Numerical Linear Algebra in Applications to Data Sciences

Hands on: Low-rank approximations and Hierarchical matrices

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In this session, we will try to get hands on low-rank approximation and Hierarchical matrices through following questions. Questions/Statements marked with (⧫) is for your further thinking and won't be discussed.

1 Matrix-Matrix multiplication

- Consider three matrices A,B,C of size 500x100, 100x700,700x900, then compute the computational complexity required for
  - A(BC)
  - (AB)C
- Is it possible to arrive at a condition with matrices A,B,C with respective sizes $m \times n$,$n \times p$,$p \times l$ in terms of m,n,p,l such that it defines which order is better?
- What happens if the size of C is $p \times 1$ and $n \ll min(m, p)$?
  ⧫ What happens to the condition A,C are vectors of size $1 \times n$ and $p \times 1$ respectively?

2 Low-Rank Algebra

- Calculate the matrix-vector product, $x = \begin{bmatrix} 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 5 & 7 & 9 & 11 & 13 \\ 4 & 7 & 10 & 13 & 16 & 19 \\ 5 & 9 & 13 & 17 & 21 & 25 \\ 6 & 11 & 16 & 21 & 26 & 31 \\ 7 & 13 & 19 & 25 & 31 & 37 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.
- Find the computational cost of following operations:
  - Matrix-Matrix multiplication Y =AB , where A and B have low-rank decomposition.
  - Matrix-Matrix multiplication Y =AB , where either A or B have low-rank decomposition.
  - Matrix-Vector multiplication y = Ax, where A has low-rank decomposition.
  - Matrix-Matrix Addition C = A + B, where A and B have low-rank decomposition.
    ⧫ In the previous we interpreted the matrix as sum of two low rank matrices!!

3 Low-rank approximations

3.1 Interpolation

- Use Chebyshev nodes to interpolate the matrix A, whose element $A_{ij} = \log r_{ij}$ and the location of charges are in 1D.
  ♦ Why Chebyshev nodes?

3.2 Singular Value Decomposition(SVD)

- Implementation of Randomized Singular Value Decomposition
  ♦ Given the concept of Randomized SVD, for a symmetric matrix A, can we find Eigen value decomposition?
  ♦ What step needs to be modified to get Eigen value decomposition?

4 Hierarchical matrices

- What is well-separateness?
- How is Hierarchical matrix($\mathcal{H}$-matrix) stored?

5 Further Readings

Find below some useful resources with hyperlinks:

- Matrix-Chain multiplication
- Hierarchical matrices
- Randomized Methods for Linear Algebra
- Fast Multipole method
- HODLRlib
- Why Are Big Data Matrices Approximately Low Rank?