Generalized Hyperbolic Secant Distributions

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Source: *Journal of the American Statistical Association*, Vol. 63, No. 321 (Mar., 1968), pp. 329-337
Published by: Taylor & Francis, Ltd. on behalf of the American Statistical Association
Stable URL: https://www.jstor.org/stable/2283852
Accessed: 01-05-2025 19:53 UTC

GENERALIZED HYPERBOLIC SECANT DISTRIBUTIONS

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The two-parameter family of distributions having characteristic functions given by $\varphi(t) = (\text{sech } \alpha t)^\rho$, $\alpha > 0$ $\rho > 0$, is introducted and their basic structural properties are derived. Examples of some random variables having this distribution are given and, as an application, the distribution of the geometric mean of the absolute values of Cauchy variables is obtained. Finally, estimation of the parameters α and ρ is considered.

1. INTRODUCTION

WE SHALL be concerned here with a study of some properties of the two-parameter family of probability distributions having characteristic functions (ch.f.'s) given by

$$\varphi(t) = \varphi(t; \alpha, \rho) = (\text{sech}\, \alpha t)^\rho, \qquad \alpha > 0, \qquad \rho > 0. \tag{1}$$

Distributions having ch.f.'s of this form will be called generalized hyperbolic secant distributions (*ghsd*'s) with parameters α and ρ (the term "hyperbolic secant distribution" is already in use in the literature as a name for the distribution corresponding to ρ being equal to unity). Our motivation for this study is derived from the increasing frequency with which these probability distributions are occurring in the statistical literature. In an early paper, Baten [2] obtained the distribution of the sample mean for samples of size n from a *ghsd* with parameters α and $\rho = 1$. Talacko [12] discussed certain properties of the hyperbolic secant distribution and showed [13] that this distribution has a role in the theory of Wiener's stochastic function. This class of distributions is mentioned (in various examples and problems) on several occasions by Feller [7, pp. 63, 476, 501, 538, 559]. Finally, we note that *ghsd*'s have arisen at least twice in studies of characterizations of distributions. On one occasion Laha and Lukacs [10] characterized the class of proper distributions for which a quadratic form $Q = X'AX$ has quadratic regression on a linear form $\Lambda = a'X$ $[X' = (X_1 X_2 \cdots X_n)]$, and showed that this class contains the normal, Poisson, gamma, binomial, negative binomial, and the *ghsd*'s. Finally, Bolger and Harkness [3] characterized the family of distributions of independent random variables (r.v.'s) X_1 and X_2 having the property that $E(X_1 | Y = y) = \lambda_1 y/\lambda$ and $V(X_1 | Y = y) = (\lambda_1\lambda_2/\lambda^2)u(y)$, where λ_1 and λ_2 are positive constants with $\lambda = \lambda_1 + \lambda_2$ and $u(y)$ is a non-negative function of y and found (in the special case when $u(y)$ is a polynomial in y of degree at most two) that the family consisted of the Cauchy distribution and the distributions characterized by Lukacs and Laha.

2. PRELIMINARY PROPERTIES

Lukacs [11, p. 74] has shown that $\varphi(t) = \operatorname{sech} \alpha t$ is a ch.f. by noting that

$$\operatorname{sech} \alpha t = \prod_{j=1}^{\infty} [1 + 4\alpha^2 t^2 (2j - 1)^{-2} \pi^{-2}]^{-1}$$

and noting further that each factor in this infinite product is the ch.f. of a Laplace distribution. More generally, Laha and Lukacs [10] showed that

$$\varphi(t) = [\cosh \alpha t + i\lambda \sinh \alpha t]^{-\rho}$$

is an (infinitely divisible) ch.f. for α, ρ, and λ non-negative real numbers. For $\lambda = 0$ we get the ch.f. given by (1); in this case, the ch.f. is real-valued and therefore the corresponding distribution function (d.f.) is symmetric. Furthermore, since $(\operatorname{sech} \alpha t)^\rho \leq 2^\rho e^{-\rho\alpha t}$, we see that $(\operatorname{sech} \alpha t)^\rho$ is absolutely integrable and hence (using the inversion for such ch.f.'s) the density function corresponding to (1) is given by

$$f(x) = f(x; \alpha, \rho) = \pi^{-1} \int_0^\infty (\operatorname{sech} \alpha t)^\rho \cos txdt. \tag{2}$$

Using Bateman's tables [1, p. 30] we find that (for all real x)

$$f(x) = [\alpha\pi\Gamma(\rho)]^{-1} 2^{\rho-2} \Gamma\left(\frac{\rho}{2} + \frac{ix}{2\alpha}\right) \Gamma\left(\frac{\rho}{2} - \frac{ix}{2\alpha}\right) \tag{3}$$

for arbitrary $\alpha > 0$ and $\rho > 0$. In particular,

$$f(x; \alpha, 1) = \frac{1}{2\alpha} \operatorname{sech} \frac{\pi x}{2\alpha} = \alpha^{-1}(e^{\pi x/2\alpha} + e^{-\pi x/2\alpha})^{-1} \tag{4}$$

and

$$f(x; \alpha, 2) = \frac{x}{2\alpha^2} \operatorname{csch} \frac{\pi x}{2\alpha} = \alpha^{-2} x (e^{\pi x/2\alpha} - e^{-\pi x/2\alpha})^{-1}. \tag{5}$$

More generally, for positive integral values n of ρ we have

$$f(x; \alpha, 2n + 1) = \left[\frac{2^{2n-1}}{(2n)!\alpha} \operatorname{sech} \frac{\pi x}{2\alpha}\right] \left[\prod_{r=1}^{n} \left\{\frac{x^2}{4\alpha^2} + \left(\frac{2r - 1}{2}\right)^2\right\}\right] \tag{6}$$

and

$$f(x; \alpha, 2n) = \left[\frac{4^{n-1} x}{(2n - 1)!2\alpha^2} \operatorname{csch} \frac{\pi x}{2\alpha}\right] \left[\prod_{r=1}^{n-1} \left\{\frac{x^2}{4\alpha^2} + r^2\right\}\right]. \tag{7}$$

Closed form expressions for the d.f.'s are not apparently available in general; however, for $\rho = 1$

$$F(x; \alpha, 1) = \int_{-\infty}^{x} f(y; \alpha, 1)\, dy = 1 - \frac{2}{\pi} \tan^{-1} e^{-\pi x/2\alpha}.$$

In section four, we will consider approximations for $F(x; \alpha, \rho)$ for arbitrary ρ.

If Y is a r.v. having a Cauchy distribution with density function $[\pi(1+y^2)]^{-1}$, then the r.v. $X=\ln|Y|$ has a density function given by (4) with $\alpha=1$. It follows that if X_1 and X_2 are independent r.v.'s, each normally distributed with mean zero and variance one, then $X=\ln|X_1/X_2|$ will have a *ghsd* with parameters $\alpha=1$, and $\rho=1$. More generally, if X_1 and X_2 have a bivariate normal distribution with zero means, unit variance, and correlation ρ, then it can be shown that

$$X = \ln\left[\left|\frac{X_1}{X_2} - \rho\right| \Big/ (1-\rho^2)^{1/2}\right]$$

is distributed according to (4), with $\alpha=1$.

We observe also that the family of *ghsd*'s is reproductive in ρ for fixed α, i.e., if X_1 and X_2 are independent r.v.'s having densities $f(x_1;\alpha,\rho_1)$ and $f(x_2;\alpha,\rho_2)$, then $X=X_1+X_2$ has density $f(x;\alpha,\rho_1+\rho_2)$. This is clear from the fact that the ch.f. of $X=X_1+X_2$ is the product of the ch.f.'s of X_1 snd X_2. Hence, if $X_1, X_2, \cdots, X_n$ is a random sample from a *ghsd* with parameters α and ρ, then the sum $n\overline{X}=X_1+\cdots+X_n$ has a *ghsd* with parameters α and $n\rho$. These observations may be combined to find the geometric mean Z of the absolute values of Cauchy r.v.'s. Thus suppose $Y_1, Y_2, \cdots, Y_n$ is a random sample from the Cauchy distribution with density $[\pi(1+y^2)]^{-1}$. Then it follows from above that

$$\ln Z = \frac{1}{n}\sum_{i=1}^{n}\ln|Y_i| = \ln\left[\prod_{i=1}^{n}|Y_i|\right]^{1/n}$$

has density $f(x;1,n)$. Therefore, changing variables, the density g of Z is given by

$$g(z) = (\ln z)f(\ln z;1,n), \qquad \text{for } z>0$$

and zero for $z<0$.

3. MOMENTS AND CUMULANTS

Since $f(x;\alpha,\rho)$ is symmetric (about the origin) the moments about the origin and the mean (which is zero) coincide. Let $\mu_k(\rho)$ denote the kth moment of a r.v. X having density $f(x;\alpha,\rho)$. Clearly, for all non-negative integers k, $\mu_{2k+1}(\rho)=0$. Since $\varphi(t;\alpha,\rho)$ is an analytic ch.f., all moments exist, are finite, and are given by

$$\mu_k(\rho) = i^k\varphi^{(k)}(0;\alpha,\rho), \qquad k = 1,2,\cdots. \tag{8}$$

Evaluating (8) for $k=1$, 2, 3, and 4, we get

$$\begin{aligned}
\mu_2(\rho) &= \alpha^2\rho \qquad \mu_4(\rho) = \alpha^4(3\rho^2+2\rho),\\
\mu_6(\rho) &= \alpha^6(15\rho^3+30\rho^2+16\rho)\\
\mu_8(\rho) &= \alpha^8(105\rho^4+420\rho^3+588\rho^2+272\rho).
\end{aligned}$$

Expressions for the cumulants $\lambda_k(\rho)$, defined by

$$c(t;\alpha,\rho) \equiv \ln\varphi(t;\alpha,\rho) = \sum_{k=1}^{\infty}\lambda_k(it)^k/k!,$$

are more easily obtained. Here, $c(t;\alpha,\rho)$ is the cumulant generating function. Noting the infinite product expansion of sech αt, we have

$$c(t;\alpha,\rho) = \rho \ln \operatorname{sech} \alpha t = \rho \ln \prod_{j=1}^{\infty}\left[1+\frac{4\alpha^2t^2}{(2j-1)^2\pi^2}\right]^{-1}$$

$$= -\rho\sum_{j=1}^{\infty}\ln\left[1+\left(\frac{2\alpha t}{(2j-1)\pi}\right)^2\right]$$

$$c(t;\alpha,\rho) = -\rho\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\left[\frac{2\alpha t}{(2j-1)\pi}\right]^{2k}$$

$$= \rho\sum_{k=1}^{\infty}\left\{\frac{2^{2k}\alpha^{2k}(2k)!}{k\pi^{2k}}\sum_{j=1}^{\infty}(2j-1)^{-2k}\right\}\frac{(it)^{2k}}{(2k)!}\,.$$

But [14, p. 805]

$$\sum_{j=1}^{\infty}(2j-1)^{-2k} = \frac{(-1)^{k+1}(2^{2k}-1)\pi^{2k}}{2(2k)!}B_{2k}$$

where B_{2k} is the $2k$th Bernoulli number, so that for $k=1, 2, \cdots$

$$\lambda_{2k}(\rho) = (-1)^{k+1}(2^{2k}-1)2^{2k-1}\alpha^{2k}B_{2k}\rho/k, \tag{9}$$

and

$$\lambda_{2k-1}(\rho) = 0.$$

In particular,

$$\lambda_2(\rho) = \alpha^2\rho, \qquad \lambda_4(\rho) = 2\alpha^4\rho,$$
$$\lambda_6(\rho) = 16\alpha^6\rho, \qquad \lambda_8(\rho) = 272\alpha^8\rho.$$

The coefficient γ_2 of kurtosis (or of excess), defined by

$$\gamma_2 = \frac{\mu_4(\rho)}{[\mu_2(\rho)]^2} - 3 = \frac{\lambda_4(\rho)}{\lambda_2(\rho)} \tag{10}$$

is equal to $2\rho^{-1}$, which varies between zero and infinity, as ρ varies over the positive real numbers.

4. APPROXIMATIONS

Possible application of the family of *ghsd*'s is (at least) contingent upon the availability of a means of adequately approximating the d.f. $F(x;\alpha,\rho)$. Here we consider first the asymptotic behavior of this family of distributions (as $\rho\rightarrow\infty$) and then present an approximation to $F(x;\alpha,\rho)$ which appears to be quite satisfactory.

Let X be a r.v. having d.f. $F(\mathrm{x};\alpha,\rho)$ and let $Y=X/\alpha\rho^{\frac{1}{2}}$. Then the ch.f. of Y is $\varphi(t/\alpha\rho^{\frac{1}{2}};\alpha,\rho)$ and

$$\varphi(t/\alpha\rho^{1/2};\alpha,\rho) = [(e^{\rho^{-1/2}}t + e^{-\rho^{-1/2}}t)/2]^{-\rho}$$
$$= [1+(t^2/2\rho)+o(t^2/\rho)]^{-\rho}$$
$$\rightarrow e^{-t^2/2} = g(t),$$

as $\rho \to \infty$. Since $g(t)$ is the ch.f. of the standard normal distribution, by the continuity theorem for ch.f.'s it follows that

$$\lim_{\rho \to \infty} F(\alpha\rho^{1/2}y;\, \alpha,\, \rho) = (2\pi)^{-1} \int_{-\infty}^{y} e^{-x^2/2} dx.$$

It can also be shown that the density $h(y;\, \alpha,\, \rho) = (\alpha\rho^{1/2})^{-1} f(\alpha\rho^{1/2}y;\, \alpha,\, \rho)$ of Y converges to the standard normal density. In fact, using the inequality

$$\left| \operatorname{costy} \left[\operatorname{sech}\ \alpha t \rho^{-1/2}\right]^{\rho} \right| \le \left(1 + \frac{\alpha^2 t^2}{2}\right)^{-1}$$

and the dominated convergence theorem, it follows that

$$\lim_{\rho \to \infty} h(y;\, \alpha,\, \rho) = \lim_{\rho \to \infty} \pi^{-1} \int_{0}^{\infty} \operatorname{costy} \left[\operatorname{sech} \alpha t \rho^{-1/2}\right]^{\rho} dt$$

$$= \pi^{-1} \int_{0}^{\infty} \operatorname{costy}\ e^{-t^2/2} dt = (2\pi)^{-1/2} e^{-y^2/2}.$$

In a similar way, it can be shown that

$$F(x;\ \alpha,\ \rho) \to \int_{-\infty}^{x} (2\pi)^{-1/2} e^{-y^2/2} dy$$

and

$$f(x;\ \alpha,\ \rho) \to (2\pi)^{-1/2} e^{-y^2/2}$$

as $\rho \to \infty$, $\alpha \to 0$, with $\alpha^2\rho \to 1$.

These results suggest the use of an Edgeworth expansion to approximate $F(x;\, \alpha,\, \rho)$. For an arbitrary d.f. $F(x)$, with variance σ^2 and coefficients of skewness γ_1 and of kurtosis γ_2, the Edgeworth expansion of $F(x)$ (to three terms) is given by

$$F(x) \doteq \Phi(x/\sigma) + \frac{1}{6}\ \gamma_1 \Phi^{(2)}(x/\sigma) + \frac{1}{24} \gamma_2 \Phi^{(3)}(x/\sigma)$$

where $\Phi(x)$ is the standard normal d.f. and $\Phi^{(k)}(x)$ is the k^{th} derivative of $\Phi(x)$. For the family of *ghsd*'s, $\sigma^2 = \alpha^2\rho$, $\gamma_1 = 0$, and $\gamma_2 = 2/\rho$, leading to the approximation

$$F(x;\, \alpha,\, \rho) \doteq \Phi(x/\alpha\rho^{1/2}) + \frac{1}{12\rho} \Phi^{(3)}(x/\alpha\rho^{1/2}). \tag{11}$$

It can be seen by inspection of Table 1, in which the exact values of $F(x;\, 1,\, \rho)$, for $\rho = 1$, 2, and 4, are compared with the approximation given by (11), that the approximation improves rapidly with increasing ρ and is quite satisfactory even for these values of ρ.

5. ESTIMATION

Let $X_1, X_2, \cdots, X_n$ be a random sample from a population having the density function $f(x;\, \alpha,\, \rho)$. We shall consider the estimation of the parameters

TABLE I. COMPARISON OF EXACT DISTRIBUTION FUNCTIONS WITH THE EDGEWORTH EXPANSION (11)

x	$F_1(x)$	(11)	x	$F_2(x)$	(11)	x	$F_4(x)$	(11)
.200	.5984	.5985	.400	.6246	.6246	.400000	.5839	.5841
.400	.6880	.6902	.800	.7345	.7358	.500000	.6042	.6046
.600	.7635	.7747	1.200	.8213	.8243	1.000000	.7011	.7016
.800	.8235	.8337	1.600	.8845	.8882	1.500000	.7843	.7849
1.000	.8695	.8826	$1.282\sqrt{2}$	.9095	.9130	2.000000	.8505	.8514
1.282	.9145	.9260	2.000	.9274	.9300	1.282(2)	.9060	.9065
1.645	.9520	.9542	$1.645\sqrt{2}$	.9512	.9521	1.645(2)	.9507	.9510
1.960	.9707	.9670	$1.96\sqrt{2}$	.9721	.9718	1.96(2)	.9733	.9730
2.326	.9835	.9776	$2.326\sqrt{2}$	.9858	.9838	2.326(2)	.9875	.9869
2.576	.9892	.9837	$2.576\sqrt{2}$	.9911	.9894	2.576(2)	.9927	.9922
3.091	.9949	.9933	$3.091\sqrt{2}$	.9967	.9962	3.091(2)	.9977	.9976

α and ρ. We begin first by finding the Cramer-Rao lower bound for the variance of an unbiased estimator of α for $\rho=1$, 2, 3, and 4, and then examine the lower bound as $\rho\to\infty$.

For arbitrary ρ, the Cramer-Rao lower bound for the variance $\sigma_{\hat{\alpha}}^2(\alpha)$ of an unbiased estimator $\hat{\alpha}$ of α is given by $\sigma_{\hat{\alpha}}^2(\alpha)\geq(n\beta)^{-1}$, where

$$\beta = E\left[\frac{\partial \ln f(x;\ \alpha,\ \rho)}{\partial\alpha}\right]^2 = -E\left[\frac{\partial^2 \ln f(x;\ \alpha,\ \rho)}{\partial\alpha^2}\right]. \tag{12}$$

If $\rho=1$, then by straight forward calculations

$$\begin{aligned}\left[\frac{\partial}{\partial\alpha}\ln f(x;\ \alpha,\ 1)\right]^2 &= \alpha^{-2} - \pi x\alpha^{-3}\tanh\frac{\pi x}{2\alpha} + \pi^2x^2(4\alpha^4)^{-1}\\ &\quad - \frac{\pi^2x^2}{4\alpha^4}\operatorname{sech}^2\frac{\pi x}{2\alpha}\end{aligned} \tag{13}$$

and

$$\frac{\partial^2}{\partial\alpha^2}\ln f(x;\ \alpha,\ 1) = \alpha^{-2} - \frac{\pi x}{\alpha^3}\tanh\frac{\pi x}{2\alpha} - \frac{\pi^2x^2}{4\alpha^2}\operatorname{sech}^2\frac{\pi x}{2\alpha}\cdot \tag{14}$$

Thus, (12), (13) and (14) imply that

$$\beta = -\beta + E\left(\frac{\pi^2x^2}{4\alpha^4}\right) = -\beta + \pi^2/4\alpha^2$$

so that

$$\beta = \pi^2/8\alpha^2.$$

Therefore, the variance $\sigma_{\hat{\alpha}}^2(\alpha)$ of $\hat{\alpha}$ satisfies

$$\sigma_{\hat{\alpha}}^2(\alpha) \geq 8\alpha^2/n\pi^2 = \alpha^2/1.2337n. \tag{15}$$

The technique used for $\rho=1$ works for $\rho=2$, 3, and 4, with the following results

$$\sigma_{\hat{\alpha}}^{2}(\alpha) \geq \begin{cases} \alpha^2/1.4674n & \text{for } \rho = 2 \\ \alpha^2/1.587n & \text{for } \rho = 3 \\ \alpha^2/1.6617n & \text{for } \rho = 4 \end{cases} \tag{16}$$

As $\rho \to \infty$ (independently of α), $f(x\rho^{-1/2};\ \alpha,\ \rho)$ converges to the normal density $(2\pi\alpha^2)^{-1/2} \exp(-x^2/2\alpha^2) = n(x;\ \alpha)$. This fact may be used (in conjunction with the Lebesgue dominated convergence theorem and some elementary analysis) to show that

$$\lim_{\rho\to\infty} \sigma_{\hat{\alpha}}^{2}(\alpha) \geq -\left[nE\frac{\partial^2}{\partial\alpha^2} \ln n(x;\ \alpha)\right]^{-1} = \alpha^2/2n \tag{17}$$

Estimation of α and ρ by the method of maximum likelihood is not computationally feasible, so we consider "ad hoc" and moment estimators. We consider first the case when ρ is a known integer. For $\alpha>0$, the statistic

$$\hat{\alpha} = \sum_{j=1}^{n} |\ x_i\ | / nc_\rho \tag{18}$$

is an unbiased estimator of α for appropriately chosen constants c_ρ. Calculation of c_ρ involves computing the expected value of $|x|$, which may be carried out using Bateman's tables [1] (see p. 322). We find that $c_1=.74245$, $c_2=1.0855$, $c_3=1.34587$, and $c_4=1.5640$. The variance $\sigma_{\hat{\alpha}}^2(\alpha)$ of $\hat{\alpha}$ may be computed directly and is given by

$$\sigma_{\hat{\alpha}}^{2}(\alpha) = \alpha^{2}(\rho - c_\rho^{2})/nc_\rho^{2}. \tag{19}$$

The efficiency $e(\hat{\alpha})$ of $\hat{\alpha}$ in estimating α, obtained as the ratio of the Cramer-Rao lower bound and $\sigma_{\hat{\alpha}}^2(\alpha)$, is given by

$$e(\hat{\alpha}) = \begin{Bmatrix} .996, & \rho = 1 & \quad .960, & \rho = 3 \\ .977, & \rho = 2 & \quad .947, & \rho = 4 \end{Bmatrix} \tag{20}$$

We note that $e(\hat{\alpha})$ is monotonically decreasing as a function of ρ.

Next we consider the moment estimator

$$\tilde{\alpha} = \left\{ \sum_{i=1}^{n} x_i^{2}/n\rho \right\}^{1/2} \tag{21}$$

assuming ρ is known (but not necessarily an integer). From well-known properties of moment estimators it follows easily that $\tilde{\alpha}$ has a bias of order $0(n^{-1/2})$, so that for large n, $E(\tilde{\alpha})\simeq\alpha$. Using the delta method to approximate the variance $\sigma_{\tilde{\alpha}}^2(\alpha)$ of $\tilde{\alpha}$, we find that

$$\sigma_{\tilde{\alpha}}^{2}(\alpha) \simeq \alpha^2(\rho + 1)/2n\rho \tag{22}$$

so that the approximate efficiency of $\tilde{\alpha}$ is given by

$$e(\tilde{\alpha}) = \begin{Bmatrix} .811, & \rho = 1 & \quad .945, & \rho = 3 \\ .909, & \rho = 2 & \quad .963, & \rho = 4. \end{Bmatrix} \tag{23}$$

We see that $e(\tilde{\alpha})$ is a monotonically increasing function of ρ; it can be shown with no difficulty that $\tilde{\alpha}$ is an asymptotically efficient estimator of α as $\rho \to \infty$.

Confidence intervals for α (if ρ is known) can be constructed by using the Central Limit Theorem to show that $\hat{\alpha}$ and $\tilde{\alpha}$ are asymptotically normally distributed each with mean α, but with variances $\sigma^2_{\hat{\alpha}}(\alpha)$ and $\sigma^2_{\tilde{\alpha}}(\alpha)$ given, respectively, by (19) and (22).

If ρ is unknown, we proceed as follows. Since the coefficient of excess γ_2 (given by (10)) is equal to $2/\rho$, the moment estimator of ρ is $\tilde{\rho} = 2/\hat{\gamma}_2$, where

$$\hat{\gamma}_2 = \left[n^{-1} \sum_{i=1}^{n} x_i^4 - 3\left(n^{-1} \sum_{i=1}^{n} x_i^2 \right)^2 \right] \Big/ \left[n^{-1} \sum_{i=1}^{n} x_i^2 \right]^2$$

is the sample estimator of kurtosis. The moment estimator of α (if α is also unknown), would, in this case, be given by (21) with ρ replaced by $\tilde{\rho}$. We note that $\tilde{\rho}$ is invariant under change of scale, so that its distribution is independent of the scale parameter α. Further from well-known properties of moment estimators, it follows that for large n, $E(\tilde{\rho}) \simeq \rho$, $E(\tilde{\alpha}) \simeq \alpha$; using the delta method to approximate the second order moments of $\tilde{\alpha}$ and $\tilde{\rho}$ one finds that the covariance between $\tilde{\alpha}$ and $\tilde{\rho}$ is approximately equal to zero, that $\sigma^2_{\tilde{\alpha}}(\alpha)$ is given still by (22), while

$$\sigma^2_{\tilde{\rho}}(\rho) \simeq \rho[8\mu_4(\mu_4^2 - \mu_2\mu_6) + 2\mu_2^2(\mu_8 - \mu_4^2)]/n[\mu_4 - 3\mu_2^2]^3. \tag{24}$$

Substituting values for μ_2, μ_4, μ_6, and μ_8 [given explicitly following (8)] we conclude that

$$\sigma^2_{\tilde{\rho}}(\rho) \simeq 2\rho(\rho + 1)(3\rho^2 + 15\rho + 22)/n. \tag{25}$$

Confidence intervals for α and/or ρ can be constructed by again using the Central Limit Theorem to show that $\tilde{\alpha}$ and $\tilde{\rho}$ are asymptotically independently normally distributed with means α and ρ and variances $\sigma^2_{\tilde{\alpha}}(\alpha)$ and $\sigma^2_{\tilde{\rho}}(\rho)$ given, respectively, by (22) and (25).

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