Indian Institute of Technology, Madras

Computational Tools: Algorithms, Data Structures and Programs - ID6105

Kanak Agarwal

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1 Problem 1

Problem Statement: Reverse a singly linked list with $\mathcal{O}(n)$ time complexity.

1.1 Python Implementation

```
class Node:
    def __init__(self, data):
        self.data = data
        self.next = None

class LinkedList:
    def __init__(self):
        self.head = None

    def push(self, new_data):
        new_node = Node(new_data)
        new_node.next = self.head
        self.head = new_node

    def print_list(self):
        temp = self.head
        while temp:
            print(temp.data, end=" -> ")
            temp = temp.next
        print("None")

    def reverse(self):
        prev = None
        current = self.head

        while current:
            while current:
                current.next, prev, current = prev, current, current.next

        self.head = prev

if __name__ == "__main__":
    llist = LinkedList()
    for val in [5, 4, 3, 2, 1]:
        llist.push(val)

    print("Original list:")
    llist.print_list()
```

```python
    llist.reverse()
    print()

    print("Reversed list:")
    llist.print_list()
```

1.2 Python Output

```
Original list:
1 -> 2 -> 3 -> 4 -> 5 -> None

Reversed list:
5 -> 4 -> 3 -> 2 -> 1 -> None
```

2 Problem 2

Problem Statement: Tower of Hanoi.

2.1 Python Implementation

```python
def TowerOfHanoi(n, from_rod, to_rod, aux_rod):
    if n == 0:
        return
    TowerOfHanoi(n-1, from_rod, aux_rod, to_rod)
    print("Move disk", n, "from rod", from_rod, "to rod", to_rod)
    TowerOfHanoi(n-1, aux_rod, to_rod, from_rod)

TowerOfHanoi(3, 'A', 'C', 'B')
```

2.2 Python Output

```
Move disk 1 from rod A to rod C
Move disk 2 from rod A to rod B
Move disk 1 from rod C to rod B
Move disk 3 from rod A to rod C
Move disk 1 from rod B to rod A
Move disk 2 from rod B to rod C
Move disk 1 from rod A to rod C
```

2.3 Time Complexity

The Tower of Hanoi problem consists of moving n disks from a source rod to a destination rod, using an auxiliary rod, under the constraints,

- Only one disk can be moved at a time.
- A larger disk cannot be placed on top of a smaller disk.

To move n disks:

1. Move the top $n - 1$ disks from the source rod to the auxiliary rod.

2. Move the largest disk (the n^{th} disk) from the source rod to the destination rod.
3. The auxiliary rod now becomes the source rod and the process repeats.

If $T(n)$ denotes the minimum number of moves required to solve the problem with n disks, then the recurrence relation is:

$$T(n) = 2T(n-1) + 1$$

with the base case:

$$T(1) = 1$$

Unrolling the recurrence:

$$\begin{aligned} T(n) &= 2T(n-1) + 1 \\ &= 2\big(2T(n-2) + 1\big) + 1 = 2^2T(n-2) + 2 + 1 \\ &= 2^3T(n-3) + 2^2 + 2 + 1 \\ &\vdots \\ &= 2^kT(n-k) + (2^{k-1} + \cdots + 2 + 1) \end{aligned}$$

When $k = n - 1$:

$$\begin{aligned} T(n) &= 2^{n-1}T(1) + (2^{n-1} - 1) \\ &= 2^{n-1}(1) + (2^{n-1} - 1) \\ &= 2^n - 1 \end{aligned}$$

Thus, the exact number of moves required is:

$$T(n) = 2^n - 1 \approx \mathcal{O}(2^n)$$

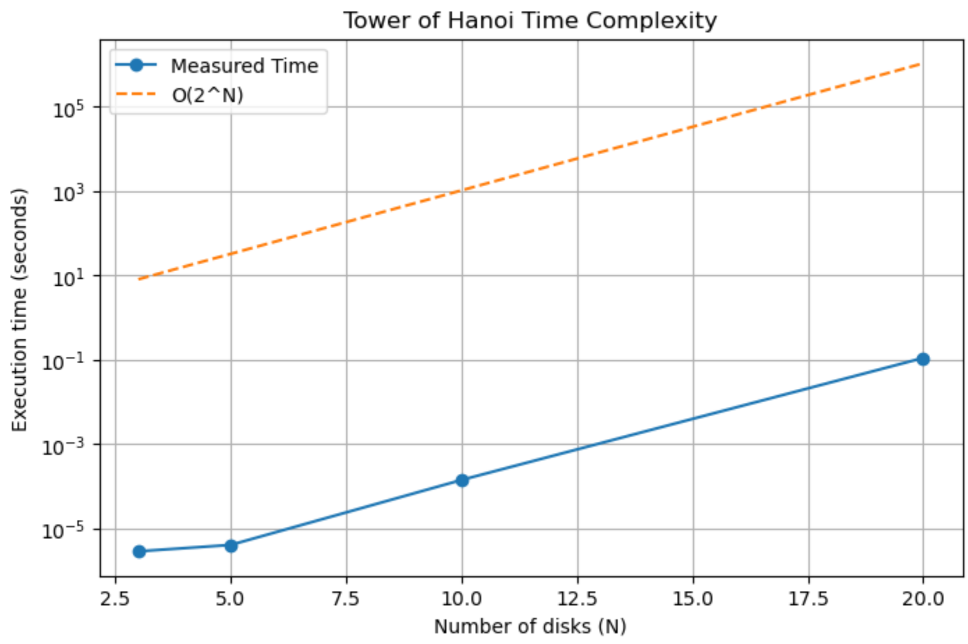
Python Implementation:

```
import time
import matplotlib.pyplot as plt

def TowerOfHanoi(n, from_rod, to_rod, aux_rod):
    if n == 0:
        return
    TowerOfHanoi(n-1, from_rod, aux_rod, to_rod)
    TowerOfHanoi(n-1, aux_rod, to_rod, from_rod)

N = [3, 5, 10, 20]

times = []

for n in N:
    start_time = time.time()
    TowerOfHanoi(n, 'A', 'C', 'B')
    end_time = time.time()
    times.append(end_time - start_time)
    print(f"N={n}, Time={end_time - start_time:.6f} sec")

# Plotting
plt.figure(figsize=(8,5))
plt.plot(N, times, marker='o',label='Measured Time')
plt.plot(N, [2**n for n in N], linestyle='--', label='O(2^N)')
plt.xlabel("Number of disks (N)")
plt.ylabel("Execution time (seconds)")
plt.title("Tower of Hanoi Time Complexity")
plt.yscale("log")
plt.grid(True)
```

```
plt.legend()
plt.show()
```

Python Output:

```
N=3, Time=0.000003 sec
N=5, Time=0.000004 sec
N=10, Time=0.000140 sec
N=20, Time=0.109444 sec
```

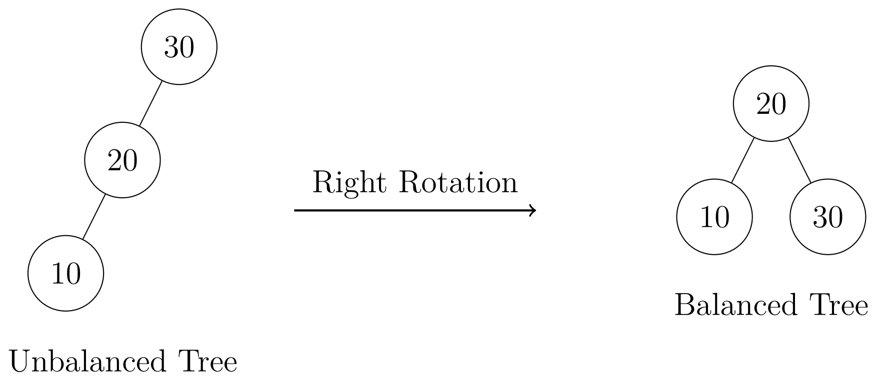


3 Problem 3

Problem Statement: Rotation in a Binary Search Tree.

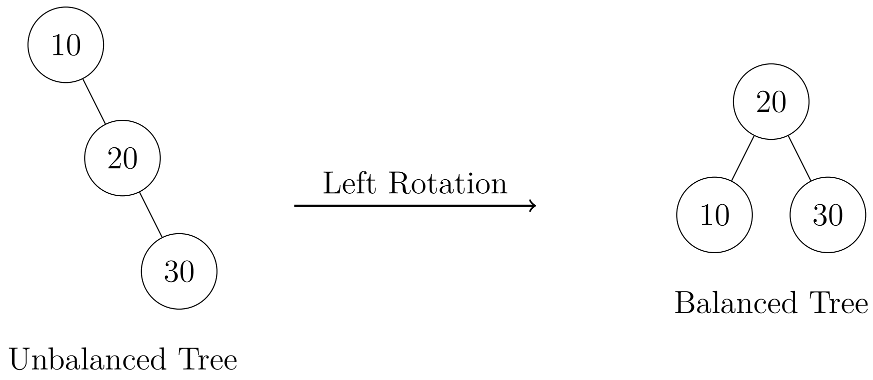
3.1 Right Rotation

Used when a left-heavy imbalance occurs (left subtree is taller than right). The left child becomes the new root, and the previous root becomes the right child of the new root.

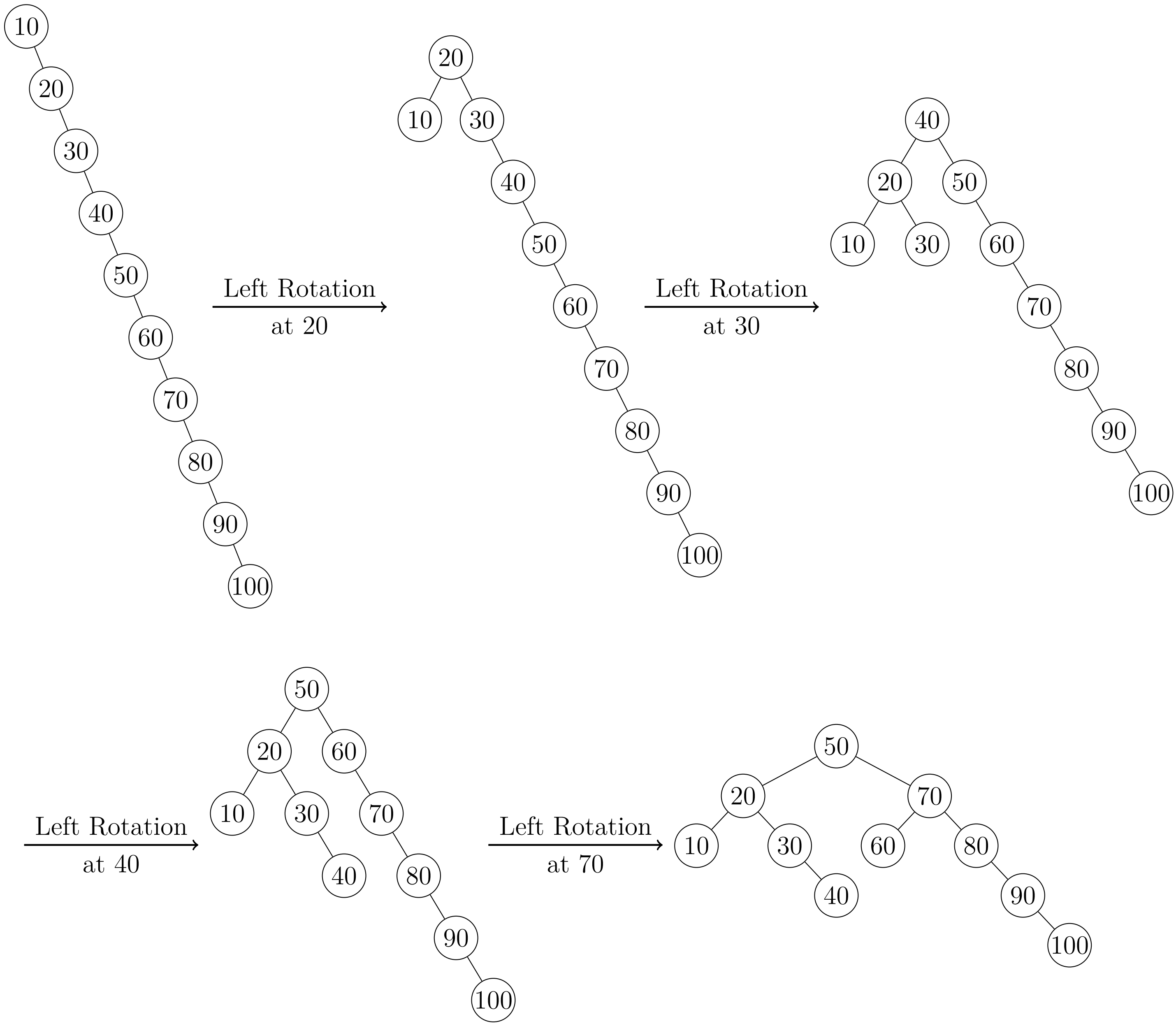


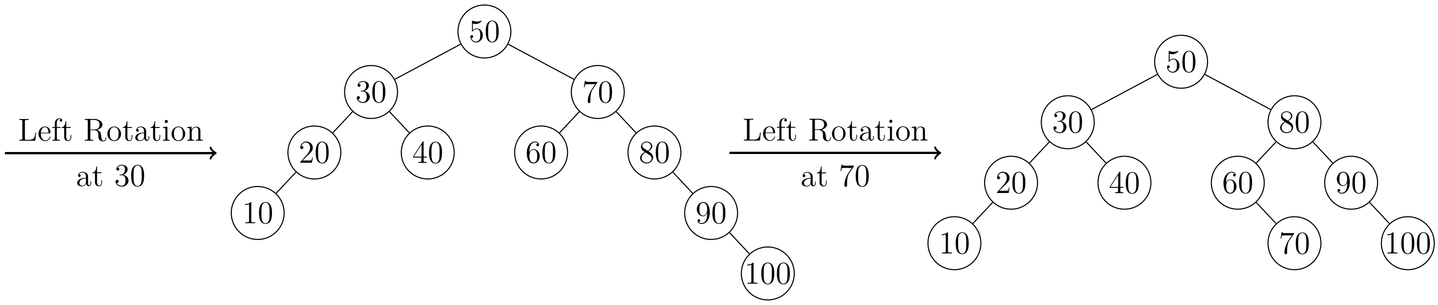
3.2 Left Rotation

Used when a right-heavy imbalance occurs (right subtree is taller than left). The right child becomes the new root, and the previous root becomes the left child of the new root the new root.



4 Balancing the Given Tree





5 Problem 4

Local entropy of a node is given by,

$$H(v_i) = -\sum_{c \in children(v_i)} \left(\frac{W_c}{W_v} \times \log_s \left(\frac{W_c}{W_v} \right) \right)$$

where each node v_i has a positive weight $w_i(v_i)$. The global entropy is given by,

$$S = \sum_{v_i \in T} H(v_i)$$

where T represents the binary tree with n nodes.

Python Implementation:

```
import math
import random
class Node:
    def __init__(self, data):
        self.data = data
        self.left = None
        self.right = None

def insert(root, value):
    if root is None:
        return Node(value)
    if value < root.data:
        root.left = insert(root.left, value)
    else:
        root.right = insert(root.right, value)
    return root

def subtree_weight(root):
    if root is None:
        return 0
    return root.data + subtree_weight(root.left) + subtree_weight(root.right)

def local_entropy(node):
    if node is None:
        return 0
    if node.left is None and node.right is None: # Leaf Node
        return 0

    Wv = subtree_weight(node)
    entropy = 0
    for child in [node.left, node.right]:
        if child:
            Wc = subtree_weight(child)
            p = Wc / Wv
```

```python
            entropy -= p * (0 if p == 0 else math.log2(p))
    return entropy

def total_entropy(node):
    if node is None:
        return 0
    return local_entropy(node) + total_entropy(node.left) + total_entropy(node.right)

def print_tree(node, prefix="", is_left=True):
    if node is not None:
        print_tree(node.right, prefix + ("│   " if is_left else "    "), False)

        # Print current node
        print(prefix + ("└── " if is_left else "┌── ") + str(node.data))

        # Print left subtree
        print_tree(node.left, prefix + ("    " if is_left else "│   "), True)

def print_entropies(node):
    if node is not None:
        print_entropies(node.left)
        print(f"Node {node.data}: Local Entropy = {local_entropy(node):.4f}")
        print_entropies(node.right)

random.seed(0) # Fixes random values generated
numbers = random.sample(range(10, 100), 15)
root = None
for n in numbers:
    root = insert(root, n)
print("Numbers:", numbers,"\n")
print("Binary Search Tree:\n")
print_tree(root)
print()
print("Local Entropies:")
print_entropies(root)
print(f"\nTotal Entropy of the tree: {total_entropy(root):.4f}")
```

Python Output:

```
Numbers: [59, 63, 15, 43, 75, 72, 61, 48, 71, 55, 84, 37, 74, 27, 46]

Binary Search Tree:

│           ┌── 84
│       ┌── 75
│       │   │   ┌── 74
│       │   └── 72
│       │       └── 71
│   ┌── 63
│   │   └── 61
└── 59
    │           ┌── 55
    │       ┌── 48
    │       │   └── 46
    │   ┌── 43
    │   │   └── 37
    │   │       └── 27
    └── 15
```

```
Local Entropies:
Node 15: Local Entropy = 0.0776
Node 27: Local Entropy = 0.0000
Node 37: Local Entropy = 0.5253
Node 43: Local Entropy = 0.9545
Node 46: Local Entropy = 0.0000
Node 48: Local Entropy = 1.0542
Node 55: Local Entropy = 0.0000
Node 59: Local Entropy = 0.9677
Node 61: Local Entropy = 0.0000
Node 63: Local Entropy = 0.6795
Node 71: Local Entropy = 0.0000
Node 72: Local Entropy = 1.0567
Node 74: Local Entropy = 0.0000
Node 75: Local Entropy = 0.9407
Node 84: Local Entropy = 0.0000

Total Entropy of the tree: 6.2562
```

The entropy calculated is a measure of how balanced the binary tree is. A tree with high entropy implies that the weights of the children are distributed roughly evenly under each parent node. Conversely, a binary tree with low entropy has the weights of the children skewed towards one of the subtrees. An important point to note is that while the binary tree may be balanced structurally it could be still skewed in terms of the weights of its children.

Python Implementation for Min/Max Tree:

Note: Few of the functions are the same as the previous code.

```python
def build_tot(numbers):
    # Build tree for a given insertion order and
    # calculate its total entropy
    root = None
    for n in numbers:
        root = insert(root, n)
    return root, total_entropy(root)

def find_min_max_entropy(numbers, trials=1000):
    # Initialisation
    min_entropy, max_entropy = float("inf"), float("-inf")
    min_order, max_order = None, None

    for _ in range(trials):
        shuffled = numbers[:]
        random.shuffle(shuffled)
        _ , entropy = build_tot(shuffled)

        if entropy < min_entropy:
            min_entropy = entropy
            min_order = shuffled[:]
            min_tree = root
        if entropy > max_entropy:
            max_entropy = entropy
            max_order = shuffled[:]
            max_tree = root

    return (min_entropy, min_order), (max_entropy, max_order)

(min_entropy, min_order), (max_entropy, max_order) = find_min_max_entropy(number)

print(f"\nMinimum Entropy: {min_entropy:.4f}, Order: {min_order}")
```

```python
print(f"Maximum Entropy: {max_entropy:.4f}, Order: {max_order}")

max_root, _ = build_tot(max_order)
min_root, _ = build_tot(min_order)

print("\nMaximum Entropy Tree:")
print_tree(max_root)
print(f"Total Entropy = {total_entropy(max_root):.4f}")

print("\nMinimum Entropy Tree:")
print_tree(min_root)
print(f"Total Entropy = {total_entropy(min_root):.4f}")
```

Python Output:

```
Minimum Entropy: 3.6719, Order: [27, 43, 48, 59, 63, 37, 84, 55, 75, 71, 46, 61, 72,
↪  74, 15]
Maximum Entropy: 7.1540, Order: [59, 72, 75, 46, 63, 55, 48, 61, 74, 84, 37, 43, 71,
↪  15, 27]

Maximum Entropy Tree:
│           ┌── 84
│       ┌── 75
│       │   └── 74
│   ┌── 72
│   │   │   ┌── 71
│   │   └── 63
│   │       └── 61
└── 59
    │   ┌── 55
    │   │   └── 48
    └── 46
        │   ┌── 43
        └── 37
            │   ┌── 27
            └── 15
Total Entropy = 7.1540

Minimum Entropy Tree:
│                   ┌── 84
│                   │   └── 75
│                   │       │       ┌── 74
│                   │       │   ┌── 72
│                   │       └── 71
│               ┌── 63
│               │   └── 61
│           ┌── 59
│           │   └── 55
│       ┌── 48
│       │   └── 46
│   ┌── 43
│   │   └── 37
└── 27
    └── 15
Total Entropy = 3.6719
```
