Indian Institute of Technology, Madras

Computational Tools: Algorithms, Data Structures and Programs - ID6105

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1 Problem 1

A $n \times n$ Toeplitz matrix is given by,

$$T[i,j] = T[i-1,j-1] = t_{i-j} \quad \forall \quad 2 \leq \ i,j \ \leq n$$

It can be completely determined by the first row and the first column resulting in $2n-1$ elements. Consider its multiplication with a matrix of size $n \times 1$, this would require n multiplications and $n-1$ addition operations, thus,

$$n(n-1) = \mathcal{O}(n^2)$$

This is a naive multiplication, consider a Strassen's like approach to this problem where the matrix is reduced to 4 subproblems, which reduces to 3 operations. Thus,

$$T(n) = 3T\left(\frac{n}{2}\right) + \mathcal{O}(n)$$

from the master theorem,

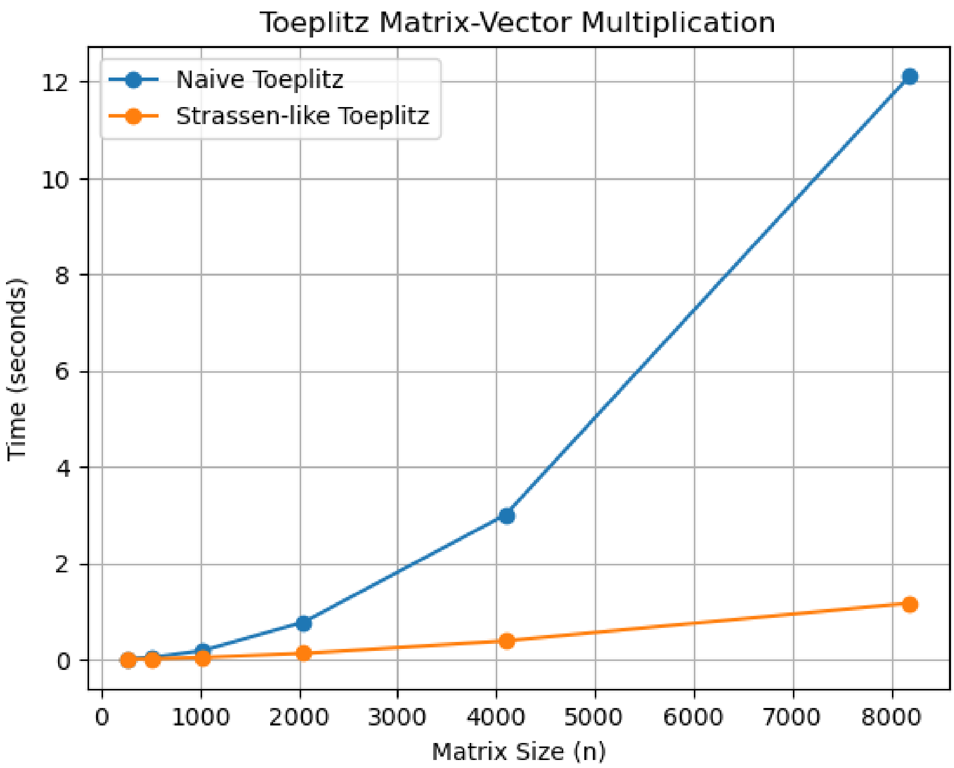
$$T(n) = \mathcal{O}(n^{\log_2 3}) \approx \mathcal{O}(n^{1.585})$$

Python Implementation:

```
import numpy as np
import time
import matplotlib.pyplot as plt

# Naive Toeplitz
def naive_toeplitz_mult(t_col, t_row, x):
    n = len(x)
    T = np.zeros((n,n))
    for i in range(n):
        for j in range(n):
            if i >= j:
                T[i,j] = t_col[i-j]
            else:
                T[i,j] = t_row[j-i]
    return T @ x

# Recursive Strassen-like Toeplitz
def toeplitz_strassen(t_col, t_row, x):
    n = len(x)
    mid = n // 2

    if n <= 32:  # base case - just use naive
        return naive_toeplitz_mult(t_col, t_row, x)
```

```python
    # Split vectors
    x1 = x[:mid]
    x2 = x[mid:]

    # Split into submatrices
    A_col = t_col[:mid]
    A_row = t_row[:mid]

    B_col = t_col[:mid]
    B_row = t_row[mid:]

    C_col = t_col[mid:]
    C_row = t_row[:mid]

    # Recursively compute three multiplications
    P1 = toeplitz_strassen(A_col, A_row, x1 + x2)
    P2 = toeplitz_strassen(A_col - C_col, A_row - C_row, x1)
    P3 = toeplitz_strassen(A_col + B_col, A_row + B_row, x2)

    # Combine
    y1 = P1 - P3
    y2 = P1 + P2
    return np.concatenate([y1, y2])

time_naive = []
time_strassen = []

n = [256, 512, 1024, 2048, 4096, 8192]  # Change index for different sizes
for i in n:
    t_col = np.random.rand(i)
    t_row = np.random.rand(i)
    t_row[0] = t_col[0]  # Ensure first elements are the same
    x = np.random.rand(i)

    start = time.time()
    y = naive_toeplitz_mult(t_col, t_row, x)
    end = time.time()
    time_naive.append(end - start)

    start = time.time()
    y = toeplitz_strassen(t_col, t_row, x)
    end = time.time()
    time_strassen.append(end - start)

# Plotting results
plt.plot(n, time_naive, label='Naive Toeplitz', marker='o')
plt.plot(n, time_strassen, label='Strassen-like Toeplitz', marker='o')
plt.xlabel('Matrix Size (n)')
plt.ylabel('Time (seconds)')
plt.title('Toeplitz Matrix-Vector Multiplication')
plt.legend()
plt.grid()
plt.show()
```

Python Output:



2 Problem 2

Problem Statement: Find the optimum path through a weighted grid.

Brute Force Approach: The Python package itertools is used to generate a set of all possible combinations in one go.

```python
import itertools

class BruteForceSolver:
    def __init__(self, maze):
        self.maze = maze
        self.N, self.M = len(maze), len(maze[0])

    def solve(self):
        moves = ['D'] * (self.N - 1) + ['R'] * (self.M - 1)

        best_cost = float('inf')
        best_path = None

        # Use set to avoid duplicate permutations (since moves contain repeats)
        for perm in set(itertools.permutations(moves)):
            x, y = 0, 0
            cost = self.maze[0][0]
            path = [(0,0)]
            valid = True

            for move in perm:
                if move == 'D':
                    x += 1
                else:  # move == 'R'
                    y += 1
                if x >= self.N or y >= self.M:
```

```
                    valid = False
                    break
                cost += self.maze[x][y]
                path.append((x,y))

            if valid and cost < best_cost:
                best_cost = cost
                best_path = path

        return best_path, best_cost

if __name__ == "__main__":
    maze = [
        [1, 3, 1],
        [1, 5, 1],
        [4, 2, 1]
    ]
    solver = BruteForceSolver(maze)
    path, cost = solver.solve()
    print("Optimal Path (Brute Force):", path)
    print("Minimum Cost:", cost)
```

Python Output:

```
Optimal Path (Brute Force): [(0, 0), (0, 1), (0, 2), (1, 2), (2, 2)]
Minimum Cost: 7
```

Dynamic Programming Approach:

```
class GridSolver:
    def __init__(self, maze):
        self.maze = maze
        self.N, self.M = len(maze), len(maze[0])

    def solve(self):
        dp = [[float('inf')] * self.M for _ in range(self.N)]
        parent = [[None] * self.M for _ in range(self.N)]

        dp[0][0] = self.maze[0][0]

        # Fill DP table
        for i in range(self.N):
            for j in range(self.M):
                if i > 0 and dp[i-1][j] + self.maze[i][j] < dp[i][j]:
                    dp[i][j] = dp[i-1][j] + self.maze[i][j]
                    parent[i][j] = (i-1, j)
                if j > 0 and dp[i][j-1] + self.maze[i][j] < dp[i][j]:
                    dp[i][j] = dp[i][j-1] + self.maze[i][j]
                    parent[i][j] = (i, j-1)

        # Backtrack to find path
        path = []
        cur = (self.N-1, self.M-1)
        while cur:
            path.append(cur)
            cur = parent[cur[0]][cur[1]]
        path.reverse()

        return path, dp[self.N-1][self.M-1]
```

```python
if __name__ == "__main__":
    maze = [
        [1, 3, 1],
        [1, 5, 1],
        [4, 2, 1]
    ]
    solver = GridSolver(maze)
    path, cost = solver.solve()
    print("Optimal Path:", path)
    print("Minimum Cost:", cost)
```

Python Output:

```
Optimal Path (Brute Force): [(0, 0), (0, 1), (0, 2), (1, 2), (2, 2)]
Minimum Cost: 7
```

Larger Grid:

```python
maze = [
    [ 1,  4,  8,  5,  7, 10,  3,  7],
    [ 8,  5,  4,  8,  8,  3,  6,  5],
    [ 2,  8,  6,  2,  5,  1, 10,  6],
    [ 9,  1, 10,  3,  7,  4,  9,  3],
    [ 5,  3,  7,  5,  9,  7,  2,  4],
    [ 9,  2, 10,  9, 10,  5,  2,  4],
    [ 7,  8,  3,  1,  4,  2,  8,  4],
    [ 2,  6,  6, 10,  4,  6,  2,  1]
]

solver = GridSolver(maze)
path, cost = solver.solve()
print("Optimal Path (Dynamic Programming):", path)
print("Minimum Cost:", cost)
```

Python Output:

```
Optimal Path (Dynamic Programming): [(0, 0), (0, 1), (1, 1), (2, 1), (3, 1), (4, 1),
↪ (5, 1), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (7, 5), (7, 6), (7, 7)]
Minimum Cost: 51
```

2.1 Bellman's Principle

To Prove: For the global optimal path solution, the subpaths are also optimal.

Python Implementation:

```python
class GridSolver:
    def __init__(self, maze):
        self.maze = maze
        self.N, self.M = len(maze), len(maze[0])

    def solve(self):
```

```
        dp = [[float('inf')] * self.M for _ in range(self.N)]
        parent = [[None] * self.M for _ in range(self.N)]

        dp[0][0] = self.maze[0][0]

        # Fill DP table and track multiple candidate predecessors
        candidates = [[[] for _ in range(self.M)] for _ in range(self.N)]
        for i in range(self.N):
            for j in range(self.M):
                if i > 0:
                    candidates[i][j].append((dp[i-1][j] + self.maze[i][j], (i-1, j)))
                    if dp[i-1][j] + self.maze[i][j] < dp[i][j]:
                        dp[i][j] = dp[i-1][j] + self.maze[i][j]
                        parent[i][j] = (i-1, j)
                if j > 0:
                    candidates[i][j].append((dp[i][j-1] + self.maze[i][j], (i, j-1)))
                    if dp[i][j-1] + self.maze[i][j] < dp[i][j]:
                        dp[i][j] = dp[i][j-1] + self.maze[i][j]
                        parent[i][j] = (i, j-1)

        # Reconstruct optimal path
        path = []
        cur = (self.N-1, self.M-1)
        while cur:
            path.append(cur)
            cur = parent[cur[0]][cur[1]]
        path.reverse()

        print("Optimal Path:")
        print(path)
        print("Total Minimum Cost:", dp[self.N-1][self.M-1])
        print("\nSubpaths, costs, and candidate options (Bellman principle):")

        # Print subpaths and candidates at each step
        for idx, (i,j) in enumerate(path):
            subpath = path[:idx+1]
            cost = sum([self.maze[x][y] for x,y in subpath])
            print(f"\nSubpath to ({i},{j}): {subpath}")
            print(f"  Cost along path: {cost}")
            print(f"  DP value at cell: {dp[i][j]}")
            print(f"  Candidate options at this cell:")
            for cval, cparent in candidates[i][j]:
                print(f"    From {cparent} with cost {cval}")

        return path, dp[self.N-1][self.M-1]

# Column-wise 8x8 grid
maze = [
    [ 1,  4,  8,  5,  7, 10,  3,  7],
    [ 8,  5,  4,  8,  8,  3,  6,  5],
    [ 2,  8,  6,  2,  5,  1, 10,  6],
    [ 9,  1, 10,  3,  7,  4,  9,  3],
    [ 5,  3,  7,  5,  9,  7,  2,  4],
    [ 9,  2, 10,  9, 10,  5,  2,  4],
    [ 7,  8,  3,  1,  4,  2,  8,  4],
    [ 2,  6,  6, 10,  4,  6,  2,  1]
]

solver = GridSolver(maze)
path, cost = solver.solve()
```

Python Output:

```
Optimal Path:
[(0, 0), (0, 1), (1, 1), (2, 1), (3, 1), (4, 1), (5, 1), (6, 1), (6, 2), (6, 3), (6,
 ↪  4), (6, 5), (7, 5), (7, 6), (7, 7)]
Total Minimum Cost: 51

Subpaths, costs, and candidate options (Bellman principle):

Subpath to (0,0): [(0, 0)]
  Cost along path: 1
  DP value at cell: 1
  Candidate options at this cell:

Subpath to (0,1): [(0, 0), (0, 1)]
  Cost along path: 5
  DP value at cell: 5
  Candidate options at this cell:
    From (0, 0) with cost 5

Subpath to (1,1): [(0, 0), (0, 1), (1, 1)]
  Cost along path: 10
  DP value at cell: 10
  Candidate options at this cell:
    From (0, 1) with cost 10
    From (1, 0) with cost 14

Subpath to (2,1): [(0, 0), (0, 1), (1, 1), (2, 1)]
  Cost along path: 18
  DP value at cell: 18
  Candidate options at this cell:
    From (1, 1) with cost 18
    From (2, 0) with cost 19

Subpath to (3,1): [(0, 0), (0, 1), (1, 1), (2, 1), (3, 1)]
  Cost along path: 19
  DP value at cell: 19
  Candidate options at this cell:
    From (2, 1) with cost 19
    From (3, 0) with cost 21

Subpath to (4,1): [(0, 0), (0, 1), (1, 1), (2, 1), (3, 1), (4, 1)]
  Cost along path: 22
  DP value at cell: 22
  Candidate options at this cell:
    From (3, 1) with cost 22
    From (4, 0) with cost 28

Subpath to (5,1): [(0, 0), (0, 1), (1, 1), (2, 1), (3, 1), (4, 1), (5, 1)]
  Cost along path: 24
  DP value at cell: 24
  Candidate options at this cell:
    From (4, 1) with cost 24
    From (5, 0) with cost 36

Subpath to (6,1): [(0, 0), (0, 1), (1, 1), (2, 1), (3, 1), (4, 1), (5, 1), (6, 1)]
  Cost along path: 32
  DP value at cell: 32
  Candidate options at this cell:
    From (5, 1) with cost 32
    From (6, 0) with cost 49
```

```
Subpath to (6,2): [(0, 0), (0, 1), (1, 1), (2, 1), (3, 1), (4, 1), (5, 1), (6, 1), (6,
 ↪  2)]
  Cost along path: 35
  DP value at cell: 35
  Candidate options at this cell:
    From (5, 2) with cost 37
    From (6, 1) with cost 35

Subpath to (6,3): [(0, 0), (0, 1), (1, 1), (2, 1), (3, 1), (4, 1), (5, 1), (6, 1), (6,
 ↪  2), (6, 3)]
  Cost along path: 36
  DP value at cell: 36
  Candidate options at this cell:
    From (5, 3) with cost 40
    From (6, 2) with cost 36

Subpath to (6,4): [(0, 0), (0, 1), (1, 1), (2, 1), (3, 1), (4, 1), (5, 1), (6, 1), (6,
 ↪  2), (6, 3), (6, 4)]
  Cost along path: 40
  DP value at cell: 40
  Candidate options at this cell:
    From (5, 4) with cost 53
    From (6, 3) with cost 40

Subpath to (6,5): [(0, 0), (0, 1), (1, 1), (2, 1), (3, 1), (4, 1), (5, 1), (6, 1), (6,
 ↪  2), (6, 3), (6, 4), (6, 5)]
  Cost along path: 42
  DP value at cell: 42
  Candidate options at this cell:
    From (5, 5) with cost 46
    From (6, 4) with cost 42

Subpath to (7,5): [(0, 0), (0, 1), (1, 1), (2, 1), (3, 1), (4, 1), (5, 1), (6, 1), (6,
 ↪  2), (6, 3), (6, 4), (6, 5), (7, 5)]
  Cost along path: 48
  DP value at cell: 48
  Candidate options at this cell:
    From (6, 5) with cost 48
    From (7, 4) with cost 50

Subpath to (7,6): [(0, 0), (0, 1), (1, 1), (2, 1), (3, 1), (4, 1), (5, 1), (6, 1), (6,
 ↪  2), (6, 3), (6, 4), (6, 5), (7, 5), (7, 6)]
  Cost along path: 50
  DP value at cell: 50
  Candidate options at this cell:
    From (6, 6) with cost 52
    From (7, 5) with cost 50

Subpath to (7,7): [(0, 0), (0, 1), (1, 1), (2, 1), (3, 1), (4, 1), (5, 1), (6, 1), (6,
 ↪  2), (6, 3), (6, 4), (6, 5), (7, 5), (7, 6), (7, 7)]
  Cost along path: 51
  DP value at cell: 51
  Candidate options at this cell:
    From (6, 7) with cost 52
    From (7, 6) with cost 51
```

From the above Python output it is evident that at each subpath the algorithm chooses the one that is the local optimum and eventually converges to the global optimum. This proves that Bellman's principle is indeed valid and true.

3 Problem 3

Problem Statement: Find the optimum set of change coins for any arbitrary amount.

Brute Force Approach: The Python package itertools is used to generate a set of all possible combinations in one go.

```
import itertools

def brute_force_coin_iter(coins, amount):
    for r in range(1, amount+1):  # max r = amount (worst case: all 1's)
        for combo in itertools.combinations_with_replacement(coins, r):
            if sum(combo) == amount:
                return list(combo)  # first found is minimal
    return None

coins = [1, 2, 5, 10, 20]
amount = 38
result = brute_force_coin_iter(coins, amount)
print("Brute force result:", result)
```

Python Output:

```
Brute force result: [1, 2, 5, 10, 20]
```

Greedy Approach:

```
def greedy_coin(coins, amount):
    coins.sort(reverse=True)  # Sort coins in descending order
    result = []
    for coin in coins:
        while amount >= coin:
            amount -= coin
            result.append(coin)
    return result

coins = [1,2,5,10,20]
amount = 38

result = greedy_coin(coins, amount)
print("Coins used to make", amount, "are:", result)
```

Python Output:

```
Coins used to make 38 are: [20, 10, 5, 2, 1]
```

3.1 The Greedy Approach May Not be Optimal!

```
import itertools

def greedy_coin(coins, amount):
    coins.sort(reverse=True)  # Sort coins in descending order
    result = []
```

```python
    for coin in coins:
        while amount >= coin:
            amount -= coin
            result.append(coin)
    return result

def brute_force_coin_iter(coins, amount):
    for r in range(1, amount+1):  # max r = amount (worst case: all 1's)
        for combo in itertools.combinations_with_replacement(coins, r):
            if sum(combo) == amount:
                return list(combo)  # first found is minimal since r increases
    return None

coins = [1,3,4]
amount = 38
resultb = brute_force_coin_iter(coins, amount)
print("Brute Force Approach:", resultb)

resultg = greedy_coin(coins, amount)
print("Greedy Approach:", resultg)
```

```
Brute Force Approach: [3, 3, 4, 4, 4, 4, 4, 4, 4, 4]
Greedy Approach: [4, 4, 4, 4, 4, 4, 4, 4, 4, 1, 1]
```

As evidenced by the output above the brute force approach does a better job of finding the optimal solution than the greedy algorithm. This can be attributed to the nature of the greedy algorithm to seek out a local maximum which may not coincide with the global optimum solution.

4 Problem 4

Problem Statement: Find the area of an arbitrary shape/polygon.

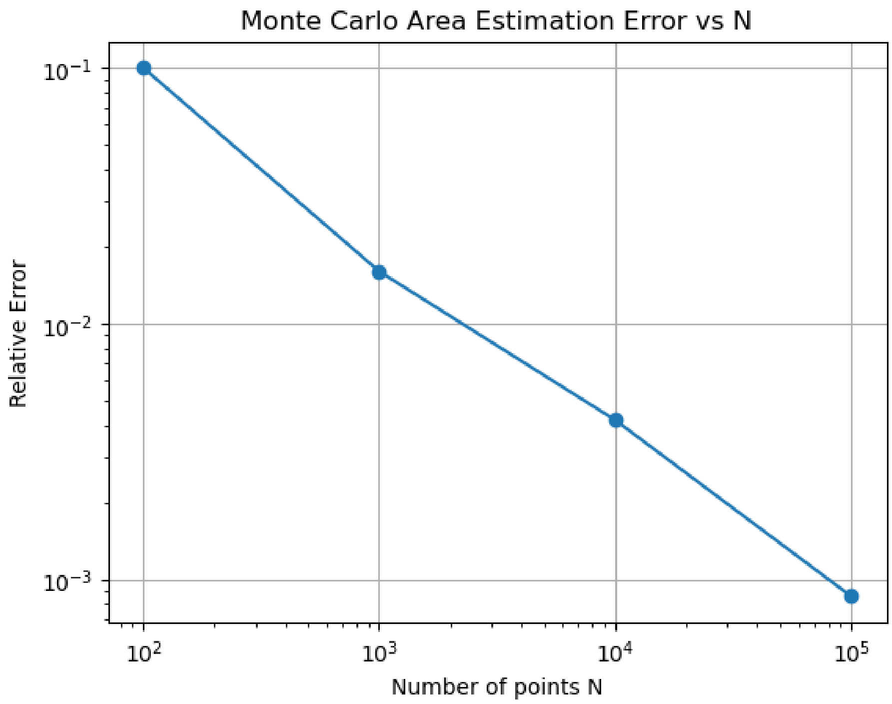
Python Implementation (Polygon):

```python
import numpy as np
import matplotlib.pyplot as plt

# Example polygon (a triangle)
polygon = np.array([[0,0], [2,0], [1,1.5]])

def point_in_polygon(x, y, poly):
    # Ray-casting method
    # Number of intersections - even/odd rule
    # Even - point is outside
    # Odd - point is inside
    n = len(poly)
    inside = False
    px, py = poly[:,0], poly[:,1]
    j = n-1
    for i in range(n):
        if ((py[i] > y) != (py[j] > y)) and \
           (x < (px[j]-px[i]) * (y - py[i]) / (py[j]-py[i] + 1e-12) + px[i]):
            inside = not inside
        j = i
    return inside

# Bounding rectangle
```

```
xmin, ymin = polygon[:,0].min(), polygon[:,1].min()
xmax, ymax = polygon[:,0].max(), polygon[:,1].max()
A_rect = (xmax - xmin) * (ymax - ymin)

N_list = [100, 1000, 10000, 100000]
errors = []

print("Actual Area of Polygon:", 0.5 * 2 * 1.5)

for N in N_list:
    X = np.random.uniform(xmin, xmax, N)
    Y = np.random.uniform(ymin, ymax, N)
    M = sum(point_in_polygon(x, y, polygon) for x, y in zip(X,Y))

    A_est = A_rect * M / N
    A_exact = 0.5 * 2 * 1.5  # exact area of triangle
    error = abs(A_est - A_exact) / A_exact
    errors.append(error)
    print(f"N={N}, Estimated Area={A_est:.4f}, Error={error:.4f}")

# Plot error vs N
plt.figure()
plt.loglog(N_list, errors, marker='o')
plt.xlabel("Number of points N")
plt.ylabel("Relative Error")
plt.title("Monte Carlo Area Estimation Error vs N")
plt.grid(True)
plt.show()
```

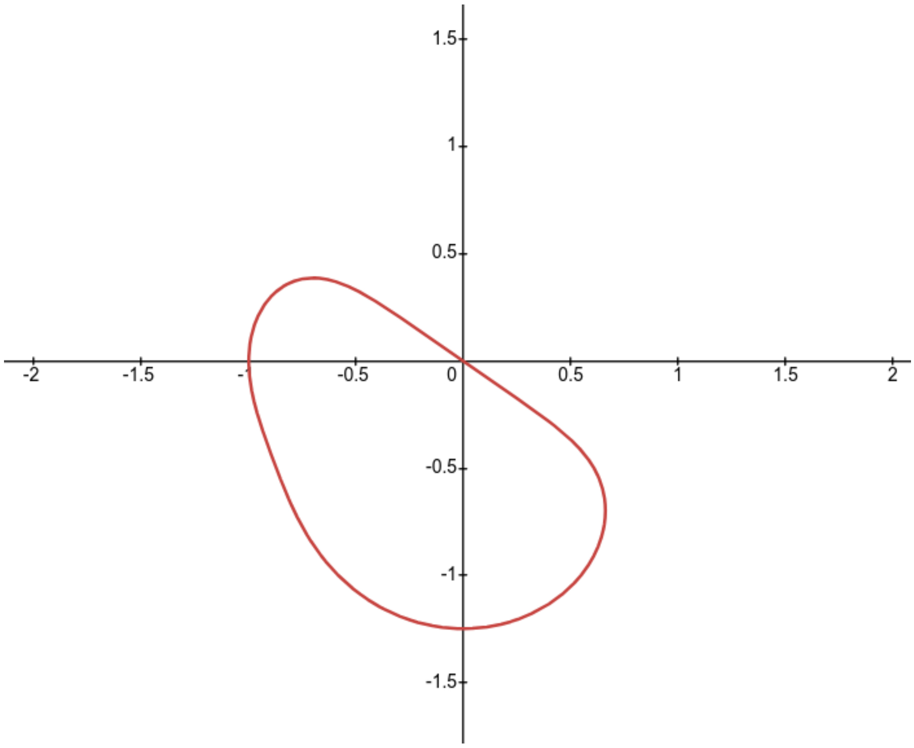
Output:

```
Actual Area of Polygon: 1.5
N=100, Estimated Area=1.3500, Error=0.1000
N=1000, Estimated Area=1.5240, Error=0.0160
N=10000, Estimated Area=1.4937, Error=0.0042
N=100000, Estimated Area=1.5013, Error=0.0009
```



Python Implementation (Arbitrary Shape): The arbitrary shape is given by the function,

$$(x^2 + y^2)^4 + \ x^3 \ + \ 3y^3 \ \leq 0$$

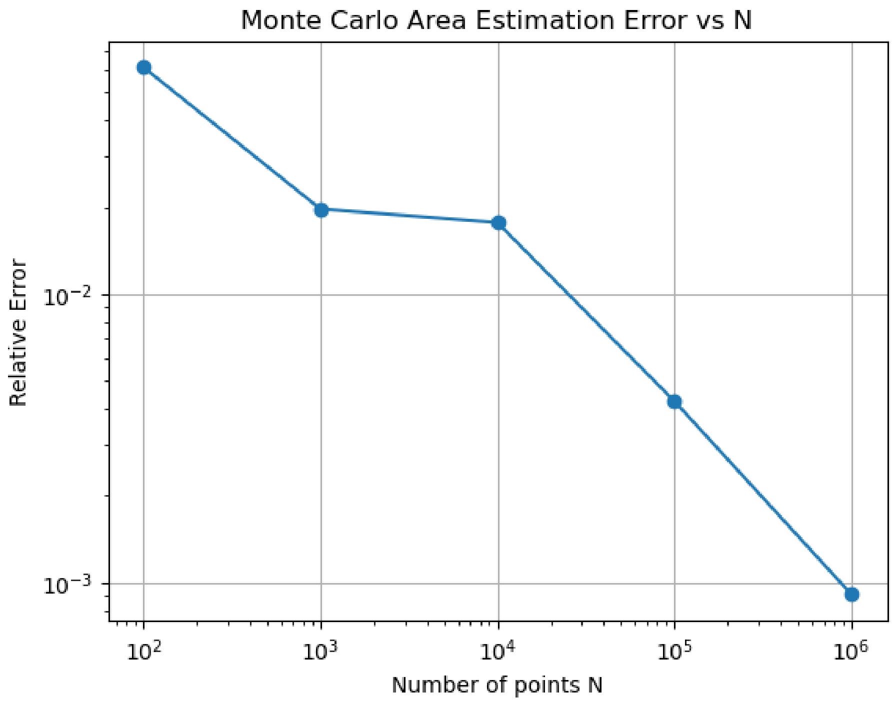


```
import numpy as np
import matplotlib.pyplot as plt

def inside_shape(x, y):
    return (x**2 + y**2)**4 + x**3 + 3*y**3 <= 0

# Bounding rectangle
xmin, xmax = -1.5, 1.5
ymin, ymax = -1.5, 1.5
A_rect = (xmax - xmin) * (ymax - ymin)

# Different numbers of points
N_list = [100, 1000, 10000, 100000, 1000000]
errors = []

N_exact = 10**7 # Assume that the area computed with 10 million points is the "exact"
 ↪ area
X_exact = np.random.uniform(xmin, xmax, N_exact)
Y_exact = np.random.uniform(ymin, ymax, N_exact)
M_exact = sum(inside_shape(x, y) for x, y in zip(X_exact, Y_exact))
A_exact = A_rect * M_exact / N_exact
print("Approximate exact area:", A_exact)


for N in N_list:
    X = np.random.uniform(xmin, xmax, N)
    Y = np.random.uniform(ymin, ymax, N)
    M = sum(inside_shape(x, y) for x, y in zip(X,Y))

    A_est = A_rect * M / N
    error = abs(A_est - A_exact) / A_exact
    errors.append(error)
    print(f"N={N}, Estimated Area={A_est:.4f}, Error={error:.4f}")
```

```
# Plot error vs N
plt.figure()
plt.loglog(N_list, errors, marker='o')
plt.xlabel("Number of points N")
plt.ylabel("Relative Error")
plt.title("Monte Carlo Area Estimation Error vs N")
plt.grid(True)
plt.show()
```

Output:

```
Approximate exact area: 1.7811099
N=100, Estimated Area=1.8900, Error=0.0611
N=1000, Estimated Area=1.7460, Error=0.0197
N=10000, Estimated Area=1.8126, Error=0.0177
N=100000, Estimated Area=1.7735, Error=0.0043
N=1000000, Estimated Area=1.7827, Error=0.0009
```



5 Problem 5

5.1 Partial Dynamics

This problem is a finite-horizon optimal horizon control problem. The dynamics are given by,

$$x_{k+1} = x_k + dt\ u_k, \quad x_0 = 0$$

The terinal constraint is as follows,

$$x_N = 1 \Rightarrow dt \sum_{k=0}^{N-1} u_k = 1$$

The cost function is given by,

$$J(u) = \sum_{k=0}^{N-1} u_k^2$$

Hence the objective is to minimise a quadratic cost (fuel spent) function subject to linear dynamics and a terminal constraint. By convexity, the unique optimal solution is to spread the control equally across all the timesteps,

$$u_k^* = \frac{1}{Ndt}, \quad k = 0, 1, ..., N-1$$

Thus the minimum cost is,

$$J_{min} = \frac{1}{Ndt^2}$$

Further, by Cauchy-Schwarz inequality, the sum of squares is the minimum when all the terms are equal.

5.2 Full Dynamics

The full dynamics (position and velocity vectors) are given by,

$$x_{k+1} = x_k + dt\ u_k, \quad u_{k+1} = u_k + dt\ a_k \quad \forall \quad x_0 = 0, u_0 = 0 \ \&\ x_N = 1, u_N = 0$$

The terinal constraint is as follows,

$$x_N = 1 \Rightarrow dt \sum_{k=0}^{N-1} u_k = 1$$

The cost function is given by,

$$J(a) = (x_N - 1)^2 + u_N^2 + \sum_{k=0}^{N-1} a_k^2$$

Hence the objective is to minimise a quadratic cost (fuel spent) function subject to the full linear dynamics and a terminal constraint. We start off by expressing x_N and u_N in terms of the control sequence $a = [a_0, a_1, ...a_{N-1}]^T$. From the velocity component of the full dynamics,

$$u_N = dt \sum_{k=0}^{N-1} a_i = dt 1^T a$$

where 1 is a vector of ones of length N. Similarly from the position part fo the full dynamics,

$$x_N = dt \sum_{k=0}^{N-1} u_k = dt^2 \sum_{k=0}^{N-1} \sum_{i=0}^{k-1} a_i = dt^2 \sum_{k=0}^{N-1} (N-1-i) a_i = dt^2 w^T a$$

where, w is a vector of length N with $w_i = N - 1 - i$. Hence,

$$u_N = dt 1^T a \quad \& \quad x_N = dt^2 w^T a$$

Plugging this into the cost function,

$$J(a) = a^T a + (dt^2 w^T a - 1)^2 + (dt 1^T a)^2,$$

with

$$1 = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}^T, \qquad w = \begin{bmatrix} N-1 & N-2 & \cdots & 0 \end{bmatrix}^T.$$

Expanding,

$$J(a) \;=\; a^T a \;+\; \left(\tfrac{dt}{2} w^T a - 1\right)^2 \;+\; \left(dt\, 1^T a\right)^2.$$

Taking gradient with respect to a and setting it to zero,

$$\nabla_a J(a) = 0 \Rightarrow (I + \alpha w w^T + \beta 1 1^T) a^* = dt^2 w$$

where,

$$\alpha = dt^4, \quad \beta = dt^2.$$

So the solution is,

$$a^* = (I + \alpha w w^T + \beta 1 1^T)^{-1} dt^2 w$$

This is a well defined positive definite linear system which has a unique solution.

Python Implementation:

```
import numpy as np

def compute_optimal_a(N, dt):
    w = np.array([N-1-i for i in range(N)], dtype=float)
    I = np.eye(N)
    A = I + (dt**4) * np.outer(w,w) + (dt**2) * np.outer(np.ones(N), np.ones(N))
    b = dt**2 * w
    a_opt = np.linalg.solve(A, b)
    return a_opt

# Sample Inputs
N = 10
dt = 1.0
a_opt = compute_optimal_a(N, dt)
print("a_opt:", a_opt)
print("u_N:", dt * a_opt.sum())
print("x_N:", dt**2 * np.dot(np.array([N-1-i for i in range(N)]), a_opt))
print("J:", a_opt.dot(a_opt) + (dt**2 * np.dot(np.array([N-1-i for i in range(N)]),
↪  a_opt) - 1)**2 + (dt * a_opt.sum())**2)
```

Output:

```
a_opt: [ 0.04817128  0.03835861  0.02854594  0.01873327  0.00892061 -0.00089206
 -0.01070473 -0.0205174  -0.03033006 -0.04014273]
u_N: 0.0401427297056199
x_N: 0.9901873327386262
J: 0.009812667261373769
```

5.3 Full Dynamics with Upper Bound

The bounds on a_{max} are enforced using the sequential least squares quadratic programming (SLSQP) implemented using *scipy*. The initial guess is made from the unconstrained solution.

Python Implementation:

```
import numpy as np
from scipy.optimize import minimize

def compute_optimal_a(N, dt):
    w = np.array([N-1-i for i in range(N)], dtype=float)
    I = np.eye(N)
    A = I + (dt**4) * np.outer(w,w) + (dt**2) * np.outer(np.ones(N), np.ones(N))
    b = dt**2 * w
```

```python
    a_opt = np.linalg.solve(A, b)
    return a_opt


def compute_optimal_a_bounded(N, dt, amax):
    # Weight vector
    w = np.array([N-1-i for i in range(N)], dtype=float)

    # Cost function
    def J(a):
        u_N = dt * np.sum(a)
        x_N = dt**2 * np.dot(w, a)
        return np.sum(a**2) + (x_N - 1)**2 + u_N**2

    # bounds on each a_k
    bounds = [(-np.inf, amax) for _ in range(N)]

    a0 = compute_optimal_a(N, dt) # Initial guess

    res = minimize(J, a0, bounds=bounds, method='SLSQP', options={'ftol':1e-12})

    a_opt = res.x
    u_N = dt * np.sum(a_opt)
    x_N = dt**2 * np.dot(w, a_opt)
    J_min = J(a_opt)

    return a_opt, u_N, x_N, J_min

# Sample Inputs
N = 10
dt = 1.0
amax = 0.04
a_opt, u_N, x_N, J_min = compute_optimal_a_bounded(N, dt, amax)

print("Optimal a:", a_opt)
print("u_N:", u_N)
print("x_N:", x_N)
print("Cost J:", J_min)
```

Output:

```
Optimal a: [ 0.04  0.04  0.03216462  0.02160826  0.01105175  0.00049501 -0.01006175
 ↪  -0.02061852 -0.0311753  -0.04173209]
u_N: 0.041731984977793186
x_N: 0.9894431066839109
Cost J: 0.009916701031173562
```

5.4 Full Dynamics with Upper Bound and Debris

The redefined cost function with a penalty at the debris locations is as follows,

$$J(a) = a^T a + (dt^2 w^T a - 1)^2 + (dt 1^T a)^2 + \sum_{x_s} P \sum_{k=0}^{N-1} \exp\left(- \frac{(x_k - x_s)^2}{2\sigma^2} \right)$$

where P is the weight of the penalty and σ is the threshold which controls how far from x_s the penalty is imposed.

Python Implementation:

```python
import numpy as np
from scipy.optimize import minimize

def compute_optimal_a(N, dt):
    w = np.array([N-1-i for i in range(N)], dtype=float)
    I = np.eye(N)
    A = I + (dt**4) * np.outer(w,w) + (dt**2) * np.outer(np.ones(N), np.ones(N))
    b = dt**2 * w
    a_opt = np.linalg.solve(A, b)
    return a_opt

def compute_optimal_a_bounded_debris(N, dt, amax, unsafe_positions, P=100, sigma=0.01):
    w = np.array([N-1-i for i in range(N)], dtype=float)

    # Cost function with debris penalty
    def J(a):
        u_N = dt * np.sum(a)
        x_N = dt**2 * np.dot(w, a)
        # Terminal cost
        cost = np.sum(a**2) + (x_N - 1)**2 + u_N**2

        # Compute all positions x_k
        u_k = np.zeros(N)
        x_k = np.zeros(N)
        for k in range(N):
            u_k[k] = dt * np.sum(a[:k+1])
            x_k[k] = dt * np.sum(u_k[:k+1])

        # Add penalties
        for xs in unsafe_positions:
            cost += P * np.sum(np.exp(-((x_k - xs)**2)/(2*sigma**2)))
        return cost

    # Bounds
    bounds = [(-np.inf, amax) for _ in range(N)]

    a0 = compute_optimal_a(N, dt) # Initial guess

    # Solve
    res = minimize(J, a0, bounds=bounds, method='SLSQP', options={'ftol':1e-12})

    a_opt = res.x
    u_N = dt * np.sum(a_opt)
    x_N = dt**2 * np.dot(w, a_opt)
    J_min = J(a_opt)

    return a_opt, u_N, x_N, J_min

# Example usage
N = 10
dt = 1.0
amax = 0.05
unsafe_positions = [0.4, 0.6]

a_opt, u_N, x_N, J_min = compute_optimal_a_bounded_debris(N, dt, amax,
 ↪  unsafe_positions)
print("Optimal a:", a_opt)
print("u_N:", u_N)
print("x_N:", x_N)
print("J:", J_min)
```

Output:

```
Optimal a: [ 0.01838171  0.02197463  0.02556725  0.02915991  0.03275231  0.036344
 ↪  0.01163681 -0.01307027 -0.03777719 -0.06248458]
u_N: 0.06248456623538016
x_N: 0.9752928077403589
J: 0.014978781222748932
```