Chapter 3 Solutions

1. A patient is considering a number of treatment options available through her general practitioner (GP) between receiving medication or having a surgery. The costs of the different treatments vary as well as their likelihood of success. The GP has discussed with the patient the success rates of each treatment when used in other patients.

   Describe the parameter, data, actions, and loss function for this problem.

   The **parameter** corresponds to the probability of success of each treatment.

   The **data** consist of the success rates provided by the GP.

   The available **actions** are: no treatment, medication, surgery.

   The **loss** function incorporates the cost and improvement in quality of life for the patient.

2. An investor is considering whether or not to buy certain risky bonds. If he buys the bonds, they can be redeemed at maturity for a net gain of £500. There is probability θ that there will be a default on the bonds, in which case the investor is set to lose his investment of £1000. If the investor instead puts his money in a “safe” investment, he will receive a net gain of £300 over the same period.

   (a) Define appropriate actions, parameter, and parameter space for the problem.

   The two **actions** are: “Invest in bonds” and “Invest in safe assets”. The **parameter** is the probability that there is a default on the bonds. The **parameter space** is $[0, 1]$.

   (b) Derive the loss function for the problem.

   Let y denote the event of default on the bonds. We set $y = 1$ if a default occurs and $y = 0$ if a default does not occur. Then $y \sim \text{Bernoulli}(\theta)$. Let $a = 1$ if the investor invests in bonds and $a = 0$ if the investor invests in safe assets. Then, the loss from the outcome y is

$$l(y, a) = \begin{cases} -300 & \text{if } a = 0 \text{ and } y = 0 \text{ or } 1, \\ -500 & \text{if } a = 1 \text{ and } y = 0, \\ 1000 & \text{if } a = 1 \text{ and } y = 1. \end{cases}$$

This can also be written as a 2×2 table:

$$
\begin{array}{c|c|c|}
 & a=0 & a=1 \\
\hline
y=0 & -300 & -500 \\
\hline
y=1 & -300 & 1000 \\
\hline
\end{array}.
$$

Then,

$$
\begin{aligned}
L(\theta, a=0) &= \mathsf{E}[l(y, a=0)] && (1)\\
&= l(y=0, a=0)\mathbb{P}(y=0) + l(y=1, a=0)\mathbb{P}(y=1) && (2)\\
&= -300 \times (1-\theta) - 300 \times \theta && (3)\\
&= -300 && (4)\\
L(\theta, a=1) &= \mathsf{E}[l(y, a=1)] && (5)\\
&= l(y=0, a=1)\mathbb{P}(y=0) + l(y=1, a=1)\mathbb{P}(y=1) && (6)\\
&= -500 \times (1-\theta) + 1000 \times \theta && (7)\\
&= -500 + 1500\theta && (8)
\end{aligned}
$$

So,

$$
L(\theta, a) = \begin{cases} -300 & \text{if } a = 0, \\ -500 + 1500\theta & \text{if } a = 1. \end{cases}
$$

(c) Describe all randomised decision rules and find the minimax decision among them.

There are no data. All randomised decision rules are of the form

$$
d_p = \begin{cases} 1 & \text{with probability } p, \\ 0 & \text{with probability } 1-p, \end{cases}
$$

i.e., invest in bonds with probability p, and invest in safe assets with probability $1-p$. As there are no data, the risk is the loss, i.e.,

$$
\begin{aligned}
R(\theta, d_p) &= L(\theta, d_p) \\
&= -300 \times (1-p) + (-500 + 1500 \times \theta) \times p \\
&= -300 - 200 \times p + 1500 \times \theta \times p.
\end{aligned}
$$

This is maximised when $\theta = 1$ for all p, so

$$\begin{aligned}\bar{R}(d_p) &= \max_{0\le\theta\le 1} R(\theta, d_p)\\ &= -300 - 200 \times p + 1500 \times 1 \times p\\ &= -300 + 1300 \times p.\end{aligned}$$

The value of p that minimises the maximum risk, $\bar{R}(d_p)$ is $p = 0$, so the optimal decision is in fact a deterministic decision,

$$d_{\text{MM}} = d_0 = \begin{cases} 1 & \text{with probability } 0,\\ 0 & \text{with probability } 1 \end{cases} = 0 \text{ always.}$$

In other words, the minimax strategy is to invest in safe assets.

3. A coin has probability $\theta \in [0, 1]$ of coming up heads ($y = 1$), and $1 - \theta$ of coming up tails ($y = 0$). You are playing a game where if you guess the outcome of a coin flip correctly you receive a payment of £1, but if you guess wrongly, you loose £1.

   (a) What are the parameter and parameter space for this problem?

   The parameter is the probability of the coin coming up heads, θ, and the parameter space is $\Theta = [0, 1]$.

   (b) What is the action space for this problem?

   The actions that can be taken are: "guess heads" ($a = 1$) or "guess tails" ($a = 0$), so the action space is $\mathcal{A} = \{0, 1\}$.

   (c) Show that the loss function, $L(\theta, a)$, for this problem is given by

   $$L(\theta, a) = \begin{cases} 2\theta - 1 & \text{if guessing "tails"},\\ 1 - 2\theta & \text{if guessing "heads"}. \end{cases}$$

   Let y be the outcome of a future coin flip such that $\mathbb{P}(y = 1) = \theta$ and $\mathbb{P}(y = 0) = 1 - \theta$. Then the loss from that outcome is

   $$l(y, 0) = \begin{cases} -1 & \text{if } y = 0,\\ 1 & \text{if } y = 1, \end{cases} \qquad l(y, 1) = \begin{cases} 1 & \text{if } y = 0,\\ -1 & \text{if } y = 1. \end{cases}$$

Then, $L(\theta,0) = -1 \times \mathbb{P}(y=0) + 1 \times \mathbb{P}(y=1) = -(1-\theta) + \theta = 2\theta - 1$,
and $L(\theta,1) = 1 \times \mathbb{P}(y=0) - 1 \times \mathbb{P}(y=1) = (1-\theta) - \theta = 1 - 2\theta$.

4. Let x be the outcome the coin flip from an earlier game. Consider the following two strategies for guessing the outcome of a future coin flip:

   - **Strategy 1:** Guess the same as the outcome of the earlier coin flip.
   - **Strategy 2:** Guess "heads" regardless of the outcome of the earlier coin flip.

   (a) Write a mathematical expression for the decision rules corresponding to these two strategies and compute their risks.

The decision rules corresponding to the two strategies are $d_1(x) = x$ and $d_2(x) = 1$.

Since $x = 1$ with probability θ, and $x = 0$ with probability $1-\theta$, $R(\theta, d_1) = \mathsf{E}\, L(\theta, x) = L(\theta,0)(1-\theta) + L(\theta,1)\theta = (2\theta-1)(1-\theta) + (1-2\theta)\theta = (2\theta-1)(1-\theta) - (2\theta-1)\theta = (2\theta-1)(1-\theta-\theta) = (2\theta-1)(1-2\theta) = -(2\theta-1)^2$.

$R(\theta, d_2) = \mathsf{E}\, L(\theta,1) = \mathsf{E}(1-2\theta) = 1-2\theta$.

   (b) Between the two strategies, which one is the minimax decision rule?

It can be seen that for d_1, the value of θ that attains the maximum risk is $\theta = \frac{1}{2}$, in which case $\bar{R}(d_1) = -(2 \times \frac{1}{2} - 1)^2 = 0$.
For d_2, the risk is maximised when $\theta = 0$, in which case $\bar{R}(d_2) = 1 - 2 \times 0 = 1$.
So, between the two decision rules, d_1 is minimax as $\bar{R}(d_1) < \bar{R}(d_2)$.