4/4/25 Assignments

Kanak Nilesh Sanghvi
Bmat 2416

Exercise 5.1 : 9. $f(x) = \frac{1}{1+x^2}$

$P = \{0, 1/2, 1\}$, $h = 1/2$

Now $\int_a^b f(x)\,dx \approx \frac{h}{2}\left[f(x_0) + f(x_n)\right] + h\sum_{i=1}^{n-1} f(x_i)$

Here $\int_0^1 \frac{1}{1+x^2}\,dx \approx \frac{1}{4}\left[1+\frac{1}{2}\right] + \frac{1}{2}\left[\frac{1}{1+1/4}\right] = \frac{1}{4}\left(\frac{3}{2}\right) + \frac{1}{2}\left(\frac{4}{5}\right) = 0.775$

Actualvalue $= \int_0^1 \frac{1}{1+x^2}\,dx = \tan^{-1}x\Big|_0^1 = \frac{\pi}{4} \approx 0.7854$

$\therefore$ |Actual value − Approximate| ≈ 0.0104

Error formula $= -\frac{1}{12}(b-a)h^2 f''(\xi) = -\frac{1}{12}(1)\left(\frac{1}{2}\right)^2 f''(\xi) = -\frac{1}{48} f''(\xi)$

$f(x) = \frac{1}{1+x^2}$, $f'(x) = \frac{-2x}{(1+x^2)^2}$, $f''(x) = \frac{(1+x^2)^2(-2) + 2x(2x)\,2(1+x^2)}{(1+x^2)^4}$

$= \frac{2(3x^2-1)}{(1+x^2)^3}$

We need an upper bound on $|f''(x)|$ in $[0,1]$

$f'''(x) = \frac{(1+x^2)^3\,2(6x) - 2(3x^2-1)\,3(1+x^2)^2(2x)}{(1+x^2)^6} = \frac{-24x(x^2-1)}{(x^2+1)^4}$

Maxima/minima occurs: $f'''(x) = 0$ at $x=0$, $x=1$

$f''(0) = -2$, $f''(1) = \frac{4}{8} = 1/2$

$\therefore |f''(x)| \le 2$

$\therefore$ Bound on error $= \frac{1}{48}\cdot 2 = \frac{1}{24} \approx 0.0417$

(Note, $0.0104 < 0.0417$)

PTO

10. $\cancel{R(2,0) = h \sum_{i=1}^{2^2-1} f(a+ih) + \frac{h}{2}[f(a)+f(b)]}$, $\cancel{R(0,0) = \frac{1}{2}(b-a)[f(a)+f(b)]}$ 10.

$\cancel{= \frac{1}{2} \sum_{i=1}}$

Recassive formula : $R(n,0) = \frac{1}{2}R(n-1,0) + h\sum_{k=1}^{2^{n-1}} f(a+(2k-1)h)$.

$\therefore R(2,0) = \frac{1}{2}R(1,0) + \frac{1}{4}\sum_{k=1}^{2} f\left(\frac{2k-1}{4}\right)$

(h becomes 1/4 as every iteration, number of subintervals doubles)

$= \frac{1}{2}(0.775) + \frac{1}{4}\left[f(1/4) + f(3/4)\right]$

$= 0.3875 + 0.25\left[\frac{1}{1+0.25^2} + \frac{1}{1+0.75^2}\right] \approx 0.3875 + 0.25(0.941 + 0.64)$

$\therefore R(2,0) \approx 0.78275$

Exercise 5.2 : 5. $\int_0^2 \frac{4\,dx}{1+x^2}$

$R(0,0) = \frac{1}{2}(b-a)[f(a)+f(b)] = \frac{1}{2}(2)\left[4+\frac{4}{5}\right] = \frac{24}{5} = 4.8$

$R(1,0) = \frac{1}{2}R(0,0) + \left(\frac{2-0}{2^1}\right)\sum_{k=1}^{1} f(2k-1)$ $\qquad h = \frac{b-a}{2^n}$

$= \frac{1}{2}(4.8) + f(1)$

$= 2.4 + 2$

$= 4.4$

~~R(1,1) =~~ $R(n,m) = R(n,m-1) + \frac{1}{4^m-1}\left[R(n,m-1) - R(n-1,m-1)\right]$

$\therefore R(1,1) = R(1,0) + \frac{1}{4-1}\left[R(1,0) - R(0,0)\right]$

$= 4.4 + \frac{1}{3}(4.4-4.8)$

≈ 4.2667

PTO

10. 1st column of Romberg array is calculated using

$$R(n,0) = \frac{1}{2}R(n-1,0) + h\sum_{k=1}^{2^{n-1}} f((2k-1)h)$$

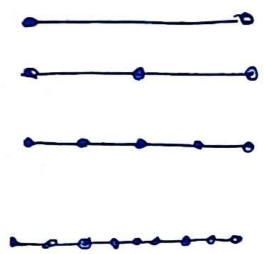
3 total, 5 total, 9 total

$R(0,0)$ uses 2 function evaluation (at a & b)

$R(1,0)$ Adds 1 more evaluation. (at midpoint)

$R(2,0)$ " 2 " "

$R(3,0)$ " 4 " "



$\therefore$ Calculations of function upto $R(m,0) = 2 + \sum_{i=1}^{m} 2^{i-1} = 2 + \frac{1(2^m - 1)}{2-1}$

$= 1 + 2^m$

2nd column $R(n,1)$ is computed using $R(n,1) = \underbrace{R(n,0) + \frac{1}{4^1-1}\left[R(n,0) - R(n-1,0)\right]}_{\text{These are already calculated}}$

$\therefore$ No new calculations

Similarly for $R(n,2)$ & other columns.

~~Also last $R(n,n) = R(n,n-1)$~~

$\therefore$ We need $1+2^{n-1}$ calculations of function value ~~[$\because$ n columns $\Rightarrow R(n,0)$ to $R(n,n-1)$]~~ [$\because$ n rows $\Rightarrow R(0,0)$ to $R(n-1,0)$]

Exercise 5-3 :

1. Basic ~~Sips~~ Simpson's Rule : $\int_a^b f(x)dx \approx \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$

$$\therefore \int_0^1 \frac{dx}{1+x^2} \approx \frac{1/2}{3}\left[1 + \frac{1}{1+0.5^2} + \frac{1}{2}\right] \approx 0.7833$$

True value $= \int_0^1 \frac{dx}{1+x^2} = \tan^{-1}x\Big|_0^1 = \pi/4 \approx 0.7854$

Error $\approx |0.7854 - 0.7833| = 0.0021$

2\. (a) For trapezoid basic rule, Error formula $= -\frac{1}{12}\overset{(1-0)=1}{\cancel{(b-a)}}h^2 f''(\xi_1)$

$$f(x) = \sin\left(\frac{\pi}{2}x^2\right)$$

$$f'(x) = \cos\left(\frac{\pi}{2}x^2\right)\cdot \pi x$$

$$f''(x) = \pi\cos\left(\pi/_2\, x^2\right) - \pi x\sin\left(\frac{\pi}{2}x^2\right)\cdot \pi x$$

$$= \pi\cos\left(\frac{\pi}{2}x^2\right) - \pi^2x^2\sin\left(\frac{\pi}{2}x^2\right)$$

$$\cancel{f'''(x) = -\pi^2 x\sin\left(\frac{\pi}{2}x^2\right) - \pi^3x^3\cos\left(\frac{\pi}{2}x^2\right) = 0}$$

$$\cancel{\Rightarrow -\sin\left(\frac{\pi}{2}x^2\right) = \pi x^2\cos\left(\frac{\pi}{2}x^2\right)}$$

$$\cancel{\Rightarrow \tan\left(\frac{\pi}{2}x^2\right) = -\pi x^2}$$

$$f'''(x) = -\pi^2 x\sin\left(\frac{\pi}{2}x^2\right) - 2\pi^2 x\sin\left(\frac{\pi}{2}x^2\right) - \pi^3x^3\cos\left(\frac{\pi}{2}x^2\right) = 0$$ [to get bound on $f''(x)$]

$\Rightarrow x = 0$ or $\tan\left(\frac{\pi}{2}x^2\right) = -\frac{\pi x^2}{3}$ → Solving this using Newton-Raphson $x \approx 0.4$ (used python)

$f''(0) = \pi$, $f''(1) = -\pi^2$, $|f''(0.4)| \approx 11.05$

$$\therefore |f''(x)| \le 11.05$$

$$\therefore \frac{h^2}{12}\times 11.05 \le 10^{-3} \Rightarrow h \approx 0.033 \Rightarrow h \le 0.033$$

(b) For basic simpsons, Error formula $= \frac{h^4}{180}(b-a)f''''(\xi)$

Same procedure, $\max|f''''(x)| \approx 38.8$

$$\Rightarrow \frac{h^4}{180}\times 38.8 \le 10^{-3} \Rightarrow h \approx 0.17 \Rightarrow h \le 0.17$$

(c) For Simpson's $\frac{3}{8}$ rule, Error formula $= \frac{3}{80}h^4(b-a)f''''(\xi)$

$$\therefore \frac{3h^4}{80}\times 38.8 \le 10^{-3} \Rightarrow h \approx 0.11 \Rightarrow h \le 0.11$$