Chebyschev IIR Filter Design Report

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1 Design

1.1 Un-normalized Discrete Time Filter

The un-normalized discrete time filter specifications are uniquely assigned to each student using the filter number assigned to them.

The Chebyschev filter to be designed is a Bandstop filter for filter numbers 1 to 80 and a Bandpass filter for filter numbers 81 to 160.

Hence, we would be designing a Chebyschev IIR Bandstop Filter.

The stopband for the Bandstop filter is BL(m) kHz to BH(m) kHz, which are defined as:

$$q(m) = r(m) = 5$$

$$\begin{aligned}
BL(m) &= 20 + 3q(m) + 11r(m), \\
&= 20 + 3(5) + 11(5) \\
&= 90, \\
BH(m) &= BL(m) + 40, \\
&= 90 + 40 \\
&= 130.
\end{aligned}$$

Therefore, the stopband edges are:

$$\begin{aligned}
f_{s1} &= 90kHz, \\
f_{s2} &= 130kHz.
\end{aligned}$$

The transition band width is given to be

$$\Delta_f = 5kHz.$$

Therefore, the passband edges are:

$$\begin{aligned}
f_{p1} &= f_{s1} - \Delta_f = 85kHz, \\
f_{p2} &= f_{s2} + \Delta_f = 135kHz.
\end{aligned}$$

Now we can state the specifications of the discrete-time Bandstop filter.

Specifications:

- Passband: 0 to 85 kHz and 135 kHz to 212.5 kHz (since $f_s = 425$ kHz),
- Stopband: 100 kHz to 175 kHz,
- Passband and Stopband tolerance: 0.15,
- Passband and Stopband nature: Equiripple and Monotonic respectively.

1.2 Normalized Digital Filter

For normalizing the frequencies, we need to make the sampling frequency map to 2π. Thus,

$$\omega = \frac{2\pi f}{f_s}$$

This gives us the normalized digital frequencies as follows:

Discrete time frequency f (kHz)	Normalized digital frequency ω (radians)
0	0
85	$\omega_{p1} = 1.2566371$
90	$\omega_{s1} = 1.3305569$
130	$\omega_{s2} = 1.9219155$
135	$\omega_{p2} = 1.9958353$
212.5	π

Table 1: Normalizing frequency.

Now we can state the specifications of the digital Bandstop filter.

Specifications:

- Passband: 0 to 1.2566371 and 1.9958353 to π,
- Stopband: 1.3305569 to 1.9219155,
- Passband and Stopband tolerance: 0.15,

- Passband and Stopband nature: Equiripple and Monotonic respectively.

1.3 Analog Bandstop Filter

For converting the normalized digital specifications to the specifications for an analog filter of the same type (Bandstop), we use the bilinear transformation:

$$\Omega = \tan(\omega/2)$$

This gives us the analog Bandstop frequencies as:

Normalized digital frequency ω (radians)	Analog frequency Ω
0	0
1.2566371	$\Omega_{p1} = 0.7265425$
1.3305569	$\Omega_{s1} = 0.7845976$
1.9219155	$\Omega_{s2} = 1.4312732$
1.9958353	$\Omega_{p2} = 1.5502977$
π	∞

Table 2: Applying the bilinear transformation.

Now we can state the specifications of the digital Bandstop filter.

Specifications:

- Passband: 0 to 0.7265425 and 1.5502977 to ∞,
- Stopband: 0.7845976 to 1.4312732,
- Passband and Stopband tolerance: 0.15,
- Passband and Stopband nature: Equiripple and Monotonic respectively.

1.4 Frequency Transformation

We need to employ a frequency transformation to convert our Bandstop filter specifications into those of a lowpass filter. For this, we use the frequency transformation derived from the impedance of a parallel LC circuit:

$$s_L = \frac{Bs}{s^2 + \Omega_0^2},$$

where the subscript L stands for Lowpass. Ω_0 is the resonant frequency and B is the bandwidth of the parallel LC circuit. Substituting the values of s_L and s, we get

$$\begin{aligned} j\Omega_L &= \frac{B(j\Omega)}{(j\Omega)^2 + \Omega_0^2} \\ &= j\frac{B\Omega}{(-\Omega^2 + \Omega_0^2)} \\ \Omega_L &= \frac{B\Omega}{\Omega_0^2 - \Omega^2} \end{aligned}$$

Now, we have two degrees of freedom, Ω_0 and B. We also decide to follow the convention that the passband edge of the lowpass filter, in both the positive and negative half of the Ω_L axis, has a magnitude of one. In other words, we want Ω_{p1} and Ω_{p2} to map to +1 and -1 respectively on the Ω_L axis. To achieve this, we solve the following two equations

$$+1 = \frac{B\Omega_{p1}}{\Omega_0^2 - \Omega_{p1}^2}$$

and

$$-1 = \frac{B\Omega_{p2}}{\Omega_0^2 - \Omega_{p2}^2}$$

Solving we get

$$\begin{aligned} B &= \Omega_{p2} - \Omega_{p1} \\ \Omega_0 &= \sqrt{\Omega_{p2}\Omega_{p1}} \end{aligned}$$

By substituting the value of Ω_{p1} and Ω_{p2}, we get

$$\begin{aligned} B &= 1.5502977 - 0.7265425 = 0.8237552 \\ \Omega_0 &= \sqrt{(1.5502977)(0.7265425)} = \sqrt{1.1263572} \approx 1.0612998 \end{aligned}$$

Transformation and Parameters

- Transformation:
$$\Omega_L = \frac{B\Omega}{\Omega_0^2 - \Omega^2}$$
- Parameters:
  - $\Omega_0 = 1.0612998$,
  - $B = 0.8237552$

1.5 Analog Lowpass Filter

Substituting the values of Ω_o and B, we get the frequency transformation to be employed as:

$$\Omega_L = \frac{(0.8237552)\Omega}{(1.1263572) - \Omega^2}$$

This gives us the analog lowpass frequencies as:

Analog frequency Ω	Analog Lowpass Frequency Ω_L
0^+	0^+
0.7265425	$\Omega_{Ls1} = 1.2653920$
0.7845976	$\Omega_{Lp1} = 1$
Ω_0^-	∞^+
Ω_0^+	∞^-
1.4312732	$\Omega_{Lp2} =$ -1
1.5502977	$\Omega_{Ls2} =$ -1.2785044
∞	0^-

Table 3: Analog frequency transformation.

The passband edge of the lowpass filter was specified to be one while employing the frequency transformation but, the stopband edge now has two possible values of which, we choose the more stringent value (the value with a smaller magnitude) since satisfying the stronger specifications will automatically satisfy the weaker specifications. Thus

$$
\begin{aligned}
\Omega_p &= 1 \\
\Omega_s &= min(abs(\Omega_{Ls1}), abs(\Omega_{Ls1})) \\
&= min(1.2653920, 1.2785044) \\
\Omega_s &= 1.2653920
\end{aligned}
$$

Now we can state the specifications of the analog lowpass filter.

Specifications:

- Passband: 0 to 1,
- Stopband: 1.2653920 to ∞,
- Passband and Stopband tolerance: 0.15,
- Passband and Stopband nature: Equiripple and Monotonic respectively.

1.6 Chebyschev Analog Lowpass Transfer Function

The Chebyschev analog lowpass transfer function is given as:

$$\mathrm{H}_{\text{analog, LPF}}(\mathrm{s_L}) = \frac{C}{\prod_{k \epsilon LHP}(s - s_k)}.$$

N is given by the specifications as:

$$N \geq \frac{\cosh^{-1}\sqrt{\frac{D_2}{D_1}}}{\cosh^{-1}\frac{\Omega_s}{\Omega_p}},$$

where D_1 and D_2 are given by:

$$
\begin{aligned}
D_1 &= \frac{1}{(1-\delta_1)^2} - 1, \\
D_2 &= \frac{1}{\delta_2^2} - 1.
\end{aligned}
$$

δ_1 and δ_2 are the passband and stopband tolerances.

$$D_1 = \frac{1}{(1-0.15)^2} - 1 = 0.3840830$$
$$D_2 = \frac{1}{0.15^2} - 1 = 43.444444$$

This gives us

$$N \geq \frac{\cosh^{-1}(\sqrt{113.11211})}{\cosh^{-1}(1.1430597)} \approx 4.2829034,$$

Hence, we choose

$$N = 5.$$

Since N is odd,

$$C = \prod_{k \epsilon LHP} |s_k|$$

to satisfy,

$$|\mathrm{H}_{\text{analog, LPF}}(0)| = 1,$$

since for odd N,

$$C_N(0) = 0.$$

Now the poles can be found by setting the denominator of the magnitude squared form of the Chebyschev lowpass filter transfer function to zero.

$$|\mathrm{H}_{\text{analog, LPF}}(\mathrm{s_L})|^2 = \frac{1}{1 + \epsilon^2 C_N^2(\frac{s}{j\Omega_p})}$$

where,

$$\epsilon = \sqrt{D_1} = 0.6197443.$$

Now,

$$1 + \epsilon^2 C_N^2 \left(\frac{s_k}{j\Omega_p} \right) = 0$$
$$C_N^2 \left(\frac{s_k}{j\Omega_p} \right) = \frac{-1}{\epsilon^2}$$
$$C_N \left(\frac{s_k}{j\Omega_p} \right) = \pm \frac{j}{\epsilon}$$

To solve this, let us take

$$\cos^{-1}(\frac{s_k}{j\Omega_p}\Bigg) = A_k + jB_k$$

$$\frac{s_k}{j\Omega_p} = \cos(A_k + jB_k)$$

$$s_k = j\Omega_p(cos(A_k)cos(jB_k) - sin(A_k)sin(jB_k))$$

$$s_k = \Omega_p sin(A_k)sinh(B_k) + j\Omega_p cos(A_k)cosh(B_k)$$

It is interesting to see that while the poles of the Butterworth lowpass filter transfer function lie on the circle with magnitude ω_c, the poles of the Chebyschev lowpass filter transfer function lie on an ellipse with the imaginary axis as its major axis. The equation of the ellipse is given by,

$$\left(\frac{\Sigma_k}{\Omega_p sinh(B_k)}\right)^2 + \left(\frac{\Omega_k}{\Omega_p cosh(B_k)}\right)^2 = 1,$$

where,

$$s_k = \Sigma_k + j\Omega_k.$$

Now, substituting this, to find A_k and B_k,

$$cos(NA_k + jNB_k) = \pm\frac{j}{\epsilon}$$

$$cos(NA_K)cos(jNB_k) - sin(NA_k)sin(jNB_k) = \pm\frac{j}{\epsilon}$$

$$cos(NA_K)cosh(NB_k) - jsin(NA_k)sinh(NB_k) = \pm\frac{j}{\epsilon}$$

By comparing the real and imaginary parts on both sides we get,

$$cos(NA_K)cosh(NB_k) = 0 \implies cos(NA_k) = 0$$

$$A_k = \frac{\pi}{2N} + \frac{k\pi}{N}.$$

And,

$$sin(NA_k)sinh(NB_k) = \frac{1}{\epsilon} \implies sinh(NB_k) = \frac{1}{\epsilon}$$

$$B_k = \frac{1}{N}sinh^{-1}(\frac{1}{\epsilon})$$

$$B_k = 0.2512306$$

Now, we can take any sign for the square root provided that we vary k over 2N contiguous integers and get the poles as:

$$s_k = \Omega_p sin(A_k) sinh(B_k) + j\Omega_p cos(A_k) cosh(B_k)$$

To obtain the LHP (Left Half Plane) poles with the positive square root taken, we need to take the values of k ranging from N to 2N-1.

Now we can find the Chebyschev analog lowpass filter transfer function. $\mathbf{H_{analog,\ LPF}(s_L)}$:

$$\frac{0.100848}{0.100848 + 0.5146103s + 0.8288087s^2 + 1.5874956s^3 + 0.8215785s^4 + s^5}$$

Figure 1: Screenshot from SCILAB Console.

1.7 Analog Bandstop Transfer Function

This can be obtained by replacing s_L with F(s), the frequency transformation that we had employed earlier:

$$s_L \leftarrow F(s) = \frac{Bs}{s^2 + \Omega_0^2}$$

$\mathbf{H_{analog,\ BSF}(s)}$

$$\frac{0.1828303 + 0.8116s^2 + 1.4411058s^4 + 1.2794394s^6 + 0.5679546s^8 + 0.100848s^{10}}{0.1828303 + 0.6823091s + 1.6152724s^2 + 3.5488597s^3 + 4.0077561s^4 + 5.6051686s^5 + 3.5581572s^6 + 2.7972838s^7 + 1.1303615s^8 + 0.4239129s^9 + 0.100848s^{10}}$$

Figure 2: Screenshot from SCILAB Console.

1.8 Discrete Time Bandstop Transfer Function

This can be obtained by applying the bilinear transformation to s:

$$s \leftarrow \frac{1 - z^{-1}}{1 + z^{-1}}$$

$\mathbf{H_{discrete\ time,\ BSF}(z)}$

$$\frac{\begin{gathered}0.1853436 + 0.1101339z + 0.9528815z^{2} + 0.4436571z^{3} + 1.9321166z^{4} \\ + 0.6670546z^{5} + 1.9321137z^{6} + 0.4436664z^{7} + 0.9528745z^{8} + 0.1101392z^{9} \\ + 0.185339z^{10}\end{gathered}}{\begin{gathered}-0.1041035 - 0.0188127z + 0.2063441z^{2} + 0.1576915z^{3} + 0.9554288z^{4} \\ + 0.5275205z^{5} + 2.1110688z^{6} + 0.6980531z^{7} + 1.9719308z^{8} + 0.4101986z^{9} + z^{10}\end{gathered}}$$

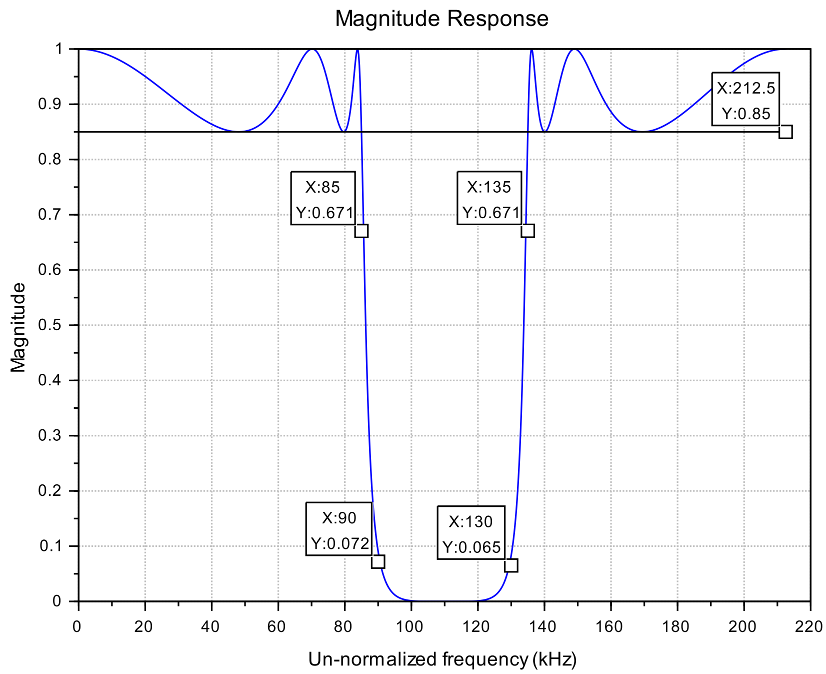
Figure 3: Screenshot from SCILAB Console.

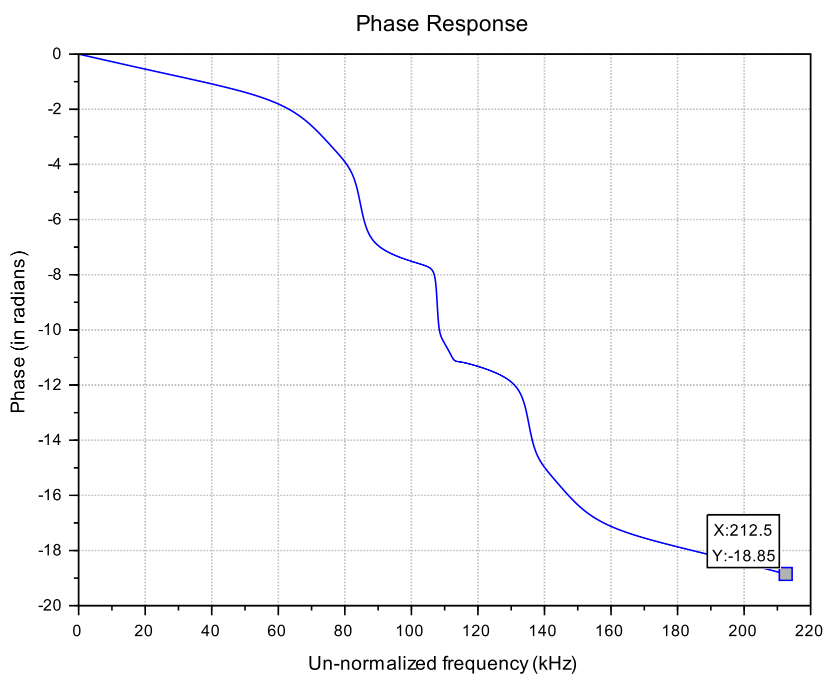
2 Review

I have reviewed Abhijeet's report and certify it to be correct.

3 Plots

Here are some plots generated using SCILAB:


Magnitude Response
Magnitude
Un-normalized frequency (kHz)
X:212.5
Y:0.85
X:85
Y:0.671
X:135
Y:0.671
X:90
Y:0.072
X:130
Y:0.065
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
20
40
60
80
100
120
140
160
180
200
220


Phase Response
Phase (in radians)
Un-normalized frequency (kHz)
X:212.5
Y:-18.85
0
-2
-4
-6
-8
-10
-12
-14
-16
-18
-20
0
20
40
60
80
100
120
140
160
180
200
220

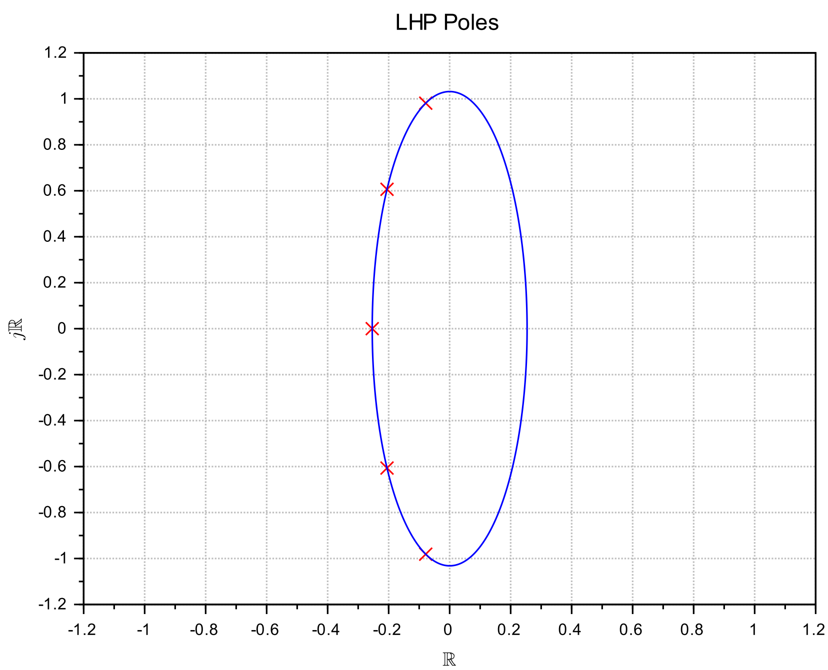


Figure 4: Poles of the Chebyschev lowpass transfer function.