MAT315 Combinatorial Enumeration
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Contents

Preface

These are the lecture notes of MAT315 Combinatorial Enumeration, offered in Monsoon 2024 semester at Ahmedabad University, India. The notes were written down by the TA for the course, Kanak Dhotre and is as close to the classroom teaching as possible.

There are several very good textbooks in combinatorics, however the material that I wish to cover in this course is not available in a single source to my liking. So, I decided to make my own notes for this iteration, as well as for any future iterations of this course.

The background required to enroll for the course is very minimal, in fact, it is not even mandatory for a student to have done a first course in Discrete Mathematics. So, we introduce several basic concepts along the way and if there is a scope for some digression then we will take it. The course is supplemented by some homework assignments, some of the problems in the text were set in those assignments, some even appeared in the examinations.

I am thankful to Kanak Dhotre for typing these notes. Any errors that remain are mine. If there are any errors, comments, or corrections, please write to me via email.

Manjil Saikia

CHAPTER 1

What is Combinatorics?

1. Introduction

This course aims to delve into the study of discrete mathematical structures, a field which traces its roots back to the 1700s with the work of Leonhard Euler and has gained much attention between the 1960s and 1970s with the advent of computer science. Notably, Euler answered the following question posed by Philip Naude in the year 1741: "In how many ways can the number 50 be written as a sum of seven different positive integers?". We shall understand the outline of Euler's solution to the problem later in this course. A few important personalities (some of whose work we will study eventually) in the subject include Gian Carlo Rota, Donald Knuth, Richard Stanley, Srinivasa Ramanujan, and Pinagala.

Combinatorics is the science of patterns and arrangements. More concretely, it deals with the study of the existence and the number of arrangements possible for a given mathematical structure. We start our discussion with a few motivating questions which will make our statement clearer.

QUESTION 1.1. *In how many ways can you arrange the elements of the set* $[n] := \{1, 2, 3, \dots, n\}$ *such that the first entry in the arrangement is an even number?*

Notice how when n is an even number we have $n/2$ choices of even numbers to make for the first entry in our arrangement. Once a choice for the said even number is made the remaining $n-1$ choices can be made in $(n-1)!$ ways. Hence, in the case where n is an even number we have $n/2 \cdot (n-1)!$ possible arrangements. Can you see why we will have $(n-1)/2 \cdot (n-1)!$ arrangements for the case where n is an odd number?

QUESTION 1.2. *In how many ways can you arrange elements from the set* $[n] := \{1, 2, 3, \dots, n\}$ *on a grid with* n *columns and* n *rows?*

Notice how for each one of the n^2 spaces we have n choices to make. Hence, there are a total of $\underbrace{n \cdots n}_{n^2 \text{ times}} = n^{n^2}$ possible arrangements.

QUESTION 1.3. *In how many ways can you arrange elements from the set* $[n]$ *(as defined in the previous two examples) on a grid with* n *columns and* n *rows such that each element appears atleast(/exactly) once in each row?*

Since for each row in the grid we have $n!$ possible arrangements and the grid has n rows there are a total of $n \cdot n!$ possible arrangements.

QUESTION 1.4. *How many matrices of order* $n \times n$ *exist given that the entries must be from the set* $\{0, 1\}$*?*

Since we have 2 choices for each one of the n^2 entries there are a total of 2^{n^2} such matrices.

QUESTION 1.5. *How many matrices of order $n \times n$ exist given that the entries must be from the set $\{0, 1\}$ and each row and column must have exactly one 1.*

Notice how in the first row of our matrix we have n ways to fix the occurrence of 1. This forces $n-1$ ways to fix the occurrence of 1 in the second row and so on. Hence, in all there are a total of $n \cdot (n-1) \cdots 1 = n!$ such matrices.

REMARK 1.1. *Question 1.5 can also be re-stated as counting the number of order $n \times n$ matrices which have row-sum and column-sum equal to 1.*

With the following definition we shall now look at a generalization of sorts of the kind of matrices we were dealing with in Question 1.5.

DEFINITION 1.1 (Alternating Sign Matrix (ASM)). *A matrix of order $n \times n$ is called an alternating sign matrix if the following conditions hold:*

(1) *All the entries of the matrix come from the set $\{-1, 0, 1\}$.*
(2) *Each row-sum and column-sum is 1.*
(3) *The non-zero entries (both row-wise and column-wise) alternate in sign.*

A result first proved by Doron Zeilberger in the year 1992 states that there are precisely

$$\prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!}$$

number of ASMs of order $n \times n$. A proof of this result is beyond the scope of these lectures and is mentioned only for the sake of completeness. We shall, however, count ASMs of order 3 now.

QUESTION 1.6. *How many ASMs of order 3 exist?*

Notice how the set of matrices we counted in Question 1.5 are a subset of the set of ASMs of order n (verify each one of the three defining properties of an ASM). Next, we notice a pattern; an ASM (of any order) can't have a -1 in the first row. Why? To the contrary, assume there is an ASM with a -1 in the first row. Since the immediate non-zero entry below it must be a 1, the column sum cannot be 1 without violating the alternativity condition. A similar argument shows that ASMs cannot have a -1 in the last row, the first column, or the last column either. This pattern allows us to easily list all ASMs of order 3.

First we list all the ASMs counted in Question 1.5:

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}, \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}, \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}, \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}, \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}.$$

Now, the only ASM of order 3 with a negative entry is

$$\begin{pmatrix}0&1&0\\1&-1&1\\0&1&0\end{pmatrix}.$$

2. Counting Principles

3. The Pigeon-hole Principle

4. The Principle of Inclusion-Exclusion

CHAPTER 2

The Art of Bijections

1. Binomial Coefficients

2. Catalan Numbers

CHAPTER 3

Generating Functions

1. Ordinary Generating Functions

2. Exponential Generating Functions

CHAPTER 4

Partitions

1. Set Partitions

2. Integer Partitions

CHAPTER 5

Lattice Path Combinatorics

CHAPTER 6

A Combinatorial Miscellany

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