# MAT315 Combinatorial Enumeration
# Monsoon 2024

Instructor: Manjil Saikia

TA: Kanak Dhotre

Mathematical and Physical Sciences division, School of Arts and Sciences, Ahmedabad University, Ahmedabad 380009, Gujarat, India

*Email address*: `manjil.saikia@ahduni.edu.in`

2020 *Mathematics Subject Classification.* Primary 05-01
*Key words and phrases.* Combinatorics, Discrete Mathematics.

# Contents

# Preface

These are the lecture notes of MAT315 Combinatorial Enumeration, offered in the Monsoon 2024 semester at Ahmedabad University, India. The notes were written down by the TA for the course, Kanak Dhotre, and is as close to the classroom teaching as possible.

There are several very good textbooks in combinatorics, however, the material that I wish to cover in this course is not available in a single source to my liking. So, I decided to make my own notes for this iteration, as well as for any future iterations of this course.

The background required to enroll for the course is very minimal, it is not even mandatory for a student to have done a first course in Discrete Mathematics. So, we introduce several basic concepts along the way and if there is a scope for some digression then we will take it. The course is supplemented by some homework assignments, some of the problems in the text were set in those assignments, and some even appeared in the examinations.

I am thankful to Kanak Dhotre for typing these notes. Any errors that remain are mine. If there are any errors, comments, or corrections, please write to me via email.

Manjil Saikia

CHAPTER 1

# What is Combinatorics?

## 1. Introduction

This course aims to delve into the study of discrete mathematical structures, a field that traces its roots back to the 1700s with the work of Leonhard Euler and has gained much attention between the 1960s and 1970s with the advent of computer science. Notably, Euler answered the following question posed by Philip Naude in the year 1741: "In how many ways can the number 50 be written as a sum of seven different positive integers?". We shall understand the outline of Euler's solution to the problem later in this course. A few important personalities (some of whose work we will study eventually) in the subject include Gian Carlo Rota, Donald Knuth, Richard Stanley, Srinivasa Ramanujan, and Pinagala.

Combinatorics is the science of patterns and arrangements. More concretely, it deals with the study of the existence and the number of arrangements possible for a given mathematical structure. We start our discussion with a few motivating questions which will make our statement clearer.

QUESTION 1.1. *In how many ways can you arrange the elements of the set $[n] := \{1, 2, 3, \ldots, n\}$ such that the first entry in the arrangement is an even number?*

SOLUTION. Notice how when $n$ is an even number we have $n/2$ choices of even numbers to make for the first entry in our arrangement. Once a choice for the said even number is made the remaining $n-1$ choices can be made in $(n-1)!$ ways. Hence, in the case where $n$ is an even number we have $n/2 \cdot (n-1)!$ possible arrangements. Can you see why we will have $(n-1)/2 \cdot (n-1)!$ arrangements for the case where $n$ is an odd number?

QUESTION 1.2. *In how many ways can you arrange elements from the set $[n] := \{1, 2, 3, \ldots, n\}$ on a grid with $n$ columns and $n$ rows?*

SOLUTION. Notice how for each one of the $n^2$ spaces we have $n$ choices to make. Hence, there are a total of $\underbrace{n \ \cdots \ n}_{n^2 \text{ times}} = n^{n^2}$ possible arrangements.

QUESTION 1.3. *In how many ways can you arrange elements from the set $[n]$ (as defined in the previous two examples) on a grid with $n$ columns and $n$ rows such that each element appears at least (/exactly) once in each row?*

SOLUTION. Since for each row in the grid, we have $n!$ possible arrangements and the grid has $n$ rows there are a total of $n \cdot n!$ possible arrangements.

QUESTION 1.4. *How many matrices of order $n \times n$ exist given that the entries must be from the set $\{0, 1\}$?*

Solution. Since we have 2 choices for each one of the $n^2$ entries there are a total of $2^{n^2}$ such matrices.

Question 1.5 (Permutation Matrices). *How many matrices of order $n \times n$ exist given that the entries must be from the set $\{0, 1\}$ and each row and column must have exactly one 1.*

Solution. Notice how in the first row of our matrix we have $n$ ways to fix the occurrence of 1. This forces $n-1$ ways to fix the occurrence of 1 in the second row and so on. Hence, in all, there are a total of $n \cdot (n-1) \cdots 1 = n!$ such matrices.

Remark 1.1. *Question 1.5 can also be restated as counting the number of order $n \times n$ matrices which have row-sum and column-sum equal to 1.*

With the following definition, we shall now look at a generalization of sorts of the kind of matrices we were dealing with in Question 1.5.

Definition 1.1 (Alternating Sign Matrix (ASM)). *A matrix of order $n \times n$ is called an alternating sign matrix if the following conditions hold:*

1. *All the entries of the matrix come from the set $\{-1, 0, 1\}$.*
2. *Each row-sum and column-sum is 1.*
3. *The non-zero entries (both row-wise and column-wise) alternate in sign.*

A result first proved by Doron Zeilberger in the year 1992 states that there are precisely

$$\prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!}$$

number of ASMs of order $n \times n$. A proof of this result is beyond the scope of these lectures and is mentioned only for the sake of completeness. We shall, however, count ASMs of order 3 now.

Question 1.6. *How many ASMs of order 3 exist?*

Solution. Notice how the set of matrices we counted in Question 1.5 are a subset of the set of ASMs of order $n$ (verify each one of the three defining properties of an ASM). Next, we notice a pattern; an ASM (of any order) can't have a $-1$ in the first row. Why? To the contrary, assume there is an ASM with a $-1$ in the first row. Since the immediate non-zero entry below it must be a 1, the column sum cannot be 1 without violating the alternativity condition. A similar argument shows that ASMs cannot have a $-1$ in the last row, the first column, or the last column either. This pattern allows us to easily list all ASMs of order 3. First, we list all the ASMs counted in Question 1.5:

$$\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0 & 0\\0 & 0 & 1\\0 & 1 & 0\end{pmatrix}, \begin{pmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1 & 0\\0 & 0 & 1\\1 & 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0 & 1\\1 & 0 & 0\\0 & 1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0 & 1\\0 & 1 & 0\\1 & 0 & 0\end{pmatrix}.$$

Now, the only ASM of order 3 with a negative entry is

$$\begin{pmatrix}0 & 1 & 0\\1 & -1 & 1\\0 & 1 & 0\end{pmatrix}.$$

We shall now explore a combinatorial problem that involves counting the number of checkerboard tilings (also known as dimer models) which is of great interest to physicists.

QUESTION 1.7. *Consider an $m \times n$ board. In how many ways can you cover all the squares of the said board with no overlaps, no diagonal placements, and no protrusions off the board, using dominoes (blocks of order $2 \times 1$)?*

The existence of at least one tiling is guaranteed if and only if $m$ and $n$ are not simultaneously odd. However, counting the number of such tilings is not a trivial task. A result due to the famous Dutch physicist Pieter Kasteleyn in the 1960s states that the number of such tilings is

$$\prod_{v=1}^{m}\prod_{h=1}^{n}\left(4\cos^2\left(\frac{v\pi}{m+1}\right)+4\cos^2\left(\frac{h\pi}{n+1}\right)\right)^{\frac{1}{4}}.$$

Once again, proof of the result is beyond the scope of these lectures and is stated only for the sake of completeness.

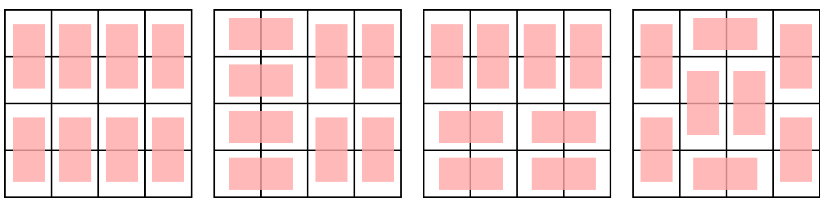

FIGURE 1. 4 out of the 36 possible domino-tilings of a $4 \times 4$ board.

Next, we state an equivalent formulation of Question 1.7.

DEFINITION 1.2 (Graph). *A graph is a pair $G = (V, E)$ where $V$ is a set whose elements are called vertices and $E$ is a set of unordered pairs of vertices whose elements are called edges.*

DEFINITION 1.3 (Perfect Matching). *Let $G = (V, E)$ be a graph. An $M \subseteq E$ is called a perfect matching of $G$ if no two edges in $M$ share a common vertex and every vertex of $G$ is incident to at least one edge in $M$.*

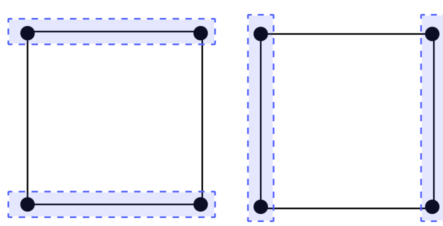

FIGURE 2. The only possible perfect matchings of the grid graph of order 2

Now consider the following construction. To each square in a given board of order $m \times n$, we assign a vertex. Additionally, there is an edge between the said vertices if and only if the corresponding squares are adjacent to one another. Figure 3 is an example of this construction.

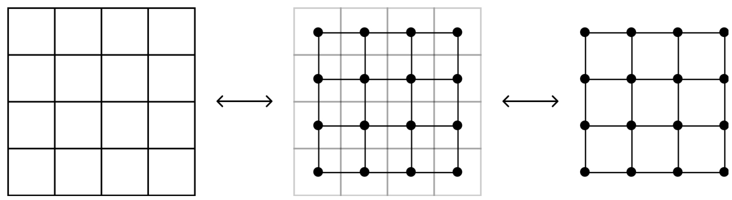

FIGURE 3. A board of order $4 \times 4$ and it's corresponding graph

Our construction gives a one-to-one correspondence between the set of possible domino-tilings of a board and the set of perfect matchings of its corresponding graph.

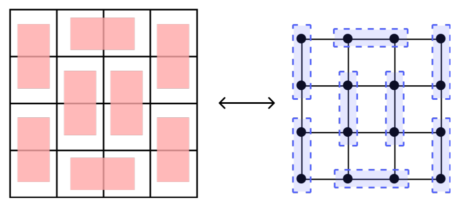

FIGURE 4. A domino tiling of a board of order $4 \times 4$ and the perfect matching that it admits

We shall revisit this formulation of the problem much later in the course. For now, restrict ourselves to a board of order $2 \times n$ and count the number of possible domino tilings, say $t_n$ as $n$ varies over the set of natural numbers. It is not too difficult to convince yourself of the fact that $t_0 = 1$, $t_1 = 1$, $t_2 = 2, t_3 = 3$, and $t_4 = 5$. Since the first five terms in the sequence $t_n$ are the same as the five terms of the Fibonacci sequence (say $F_n$), one might guess that $t_n = F_n$. This is indeed true, and we shall now give a proof of our claim using a counting argument.

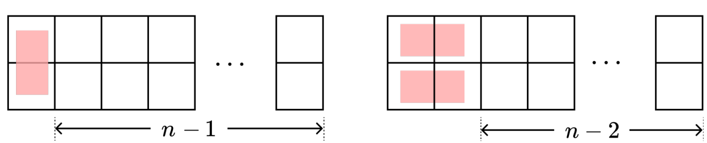


FIGURE 5. Two ways to tile the first column of board of order $2 \times n$

PROOF. Notice how there are exactly two ways to tile the first column of any board of order $2 \times n$ (refer to Figure 5). It can either be tiled using a single vertical domino, in which case it remains to tile the sub-board of order $2 \times (n-1)$, or using two horizontal dominos, in which case it remains to tile the sub-board of order $2 \times (n-2)$. Since both of these cases are valid, it follows that

$$t_n = t_{n-1} + t_{n-2}.$$

Which is the same as the Fibonacci recurrence. $\square$

Now that we have $F_n = t_n$, we prove two standard Fibonacci identities using counting arguments similar to the ones used in the above proof.

CLAIM 1.1. $F_{m+n} = F_m F_n + F_{m-1} F_{n-1}$

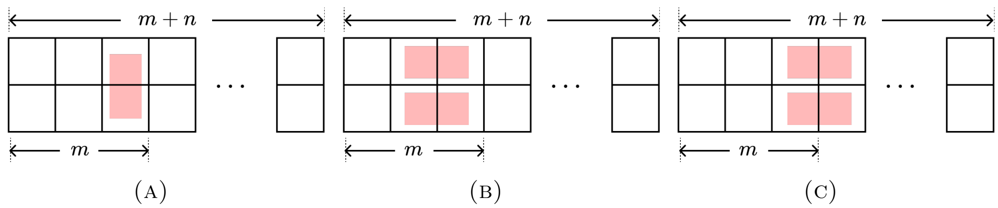


FIGURE 6. The only 3 ways to tile the $m$-th row of a $2 \times (m+n)$ board.

PROOF. Notice how we have exactly 3 ways to tile the $m$-th row of a $2\times(m+n)$ board (refer to Figure 6). Cases (a) and (b) account for when we have $F_m$ ways to tile the sub-block of order $2 \times m$ **and** $F_n$ ways to tile the sub-board which occurs immediately after. Case (c), on the other hand, accounts for when we have $F_{m-1}$ ways to tile the sub-board of order $2\times(m-1)$ which occurs before the already tiled $m$-th row **and** $F_{n-1}$ ways to tile the sub-board of order $2 \times (n-1)$ which occurs after the already tiled $m$-th row. This proves the required identity. □

CLAIM 1.2. $F_0 + \cdots + F_n = F_{n+2} - 1$

PROOF. Notice how every board of order $2\times k$ has a trivial tiling; the one which only uses vertical dominos and no horizontal ones. On a board of order $2 \times (n+2)$ if this trivial tiling is ignored, every other possible tiling must have the occurrence of at least one pair of horizontal dominos. If the last such pair occurs at the $n+1$-th column, the sub-board of order $2 \times n$ preceding it can be tiled in $F_n$ ways. If the last such pair occurs at the $n$-th column, the sub-board of order $2 \times (n-1)$ can be tiled in $F_{n-1}$ ways, and so on. This gives us the required identity. □

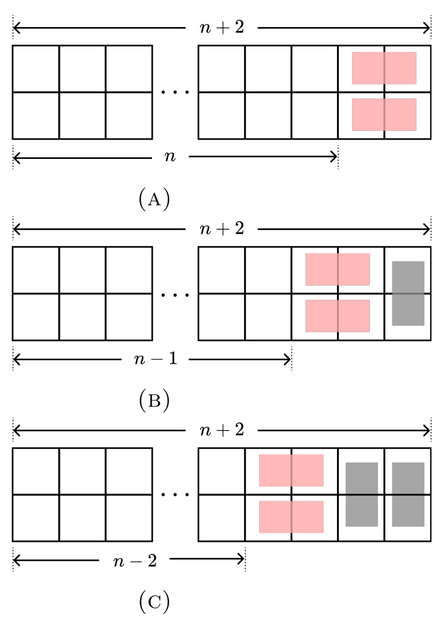


FIGURE 7. Examples of occurrences of the last pair of horizontal dominos in 3 non-trivial tilings of a board of order $2 \times (n+2)$

Recall how $\binom{n}{k}$ counts the number of ways one can choose $k$ elements from a set of $n$ elements. We shall now state a rather interesting re-interpretation of binomial coefficients.

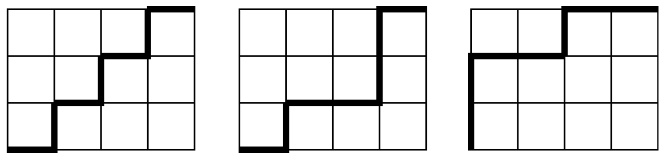

FIGURE 8. 3 examples of lattice paths from $(0,0)$ to $(4,3)$

QUESTION 1.8. *Given that the only moves allowed are North (N), corresponding to $(i,j) \to (i,j+1)$, or East (E), corresponding to $(i,j) \to (i+1,j)$, count the number of paths from $(0,0)$ to $(m,n)$ on a grid of order $m \times n$ (refer to Figure 8 for examples of such paths).*

SOLUTION. Notice how we require exactly $m$ E-moves and $n$ N-moves to reach $(m,n)$ from $(0,0)$. However, every lattice path is determined completely by a choice of $m$ E-moves (or equivalently $n$ N-moves) which can be made in $\binom{m+n}{m}$ ways (or equivalently $\binom{m+n}{n}$ ways).

REMARK 1.2. *As a consequence of this counting exercise, we have not only given a re-interpretation of binomial coefficients but have also proved that they are symmetric, that is,* $\binom{n}{k} = \binom{n}{n-k}$.

Per usual, we are now interested in giving a counting argument for a standard identity concerning binomial coefficients called Pascal's identity.

CLAIM 1.3. $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$

PROOF. Notice how there are $\binom{n+1}{k+1}$ choices of a lattice paths from $(0,0)$ to $(n-k,k+1)$. The last step in each one of these paths is either an N-move or an E-move. If it is an N-move then it suffices to choose one lattice path from $(0,0)$ to $(n-k-1,k+1)$ out of the

$$\binom{(n-k-1)+(k+1)}{k+1} = \binom{n}{k+1}$$

choices. Alternatively, if it is an E-move then it suffices to choose one lattice path from $(0,0)$ to $(n-k,k)$ out of the

$$\binom{(n-k)+k}{k} = \binom{n}{k}$$

choices. □

Let's remind ourselves that the central objective of this course is to count. From Question 1.1 through Question 1.6, we encountered examples where we derived exact closed-form expressions for counting. During our discussion of the dimer model, we first stated a closed-form expression and then constructed a bijection between the tiling configuration and the perfect matching configuration. By restricting ourselves to boards of size $2 \times n$, we demonstrated that finding a recurrence relation is also a valid counting technique. However, as we will see, the methods we have discussed so far are not always applicable. In such cases, we turn to the idea of generating functions, which we will introduce next.

Definition 1.4 (Ordinary Generating Function). *Given a sequence* $\{a_n\}_{n\geq 0}$ *the formal power series,* $\sum_{k=0}^{\infty} a_k x^k$*, in an indeterminate* $x$ *is called the generating function of* $\{a_n\}_{n\geq 0}$*.*

Since our definition works with a *formal* power series we need not worry about the divergence of the involved infinite sum. That being said, we can always be cautious and assume $|x| < 1$ to guarantee convergence.

Question 1.9. *Find the generating function for the sequence of Fibonacci numbers.*

Solution. Let $\{f_n\}_{n\geq 0}$ denote the sequence of Fibonacci numbers and let $F(x)$ be the corresponding generating function. Notice how

$$\begin{aligned} F(x) &= \sum_{k=0}^{\infty} f_k x^k \\ &= f_0 + f_1 x + \sum_{k=2}^{\infty} f_k x^k \\ &= 1 + x + \sum_{k=2}^{\infty} f_k x^k \\ &= 1 + x + \sum_{k=2}^{\infty} (f_{k-1} + f_{k-2}) x^k \\ &= 1 + x + \sum_{k=2}^{\infty} f_{k-1} x^k + \sum_{k=2}^{\infty} f_{k-2} x^k \\ &= 1 + x + x(F(x) - 1) + x^2 F(x) \\ &= 1 + xF(x) + x^2 F(x) \end{aligned}$$

Solving which we obtain

$$F(x) = \frac{1}{1 - x - x^2}$$

.

Throughout this course, we will explore how and why generating functions are extensively used in combinatorics. We introduce one such instance now, which will be discussed in greater depth later.

Definition 1.5. *An integer partition of a natural number* $n$ *is a non-increasing sequence of natural numbers,* $\{\lambda_i\}_{i\geq 1}$*, whose sum is* $n$*.*

Question 1.10. *In how many ways can a natural number* $n$ *be partitioned?*

Let $p(n)$ denote the number of partitions of $n$. Interestingly, no "nice" closed-form expression for $p(n)$ has been found. However, we will derive the generating function

$$\prod_{n=1}^{\infty}\left(1+x^{n}+x^{2n}+x^{3n}+\cdots\right)=\sum_{n=0}^{\infty}p\left(n\right)x^{n}$$

which was given by Euler, later in the course.

REMARK 1.3. *In the year 1918, G.H Hardy and Srinivasa Ramanujan obtained an asymptotic expression (an expression which describes the limiting behavior) for* $p\left(n\right)$ *which is given by*

$$p\left(n\right) \quad \frac{1}{4n\sqrt{3}}\exp\left(\pi\sqrt{\frac{2n}{3}}\right).$$

*Oftentimes stating an asymptotic expression is also a valid counting technique.*

## 2. Counting Principles

We state the fundamental counting principles now.

(1) (Addition Principle): If $\{A_k\}_{k\geq 1}$ is a family of finite and pairwise disjoint sets then $|\bigcup_{k\geq 1}A_k|=\sum_{k\geq 1}|A_k|$.
(2) (Substraction Principle): If $A$ and $B$ are finite sets such that $B\subseteq A$ then $|A\backslash B|=|A|-|B|$.
(3) (Product Rule): If $A$ and $B$ are finite sets, then $|A\times B|=|A|\cdot|B|$.
(4) (Division Rule): For finite sets $A$ and $B$ if there exists a $d$-to-many function $f:A\to B$ then $|B|=|A|/d$.

In some sense, we've already been using these principles in disguise thus far. For instance;

QUESTION 2.1. *How many $k$-digit positive numbers are there?*

SOLUTION. Since we have 10 choices for the first $(k-1)$ digits and 9 choices for the last digit, by the product rule there are $9\cdot 10^{k-1}$ $k$ digit numbers.

This is a good point to recall the binomial theorem and prove it using a counting argument instead of the standard induction argument.

THEOREM 2.1 (Binomial Theorem).

$$\left(x+y\right)^{n}=\sum_{k=0}^{n}\left(\begin{array}{c}n\\ k\end{array}\right)x^{n-k}y^{k}.$$

PROOF. Consider the expansion of

$$\left(x+y\right)^{n}=\underbrace{\left(x+y\right)\cdots\left(x+y\right)}_{n\text{ factors}}.$$

In this expansion, there are $\left(\begin{array}{c}n\\ k\end{array}\right)$ ways to choose $x$ from exactly $k$ of the $n$ factors (which forces the choice of $y$ from the remaining $(n-k)$ factors). Since $k$ can range from 0 to $n$, applying the addition principle gives us the binomial theorem. $\square$

Now we state the multinomial theorem which generalizes the binomial theorem to expressions with more than two terms, allowing us to expand any power of a sum of multiple terms as a sum of products of those terms, each multiplied by what is called a multinomial coefficient which is denoted and defined as

$$\binom{n}{k_1, \cdots, k_m} := \frac{n}{k_1! \cdots k_m!}.$$

Next, we state an identity concerning multinomial coefficients which would help us prove the multinomial theorem.

Claim 2.1.

$$\binom{n}{k_1, \cdots, k_r} = \binom{n}{k_1}\binom{n-k_1}{k_2}\cdots\binom{n-k_1-\cdots-k_{r-1}}{k_r}$$

Proof. The proof follows simply by expanding the R.H.S. out.

$$\begin{aligned}\binom{n}{k_1}\cdots\binom{n-k_1-\cdots-k_{r-1}}{k_r} &= \frac{n!}{(n-k_1)!k_1!}\frac{(n-k_1)!}{(n-k_1-k_2)!k_2!}\cdots \\ &\quad \times \cdots \frac{(n-k_1-\cdots-k_{r-1})!}{(n-k_1-\cdots-k_r)!k_r!} \\ &= \frac{n!}{k_1!\cdots k_r!}\frac{(n-k_1)!\cdots(n-k_1-\cdots-k_{r-1})!}{(n-k_1)!\cdots(n-k_1-\cdots-k_{r-1})!} \\ &= \frac{n!}{k_1!\cdots k_r!} \\ &= \binom{n}{k_1, \cdots, k_r}\end{aligned}$$

□

We would expect the multinomial coefficient to count something. That is indeed the case, it counts the number of ways to arrange $n$ distinct objects into $m$ distinct bins such that the $i$-th bin contains $k_i$ elements. With this interpretation of multinomial coefficients, and Claim 2.1, mimicking the steps involved in the proof of the binomial theorem we get the multinomial theorem which is stated below.

Theorem 2.2 (Multinomial Theorem).

$$(x_1 + \cdots + x_r)^n = \sum_{k_1+\cdots+k_r=n} \binom{n}{k_1, \cdots, k_r} x_1^{k_1} \cdots x_r^{k_r}.$$

It is natural to expect a Pascal-like identity for multinomial coefficients as well. In this spirit, we state the following claim.

Claim 2.2.

$$\binom{n}{k_1, \cdots, k_r} = \binom{n-1}{k_1-1, k_2, \cdots, k_r} + \cdots + \binom{n-1}{k_1, k_2, \cdots, k_r - 1}.$$

Remark 2.1. *Notice how when $k_1 = k$, $k_2 = n-k$ and all the other $k_i$s are 0, we retrieve the Pascal's identity (Claim 1.3)*

This is a good point to introduce ourselves to yet another generalization of the binomial theorem which deals with the expansion of $(x+y)^n$ where $n$ is allowed to take complex values.

Theorem 2.3 (Generalized Binomial Theorem). *For all $n \in \mathbb{C}$ we have*

$$(1+x)^n = \sum_{k \geq 0} \binom{n}{k} x^k$$

Remark 2.2. *There are two well-known proofs of Theorem 2.3. One uses the idea of Taylor's series expansion, and the other uses the solution to a differential equation called the Legendre's equation. We state neither one of them here.*

In addition to the fundamental counting principles, there are a few more principles/techniques that we will use quite often in this course. One of them is the bijection principle. We state the principle formally and then give an example.

Theorem 2.4 (Bijection Principle). *If $S$ and $T$ are finite sets then $|S| = |T|$ if and only if there exists a bijection $f : S \to T$.*

Question 2.2. *Let $S$ be a finite set. A* ***permutation*** *of $S$ is a bijection from and to $S$. How many permutations of $[n] = \{1, 2, 3, \ldots, n\}$ exist?*

We introduce the cycle notation before outlining a solution. In the case where $n = 1$, it is clear that there is only one possible bijection, namely, $f_1 : \{1\} \to \{1\}$ such that $f_1(1) = 1$. We shall write $f_1$ as

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

In the case where $n = 2$, it is clear that there are two possible bijections. These are $g_1 : \{1, 2\} \to \{1, 2\}$ such that $g_1(1) = 1$, $g_1(2) = 2$, and $g_2 : \{1, 2\} \to \{1, 2\}$ such that $g_2(1) = 2$, $g_2(2) = 1$. We shall write $g_1$ and $g_2$ as

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

respectively. In essence, the first row in our notation lists the elements of the domain and the second row lists where the bijection sends these elements. Do you see why in the case where $n = 3$ the bijections can be written as

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$

Since the first row of any bijection in our notation is fixed and we have

$$n \cdot (n-1) \cdots 1 = n!$$

ways to fill out the second row, it clear that there are $n!$ permutations of $[n]$. Recall from Question 1.5 that we also have exactly $n!$ permutation matrices. By the bijection principle, we must have a bijection between the set of permutation matrices of order $n \times n$ and the set of permutations of $[n]$. The bijection is as follows. Corresponding to every permutation

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ f(1) & f(2) & \cdots & f(n) \end{pmatrix}$$

of $[n]$ we assign an $n \times n$ matrix where the entry at $(i, f(i))$ for $1 \le i \le n$ is 1, and all other entries are 0. More explicitly,

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \longleftrightarrow \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \longleftrightarrow \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix},$$

$$\vdots$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \longleftrightarrow \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}.$$

We outline a proof of an identity involving binomial coefficients using the bijection principle now.

Claim 2.3.

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n$$

Proof. Let $S = \{a_1, \dots, a_n\}$ be a set with $n$ elements. For each of the $2^n$ possible subsets $T = \{a'_1, \dots, a'_k\}$ of $S$ (where $k$ can be any value from 0 to $n$), there is a corresponding binary string of length $n$. In this string, the $i$th bit is 1 if the element $a_i$ is in the subset $T$, and 0 if it is not. For instance in the case where $S = \{0, 1\}$ we have the following correspondence

$$\begin{aligned} \emptyset &\longleftrightarrow 00 \\ \{0\} &\longleftrightarrow 10 \\ \{1\} &\longleftrightarrow 01 \\ \{0,1\} &\longleftrightarrow 11 \end{aligned}$$

□

Another counting technique that we've seen before in the proof of Pascal's identity (Claim 1.3) is called double counting. The broad idea is to count the same combinatorial object in two different ways. As an example, we give a proof of Claim 2.3 using the double counting technique.

Proof. It is clear that $2^n$ counts the number of $n$-bit binary strings. To prove the claimed identity, we count the number of $n$-bit binary strings once again in a different way. Notice how an arbitrary$n$-bit binary string has $0 \le i \le n$ number of 0s. Since there are $\binom{n}{0}$ choices when $i = 0$, $\binom{n}{1}$ choices when $i = 1$, and so on, by the addition principle we have proved the required identity. □

We have all seen a proof by induction of the following identity. Now we give a proof using a double-counting argument.

Claim 2.4. $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2} = \dbinom{n+1}{2}$

Proof. Consider the problem of counting the number of handshakes in a group of $n+1$ people, where each person shakes hands with every other person. The first person in the group shakes hands with everyone except themselves; this gives $n$ handshakes. The second person in the group shakes hands with everyone else; this gives $(n-1)$ handshakes (not $n$ because we've already counted the handshake between the first and second person). The third person in the group shakes hands with everyone; this gives $(n-2)$, and so on. In all we have $n+(n-1)+(n-2)+\cdots+1$ handshakes. Let's count again in a different way now. Notice how for each pair of people there is one handshake. Hence, there are $\binom{n+1}{2}$ handshakes in total. Putting the two observations together proves the claim. $\square$

Next, we introduce the "Stars-and-Bars" method. We shall outline how this method works with the help of a standard example.

Question 2.3. *For a given $n \in \mathbb{N}$ find the number of solutions to the equation $x_1 + \cdots + x_k = n$ where $n \in \mathbb{N}$ and $x_i \in \mathbb{Z}$?*

Solution. Recall how in the definition of partition (Definition 1.5), the order of the parts did not matter, i.e., the partitions $2+1$ and $1+2$ of 3 are the same to us. Partitions where the order does matter are called compositions. Compositions where 0s are allowed to be a part are called weak compositions. With these two definitions at hand, notice how there are two cases to deal with in *Question* 2.3.

(1) All $x_i > 0$; which is equivalent to counting the number of compositions of $n$ into $k$ parts.
(2) All $x_i \geq 0$; which is equivalent to counting the number of weak compositions of $n$ into $k$ parts.

Let $x_1 + \cdots + x_k = n$ be a composition (may or may not be weak) of $n$ into $k$ parts. Now we replace each $x_i$ with $x_i$ consecutive stars and each $+$ with a bar. For instance, the composition $5+3+1$ would be written as

$$* * * * * \mid * * * \mid *$$

and the (weak) composition $5+3+1+0$ would be written as

$$* * * * * \mid * * * \mid * \mid |.$$

Since every composition (may or may not be weak) of $n$ into $k$ parts is completely determined by an arrangement of $n$ stars and $k-1$ bars, it is clear that we have $\binom{n+k-1}{k-1}$ weak compositions and $\binom{n-1}{k-1}$ compositions.

Remark 2.3. *Notice how every weak composition $n = d_1 + \cdots + d_k$ of $n$ into $k$ parts naturally corresponds to the monomial degree $n$ in $k$ variables given by $x_1^{d_1} \cdots x_k^{d_k}$.*

We conclude this section by stating an informal technique called proof by pictures (or proof without words). This technique involves the use of visual representations to demonstrate the validity of a mathematical statement. Although in the strictest sense of things a proof by pictures is not a counting principle, it often helps develop intuition. For instance, we state such a proof for Claim 2.4.

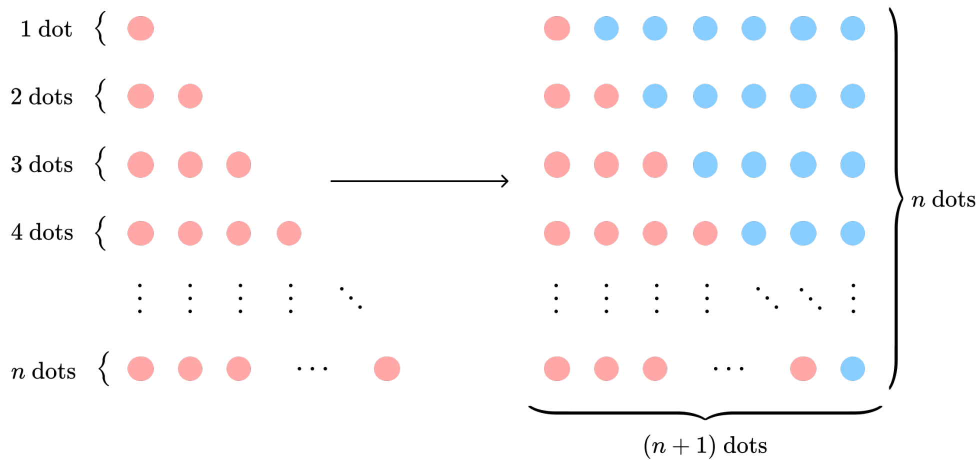


FIGURE 9

PROOF. In Figure 9 notice how there are $1+2+3+\cdots+n$ dots colored in red, $1+2+3+\cdots+n$ dots colored in blue, and hence $2(1+2+3+\cdots+n)$ total. It is also true that in our arrangement there are $n(n+1)$ dots in total. Now the identity follows. $\square$

Yet another fundamental counting principle is the pigeon-hole principle. Given its importance, we will dedicate the upcoming section to explore it in detail.

REMARK 2.4. *We have seen how the principles we've discussed so far can be useful in proving combinatorial identities. Determining how to derive these identities is not always straightforward. However, a common technique involves running computer simulations to generate a sequence of values corresponding to the first few natural numbers. Tools like OEIS, RATE (in Mathematica), and Guess (in SageMath and Maple) can then be used to conjecture a closed form for the sequence.*

## 3. The Pigeon-hole Principle

We state what is known as the pigeon-hole principle.

THEOREM 3.1 (Pigeon-hole Principle). *Let $A_1, \cdots, A_k$ be pair-wise disjoint finite sets. If*

$$|A_1 \cup A_2 \cup \cdots \cup A_k| > kr$$

*then there exists an $1 \leq i \leq k$ such that $|A_i| > r$.*

Even though the statement sounds trivial, it has immense applications. We state a few examples now.

CLAIM 3.1. *Given $b \in \mathbb{Z}$, the decimal expansion of $1/b$ is either finite or eventually periodic.*

PROOF. Each step in the long division of 1 by $b$ involves subtracting multiples of $b$ from powers of 10 until the remainder is 0 or repeats itself. Since each one of these remainders must be strictly less than $b$, we have $b$ possible remainders, namely, $0, 1, 2, \cdots, b-1$. If the remainder becomes 0 at some step, the division terminates, in which case the decimal expansion is finite. Next, consider the case where the division does not terminate. By the pigeon-hole principle, since there are only $b$ possible remainders and more than $b$ steps, at least one of the remainders must repeat. This proves our claim because when a remainder repeats, the sequence of digits that follow in the decimal expansion must repeat as well. $\square$

Claim 3.2. *Let $S = \{1, 2, \cdots, 2n\}$. Any $n+1$ element subset $K$ of $S$ has two co-prime numbers and two numbers such that one is an integer multiple of the other.*

Proof. Consider the $n$-element subset $\{(1,2), (3,4), \cdots (2n-1, 2)\}$ of $S \times S$ and notice how if we choose $n+1$ numbers from $\{0, \cdots, 2n\}$ then by the pigeon-hole principle we are guaranteed to pick two numbers from the same pair in $S \times S$. Since each element in the said subset is a pair of consecutive numbers, their gcd is necessarily 1. Next, notice how every element in the set $\{1, 2, \cdots, 2n\}$ can be written in the form $2^a b$ where $b$ is odd by factoring out as many 2s as possible. Since there are only $n$ odd numbers in these set $\{1, 2, \cdots, 2n\}$, we are left with $n$ choices of $b$. Next, we construct $b$ subsets of $S$, say $S_k \subseteq S$ such that $i \in S_k$ if and only if $i = 2^k b$. Finally, by the pigeon-hole principle, if one were to pick $n+1$ elements from $S$ at least 2 of them would be in the same $S_k$, i.e, there always exists a pair of numbers such that one divides the other. $\square$

Claim 3.3. *If $S$ is an $n+1$-element set of natural numbers, then there exists a pair of numbers in $S$ whose difference is divisible by $n$.*

Proof. Notice how the remainder obtained upon dividing any element $a_i \in S = \{a_1, \cdots, a_{n+1}\}$ by $n$ must be one of $0, 1, \cdots, n-1$. Since there are $n+1$ $a_i$s but only $n$ possible remainders, by the pigeon-hole principle, at least two members (say $a_i$ and $a_j$)in $S$ must have the same remainder, i.e, $a_i = nq_1 + r$ and $a_j = nq_2 + r$ for some natural numbers $q_1$ and $q_2$ and $0 \leq r \leq n-1$. Clearly, $a_i - a_j = n(q_1 - q_2)$ is divisible by $n$. $\square$

Claim 3.4. *Given $m$ integers say $a_1, \cdots, a_m$ there exists $k$ and $l$ with $0 \leq k \leq l \leq m$ such that $m$ divides $a_{k+1} + a_{k+2} + \cdots + a_l$*

Proof. Consider sums of the form $\sum_{i=1}^{k} a_i$ for $1 \leq k \leq m$. If one of these sums is divisible by $m$, i.e, their remainder upon division by $m$ is 0 then we are done. If not, we have $m-1$ possible remainders for the said sums, namely, $1, 2, \cdots m-1$. Since there are $m$ possible sums and only $m-1$ possible remainders at least two of these sums share the same remainder. More specifically, for some $1 \leq r < s \leq m$ and integers $q_1, q_2$ we have $\sum_{i=1}^{r} a_i = nq_1 + r$ and $\sum_{i=1}^{s} a_i = nq_2 + r$. But then clearly, $\sum_{i=r+1}^{s} a_j = n(q_1 - q_2)$ is divisible by $n$ and we are done. $\square$

Question 3.1. *In a party of $n$ people, where some (maybe all, maybe none) shake hands with one another, show that there are at least two people who shake the same number of hands.*

Solution. Consider the case where each one of the $n$ people shakes hands at least once. The number of handshakes possible for everybody then is $n-1$. Since there are $n$ people in the party, by the pigeon-hole principle our claim holds true. If we don't assume that everyone shakes at least one hand then we can separate the $n$ people between $k$ people who don't shake any hands and $m$ people who do, and the problem is reduced to the last case with $n = m$.

Question 3.2. *In a group of* 6 *people where each pair of people are either friends or enemies, show that there are* 3 *people who are either all friends or all enemies.*

Solution. Let $A, B, C, D, E$ and $F$ be a group of 6 friends. Pick $A$, and notice how each one of the remaining 5 people is either a friend of $A$ or an enemy of $A$. Now, by the pigeon-hole at least three of them (say $B, C$ and $D$) are either friends of $A$, or enemies of $A$. We consider the case where they are friends of $A$ and the other case will follow similarly. If any one of the pairs $B$-and-$C$, $C$-and-$D$, or, $B$-and-$D$ are friends, then we have found a group of 3 people who are all friends. If neither one of the pairs $B$-and-$C$, $C$-and-$D$, or, $B$-and-$D$ are friends, then we have found a group of 3 people who are all enemies.

We state an interesting graph-theoretic (recall Definition 1.2) restatement of Question 3.2 now. Let the group of 6 be represented as vertices in a graph. For each pair of these vertices, let there be an edge connecting them, which is colored red if the two represented people are enemies and blue if they are friends. Now Question 3.2 asserts that no matter how you color the (edges of the) graph, there is always a "triangle" of the same color in the graph.

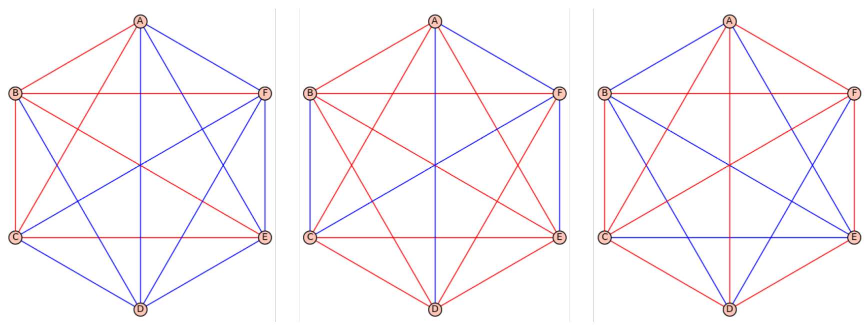


Figure 10. 3 randomly colored examples, each of which has either a red triangle or a blue triangle.

Remark 3.1. *Figure 10 makes it clear that in some sense we are looking for the appearance of a uniformly colored triangle sitting inside the (bigger) graph. Ramsey theory, a branch in the field of combinatorics, focuses precisely on questions of this kind.*

We conclude this section by stating two definitions that will turn out to be useful in the upcoming sections.

Definition 3.1 ($k$-Permutation). *A $k$-permutation on $[n] = \{1, 2, \cdots, n\}$ is a bijection between two $k$-element subsets of $[n]$.*

Definition 3.2 (Circular $k$-Permutation). *$k$-permutations $\pi$ and $\sigma$ are called circular equivalents if there exists a shift $s \in [k]$ such that $i + s \equiv j (mod\ k)$.*

## 4. The Principle of Inclusion-Exclusion

Recall that for finite sets $A$ and $B$, the cardinality of their union is given by $|A \cup B| = |A| + |B| - |A \cap B|$. This formula, which adds (includes) the sizes of $A$ and $B$ while subtracting (excluding) the overcount in $|A \cap B|$ encapsulates the essence of the inclusion-exclusion principle. We start with a useful generalization of the motivating formula we just discussed.

THEOREM 4.1 (Inclusion-Exclusion Principle). *Given finite sets* $A_1, \cdots, A_k$,

$$|\bigcup_{i=1}^{k} A_i| = \sum_{n=1}^{k} (-1)^{n+1} \left( \sum_{1 \leq i_1 < \ldots < i_n \leq k+1} |A_{i_1} \cap \cdots A_{i_n}| \right).$$

To see how useful this result is, we state two examples.

QUESTION 4.1. *How many of the first* 1000 *natural numbers are not divisible by* $2, 3$ *or* 6*?*

SOLUTION. Out of the first 1000 natural numbers, notice how there are $\lfloor 1000/n \rfloor$ numbers which are divisible by $n$. Now, by the inclusion-exclusion principle we have

$$1000 - 500 - 333 - 200 + \lfloor 1000/6 \rfloor + \lfloor 1000/15 \rfloor + \lfloor 1000/10 \rfloor - \lfloor 1000/30 \rfloor = 266$$

natural numbers which are not divisible by $2, 3$ or 6.

Question 4.1 motivates an equivalent version of Theorem 4.1 which we shall now state and prove.

THEOREM 4.2 (Inclusion-Exclusion Principle). *For* $1 \leq i \leq$, *let* $A_i$ *be subsets of a set* $X$. *Then the number of elements of* $X$ *that do not lie in any of the* $A_i$*s is given by*

$$\sum_{I \subseteq \{1, \cdots, n\}} (-1)^{|I|} |A_I|$$

*where* $A_I = A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}$ *for* $I = \{i_1, \cdots, i_k\}$.

PROOF. We argue combinatorially. Let $S$ be a finite set. Let $P_1, P_2, \ldots, P_n$ be $n$ properties of objects in $S$. Additionally, let $A_i$ denote that subset of $S$ whose elements satisfy the $P_i$-th property. Consider

$$|A_1^c \cap \cdots \cap A_n^c|,$$

i.e, the number of objects in $S$ with none of the properties. Let $x$ be an object satisfying none of the properties. The net contribution of such an $x$ to

$$|S| - \sum_{I \subseteq \{1, \ldots, n\}} (-1)^{|I|} |A_I|$$

is 1. Similarly, the contribution of an element satisfying exactly $0 < k \leq n$ properties is 0. More precisely,

$$|A_1^c \cap \cdots \cap A_n^c| = |S| - \sum_{I \subseteq \{1, \ldots, n\}} (-1)^{|I|} |A_I|.$$

Now the proof boils down to applying De-Morgan's laws. Notice how

$$\begin{aligned} |A_1 \cup \cdots \cup A_n| &= |S| - |(A_1 \cup \cdots A_n)^c| \\ &= |S| - |A_1^c \cap \cdots A_n^c|. \end{aligned}$$

Putting both of our observations together proves our claim. $\square$

Now we state a corollary of Theorem 4.2.

CLAIM 4.1. *The number of surjective mappings from* $[n] = \{1, 2, 3, \cdots, n\}$ *to* $[k] = \{1, 2, 3, \cdots, k\}$ *is given by*

$$\sum_{i=0}^{k}(-1)^i \binom{k}{i} (k-i)^n$$

PROOF. We mimic the setting of Theorem 4.2. Choose $X$ to be the set of all mappings from $[n]$ to $[k]$, and $A_i$ to be the set of all maps from $[n]$ to $[k]$ such that $i$ is not in the range. Notice how $|X| = k^n$, $|A_i| = (k-1)^n$, and $|A_I| = |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_j}| = (k-j)^n$. Now the claim follows. $\square$

REMARK 4.1. *Since every surjection from* $[n]$ *to* $[n]$ *is necessarily a bijection, and by Question 1.5 we know that there are* $n!$ *bijections from* $[n]$ *to* $[n]$*, we've proved a combinatorial identity, namely that*

$$\sum_{i=0}^{n} \binom{n}{i} (n-i)^n = n!.$$

We now give an application of the inclusion-exclusion principle in number theory. More specifically, consider Euler's $\varphi$ function which is defined as

$$\begin{aligned}\varphi :&\mathbb{N} \to \mathbb{N}\\ n &\rightsquigarrow |\{x \in \mathbb{N} : x < n, \gcd(x, n) = 1\}|.\end{aligned}$$

For instance, $\varphi(1) = |\{1\}| = 1$, $\varphi(2) = |\{1\}| = 1$, $\varphi(3) = |\{1, 2\}| = 3$, and so on. In essence, $\varphi(n)$ counts the positive integers which are less $n$ and are co-prime to $n$. It is clear that for a prime $p$, $\varphi(p) = p - 1$. More generally for a natural number $r > 0$ we also have

$$\varphi(p^r) = p^r - \left\lfloor \frac{p^r}{p} \right\rfloor = p^r - p^{r-1}.$$

THEOREM 4.3. *For* $n = p_1^{a_1} \cdots p_k^{a_k}$ *we have*

$$\varphi(n) = n\left(1 - \frac{1}{p_1}\right) \cdots \left(1 - \frac{1}{p_k}\right)$$

PROOF. We are interested in counting all the numbers $a$ in the set $[n] = \{1, 2, 3, \cdots, n\}$ is relatively prime to $n$. This happens if and only if $a$ is not divisible by any of the primes $p_1, \cdots, p_k$. Now, let $A_i$ be the subset of $[n]$ that contains all numbers which are divisible $p_i$. By our setup, it is clear that

$$\begin{aligned}\varphi(n) &= |[n] \backslash \bigcup_{i=1}^{n} A_i|\\ &= n - |\bigcup_{i=1}^{n} A_i|\\ &= n - \sum_{I_{\neq\emptyset} \subseteq [m]} (-1)^{|I|+1} |\bigcap_{i \in I} A_i| \quad \text{(By Theorem 4.2)}\end{aligned}$$

Notice how $\cap_{i\in I} A_i$ contains elements of $[n]$ which are not divisible by any $p_i$ where $i \in I$, and hence also not divisible by $\prod_{i\in I} p_i$. It follows that

$$\begin{aligned}
&|\cap_{i\in I} A_i| = \frac{n}{\prod_{i\in I} p_i} \\
&\implies \varphi(n) = n - \sum_{I\neq\emptyset\subseteq[m]} (-1)^{|I|+1} \frac{n}{\prod_{i\in I} p_i} \\
&\implies \varphi(n) = n\left(1 + \sum_{I\neq\emptyset\subseteq[m]} (-1)^{|I|} \frac{1}{\prod_{i\in I} p_i}\right) \\
&\implies \varphi(n) = n\left(1 + \sum_{I\neq\emptyset\subseteq[m]} \prod_{i\in I} \frac{-1}{p_i}\right) \\
&\implies \varphi(n) = n\prod_{i=1}^{m}\left(1 - \frac{1}{p_i}\right)
\end{aligned}$$

□

We conclude this section by introducing derangements.

QUESTION 4.2. *Suppose $n$ people go to a party and each person is wearing a different hat. In how many ways can the $n$ people return from the party such that nobody is wearing their original hat?*

SOLUTION. We are counting the number of permutations on $[n]$ with no fixed points. Let $A_j$ be the set of permutations which consist of all permutations such that $j$ is a fixed point. Now the inclusion-exclusion principle the number of permutations with no fixed points are

$$D_n := |\underbrace{A_1^c}_{1 \text{ is not fixed}} \cup \underbrace{A_2^c}_{2 \text{ is not fixed}} \cup \underbrace{A_3^c}_{3 \text{ is not fixed}} \cup \cdots \underbrace{\cup A_n^c}_{n \text{ is not fixed}}|.$$

Since it is clear that $|A_j| = (n-1)!$, we have

$$|A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_k}| = (n-k)!.$$

Once again by the inclusion-exclusion principle

$$\begin{aligned}
D_n &= n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \cdots + (-1)^n \binom{n}{n} 0! \\
&= n! - \frac{n!}{1!} + \frac{n!}{2!} - \cdots + (-1)^n \frac{n!}{n!}
\end{aligned}$$

CHAPTER 2

# Generating Functions

This chapter builds on our earlier discussion of generating functions from Chapter 1 (see Definition 1.4). Among the various types used in combinatorics and other fields like number theory, we will focus on two: ordinary generating functions and exponential generating functions

## 1. Ordinary Generating Functions

We start with a few examples.

EXAMPLE 1.1. *Corresponding to the constant sequence of* $1$*s, the generating function is* $1 + x + x^2 + x^3 + \cdots = \dfrac{1}{1-x}$

EXAMPLE 1.2. *Corresponding to the sequence* $1, 2, 3, \cdots$*, the generating function is* $1 + 2x + 3x^2 + 4x^3 + \cdots = 1/(1-x)^2$*. Notice how equality follows from the fact that* $1 + 2x + 3x^2 + 4x^3 + \cdots$ *is the formal derivative of the generating function we obtained in the previous example.*

More often than not generating functions are used to solve recurrences. For instance, consider the following question.

QUESTION 1.1. *Find a closed form expression for the recurrence given by* $a_{n+1} = 2a_n + 1$ *where* $a_0 = 0$.

SOLUTION. Let $A(x) = a_0 + a_1 x + a_2 x^2 + \cdots$ be the generating function corresponding to the sequence. Notice how

$$\begin{aligned}
&\sum_{n\geq 0} a_{n+1}x^n = 2\sum_{n\geq 0} a_n x^n + \sum_{n\geq 0} x^n \\
&\implies \frac{A(x)}{x} = 2A(x) + \frac{1}{1-x} \\
&\implies A(x) = \frac{x}{(1-x)(1-2x)} \\
&\implies A(x) = \frac{1}{1-2x} - \frac{1}{1-x} \\
&\implies A(x) = (1 + 2x + 4x^2 + 8x^3 + \cdots) - (1 + x + x^2 + x^3 + \cdots) \\
&\implies A(x) = x + 3x^2 + 7x^3 + \cdots
\end{aligned}$$

Now $a_n$ is just the coefficient of $x^n$ in $A(x)$.

We state one more example. Recall how with Question 1.9, we found the generating function for the sequence of Fibonacci numbers. We are now interested in finding a closed form of numbers in this sequence.

SOLUTION. Let $r_1, r_2 = (-1 \pm \sqrt{5})/2$ be the roots of the polynomial $1 - x - x^2$ and notice how

$$\begin{aligned}
F(x) &= \frac{1}{1-x-x^2} \\
&= \frac{1}{(r_1 - x)(r_2 - x)} \\
&= \frac{1}{(r_1 - x)(r_2 - r_1)} + \frac{1}{(r_2 - x)(r_1 - r_2)} \\
&= \frac{1}{\sqrt{5}} \left( \frac{1}{r_2 - x} - \frac{1}{r_1 - x} \right) \\
&= \frac{1}{\sqrt{5}} \left( \frac{1/r_2}{1 - (x/r_2)} - \frac{1/r_1}{1 - (x/r_1)} \right) \\
&= \frac{1}{\sqrt{5}} \left( \frac{1}{r_2} \left( 1 + \frac{x}{r_2} + \frac{x^2}{r_2^2} + \cdots \right) - \frac{1}{r_1} \left( 1 + \frac{x}{r_1} + \frac{x^2}{r_1^2} + \cdots \right) \right) \\
&= \frac{1}{\sqrt{5}} \left( \frac{1}{r_2} + \frac{x}{r_2^2} + \frac{x^2}{r_2^3} + \cdots - \frac{1}{r_1} - \frac{x}{r_1^2} - \frac{x^2}{r_1^3} - \cdots \right) \\
&= \frac{1}{\sqrt{5}} \left( \left( \frac{1}{r_2} - \frac{1}{r_1} \right) + \left( \frac{1}{r_2^2} - \frac{1}{r_1^2} \right) x + \left( \frac{1}{r_2^3} - \frac{1}{r_1^3} \right) x^2 + \cdots \right)
\end{aligned}$$

Now, the $n$-th Fibbonaci number is just the coefficient of $x^n$ in $F(x)$.

Oftentimes we are interested in computing the product of two generating functions. To this end, consider the following result due to Cauchy.

CLAIM 1.1 (Cauchy Product). *Let* $A(x) = \sum_{n \geq 0} a_n x^n$ *and* $B(x) = \sum_{n \geq 0} b_n x^n$ *be two ordinary generating functions. Their product* $C(x)$, *is then given by* $A(x)B(x) = \sum_{n \geq 0} c_n x^n$ *where*

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

CHAPTER 3

# The Art of Combinatorial Thinking

## 1. Binomial Coefficients

This section aims to push forward our familiarity with binomial coefficients which we have gained from the previous chapter. We do so by outlining proofs of a few standard combinatorial identities.

Claim 1.1.

$$\sum_{k=1}^{n} k \binom{n}{k}^2 = n \binom{2n-1}{n-1}$$

Proof. Let $J$ be a collection of $2n$ objects which is partitioned into 2 equal sub-collections $J_1$ and $J_2$. Now, we want to make a choice of $n$ elements from $J$ which includes a distinguished element, say $x$ from $J_1$ (or equivalently $J_2$). Since there are $n$ ways to choose the said distinguished element, and $\binom{2n-1}{n-1}$ ways to choose the rest of the $n-1$ elements, in all we have

$$n \binom{2n-1}{n-1}$$

ways to make the said choice. Alternatively, we can also choose $k$ elements from $J_1$ in $\binom{n}{k}$ ways, $n-k$ elements from $J_2$ in $\binom{n}{n-k}$ ways, and then choose the distinguished element from $J_1$ (or equivalently $J_2$) in $1 \leq k \leq n$ ways. Thus far, we have proved

$$\sum_{k=1}^{n} k \binom{n}{k} \binom{n}{n-k} = n \binom{2n-1}{n-1}.$$

Since by Remark 1.2

$$\binom{n}{k} = \binom{n}{n-k},$$

we are done. □

Claim 1.2.

$$\sum_{k=1}^{n} k \binom{n}{k} = n2^{n-1}$$

Proof. In a class of $n$ students, a football coach wants to choose students to form a football team of $1 \leq k \leq n$ players. Additionally, the coach also wants one of $k$ students to be the captain of the team. Since the representative can be chosen in $n$ ways and the remaining set of students can be chosen in $2^{n-1}$ ways, the team can be chosen in $n2^{n-1}$ ways. Alternatively, the coach can select $k$ players out of

$n$ in $\binom{n}{k}$ ways, and then choose the captain in $k$ ways. Since $0 < k \leq n$, we are done. □

Claim 1.3.

$$\sum_{i=1}^{n} i(n-1) = \sum_{i=1}^{n} \binom{i}{2} = \sum_{i=0}^{n-2} \binom{n-i}{2} = \binom{n+1}{3}$$

Proof. It is clear that $\binom{n+1}{3}$ is the number of ways to choose 3 numbers, say $a_1 < a_2 < a_3$, from the $n+1$-element set $\{a_1, \cdots, a_{n+1}\}$. Now we argue based on which one of these three is chosen first. If □

Claim 1.4. *For all $n \geq r \geq 0$ we have*

$$\binom{r}{r} + \binom{r+1}{r} + \cdots + \binom{n}{r} = \binom{n+1}{r+1}.$$

Proof. We give a double-counting argument. Noitce how $\binom{n+1}{r+1}$ counts the number of ways to choose $(r+1)$-element subsets of $[n+1] = \{1, \ldots, n+1\}$. A choice of the said subsets can also be made by first fixing the largest of the $r+1$ numbers. This is given by $\binom{r}{r} + \cdots + \binom{n}{r}$. □

Claim 1.5. $\binom{2n}{n} < 4^n$

Proof. The claim follows from the simple observation that

$$4^n = (1+1)^{2n} = \sum_{i=0}^{2n} \binom{2n}{i} = \binom{2n}{n} + \underbrace{\sum_{0 \leq i \leq 2n, i \neq n} \binom{2n}{i}}_{>0}.$$

□

Claim 1.6 (Chu-Vandermonde Identity).

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$$

Proof. Once again, by Remark 1.2, notice how

$$\sum_{k=1}^{n} \binom{n}{k}^2 = \sum_{k=1}^{n} \binom{n}{k} \binom{n}{n-k}.$$

Now we argue combinatorially. Let $J$ be a collection of $2n$ objects which is partitioned into 2 equal sub-collections $J_1$ and $J_2$. Any choice of $n$ elements from $J$ involves choosing $k$ elements from $J_1$ and $n-k$ elements from $J_2$ where $0 \leq k \leq n$. □

Claim 1.7. *For all $n > 0$.*

$$\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$$

PROOF. We give a double-counting argument. Notice how $3^n$ is the number of $n$-bit ternary strings. Each such string will have $n-k$-many 0s in it where $0 \leq k \leq n$. Such a string can be chosen in $\binom{n}{n-k} = \binom{n}{k}$ ways, and the remaining $k$ bits (which can now only either be 1s or 2s) can be chosen in $2^k$ ways. This completes the proof. $\square$

## 2. Catalan Numbers

This section introduces a set of problems that are seemingly distinct but share a common underlying sequence of numbers called the Catalan numbers. We shall show this commonality by setting up appropriate bijections between the said problems.

QUESTION 2.1. *Recall how with Question 1.8 we counted the number of lattice paths from* $(0,0)$ *to* $(m,n)$. *Count the number of lattice paths,* $a_1(n)$, *from* $(0,0)$ *to* $(n,n)$ *which never go below the* $x=y$ *diagonal.*

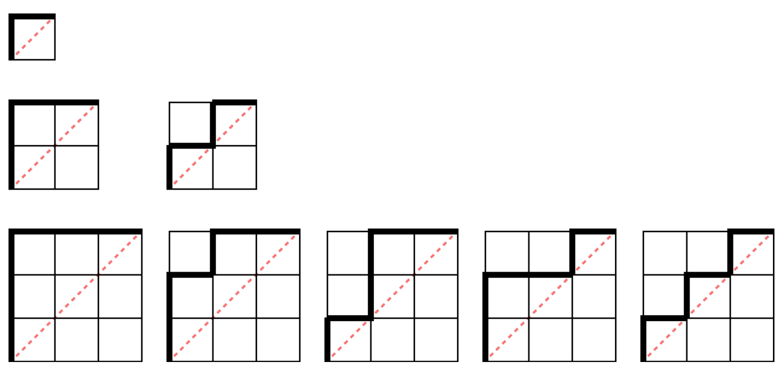

FIGURE 1. All possible lattice paths which don't go below the diagonal for cases $n = 1, 2, 3$

REMARK 2.1. *Notice that reflecting each path in Figure 1 across the main diagonal produces a lattice path that remains strictly below it. This bijection clearly shows that* $a_1(n)$, *the number of lattice paths that do not fall below the diagonal is the same as* $a_2(n)$, *the number of those that do not rise above it.*

QUESTION 2.2. *Count the number of ways* $a_3(n)$, *of filling a* $2 \times n$ *grid with elements from the set* $\{0, 1, 2, 3, \cdots, 2n\}$ *such that all elements are unique, increasing row-wise, and decreasing column-wise.*

| |
|---|
| 1 |
| 2 |

| | |
|---|---|
| 1 | 2 |
| 3 | 4 |

| | |
|---|---|
| 1 | 3 |
| 2 | 4 |

| | | |
|---|---|---|
| 1 | 2 | 3 |
| 4 | 5 | 6 |

| | | |
|---|---|---|
| 1 | 2 | 4 |
| 3 | 5 | 6 |

| | | |
|---|---|---|
| 1 | 3 | 4 |
| 2 | 5 | 6 |

| | | |
|---|---|---|
| 1 | 2 | 5 |
| 3 | 4 | 6 |

| | | |
|---|---|---|
| 1 | 3 | 5 |
| 2 | 4 | 6 |

FIGURE 2. All possible such arrangements of order $2 \times n$ for cases $n = 1, 2, 3$

SOLUTION. A bijection presents itself when Figure 1 and Figure 2 are compared. Namely, the entries in the first row of the $2 \times n$ grid correspond to when an $N$-step in our lattice path occurs, and the entries in the second row correspond to when an $E$-step in our lattice path occurs.

QUESTION 2.3. *Count the number of Dyck paths $a_4(n)$, from $(0, 0)$ to $(2n, 0)$. Where, by a Dyck path we refer to the path admitted by a sequence of up-moves, corresponding to $(i, j) \to (i + 1, j + 1)$ and down-moves, corresponding to $(i, j) \to (i - 1, j - 1)$, which does not go below the $x$-axis.*

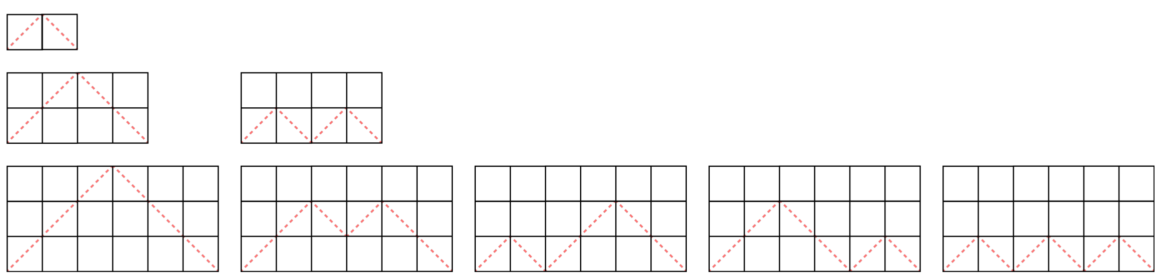

FIGURE 3. All possible Dyck paths from $(0, 0) \to (2n, 0)$ for $n = 1, 2, 3$

SOLUTION. Once again, a bijection presents itself when Figure 1 and Figure 3 are compared. Namely, the entries in the first row of the $2 \times n$ grid correspond to when an up-step in our Dyck path occurs, and the entries in the second row correspond to when a down-step in our Dyck path occurs.

QUESTION 2.4. *Count the number of ways $a_5(n)$, of joining $n$ non-intersecting chords on a circle marked with $2n$ points.*

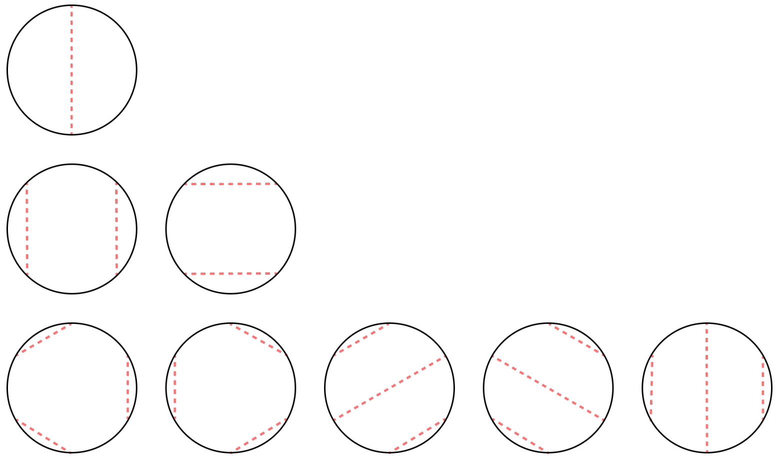

FIGURE 4. All possible such configurations for cases $n = 1, 2, 3$

SOLUTION. Once again, a bijection presents itself when Figure 2 and Figure 4 are compared. Namely, if $p_1, \cdots, p_{2n}$ are the marked points on the circle, then we join $p_i$ and $p_j$ if and only if $i$ occurs below $j$ in the arranged grid of numbers.

QUESTION 2.5. *Count $a_6(n)$, the number of legal sequences of $2n$ parentheses. Where, by a legal sequence of parentheses we mean one in which the parentheses can be properly matched, i.e., each opening parenthesis should be matched to a closing one that lies further to its right. We count these for $n = 1, 2, 3$.*

$$\begin{aligned}
n = 1: &\quad () \\
n = 2: &\quad (()) \,, () () \\
n = 3: &\quad ((())) \,, () (()) \,, (()) () \,, () () ()
\end{aligned}$$

*Once again, notice how corresponding to each grid in Figure 2 there is a legal sequence of parentheses. Namely, for each entry in the first row, we open a parenthesis, and close it for each entry in the second row.*

QUESTION 2.6. *Count $a_7(n)$, the number of non-crossing partitions on the set $[2n] := \{1, 2, 3, \cdots, 2n\}$. Where by a non-crossing partition on $[2n]$ we refer to an arrangement of $2n$ points on a line, with $n$ non-intersecting arcs joining them.*

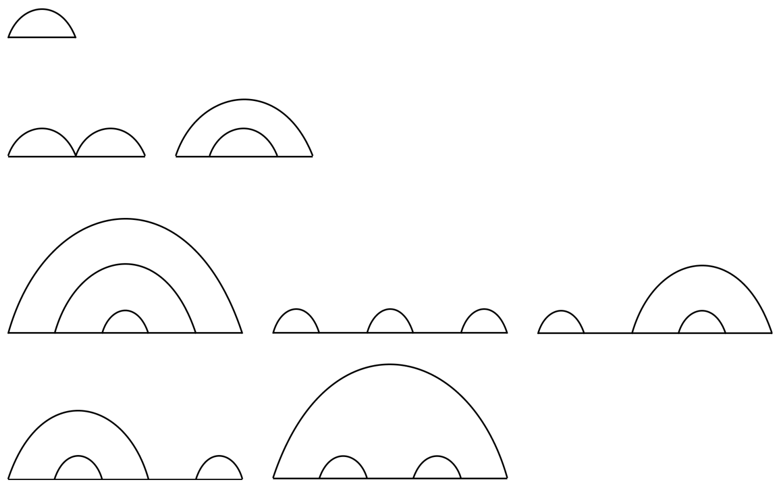

FIGURE 5. All possible non-crossing partitions on $[2n]$ for $n = 1, 2, 3$

SOLUTION. Once again, a bijection presents itself when Figure 2 and Figure 5 are compared. Namely, if $p_1, \cdots, p_{2n}$ are the marked points on the line, then we join $p_i$ and $p_j$ if and only if $i$ occurs below $j$ in the arranged grid of numbers.

To summarize, by the many bijections we have set up, $C(n) := a_1(n) = a_2(n) = \cdots = a_7(n)$ for all $n \geq 0$. Additionally, we have also seen that $A(n)$ takes values $1, 1, 2$ and $5$ for when $n = 0, 1, 2$ and $3$ respectively. Motivated readers may check why and how $C(4) = 14$, $C(5) = 42$, and so on, but this is not how we want to proceed. Consider the following definitions.

DEFINITION 2.1 (Triangulation). *A triangulation of a convex polygon with $n+2$ vertices $\mathcal{P}_{n+2}$, is a set of $n-1$ diagonals that do not cross each other in the interior of $\mathcal{P}_{n+2}$.*

REMARK 2.2. *For instance, the 3-gon (triangle) has no triangulations, but the 4-gon (square/rectangle) has 2.*

DEFINITION 2.2 (Catalan Numbers). *The sequence $C(n)$ which counts the the number of triangulations of $\mathcal{P}_{n+2}$ is called the sequence of Catalan numbers.*

THEOREM 2.1. *For all $n \geq 0$ we have*

$$C(n+1) = \sum_{k=0}^{n} C(k)C(n-k)$$

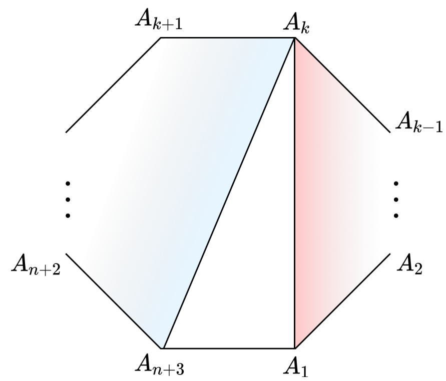


FIGURE 6

PROOF. Let $\mathcal{P}_{n+3}$ be an $n+3$ convex polygon with vertices $A_1, \cdots, A_{n+3}$. Next, pick an arbitrary vertex $A_k$ and consider the triangle $\Delta A_1 A_{n+3} A_k$. See Figure 6 and notice how this choice splits $\mathcal{P}_{n+3}$ into two convex polygons. One with $k$ vertices (marked red) and the one with $n-k+4$ vertices (marked blue). Since $k$ is allowed to vary from 2 to $n+2$, by Definition 2.2 we have $\sum_{k=2}^{n+2} C(k-2)C(n-k+2)$ ways to triangulate $\mathcal{P}_{n+3}$. Finally, shifting the index of summation by 2 grants the required result. □

CLAIM 2.1.

$$C(n) = \frac{1}{n+1}\binom{2n}{n}$$

PROOF. Let

$$f(x) = \sum_{n=0}^{\infty} C(n)x^n$$

be the generating function corresponding to the Catalan numbers. By Theorem 2.1,

$$\begin{aligned} f(x)^2 &= C(0)^2 + (C(0)C(1) + C(1)C(0))x + \cdots + (C(0)C(n) \\ &\quad + C(1)C(n-1) + \cdots + C(n)C(0))x^n + \cdots \\ &= C(1) + C(2)x + \cdots + C(n+1)x^n + \cdots \\ &= \frac{C(x) - C(0)}{x}. \end{aligned}$$

Solving for $C(x)$ gives

$$C(x) = \frac{1 \pm \sqrt{1-4x}}{2x}.$$

Since $C(n)$ is always positive, we discard

$$C(x) = \frac{1 + \sqrt{1-4x}}{2x}.$$

and expand $\sqrt{1-4x}$ using Theorem 2.3 to see why

$$C(x) = \sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n} x^n.$$

□

Remark 2.3. *Catalan numbers come up in a huge class of counting problems. Richard Stanley, in his book, "Catalan Numbers" outlines* 214 *such problems. That being said, the many bijections we have outlined in this section are not always as easy to find. In such cases, the defining recurrence (Theorem 2.1) we have obtained is particularly useful. Consider the following example for instance.*

Question 2.7. *Count the number of binary trees with $n$ vertices.*

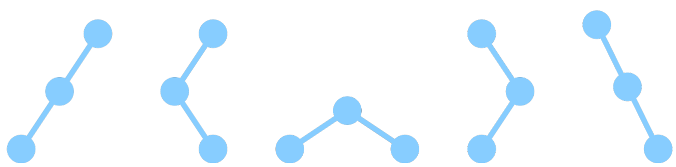

Figure 7. The 5 binary trees on 3 vertices.

Solution. Let $a_n$ denote the number of binary trees with $n$ vertices, where $n \geq 0$. Since the empty tree is the only binary tree with 0 vertices, it is clear why $a_0 = 1$. Similarly, it is also clear why $a_1 = 1$. Consider a binary tree $T$ (say) with $n$ vertices. The root of $T$ has $n-1$ children. For a choice of $0 \leq i \leq n-i-1$ let $T$ have $i$ children on the left sub-tree and $n-i-1$ children on the right. By our set-up, there are $a_i$ binary trees with $i$ vertices and $a_{n-i-1}$ binary trees with $n-i-1$ vertices. It follows that there are $a_i a_{n-i-1}$ binary trees with $i$ children on their left subtrees and $n-i-1$ children on their right. Thus, the total number of binary trees with $n$ vertices is given by

$$a_n = \sum_{i=0}^{n-1} a_i a_{n-i-1}.$$

Comparing with Theorem 2.1 we get that $a_n$ is counted by the Catalan numbers.

We are now interested in stating a recurrence for Catalan numbers which is different from the one we already stated. To this end consider the following theorem and its corollary.

Theorem 2.2. *Let $T_n$ be the number of triangulations of an $n$-gon, then for $n \geq 4$,*

$$(2n-6)T_n = n \sum_{k=3}^{n-1} T_k T_{n-k+2}$$

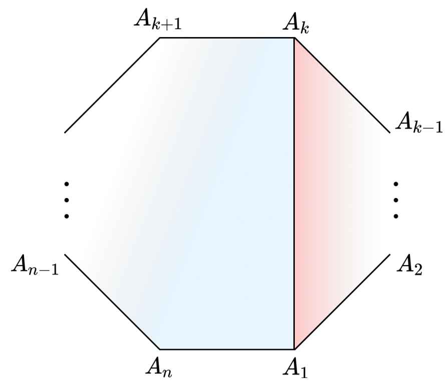


FIGURE 8

PROOF. Consider an $n$-gon with vertices $A_1, A_2, \ldots, A_n$. Conisder an arbitrary diagonal $\overline{A_1A_k}$ leaving $A_1$, where $2 < k < n$. See Figure 8 and notice how this partitions the $n$-gon into the $k$-gon $A_1A_2 \ldots A_k$ (marked red) and the $(n-k+2)$-gon $A_1A_kA_{k+1} \ldots A_n$ (marked blue). The $k$-gon can be triangulated in $T_k$ ways and the $(n-k+2)$-gon in $T_{n-k+2}$ ways. It follows that the $n$-gon can be triangulated in a total of $T_kT_{n-k+2}$ ways. Since $k$ is allowed to vary between 3 and $n-1$, it also follows that the total number of triangulations based at $A_1$ are given by

$$\sum_{k=3}^{n-1} T_kT_{n-k+2}.$$

Our choice of $A_1$ is not special. Since there $n$ choices of vertices where the triangulation can be based one might think the count is

$$n\sum_{k=3}^{n-1} T_kT_{n-k+2}.$$

However, we have overcounted. More specifically, the diagonals $\overline{A_iA_j}$ are counted twice. This gives;

$$n/2\sum_{k=3}^{n-1} T_kT_{n-k+2}.$$

Finally, since the $n$-gon has $n-3$ diagonals at every vertex and every triangulation uses $n-3$ diagonals, it follows that

$$(n-3)T_n = n/2\sum_{k=3}^{n-1} T_kT_{n-k+2}.$$

as required. □

A corollary follows naturally.

COROLLARY 2.1.

$$(2n-2)C_n = (n+2)\sum_{k=1}^{n-1} C_kC_{n-k}$$

CHAPTER 4

# Partitions

A first course in combinatorics typically focuses on two types of partitions: set partitions and integer partitions. We will begin with a brief discussion of set partitions, followed by a more in-depth exploration of integer partitions.

## 1. Set Partitions

Notice how there are 6 ways to partition the set $\{1, 2, 3, 4\}$ into 3 blocks. These are,

(1) $[1], [2], [3, 4]$
(2) $[1], [2, 3], [4]$
(3) $[1, 2], [3], [4]$
(4) $[1, 4], [2], [3]$
(5) $[1, 3], [2], [4]$
(6) $[2, 4], [1], [3]$

This kind of counting is generalized by what are called a Stirling numbers of the $2nd$ kind. More formally,

Definition 1.1. *A set partition of a finite set $B$ into $k$ "blocks" is a collection of $k$ subsets of $B$ say $B_1, \cdots, B_k$ such that*

(1) $\bigcup_{i=1}^{k} B_i = B$,
(2) $B_i \cap B_j = \emptyset$ *for all* $i \neq j$,
(3) *and none of the $B_i$'s are empty.*

Definition 1.2. *If $B = [n] = \{1, \cdots, n\}$ then a Stirling number of the 2nd kind $S(n, k)$, is the number of set partitions of $B$ into $k$ blocks.*

We take $S(0, 0)$ to be 1 by convention. Additionally, the fact that $S(n, n) = 1$, $S(n, 0) = 0$, and $S(n, 1) = 1$ follow immediately. Per usual we state a few identities concerning these numbers.

Claim 1.1. *For all $n, k \geq 0$, with $n \geq k$ we have $S(n, k) = S(n-1, k-1) + kS(n-1, k)$*

Proof. By the Definition 1.2, the L.H.S of our claim counts the set of partitions of $[n]$ into $k$ blocks. We give a double-counting argument; one, the partitions where $n$ is itself a block, two, the partitions where the block containing $n$ has a size of at least two. To count the partitions where $n$ is a block by itself, we can take $n$ out, choose a partition of $[n-1]$ into $k-1$ blocks in $(n-1, k-1)$ ways, and enlarge the chosen partition to obtain a partition of $[n]$ into $k$ blocks by adding $\{n\}$ as the $n$-th block. To count the partitions where the block containing $n$ has a size at least two, choose a partition of $[n-1]$ into $k$ blocks in $S(n-11, k)$ different ways, and for each of such choices create a partition of $[n]$ into $k$ blocks in $k$ different ways by placing $n$ inside one of the $k$ blocks. Putting all together, we see that the number of partitions of $[n]$ into $k$ blocks is $S(n-1,k-1)+kS(n-1,k)$, as required. $\square$

Claim 1.2. $S(n,2) = 2^{n-1}-1$

Proof. Each partition of $[n]$ into two blocks, say $\{B_1, B_2\}$, can be constructed by first choosing the subset $B_1$ in $2^n-2$ ways (since $B_1$ cannot be neither empty nor the whole set $[n]$), which forces $B_2$ to be equal to $[n]\backslash B_1$. Next, we divide our number of choices, $2^n-2$, by 2 to account for the fact that the order of the blocks inside the partition is irrelevant to see why the claim is true. $\square$

Claim 1.3. $S(n,n-1) = \left( \begin{array}{c} n \\ 2 \end{array} \right)$

Proof. Each partition of $[n]$ into $n-1$ blocks must contain exactly one block of size 2, which completely determines the rest of the blocks, namely the remaining $n-2$ blocks of size 1. Therefore the set of partitions of $[n]$ into two blocks is in bijection with the set of 2 element subsets of $[n]$. Now the claim follows. $\square$

Claim 1.4.

$$S(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\left( \begin{array}{c} k \\ j \end{array} \right) j^n$$

Proof. To count the number of surjective functions $f:[n]\to[k]$, we can first fix a set partition $\{B_1,\ldots,B_k\}$ of $[n]$ into $k$ blocks in $S(n,k)$ ways, then make a linear arrangement $w_1w_2\cdots w_k$ with the elements of $[k]$ in $k!$ ways, and then finally the set $f^{-1}(w_i)=B_i$ to see why there are $S(n,k)k!$ choices of $f$. By the virtue of Claim 4.1 we are done. $\square$

Definition 1.3 (Bell Numbers). *The number of all partitions of $[n]$ is called a bell number and is denoted by $B(n)$. More specifically,*

$$B(n) = \sum_{k=1}^{n} S(n,k)$$

We are interested in coming up with a recurrence of $B(n)$ which is independent of any $S(n,k)$s.

Claim 1.5.

$$B(n+1) = \sum_{k=0}^{n}\left( \begin{array}{c} n \\ n-k \end{array} \right) B(k)$$

Proof. By Definition 1.3, the L.H.S of our claim counts the set of partitions of $[n+1]$. We give a double-counting argument. For each $s$ in $[1,n+1]$, we can count the partitions of $[n+1]$ where the block $B$ containing $\{n+1\}$ has size $s$; first we choose in $\left( \begin{array}{c} n \\ s-1 \end{array} \right)$ ways the elements in $B$ that are different from $n+1$, and then create a partition of $[n+1]\backslash B$ in $B(n+1-s)$ ways. It follows that

$$\begin{aligned} B(n+1) &= \sum_{s=1}^{n+1} B(n+1-s) \\ &= \sum_{j=0}^{n} \binom{n}{n-j} B(j) \\ &= \sum_{j=0}^{n} \binom{n}{j} B(j) \end{aligned}$$

as required. □

## 2. Integer Partitions

Recall how with Definition 1.5 we defined integer partitions. The study of these partitions dates back to the work of Leonard Euler in the 18th century, who introduced generating functions to analyze them. Their study was then developed further through the work of mathematicians like Srinivasa Ramanujan and Major MacMachon, who revealed deep arithmetic and combinatorial properties. For instance, if $p(n)$ denotes the number of partitions of $n$, then the celebrated Ramanujan congruences state that

$$\begin{aligned} P(5n+4) &\equiv 0 \mod (5) \\ P(7n+5) &\equiv 0 \mod (7) \\ P(11n+6) &\equiv 0 \mod (11) \end{aligned}$$

In the absence of a closed form (recall Remark 1.3), we are interested in finding the generating function of $p(n)$.

Claim 2.1.

$$\prod_{n\geq 1}(1+q^n+q^{2n}+\cdots) = \sum_{n\geq 0} p(n)q^n.$$

Proof. We can expand the product on the L.H.S,

$$(1+q+q^2+q^3+\cdots)(1+q^2+q^4+q^6+\cdots)(1+q^3+q^6+q^9+\cdots)\cdots,$$

out by choosing one term from each factor in all possible ways. If we then collect like terms, the coefficient of $q^k$ will be the number of ways to choose one term from each factor so that the exponents of the said terms sum up to $k$. This is also what the R.H.S. counts. For instance $q^3$ can be obtained in the following ways

(1) Choose $q^3$ from the first bracket and 1 from every other bracket. This corresponds to the partition $3 = 1+1+1$.
(2) Choose $q$ from the first bracket, $q^2$ from the second bracket, and 1 from every other bracket. This corresponds to the partition $3 = 2+1$.
(3) Chose 1 from the first two brackets and $q^3$ from the third bracket. This corresponds to the partition $3 = 3$.

□

Next, we are interested in looking at two refinements of partitions which follow quite naturally from the arguments we used in the previous proof.

Claim 2.2. *Let $p_E(n)$ and $p_O(n)$ denote the number of partitions of $n$ into even and odd parts respectively. Then,*

$$\prod_{n=1,3,5,\cdots}(1+q^n+q^{2n}+\cdots)=\sum_{n\geq 0}p_O(n)q^n$$
$$\prod_{n=2,4,6,\cdots}(1+q^n+q^{2n}+\cdots)=\sum_{n\geq 0}p_E(n)q^n$$

Next, we state an interesting result, once again due to Euler.

Theorem 2.1 (Euler's Gem). *The number of partitions of $n$ into distinct parts say $p_D(n)$, is the same as the number of partitions of $n$ into odd parts, say $p_O(n)$.*

We will outline two different proofs of the result.

Proof. By Claim 2.2 we already know that the generating function for $p_O(n)$ is

$$\prod_{i=1}^{\infty}\frac{1}{1-q^{2i-1}}.$$

It is also clear (by arguments similar to the ones which were involved in the proof of *Claim* 2.1) that the generating function for $p_D(n)$ is

$$(1+q)(1+q^2)(1+q^3)\cdots=\prod_{i=1}^{\infty}(1+q^i).$$

Now to complete the proof it suffices to show that the two are equal. To this end, notice how

$$\begin{aligned}\prod_{i=1}^{\infty}(1+q^i)&=\prod_{i=1}^{\infty}(1+q^i)\frac{1-q^i}{1-q^i}\\&=\prod_{i=1}^{\infty}\frac{1-q^{2i}}{1-q^i}\\&=\frac{(1-q^2)(1-q^4)(1-q^6)\cdots}{(1-q)(1-q^2)(1-q^3)(1-q^4)\cdots}\\&=\prod_{i=1}^{\infty}\frac{1}{1-q^{2i-1}}\end{aligned}$$

□

In the spirit of giving a combinatorial proof, we want to setup a bijection called Glashier's bijection between the two types of partitions.

Proof. First, we setup a map that sends a partition into odd parts to a partition into distinct parts. The most natural thing to do is to merge any pairs of repeating parts into one part of double the size. We can repeat this procedure until all the parts are distinct. For instance $3+3+3+1+1+1+1\rightarrow(3+3)+3+(1+1)+(1+1)=6+3+2+2=6+3+(2+2)=6+4+3$. Next, we set up a map that sends a partition into distinct parts to a partition into odd parts. Once again, the most natural thing to do is split every occurrence of an even part into two equal parts. We can repeat this procedure until all the parts are odd. For instance $6+4+3\rightarrow(3+3)+(2+2)+3\rightarrow 3+3+3+1+1+1+1$. □

We state a generalization of Euler's gem now.

THEOREM 2.2. *The number of partitions where no part appears $d$ or more times is the same as the number of partitions where no part is divisible by $d$.*

PROOF. Let $p_1(n)$ denote the number of partitions of $n$ with no parts divisible by $d$. Let $p_2(n)$ denote the number of partitions of $n$ where no part appears $d$ or more times. Notice how the generating function for $p_1(n)$ is given by

$$\sum_{n=0}^{\infty} p_1(n)q^n = \prod_{n=1, d\nmid n}^{\infty} \frac{1}{1-q^n}$$

and that of $p_2(n)$ is given by

$$\begin{aligned}\sum_{n=0}^{\infty} p_2(n)q^n &= \prod_{n=1}^{\infty} \frac{1-q^{dn}}{1-q^n} \\ &= \frac{1-q^d}{1-q}\frac{1-q^{2d}}{1-q^2}\cdots\frac{1-q^{kd}}{1-q^k}\cdots .\end{aligned}$$

Finally, each term in the numerator cancels with the corresponding multiple of $d$ in the denominator and we are left with the generating function for $p_1(n)$. This completes the proof. ☐

REMARK 2.1. *One might ask in what sense is Theorem 2.2 a generalization of Theorem 2.1 To see this, notice how setting $d = 2$ returns the Euler's gem.*

A rather interesting corollary of Theorem 2.2 can be used to see why the binary representation of a number is unique. Consider the set of all partitions of $n$ which has parts only of size 1. The only such partition is $\underbrace{1 + \cdots + 1}_{n \text{ times}}$. Next, we are interested in merging the said parts to the point where the resultant partition has all parts distinct. It is also clear that any such sequence of merges will result in a partition where the parts occur as powers of 2. On the other hand, any partition that only has powers of 2 as its parts can be repeatedly split repeatedly to the point where the resultant partition has only 1s in it. This completes a neat proof of the uniqueness of a binary representation.

Although the proof of Euler's Gem is fairly straightforward, it is not clear how one might come up with such an identity. To this end, we shall try and guess an identity due to Leonard Rogers and Srinivasa Ramanujan. Consider those partitions of $1 \leq n \leq 10$ which have parts differing by at least 2. We list these in the table below.

| $n$ | # | Admissible partitions of $n$ |
|---|---|---|
| 1 | 1 | 1 |
| 2 | 1 | 2 |
| 3 | 1 | 3 |
| 4 | 2 | 4, 3+1 |
| 5 | 2 | 5, 4+1 |
| 6 | 3 | 6, 5+1, 4+2 |
| 7 | 3 | 7, 6+1, 5+2 |
| 8 | 4 | 8, 7+1, 6+2, 5+3 |
| 9 | 5 | 9, 8+1, 7+2, 6+3, 5+3+1 |
| 10 | 6 | 10, 9+1, 8+2, 7+3, 6+4, 6+3+1 |

TABLE 1. Partitions of $n$ into what are called 2-distinct parts.

Next, we attempt to construct a set $X$ such that the number of partitions of $n$ with parts in $X$ is the same as the one we have counted in Table 1. To this end, we make a series of observations.

(1) Since there should be one partition of 1 with parts in $X$, 1 must necessarily be a part of $X$.
(2) Since there should be one partition of 2 with parts in $X$, 1 is already in $X$, and $2 = 1+1$, we need not add 2 to $X$. Similarly, we have no additions to $X$ corresponding to 3 either.
(3) Since there should be two partitions of 4 with parts in $X$, 1 is in $X$, and $4 = 1 + 1 + 1 + 1$, we need one more partition. Hence, we add 4 to $X$ to take care of the partition $4 = 4$.
(4) Since there should be two partitions of 5 with parts in $X$, 1 and 4 are in $X$, and $5 = 1 + 1 + 1 + 1 + 1 = 1 + 4$, we need not add 5 to $X$.
(5) $\cdots$

Doing this exercise for the remaining numbers ($6 \leq n \leq 10$) makes it clear that we must also add 6 and 9 must also be added to $X$. A pattern presents itself, namely, that all the members have to admit 1 or 4 as a remainder upon division by 5. In summary, we have guessed (not proved) that the number of partitions of $n$ into 2-distinct parts is the same as the number of partitions of $n$ into parts which are congruent to $1, 4 \mod 5$.

As it turns out many partition identities are best explained using pictures. To this end we introduce a tool called Ferrer's diagrams. In the said diagram the parts of a partition are shown as rows of dots/squares.

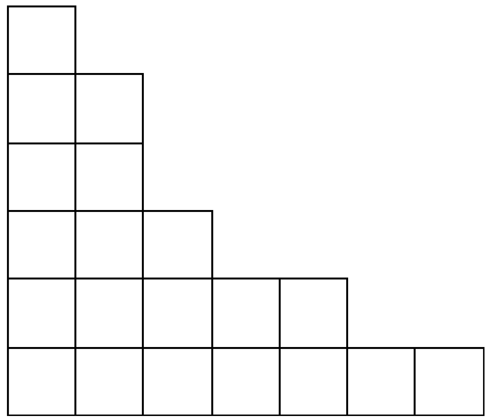

FIGURE 1. The Ferrer's diagram for $7 + 5 + 3 + 2 + 2 + 1$

It is clear that height of the Ferrer's diagram corresponds to the number of parts in the partition and that the width corresponds to the size of the largest part in the partition. To this end, consider the following result.

THEOREM 2.3. *The number of partitions of $n$ into atmost $i$ parts, each of which is atmost $j$ is the same as the number of partitions of $n$ into atmost $j$ parts, each of which is atmost $i$.*

PROOF. Let $\pi$ a partition of $n$ into at most $i$ parts each of which is at most $j$. The Ferrer's diagram corresponding to $\pi$ now has height at mmost $i$ and width at most $j$. Flipping the said diagram about its main diagonal results in a diagram that has height at most $j$ and width at most $i$. Finally, the partition corresponding to this flipped diagram (called the conjugate partition) is a partition of $n$ into atmost $j$ parts, each of which is atmost $i$. $\square$

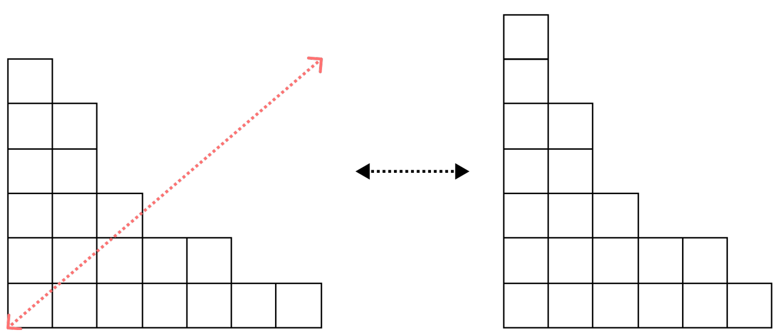

FIGURE 2. An example of the conjugate of $7 + 5 + 3 + 2 + 2 + 1$

Another useful pictorial construct is that of a Durfee square. We define the said square to be the largest one which can fit in the Ferrer's diagram corresponding to a partition.

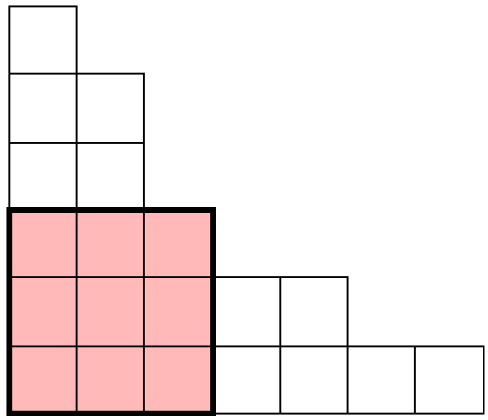

FIGURE 3. The Durfee square of order 3 in the partition $7+5+3+2+2+1$

(Keeping in mind Figure 3) notice how

(1) Every non-empty partition has a Durfee square of order $k$ (if nothing, you always have the Durfee square of order $k=1$).
(2) Partitions corresponding to the triangle above the Durfee square are the ones where each part is at most $k$.
(3) Partitions corresponding to the triangle below the Durfee square are the ones where the number of parts is at most $k$.

With these three observations at hand, an identity presents itself almost immediately. Namely,

CLAIM 2.3 (Jacobi's Identity).

$$\prod_{i=1}^{\infty} \frac{1}{1-q^i} = \sum_{i=0}^{\infty} \frac{q^{i^2}}{(1-q)^2(1-q^2)^2\cdots(1-q^i)^2}$$

Next, we state a remarkable result first proved by Euler in the year 1785.

THEOREM 2.4 (Euler's Pentagonal Number Theorem).

$$\begin{aligned}
\prod_{i=1}^{\infty}(1-q^i) &= \sum_{k=-\infty}^{\infty} (-1)^k q^{\frac{k(3k-1)}{2}} \\
&= 1 + \sum_{k=1}^{\infty}(-1)^k q^{\frac{k(3k-1)}{2}} + \sum_{k=-\infty}^{-1} (-1)^k q^{\frac{k(3k-1)}{2}} \\
&= 1 + \sum_{k=1}^{\infty}(-1)^k q^{\frac{k(3k-1)}{2}} + \sum_{k=1}^{\infty}(-1)^k q^{\frac{k(3k+1)}{2}} \\
&= 1 + \sum_{k=1}^{\infty}(-1)^k \left( q^{\frac{k(3k-1)}{2}} + q^{\frac{k(3k+1)}{2}} \right)
\end{aligned}$$

We are interested in starting three different proofs of the theorem. Before moving on to them, it would help to understand what pentagonal numbers are. Simply put, these are numbers of the form $n(3n-1)/2$. Why they are called pentagonal is clear from the following figure.

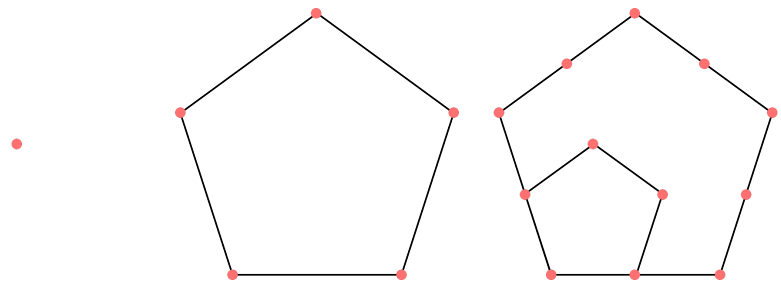

FIGURE 4. Counting the number of vertices (marked in red) at each iteration gives the sequence of pentagonal numbers, i.e, $1, 5, 12, \ldots$

Notice how the vertices (marked in red) in Figure 4 admit a natural partition and hence a Ferrer's diagram.

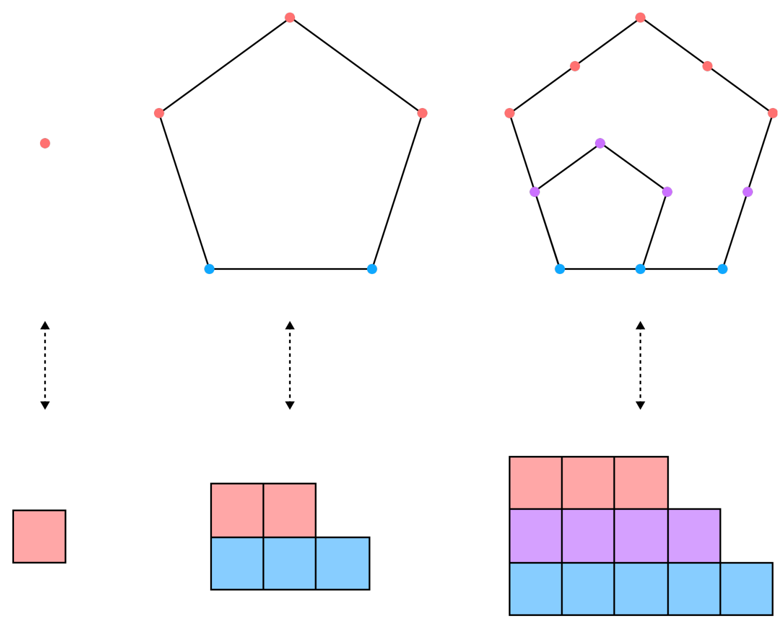

FIGURE 5. The first 3 pentagonal numbers and the natural partitions they admit, i.e, $1 = 1, 5 = 3 + 2$, and $12 = 5 + 4 + 3$.

These partitions of pentagonal numbers we have obtained are partitions into distinct parts. To this end, we will try and set up a bijection between partitions of $n$ into an odd number of distinct parts and the partitions of $n$ into an even number of distinct parts. Consider the following auxiliary claim.

CLAIM 2.4. *Let $p_e(n)$ and $p_o(n)$ denote the number of partitions of $n$ into an even and odd number of distinct parts respectively. Then*

$$p_e(n) - p_o(n) = \begin{cases} (-1)^m & \text{if } n = \dfrac{m(3m \pm 1)}{2} \\ 0 & \text{otherwise} \end{cases}$$

PROOF. Let $\lambda$ be a partition of $n$. Let $s(\lambda)$ and $\sigma(\lambda)$ denote the smallest part of $\lambda$ and the part obtained by considering elements on the rightmost diagonal respectively. If $s(\lambda) \le \sigma(\lambda)$, we add one to each of the $s(\lambda)$ largest parts of $\lambda$ and we delete the smallest part. Consider the following example for instance.

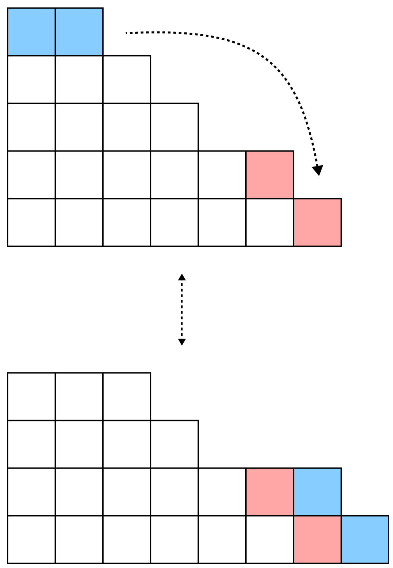

FIGURE 6. The partition $\lambda = 7 + 6 + 4 + 3 + 2$ with $s(\lambda) = 2$ (marked blue) and $\sigma(\lambda) = 2$ (marked red) admits the partition $\lambda' = 8 + 7 + 4 + 3$.

On the other hand, if $s(\lambda) > \sigma(\lambda)$ then we subtract one from each of the $\sigma(\lambda)$ largest parts of $\lambda$ and insert a new smallest part of size $\sigma(\lambda)$. Consider the following example for instance.

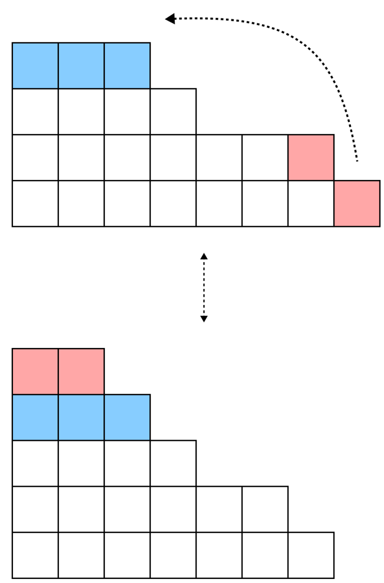

FIGURE 7. The partition $\lambda = 8+7+4+3$ with $s(\lambda) = 3$ (marked blue) and $\sigma(\lambda) = 2$ (marked red) admits the partition $\lambda' = 7+6+4+3+2$.

In summary, our map changes the parity of the number of parts of the partition, and noting that exactly one case applies to any partition, it is clear why our map is a bijection. However, we are not done yet! Our map fails in two cases. One, where $s(\lambda) = m+1$ is exactly one more than $\sigma(\lambda) = m$, i.e, the case where the number being partitioned is

$$(m+1) + (m+2) + \cdots + 2m = m(3m+1)/2.$$

Two, where $s(\lambda) = \sigma(\lambda)$, i.e, the case where the number being partitioned is

$$m + (m+1) + \cdots + (2m-1) = m(3m-1)/2.$$

This completes the proof. $\square$

Now, Theorem 2.4 follows as a direct consequence of Claim 2.4.

To see the usefulness of Theorem 2.4 consider the following remark.

REMARK 2.2. *Putting Theorem 2.4 and Claim 2.1 together gives us*

$$\left(\sum_{n\geq 0} p(n)q^n\right)\left(\prod_{i\geq 1}(1-q^i)\right) = 1.$$

*A recurrence of sorts presents itself. Namely,*

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) + \cdots.$$

*The reason this recurrence is useful is two-pronged. One, the sum on the right is not infinite because $p(k) = 0$ for all choices of $k < 0$. Two, to compute $p(n)$ we need not compute $p(n-i)$ for all choices of $1 \leq i \leq n-1$.*

Next, we present a result known as Jacobi's triple product identity. Interestingly, this identity is easier to prove than Euler's pentagonal number theorem. Not only that, but, Euler's theorem emerges as a corollary of Jacobi's identity. Before stating the identity, we introduce some useful notation. For formal variables $x, q$ and a natural number $n > 0$ we define

$$(x;q)_n := (1-x)(1-xq)(1-xq^2)\cdots(1-xq^{n-1}) = \prod_{i=0}^{n-1}(1-xq^i).$$

The symbol $(x;q)_n$ is referred to as the $q$-shifted factorial. We extend our notation to allow for $n$ to be $\infty$ by defining

$$(x;q)_\infty := \lim_{n\to\infty}\left(\prod_{i=0}^{n-1}(1-xq^i)\right) = \prod_{i=0}^{\infty}(1-xq^i) = (1-x)(1-xq)(1-xq^2)\cdots.$$

It follows (by setting $x$ to $q$) that

$$(q;q)_n = (1-q)(1-q^2)\cdots(1-q^n) = \prod_{i\geq 1}^{n}(1-q^i)$$

$$(q;q)_\infty = (1-q)(1-q^2)\cdots = \prod_{i\geq 1}(1-q^i)$$

In the spirit of getting comfortable with the newly introduced notation it is a good exercise to verify the following re-statements of the claims we have made so far;

$$\sum_{n\geq 0} p(n)q^n = \frac{1}{(q;q)_\infty} \quad \text{(Refer to } \textit{Claim } 2.1)$$

$$\sum_{n\geq 0} p_D(n)q^n = (-q;q)_\infty \quad \text{(Refer to } \textit{Theorem } 2.1)$$

$$\sum_{n\geq 0} \frac{q^{n^2}}{(q;q)_n^2} = \frac{1}{(q;q)_\infty} \quad \text{(Refer to } \textit{Claim } 2.3)$$

To see the usefulness of our notation, we introduce ourselves to a bi-variate generating function. Consider the expansion of $1/(xq;q)_\infty$ in which the coefficient of $x^k$ is $1/(q;q)_k$. More succinctly put we get;

Claim 2.5.

$$\sum_{n\geq 0}^{\infty} \frac{x^n}{(q;q)_n} = \frac{1}{(x;q)_\infty}.$$

A more natural interpretation of the identity comes from setting $x \to xq$ to arrive at

$$\sum_{n\geq 0}^{\infty} \frac{x^k q^k}{(q;q)_k} = \frac{1}{(xq;q)_\infty}.$$

Now the coefficient of $x^k$ is the generating function for partitions with exactly $k$ parts.

Now we are at a stage to introduce ourselves to Jacobi's triple product identity.

THEOREM 2.5 (Jacobi Triple Product). *For $|q| < 1$ and $z \neq 0$ we have*

$$\sum_{n=-\infty}^{\infty} q^{n^2} z^n = (q^2;q^2)_\infty(-q/z;q^2)_\infty(-zq;q^2)_\infty$$

Before giving a proof of the identity, we remark that by setting $q \to q^{3/2}$ and $z \to -\sqrt{q}$ we retrieve Euler's pentagonal number theorem back.

PROOF. From Claim 2.5 it is clear why

$$\frac{1}{(-x;q)_\infty} = \sum_{n\geq 0} \frac{(-1)^n x^n}{(q;q)_n}.$$

Additionally, since $q^{\left(\begin{array}{c}k\\2\end{array}\right)}/(q;q)_k$ is the generating function for partitions into exactly $k$-many distinct parts, it is also true that

$$\sum_{k=0}^{\infty} \frac{q^{\left(\begin{array}{c}k\\2\end{array}\right)} x^k}{(q;q)_k} = (-x;q)_\infty.$$

Keeping these two identities at hand, notice how

$$\begin{aligned}
(-xq;q^2)_\infty &= \sum_{k=0}^{\infty} \frac{q^{k^2} x^k}{(q^2;q^2)_k} \\
&= \frac{1}{(q^2;q^2)_\infty} \sum_{k=-\infty}^{\infty} q^{k^2} x^k (q^{2n+2};q^2)_\infty. \\
&= \frac{1}{(q^2;q^2)_\infty} \sum_{k=-\infty}^{\infty} q^{k^2} x^k \sum_{m=0}^{\infty} \frac{(-1)^m q^{m^2+m+2mn}}{(q^2;q^2)_m} \\
&= \frac{1}{(q^2;q^2)_\infty} \sum_{m=0}^{\infty} \frac{(-1)^m (q/x)^m}{(q^2;q^2)_m} \sum_{n=-\infty}^{\infty} q^{(n+m)^2} x^{n+m} \\
&= \frac{1}{(q^2;q^2)_\infty} \frac{1}{(-q/x;q^2)_\infty} \sum_{n=-\infty}^{\infty} q^{n^2} x^n
\end{aligned}$$

as required. □

Next, we turn our attention to an identity due to Sylvester. One reason why this identity is important is because Euler's pentagonal theorem presents itself as a corollary of Sylvester's identity.

THEOREM 2.6 (Sylvester's Identity).

$$(-xq;q)_\infty = 1 + \sum_{k=1}^{\infty} x^k q^{\frac{k(3k-1)}{2}} \frac{(-xq;q)_{k-1}}{(q;q)_k} \left(1 + xq^{2k}\right)$$

Before giving a proof of the identity, we remark that by setting $x \to -1$ we retrieve Euler's pentagonal number theorem back. Next, we make a few observations about the Ferrer's diagram corresponding to a partition which has distinct parts which will turn out to be important while writing a combinatorial proof of Sylvester's Identity. Refer to Figure 8 and notice how if the said diagram has a

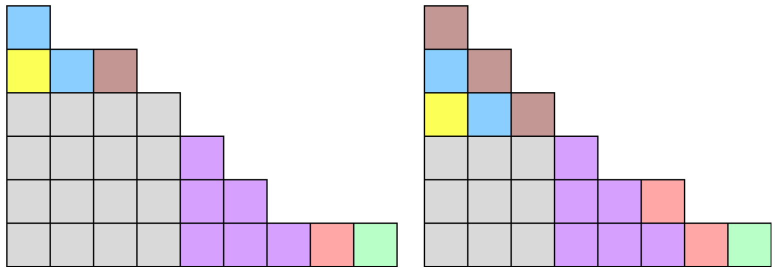

FIGURE 8. Franklin Triangles

Durfee square (colored grey) of order $k$, then to the right of this square, there must be an isoceles triangle (colored purple, and called the **Franklin** triangle) of length either $k$, or $k-1$, depending on where the Durfee square ends. In the case where the Franklin triangle is of order $k-1$;

(1) The Durfee square and the Franklin triangle correspond to a unique partition of $k^2 + \begin{pmatrix} k \\ 2 \end{pmatrix}$
(2) The pieces to the right of the Franklin triangle, when counted diagonally, correspond to partitions where each part is atmost $k-1$.
(3) The pieces below the Franklin triangle, when counted diagonally, correspond to partitions into atmost $k-1$ parts.

Similarly, in the case where the Franklin triangle is of order $k$;

(1) The Durfee square and the Franklin triangle correspond to a unique partition of $k^2 + \begin{pmatrix} k+1 \\ 2 \end{pmatrix}$
(2) The pieces to the right of the Franklin triangle, when counted diagonally, correspond to partitions where each part is atmost $k$.
(3) The pieces below the Franklin triangle, when counted diagonally, correspond to partitions into atmost $k$ parts.

PROOF. We know that the coefficient of $x^i q^j$ in $(-xq;q)_\infty$ is the number of partitions of $j$ into exactly $i$ parts each of which is distinct. But every such partition has a Durfee square of order atleast $k=1$, and a Franklin triangle of order either

$k-1$, or $k$ in it's Ferrer's diagram. From the observations made above, we get

$$(-xq;q)_\infty = \underbrace{\sum_{k=1}^{\infty} x^k q^{k^2+\binom{k}{2}} \frac{(-xq;q)_{k-1}}{(q;q)_{k-1}}}_{\text{Order of Franklin triangle: } k-1} + \underbrace{\sum_{k=0}^{\infty} x^k q^{k^2+\binom{k+1}{2}} \frac{(-xq;k)_k}{(q;q)_k}}_{\text{Order of Franklin triangle: } k}$$

$$= \sum_{k=1}^{\infty} x^k q^{k^2+\binom{k}{2}} \frac{(-xq;q)_{k-1}}{(q;q)_{k-1}} + 1 + \sum_{k=1}^{\infty} x^k q^{k^2+k+\binom{k}{2}} \frac{(-xq;q)_{k-1}}{(q;q)_{k-1}} \frac{\left(1+xq^k\right)}{1-q^k}$$

$$= 1 + \sum_{k=1}^{\infty} x^k q^{k^2+\binom{k}{2}} \frac{(-xq;q)_{k-1}}{(q;q)_{k-1}} \left(1 + \frac{q^k\left(1+xq^k\right)}{\left(1-q^k\right)}\right)$$

$$= 1 + \sum_{k=1}^{\infty} x^k q^{k^2+\binom{k}{2}} \frac{(-xq;q)_{k-1}}{(q;q)_{k-1}\left(1-q^k\right)} \left(1+xq^{2k}\right)$$

$$= 1 + \sum_{k=1}^{\infty} x^k q^{k^2+\binom{k}{2}} \frac{(-xq;q)_{k-1}}{(q;q)_k} \left(1+xq^{2k}\right)$$

□

Next, we give an analytic proof of the same identity.

PROOF. Notice how

$$\begin{aligned} f(x) := {} & 1 + \sum_{k=1}^{\infty} x^k q^{k(3k-1)/2} \frac{(-xq;q)_{k-1}}{(q;q)_k} (1+xq^{2k}) \\ = {} & 1 + \sum_{k=1}^{\infty} x^k q^{k(3k-1)/2} \frac{(-xq;q)_{k-1}}{(q;q)_k} \left((1-q^k) + q^k(1+xq^k)\right) \\ = {} & 1 + \sum_{k=1}^{\infty} x^k q^{k(3k-1)/2} \frac{(-xq;q)_{k-1}}{(q;q)_{k-1}} + \sum_{k=1}^{\infty} x^k q^{k(3k+1)/2} \frac{(-xq;q)_k}{(q;q)_k} \\ = {} & \sum_{k=0}^{\infty} x^k q^{k(3k+1)/2} \frac{(-xq;q)_k}{(q;q)_k} + \sum_{k=0}^{\infty} x^{k+1} q^{(k+1)(3k+2)/2} \frac{(-xq;q)_k}{(q;q)_k} \\ = {} & \sum_{k=0}^{\infty} x^k q^{k(3k+1)/2} q^{k(3k+1)/2} \frac{(-xq;q)_k}{(q;q)_k} (1+xq^{2k+1}). \end{aligned}$$

Our observation allows us to conclude that

$$f(x) = (1+xq)\left(1 + \sum_{k=1}^{\infty} x^k q^{k(3k+1)/2} \frac{(-xq^2;q)_{k-1}}{(q;q)_k} (1+xq^{2k+1}).\right)$$

More succinctly, $f(x) = (1+xq)f(xq)$, iterating which we get $f(x) = (-xq;q)_\infty$ as required. □

We conclude this section on integer partitions by discovering a Rogers-Ramanujan identities. Namely,

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty},$$
$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}.$$

Doing so requires some familiarity with continued fractions. As an example consider the fraction

$$C = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}.$$

Truncating the fraction at various depths gives us a sequence of what are called convergents. Observing the convergents of $C$,

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \dots$$

makes it clear that the $n$-th convergent is given by $f_n/f_{n-1}$ where $f_n$ denote the $n$-th Fibonacci number. Next, we give a two-parameter generalization of $C$, called the Rogers-Ramanujan continued fraction, in the variables $z$ and $q$.

$$C(z,q) = 1 + \cfrac{zq}{1 + \cfrac{zq^2}{1 + \cfrac{zq^3}{1 + \ddots}}}.$$

Observing the convergents of $C(z,q)$,

$1, 1+zq, (1+zq+zq^2)/(1+zq^2), (1+zq+zq^2+zq^3+z^2q^4)/(1+zq^2+zq^3), \cdots,$

makes it clear that the $n$-th convergent is of the form

$$H_n(z,q)/H_{n-1}(zq,q) = 1 + \frac{zqH_{n-2}(zq^2,q)}{H_{n-1}(zq,q)}.$$

Additionally, it is also not difficult to verify that $C(z,q) = 1 + zq/C(zq,q)$. Putting both of our observations together, it is seems logical to assume that $C(z,q)$ must be of the form $H(z,q)/H(zq,q) = 1 + zqH(z^q,q)/H(zq,q)$. This gives the recurrence $H(z,q) = H(zq,q) + zqH(z^2q,q)$ which is solve using the method of generating functions to see that if $H(z,q) = \sum_{k=0}^{\infty} a_k z^k$, then

$$\begin{aligned} a_k &= a_k q^k + a_{k-1} q^{2k-1} \\ \implies a_k &= (q^{2k-1}/1 - q^k) a_{k-1} \\ \implies a_k &= (q^{k^2}/(q;q)_k) a_0. \end{aligned}$$

Since $a_0 = 1$, it follows that

$$H(z,q) = \sum_{k-0}^{\infty} \frac{q^{k^2}}{(q;q)_k} z^k.$$

Finally, since $C(1,q) = H(1,q)/H(q,q)$ we get the two Rogers-Ramanujan identities.

Notice how $1/(q;q^5)_\infty(q^4;q^5)_\infty$ and $1/(q^2;q^5)_\infty(q^3;q^5)_\infty$ are the generating functions for partitions into parts which are congruent to $1,4 \mod 5$ and $2,3 \mod 5$ respectively. Additionally, $q^{n^2}/(q;q)_n$ is the generating functions for partitions with exactly $n$ parts having a difference of atleast 2. Finally, $q^{n^2+n}/(q;q)_n$ is the same generating function but the smallest part is not allowed to be less than 2. Putting all these observations together, we draw the following combinatorial interpretations. One, the number of partitions of $n$ with $k$ parts, where the smallest part is at least $k$, is equal to the number of partitions of $n$ in which each part is congruent to either 1 or 4 modulo 5. Two, the number of partitions of $n$ with $k$ parts, where the smallest part is at least $k+1$, is equal to the number of partitions of $n$ in which each part is congruent to either 2 or 3 modulo 5.

CHAPTER 5

# $q-$Combinatorics

## 1. $q-$analogs

The idea here is to count objects with weights associated with them. For instance in a lattice path, one might be interested in assigning the number of blocks spanned below the said path and/or the number of east steps below the main diagonal. We shall start our discussion by counting all possible (what are called) inversions on the set $[n]$.

DEFINITION 1.1 (Inversion). *Let $\sigma$ be a bijection on $[n]$. An inversion of $\sigma$ is a tuple $(\sigma(i), \sigma(j))$ such that $1 \leq i < j \leq n$ and $\sigma(i) > \sigma(j)$.*

As an example, we count the number of inversions on the set [3]

| **Permutation** | **Inversions** | **Remark** |
|---|---|---|
| (123) | No Inversions | Trivial |
| (132) | (32) | $2 < 3$ but $3 > 2$ |
| (213) | (21) | $1 < 2$ but $2 > 1$ |
| (231) | (21), (31) | Same As Above |
| (312) | (31), (32) | Same As Above |
| (321) | (32), (31), (21) | Notice a pattern here when the permutation is "decreasing" |

CLAIM 1.1. *If $I_n$ denote the number of inversions on $[n]$, then $I_n = \begin{pmatrix} n \\ 2 \end{pmatrix} \frac{n!}{2}$*

PROOF. We attempt to come up with a recursive formula first. Let $I_n$ be known. Now consider the addition of a new symbol (indexed by $n+1$). Notice how the symbol $n+1$ can be placed at a total of $n+1$ places. Placing $n+1$ at the $1st$ position grants $n$ new inversions. Similarly, placing $n+1$ at the $2nd$ position grants $n-1$ new inversions, and so on. Adding all of these together grants

$$
\begin{aligned}
I_{n+1} &= \underbrace{n!n + I_n}_{\substack{I_n \text{ previously counted inversions} \\ +n \text{ new ones for each one of the } n! \text{ permutations}}} + n!(n-1) + I_n + \cdots + n!(1) + I_n + n!(0) + I_n \\
&= n!\,(n + (n-1) + (n-2) + \cdots + 2 + 1 + 0) + (n+1)I_n \\
&= n!\left(\frac{n(n+1)}{2}\right) + (n+1)I_n \\
&= n!\begin{pmatrix} n+1 \\ 2 \end{pmatrix} + (n+1)I_n
\end{aligned}
$$

From here on end, the proof can also be completed using the method of induction. However, this is not the way we want to go about the proof. Notice how the formula we've obtained is a recursion which can be solved using more than one technique to arrive at an explicit expression. We start off by using the method of back-substitution and re-write the obtained recurrence as

$$I_n = nI_{n-1} + (n-1)!\begin{pmatrix} n \\ 2 \end{pmatrix} = nI_{n-1} + n!\frac{n-1}{2}$$

Next, we set $n \to n-1$ and $n \to n-2$ in the obtained equations to arrive at

$$I_{n-1} = (n-1)I_{n-2} + (n-1)!\frac{n-2}{2}$$
$$I_{n-2} = (n-2)I_{n-3} + (n-2)!\frac{n-3}{2}$$

Making appropriate substitutions gives

$$\begin{aligned} I_n &= n\left\{(n-1)I_{n-2} + (n-1)!\left(\frac{n-2}{2}\right)\right\} + n!\frac{n-1}{2} \\ &= n(n-1)I_{n-2} + \frac{n!}{2}\left\{(n-2) + (n-1)\right\}. \end{aligned}$$

More specifically,

$$I_n = n(n-1)(n-2)I_{n-3} + \frac{n!}{2}\left\{(n-3) + (n-2) + (n-1)\right\}.$$

It is easy to notice a pattern, that is, after $k$-many such back-substitutions we get

$$I_n = n(n-1)(n-2)\cdots(n-k+1)I_{n-k} + \frac{n!}{2}\left\{(n-1) + (n-2) + \cdots + (n-k)\right\}.$$

Setting $n \to n-k$ gives

$$I_{n-k} = (n-k)I_{n-k-1} + (n-k)!\frac{n-k-1}{2}.$$

Once again, making appropriate substitutions we get.

$$\begin{aligned} I_n &= n(n-1)\cdots(n-k+1)(n-k)I_{n-k-1} \\ &\quad + \frac{n!}{2}\left\{(n-k-1) + (n-1) + (n-2) + \cdots + (n-k)\right\}. \end{aligned}$$

Finally, to see why the claim is true it suffices to set $k \to n-1$ because then we would have

$$I_n = n(n-1)(n-2)\cdots(2)(1)I_0 + \frac{n!}{2}\left\{(n-1) + (n-2) + (n-3) + \cdots + 1\right\}.$$

Since $I_0 = 0$, we get

$$I_n = \frac{n!}{2}\begin{pmatrix} n \\ 2 \end{pmatrix}$$

as required. □

Next, we give a combinatorial proof.

Proof. For an $n$ given to us, consider all the $n!$ possible permutations on the set $[n]$ arranged in pairs like $(\sigma(1)\sigma(2)\cdots\sigma(n)), \underbrace{(\sigma(n), \sigma(n-1), \cdots, \sigma(1))}_{\text{Called the mate of } \sigma}$. This arrangement separates the $n!$ permutations into $n!/2$ pairs. Now, by the following observations we are done.

(1) If $(\sigma(i), \sigma(j))$ is an inversion of $\sigma$, then it's not an inversion of $\sigma$'s mate.

(2) Each one of the $\binom{n}{2}$ pairs is an inversion exactly once in each couple.

$\square$

Corresponding to a given $n$ we know that there are $n!$ possible permutations on the set $[n]$. In the formal variable $q$, we define the inversion polynomial on $[n]$ as

$$\sum_{\sigma \in \text{Bijections on } [n]} q^{\text{inv}(\sigma)}$$

where $\text{inv}(\sigma)$ denotes the number of inversions of $\sigma$. As an example, we compute the inversion polynomial on the set [3].

| **Permutation** | **Inversions** | $q^{\text{inv}(\sigma)}$ |
|---|---|---|
| (123) | No Inversions | $q^0$ |
| (132) | (32) | $q^1$ |
| (213) | (21) | $q^1$ |
| (231) | (21), (31) | $q^2$ |
| (312) | (31), (32) | $q^2$ |
| (321) | (32), (31), (21) | $q^3$ |

It is clear that the inversion polynomial corresponding to [3] is given by $1+q+q+q^2+q^2+q^3 = (1+q)(1+q+q^2)$.

Question 1.1. *What is the inversion polynomial corresponding to* $[n]$ *for an arbitrary choice of* $n$*?*

Solution. We know that the inversion polynomial for [1] is $q^0 = 1$, for [2] is $q^0 + q^1 = 1 + q$, for [3] as shown above is $(1+q)(1+q+q^2)$. One might be tempted to (correctly) assume that the inversion polynomial for $[n]$ is $(1+q)(1+q+q^2)\cdots(1+q+q^2+\cdots+q^{n-1})$. Notice how the addition of a symbol indexed by 4 in the first place raises the degree by 3.

| **Permutation** | **Inversions** | $q^{\text{inv}(\sigma)}$ |
|---|---|---|
| (4123) | (41), (42), (43) | $q^{0+3}$ |
| (4132) | (41), (43), (42), (32) | $q^{1+3}$ |
| (4213) | (42), (41), (43), (21) | $q^{1+3}$ |
| (4231) | (42), (43), (41), (21), (31) | $q^{2+3}$ |
| (4312) | (43), (41), (42), (31), (32) | $q^{2+3}$ |
| (4321) | (43), (42), (41), (32), (31), (21) | $q^{3+3}$ |

Hence, the sum corresponding to the addition of a symbol indexed by 4 at the first position becomes

$$q^3 \underbrace{((1+q)(1+q+q^2))}_{\text{Inversion polynomial of } S_3}.$$

Similarly, the addition of this symbol at the second place raises the degree by 2, the addition of this symbol at the third raises the degree by 1, and so on. Finally, we get

$$\begin{aligned}\sum_{\sigma \in \text{Bijections on } [4]} q^{\text{inv}(\sigma)} &= q^3((1+q)(1+q+q^2)) + q^2((1+q)(1+q+q^2)) \\ &\quad + q^1((1+q)(1+q+q^2)) + q^0((1+q)(1+q+q^2)) \\ &= (1+q)(1+q+q^2)(1+q+q^2+q^3)\end{aligned}$$

This allows us to conclude that

$$\sum_{\sigma \in \text{Bijections on } [n]} q^{\text{inv}(\sigma)} = (1+q)(1+q^2)\cdots(1+q^{n-1})$$

Polynomials of the form involved in our solution keep coming up in the study of $q$-combinatorics. For this reason, we introduce some notation for succinct writing (amongst other reasons which will be explained soon).

Definition 1.2 ($q$-analogue of numbers). *For a real number $n$, we denote it's $q$-analogue by*

$$[n]_q := \begin{cases} \dfrac{1-q^n}{1-q} & \text{if } q \neq 1 \\ n & \text{if } q = 1 \end{cases}.$$

The following definition now follows quite naturally.

Definition 1.3 ($q$-analogue of factorials). *For a real number $n$, we denote the $q$-analogue of its factorial by*

$$n!_q = [1]_q[2]_q \cdots [n]_q$$

.

Remark 1.1. *The introduction of Definition 1.3 allows us to conclude that the inversion polynomial corresponding to $[n]$ is $n!_q$.*

In fact, yet another definition follows quite naturally.

Definition 1.4 ($q$-analogue of binomial coefficients). *For appropriate choices of $n$ and $k$, we denote the $q$-analogue of $\binom{n}{k}$ by*

$$\binom{n}{k}_q = \frac{n!_q}{(n-k)!_q k!_q}.$$

However, a combinatorial interpretation of Definition 1.4 is not immediately clear. To this end, consider the following problem.

Question 1.2. *Let $S(k, n-k)$ denote the set of all $n$-bit sequences with $k$ zeros and $n-k$ ones. What is the inversion polynomial corresponding to $S(k, n-k)$?*

The following table lists all the possible 4-bit sequences with 2-zeros along with their contributions to the inversion polynomial.

| | | |
|---|---|---|
| (0011) | 0 inversions | $q^0$ |
| (0101) | (10) once | $q^1$ |
| (0110) | (10) twice | $q^2$ |
| (1001) | (10) twice | $q^2$ |
| (1010) | (10) thrice | $q^3$ |
| (1100) | (10) four times | $q^4$ |

From the table, it is clear that the inversion polynomial corresponding to $S(k,n-k)$ is given by

$$\begin{aligned}\sum_{\sigma} q^{\mathrm{inv}(\sigma)} &= 1+q+2q^2+q^3+q^4\\ &= (1+q+q^2)(1+q^2)\\ &= [3]_q(1+q^2)\\ &= [3]_q(1+q^2)\frac{1+q}{1+q}\\ &= [3]_q\frac{(1+q+q^2+q^3)}{1+q}\\ &= [3]_q\frac{[4]_q}{[2]_q}\\ &= \frac{[4]_q[3]_q[2]_q[1]_q}{[2]_q[1]_q[2]_q[1]_q}\\ &= \frac{4!_q}{2!_q 2!_q}\\ &= \frac{4!_q}{2!_q(4-2)!_q}.\end{aligned}$$

More generally, we have the following theorem.

Theorem 1.1. *Let $S(k,n-k)$ denote the set of all $n$-bit sequences of $k$ zeros and $n-k$ ones. Then the inversion polynomial corresponding to $S(k,n-k)$ is*

$$\sum_{\sigma\in S(k,n-k)} q^{inv(\sigma)} = \left(\begin{array}{c} n \\ k \end{array}\right)_q = \sum_{j=0}^{k(n-k)} c_j(k,n-k)q^j$$

*where $c_j(k,n-k)$ counts the number of $n$-bit string with exactly $k$ zeros and $j$ inversions.*

Claim 1.2 (A $q$-analogue of Pascal's identity).

$$\left(\begin{array}{c} n+1 \\ k \end{array}\right)_q = \left(\begin{array}{c} n \\ k \end{array}\right)_q + q^{n-k+1}\left(\begin{array}{c} n \\ k-1 \end{array}\right)_q$$

Proof. We give a double counting argument. Notice how the L.H.S is the inversion polynomial corresponding to $S(k,n+1-k)$. It is also true that every $n+1$-bit sequence in $S(k,n+1-k)$ either ends with a 1 in which case it is not inverted with any of the preceding symbols - this explains the first term of the R.H.S, or ends with a 0 in which case it is inverted with all the $n-k+1$-many 1s - this explains the second term of the R.H.S. $\square$

Given our introduction to $q$-binomial coefficients, it is natural to ask if there is such a thing as $q$-multinomial coefficients as well. To this end, consider,

Definition 1.5. *The multinomial coefficient corresponding to non-negative $k_1,\ldots,k_m$ adding up to $n$ is denoted by*

$$\left(\begin{array}{c} n \\ k_1,\ldots,k_m \end{array}\right) := \frac{n!_q}{k_1!_q\cdots k_m!_q}.$$

As one might expect $\binom{n}{k_1,\ldots,k_m}$ is the inversion polynomial corresponding to $S_n(k_1,\ldots,k_m)$, the set of all $n$-length permutations with $k_i$-many $i$s ($i=1,\ldots,m$). We shall omit the proof, however a proof by induction is not too difficult to work out. Once again, it is also natural to ask if there is such a thing as the $q$-binomial theorem. Infact there are several of them. We state one of them here.

THEOREM 1.2.

$$\prod_{i=1}^{n}(1+xq^i)=\sum_{i=0}^{n}\binom{n}{i}_q q^{i(i+1)/2}x^i$$

PROOF. Notice how $\prod_{i=1}^{n}(1+xq^i)$ can be written as

$$\sum_{i=0}^{n}a_i(q)x^i$$

where $a_i(q)$ is the generating function of partitions into distinct parts with exactly $i$ part, where each part is $\leq n$. With this observation at hand it suffices to show that $a_i(q)$ is

$$\binom{n}{i}_q q^{i(i+1)/2}=\binom{n}{i}q^{1+2+3+\cdots+i}.$$

To this end let $\lambda=\lambda_1+\cdots+\lambda_i$ be a partition into distinct parts and consider the partition into exactly $i$ parts where each part is $\leq n-i$ which is constructed by removing $i$ from the first part, $i-1$ from the the second part, and so on. More specifically consider $\lambda'=(\lambda_1-i)+(\lambda_2-(i-1)+\cdots+\lambda_i-1)$. The generating function for such partitions, as we know, is

$$\binom{n-i+1}{i}_q=\binom{n}{i}_q.$$

This completes the proof. □

## 2. $q-$Counting of lattice paths

We are interested, once again, in the counting of lattice paths, but with weights this time. More specifically to each step in a lattice path we assign the number of unit blocks right below it as it's weight.

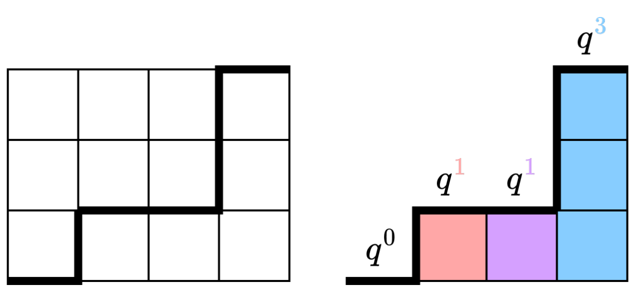


FIGURE 1. A lattice path with weight $q^0q^1q^1q^3=q^5$.

We want to give a $q$-analogue of the double counting argument we used in Question 1.8 to come up with a useful recurrence.

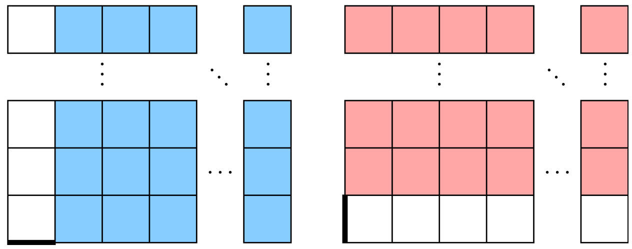

FIGURE 2

Refer to Figure 2 and notice how if a lattice path starts with an $E$-step then we only have to care about the weights due the "smaller" path over the remaining blocks (colored blue). On the other hand, if a lattice path starts with an $N$-step then we first row (colored white) always adds a weight of 1 block and we only have to take care of the "smaller" lattice path over the remaining blocks (colored red). Putting Claim 1.2 and our observation together allows us to deduce the following result due to George Polya.

THEOREM 2.1. *Let $A_{n,k}(r)$ denote the number of $n$-step lattice paths from $(0,0)$ to $(k, n-k)$ which span $r$ unit blocks under them. Then*

$$\left(\begin{array}{c} n \\ k \end{array}\right)_q = \sum_{r=0}^{k(n-k)} A_{n,k}(r) q^r$$

It is not difficult to see that every lattice path corresponds to a Ferrer's diagram and hence a partition. For instance, the lattice path in Figure 1 corresponds to the partition $3+1+1$ of 5. To this end, consider the following theorem due to Cayley.

THEOREM 2.2. *Let $p_{i,j}(n)$ denote the number of partitions of $n$ into atmost $i$ parts each of which is atmost $j$. Then*

$$\sum_{n=0}^{ij} p_{i,j}(n)\, q^n = \left(\begin{array}{c} i+j \\ i \end{array}\right)_q$$

PROOF. Let $c_{i,j}(n)$ denote the number of sequences with $i$-many 0s and $j$-many 1s having exactly $n$-many inversions. We know that

$$\left(\begin{array}{c} i+j \\ i \end{array}\right)_q = \sum_{n=0}^{ij} c_{i,j}(n)\, q^n.$$

Also, if $A_{i,j}(n)$ denotes the number of lattice paths from $(0,0)$ to $(i,j)$ we also know that

$$\left(\begin{array}{c} i+j \\ i \end{array}\right)_q = \sum_{n=0}^{ij} A_{i,j}(n)\, q^n.$$

Hence, to prove our claim it suffices to construct a bijection between $p_{i,j}(n)$ and $A_{i,j}(n)$ and/or $c_{i,j}(n)$. To this end, consider the following partition

$$n = \pi_1 + \pi_2 + \cdots + \pi_i$$

where $\pi_k \leq j$ for all possible choices of $k$. For every such partition, it is possible to construct a sequence of $j$-many 0s and $i$-many 1s, say $\sigma$, which has exactly $\pi_1$-many 0s followed by the first occurrence of 1 in $\sigma$, $\pi_2$-many 0s followed by the second occurrence of 1 in $\sigma$, and so on, all the way up to $\pi_i$-many 0s followed by the last occurrence of 1 in $\sigma$. This gives a bijection between $p_{i,j}(n)$ and $c_{j,i}(n)$. Next, we give a bijection between $p_{i,j}(n)$ and $A_{i,j}(n)$. Let $\sigma \in S(j,i)$ be the sequence of $j$-many 0s and $i$-many 1s corresponding to the partition of $n$ considered above. Now, setting the occurrence of a 1 in $\sigma$, and the occurrence of a 0 in $\sigma$ to an $N$ move, and an $E$ move respectively in the Ferrers diagram of $\pi_1 + \cdots + \pi_i$ gives a path from $(0,0)$ to $(i,j)$ which spans exactly $n$ blocks under it. □

We conclude this section by introducing two different $q$-analogues of Catalan numbers. Before doing so we introduce a new statistic on permutations.

Definition 2.1. *Let $\sigma$ be a permutation on $[n]$. An integer $1 \leq i \leq n-1$ is called a descent of $\sigma$ if $\sigma(i) > \sigma(i+1)$.*

The set of all descents of a permutation is called it's descent set. For instance the descent set of (613524) is $\{1,4\}$. Finally, the major index of a permutation is the sum of all the elements in the descent set. These definitions might seem unmotivated. However, consider the following result which we state without a proof (one proof is outlined as an exercise in Assignment 3).

Theorem 2.3. *Let $S_n$ denote the set of all permutations on $[n]$. Let maj$(\sigma)$ denote the major index of a permutation $\sigma$ in $S_n$. Then*

$$\sum_{\sigma \in S_n} q^{maj(\sigma)} = n!_q.$$

With this background at hand, we are ready to introduce a $q$-analogue of Catalan numbers first given by MacMahon.

Theorem 2.4. *Let $L^+$ denote the set of all lattice paths from $(0,0)$ to $(n,n)$ which never go below the main diagonal. For an arbitrary choice of $\pi \in L^+$, let $\sigma(\pi)$ denote the $2n$-length sequence obtained by replacing each occurrence of an $N$ with 0 and each occurrence of an $E$ with a 1. Then*

$$\sum_{\pi \in L^+} q^{maj(\sigma(\pi))} = \frac{1}{[n+1]_q} \binom{2n}{n}_q$$

We omit the proof and present yet another natural $q$-analogue of Catalan numbers, one which satisfies a recurrence relation similar to Theorem 2.1. Per usual, we start with a definition.

Definition 2.2. *Let $L^+$, $\pi$ and $\sigma(\pi)$ be as defined in the setting of Theorem 2.4. By area$(\pi)$ we refer to sum of the components of the vector obtained by counting the number of complete unit squares to the right at the occurrence of each $N$ step in $\pi$.*

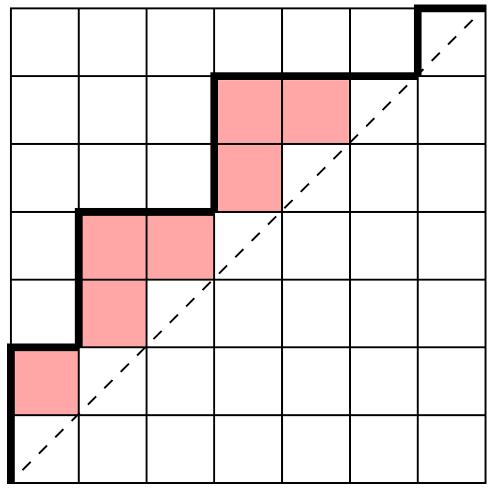

FIGURE 3. An example of a path in $L^+$ with area vector $(1, 1, 2, 1, 2)$ and area $1 + 1 + 2 + 1 + 2 = 7$.

Now, by a result due to Carlitz and Riordan we have.

THEOREM 2.5. *If we set* $C_n(q) = \sum_{\pi \in L^+} q^{area(\pi)}$, *then*

$$C_n(q) = \sum_{k=1}^{n} q^{k-1} C_k(q) C_{n-k}(q).$$

PROOF. The proof follows from the following trick. We decompose the path $\pi$ by identifying the point of first return to the main diagonal. Suppose the said point is $(k, k)$. Then, the segment of $\pi$ from $(0, 1)$ to $(k - 1, k)$, when treated as an element of the set of all lattice paths from $(0, 0)$ to $(k - 1, k - 1)$ which remain above the main diagonal, has an area that is $k - 1$ less than when the same segment is treated as part of $L^+$. $\square$

CHAPTER 6

# A Combinatorial Miscellany

## 1. Perfect Matchings and Pfaffian Orientations

Refer to https://youtu.be/ydCWu6aiAxE?si=v91XmPDjM3qkfGKw.

## 2. Graphs and Trees

We start with a standard result on graphs.

THEOREM 2.1. *Let $G$ be a simple, connected graph on $n$ vertices. Then the following are equivalent.*

(1) *$G$ is minimally connected.*
(2) *$G$ has $n-1$ edges.*
(3) *$G$ has no cycles.*

PROOF. If $G$ is minimally connected, then removing any edge of $G$ would disconnect it. This means that each edge of $G$ is a bridge. Since G is connected, it must have at least $n-1$ edges. If G had more than $n-1$ edges, then removing any additional edge would not disconnect $G$, contradicting the minimality assumption. Therefore, $G$ must have exactly $n-1$ edges. This proves that (1) $\implies$ (2). If G has $n-1$ edges and $n$ vertices, then it is a tree. Trees are acyclic by definition, so $G$ has no cycles. This proves (2) $\implies$ (3). Finally, if G has no cycles, then it is a tree. Trees are minimally connected, meaning that removing any edge disconnects the graph. Therefore, $G$ is minimally connected. This proves (3) $\implies$ (1). Having formed a loop of implications, we are done. $\square$

Recall the following definitions. Graphs satisfying any one of the properties in Theorem 2.1 are called trees. Additionally, graphs whose connected components are trees are called forests. More importantly, a tree on $n$ vertices is called a labeled tree if each vertex gets a label from the set $[n]$.

With this background at hand, we are interested in counting $T_n$, the number of labeled trees on $n$ vertices. From the figure below it is clear that $T_1, T_2, T_3$ and $T_4$ are $1, 1, 3$ and 16. One might guess that $T_n = n^{n-2}$. In fact this is precisely what we want to prove.

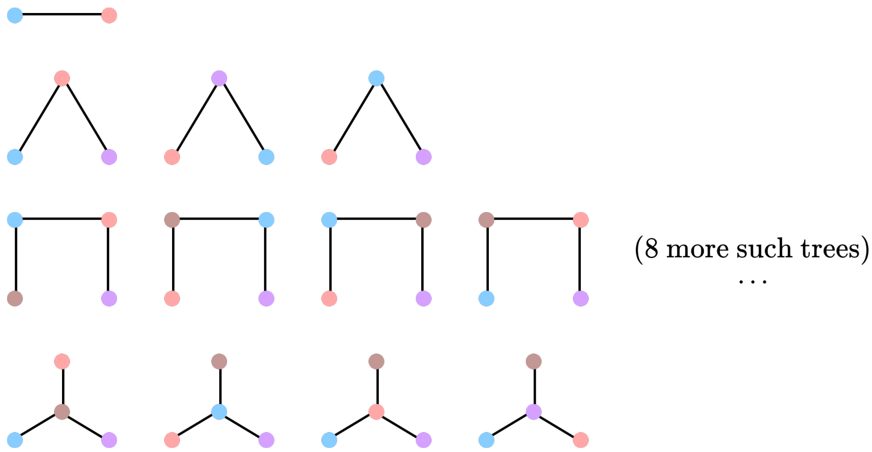


FIGURE 1

THEOREM 2.2 (Cayley's Formula). *If $T_n$ denotes the number of labeled trees on $n$ vertices, then*

$$T_n = n^{n-2}.$$

PROOF. Let $A = [k]$ be a set of vertices. Let $T_{n,k}$ count the number of labeled forests on $[n]$ consisting of $k$ trees where the vertices of $A$ appear in different trees. Let $F$ be a forest in this setting.

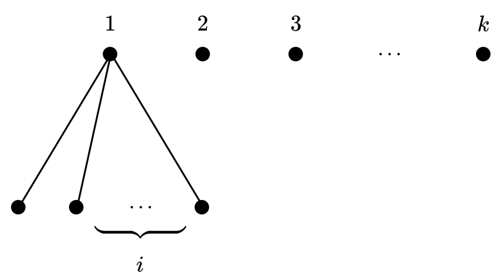


FIGURE 2

As is clear by Figure 2, if 1 is adjacent to $i$ vertices, deleting it, the $i$ neighbors together with $2, \ldots, k$ yield one vertex each in the components of a forest that consists of $k - 1 + i$ trees. Since $F$ can also be constructed by fixing $i$, then choosing the $i$ neighbors of 1 and then the forest $F \backslash \{1\}$ gives

$$T_{n,k} = \sum_{i=0}^{n-k} \binom{n-k}{i} T_{n-1,k-1+i}$$

for all choices of $n \geq k \geq 1$ where $T_{0,0} = 1$ and $T_{n,0} = 0$ as expected. By induction and the expression for $T_{n,k}$ which we have found, it follows that

$$\begin{aligned}
T_{n,k} &= \sum_{i=0}^{n-k} \binom{n-k}{i} (k-1+i)(n-1)^{n-1-k-i} \\
&= \sum_{i=0}^{n-k} (n-1)^i - \sum_{i=1}^{n-k} \binom{n-k}{i} i(n-1)^{i-1} \\
&= n^{n-k} - (n-k) \sum_{i=0}^{n-1-k} \binom{n-1-k}{i} (n-1)^i \\
&= n^{n-k} - (n-k)n^{n-1-k} \\
&= kn^{n-1-k}.
\end{aligned}$$

Setting $k \to 1$ gives us the required result. □

THEOREM 2.3. *The number of rooted forests on* $[n]$ *is* $(n+1)^{n-1}$.

PROOF. The proof follows from the following observation. For any rooted forest with $n$ vertices, add a new vertex $v$ and connect it to all the roots as their parent. This forms a tree with $n+1$ vertices. Conversely, for any tree with $n+1$ vertices, where one vertex is labeled $v$, remove $v$ along with its edges. The resulting structure is a rooted forest, with the former children of $v$ serving as the roots. □

We conclude this section with a result whose proof we shall omit but see Theorem 2.2 as a consequence of.

THEOREM 2.4. *The number of rooted forests on* $[n]$ *with* $k$ *components is*

$$\binom{n-1}{k-1} n^{n-k}.$$

Notice how setting $k \to 1$ and multiplying the result by $n$ (to account for the $n$ choices of roots) gives Theorem 2.2 back.

## 3. Permutations Revisited

We are interested in counting $P_n$, the number of permutations on $[n]$ that can be generated using a single stack. We start by computing $P_n$ by hand for $n = 1, 2$ and 3. When $n = 1$, we can push 1 on to the empty stack and pop it immediately. The stack is now empty, and there are no more input values. So we are done and the output is the permutation (1). Next, when $n = 2$, we have two possible permutations, (12) and (21). To obtain the permutation (12), first push 1 onto the stack, then pop it, then push 2 onto the stack and then pop it. To obtain the permutation (21), first push 1 onto the stack, then push 2, then pop 2, and then pop 1. Similarly, the permutations $(123), (132), (213), (231)$ and $(321)$ are obtainable. One might (correctly) guess from here one that this is the sequence of Catalan numbers. What follows is a proof of the same.

PROOF. To this end, let $i$ denote the position of the element 1 in a valid permutation produced by the stack, where $1 \leq i \leq n$. Then, $P_{i-1}$ counts the number of valid permutations with $i-1$ elements to the left of 1, and $P_{n-i-1}$ counts the number of valid permutations with $n-i-1$ elements to the right of 1.

If we set $P_0 = 1$, then by the multiplication and addition principles, we obtain the recurrence

$$P_n = \sum_{i=1}^{n} P_{i-1}P_{n-i-1},$$

which, more explicitly is written as

$$P_n = P_0P_{n-1} + P_1P_{n-2} + \cdots + P_{n-1}P_0.$$

Recall how this is precisely the recurrence relation we derived in Theorem 2.1. Therefore, $P_n = C_n$ for all $n \geq 0$, where $C_n$ denotes the $n$-th Catalan number. $\square$