Indian Institute of Technology, Madras

Computational Tools: Algorithms, Data Structures and Programs - ID6105

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1 Problem 1

Consider a graph $G = (V, E)$,

1. **Prove:** The number of vertices of odd degree in a finite graph is even.

   Proof: Let $\mathcal{V} = \{v_1, v_2, ..., v_n\}$. By the Handshake Lemma,

   $$\sum_{v \in \mathcal{V}} deg(v) = 2|E|$$

   which is even. The sum of integers is even if and only if there are an even number of odd terms. Hence the number of vertices of odd degree is even.

2. **Prove:** Every graph contains two vertices of equal degree.

   Proof: The possible degrees of a vertex in a simple graph on n vertices are 0, 1, ..., $n-1$ (n values). If there no vertices of degree 0 then the possible degrees are 1, 2, ..., $n-1$ ($n-1$ values). If there is a vertex of degree $n-1$, then no vertes can have degree 0, since the $n-1$ dgree vertex is adjacent to all vertices. Thus, it is not possible to simultaneously reallize all n degrees in a simple graph and there are atmost $n-1$ distinct degree values available for n vertices and by the pigeonhole principle two vertices must share the same degree.

3. **Prove:** For a simple connected graph with n number of vertices and e number of edges, $(n–1) \leq e \leq (n + n^2)/2$.

   Proof: The lower bound is $(n-1)$ since any connected graph has a spanning tree and a tree on n edges has $n-1$ edges. The upper bound is invalid it should be $n(n-1)/2$ since a simple graph has atmost one edge between any unordered pair of vertices and there are $\binom{n}{2}$ unordered pairs. Equality occurs when the graph is the complete graph K_n. The lower bound is attained exactly for trees (connected acyclic graphs). A tree has a unique simple path between any pair of vertices.

4. **Prove:** If a simple graph has more than $(n–1)(n–2)/2$ edges, then it must be connected.

   Proof: If a graph is disconnected its vertex set splits into components of sizes $s_1, s_2, ..., s_k$ with $\Sigma s_i = n$ and $k \geq 2$. The number of edges is atmost the sum of complete graphs on those components,

   $$e \leq \sum_{i=1}^{k} \binom{s_i}{2}$$

   For fixed $\Sigma s_i = n$ this sum is maximized when one component has size $n–1$ and the other has size 1 (extreme distribution maximizes sum of squares). That maximum is,

$$\binom{n-1}{2} + \binom{1}{2} = \binom{n-1}{2} = \frac{(n-1)(n-2)}{2}$$

Thus any disconnected graph has $e \leq (n-1)(n-2)/2$. Contrapositive, if $e > (n-1)(n-2)/2$, the graph cannot be disconnected, so it is connected.

5. **Prove:** Given an arbitrary simple planar graph with n number of vertices and e number of edges, the maximum number of edges,M, that can be added to the graph, subject to it remaining planar is given by $M = 3n–e–6$.

   Proof: Use Euler's formula for a connected planar simple graph,

   $$n–e + f = 2$$

   where f is the number of faces. In a simple planar graph every face is bounded by at least 3 edges, and every edge borders at most 2 faces, so

   $$3f \geq 2e \rightarrow fle\frac{2e}{3}$$

   from Euler,

   $$3(2–n + e) \leq 2e \rightarrow 6–3n + 3e \leq 2e \rightarrow e \leq 3n–6$$

   So the maximum number of edges a planar simple graph with n vertices can have is $3n–6$. If the graph currently has e edges, the maximum number M of additional edges you can add while keeping it planar is,

   $$M = (3n–6)–e = 3n–e–6$$

2 Problem 2

Problem Statement: Adjacency matrix for a given simple graph.

```python
import numpy as np

def graph_to_adjacency(V, E):
    # Convert a graph G = (V, E) to its adjacency matrix.
    # V: list of vertices
    # E: list of edges (each edge is a tuple (u, v))
    n = len(V)
    index = {v: i for i, v in enumerate(V)}  # Map vertex to index
    A = np.zeros((n, n), dtype=int)

    for (u, v) in E:
        if u in index and v in index:
            i, j = index[u], index[v]
            A[i][j] = 1
            A[j][i] = 1  # undirected graph
    return A


def adjacency_to_graph(A):
    # Convert an adjacency matrix A to graph G = (V, E)
    n = len(A)
    V = list(range(1, n + 1))
```

```python
    E = []

    for i in range(n):
        for j in range(i + 1, n):  # only the upper triangle
            if A[i][j] == 1:
                E.append((V[i], V[j]))

    return V, E


def complete_graph(n):
    # Generate a complete graph Kn
    V = list(range(1, n + 1))
    E = [(i, j) for i in V for j in V if i < j]
    return V, E


print("============ Complete Graphs =============")
for n in [4, 5, 6, 10]:
    V, E = complete_graph(n)
    A = graph_to_adjacency(V, E)
    print(f"\nK{n}:")
    print("Adjacency Matrix:\n", A)
    V2, E2 = adjacency_to_graph(A)
    print("Recovered E edges:", len(E2))

# The Heawood graph has 14 vertices and 21 edges.
V_heawood = list(range(14))
E_heawood = [
    (0,1), (0,9), (0,13),
    (1,2), (1,10),
    (2,3), (2,11),
    (3,4), (3,12),
    (4,5), (4,13),
    (5,6), (5,10),
    (6,7), (6,11),
    (7,8), (7,12),
    (8,9), (8,10),
    (9,11),
    (10,12),
    (11,13)
]

A_heawood = graph_to_adjacency(V_heawood, E_heawood)
print("\n========== Heawood Graph ================")
print("Adjacency Matrix (14×14):\n", A_heawood)
print("Number of edges:", len(E_heawood))

V_back, E_back = adjacency_to_graph(A_heawood)
print("Recovered vertices:", len(V_back))
print("Recovered edges:", len(E_back))
```

Output:

```
============ Complete Graphs =============

K4:
Adjacency Matrix:
 [[0 1 1 1]
 [1 0 1 1]
 [1 1 0 1]
```

```
 [1 1 1 0]]
Recovered E edges: 6

K5:
Adjacency Matrix:
 [[0 1 1 1 1]
 [1 0 1 1 1]
 [1 1 0 1 1]
 [1 1 1 0 1]
 [1 1 1 1 0]]
Recovered E edges: 10

K6:
Adjacency Matrix:
 [[0 1 1 1 1 1]
 [1 0 1 1 1 1]
 [1 1 0 1 1 1]
 [1 1 1 0 1 1]
 [1 1 1 1 0 1]
 [1 1 1 1 1 0]]
Recovered E edges: 15

K10:
Adjacency Matrix:
 [[0 1 1 1 1 1 1 1 1 1]
 [1 0 1 1 1 1 1 1 1 1]
 [1 1 0 1 1 1 1 1 1 1]
 [1 1 1 0 1 1 1 1 1 1]
 [1 1 1 1 0 1 1 1 1 1]
 [1 1 1 1 1 0 1 1 1 1]
 [1 1 1 1 1 1 0 1 1 1]
 [1 1 1 1 1 1 1 0 1 1]
 [1 1 1 1 1 1 1 1 0 1]
 [1 1 1 1 1 1 1 1 1 0]]
Recovered E edges: 45

========== Heawood Graph ================
Adjacency Matrix (14×14):
 [[0 1 0 0 0 0 0 0 0 1 0 0 0 1]
 [1 0 1 0 0 0 0 0 0 0 1 0 0 0]
 [0 1 0 1 0 0 0 0 0 0 0 1 0 0]
 [0 0 1 0 1 0 0 0 0 0 0 0 1 0]
 [0 0 0 1 0 1 0 0 0 0 0 0 0 1]
 [0 0 0 0 1 0 1 0 0 0 1 0 0 0]
 [0 0 0 0 0 1 0 1 0 0 0 1 0 0]
 [0 0 0 0 0 0 1 0 1 0 0 0 1 0]
 [0 0 0 0 0 0 0 1 0 1 1 0 0 0]
 [1 0 0 0 0 0 0 0 1 0 0 1 0 0]
 [0 1 0 0 0 1 0 0 1 0 0 0 1 0]
 [0 0 1 0 0 0 1 0 0 1 0 0 0 1]
 [0 0 0 1 0 0 0 1 0 0 1 0 0 0]
 [1 0 0 0 1 0 0 0 0 0 0 1 0 0]]

Number of edges: 22
Recovered vertices: 14
Recovered edges: 22
```

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3 Problem 3

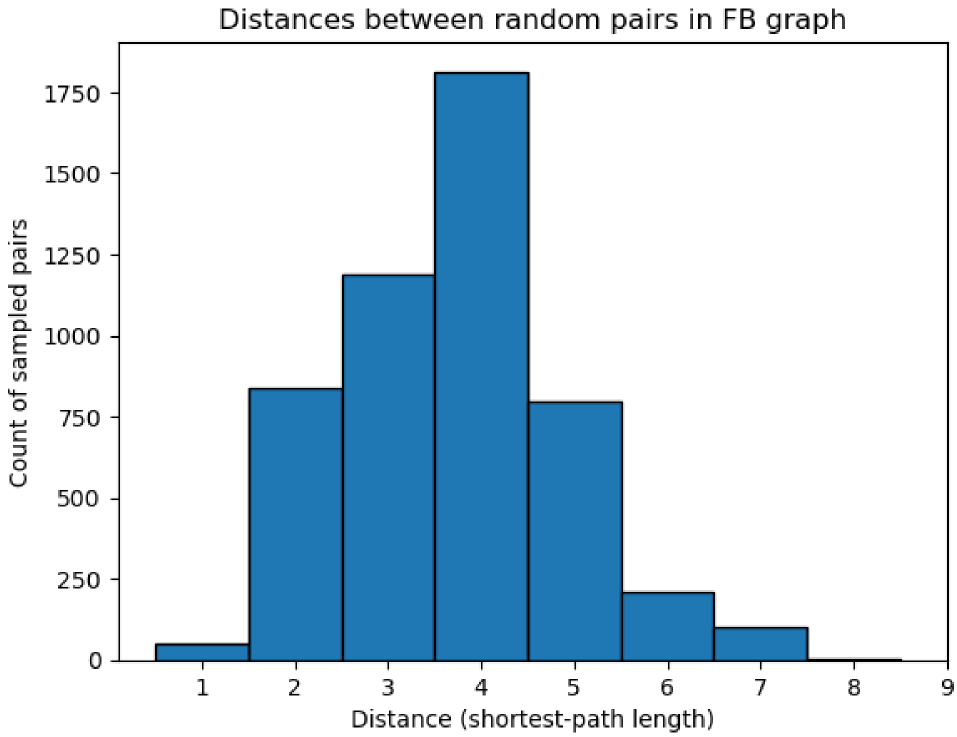
<u>**Problem Statement:**</u> Six degrees of separation.

```python
import random
import networkx as nx
import matplotlib.pyplot as plt
import numpy as np
from statistics import mean, median, stdev

def load_snap_facebook(edgefile):
    # Load graph from SNAP facebook_combined.txt
    G = nx.Graph()
    with open(edgefile, 'rt') as f:
        for line in f:
            if line.startswith('#') or line.strip()=='':
                continue
            u, v = line.split()
            u = int(u); v = int(v)
            G.add_edge(u, v)
    return G

def average_degree(G):
    # Compute the average degree of the Graph
    n = G.number_of_nodes()
    m = G.number_of_edges()
    return 2*m / n

def sample_avg_separation(G, num_samples=1000):
    # Pick random node pairs and compute average shortest-path distance
    nodes = list(G.nodes())
    dists = []
    for _ in range(num_samples):
        u, v = random.sample(nodes, 2)
        try:
            d = nx.shortest_path_length(G, u, v)
        except nx.NetworkXNoPath:
            # you may want to skip unreachable pairs or assign inf
            continue
        dists.append(d)
    return dists  # list of path lengths for sampled pairs

def plot_histogram(distances, title="Histogram of shortest-path distances"):
    maxd = max(distances)
    bins = list(range(1, maxd+2))
    plt.hist(distances, bins=bins, align='left', edgecolor='black')
    plt.xlabel("Distance (shortest-path length)")
    plt.ylabel("Count of sampled pairs")
    plt.title(title)
    plt.xticks(bins)
    plt.show()

def stats_distances(distances):
    # Return (mean, median, std_dev)
    return mean(distances), median(distances), stdev(distances)

if __name__ == "__main__":
    edgefile = "facebook_combined.txt"
    G = load_snap_facebook(edgefile)
    print("Loaded graph: n =", G.number_of_nodes(), "m =", G.number_of_edges())
    print("Average degree:", average_degree(G))

    dists = sample_avg_separation(G, num_samples=5000)
    print("Sample size:", len(dists))
    print("Stats (mean, median, std):", stats_distances(dists))
```

```
    plot_histogram(dists, title="Distances between random pairs in FB graph")
```

Output:

```
Loaded graph: n = 4039 m = 88234
Average degree: 43.69101262688784
Sample size: 5000
Stats (mean, median, std): (3.7054, 4.0, 1.1962010781653227)
```



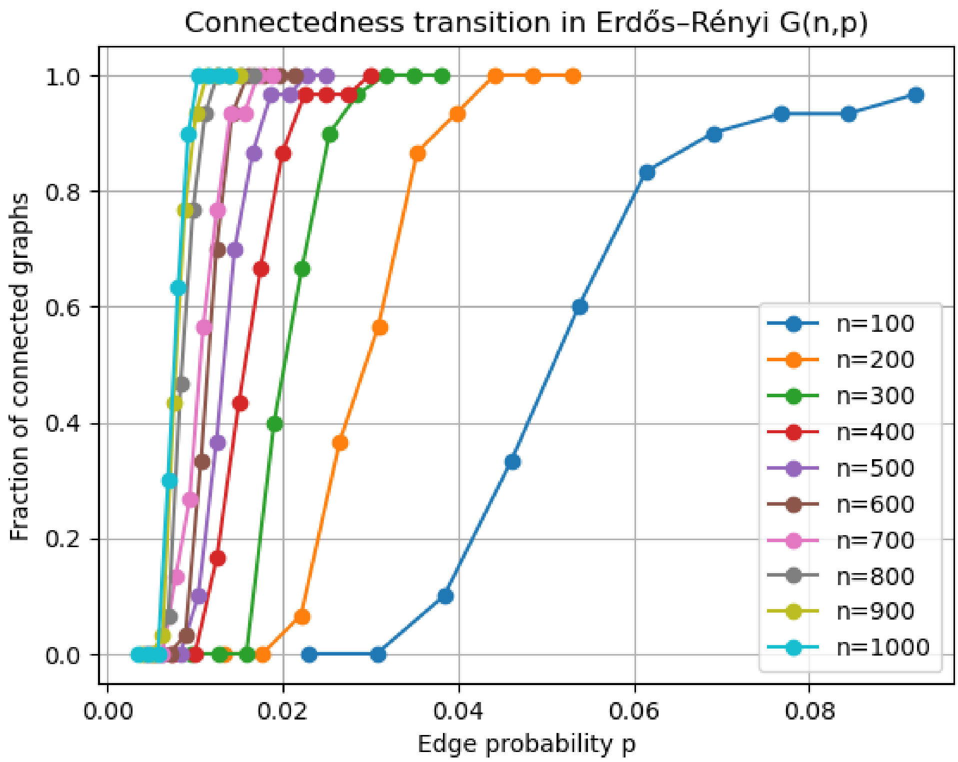
4 Problem 4

Problem Statement: Connectedness of a Erdos-Renyi graph

```
import numpy as np
import networkx as nx
import matplotlib.pyplot as plt
import random

def is_connected(G):
    # Check if a graph is connected using BFS
    return nx.is_connected(G)

def test_connectedness_erdos_renyi():
    n_values = np.arange(100, 1100, 100)
    num_trials = 30
    results = {}

    for n in n_values:
        logn = np.log(n)
        p_values = np.linspace(0.5 * logn / n, 2 * logn / n, 10)
        connected_frac = []
```

```
        for p in p_values:
            count_connected = 0
            for seed in range(num_trials):
                random.seed(seed)
                G = nx.erdos_renyi_graph(n, p, seed=seed)
                if is_connected(G):
                    count_connected += 1
            connected_frac.append(count_connected / num_trials)

        results[n] = (p_values, connected_frac)
        plt.plot(p_values, connected_frac, marker='o', label=f"n={n}")

    plt.xlabel("Edge probability p")
    plt.ylabel("Fraction of connected graphs")
    plt.title("Connectedness transition in Erdős–Rényi G(n,p)")
    plt.legend()
    plt.grid(True)
    plt.show()

    return results

results = test_connectedness_erdos_renyi()
```

Output:



As p increases, graphs move from disconnected (many small components) to almost surely connected. The critical threshold for connectivity is roughly:

$$p_c \approx \frac{\log n}{n}$$

For larger n, the transition becomes sharper (steeper slope).

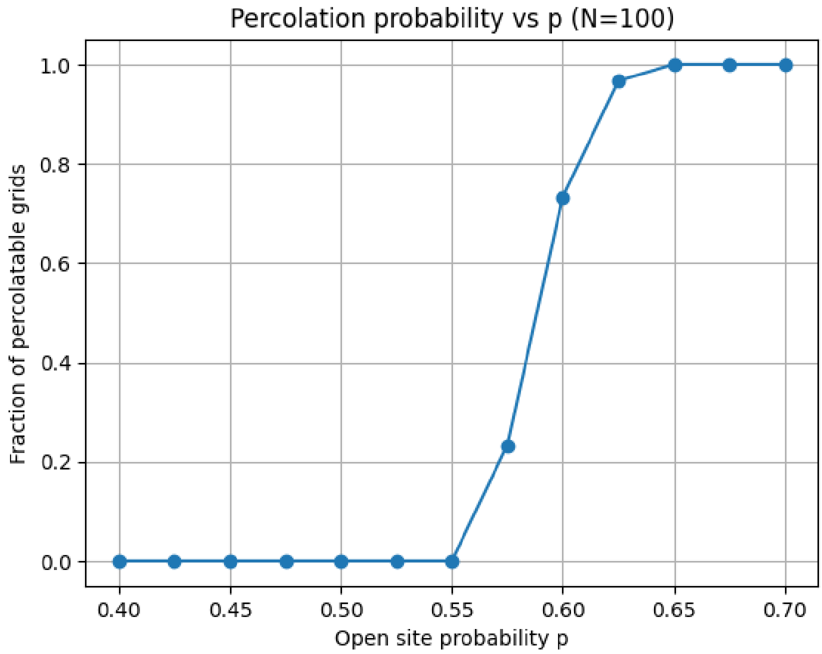
```
import numpy as np
import matplotlib.pyplot as plt
```

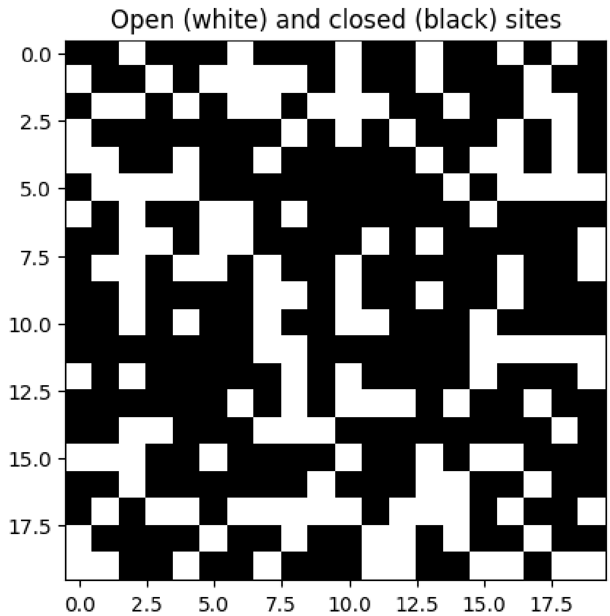
```python
from collections import deque
import random

def generate_grid(N, p):
    # Generate NxN grid with open=1, closed=0
    grid = np.random.rand(N, N) < p
    return grid.astype(int)

def plot_grid(grid):
    plt.imshow(grid, cmap='Greys', origin='upper')
    plt.title("Open (white) and closed (black) sites")
    plt.show()

def percolates(grid):
    # Check if there's a path from top to bottom through open sites
    N = len(grid)
    visited = np.zeros_like(grid, dtype=bool)
    q = deque()

    # enqueue all open cells in the top row
    for j in range(N):
        if grid[0, j] == 1:
            q.append((0, j))
            visited[0, j] = True

    dirs = [(1, 0), (-1, 0), (0, 1), (0, -1)]

    while q:
        i, j = q.popleft()
        if i == N - 1:  # reached bottom row
            return True
        for di, dj in dirs:
            ni, nj = i + di, j + dj
            if 0 <= ni < N and 0 <= nj < N:
                if grid[ni, nj] == 1 and not visited[ni, nj]:
                    visited[ni, nj] = True
                    q.append((ni, nj))
    return False

def test_percolation(N=100, p_values=np.linspace(0.4, 0.7, 13), trials=30):
    percolation_prob = []

    for p in p_values:
        count = 0
        for _ in range(trials):
            grid = generate_grid(N, p)
            if percolates(grid):
                count += 1
        percolation_prob.append(count / trials)
        print(f"p={p:.3f}, percolation prob={percolation_prob[-1]:.2f}")

    plt.plot(p_values, percolation_prob, marker='o')
    plt.xlabel("Open site probability p")
    plt.ylabel("Fraction of percolatable grids")
    plt.title(f"Percolation probability vs p (N={N})")
    plt.grid(True)
    plt.show()

    return p_values, percolation_prob

p_values, perc_prob = test_percolation()
```

```
plot_grid(generate_grid(20, 0.6))
```

Output:

```
p=0.400, percolation prob=0.00
p=0.425, percolation prob=0.00
p=0.450, percolation prob=0.00
p=0.475, percolation prob=0.00
p=0.500, percolation prob=0.00
p=0.525, percolation prob=0.00
p=0.550, percolation prob=0.00
p=0.575, percolation prob=0.20
p=0.600, percolation prob=0.60
p=0.625, percolation prob=1.00
p=0.650, percolation prob=1.00
p=0.675, percolation prob=1.00
p=0.700, percolation prob=1.00
```





The grid will be mostly black (closed) for small p, and increasingly white/open for large p. For $p < p_c \approx 0.59$, percolation rarely occurs. Around $p \approx 0.59$, the probability of percolation rises sharply. For $p > 0.6$, almost all grids percolate.

5 Problem 5

Problem Statement: Connectedness and a Minimal Spanning Tree.

```python
import os
import sys
import pandas as pd
import networkx as nx

EDGE_FILE = "Edges.txt"  # change this path if your file is named differently

def load_edges(edge_file):
    # Try to read; whitespace or comma separated
    try:
        df = pd.read_csv(edge_file, comment='#', header=None, sep=None,
         ↪  engine='python',
                         names=['u', 'v', 'w'])
    except Exception:
        df = pd.read_csv(edge_file, comment='#', header=None, sep=r'\s+', names=['u',
         ↪  'v', 'w'])

    df = df.dropna()
    df['w'] = pd.to_numeric(df['w'], errors='coerce')
    if df['w'].isnull().any():
        print("Warning: some weights could not be parsed as numeric. They will be set
         ↪  to 0.0")
        df['w'] = df['w'].fillna(0.0)
    return df

def build_graph_from_df(df):
    G = nx.Graph()
    for _, row in df.iterrows():
        u, v, w = row['u'], row['v'], float(row['w'])
        G.add_edge(u, v, weight=w)
    return G

def largest_connected_subgraph(G):
    if nx.is_connected(G):
        return G, None
    comps = list(nx.connected_components(G))
    comps.sort(key=len, reverse=True)
    largest = comps[0]
    Gs = G.subgraph(largest).copy()
    return Gs, comps

def compute_mst(G):
    T = nx.minimum_spanning_tree(G, weight='weight', algorithm='kruskal')
    total_w = sum(d['weight'] for _, _, d in T.edges(data=True))
    return T, total_w

try:
    edges = load_edges(EDGE_FILE)
    print(f"Loaded edges: {len(edges)} rows. Sample:")
    print(edges.head(5).to_string(index=False))
except FileNotFoundError:
    sys.exit(1)
```

```python
# Build undirected graph
G = build_graph_from_df(edges)
print(f"\nConstructed undirected graph: n = {G.number_of_nodes()}, m =
↪  {G.number_of_edges()}")

# Connectedness check
if nx.is_connected(G):
    print("Graph is connected.")
    Gcc = G
else:
    print("Graph is NOT connected.")
    Gcc, comps = largest_connected_subgraph(G)
    print(f" - Number of connected components: {len(comps)}")
    print(f" - Largest connected component size: {Gcc.number_of_nodes()} nodes,
     ↪  {Gcc.number_of_edges()} edges")

# Minimal spanning tree
T, total_weight = compute_mst(Gcc)
print(f"\nMinimum Spanning Tree:")
print(f" - Nodes: {T.number_of_nodes()}, Edges: {T.number_of_edges()}")
print(f" - Total weight: {total_weight:.6g}")

pd.DataFrame([(u, v, d['weight']) for u, v, d in T.edges(data=True)], columns=['u',
↪  'v', 'weight']).to_csv("MST_edges.csv", index=False)
print("Saved MST edges to 'MST_edges.csv'.")
```

Output:

```
Loaded edges: 19094 rows. Sample:
   u    v            w
2903 2903 19093.000000
   1    1     0.066908
   2    1    -0.029737
  17    1    -0.029737
 225    1    -0.003717

Constructed undirected graph: n = 2903, m = 10998
Graph is connected.

Minimum Spanning Tree:
 - Nodes: 2903, Edges: 2902
 - Total weight: -232.263
Saved MST edges to 'MST_edges.csv'.
```

---

6 Problem 6

Problem Statement: Connectedness and a Minimal Spanning Tree.

```python
import osmnx as ox
import networkx as nx
import matplotlib.pyplot as plt

city_name = "Chennai, India"

G = ox.graph_from_place(city_name, network_type='drive')
G_undirected = G.to_undirected()
```

```python
print(f"Graph loaded: {G_undirected.number_of_nodes()} nodes,
↪ {G_undirected.number_of_edges()} edges")

if nx.is_connected(G_undirected):
    print("Graph is connected.")
else:
    # Get largest connected component
    largest_cc = max(nx.connected_components(G_undirected), key=len)
    G_undirected = G_undirected.subgraph(largest_cc).copy()
    print(f"Largest connected component has {G_undirected.number_of_nodes()} nodes")

T = nx.minimum_spanning_tree(G_undirected, weight='length')
print(f"MST has {T.number_of_edges()} edges, total length = {sum(d['length'] for _,_,d
↪ in T.edges(data=True)):.2f} meters")

# node with highest degree
source = max(dict(G_undirected.degree()).items(), key=lambda x: x[1])[0]

# Shortest paths
lengths = nx.single_source_shortest_path_length(G_undirected, source)
print(f"Computed shortest paths to {len(lengths)} nodes")

for k in list(lengths.keys())[:10]:
    print(f"{source} -> {k}: {lengths[k]} hops")

import numpy as np
import matplotlib.pyplot as plt

# Degree
degrees = [d for n, d in G_undirected.degree()]
print(f"Average degree: {np.mean(degrees):.2f}")

import random
nodes = list(G_undirected.nodes())
sample_size = 5000
distances = []
for _ in range(sample_size):
    u, v = random.sample(nodes, 2)
    try:
        distances.append(nx.shortest_path_length(G_undirected, u, v))
    except nx.NetworkXNoPath:
        distances.append(np.nan)

distances = [d for d in distances if not np.isnan(d)]
print(f"Mean: {np.mean(distances):.2f}, Median: {np.median(distances):.2f}, Std:
↪ {np.std(distances):.2f}")

# Histogram
plt.hist(distances, bins=range(1, max(distances)+2), edgecolor='black')
plt.xlabel("Shortest path length (hops)")
plt.ylabel("Number of pairs")
plt.show()
```

Output:

```
Graph loaded: 68301 nodes, 90794 edges
Graph is connected.
MST has 68300 edges, total length = 3492451.67 meters

Computed shortest paths to 68301 nodes
```

```
254142574 -> 254142574: 0 hops
254142574 -> 11185535643: 1 hops
254142574 -> 289176728: 1 hops
254142574 -> 6239051390: 1 hops
254142574 -> 254143110: 1 hops
254142574 -> 289176907: 2 hops
254142574 -> 289176908: 2 hops
254142574 -> 289176727: 2 hops
254142574 -> 289176732: 2 hops
254142574 -> 311310222: 2 hops

Average degree: 2.66
Mean: 103.00, Median: 96.00, Std: 47.05
```