MAT315 Combinatorial Enumeration
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Instructor: Manjil Saikia

TA: Kanak Dhotre

MATHEMATICAL AND PHYSICAL SCIENCES DIVISION, SCHOOL OF ARTS AND SCIENCES, AHMEDABAD UNIVERSITY, AHMEDABAD 380009, GUJARAT, INDIA
Email address: `manjil.saikia@ahduni.edu.in`

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Preface

These are the lecture notes of MAT315 Combinatorial Enumeration, offered in Monsoon 2024 semester at Ahmedabad University, India. The notes were written down by the TA for the course, Kanak Dhotre and is as close to the classroom teaching as possible.

There are several very good textbooks in combinatorics, however the material that I wish to cover in this course is not available in a single source to my liking. So, I decided to make my own notes for this iteration, as well as for any future iterations of this course.

The background required to enroll for the course is very minimal, in fact, it is not even mandatory for a student to have done a first course in Discrete Mathematics. So, we introduce several basic concepts along the way and if there is a scope for some digression then we will take it. The course is supplemented by some homework assignments, some of the problems in the text were set in those assignments, some even appeared in the examinations.

I am thankful to Kanak Dhotre for typing these notes. Any errors that remain are mine. If there are any errors, comments, or corrections, please write to me via email.

Manjil Saikia

CHAPTER 1

What is Combinatorics?

1. Introduction

This course aims to delve into the study of discrete mathematical structures, a field which traces its roots back to the 1700s with the work of Leonhard Euler and has gained much attention between the 1960s and 1970s with the advent of computer science. Notably, Euler answered the following question posed by Philip Naude in the year 1741: "In how many ways can the number 50 be written as a sum of seven different positive integers?". We shall understand the outline of Euler's solution to the problem later in this course. A few important personalities (some of whose work we will study eventually) in the subject include Gian Carlo Rota, Donald Knuth, Richard Stanley, Srinivasa Ramanujan, and Pinagala.

Combinatorics is the science of patterns and arrangements. More concretely, it deals with the study of the existence and the number of arrangements possible for a given mathematical structure. We start our discussion with a few motivating questions which will make our statement clearer.

Question 1.1. *In how many ways can you arrange the elements of the set* $[n] := \{1, 2, 3, \ldots, n\}$ *such that the first entry in the arrangement is an even number?*

Notice how when n is an even number we have $n/2$ choices of even numbers to make for the first entry in our arrangement. Once a choice for the said even number is made the remaining $n-1$ choices can be made in $(n-1)!$ ways. Hence, in the case where n is an even number we have $n/2 \cdot (n-1)!$ possible arrangements. Can you see why we will have $(n-1)/2 \cdot (n-1)!$ arrangements for the case where n is an odd number?

Question 1.2. *In how many ways can you arrange elements from the set* $[n] := \{1, 2, 3, \ldots, n\}$ *on a grid with* n *columns and* n *rows?*

Notice how for each one of the n^2 spaces we have n choices to make. Hence, there are a total of $\underbrace{n \ \cdots \ n}_{n^2 \text{ times}} = n^{n^2}$ possible arrangements.

Question 1.3. *In how many ways can you arrange elements from the set* $[n]$ *(as defined in the previous two examples) on a grid with* n *columns and* n *rows such that each element appears atleast(/exactly) once in each row?*

Since for each row in the grid we have $n!$ possible arrangements and the grid has n rows there are a total of $n \cdot n!$ possible arrangements.

Question 1.4. *How many matrices of order* $n \times n$ *exist given that the entries must be from the set* $\{0, 1\}$*?*

Since we have 2 choices for each one of the n^2 entries there are a total of 2^{n^2} such matrices.

QUESTION 1.5 (Permutation Matrices). *How many matrices of order $n \times n$ exist given that the entries must be from the set $\{0, 1\}$ and each row and column must have exactly one* 1.

Notice how in the first row of our matrix we have n ways to fix the occurrence of 1. This forces $n - 1$ ways to fix the occurrence of 1 in the second row and so on. Hence, in all there are a total of $n \cdot (n-1) \cdots 1 = n!$ such matrices.

REMARK 1.1. *Question 1.5 can also be re-stated as counting the number of order $n \times n$ matrices which have row-sum and column-sum equal to* 1.

With the following definition we shall now look at a generalization of sorts of the kind of matrices we were dealing with in Question 1.5.

DEFINITION 1.1 (Alternating Sign Matrix (ASM)). *A matrix of order $n \times n$ is called an alternating sign matrix if the following conditions hold:*

(1) *All the entries of the matrix come from the set $\{-1, 0, 1\}$.*
(2) *Each row-sum and column-sum is* 1.
(3) *The non-zero entries (both row-wise and column-wise) alternate in sign.*

A result first proved by Doron Zeilberger in the year 1992 states that there are precisely

$$\prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!}$$

number of ASMs of order $n \times n$. A proof of this result is beyond the scope of these lectures and is mentioned only for the sake of completeness. We shall, however, count ASMs of order 3 now.

QUESTION 1.6. *How many ASMs of order* 3 *exist?*

Notice how the set of matrices we counted in Question 1.5 are a subset of the set of ASMs of order n (verify each one of the three defining properties of an ASM). Next, we notice a pattern; an ASM (of any order) can't have a -1 in the first row. Why? To the contrary, assume there is an ASM with a -1 in the first row. Since the immediate non-zero entry below it must be a 1, the column sum cannot be 1 without violating the alternativity condition. A similar argument shows that ASMs cannot have a -1 in the last row, the first column, or the last column either. This pattern allows us to easily list all ASMs of order 3.

First we list all the ASMs counted in Question 1.5:

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}, \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}, \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}, \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}, \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}.$$

Now, the only ASM of order 3 with a negative entry is

$$\begin{pmatrix}0&1&0\\1&-1&1\\0&1&0\end{pmatrix}.$$

We shall now explore a combinatorial problem which involves counting the number of checkerboard tilings (also known as dimer models) which is of great interest to physicists.

QUESTION 1.7. *Consider an $m \times n$ board. In how many ways can you cover all the squares of the said board with no overlaps, no diagonal placements, and no protrusions off the board, using dominoes (blocks of order 2×1)?*

It is clear that the existence of atleast one tiling is guaranteed if and only if m and n are not simultaneously odd. However, to count the number of such tilings is not a trivial task. A result due to the famous Dutch physicist Pieter Kasteleyn in the 1960s states that the number of such tilings is

$$\prod_{v=1}^{m}\prod_{h=1}^{n}\left(4\cos^2\left(\frac{v\pi}{m+1}\right)+4\cos^2\left(\frac{h\pi}{n+1}\right)\right)^{\frac{1}{4}}.$$

Once again, a proof of the result is beyond the scope of these lectures and is stated only for the sake of completeness.

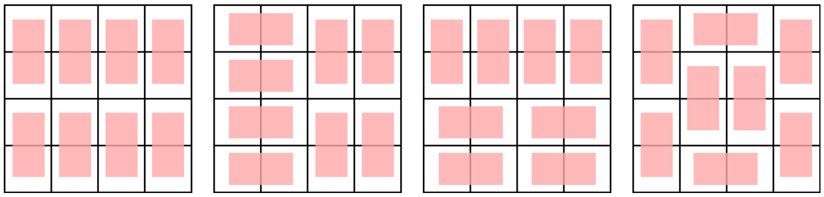


FIGURE 1. 4 out of the 36 possible domino-tilings of a 4×4 board.

Next, we state an equivalent formulation of Question 1.7.

DEFINITION 1.2 (Graph). *A graph is a pair $G = (V, E)$ where V is a set whose elements are called vertices and E is a set of unordered pairs of vertices whose elements are called edges.*

DEFINITION 1.3 (Perfect Matching). *Let $G = (V, E)$ be a graph. An $M \subseteq E$ is called a perfect matching of G if no two edeges in M share a common vertex and every vertex of G is incident to atleast one edge in M.*

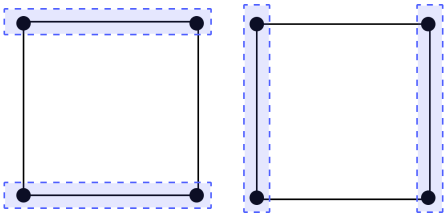


FIGURE 2. The only possible perfect matchings of the grid graph of order 2

Now consider the following construction. To each square in a given a board of order $m \times n$, we assign a vertex. Additionally, there is an edge between the said vertices if and only if the corresponding squares are adjacent to one another. Figure 3 is an example of this construction.

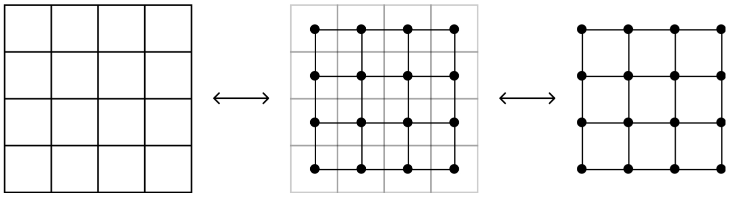


FIGURE 3. A board of order 4×4 and it's corresponding graph

It is clear that our construction gives a one-to-one correspondence between the set of possible domino-tilings of a board and the set of perfect matchings of it's corresponding graph.

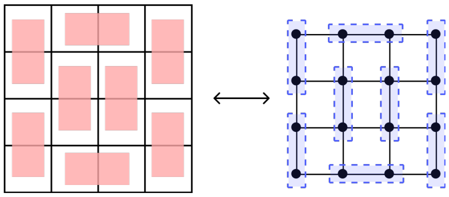


FIGURE 4. A domino tiling of a board of order 4×4 and the perfect matching that it admits

We shall revisit this formulation of the problem much later in the course. For now, restrict ourselves to a boards of order $2 \times n$ and count the number of possible domino tilings, say t_n as n varies over the set of natural numbers. It is not too difficult to convince yourself of the fact that $t_0 = 1$, $t_1 = 1$, $t_2 = 2, t_3 = 3$, and $t_4 = 5$. Since the first five terms in the sequence t_n are the same as the five terms of the Fibonacci sequence (say F_n), one might guess that $t_n = F_n$. This is indeed true, and we shall now give a proof of our claim using a counting argument.

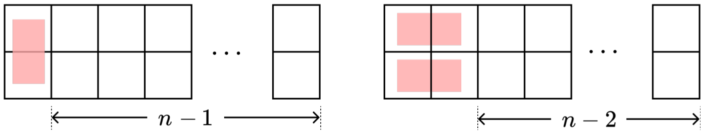


FIGURE 5. Two ways to tile the first column of board of order $2 \times n$

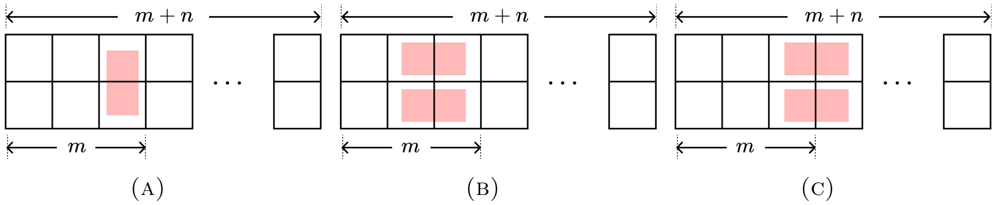
PROOF. Notice how there are exactly two ways to tile the first column of any board of order $2 \times n$ (refer to Figure 5). It can either be tiled using a single vertical domino, in which case it remains to tile the sub-board of order $2 \times (n-1)$, or using two horizontal dominos, in which case it remains to tile the sub-board of order $2 \times (n-2)$. Since both of these cases are valid, it follows that

$$t_n = t_{n-1} + t_{n-2}.$$

Which is the same as the Fibonacci reccurence. □

Now that we have $F_n = t_n$, we prove two standard Fibonacci identities using counting arguments similar to the ones used in the above proof.

CLAIM 1.1. $F_{m+n} = F_m F_n + F_{m-1} F_{n-1}$



(A) (B) (C)

FIGURE 6. The only 3 ways to tile the m-th row of a $2 \times (m+n)$ board.

PROOF. Notice how we have exactly 3 ways to tile the m-th row of a $2 \times (m+n)$ board (refer to Figure 6). Cases (a) and (b) account for when we have F_m ways to tile the sub-block of order $2 \times m$ **and** F_n ways to tile the sub-board which occurs immediately after. Case (c), on the other hand, accounts for when we have F_{m-1} ways to tile the sub-board of order $2 \times (m-1)$ which occurs before the already tiled m-th row **and** F_{n-1} ways to tile the sub-board of order $2 \times (n-1)$ which occurs after the already tiled m-th row. This proves the required identity. □

CLAIM 1.2. $F_0 + \cdots + F_n = F_{n+2} - 1$

PROOF. Notice how every board of order $2 \times k$ has a trivial tiling; the one which only uses vertical dominos and no horizontal ones. On a board of order $2 \times (n+2)$ if this trivial tiling is ignored, every other possible tiling must have the occurence of atleast one pair of horizontal dominos. If the last such pair occurs at the $n+1$-th column, the sub-board of order $2 \times n$ preceeding it can be tiled in F_n ways. If the last such pair occurs at the n-th column, the sub-board of order $2 \times (n-1)$ can be tiled in F_{n-1} ways, and so on. This gives us the required identity. □

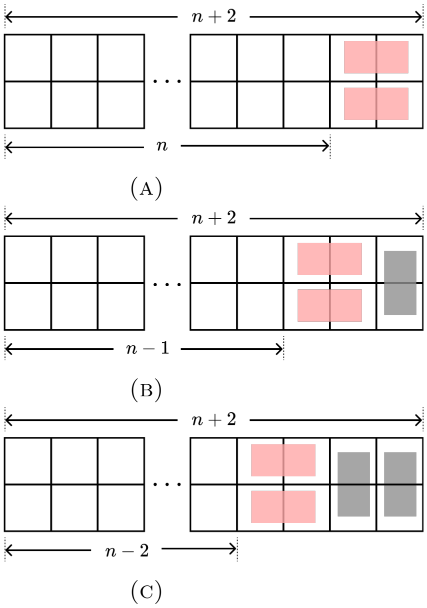


FIGURE 7. Examples of occurences of the last pair of horizontal dominos in 3 non-trivial tilings of a board of order $2 \times (n+2)$

Recall how $\binom{n}{k}$ counts the number of ways one can choose k elements from a set of n elements. We shall now state a rather interesting re-interpretation of binomial coefficients.

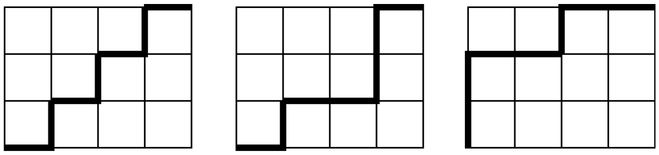


FIGURE 8. 3 examples of lattice paths from $(0,0)$ to $(4,3)$

QUESTION 1.8. *Given that the only moves allowed are North (N), corresponding to* $(i,j) \to (i,j+1)$, *or East (E), corresponding to* $(i,j) \to (i+1,j)$, *count the number of paths from* $(0,0)$ *to* (m,n) *on a grid of order* $m \times n$ *(refer to Figure 8 for examples of such paths).*

Notice how we require exactly m E-moves and n N-moves to reach (m,n) from $(0,0)$. However, every lattice path is determined completely by a choice of m E-moves (or equivalently n N-moves) which can be made in $\binom{m+n}{m}$ ways (or equivalently $\binom{m+n}{n}$ ways). As a consequence of this counting exercise we have not only given a re-interpretation of binomial coefficients, but have also proved that they are symmetric, i.e, $\binom{n}{k} = \binom{n}{n-k}$.

Per usual, we are now interested in giving a counting argument for a standard identity concerning binomial coefficients called Pascal's identity.

CLAIM 1.3. $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$

PROOF. Notice how there are $\binom{n+1}{k+1}$ choices of a lattice paths from $(0,0)$ to $(n-k, k+1)$. The last step in each one of these paths is either an N-move or an E-move. If it is an N-move then it suffices to choose one lattice path from $(0,0)$ to $(n-k-1, k+1)$ out of the

$$\binom{(n-k-1)+(k+1)}{k+1} = \binom{n}{k+1}$$

choices. Alternatively, if it is an E-move then it suffices to choose one lattice path from $(0,0)$ to $(n-k, k)$ out of the

$$\binom{(n-k)+k}{k} = \binom{n}{k}$$

choices. □

Let's remind ourselves that the central objective of this course is to count. From Question 1.1 through Question 1.6, we encountered examples where we derived exact closed-form expressions for counting. During our discussion of the dimer model, we first stated a closed-form expression and then constructed a bijection between the tiling configuration and the perfect matchings configuration. By restricting ourselves to boards of size $2 \times n$, we demonstrated that finding a recurrence relation is also a valid counting technique. However, as we will see, the methods we have discussed so far are not always applicable. In such cases, we turn to the idea of generating functions, which we will introduce next.

DEFINITION 1.4 (Generating Function). *Given a sequence* $\{a_n\}_{n\geq 0}$ *the formal power series,* $\sum_{k=0}^{\infty} a_k x^k$*, in an indeterminate* x *is called the generating function of* $\{a_n\}_{n\geq 0}$.

Since our definition works with a *formal* power series we need not worry about the divergence of the involved infinite sum. That being said, we can always be cautious and assume $|x| < 1$ to guarantee convergence.

QUESTION 1.9. *Find the generating function for the sequence of Fibonacci numbers*

Let $\{f_n\}_{n\geq 0}$ denote the sequence of Fibonacci numbers and let $F(x)$ be the corresponding generating function. Notice how

$$\begin{aligned}
F(x) &= \sum_{k=0}^{\infty} f_k x^k \\
&= f_0 + f_1 x + \sum_{k=2}^{\infty} f_k x^k \\
&= 1 + x + \sum_{k=2}^{\infty} f_k x^k \\
&= 1 + x + \sum_{k=2}^{\infty} (f_{k-1} + f_{k-2}) x^k \\
&= 1 + x + \sum_{k=2}^{\infty} f_{k-1} x^k + \sum_{k=2}^{\infty} f_{k-2} x^k \\
&= 1 + x + x(F(x) - 1) + x^2 F(x) \\
&= 1 + xF(x) + x^2 F(x)
\end{aligned}$$

Solving which we obtain

$$F(x) = \frac{1}{1 - x - x^2}$$

.

Throughout this course, we will explore how and why generating functions are extensively used in combinatorics. We introduce one such instance now, which will be discussed in greater depth later.

DEFINITION 1.5. *An integer partition of a natural number* n *is a non-increasing sequence of natural numbers,* $\{\lambda_i\}_{i\geq 1}$*, whose sum is* n.

QUESTION 1.10. *In how many ways can a natural number* n *be partitioned?*

Let $p(n)$ denote the number of partitions of n. Interestingly, no "nice" closed-form expression for $p(n)$ has been found. However, we will derive the generating function

$$\prod_{n=1}^{\infty}\left(1+x^{n}+x^{2n}+x^{3n}+\cdots\right)=\sum_{n=0}^{\infty}p\left(n\right)x^{n}$$

which was given by Euler, later in the course.

REMARK 1.2. *In the year 1918, G.H Hardy and Srinivasa Ramanujan obtained an asymptotic expression (an expression which describes the limiting behaviour) for* $p\left(n\right)$ *which is given by*

$$p\left(n\right)\quad\frac{1}{4n\sqrt{3}}\exp\left(\pi\sqrt{\frac{2n}{3}}\right).$$

Oftentimes stating an asymptotic expression is also a valid counting technique.

2. Counting Principles

We state the fundamental counting principles now.

1. (Addition Principle): If $\{A_k\}_{k\geq 1}$ is a family of finite and pairwise disjoint sets then $|\bigcup_{k\geq 1}A_k|=\sum_{k\geq 1}|A_k|$.
2. (Substraction Principle): If A and B are finite sets such that $B\subseteq A$ then $|A\backslash B|=|A|-|B|$.
3. (Product Rule): If A and B are finite sets, then $|A\times B|=|A|\cdot|B|$.
4. (Division Rule): For finite sets A and B if there exists a d-to-many function $f:A\to B$ then $|B|=|A|/d$.

In some sense we've already been using these principles in disguise thus far. For instance;

QUESTION 2.1. *How many k-digit positive numbers are there?*

Since we have 10 choices for the first $(k-1)$ digits and 9 choices for the last digit, by the product rule there are $9\cdot 10^{k-1}$ k digit numbers.

This is a good point to recall the binomial theorem and prove it using a counting argument instead of the standard induction argument.

THEOREM 2.1 (Binomial Theorem).

$$(x+y)^{n}=\sum_{k=0}^{n}\left(\begin{array}{c}n\\k\end{array}\right)x^{n-k}y^{k}.$$

PROOF. Consider the expansion of

$$(x+y)^{n}=\underbrace{(x+y)\cdots(x+y)}_{n\text{ factors}}.$$

In this expansion, there are $\left(\begin{array}{c}n\\k\end{array}\right)$ ways to choose x from exactly k of the n factors (which forces the choice of y from the remaining $(n-k)$ factors). Since k can range from 0 to n, applying the addition principle gives us the binomial theorem. ☐

Now we state a the multinomial theorem which generalizes the binomial theorem to expressions with more than two terms, allowing us to expand any power of a sum of multiple terms as a sum of products of those terms, each multiplied by a what is called a multinomial coefficient which is denoted and defined as

$$\left(\begin{array}{c}n\\k_1,\cdots,k_m\end{array}\right):=\frac{n}{k_1!\cdots k_m!}.$$

We would expect the multinomial coefficient to count something. That is indeed the case, it counts the number of ways to arrange n distinct objects into m distinct bins such that the i-th bin contains k_i elements. With this interepretation of multinomial coefficients, mimicing the steps involved in the proof of the binomial theorem we get the multinomial theorem which is stated below.

THEOREM 2.2 (Multinomial Theorem).

$$(x_1 + \cdots + x_r)^n = \sum_{k_1+\cdots+k_r=n} \left(\begin{array}{c} n \\ k_1, \cdots, k_r \end{array} \right) x_1^{k_1} \cdots x_r^{k_r}.$$

It is natural to expect a Pascal-like identity for multinomial coefficients as well. In this spirit consider the following claim.

CLAIM 2.1.

$$\left(\begin{array}{c} n \\ k_1, \cdots, k_r \end{array} \right) = \left(\begin{array}{c} n-1 \\ k_1 - 1, k_2, \cdots, k_r \end{array} \right) + \cdots + \left(\begin{array}{c} n-1 \\ k_1, k_2, \cdots, k_r - 1 \end{array} \right).$$

PROOF. □

Notice how when $k_1 = k$, $k_2 = n - k$ and all the other k_is are 0, we retrieve the Pascal's identity (Claim 1.3)

In addition to the fundamental counting principles, there are a few more principles/techiniques which we will use quite often in this course. One of them is the bijection principle. We state the principle formally and then given an example.

THEOREM 2.3 (Bijection Principle). *If S and T are finite sets then $|S| = |T|$ if and only if there exists a bijection $f : S \to T$.*

QUESTION 2.2. *Let S be a finite set. A* ***permutation*** *of S is a bijection from and to S. How many permutations of $[n] = \{1, 2, 3, \ldots, n\}$ exist?*

We introduce the cycle notation before outlining a solution. In the case where $n = 1$, it is clear that there is only possible bijection, namely, $f_1 : \{1\} \to \{1\}$ such that $f_1(1) = 1$. We shall write f_1 as

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

In the case where $n = 2$, it is clear that there are two possible bijections. These are $g_1 : \{1, 2\} \to \{1, 2\}$ such that $g_1(1) = 1$, $g_1(2) = 2$, and $g_2 : \{1, 2\} \to \{1, 2\}$ such that $g_2(1) = 2$, $g_2(2) = 1$. We shall write g_1 and g_2 as

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

respectively. In essence, the first row in our notation lists the elements of the domain, and the second row lists where the bijection sends these elements. Do you see why in the case where $n = 3$ the bijections can be written as

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$

Since the first row of any bijection in our notation is fixed and we have

$$n \cdot (n-1) \cdots 1 = n!$$

ways to fill out the second row, it clear that there are $n!$ permutations of $[n]$. Recall from Question 1.5 that we also have exactly $n!$ permutation matrices. By the bijection principle we must have a bijection between the set of permutation matrices of order $n \times n$ and the set of permuations of $[n]$. The bijection is as follows. Corresponding to every permutation

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ f(1) & f(2) & \cdots & f(n) \end{pmatrix}$$

of $[n]$ we assign an $n \times n$ matrix where the entry at $(i, f(i))$ for $1 \leq i \leq n$ is 1, and all other entries are 0. More explicity,

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \longleftrightarrow \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \longleftrightarrow \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix},$$

$$\vdots$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \longleftrightarrow \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}.$$

We outline a proof of an identity involving binomial coefficients using the bijection principle now.

Claim 2.2.

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n$$

Proof. Let $S = \{a_1, \ldots, a_n\}$ be a set with n elements. For each of the 2^n possible subsets $T = \{a'_1, \ldots, a'_k\}$ of S (where k can be any value from 0 to n), there is a corresponding binary string of length n. In this string, the ith bit is 1 if the element a_i is in the subset T, and 0 if it is not. For instance in the case where $S = \{0, 1\}$ we have the following correspondence

$$\begin{aligned} \emptyset &\longleftrightarrow 00 \\ \{0\} &\longleftrightarrow 10 \\ \{1\} &\longleftrightarrow 01 \\ \{0,1\} &\longleftrightarrow 11 \end{aligned}$$

□

Another counting technique which we've seen before in the proof of Pascal's identity (Claim 1.3) is called double counting. The broad idea is to count the same combinatorial object in two different ways. As an example, we give a proof of Claim 2.2 using the double counting technique.

Proof. It is clear that 2^n counts the number of n-bit binary strings. To prove the claimed identity, we count the number of n-bit binary strings once again in a different way. Notice how an arbitrary$n$-bit binary string has $0 \leq i \leq n$ number of

0s. Since there are $\binom{n}{0}$ choices when $i = 0$, $\binom{n}{1}$ choices when $i = 1$, and so on, by the addition principle we have proved the required identity. ☐

We have all seen a proof by induction of the following identity. Now we give a proof using a double counting argument.

CLAIM 2.3. $1 + 2 + 3 + \cdots + n = \frac{n\,(n+1)}{2} = \binom{n+1}{2}$

PROOF. Consider the problem of counting the number of handshakes in a group of $n+1$ people, where each person shakes hands with every other person. The first person in the group shakes hands with everyone except themselves; this gives n handshakes. The second person in the group shakes hands with everyone else; this gives $(n-1)$ handshakes (not n because we've already counted the handshake between the first and second person). The third person in the group shakes hands with everyone; this gives $(n-2)$, and so on. In all we have $n + (n-1) + (n-2) + \cdots + 1$ handshakes. Let's count again in a different way now. Notice how for each pair of people there is one handshake. Hence, there are $\binom{n+1}{2}$ handshakes in total. Putting the two observations together proves the claim. ☐

Next, we introduce the "stars and bars" method. We shall outline how this method works with the help of a few standard examples.

QUESTION 2.3. *For a given* $n \in \mathbb{N}$ *how many* $x_1, \cdots, x_r \geq 0$ *exist which add upto* n*?*

We conclude this section by stating an informal technique called proof by pictures (or proof without words). This technique involves the use of visual representations to demonstrate the validity of a mathematical statement. Although in the strictest sense of things a proof by pictures is not a counting principle, it often helps develop intution. For instance, we state such a proof for Claim 2.3.

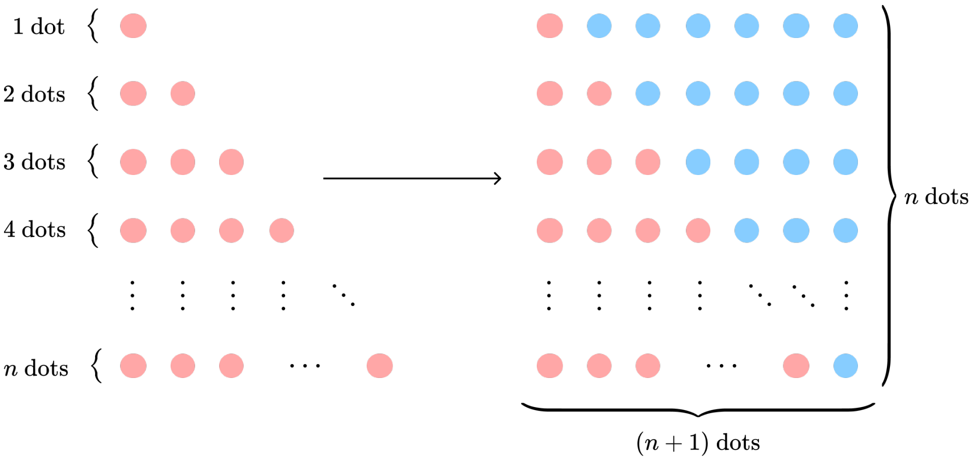


FIGURE 9

PROOF. In Figure 9 notice how there are $1 + 2 + 3 + \cdots + n$ dots colored in red, $1+2+3+\cdots+n$ dots colored in blue, and hence $2\,(1 + 2 + 3 + \cdots + n)$ total.

It is also true that in our arrangement there are $n\left(n+1\right)$ dots in total. Now the identity follows. □

Yet another fundamental counting principle is the pigeonhole principle. Given its importance, we will dedicate the upcoming section to explore it in detail.

Remark 2.1. *We have seen how the principles we've discussed so far can be useful in proving combinatorial identities. Determining how to derive these identities is not always straightforward. However, a common technique involves running computer simulations to generate a sequence of values corresponding to the first few natural numbers. Tools like OEIS, RATE (in Mathematica), and Guess (in Sage-Math and Maple) can then be used to conjecture a closed form for the sequence.*

3. The Pigeon-hole Principle

We state what is known as the pigeon-hole principle.

Theorem 3.1. *Let $A_1, \cdots, A_k$ be pair-wise disjoint finite sets. If*

$$|A_1 \cup A_2 \cup \cdots \cup A_k| > kr$$

then there exists an $1 \leq i \leq k$ such that $|A_i| > r$.

Even though the statement sounds trivial, it has immense applications. We state a few examples now.

Claim 3.1. *Given $b \in \mathbb{Z}$, the decimal expansion of $1/b$ is either finite or eventually periodic.*

Proof. Each step in the long division of 1 by b involves subtracting multiples of b from powers of 10 until the remainder is 0 or repeats itself. Since each one of these remainders must be strictly less than b, we have b possible remainders, namely, $0, 1, 2, \cdots, b-1$. If the remainder becomes 0 at some step, the division terminates, in which case the decimal expansion is finite. Next, consider the case where the division does not terminate. By the pigeon-hole principle, since there are only b possible remainders and more than b steps, at least one of the remainders must repeat. This proves our claim because when a remainder repeats, the sequence of digits that follow in the decimal expansion must repeat as well. □

Claim 3.2. *Let $S = \{1, 2, \cdots, 2n\}$. Any $n+1$ element subset K of S has two numbers a and b such that*

1. $\gcd(a, b) = 1$
2. *$a = nb$ for some $n \in \mathbb{N}$*

Proof.

1. Consider the n-element subset
$$\left\{\left(1,2\right), \left(3,4\right), \cdots \left(2n-1,2\right)\right\}$$
of $S \times S$ and notice how if we choose $n+1$ numbers from $\{0, \cdots, 2n\}$ then by the pigeonhole principle we are guaranteed to pick two numbers from the same pair in $S \times S$. Since each element in the said subset is a pair of consecutive numbers, their gcd is necessarily 1.
2. Notice how every element in the set $\{1, 2, \cdots, 2n\}$ can be written in the form $2^a b$ where b is odd by factoring out as many 2s as possible. Since there are only n odd numbers in these set $\{1, 2, \cdots, 2n\}$, we are left with n choices of b. Next, we construct b subsets of S, say $S_k \subseteq S$ such that

$i \in S_k$ if and only if $i = 2^k b$. Finally, by the pigeonhole principle, if one were to pick $n+1$ elements from S atleast 2 of them would be in the same S_k, i.e, there always exists a pair of numbers such that one divides the other.

$\square$

Claim 3.3. *If S be an $n+1$-element set of natural numbers, then there exists a pair of numbers in S whose difference is divisible by n.*

Claim 3.4. *Given m integers say $a_1, \cdots, a_m$ prove that there exists k and l with $0 \leq k \leq l \leq m$ such that m divides $a_{k+1} + a_{k+2} + \cdots + a_l$*

Claim 3.5. *In a party of n people, where some (maybe all, maybe none) shake hands with one another, show that there are atleast two people who shake the same number of hands.*

Claim 3.6. *In a group of 6 people where each pair of people are either friends or enemies, show that there are 3 people who are either all friends or all enemies.*

We conclude this section by stating a few definition which will turn out to be useful in the upcoming sections.

Definition 3.1 (k-Permutation). *A k-permutation on $[n] = \{1, 2, \cdots, n\}$ is a bijection between two k-element subsets of $[n]$.*

Definition 3.2 (Circular k-Permutation). *k-permutations π and σ are called circular equivalents if there exists a shift $s \in [k]$ such that $i + s \equiv j(mod\ k)$.*

4. The Principle of Inclusion-Exclusion

Recall that for finite sets A and B, the cardinality of their union is given by $|A \cup B| = |A| + |B| - |A \cap B|$. This formula, which adds (includes) the sizes of A and B while subtracting (excluding) the overcount in $|A \cap B|$ encapsulates the essence of the inclusion-exclusion principle. We start with a useful generalization of the motivating formula we just discussed.

Theorem 4.1. *Given finite sets $A_1, \cdots, A_k$,*

$$|\bigcup_{i=1}^{k} A_i| = \sum_{n=1}^{k} (-1)^{n+1} \left(\sum_{1 \leq i_1 < \ldots < i_n \leq k+1} |A_{i_1} \cap \cdots A_{i_n}| \right).$$

CHAPTER 2

The Art of Combinatorial Thinking

1. Binomial Coefficients

2. Catalan Numbers

CHAPTER 3

Generating Functions

1. Ordinary Generating Functions

2. Exponential Generating Functions

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Partitions

1. Set Partitions

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CHAPTER 5

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CHAPTER 6

A Combinatorial Miscellany

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2. Permutations

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4. Graphs and Trees

Bibliography