6/4/25 Assignment 11

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Exercise 6.1 : 5. $f(x) - p(x) = \frac{(x-a)(x-b)}{b-a}\left[\frac{f(x)-f(b)}{x-b} - \frac{f(x)-f(a)}{x-a}\right]$

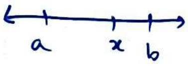
By mean value theorem $\exists\ \xi_1$ & ξ_2 with $\xi_1 \in (x,b)$ & $\xi_2 \in (a,x)$ such that

$$f'(\xi_1) = \frac{f(x)-f(b)}{x-b}, \quad f'(\xi_2) = \frac{f(x)-f(a)}{x-a}$$

$$\therefore |f(x)-p(x)| = \frac{|x-a||x-b|}{|b-a|}\,|f'(\xi_1) - f'(\xi_2)|$$

given $l = b-a$, $|f'(x)| \leq C$ on (a,b)

Maximum of $|(x-a)(x-b)| = \frac{(b-a)^2}{4} = \frac{l^2}{4}$



$$\therefore |f(x)-p(x)| \leq \frac{l^2}{4}\cdot\frac{1}{l}\cdot 2C = \frac{l}{2}C$$

9. $f(x) = \sin(100x)$. If there are n knots, $h = \frac{\pi}{n}$

$f'(x) = 100\cos(100x) \Rightarrow |f'(x)| \leq 100$

$\delta = \max_{0\leq i\leq n-1}(t_{i+1} - t_i) = h$ [in this case we have equal spacing]

given ~~$f(x)$~~ $|f(x) - S(x)| \leq C\frac{\delta}{2}$ where $|f'(x)| \leq C$

$$\Rightarrow 100 \times \frac{\delta}{2} \leq 10^{-8}$$

$$\Rightarrow \delta \leq 2\times 10^{-10}$$

$$\Rightarrow \frac{\pi}{n} \leq 2\times 10^{-10}$$

$$\Rightarrow n \geq \frac{\pi}{2}\times 10^{10} = 1.57\times 10^{10}$$

Exercise 6.2 : 1. $S(x) = \begin{cases} ax^3 + x^2 + cx, & -1 \leq x \leq 0 \\ bx^3 + x^2 + dx & 0 \leq x \leq 1 \end{cases}$

$$\lim_{x\to 0^-} S(x) = \lim_{x\to 0^+} S(x) \Rightarrow 0 = 0$$

$S(-1) = |-1| \Rightarrow -a+1-c = 1 \Rightarrow a+c=0$ —— ①

$S(0) = |0| \Rightarrow 0=0$

$S(1) = |1| \Rightarrow b+1+d = 1 \Rightarrow b+d=0$ —— ②

$$S'(x) = \begin{cases} 3ax^2+2x+c & -1 \le x \le 0 \\ 3bx^2+2x+d & 0 \le x \le 1 \end{cases}$$

$\lim_{x\to 0^-} S'(x) = \lim_{x\to 0^+} S'(x) \Rightarrow c=d$ —— ③

$$S''(x) = \begin{cases} 6ax+2 & -1 \le x \le 0 \\ 6bx+2 & 0 \le x \le 1 \end{cases}$$

$S''(-1) = 0 \Rightarrow -6a+2=0 \Rightarrow a = +1/3$ —— ④

$S''(1) = 0 \Rightarrow 6b+2=0 \Rightarrow b = -1/3$ —— ⑤

① & ④ $\Rightarrow c = -1/3$ } $c=-d \rightarrow$ contradicts ③

② & ⑤ $\Rightarrow d = 1/3$

$\therefore$ There is no such spline function.

5. $$f(x) = \begin{cases} 3+x-9x^3 & [0,1] \\ a+b(x-1)+c(x-1)^2+d(x-1)^3 & [1,2] \end{cases}$$

$$f'(x) = \begin{cases} 1-27x^2 & [0,1] \\ b+2c(x-1)+3d(x-1)^2 & [1,2] \end{cases}$$

$$f''(x) = \begin{cases} -54x & [0,1] \\ 2c+6d(x-1) & [1,2] \end{cases}$$

Continuity ~~at~~ of f at $x=1 \Rightarrow 3+1-9 = a \Rightarrow a=-5$

Continuity of f' at $x=1 \Rightarrow 1-27 = b \Rightarrow b=-26$

Continuity of f'' at $x=1 \Rightarrow -54 = 2c \Rightarrow c=-27$

Now to minimise $\int_0^2 (f''(x))^2 dx$ we only have to minimise $\int_1^2 (f''(x))^2 dx$

$[\because \int (f''(x))^2 dx$ is fixed on $[0,1]]$

$\therefore$ Minimise $\displaystyle\int_1^2 \left(-54 + 6d(x-1)\right)^2 dx := p(d) =$

Need $\dfrac{dp}{dd} = 0 \Rightarrow \cancel{\dfrac{d}{dd}}$

$$p(d) = \int_1^2 \left(-54 + 6d(x-1)\right)^2 dx \quad , \quad u = x-1, \; du = dx$$

$$= \int_0^1 \left(-54 + 6du\right)^2 du$$

$$= \int_0^1 2916 - 648 d \cdot u + 36 d^2 u^2 \, du$$

$$= 2916 - 324d + 12d^2$$

$$\frac{dp}{d \cdot d} = 0 \Rightarrow -324 + 24d = 0 \Rightarrow d = \frac{324}{24} = \frac{27}{2}$$

$\therefore$ We have $a = -5$, $b = -26$, $c = -27$, $d = 27/2$

Exercise 8.1 : 8.

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & -1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

$R_2 \leftarrow R_2 - R_1$

$R_3 \leftarrow R_3 + R_1$

$R_4 \leftarrow R_4 - R_1$

$$\Rightarrow \cancel{A} = \qquad L = \begin{bmatrix} 1 & - & - & - \\ 1 & - & - & - \\ -1 & - & - & - \\ 1 & - & - & - \end{bmatrix}$$

$$A' = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 1 & 1 & 2 \\ 0 & -1 & 1 & -2 \end{bmatrix}$$

$R_3 \leftarrow R_3 - R_2$

$R_4 \leftarrow R_4 + R_2$

$$\Rightarrow L = \begin{bmatrix} 1 & 0 & \cdot & \cdot \\ 1 & 1 & - & - \\ -1 & 1 & - & \cdot \\ 1 & -1 & - & \cdot \end{bmatrix}, \quad A'' = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & -4 \end{bmatrix}$$

$$R_4 \leftarrow R_4 - R_3 \Rightarrow L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 1 & -1 & 1 & 1 \end{bmatrix}, \quad A''' = U = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & -8 \end{bmatrix}$$

4. While doing LU decomposition (or Gaussian elimination) for matrix A, at k^{th} step you eliminate all entries below $A_{KK}^{(K)}$ taking $A_{KK}^{(K)}$ as pivot. But if $A_{KK}^{(K)} = 0$ you do this
after k steps, $A \to A^{(K)}$
So for LU to work, need $A_{KK}^{(K)} \neq 0$ at each step

Let A_K be the top-left $k \times k$ submatrix of A [the leading principal submatrix of order k]

During LU, the pivot elements $u_{11}, u_{22}, \ldots, u_{KK}$ appear on diagonal of $U \to A^{(K)}$

$$\text{But } \det A_K = u_{11} \cdot u_{22} \cdot \ldots \cdot u_{KK}$$

If $\det A_K = 0$, at least one of the pivots must be zero $\Rightarrow$ LU fails.

$\therefore$ For A to be LU decomposible, all leading principal submatrices of A must have determinant non-zero.

$\therefore$ Now $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$

Clearly $\det A_2 = \det \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} = 0$

$\therefore$ A can't be LU decomposed

But $A' = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$ can $\left[\det[3] \neq 0, \det\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \neq 0 \text{ \& } \det A' \neq 0\right]$