7/4/25 Assignment 12

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Exercise 7.1 9. General solution $x(t) = ce^t$

Error in $c = \varepsilon$

$\therefore$ Computed $\tilde{x}(t) = (c+\varepsilon)e^t$

Error $= (c+\varepsilon)e^t - ce^t = \varepsilon e^t$

When $t=10$, error $= \varepsilon \times e^{10} = \varepsilon \times 22026.5$

When $t=20$, error $= \varepsilon \times e^{20} = \varepsilon \times 4.85 \times 10^8$

Now when $x' = -x$

general solution $x(t) = ce^{-t}$

$\tilde{x}(t) = (c+\varepsilon)e^{-t}$

Error $= (c+\varepsilon)e^{-t} - ce^{-t}$

$= \varepsilon e^{-t}$

When $t=10$, error $= \varepsilon \times e^{-10} = \varepsilon \times 4.54 \times 10^{-5}$

When $t=20$, error $= \varepsilon \times e^{-20} = \varepsilon \times 2.09 \times 10^{-9}$

16. Euler's improved method (Heun's method):

$$x(t+h) = x(t) + \frac{h}{2}\left[f(t, x(t)) + f(t+h\, \hat{x}(t+h))\right]$$

where $\hat{x}(t+h) = x(t) + hf(t, x(t))$

Given $f(t, x(t)) = -x+t+1/2$, $h = 0.1$

$t=0$ to $t=1$ so 10 steps. $t_0 = 0$, $x_0 = 1$, $t_i = t_{i-1} + h$

$t_0 = 0$, $x_0 = 1 \Rightarrow f(t_0, x_0) = -1+0+0.5 = -0.5$

$\hat{x}_1 = 1 + (0.1)(-0.5) = 0.95$

$f(t_1, \hat{x}_1) = -0.95 + 0.1 + 0.5 = -0.35$

$x_1 = 1 + \frac{0.1}{2}(-0.5-0.35) = 0.9575$

$f(t_1, x_2) = -0.9575 + 0.1 + 0.5 = -0.3575$

$\hat{x}_2 = 0.9575 + 0.1(-0.3575) = 0.92175$

$f(t_2, \hat{x}_2) = -0.92175 + 0.2 + 0.5 = -0.22175$

$x_2 = 0.9875 + 0.05(-0.3575 - 0.22175) = 0.9285375$

& so on.

We get $\underline{\underline{x(1) \approx 0.6988}}$

Exercise 7.2: 4. $x' = (tx)^3 - \left(\frac{x}{t}\right)^2$, $x(1) = 1$, $h = 0.1$, $t = 1$

Using Taylor Series of order 2: $x(t+h) \approx x(t) + hx'(t) + \frac{h^2}{2}x''(t)$

$x'(1) = 1 - 1 = 0$

$$x''(t) = 3(tx)^2\left[x + tx'\right] + 2\left(\frac{x}{t}\right)\left[\frac{tx' - x}{t^2}\right]$$

$\Rightarrow x''(1) = 3 - (-2) = 5$

$\therefore x(1+0.1) = x(1.1) \approx 1 + (0.1)(0) + \frac{(0.1)^2}{2}(5) = \underline{\underline{1.025}}$

Using RK2: $x(t+h) = x(t) + \frac{1}{2}(K_1 + K_2)$

$K_1 = hf(t,x)$, $K_2 = hf(t+h, x+k_1)$

$K_1 = (0.1)(0) = 0$

$K_2 = (0.1)\, f(1.1, 1) = (0.1)\left[[(1.1)(1)]^3 - \left(\frac{1}{1.1}\right)^2\right] \approx 0.05046$

$\therefore x(1+0.1) = x(1.1) \approx 1 + \frac{1}{2}\left[0 + 0.05046\right] = \underline{\underline{1.02523}}$

→ RK2 gives slightly more accurate answer.

8. $f(x) = \int_0^x e^{-t^2}\,dt \Rightarrow$ By Fundamental theorem of calculus, $f'(x) = e^{-t^2}$

Also $f(0) = 0$. Therefore initial value problem is $\begin{cases} x' = e^{-t^2} \\ x(0) = 0 \end{cases}$

4^{th} order RK method: $x(t+h) = x(t) + \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$ where $K_1 = hf(t,x)$, $K_2 = \frac{1}{2}f(t+\frac{1}{2}h, x+\frac{1}{2}k_1)$, $K_3 = \frac{1}{2}f(t+\frac{1}{2}h, x+\frac{1}{2}K_2)$, $K_4 = hf(t+h, x+K_3)$. Interval is $[0,1]$

Steps $= 100 \Rightarrow h = 0.01$. We start from $x=0, t=0$ & do 100 steps to get till $\underline{\underline{t=1}}$.