CS168 Spring Assignment 6

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Collaborators:

By turning in this assignment, I agree by the Stanford honor code and declare that all of this is my own work.

Part 1

(a) We present the Laplacian matrix for each of the figures in turn.

(a) This matrix consists of $1, 2, 2, \cdots, 2, 2, 1$ (D) along the diagonal, with -1s surrounding the main diagonal for $i = j + 1$ and $i = j - 1$ ($-A$).

$$L = \begin{bmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & \cdots & 0 & 0 & 0 & 0 \\ & & & & \vdots & & & & \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & 1 \end{bmatrix}$$

(b) This is the same matrix as above, except the final row and column, which both consists entirely, and the diagonal. Nodes 1 and $n-1$ now have degree 2 while n has degree $n-1$ and all other have degree 3.

$$L = \begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & -1 \\ -1 & 3 & -1 & 0 & \cdots & 0 & 0 & 0 & -1 \\ 0 & -1 & 3 & -1 & \cdots & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 3 & \cdots & 0 & 0 & 0 & -1 \\ & & & & \vdots & & & & \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 3 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 3 & -1 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & -1 & \cdots & -1 & -1 & -1 & n-1 \end{bmatrix}$$

(c) This matrix is exactly the same as that in (a), except the top-right and bottom-left corners also have a -1 entry and every entry in the diagonal is 2.

$$L = \begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & \cdots & 0 & 0 & 0 & 0 \\ & & & & \vdots & & & & \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & 2 \end{bmatrix}$$

(d) This matrix is the same as in (b) except for the fact that every diagonal entry except for L_{nn} is 3, $L_{n-1,1} = L_{1,n-1} = -1$.

$$L = \begin{bmatrix} 3 & -1 & 0 & 0 & \cdots & 0 & 0 & -1 & -1 \\ -1 & 3 & -1 & 0 & \cdots & 0 & 0 & 0 & -1 \\ 0 & -1 & 3 & -1 & \cdots & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 3 & \cdots & 0 & 0 & 0 & -1 \\ & & & & \vdots & & & & \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 3 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 3 & -1 & -1 \\ -1 & 0 & 0 & 0 & \cdots & 0 & -1 & 3 & -1 \\ -1 & -1 & -1 & -1 & \cdots & -1 & -1 & -1 & n-1 \end{bmatrix}$$

(b) We plot the 8 requested figures. Please see Figure 1, Figure 2, Figure 3 and Figure 4.

These eigenvectors make perfect sense. The smaller eigenvectors are attempting to minimize the distances between neighbors in the graph, leading to smooth curves for the line and circle graphs since simply placing the points in order will decrease their neighbors distance. For the circle and line graphs with an additional point, similar curves are visible with the restriction that the n-th point is placed at 0, since this is the closest to all other points.

For the larger eigenvectors, we can see that they attempt to position neighbors as far away as possible. For the line graph, this gives rise to shapes where the closer to an endpoint a node is, the less distance it needs to be from its neighbors (because it's not connected to the other point). For the circle graph, we see that odd nodes are placed at one extreme, and even nodes at the other. When we include the additional node, we

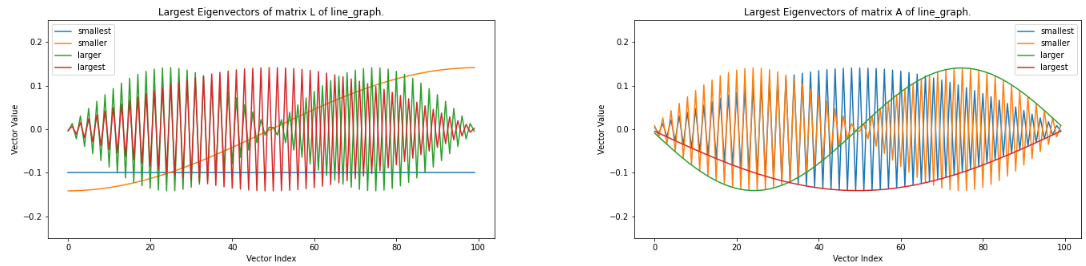


Figure 1: Eigenvectors for the laplacian and adjacency of the n-point line-graph. Vectors corresponding to the two smallest eigenvalues and vectors corresponding to the two largeest eigenvalues are shown for each case.

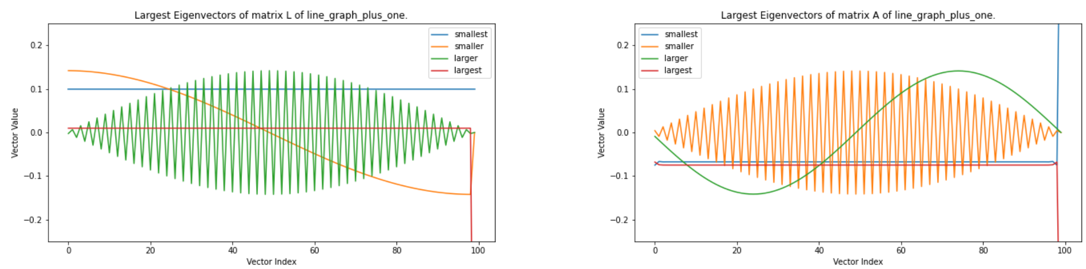


Figure 2: Eigenvectors for the laplacian and adjacency of the $n-1$-point line-graph with a final point connected to all previous points. Vectors corresponding to the two smallest eigenvalues and vectors corresponding to the two largeest eigenvalues are shown for each case.

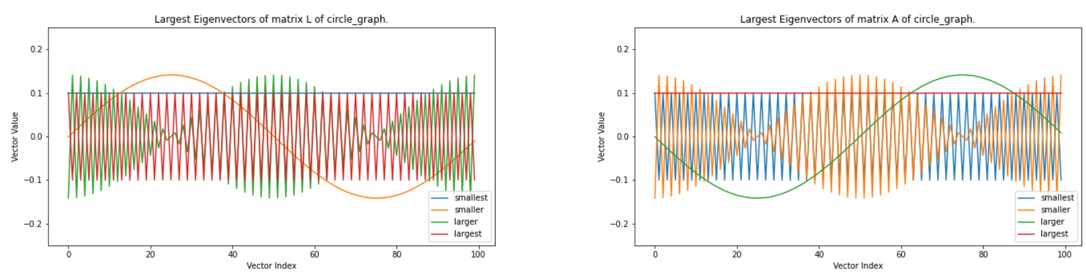


Figure 3: Eigenvectors for the laplacian and adjacency of the n-point circle-graph. Vectors corresponding to the two smallest eigenvalues and vectors corresponding to the two largeest eigenvalues are shown for each case.

observe straight-forward behavior where the main goal is to separate that single node as much as possible from all the others.

(c) See Figure 5 for the plots of the four graphs embedded onto the 2nd and 3rd smallest eignevalues.

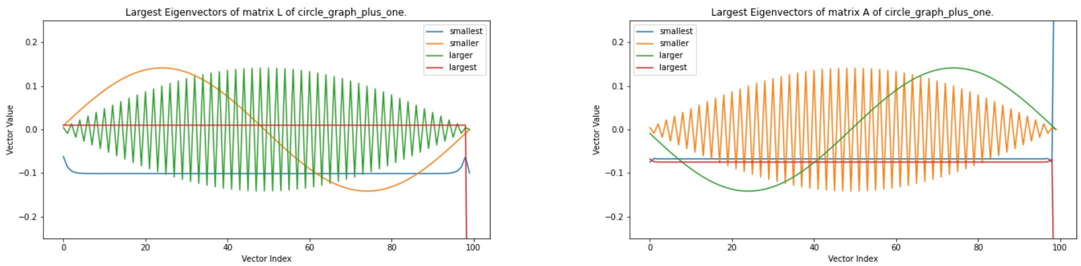


Figure 4: Eigenvectors for the laplacian and adjacency of the $n-1$-point circle-graph with a final point connected to all previous points. Vectors corresponding to the two smallest eigenvalues and vectors corresponding to the two largeest eigenvalues are shown for each case.

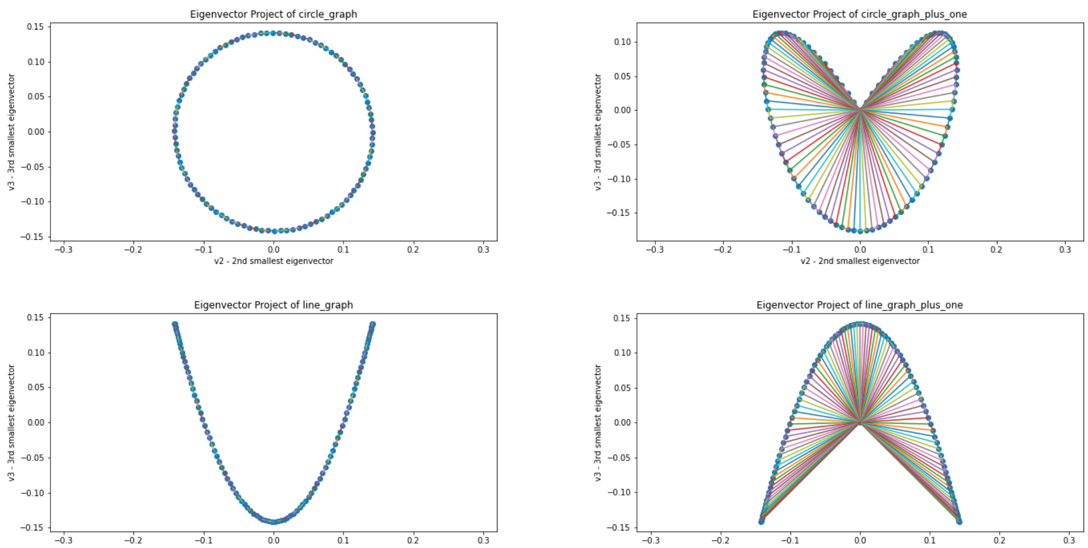
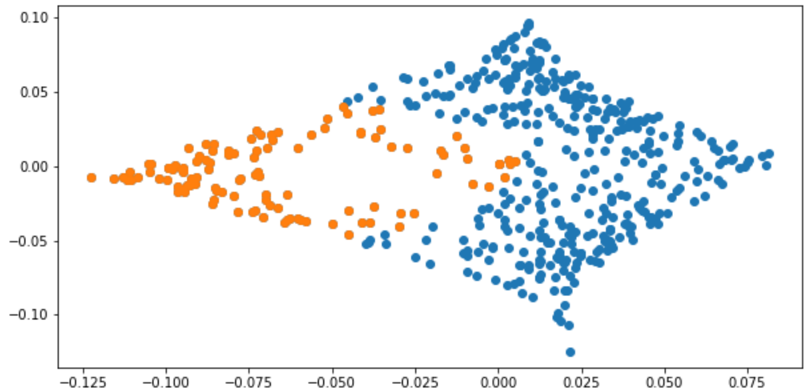


Figure 5: Projections of different types of graphs onto their second and third smallest eigenvectors.

(d) The requested plot can be seen in Figure (d). We note that the points whose x and y coordinates are less than $\frac{1}{2}$ are clustered together. The reason this makes sense is because the smaller eighvectors are trying to minimize the L_2 distance between neighbors. Since we defined the graph to form and edge between two nodes whose L_2 distance was less than $\frac{1}{4}$, it is immediately clear that all points in the lower left quadrant (whose x and y coordinates are less than $\frac{1}{2}$) will be connected (the distance between them, by definition, will be less than $\frac{1}{4}$). As such, in the projection, these points will tend to cluster so as to minimize their distance since they are neighbors in the graph.



Part 2

(a) See Appendix.

(b) Smallest 12 eigenvalues, in order:

```
0.000
0.000
0.000
0.000
0.000
0.000
0.014
0.054
0.074
0.081
0.120
0.133
```

(c) Given the results from (b), it appears that the graph has 6 connected components, since the multiplicity of the 0 eigenvalue of the Laplacian is 6. This follows from Theorem 2.1 in Lecture 11.

The sizes of the largest and second largest components are 1,484 and 3 respectively. We found this by finding 6 clusters in the eigenvectors. We repeated for a few eigenvectors to confirm.

(d) We now report the three disjoint low conductance sets. We found all three sets using the 9-th smallest eigenvector, which we plot in Figure 6. We first searched for a small set (of size between 150 and 300 that satisfied the conductance requirement by looking at points which were all within some t units of 0). Once we identified this set, the other 2 were found using the same eigenvector through a sliding window approach, where we looked for points which were shifted up or down from the previous set by some fixed amount.

We describe this process for each set in more detail below.

- S_1. This set comes from the 9-th smallest eigenvector, see Figure 6. It contains 229 elements and has a conducatance of 0.0887. 10 members are:

  ```
  2,    6,  519,    8,  521,  522, 1032, 1036,   13,  527
  ```

  This set was found by identifying the vertices whose values in the 9th eigenvector were in the range 0 ± 0.0002.

- S_2. This set comes from the 9-th smallest eigenvector, see Figure 6. It contains 561 elements and has a conducatance of 0.0586. 10 members are:

  ```
  1,  3,  4,  9, 10, 11, 14, 16, 19, 20
  ```

  This set was found by identifying the vertices whose values in the 9th eigenvector were in the range 0.00235 ± 0.00215.

- S_3. This set comes from the 9-th smallest eigenvector, see Figure 6. It contains 362 elements and has a conducatance of 0.0867. 10 members are:

  ```
  5,  7, 23, 27, 30, 42, 44, 46, 47, 50
  ```

  This set was found by identifying the vertices whose values in the 9th eigenvector were in the range 0.00469 ± 0.00019.

(e) The conductance of the randomly selected set of 150 nodes is 0.9085. This seems to indicate that the overall graph structure is not very tight (conductance is high for a random subgraph). As such, the groups identified above appear to be very tightly knit communities with little interaction outside their groups.

HW6

May 15, 2020

1 CS 168 Spring Assignment 6

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Collaborators: None

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2 Imports

[28]:
```
import collections
import matplotlib.pyplot as plt
import scipy

import numpy as np
from PIL import Image
from sklearn import decomposition
import pandas as pd
import seaborn as sns
import os
import warnings

from typing import Dict, List, Text, Tuple

# Make figure larger
plt.rcParams['figure.figsize'] = [10, 5]

# Set numpy seed for consistent results.
np.random.seed(1)
```

[29]:
```
class Globals:
    DATA_PATH = 'data/'
```

3 Part 1

3.1 Part 1b

[30]:
```
def line_graph(n=100):
    """Returns degree and adjacency matrices for line graph."""
    D = 2 * np.ones(n)
    D[0] = 1
    D[n - 1] = 1
    D = np.diag(D)
    A = np.zeros((n, n))
    for i in range(n-1):
        A[i, i + 1] = 1
        A[i + 1, i] = 1
    return A, D

def circle_graph(n=100):
    """Returns degree and adjacency matrices for circle graph."""
    D = 2 * np.ones(n)
    D = np.diag(D)
    A = np.zeros((n, n))
    for i in range(n):
        A[i, (i + 1) % n] = 1
        A[(i + 1) % n, i] = 1
    return A, D

def line_graph_plus_one(n=100):
    """
    Returns degree and adjacency matrices for an n-1
    point line graph with a final point connect to all previous points.
    """
    D = 3 * np.ones(n)
    D[0] = 2
    D[n - 2] = 2
    D[n - 1] = n - 1
    D = np.diag(D)
    A = np.zeros((n, n))
    for i in range(n-2):
        A[i, i + 1] = 1
        A[i + 1, i] = 1
    for i in range(n-1):
        A[n-1, i] = 1
        A[i, n-1] = 1
    return A, D

def circle_graph_plus_one(n=100):
    """
    Returns degree and adjacency matrices for an n-1 circle graph
```

```
    with a final point connecting to all previous points.
    """
    D = 3 * np.ones(n)
    D[n-1] = n-1
    D = np.diag(D)
    A = np.zeros((n, n))
    for i in range(n-1):
        A[i, i + 1] = 1
        A[i + 1, i] = 1
        A[i, n - 1] = 1
        A[n - 1, i] = 1
    return A, D
```

[31]:
```
def compute_eigens(graph_type, n=100):
    A, D = globals()[graph_type](n=n)
    L = D - A
    return {
        'L' : np.linalg.eig(L),
        'A' :np.linalg.eig(A)
    }
```

[32]:
```
x = compute_eigens('circle_graph')['L'][1][:, np.
↪argsort(compute_eigens('circle_graph')['L'][0])][:, 0]
```

[33]:
```
def problem1b():
    for graph_type in ['circle_graph', 'circle_graph_plus_one', 'line_graph',␣
↪'line_graph_plus_one']:
        for typ, (eig_val, eig_v) in compute_eigens(graph_type).items():
            idx = np.argsort(eig_val) # From small to large
            sorted_v = eig_v[:, idx] # Sort by eigenvalue.
            for name, i in zip(['smallest', 'smaller', 'larger', 'largest'],␣
↪[0, 1, -2, -1]):
                v = sorted_v[:, i]
                plt.ylim((-0.25, 0.25))
                plt.plot(range(len(v)), v, label=name)
            plt.legend()
            plt.xlabel('Vector Index')
            plt.ylabel('Vector Value')
            plt.title(f'{name.capitalize()} Eigenvectors of matrix {typ} of␣
↪{graph_type}.')
            plt.savefig(f'figures/eigenvectors_{graph_type}_{typ}.png',␣
↪format='png')
            plt.close()
```

[34]:
```
problem1b()
```

3.2 Problem 1c

[35]:
```
def edges(graph_type, n=100):
    """Returns an iterator over the edges (i,j) in the given graph."""
    _EDGES = {
        'line_graph': zip(range(0,n-1), range(1,n)),
        'circle_graph': zip(range(0,n), list(range(1,n)) + [0]),
        'line_graph_plus_one': list(zip(range(0,n-2), range(1,n-1))) +␣
↪list(zip(range(n-2), [n-1] * (n-1))),
        'circle_graph_plus_one': list(zip(range(0,n), list(range(1,n)) + [0]))␣
↪+ list(zip(range(n-2), [n-1] * (n-1))),
    }
    return _EDGES[graph_type]
```

[36]:
```
def ploblem1c():
    for graph_type in ['circle_graph', 'circle_graph_plus_one', 'line_graph',␣
↪'line_graph_plus_one']:
        eig_val, eig_v = compute_eigens(graph_type)['L']
        idx = np.argsort(eig_val)
        sorted_v = eig_v[:, idx]
        v2, v3 = sorted_v[:, 1], sorted_v[:, 2]
        plt.axis('equal')
        plt.scatter(v2, v3)
        for i,j in edges(graph_type):
            plt.plot([v2[i], v2[j]], [v3[i], v3[j]])
        plt.title(f'Eigenvector Project of {graph_type}')
        plt.xlabel('v2 - 2nd smallest eigenvector')
        plt.ylabel('v3 - 3rd smallest eigenvector')
        plt.savefig(f'figures/eignevector_projection_{graph_type}.png',␣
↪format='png')
        plt.close()
```

[37]:
```
ploblem1c()
```

3.3 Problem 1d

[38]:
```
def l2_dist(X, Y):
    """Computes the L2 pairwise distance between all elements in X,Y.

    Args:
        X: An (n,k) matrix where each row is an element.
        Y: An (m,k) matrix where each row is an element.

    Returns:
        D: An (n,m) matrix where D[i][j] is the distance L2 distance
            between X[i,:] and Y[j,:].
    """
```

```
    (n,k1), (m, k2) = np.shape(X), np.shape(Y)
    assert k1 == k2
    k = k1
    X2 = np.diag(np.dot(X, X.T)).reshape((n, 1))
    Y2 = np.diag(np.dot(Y, Y.T)).reshape((1, m))
    XY = np.dot(X, Y.T)
    return np.sqrt(X2 + Y2 - 2*XY)
```

[39]:
```
def problem1d():
    points = np.random.uniform(size=(500, 2))
    distances = l2_dist(points, points)
    A = (distances < 0.25).astype(int)
    assert np.allclose(A, A.T) # Matrix is symetric.
    D = np.diag(np.sum(A, axis=1))
    L = D - A
    val, eigs = np.linalg.eig(L)
    idx = np.argsort(val)
    v2, v3 = eigs[:, idx[1]], eigs[:, idx[2]]
    plt.scatter(v2, v3)

    mask = (points[:, 0] < 0.5) & (points[:, 1] < 0.5)
    plt.scatter(v2[mask], v3[mask])
    plt.savefig('figures/cluster_random.png', format='png')
    plt.title('Projection of Random Graph with Neighborhood Defined by L2 Dist␣
 ↪< 0.25')
    plt.xlabel('v2 - second smallest eigenvector of laplacian')
    plt.ylabel('v3 - third smallest eigenvector of laplacian')
    plt.close()
```

[40]:
```
problem1d()
```

4 Problem 2

4.1 Problem 2a

[14]:
```
def read_friendship_graph():
    """Returns Adjacency matrix of friendship graph."""
    data = pd.read_csv(os.path.join(Globals.DATA_PATH, 'cs168mp6.csv'),␣
 ↪header=None, names=['a', 'b'])
    assert len(data) == 61796
    assert len(set(data.a.unique()) | set(data.b.unique())) == 1495
    n = 1495
    A = np.zeros((n, n))
    for _, row in data.iterrows():
        A[row.a - 1, row.b - 1] = 1
        A[row.b - 1, row.a - 1] = 1
```

```python
    return A.astype(int)
```

4.2 Problem 2b

[25]:
```python
def problem2b():
    A = read_friendship_graph()
    D = np.diag(np.sum(A, axis=1))
    L = D - A
    vals, _ = np.linalg.eig(L)
    print('Smallest 12 eigenvalues, in order:')
    for val in sorted(vals)[:12]:
        print(f'{val:.3f}')
```

[26]:
```python
problem2b()
```

```
Smallest 12 eigenvalues, in order:
-0.000
-0.000
-0.000
-0.000
-0.000
0.000
0.014
0.054
0.074
0.081
0.120
0.133
```

4.3 Problem 2c

[446]:
```python
def problem2c():
    A = read_friendship_graph()
    D = np.diag(np.sum(A, axis=1))
    L = D - A
    vals, eighs = np.linalg.eig(L)
    sorted_eighs = eighs[:, np.argsort(vals)]
    return sorted_eighs
```

[447]:
```python
eigs = problem2c()
```

[473]:
```python
# Try a couple at different rounding factors until you get 6 distinct clusters.
for i in range(6):
    rounded = np.round(eigs[:, i], decimals=12)
    print(f'Number of components {len(np.unique(rounded))}')
    print(f'Component sizes {np.unique(rounded, return_counts=True)[1]}')
```

```
Number of components 6
Component sizes [   2    2    2    3 1484    2]
Number of components 6
Component sizes [   3    2 1484    2    2    2]
Number of components 6
Component sizes [   2    2    3 1484    2    2]
Number of components 6
Component sizes [   2    2    3 1484    2    2]
Number of components 6
Component sizes [1484    2    2    2    3    2]
Number of components 6
Component sizes [   2 1484    2    2    3    2]
```

4.4 Problem 2d

[47]:
```python
def conductance(S, A):
    n = A.shape[0]
    complementS = set(range(n)) - S
    num = 0
    for i in S:
        for j in complementS:
            num += A[i][j]
    AS = np.sum(A[list(S), :])
    AnotS = np.sum(A[list(complementS), :])
    return num / min(AS, AnotS)
```

[263]:
```python
def search_for_clusters(eigs, condutance_t=0.1, min_size=150, max_size=300,␣
↪num_unique_to_return=1):
    found = []
    prev_sets = []
    for v_idx in range(6, 40):
        print(f'Searching {v_idx}')
        for t in np.arange(0, 0.08, 0.00002):
            indexes = (np.abs(eigs[:, v_idx]) < t)
            vals = eigs[:, v_idx][indexes]
            if len(vals) >= min_size and len(vals) <= max_size:
                cond = conductance(set(indexes.nonzero()[0]), A)
                if cond < condutance_t:
                    if not prev_sets:
                        found.append((t, v_idx, cond))
                        prev_sets.append(set(indexes.nonzero()[0]))
                    else:
                        s = set(indexes.nonzero()[0])
                        skip = False
                        for prev in prev_sets:
                            if len(s & prev) > 10:
                                skip = True
```

```
                                    break
                            if not skip:
                                found.append((t, v_idx, cond))
                                prev_sets.append(s)
                        if len(found) >= num_unique_to_return:
                            return found
    return found
```

[330]:
```
# Read the graph.
A = read_friendship_graph()
D = np.diag(np.sum(A, axis=1))
L = D - A
vals, eigs = np.linalg.eig(L)
idx = np.argsort(vals)
sorted_eigs = eigs[:, idx]
```

[331]:
```
# Search to find a candidate eigenvector.
search_for_clusters(sorted_eigs, num_unique_to_return=1)
```

```
Searching 6
Searching 7
Searching 8
```

[331]: [(0.0002, 8, 0.08864468864468865)]

[342]:
```
# Try looking at different values here to identify closely clustered sets of␣
↪points.
s1 = set((np.abs(sorted_eigs[:, 8]) < 0.0002).nonzero()[0])
len(s1), conductance(s1, A)
```

[342]: (229, 0.08864468864468865)

[343]:
```
s2 = set(((sorted_eigs[:, 8] > 0.0002) & (sorted_eigs[:, 8] < 0.0045)).
↪nonzero()[0])
len(s2), conductance(s2, A)
```

[343]: (561, 0.05860035888233786)

[344]:
```
s3 = set(((sorted_eigs[:, 8] > 0.0045) & (sorted_eigs[:, 8] < 0.00488)).
↪nonzero()[0])
len(s3), conductance(s3, A)
```

[344]: (362, 0.08672401767030923)

[345]:
```
# Assert the ids are disjoint.
s1 & s2, s2 & s3, s1 & s3
```

[345]: (set(), set(), set())

[351]:
```
np.array(list(s1))[:10] + 1
```

[351]: array([2,   6, 519,   8, 521, 522, 1032, 1036,   13, 527])

[352]:
```
np.array(list(s2))[:10] + 1
```

[352]: array([1, 3, 4, 9, 10, 11, 14, 16, 19, 20])

[353]:
```
np.array(list(s3))[:10] + 1
```

[353]: array([5, 7, 23, 27, 30, 42, 44, 46, 47, 50])

[354]:
```
plt.title('Eigenvector corresponding to 9-th smallest eigenvalue.')
plt.xlabel('Vector Index')
plt.ylabel('Vector Value')
plt.plot(range(sorted_eigs.shape[0]), sorted_eigs[:, 8])
plt.savefig('figures/eigenvector_9.png', format='png')
plt.close()
```

4.5 Problem 2e

[355]:
```
def problem2e():
    A = read_friendship_graph()
    n = A.shape[0]
    selected = set(np.random.choice(range(n), size=150, replace=False))
    cond = conductance(selected, A)
    print(f'The conductance of the randomly selected set of {150} nodes is␣
 ↪{cond:.4f}.')
```

[356]:
```
problem2e()
```

The conductance of the randomly selected set of 150 nodes is 0.8975.

[ ]:
```

```

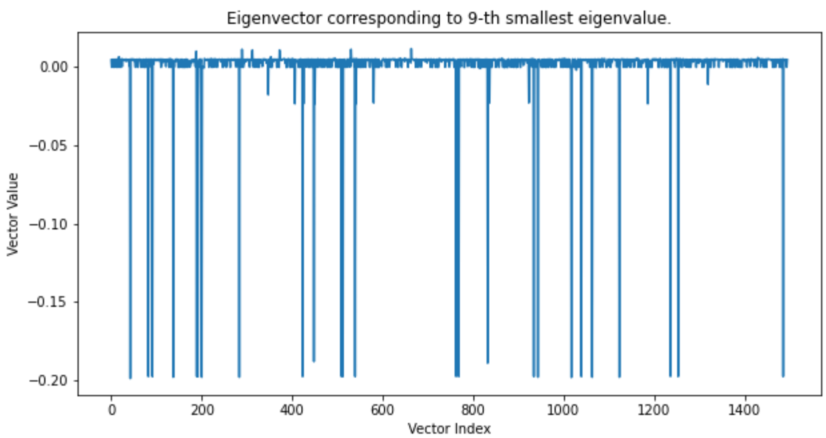

Figure 6: Eigenvector corresponding to 9th smallest eigenvalue - used to identify low conductance clusters.