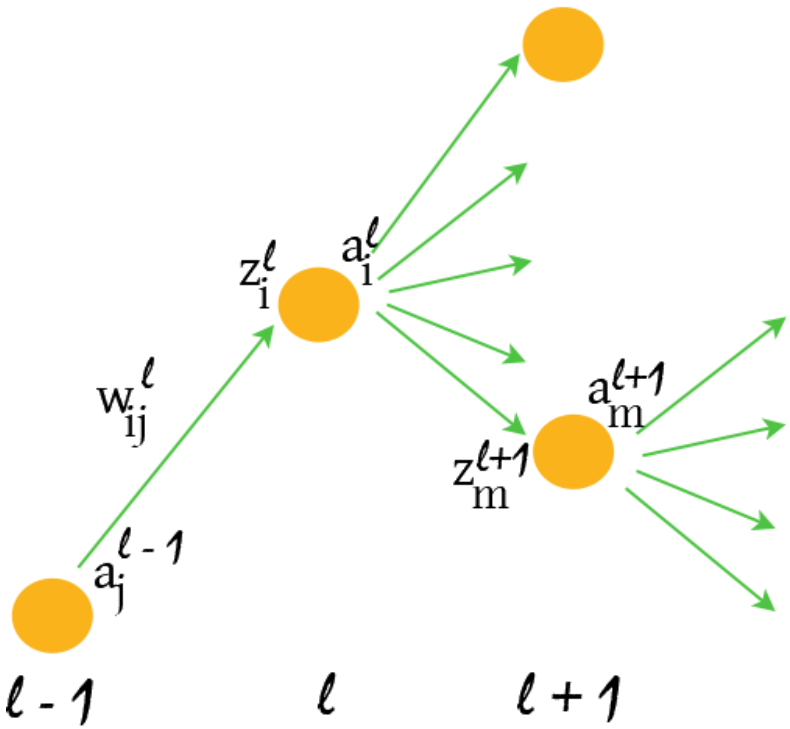
$$\frac{\partial e}{\partial w_{ij}^\ell} = \frac{\partial e}{\partial z_i^\ell} a_j^{\ell-1}$$

$$\frac{\partial e}{\partial z_i^\ell} = \frac{\partial e}{\partial a_i^\ell} \phi'(z_i^\ell)$$

$$\frac{\partial e}{\partial a_i^\ell} = \sum_{m=1}^{N_{\ell+1}} \frac{\partial e}{\partial z_m^{\ell+1}} w_{mi}^{\ell+1}$$

$$= \sum_{m=1}^{N_{\ell+1}} \frac{\partial e}{\partial a_m^{\ell+1}} \phi'(z_m^{\ell+1}) w_{mi}^{\ell+1}$$

GD/SGD for MLP: Sample-wise Gradient Computation



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$$= \begin{bmatrix} \phi'(z_1^{\ell+1}) w_{11}^{\ell+1} & \dots & \phi'(z_{N_{\ell+1}}^{\ell+1}) w_{N_{\ell+1}1}^{\ell+1} \end{bmatrix} \begin{bmatrix} \frac{\partial e}{\partial a_1^{\ell+1}} \\ \vdots \\ \frac{\partial e}{\partial a_{N_{\ell+1}}^{\ell+1}} \end{bmatrix}$$

GD/SGD for MLP: Sample-wise Gradient Computation

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$$\begin{bmatrix} \frac{\partial e}{\partial a_1^{\ell}} \\ \vdots \\ \frac{\partial e}{\partial a_{N_\ell}^{\ell}} \end{bmatrix} = \begin{bmatrix} \phi'(z_1^{\ell+1}) w_{11}^{\ell+1} & \cdots & \phi'(z_{N_{\ell+1}}^{\ell+1}) w_{N_{\ell+1}1}^{\ell+1} \\ \vdots & \cdots & \vdots \\ \phi'(z_1^{\ell+1}) w_{1N_\ell}^{\ell+1} & \cdots & \phi'(z_{N_{\ell+1}}^{\ell+1}) w_{N_{\ell+1}N_\ell}^{\ell+1} \end{bmatrix} \begin{bmatrix} \frac{\partial e}{\partial a_1^{\ell+1}} \\ \vdots \\ \frac{\partial e}{\partial a_{N_{\ell+1}}^{\ell+1}} \end{bmatrix}$$

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$$\implies \begin{bmatrix} \frac{\partial e}{\partial a_1^{\ell}} \\ \vdots \\ \frac{\partial e}{\partial a_{N_\ell}^{\ell}} \end{bmatrix} = \begin{bmatrix} w_{11}^{\ell+1} & \cdots & w_{N_{\ell+1}1}^{\ell+1} \\ \vdots & \cdots & \vdots \\ w_{1N_\ell}^{\ell+1} & \cdots & w_{N_{\ell+1}N_\ell}^{\ell+1} \end{bmatrix} \begin{bmatrix} \phi'(z_1^{\ell+1}) & & \\ & \ddots & \\ & & \phi'(z_{N_{\ell+1}}^{\ell+1}) \end{bmatrix} \begin{bmatrix} \frac{\partial e}{\partial a_1^{\ell+1}} \\ \vdots \\ \frac{\partial e}{\partial a_{N_{\ell+1}}^{\ell+1}} \end{bmatrix}$$

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$$\delta^{\ell} = (W^{\ell+1})^{\top} \text{Diag}(\phi^{\ell+1'}) \delta^{\ell+1}$$

GD/SGD for MLP: Sample-wise Gradient Computation

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$$\implies \begin{bmatrix} \frac{\partial e}{\partial a_1^{\ell}} \\ \vdots \\ \frac{\partial e}{\partial a_{N_\ell}^{\ell}} \end{bmatrix} = \begin{bmatrix} w_{11}^{\ell+1} & \cdots & w_{N_{\ell+1}1}^{\ell+1} \\ \vdots & \cdots & \vdots \\ w_{1N_\ell}^{\ell+1} & \cdots & w_{N_{\ell+1}N_\ell}^{\ell+1} \end{bmatrix} \begin{bmatrix} \phi'(z_1^{\ell+1}) & & \\ & \ddots & \\ & & \phi'(z_{N_{\ell+1}}^{\ell+1}) \end{bmatrix} \begin{bmatrix} \frac{\partial e}{\partial a_1^{\ell+1}} \\ \vdots \\ \frac{\partial e}{\partial a_{N_{\ell+1}}^{\ell+1}} \end{bmatrix}$$

$$\delta^{\ell} = (W^{\ell+1})^{\top} \text{Diag}(\phi^{\ell+1'}) \delta^{\ell+1} = V^{\ell+1} \delta^{\ell+1}$$