MA52112: Statistics for Data Science

Exercise sheet 2 (Probability)

1. Let the joint PDF of a random vector (X, Y) be given by:

$$f(x,y) = \begin{cases} 1 & 0 < x < 1\,, \quad x < y < x+1 \\ 0 & \text{otherwise} \end{cases}$$

   a) Sketch the support of the joint distribution.
   b) Verify that $f(x, y)$ is a valid joint PDF.
   c) Find the marginal PDFs of X and Y.
   d) Are X and Y independent? Justify your answer.
   e) Find the conditional distribution of Y given $X = x$.
   f) Calculate the correlation coefficient between X and Y.

2. A random point (X, Y) is distributed uniformly on the square with vertices at $(1, 1)$, $(1, -1)$, $(-1, 1)$ and $(-1, -1)$. That is, the joint PDF of (X, Y) is:

$$f(x,y) = \begin{cases} \frac{1}{4} & -1 < x < 1\,, \quad -1 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

   Determine the probabilities of the following events:

   a) $P(X^2 + Y^2 < 1)$
   b) $P(X - Y > 0)$
   c) $P(|X + Y| < 2)$

3. A pdf of a random vector (X, Y) is given by:

$$f(x,y) = \begin{cases} k(x+2y) & 0 < x < 2\,,\, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

   a) Find the value of k that makes $f(x, y)$ a valid PDF.

b) Find the marginal PDFs of X and Y.

c) Find the conditional distribution of Y given $X = x$.

4. A woman leaves for work between 8 AM and 8:30 AM and takes between 40 and 50 minutes to get there. Let the random variable X denote her time of departure, and the random variable Y the travel time. Assuming that these variables are independentand uniformly distributed, find the probability that the woman arrives at work before 9 AM.

5. Consider a continuous random variable X that follows a Normal distribution with mean $\mu = 4$ and variance $\sigma^2 = 4$. That is, $X \sim N(10, 4)$.

   a) What is the probability that X takes negative values?

   b) What is the probability that X takes values between 5 and 10?

6. Show that the Moment Generating Function (MGF) of a Geometric distribution with parameter p is given by:

$$M_X(t) = \frac{pe^t}{1 - (1-p)e^t} \quad \text{for } t < -\ln(1-p)$$

7. Show that the MFG of the Normal distribution with parameters μ and σ^2 is given by:

$$M_X(t) = \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right)$$

(Hint: Complete the square in the exponent of the integral defining the MGF.)