Chapter 4 Solutions

1. Consider the method of moments estimator of Example 4.6

   (a) The following data were observed: $x_1 = 1$, $x_2 = 1$, $x_3 = 7$ from the $\mathrm{U}(0, \theta)$ distribution. Compute the method of moments estimator for θ.

   Given the data $x_1 = 1$, $x_2 = 1$, $x_3 = 7$ we find $\bar{x} = 3$ and so $\hat{\theta} = 2\bar{x} = 6$.

   (b) Identify potential drawbacks of this estimator.

   This estimator can sometimes be nonsense. For example, using the data from the previous part, $\hat{\theta} = 6$. However we know that θ must be bigger than all x_i's so $\theta > 7$ which contradicts $\hat{\theta} < 7$.

2. Verify the formula from the mgf of the gamma distribution in Example 4.7 and use it to derive its mean and variance.

For the Gamma(α, β) distribution,

$$f(x|\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}.$$

Then

$$\begin{aligned}
M_X(t) &= \int_0^\infty e^{tx} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \, \mathrm{d}x \\
&= \int_0^\infty \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-(\beta-t)x} \, \mathrm{d}x \\
&= \frac{\beta^\alpha}{(\beta-t)^\alpha} \int_0^\infty \frac{(\beta-t)^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-(\beta-t)x} \, \mathrm{d}x \\
&= \frac{\beta^\alpha}{(\beta-t)^\alpha}, \text{ for } t < \beta. \\
&= (1 - t/\beta)^{-\alpha}, \text{ for } t < \beta. \\
\Rightarrow M_X^{(1)}(t) &= \frac{d}{dt} M_X(t) = \frac{\alpha}{\beta}(1 - t/\beta)^{-\alpha-1} \\
\Rightarrow \mu_1 &= M_X^{(1)}(0) = \frac{\alpha}{\beta} \\
\Rightarrow M_X^{(2)}(t) &= \frac{d^2}{dt^2} M_X(t) = \frac{\alpha(\alpha+1)}{\beta^2}(1 - t/\beta)^{-\alpha-2}
\end{aligned}$$

$$\Rightarrow \mu_2 = M_X^{(2)}(0) = \frac{\alpha(\alpha+1)}{\beta^2}$$
$$\Rightarrow \mathsf{Var}(X) = \mu_2 - \mu_1^2 = \frac{\alpha(\alpha+1)}{\beta^2} - \frac{\alpha^2}{\beta^2} = \frac{\alpha}{\beta^2}.$$

3. Explain why the MLE of Example 4.10 is biased but do not derive its bias. *Hint.* Think why $\max\{X_i\} < \theta$, and what this means about $\mathsf{E}[\max\{X_i\}]$.

Because each $X_i < \theta$, then $X_{(n)} = \max\{X_1, \ldots, X_n\} < \theta$, so $\mathsf{E}\,X_{(n)} < \theta$. Therefore $\text{Bias}_\theta(X_{(n)}) = \mathsf{E}\,X_{(n)} - \theta < 0$. So $\text{Bias}_\theta(X_{(n)}) \neq 0$ which means that $X_{(n)}$ is biased for θ.

4. Let $X_1, \ldots, X_n \overset{\mathsf{iid}}{\sim} \text{Bernoulli}(\theta)$, $\theta \in (0, 1)$.

(a) Derive the MLE for θ.

The pmf for X is
$$f(x|\theta) = \theta^x (1-\theta)^{1-x},\ x \in \{0,1\},\ \theta \in [0,1].$$
Taking logarithms,
$$\begin{aligned}
\log f(x|\theta) &= x \log\theta + (1-x)\log(1-\theta) \\
\Rightarrow \ell(\theta|\mathbf{x}) &= \sum_{i=1}^n \log f(x_i|\theta) \\
&= \sum_{i=1}^n \{x_i \log\theta + (1-x_i)\log(1-\theta)\} \\
&= \log\theta \sum_{i=1}^n x_i + \log(1-\theta)(n - \sum_{i=1}^n x_i) \\
\Rightarrow \ell'(\theta|\mathbf{x}) &= \frac{1}{\theta}\sum_{i=1}^n x_i - \frac{1}{1-\theta}(n - \sum_{i=1}^n x_i) \\
0 &= \frac{1}{\hat\theta}\sum_{i=1}^n x_i - \frac{1}{1-\hat\theta}(n - \sum_{i=1}^n x_i) \\
0 &= \frac{n}{\hat\theta}\bar{x} - \frac{n}{1-\hat\theta}(1-\bar{x}) \\
0 &= \frac{\bar{x}}{\hat\theta} - \frac{1-\bar{x}}{1-\hat\theta}
\end{aligned}$$

$$\hat{\theta} = \bar{x}$$

Evaluating the second derivative of $\ell(\theta|\mathbf{x})$ at $\theta = \bar{x}$ is negative which confirms this is a maximum.