Butterworth IIR Filter Design Report

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1 Design

1.1 Un-normalized Discrete Time Filter

The un-normalized discrete time filter specifications are uniquely assigned to each student using the filter number assigned to them.

The Butterworth filter to be designed is a bandpass filter for filter numbers 1 to 80 and a bandstop for filter numbers 81 to 160.

Hence, we would be designing a Butterworth IIR Bandpass Filter.

The passband for the bandpass filter is BL(m) kHz to BH(m) kHz, which are defined as:

$$\begin{aligned} BL(m) &= 10 + 5q(m) + 13r(m), \\ BH(m) &= BL(m) + 75, \end{aligned}$$

where,

$$\begin{aligned} q(m) &= \text{greatest integer} < 0.1m, \text{and} \\ r(m) &= m - 10q(m). \end{aligned}$$

For the assigned filter number m,

$$\begin{aligned} q(m) &= \text{greatest integer} < 5.5 \\ &= 5, \\ r(m) &= 55 - 10(5) \\ &= 5. \end{aligned}$$

Substituting the values of q(m) and r(m), we get

$$\begin{aligned} BL(m) &= 10 + 5(5) + 13(5) \\ &= 100, \\ BH(m) &= 100 + 75 \\ &= 175. \end{aligned}$$

Thus, the passband edges are:

$$\begin{aligned} f_{p1} &= 100kHz, \\ f_{p2} &= 175kHz. \end{aligned}$$

The transition bandwidth is given to be

$$\Delta_f = 5kHz.$$

Thus, the stopband edges are:

$$f_{s1} = f_{p1} - \Delta_f = 95kHz,$$
$$f_{s2} = f_{p1} + \Delta_f = 180kHz.$$

Now we can state the specifications of the discrete-time bandpass filter.

Specifications:

- Passband: 100 kHz to 175 kHz,
- Stopband: 0 to 95 kHz and 180 kHz to 300 kHz (since $f_s = 600$ kHz),
- Passband and Stopband tolerance: 0.15,
- Passband and Stopband nature: Monotonic.

1.2 Normalized Digital Filter

Given that our signal of interest is bandlimited to

$$f_b = 280kHz,$$

to satisfy the Nyquist Criterion,

$$f_s > 2f_b = 560kHz.$$

Given sampling frequency is

$$f_s = 600kHz > 560kHz.$$

Hence, the Nyquist criterion is satisfied.

For normalizing the frequencies, we need to make the sampling frequency map to 2π. Thus,

$$\omega = \frac{2\pi f}{f_s}$$

This gives us the normalized digital frequencies as follows:

Discrete time frequency f (kHz)	Normalized digital frequency ω (radians)
0	0
95	$\omega_{s1} = 0.9948377$
100	$\omega_{p1} = 1.0471976$
175	$\omega_{p2} = 1.8325957$
180	$\omega_{s2} = 1.8849556$
300	π

Table 1: Normalizing frequency.

Now we can state the specifications of the digital bandpass filter.

Specifications:

- Passband: 1.0471976 to 1.8325957,
- Stopband: 0 to 0.9948377 and 1.8849556 to π,
- Passband and Stopband tolerance: 0.15,
- Passband and Stopband nature: Monotonic.

1.3 Analog Bandpass Filter

For converting the normalized digital specifications to the specifications for an analog filter of the same type (bandpass), we use the bilinear transformation:

$$s = \frac{1 - z^{-1}}{1 + z^{-1}},$$

substituting the values of s and z, we get

$$\begin{aligned} j\Omega &= \frac{1 - e^{-j\omega}}{1 + e^{-j\omega}} = \frac{e^{j\omega/2} - e^{-j\omega/2}}{e^{+j\omega/2} + e^{-j\omega/2}} \\ &= \frac{2j\sin(\omega/2)}{2\cos(\omega/2)} = j\tan(\omega/2) \\ \Omega &= \tan(\omega/2) \end{aligned}$$

This gives us the analog bandpass frequencies as:

Normalized digital frequency ω (radians)	Analog frequency Ω
0	0
0.9948377	$\Omega_{s1} = 0.5429557$
1.0471976	$\Omega_{p1} = 0.5773503$
1.8325957	$\Omega_{p2} = 1.3032254$
1.8849556	$\Omega_{s2} = 1.3763819$
π	∞

Table 2: Applying the bilinear transformation.

Now we can state the specifications of the digital bandpass filter.

Specifications:

- Passband: 0.5773503 to 1.3032254,
- Stopband: 0 to 0.5429557 and 1.3763819 to ∞,
- Passband and Stopband tolerance: 0.15,
- Passband and Stopband nature: Monotonic.

1.4 Frequency Transformation

We need to employ a frequency transformation to convert our bandpass filter specifications into those of a lowpass filter. For this, we use the frequency transformation derived from the impedance of a series LC circuit:

$$s_L = \frac{s^2 + \Omega_0^2}{Bs},$$

where the subscript L stands for Lowpass. Ω_0 is the resonant frequency and B is the bandwidth of the series LC circuit. Substituting the values of s_L

and s, we get

$$
\begin{aligned}
j\Omega_L &= \frac{(j\Omega)^2 + \Omega_0^2}{B(j\Omega)} = \frac{1}{j}\frac{(-\Omega^2 + \Omega_0^2)}{B\Omega} \\
&= -j\frac{(-\Omega^2 + \Omega_0^2)}{B\Omega} = j\frac{(\Omega^2 - \Omega_0^2)}{B\Omega} \\
\Omega_L &= \frac{\Omega^2 - \Omega_0^2}{B\Omega}
\end{aligned}
$$

Now, we have two degrees of freedom, Ω_0 and B. We also decide to follow the convention that the passband edge of the lowpass filter, in both the positive and negative half of the Ω_L axis, has a magnitude of one. In other words, we want Ω_{p1} and Ω_{p2} to map to -1 and +1 respectively on the Ω_L axis. To achieve this, we solve the following two equations

$$
-1 = \frac{\Omega_{p1}^2 - \Omega_0^2}{B\Omega_{p1}}
$$

and

$$
+1 = \frac{\Omega_{p2}^2 - \Omega_0^2}{B\Omega_{p2}}
$$

By equating Ω_0^2 we get

$$
\begin{aligned}
\Omega_{p1}^2 + B\Omega_{p1} &= \Omega_{p2}^2 - B\Omega_{p2} \\
B(\Omega_{p2} + \Omega_{p1}) &= \Omega_{p2}^2 - \Omega_{p1}^2 \\
B &= \Omega_{p2} - \Omega_{p1}
\end{aligned}
$$

Substituting the value of B, we get

$$
\begin{aligned}
\Omega_0^2 &= \Omega_{p1}^2 + (\Omega_{p2} - \Omega_{p1})\Omega_{p1} \\
\Omega_0^2 &= \Omega_{p1}^2 + \Omega_{p2}\Omega_{p1} - \Omega_{p1}^2 = \Omega_{p2}\Omega_{p1} \\
\Omega_0 &= \sqrt{\Omega_{p2}\Omega_{p1}}
\end{aligned}
$$

By substituting the value of Ω_{p1} and Ω_{p2}, we get

$$
\begin{aligned}
B &= 1.3032254 - 0.5773503 = 0.7258751 \\
\Omega_0 &= \sqrt{(1.3032254)(0.5773503)} = \sqrt{0.7524175} \approx 0.8674200
\end{aligned}
$$

Transformation and Parameters

- Transformation:
$$\Omega_L = \frac{\Omega^2 - \Omega_0^2}{B\Omega}$$
- Parameters:
  - $\Omega_0 = 0.8674200$,
  - B = 0.7258751

1.5 Analog Lowpass Filter

Substituting the values of Ω_o and B, we get the frequency transformation to be employed as:
$$\Omega_L = \frac{\Omega^2 - (0.7524175)}{(0.7258751)\Omega}$$
This gives us the analog lowpass frequencies as:

Analog frequency Ω	Analog Lowpass Frequency Ω_L
0^+	$-\infty$
0.5429557	Ω_{Ls1} = -1.1611157
0.5773503	Ω_{Lp1} = -1
1.3032254	Ω_{Lp2} = 1
1.3763819	Ω_{Ls2} = 1.1430597
∞	∞

Table 3: Analog frequency transformation.

The passband edge of the lowpass filter was specified to be one while employing the frequency transformation but, the stopband edge now has two possible values of which, we choose the more stringent value (the value with a smaller magnitude) since satisfying the stronger specifications will automatically satisfy the weaker specifications. Thus

$$\begin{aligned}
\Omega_p &= 1 \\
\Omega_s &= min(abs(\Omega_{Ls1}), abs(\Omega_{Ls1})) \\
&= min(1.1611157, 1.1430597) \\
\Omega_s &= 1.1430597
\end{aligned}$$

Now we can state the specifications of the analog lowpass filter.

Specifications:

- Passband: 0 to 1,
- Stopband: 1.1430597 to ∞,
- Passband and Stopband tolerance: 0.15,
- Passband and Stopband nature: Monotonic.

1.6 Butterworth Analog Lowpass Transfer Function

The Butterworth analog lowpass transfer function is given as:

$$\mathrm{H_{analog,\,LPF}(s_L)} = \frac{\Omega_c^N}{\prod_{k\epsilon LHP}(s - s_k)}.$$

Ω_c and N are given by the specifications as:

$$N \geq \frac{\log \frac{D_2}{D_1}}{2 \log \frac{\Omega_s}{\Omega_p}},$$

$$\frac{\Omega_p}{D_1^{\frac{1}{2N}}} \leq \Omega_c \leq \frac{\Omega_s}{D_2^{\frac{1}{2N}}},$$

where D_1 and D_2 are given by:

$$\begin{aligned}
D_1 &= \frac{1}{(1 - \delta_1)^2} - 1, \\
D_2 &= \frac{1}{\delta_2^2} - 1.
\end{aligned}$$

δ_1 and δ_2 are the passband and stopband tolerances.

$$D_1 = \frac{1}{(1-0.15)^2} - 1 = 0.3840830$$
$$D_2 = \frac{1}{0.15^2} - 1 = 43.444444$$

This gives us

$$N \geq \frac{\log(113.11211)}{2\log(1.1430597)} \approx 17.681654,$$

Hence, we choose

$$N = 18.$$

Now,

$$\frac{1}{0.3840830^{\frac{1}{2(18)}}} \leq \Omega_c \leq \frac{1.143059}{43.444444^{\frac{1}{2(18)}}},$$
$$1.0269369 \leq \Omega_c \leq 1.0293682$$

We can choose any value for Ω_c, so let's choose it to be the average of the two bounds.

$$\Omega_c = 1.0281525$$

Now the poles can be found by setting the denominator of the magnitude squared form of the Butterworth lowpass filter transfer function to zero.

$$|\mathrm{H}_{\text{analog, LPF}}(\mathrm{s}_\mathrm{L})|^2 = \frac{1}{1 + (\frac{s}{j\Omega_c})^{2N}}$$

$$1 + (\frac{s_k}{j\Omega_c})^{2N} = 0$$

$$(\frac{s_k}{j\Omega_c})^{2N} = -1 = e^{j(2k+1)\pi}$$

$$\frac{s_k}{j\Omega_c} = e^{j\frac{2k+1}{2N}\pi}$$
$$s_k = j\Omega_c e^{j\frac{2k+1}{2N}\pi}$$
$$s_k = \Omega_c e^{j(\frac{k\pi}{N} + \frac{\pi}{2N} + \frac{\pi}{2})}$$

To obtain the LHP (Left Half Plane) poles, we need to take the values of k ranging from 0 to N-1.

Now we can find the Butterworth analog lowpass filter transfer function.

H$_{\text{analog, LPF}}$(s$_L$):

$$\frac{1.6482987}{\begin{aligned}&1.6482987 + 18.394261s + 102.63578s^{2} + 379.83631s^{3} \\ &+ 1043.3524s^{4} + 2256.3737s^{5} + 3977.9487s^{6} + 5841.723s^{7} \\ &+ 7240.698s^{8} + 7629.4257s^{9} + 6849.6019s^{10} + 5227.7012s^{11} \\ &+ 3367.5486s^{12} + 1806.9687s^{13} + 790.41562s^{14} \\ &+ 272.21112s^{15} + 69.581388s^{16} + 11.796727s^{17} + s^{18}\end{aligned}}$$

Figure 1: Screenshot from SCILAB Console.

1.7 Analog Bandpass Transfer Function

This can be obtained by replacing s$_L$ with F(s), the frequency transformation that we had employed earlier:

$$s_L \leftarrow F(s) = \frac{s^2 + \Omega_0^2}{Bs}$$

H$_{\text{analog, BPF}}$(s)

$$\frac{0.0051588s^{18}}{\begin{aligned}&0.0059739 + 0.0679868s + 0.5297785s^{2} + 2.9961579s^{3} \\ &+ 13.931125s^{4} + 54.460312s^{5} + 185.76551s^{6} + 559.46973s^{7} \\ &+ 1512.4135s^{8} + 3695.3152s^{9} + 8233.0276s^{10} \\ &+ 16796.959s^{11} + 31545.34s^{12} + 54675.081s^{13} \\ &+ 87737.706s^{14} + 130541.77s^{15} + 180431.25s^{16} \\ &+ 231802.97s^{17} + 277074.82s^{18} + 308077.58s^{19} \\ &+ 318708.73s^{20} + 306459.31s^{21} + 273747.82s^{22} \\ &+ 226722.66s^{23} + 173852.85s^{24} + 123032.08s^{25} \\ &+ 80147.204s^{26} + 47810.292s^{27} + 26006.479s^{28} \\ &+ 12785.823s^{29} + 5642.3274s^{30} + 2198.4391s^{31} \\ &+ 747.41466s^{32} + 213.63933s^{33} + 50.205578s^{34} \\ &+ 8.5629507s^{35} + s^{36}\end{aligned}}$$

Figure 2: Screenshot from SCILAB Console.

1.8 Discrete Time Bandpass Transfer Function

This can be obtained by applying the bilinear transformation to s:

$$s \leftarrow \frac{1 - z^{-1}}{1 + z^{-1}}$$

$\mathbf{H_{discrete\ time,\ BPF}(z)}$

```
0.0051588 -0.0928587z^2 +1.776D-15z^3 +0.789299z^4 -3.553D-14z^5
 -4.2095947z^6 -8.242D-13z^7 +15.78598z^8 +1.225D-11z^9 -44.200745z^10
 +1.633D-10z^11 +95.768281z^12 +9.093D-10z^13 -164.1742z^14 +3.515D-09z^15
 +225.73952z^16 +7.725D-09z^17 -250.82169z^18 +7.787D-09z^19 +225.73952z^20
 +3.584D-09z^21 -164.1742z^22 +9.946D-10z^23 +95.768281z^24 +1.630D-10z^25
 -44.200745z^26 +1.924D-11z^27 +15.78598z^28 -7.390D-13z^29 -4.2095947z^30
 -5.684D-14z^31 +0.789299z^32 +1.776D-15z^33 -0.0928587z^34 +0.0051588z^36
---------------------------------------------------------------------
201.33811 -1286.7002z +9379.8268z^2 -40716.191z^3 +176626.74z^4
 -590964.91z^5 +1917565z^6 -5256925.4z^7 +13860959z^8 -32236533z^9
 +71964777z^10 -1.451D+08z^11 +2.807D+08z^12 -4.973D+08z^13 +8.463D+08z^14
 -1.329D+09z^15 +2.006D+09z^16 -2.808D+09z^17 +3.779D+09z^18 -4.723D+09z^19
 +5.674D+09z^20 -6.328D+09z^21 +6.783D+09z^22 -6.724D+09z^23 +6.403D+09z^24
 -5.600D+09z^25 +4.700D+09z^26 -3.579D+09z^27 +2.614D+09z^28 -1.696D+09z^29
 +1.055D+09z^30 -5.607D+08z^31 +2.870D+08z^32 -1.154D+08z^33 +45566306z^34
 -11083732z^35 +2931076.3z^36
```

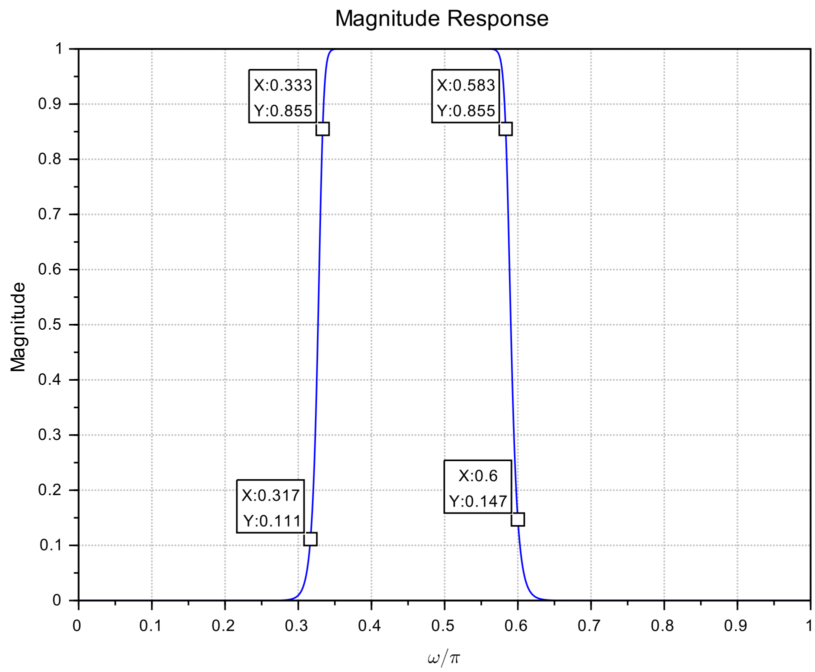
Figure 3: Screenshot from SCILAB Console.

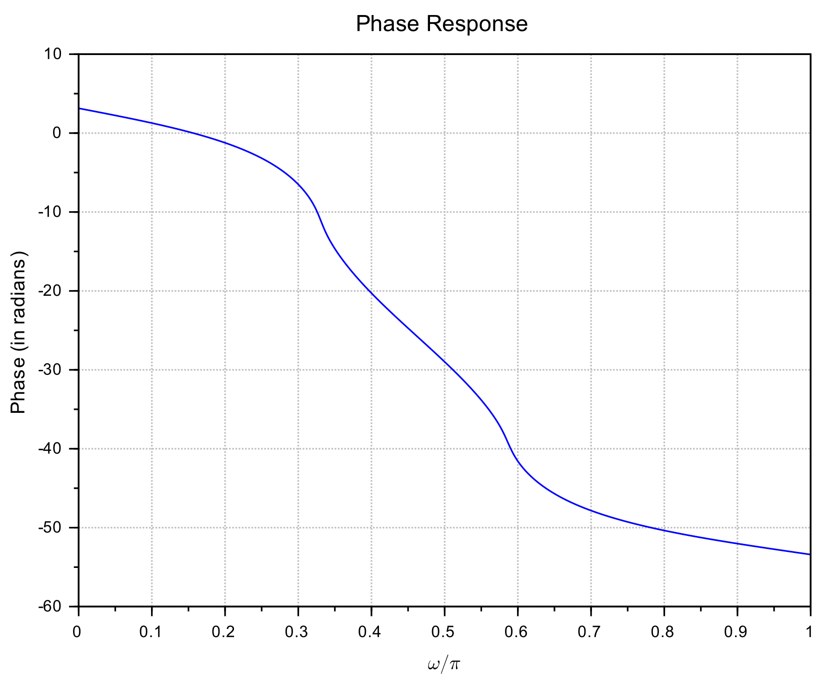
2 Review

I have reviewed Abhijeet's report and certify it to be correct.

3 Plots

Here are some plots generated using SCILAB:


Magnitude Response
Magnitude
X:0.333
Y:0.855
X:0.583
Y:0.855
X:0.317
Y:0.111
X:0.6
Y:0.147
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
ω/π


Phase Response
Phase (in radians)
10
0
-10
-20
-30
-40
-50
-60
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
ω/π

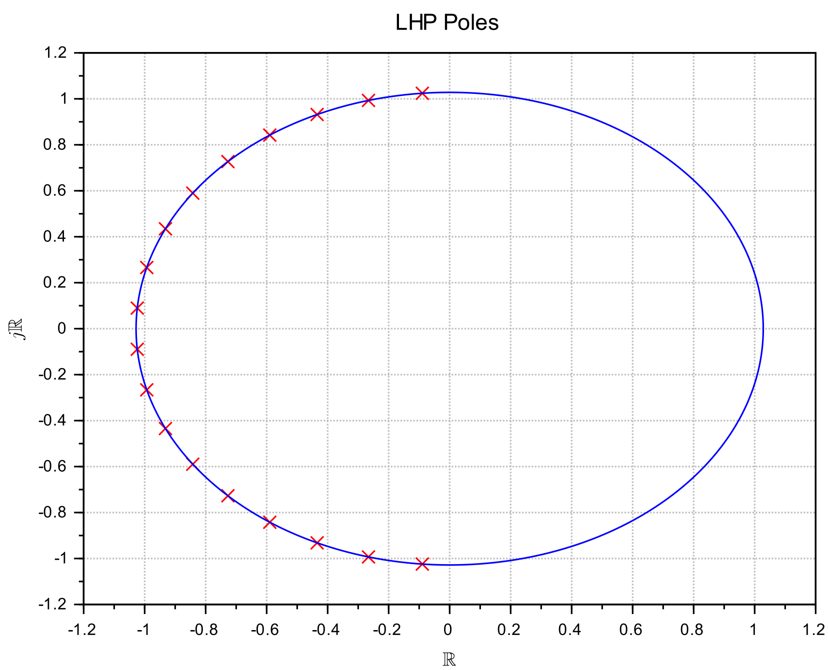


Figure 4: Poles of the Butterworth lowpass transfer function.