# HW7

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## 1 CS 168 Spring Assignment 7

SUNet ID(s): 05794739

Name(s): Luis A. Perez

Collaborators: None

By turning in this assignment, I agree by the Stanford honor code and declare that all of this is my own work.

## 2 Imports

[114]:
```python
import collections
import matplotlib.pyplot as plt
import scipy

import numpy as np
from PIL import Image
from sklearn import decomposition
import pandas as pd
import seaborn as sns
import os
import warnings

import IPython

from typing import Dict, List, Text, Tuple

# Make figure larger
plt.rcParams['figure.figsize'] = [10, 5]

# Set numpy seed for consistent results.
np.random.seed(1)
```

[2]:
```python
class Globals:
    DATA_PATH = 'data/'
```

## 3 Part 1

### 3.1 Part 1b

[105]:
```python
def circle_graph(n=10):
    """Returns transition matrix for a circle graph."""
    A = np.zeros((n, n))
    for i in range(n):
        A[i, (i + 1) % n] = 1
        A[(i + 1) % n, i] = 1
    return 0.5 * A, 1 / n * np.ones((1, n))

def circle_graph_with_extra_edge(n=10, a=1, b=5):
    """Same as above but for a circle graph with an additional edge (a,b)."""
    P, _ = circle_graph(n=n)
    P[a-1, b-1] = 0.5
    P[b-1, a-1] = 0.5
    P[a-1, :] = P[a-1, :] * 2 / 3
    P[b-1, :] = P[b-1, :] * 2 / 3
    denom = 2*(n-2) + 3*2
    pi = 2 * np.ones((1, n)) / denom
    pi[0, a - 1] = 3 / denom
    pi[0, b - 1] = 3 / denom

    return P, pi
```

[ ]:
```python
def variation_distance(x, y):
    return 1 / 2 * np.sum(np.abs(x - y))

def chain_mixing(T, pi):
    s = np.zeros((1, np.shape(T)[0]))
    s[0, 0] = 1
    distance = [variation_distance(s, pi)]
    for t in range(1, 101):
        s = np.dot(s, T)
        distance.append(variation_distance(s, pi))

    return distance
```

[110]:
```python
def problem1b():
    graphs = {
        '10-circle' : circle_graph(n=10),
        '9-circle': circle_graph(n=9),
        '9-circle-with-extra-edge': circle_graph_with_extra_edge(n=9)
    }
    for name, (T, pi) in graphs.items():
        dists = chain_mixing(T, pi)
        plt.plot(range(len(dists)), dists, label=name)
    plt.legend()
    plt.title('Distribution Mixing From Initial State on Node 1')
    plt.ylabel('Variational Distance')
    plt.xlabel('Mixing Time')
    plt.savefig('figures/mixing_time.png', format='png')
    plt.close()
```

[111]:
```
problem1b()
```

### 3.2 Part 1c

[136]:
```
def problem1c():
    graphs = {
        '10-circle' : circle_graph(n=10),
        '9-circle': circle_graph(n=9),
        '9-circle-with-extra-edge': circle_graph_with_extra_edge(n=9)
    }
    for name, (T, _) in graphs.items():
        w, _ = np.linalg.eig(T.T)
        second_max_idx = len(w) - 2
        lambda2 = np.partition(w, second_max_idx)[second_max_idx]
        print(f'{name}')
        IPython.display.display(IPython.display.Math(f'\lambda_2={lambda2}'))
```

[137]:
```
problem1c()
```

```
10-circle
```

$\lambda_2 = 0.8090169943749482$

```
9-circle
```

$\lambda_2 = 0.766044443118978$

```
9-circle-with-extra-edge
```

$\lambda_2 = 0.7675918792439985$

## 4 Part 2

[197]:
```
def route_length(D, idx):
    """Computes total length of route visiting the specified idx in order."""
    return sum([D[idx[i], idx[i+1]] for i in range(len(idx) - 1)])

def load_distance():
    """Loads a matrix containing all pair-wise distances between all parks."""
    data = pd.read_csv(os.path.join(Globals.DATA_PATH, 'parks.csv'))
    X = np.stack((data['Longitude'], data['Latitude'])).T
    n = np.shape(X)[0]
    X2 = np.diag(np.dot(X, X.T)).reshape((n, 1))
    Y2 = X2.reshape((1, n))
    XY = np.dot(X, X.T)
    D = np.sqrt(X2 + Y2 - 2*XY)
    park_to_idx = {name: i for i, name in enumerate(data['Name'])}
    idx_to_park = {i : name for name, i in park_to_idx.items()}

    # Sanity checks.
    assert np.allclose(D[park_to_idx['Acadia'], park_to_idx['Arches']], 41.75,␣
↪rtol=1e-4)
    assert np.allclose(route_length(D, list(range(n)) + [0]), 491.92)
    return D, park_to_idx, idx_to_park
```

## 4.1 Part 2a

```
[287]: def mcmc(D, n_iters, T, route_fn):
           """Implements MCMC for Eucledian Traveling Salesmen on Distance Matrix D."""
           route = np.array(range(D.shape[0]))
           np.random.shuffle(route)
           best = route.copy()

           selected = np.random.choice(range(D.shape[0]), replace=True, size=n_iters)
           selected2 = np.random.choice(range(D.shape[0]), replace=True, size=n_iters)
           route_dists = []
           for idx, idx2 in zip(selected, selected2):
               newroute = route.copy()
               route_fn(route, newroute, idx, idx2)
               route_dist = route_length(D, route.tolist() + [route[0]])
               newroute_dist = route_length(D, newroute.tolist() + [newroute[0]])
               route_dists.append(route_dist)
               delta_dist = newroute_dist - route_dist
               if delta_dist < 0 or (T > 0 and np.random.uniform() < np.
       ↪exp(-delta_dist / T)):
                   route = newroute
               if route_length(D, route.tolist() + [route[0]]) < route_length(D, best.
       ↪tolist() + [best[0]]):
                   best = route
           return best, route_dists
```

## 4.2 Part 2b

```
[225]: def sequential_modify(route, newroute, idx, idx2):
           del idx2
           next_idx = (idx + 1) % D.shape[0]
           newroute[idx] = route[next_idx]
           newroute[next_idx] = route[idx]
```

```
def problem2b():
    D, park_to_idx, idx_to_park = load_distance()
    for T in [0, 1, 10, 100]:
        best_length = None
        for trial in range(10):
            _, y = mcmc(D, n_iters=10000, T=T, route_fn=sequential_modify)
            plt.plot(range(len(y)), y, label=f'trial_idx={trial + 1}')
            if best_length is None or min(y) < best_length:
                best_length = min(y)

        plt.legend()
        plt.title(f'Current Route by Iteration for T={T}')
        plt.xlabel('Iteration')
        plt.ylabel('Route Distance')
        plt.savefig(f'figures/route_by_iteration_t={T}.png', format='png')
        plt.close()

        print(f'[T={T}]The best found route over all trials was: {best_length:.
 ↪2f}.')
```

[226]:
```
problem2b()
```

```
[T=0]The best found route over all trials was: 375.42.
[T=1]The best found route over all trials was: 320.64.
[T=10]The best found route over all trials was: 279.21.
[T=100]The best found route over all trials was: 358.89.
```

### 4.3 Problem 2c

[231]:
```
def random_modify(route, newroute, idx, idx2):
    newroute[idx] = route[idx2]
    newroute[idx2] = route[idx]

def problem2c():
    D, park_to_idx, idx_to_park = load_distance()
    for T in [0, 1, 10, 100]:
        best_length = None
        for trial in range(10):
            _, y = mcmc(D, n_iters=10000, T=T, route_fn=random_modify)
            if best_length is None or min(y) < best_length:
                best_length = min(y)
            plt.plot(range(len(y)), y, label=f'trial_idx={trial + 1}')
        plt.legend()
        plt.title(f'Current Route by Iteration for T={T}')
        plt.xlabel('Iteration')
        plt.ylabel('Route Distance')
        plt.savefig(f'figures/route_by_iteration_t={T}_random.png',␣
↪format='png')
        plt.close()

        print(f'[T={T}]The best found route over all trials was: {best_length:.
↪2f}.')
```

[232]:
```
problem2c()
```

```
[T=0]The best found route over all trials was: 200.58.
[T=1]The best found route over all trials was: 166.23.
[T=10]The best found route over all trials was: 252.79.
[T=100]The best found route over all trials was: 323.23.
```

## 5 Part 3

## 6 Part 3a

[294]:
```
import importlib
import qwop
importlib.reload(qwop)
from matplotlib import animation
```

[282]:
```
def draw_simulation(data, path):
    fig = plt.figure()
    fig.set_dpi(100)
    fig.set_size_inches(12, 3)

    ax = plt.axes(xlim=(-1, 10), ylim=(0, 3))

    joints = [5, 0, 1, 2, 1, 0, 3, 4]
    patch = plt.Polygon([[0, 0], [0, 0]], closed=None, fill=None, edgecolor='k')
    head = plt.Circle((0, 0), radius=0.15, fc='k', ec='k')

    def init():
        ax.add_patch(patch)
        ax.add_patch(head)
        return patch, head

    def animate(j):
        points = zip([data[j][0][i] for i in joints],
                     [data[j][1][i] for i in joints])
        patch.set_xy(list(points))
        head.center = (data[j][0][5], data[j][1][5])
        return patch, head
    anim = animation.FuncAnimation(fig,
                                   animate,
                                   init_func=init,
                                   frames=len(data),
                                   interval=20)


    anim.save(f'{path}.mp4', fps=50)
    plt.close()
```

[350]:
```
def mcmc_sim(n_iters):
    """Implements MCMC QWOP Simulation"""
    route = np.random.uniform(low=-1, high=1, size=40)
    route_value = qwop.sim(route, include_data=False)[0]
    best = route.copy()
    best_value = route_value
    route_dists = [route_value]
    for i in range(n_iters):
        newroute = np.clip(route + np.random.uniform(-0.1, 0.1, size=route.
 ↪shape), -1, 1)
        newroute_value = qwop.sim(newroute, include_data=False)[0]
        if newroute_value > route_value:
            route = newroute
            route_value = newroute_value
        if route_value > best_value:
            best = route
            best_value = route_value
            print(f'New best: {best_value:.2f} at iteration {i + 1}.')
    return best, best_value, route_dists
```

[ ]:
```
best_plan = None
best_value = 0
for _ in range(10):
    small_plan, small_value, _ = mcmc_sim(n_iters=20000)
    if best_plan is None or best_value < small_value:
        best_plan = plan
        best_value = small_value
```

```
New best: 1.39 at iteration 1.
New best: 3.13 at iteration 2.
New best: 3.23 at iteration 307.
New best: 3.26 at iteration 517.
New best: 3.31 at iteration 696.
New best: 3.32 at iteration 746.
New best: 3.49 at iteration 915.
New best: 3.79 at iteration 948.
New best: 3.93 at iteration 955.
New best: 4.02 at iteration 1111.
New best: 4.12 at iteration 1445.
New best: 4.69 at iteration 1479.
New best: 4.88 at iteration 1795.
```

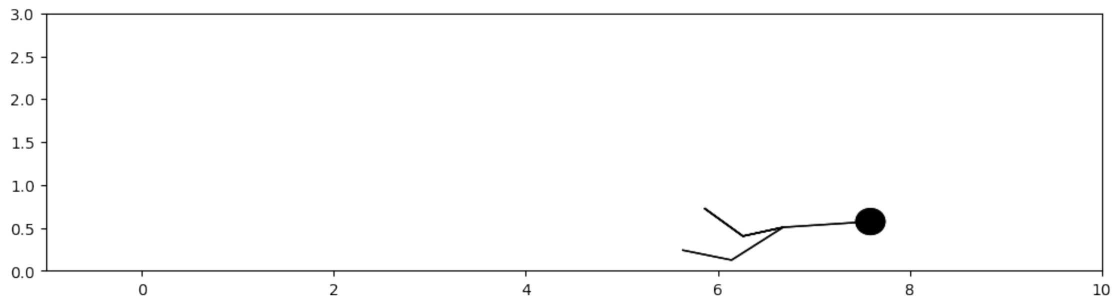
[342]:
```
plan, value, _ = mcmc_sim(n_iters=10000, T=0)
```

```
New best: 1.62 at iteration 2.
New best: 1.69 at iteration 4.
New best: 1.74 at iteration 5.
New best: 1.90 at iteration 6.
New best: 2.19 at iteration 7.
New best: 2.56 at iteration 12.
New best: 2.65 at iteration 27.
New best: 2.68 at iteration 87.
New best: 2.76 at iteration 88.
New best: 3.11 at iteration 89.
New best: 3.12 at iteration 99.
New best: 3.65 at iteration 117.
New best: 3.84 at iteration 119.
New best: 4.02 at iteration 159.
New best: 4.05 at iteration 169.
New best: 4.27 at iteration 247.
New best: 4.33 at iteration 331.
New best: 4.35 at iteration 433.
New best: 4.51 at iteration 437.
New best: 4.83 at iteration 451.
New best: 4.91 at iteration 541.
New best: 5.04 at iteration 661.
New best: 5.06 at iteration 689.
New best: 5.08 at iteration 1124.
New best: 5.13 at iteration 1337.
New best: 5.22 at iteration 1393.
New best: 5.32 at iteration 1568.
New best: 5.54 at iteration 1595.
New best: 5.77 at iteration 1759.
New best: 5.87 at iteration 1906.
New best: 6.00 at iteration 2039.
New best: 6.01 at iteration 2127.
New best: 6.05 at iteration 2190.
New best: 6.11 at iteration 2711.
New best: 6.29 at iteration 2809.
New best: 6.32 at iteration 2907.
New best: 6.41 at iteration 2951.
New best: 6.41 at iteration 3013.
New best: 6.62 at iteration 3094.
New best: 6.68 at iteration 3191.
New best: 6.73 at iteration 3309.
New best: 6.82 at iteration 3343.
New best: 6.87 at iteration 4398.
New best: 6.94 at iteration 5045.
New best: 7.06 at iteration 5134.
New best: 7.25 at iteration 5290.
New best: 7.26 at iteration 5310.
New best: 7.35 at iteration 9263.
New best: 7.38 at iteration 9364.
New best: 7.59 at iteration 9954.
```

[357]:
```
plan
```

[357]:
```
array([ 3.32046582e-01,  8.64347977e-01, -5.36958882e-01,  1.33757192e-01,
       -7.46647497e-02,  5.20644096e-02,  1.38249558e-01,  4.68027199e-01,
        2.38875307e-01,  5.97549589e-01, -1.31011744e-01,  3.00758848e-01,
       -8.75100380e-01, -1.00000000e+00,  5.33726082e-01,  7.10140454e-01,
        8.44800416e-01,  1.00000000e+00, -6.50942301e-01, -8.15635219e-01,
       -6.90576364e-01,  4.99978093e-01, -9.96742325e-01, -6.18842000e-01,
        2.73972411e-01, -5.08626797e-01, -5.25919503e-01, -3.93425059e-01,
       -8.71606742e-01, -3.69966358e-01,  8.81972954e-02, -2.36058686e-01,
       -9.45380799e-01, -4.99008952e-01,  4.76229721e-01, -6.72647790e-04,
        7.42384034e-01, -5.09676613e-01, -8.47599711e-01,  8.12253360e-01])
```

[356]:
```
_, data = qwop.sim(plan)
draw_simulation(data, 'figures/best_mcmc')
```



[298]:
```
best, routes[]
```

[298]:
```
(array([-0.78027422,  0.0309237 ,  0.01693435, -0.655999  , -0.95880106,
        -0.96315125, -0.76193845, -0.86024011,  0.75840799,  0.66953856,
         0.85208282, -0.47579035,  0.82885493,  0.73460846, -0.15858533,
         0.98560189,  0.42121195,  0.57024489, -0.79955204, -0.128188  ,
        -0.86051161, -0.77008358,  0.43200857, -0.4883419 ,  0.48698762,
         0.12144517,  0.56585249, -0.29193376,  0.53723171,  0.58063392,
         0.51053619,  0.60100035, -0.80468913, -0.43719093,  0.64147052,
         0.56480972, -0.64413858,  0.44152763, -0.33662148,  0.27307277]),
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