Elliptic Filter Design Report

Student Details:
Name: Kanak Yadav
Roll No: 20D070044
Filter Number: 55
Group Number: 37

1 Bandpass Filter Design

1.1 Un-normalized Discrete Time Filter

$$q(m) = r(m) = 5$$

Substituting the values of q(m) and r(m), we get

$$\begin{aligned} BL(m) &= 10 + 5(5) + 13(5) \\ &= 100, \\ BH(m) &= 100 + 75 \\ &= 175. \end{aligned}$$

Thus, the passband edges are:

$$\begin{aligned} f_{p1} &= 100kHz, \\ f_{p2} &= 175kHz. \end{aligned}$$

The transition bandwidth is given to be

$$\Delta_f = 5kHz.$$

Thus, the stopband edges are:

$$\begin{aligned} f_{s1} &= f_{p1} - \Delta_f = 95kHz, \\ f_{s2} &= f_{p1} + \Delta_f = 180kHz. \end{aligned}$$

Now we can state the specifications of the discrete-time bandpass filter.

Specifications:

- Passband: 100 kHz to 175 kHz,
- Stopband: 0 to 95 kHz and 180 kHz to 300 kHz (since $f_s = 600$ kHz),
- Passband and Stopband tolerance: 0.15,
- Passband and Stopband nature: Rippled.

1.2 Normalized Digital Filter

Given that our signal of interest is bandlimited to

$$f_b = 280kHz,$$

to satisfy the Nyquist Criterion,

$$f_s > 2f_b = 560kHz.$$

Given sampling frequency is

$$f_s = 600kHz > 560kHz.$$

Hence, the Nyquist criterion is satisfied.

For normalizing the frequencies, we need to make the sampling frequency map to 2π. Thus,

$$\omega = \frac{2\pi f}{f_s}$$

This gives us the normalized digital frequencies as follows:

Discrete time frequency f (kHz)	Normalized digital frequency ω (radians)
0	0
95	$\omega_{s1} = 0.9948377$
100	$\omega_{p1} = 1.0471976$
175	$\omega_{p2} = 1.8325957$
180	$\omega_{s2} = 1.8849556$
300	π

Table 1: Normalizing frequency.

Now we can state the specifications of the digital bandpass filter.

Specifications:

- Passband: 1.0471976 to 1.8325957,
- Stopband: 0 to 0.9948377 and 1.8849556 to π,
- Passband and Stopband tolerance: 0.15,
- Passband and Stopband nature: Rippled.

1.3 Analog Bandpass Filter

For converting the normalized digital specifications to the specifications for an analog filter of the same type (bandpass), we use the bilinear transformation:

$$s = \frac{1 - z^{-1}}{1 + z^{-1}},$$

substituting the values of s and z, we get

$$\begin{aligned} j\Omega &= \frac{1 - e^{-j\omega}}{1 + e^{-j\omega}} = \frac{e^{j\omega/2} - e^{-j\omega/2}}{e^{+j\omega/2} + e^{-j\omega/2}} \\ &= \frac{2j\sin(\omega/2)}{2\cos(\omega/2)} = j\tan(\omega/2) \\ \Omega &= \tan(\omega/2) \end{aligned}$$

This gives us the analog bandpass frequencies as:

Normalized digital frequency ω (radians)	Analog frequency Ω
0	0
0.9948377	$\Omega_{s1} = 0.5429557$
1.0471976	$\Omega_{p1} = 0.5773503$
1.8325957	$\Omega_{p2} = 1.3032254$
1.8849556	$\Omega_{s2} = 1.3763819$
π	∞

Table 2: Applying the bilinear transformation.

Now we can state the specifications of the digital bandpass filter.

Specifications:

- Passband: 0.5773503 to 1.3032254,
- Stopband: 0 to 0.5429557 and 1.3763819 to ∞,
- Passband and Stopband tolerance: 0.15,
- Passband and Stopband nature: Rippled.

1.4 Frequency Transformation

We need to employ a frequency transformation to convert our bandpass filter specifications into those of a lowpass filter. For this, we use the frequency transformation derived from the impedance of a series LC circuit:

$$s_L = \frac{s^2 + \Omega_0^2}{Bs},$$

where the subscript L stands for Lowpass. Ω_0 is the resonant frequency and B is the bandwidth of the series LC circuit. Substituting the values of s_L and s, we get

$$\begin{aligned} j\Omega_L &= \frac{(j\Omega)^2 + \Omega_0^2}{B(j\Omega)} = \frac{1}{j}\frac{(-\Omega^2 + \Omega_0^2)}{B\Omega} \\ &= -j\frac{(-\Omega^2 + \Omega_0^2)}{B\Omega} = j\frac{(\Omega^2 - \Omega_0^2)}{B\Omega} \\ \Omega_L &= \frac{\Omega^2 - \Omega_0^2}{B\Omega} \end{aligned}$$

Now, we have two degrees of freedom, Ω_0 and B. We also decide to follow the convention that the passband edge of the lowpass filter, in both the positive and negative half of the Ω_L axis, has a magnitude of one. In other words, we want Ω_{p1} and Ω_{p2} to map to -1 and +1 respectively on the Ω_L axis. To achieve this, we solve the following two equations

$$-1 = \frac{\Omega_{p1}^2 - \Omega_0^2}{B\Omega_{p1}}$$

and

$$+1 = \frac{\Omega_{p2}^2 - \Omega_0^2}{B\Omega_{p2}}$$

By equating Ω_0^2 we get

$$\begin{aligned} \Omega_{p1}^2 + B\Omega_{p1} &= \Omega_{p2}^2 - B\Omega_{p2} \\ B(\Omega_{p2} + \Omega_{p1}) &= \Omega_{p2}^2 - \Omega_{p1}^2 \\ B &= \Omega_{p2} - \Omega_{p1} \end{aligned}$$

Substituting the value of B, we get

$$\begin{aligned}
\Omega_0^2 &= \Omega_{p1}^2 + (\Omega_{p2} - \Omega_{p1})\Omega_{p1} \\
\Omega_0^2 &= \Omega_{p1}^2 + \Omega_{p2}\Omega_{p1} - \Omega_{p1}^2 = \Omega_{p2}\Omega_{p1} \\
\Omega_0 &= \sqrt{\Omega_{p2}\Omega_{p1}}
\end{aligned}$$

By substituting the value of Ω_{p1} and Ω_{p2}, we get

$$\begin{aligned}
B &= 1.3032254 - 0.5773503 = 0.7258751 \\
\Omega_0 &= \sqrt{(1.3032254)(0.5773503)} = \sqrt{0.7524175} \approx 0.8674200
\end{aligned}$$

Transformation and Parameters

- Transformation:
$$\Omega_L = \frac{\Omega^2 - \Omega_0^2}{B\Omega}$$
- Parameters:
  - $\Omega_0 = 0.8674200$,
  - B = 0.7258751

1.5 Analog Lowpass Filter

Substituting the values of Ω_o and B, we get the frequency transformation to be employed as:

$$\Omega_L = \frac{\Omega^2 - (0.7524175)}{(0.7258751)\Omega}$$

This gives us the analog lowpass frequencies as:

Analog frequency Ω	Analog Lowpass Frequency Ω_L
0^+	$-\infty$
0.5429557	Ω_{Ls1} = -1.1611157
0.5773503	Ω_{Lp1} = -1
1.3032254	Ω_{Lp2} = 1
1.3763819	Ω_{Ls2} = 1.1430597
∞	∞

Table 3: Analog frequency transformation.

The passband edge of the lowpass filter was specified to be one while employing the frequency transformation but, the stopband edge now has two possible values of which, we choose the more stringent value (the value with a smaller magnitude) since satisfying the stronger specifications will automatically satisfy the weaker specifications. Thus

$$\begin{aligned}
\Omega_p &= 1 \\
\Omega_s &= min(abs(\Omega_{Ls1}), abs(\Omega_{Ls1})) \\
&= min(1.1611157, 1.1430597) \\
\Omega_s &= 1.1430597
\end{aligned}$$

Now we can state the specifications of the analog lowpass filter.

Specifications:

- Passband: 0 to 1,
- Stopband: 1.1430597 to ∞,
- Passband and Stopband tolerance: 0.15,
- Passband and Stopband nature: Monotonic.

1.6 Elliptic Analog Lowpass Transfer Function

The first step was finding the poles and zeros of the Elliptic analog lowpass transfer function using the functions provided in MATLAB.

$$N = 4,$$

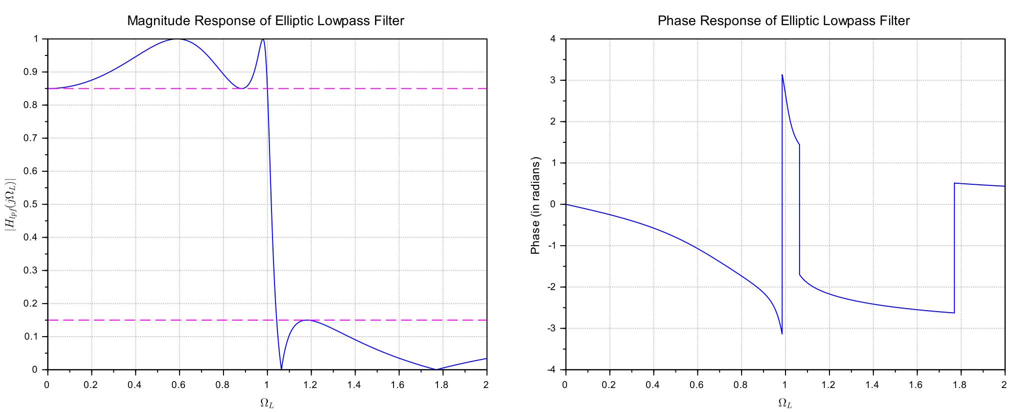
$$\text{poles} = -0.0310 + \jmath 0.9995, -0.3536 + \jmath 0.7071,$$
$$\text{zeros} = \jmath 1.0639, \jmath 1.7690.$$

Then, the Transfer function can be obtained using these poles and zeros and their complex conjugates.

$$\frac{0.85 + 1.0225818s^2 + 0.2399726s^4}{1 + 1.1934801s + 2.6701333s^2 + 1.2307217s^3 + 1.6000022s^4}$$

Figure 1: Screenshot from SCILAB Console.

Lowpass transfer function frequency response:



1.7 Analog Bandpass Transfer Function

This can be obtained by replacing s_L with F(s), the frequency transformation that we had employed earlier:

$$s_L \leftarrow F(s) = \frac{s^2 + \Omega_0^2}{Bs}$$

$\mathbf{H_{analog,\ BPF}(s)}$

$$\frac{0.0213523 + 0.1981943s^2 + 0.5168988s^4 + 0.3500849s^6 + 0.0666207s^8}{0.1423652 + 0.1056443s + 0.9779583s^2 + 0.5165665s^3 + 2.1736417s^4 + 0.6865424s^5 + 1.7274383s^6 + 0.2480101s^7 + 0.4441894s^8}$$

Figure 2: Screenshot from SCILAB Console.

1.8 Discrete Time Bandpass Transfer Function

This can be obtained by applying the bilinear transformation to s:

$$s \leftarrow \frac{1 - z^{-1}}{1 + z^{-1}}$$

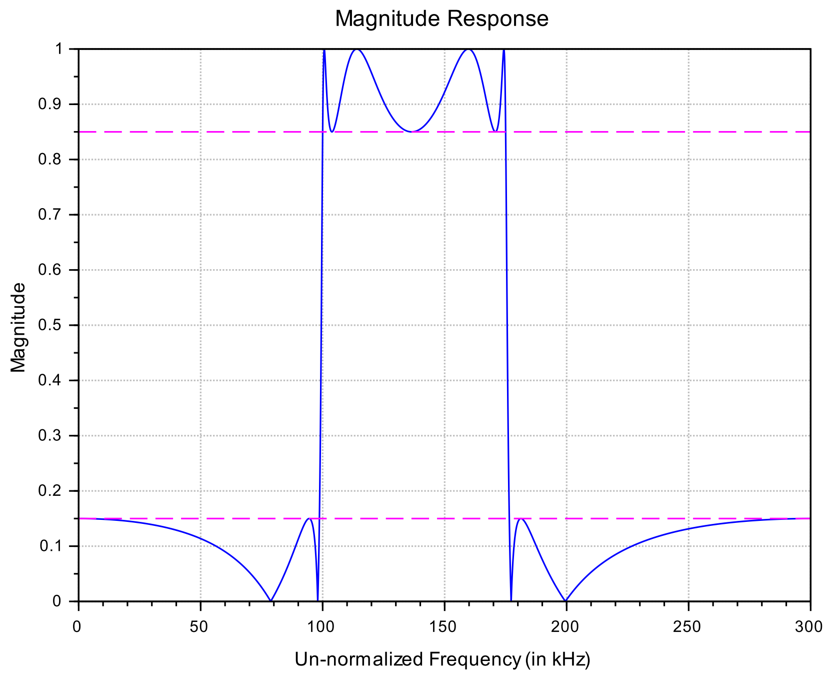
$\mathbf{H_{discrete\ time,\ BPF}(z)}$

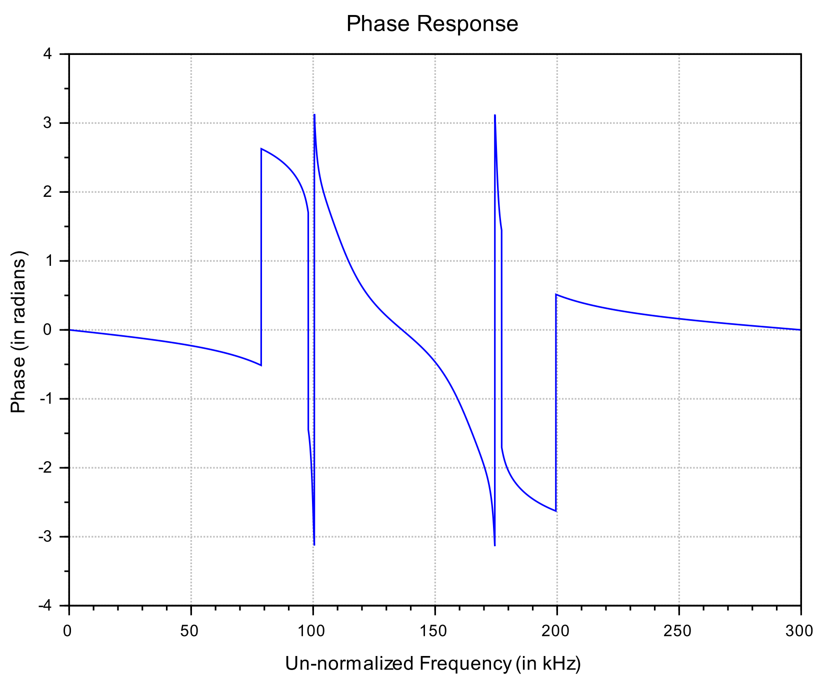
$$\frac{0.1642114 - 0.1380889z + 0.3686463z^2 - 0.274476z^3 + 0.5378125z^4 - 0.274476z^5 + 0.3686464z^6 - 0.138089z^7 + 0.1642114z^8}{0.5566264 - 0.6007054z + 2.2792351z^2 - 1.8414005z^3 + 3.8515134z^4 - 2.1185914z^5 + 3.0040472z^6 - 0.9408041z^7 + z^8}$$

Figure 3: Screenshot from SCILAB Console.

1.9 Frequency Response

The frequency response of the Elliptic Bandpass Filter with order N = 4 is provided on the next page.


Magnitude Response
Magnitude
Un-normalized Frequency (in kHz)


Phase Response
Phase (in radians)
Un-normalized Frequency (in kHz)

2 Bandstop Filter Design

2.1 Un-normalized Discrete Time Filter

$$q(m) = r(m) = 5$$

$$\begin{aligned} BL(m) &= 20 + 3q(m) + 11r(m), \\ &= 20 + 3(5) + 11(5) \\ &= 90, \\ BH(m) &= BL(m) + 40, \\ &= 90 + 40 \\ &= 130. \end{aligned}$$

Therefore, the stopband edges are:

$$\begin{aligned} f_{s1} &= 90kHz, \\ f_{s2} &= 130kHz. \end{aligned}$$

The transition bandwidth is given to be

$$\Delta_f = 5kHz.$$

Therefore, the passband edges are:

$$\begin{aligned} f_{p1} &= f_{s1} - \Delta_f = 85kHz, \\ f_{p2} &= f_{s2} + \Delta_f = 135kHz. \end{aligned}$$

Now we can state the specifications of the discrete-time Bandstop filter.

Specifications:

- Passband: 0 to 85 kHz and 135 kHz to 212.5 kHz (since $f_s = 425$ kHz),
- Stopband: 90 kHz to 130 kHz,
- Passband and Stopband tolerance: 0.15,
- Passband and Stopband nature: Rippled.

2.2 Normalized Digital Filter

Given that our signal of interest is bandlimited to

$$f_b = 200kHz,$$

to satisfy the Nyquist Criterion,

$$f_s > 2f_b = 400kHz.$$

Given sampling frequency is

$$f_s = 425kHz > 400kHz.$$

Hence, the Nyquist criterion is satisfied.

For normalizing the frequencies, we need to make the sampling frequency map to 2π. Thus,

$$\omega = \frac{2\pi f}{f_s}$$

This gives us the normalized digital frequencies as follows:

Discrete time frequency f (kHz)	Normalized digital frequency ω (radians)
0	0
85	$\omega_{p1} = 1.2566371$
90	$\omega_{s1} = 1.3305569$
130	$\omega_{s2} = 1.9219155$
135	$\omega_{p2} = 1.9958353$
212.5	π

Table 4: Normalizing frequency.

Now we can state the specifications of the digital Bandstop filter.

Specifications:

- Passband: 0 to 1.2566371 and 1.9958353 to π,
- Stopband: 1.3305569 to 1.9219155,
- Passband and Stopband tolerance: 0.15,
- Passband and Stopband nature: Rippled.

2.3 Analog Bandstop Filter

For converting the normalized digital specifications to the specifications for an analog filter of the same type (Bandstop), we use the bilinear transformation:

$$s = \frac{1 - z^{-1}}{1 + z^{-1}},$$

substituting the values of s and z, we get

$$\begin{aligned} j\Omega &= \frac{1 - e^{-j\omega}}{1 + e^{-j\omega}} = \frac{e^{j\omega/2} - e^{-j\omega/2}}{e^{+j\omega/2} + e^{-j\omega/2}} \\ &= \frac{2j\sin(\omega/2)}{2\cos(\omega/2)} = j\tan(\omega/2) \\ \Omega &= \tan(\omega/2) \end{aligned}$$

This gives us the analog Bandstop frequencies as:

Normalized digital frequency ω (radians)	Analog frequency Ω
0	0
1.2566371	$\Omega_{p1} = 0.7265425$
1.3305569	$\Omega_{s1} = 0.7845976$
1.9219155	$\Omega_{s2} = 1.4312732$
1.9958353	$\Omega_{p2} = 1.5502977$
π	∞

Table 5: Applying the bilinear transformation.

Now we can state the specifications of the digital Bandstop filter.

Specifications:

- Passband: 0 to 0.7265425 and 1.5502977 to ∞,
- Stopband: 0.7845976 to 1.4312732,
- Passband and Stopband tolerance: 0.15,
- Passband and Stopband nature: Rippled.

2.4 Frequency Transformation

We need to employ a frequency transformation to convert our Bandstop filter specifications into those of a lowpass filter. For this, we use the frequency transformation derived from the impedance of a parallel LC circuit:

$$s_L = \frac{Bs}{s^2 + \Omega_0^2},$$

where the subscript L stands for Lowpass. Ω_0 is the resonant frequency and B is the bandwidth of the parallel LC circuit. Substituting the values of s_L and s, we get

$$\begin{aligned} j\Omega_L &= \frac{B(j\Omega)}{(j\Omega)^2 + \Omega_0^2} \\ &= j\frac{B\Omega}{(-\Omega^2 + \Omega_0^2)} \\ \Omega_L &= \frac{B\Omega}{\Omega_0^2 - \Omega^2} \end{aligned}$$

Now, we have two degrees of freedom, Ω_0 and B. We also decide to follow the convention that the passband edge of the lowpass filter, in both the positive and negative half of the Ω_L axis, has a magnitude of one. In other words, we want Ω_{p1} and Ω_{p2} to map to +1 and -1 respectively on the Ω_L axis. To achieve this, we solve the following two equations

$$+1 = \frac{B\Omega_{p1}}{\Omega_0^2 - \Omega_{p1}^2}$$

and

$$-1 = \frac{B\Omega_{p2}}{\Omega_0^2 - \Omega_{p2}^2}$$

By equating Ω_0^2 we get

$$\begin{aligned} \Omega_{p1}^2 + B\Omega_{p1} &= \Omega_{p2}^2 - B\Omega_{p2} \\ B(\Omega_{p2} + \Omega_{p1}) &= \Omega_{p2}^2 - \Omega_{p1}^2 \\ B &= \Omega_{p2} - \Omega_{p1} \end{aligned}$$

Substituting the value of B, we get

$$\Omega_0^2 = \Omega_{p1}^2 + (\Omega_{p2} - \Omega_{p1})\Omega_{p1}$$
$$\Omega_0^2 = \Omega_{p1}^2 + \Omega_{p2}\Omega_{p1} - \Omega_{p1}^2 = \Omega_{p2}\Omega_{p1}$$
$$\Omega_0 = \sqrt{\Omega_{p2}\Omega_{p1}}$$

By substituting the value of Ω_{p1} and Ω_{p2}, we get

$$B = 1.5502977 - 0.7265425 = 0.8237552$$
$$\Omega_0 = \sqrt{(1.5502977)(0.7265425)} = \sqrt{1.1263572} \approx 1.0612998$$

Transformation and Parameters

- Transformation:
$$\Omega_L = \frac{B\Omega}{\Omega_0^2 - \Omega^2}$$
- Parameters:
  - $\Omega_0 = 1.0612998$,
  - B = 0.8237552

2.5 Analog Lowpass Filter

Substituting the values of Ω_o and B, we get the frequency transformation to be employed as:

$$\Omega_L = \frac{(0.8237552)\Omega}{(1.1263572) - \Omega^2}$$

This gives us the analog lowpass frequencies as:

Analog frequency Ω	Analog Lowpass Frequency Ω_L
0^+	0^+
0.7265425	Ω_{Ls1} = 1.2653920
0.7845976	Ω_{Lp1} = 1
1.4312732	Ω_{Lp2} = -1
1.5502977	Ω_{Ls2} = -1.2785044
∞	0^-

Table 6: Analog frequency transformation.

The passband edge of the lowpass filter was specified to be one while employing the frequency transformation but, the stopband edge now has two possible values of which, we choose the more stringent value (the value with a smaller magnitude) since satisfying the stronger specifications will automatically satisfy the weaker specifications. Thus

$$\begin{aligned}
\Omega_p &= 1 \\
\Omega_s &= min(abs(\Omega_{Ls1}), abs(\Omega_{Ls1})) \\
&= min(1.2653920, 1.2785044) \\
\Omega_s &= 1.2653920
\end{aligned}$$

Now we can state the specifications of the analog lowpass filter.

Specifications:

- Passband: 0 to 1,
- Stopband: 1.2653920 to ∞,
- Passband and Stopband tolerance: 0.15,
- Passband and Stopband nature: Rippled.

2.6 Elliptic Analog Lowpass Transfer Function

The first step was finding the poles and zeros of the Elliptic analog lowpass transfer function using the functions provided in MATLAB.

$$N = 3,$$

$$\text{poles} = -0.1153 + j0.9936,$$

$$\text{additional pole} = -0.6232,$$

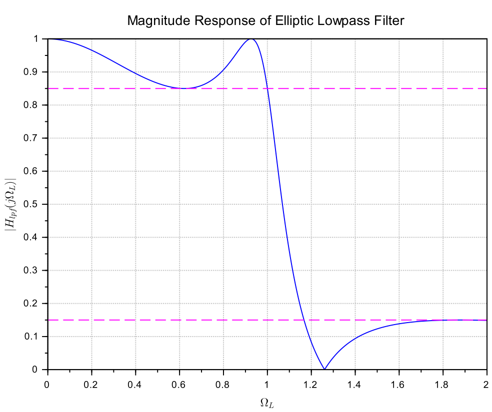
$$\text{zeros} = j1.2604.$$

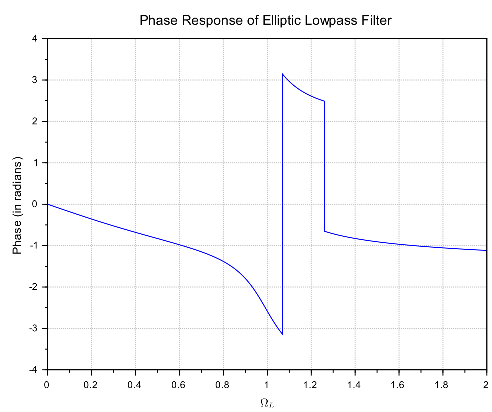
Then, the Transfer function can be obtained using these poles and zeros and their complex conjugates along with the additional pole

$$\frac{1 + 0.6294818s^2}{1 + 1.835098s + 1.369293s^2 + 1.6037632s^3}$$

Figure 4: Screenshot from SCILAB Console.

Lowpass transfer function frequency response:





2.7 Analog Bandstop Transfer Function

This can be obtained by replacing s_L with F(s), the frequency transformation that we had employed earlier:

$$s_L \leftarrow F(s) = \frac{Bs}{s^2 + \Omega_0^2}$$

$\mathbf{H_{analog,\ BSF}(s)}$

$$\frac{1.4289875 + 5.242\text{D-}14s + 4.2871641s^2 + 8.577\text{D-}14s^3 + 3.8062208s^4 + 3.622\text{D-}14s^5 + s^6}{1.4289875 + 1.9178282s + 4.8526131s^2 + 4.3018321s^3 + 4.3082364s^4 + 1.5116715s^5 + s^6}$$

Figure 5: Screenshot from SCILAB Console.

2.8 Discrete Time Bandstop Transfer Function

This can be obtained by applying the bilinear transformation to s:

$$s \leftarrow \frac{1 - z^{-1}}{1 + z^{-1}}$$

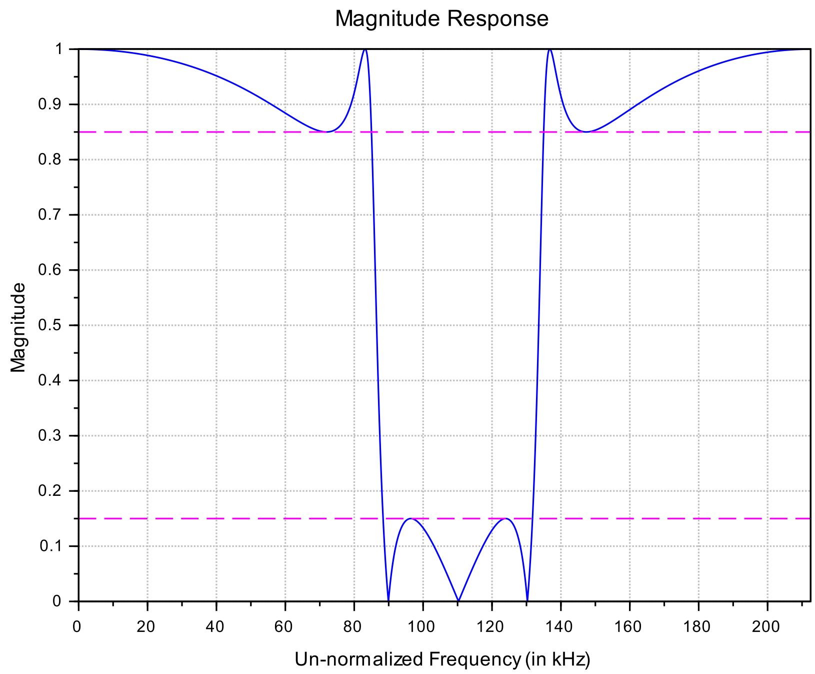
$\mathbf{H_{discrete\ time,\ BSF}(z)}$

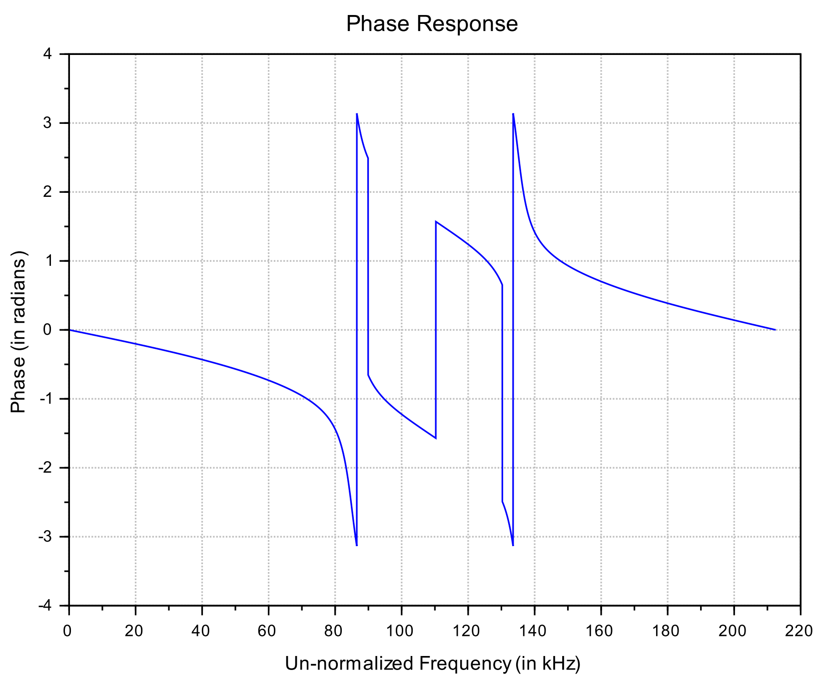
$$\frac{0.5446033 + 0.183002z + 1.4668588z^2 + 0.3444914z^3 + 1.4668588z^4 + 0.183002z^5 + 0.5446033z^6}{0.1997035 + 0.1054828z + 1.1920583z^2 + 0.331359z^3 + 1.6311624z^4 + 0.2736535z^5 + z^6}$$

Figure 6: Screenshot from SCILAB Console.

2.9 Frequency Response

The frequency response of the Elliptic Bandstop Filter with order N = 3 is provided on the next page.


Magnitude Response
Magnitude
Un-normalized Frequency (in kHz)
0 20 40 60 80 100 120 140 160 180 200
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1


Phase Response
Phase (in radians)
Un-normalized Frequency (in kHz)
0 20 40 60 80 100 120 140 160 180 200 220
-4 -3 -2 -1 0 1 2 3 4

3 References

Lecture Notes on Elliptic Filter Design - Rutgers ECE by SJ Orfanidis.