

## EE 263 Homework 7

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### Properties of symmetric matrices

**Solution:**

- (a) For the proof, note the following:

$$\begin{aligned} P &\geq 0 && \text{(Given)} \\ \implies P - 0 &\geq 0 && \text{(Zero matrix does not affect inequality)} \\ \implies P - (Q - Q) &\geq 0 && \text{(Rewriting 0 matrix as } Q - Q) \\ \implies (P + Q) - Q &\geq 0 && \text{(Matrix addition is associative)} \\ \implies P + Q &\geq Q && \text{(By definition)} \end{aligned}$$

- (b) For the proof, note the following:

$$\begin{aligned} P &\geq Q && \text{(Given)} \\ \implies P - Q &\geq 0 && \text{(By definition)} \\ \implies -(P - Q) &\leq 0 && \text{(Multiplying by } -1 \text{ makes the matrix } P - Q \text{ negative semidefinite)} \\ \implies -Q + P &\leq 0 && \text{(Distributing scalar multiplication)} \\ \implies -P &\leq -Q && \text{(By definition)} \end{aligned}$$

- (c) The proof is not as straight forward. However, note that  $P > 0$  means that all eigenvalues of  $P$  exists and are positive ( $\lambda_1, \dots, \lambda_n > 0$ ). The corresponding eigenvalues of  $P^{-1}$  are  $\frac{1}{\lambda_i}$ , and since  $\lambda_i > 0$  for all  $i$ , we have that  $\frac{1}{\lambda_i} > 0$ . As such,  $P^{-1} > 0$ .
- (d) We first prove the given hint. We wish to show that  $Z \geq I \implies Z^{-1} \leq I$ . To begin,  $Z \geq I$  means that the eigenvalues of  $Z - I$  are all non-negative. Letting  $\lambda_i$  be an eigenvalue of  $Z$ , note that we have:

$$(Z - I)v_i = Zv_i - v_i = \lambda_i v_i - v_i = (\lambda_i - 1)v_i$$

So the eigenvalues of  $Z - I$  are all of the form  $\lambda_i - 1$ . Given that  $Z - I \geq 0$ , we know that  $\lambda_i - 1 \geq 0 \implies \lambda_i \geq 1$ .

Note that  $Z^{-1}$  is defined and exists, since  $\lambda_i \geq 1 > 0$ . In fact, it has eigenvalues  $\frac{1}{\lambda_i}$ . By a similar argument as the one made above, we must have that the eigenvalues of  $Z^{-1} - I$  are all  $\frac{1}{\lambda_i} - 1$ . Since  $\lambda_i \geq 1$ , we must have that  $\frac{1}{\lambda_i} \leq 1$ , which implies  $\frac{1}{\lambda_i} - 1 \leq 0$ , which by definition means that  $Z^{-1} - I \leq 0$  (or  $Z^{-1} \leq I$ ).

Now that we've proven the hint, we focus on proving the remaining statement. First, note that  $Q > 0$  implies that  $Q^{-\frac{1}{2}}$  exists. Furthermore, we have:

$$\begin{aligned}
 Q^{-\frac{1}{2}} &= (U\Lambda U^T)^{-\frac{1}{2}} && (Q \text{ is a symmetric matrix}) \\
 &= U\Lambda^{-\frac{1}{2}}U^T && (\text{Properties of diagonalization}) \\
 &= (U^T\Lambda^{-\frac{T}{2}}U)^T && (\text{Transposing twice}) \\
 &= Q^{\frac{T}{2}}
 \end{aligned}$$

As such, we have that  $Q^{-\frac{1}{2}} = Q^{-\frac{T}{2}}$ . As such, by congruence (since  $Q$  is just a rotation), we have the following:

$$\begin{aligned}
 &P \geq Q && (\text{Given}) \\
 \implies &P - Q \geq 0 && (\text{By definition}) \\
 \implies &Q^{-\frac{T}{2}}(P - Q)Q^{-\frac{1}{2}} \geq 0 && (\text{By congruence}) \\
 \implies &Q^{-\frac{T}{2}}PQ^{-\frac{1}{2}} - Q^{-\frac{T}{2}}QQ^{-\frac{1}{2}} \geq 0 && (\text{Distributive properties}) \\
 \implies &Q^{-\frac{T}{2}}PQ^{-\frac{1}{2}} - Q^{-\frac{1}{2}}QQ^{-\frac{1}{2}} && (Q^{-\frac{T}{2}} = Q^{-\frac{1}{2}}) \\
 \implies &Q^{-\frac{T}{2}}PQ^{-\frac{1}{2}} - I \geq 0 \\
 \implies &(Q^{-\frac{T}{2}}PQ^{-\frac{1}{2}} - I) \leq 0 && (\text{By hint proven previously}) \\
 \implies &Q^{\frac{1}{2}}P^{-1}Q^{\frac{T}{2}} - I \leq 0 \\
 \implies &Q^{-\frac{1}{2}}(Q^{\frac{1}{2}}P^{-1}Q^{\frac{T}{2}} - I)Q^{-\frac{T}{2}} \leq 0 && (\text{Congruence}) \\
 \implies &P^{-1} - Q^{\frac{1}{2}}Q^{-\frac{T}{2}} \leq 0 && (\text{Distributive property}) \\
 \implies &P^{-1} - Q^{-1} \leq 0 && (Q^T = Q) \\
 \implies &P^{-1} \leq Q^{-1} && (\text{By definition})
 \end{aligned}$$

- (e) This statement is false. For the simplest counter-example, take a scalar  $1 \times 1$  matrix  $P = [-1]$  and  $Q = [-2]$ . Then note that  $P \geq Q$ , but  $P^2 \leq Q^2$ .

## Drawing a representation of a graph

### Solution:

- (a) Let us begin by rephrasing the problem a bit first. Specifically, we'll begin by focusing on our objective. Note that we can write the objective function in quadratic form (we assume that each edge is only counted once). We have:

$$\begin{aligned} J &= \sum_{(i,j) \in E} (x_i - x_j)^2 + (y_i - y_j)^2 \\ &= \sum_{(i,j) \in E} x_i^2 + \sum_{(i,j) \in E} x_j^2 - 2 \sum_{(i,j)} x_i x_j + \sum_{(i,j) \in E} y_i^2 + \sum_{(i,j) \in E} y_j^2 - 2 \sum_{(i,j)} y_i y_j \end{aligned}$$

The above is the result of just expanding out. However, note that the term  $2 \sum_{(i,j) \in E} x_i x_j$  can be written simply as  $x^T A x$  where  $A$  is the adjacency matrix. This follows from the definition of quadratic form, noting that the factor of 2 is taken care of by the fact that  $A$  is symmetric. As such, we have:

$$J = \sum_{(i,j) \in E} x_i^2 + \sum_{(i,j) \in E} x_j^2 + \sum_{(i,j) \in E} y_i^2 + \sum_{(i,j) \in E} y_j^2 - x^T A x - y^T A y$$

Now, focusing on the first two terms above, we note that they too can be written in quadratic form. We're basically summing  $x_i^2$  for  $i = 1, \dots, n$ , where  $x_i$  appears a total of  $\deg(i)$  times (the degree of node  $i$ ). As such, let us define the matrix  $B$  as a diagonal matrix as follows:

$$B = \begin{bmatrix} \deg(1) & 0 & 0 & \cdots & 0 \\ 0 & \deg(2) & 0 & \cdots & 0 \\ 0 & 0 & \deg(3) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \deg(n) \end{bmatrix} \in \mathbb{R}^{n \times n}$$

With the above, we can easily rewrite our objective as:

$$\begin{aligned} J &= x^T B x - x^T A x + y^T B y - y^T A y \\ &= x^T (B + A) x + y^T (B + A) y \\ &= \mathbf{Tr} \left( \begin{bmatrix} x^T \\ y^T \end{bmatrix} (B - A) \begin{bmatrix} x & y \end{bmatrix} \right) \\ &= \mathbf{Tr} \left( \begin{bmatrix} x & y \end{bmatrix}^T (B - A) \begin{bmatrix} x & y \end{bmatrix} \right) \\ &= \mathbf{Tr}(X^T (B - A) X) \end{aligned}$$

Similarly, we can express our centering and spreading constraints simply as:

$$\begin{aligned}
 X^T X &= \begin{bmatrix} x^T \\ y^T \end{bmatrix} \begin{bmatrix} x & y \end{bmatrix} \\
 &= \begin{bmatrix} x^T x & x^T y \\ y^T x & y^T y \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} && \text{(The given constraints)} \\
 &= I_2
 \end{aligned}$$

Putting everything together, the problem becomes:

$$\begin{aligned}
 &\text{minimize} && \text{Tr}(X^T(B - A)X) \\
 &\text{subject to} && X^T X = I_2
 \end{aligned}$$

where we know that  $B + A$  is symmetric, and  $X$  is as defined above.

We know from lecture that the solution to this problem is given simply by:

$$X = [q_n \quad q_{n-1}]$$

achieving optimal value  $\lambda_n + \lambda_{n-1}$  where  $\lambda_1 \geq \dots \geq \lambda_n$  are the eigenvalues of  $B - A$  with corresponding orthonormal  $q_1, \dots, q_n$ .

Finally, we know that shifting all the coordinates by a fixed amount won't affect our value  $J$ , so our final solution (so that we satisfy all constraints) will be given by:

$$\begin{aligned}
 x &= q_n - \frac{1}{n} \sum_{i=1}^n q_{n,i} \\
 y &= q_{n-1} - \frac{1}{n} \sum_{i=1}^n q_{n-1,i}
 \end{aligned}$$

We note that, as explained in the problem, these values of  $x$  and  $y$  are not unique (any rotation or reflection of the points will be equally valid).

(b) We apply our method to the graph described by  $A$ . Our optimal value of  $J$  is:

$$J_{opt} = 0.1072932869526993$$

We can see this connectivity structure in Figure 1.

For comparison, the corresponding value of  $J$  associated with the circle is:

$$J_{circ} = 5.32711444$$

and we can see the resulting connectivity structure in Figure 2.

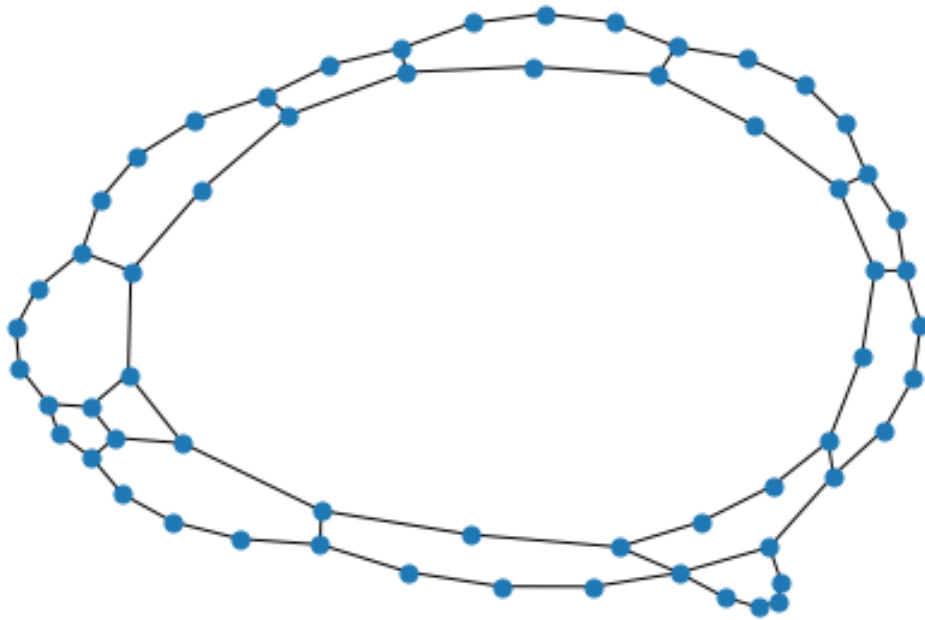


Figure 1: Optimal graph layout for the given connectivity structure

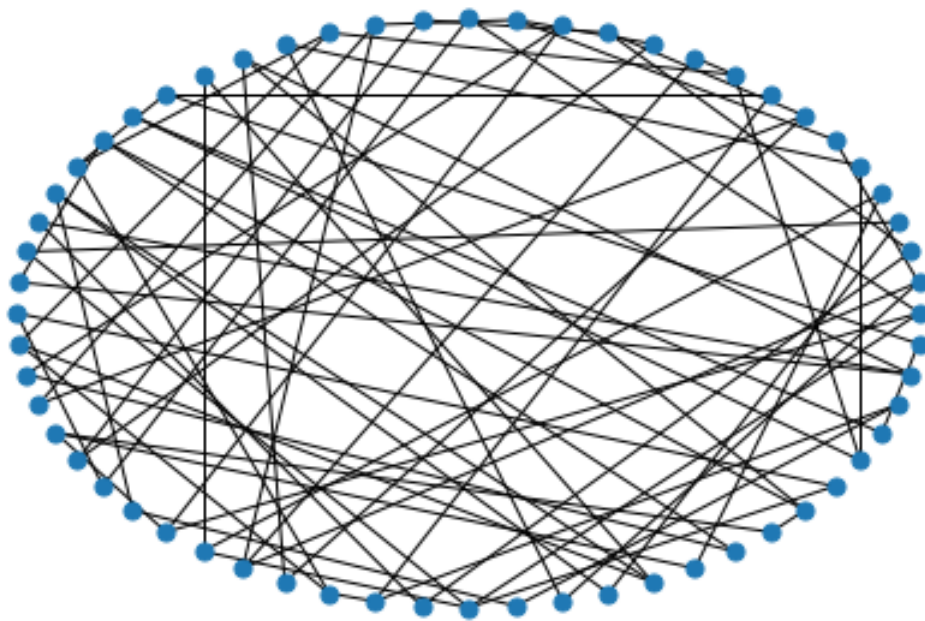


Figure 2: Graph layout with the given circle points

## Blind signal detection

**Solution:**

(a) We restate the problem in a framework we're familiar with. In fact, our goal is to:

$$\begin{array}{ll}\text{minimize} & \|w\| \\ \text{subject to} & \frac{1}{T} \|Aw\|^2 = 1\end{array}$$

Note that this is equivalent to:

$$\begin{array}{ll}\text{minimize} & \|w\|^2 \\ \text{subject to} & \frac{1}{T} \|Aw\|^2 = 1\end{array}$$

where:

$$A = \begin{bmatrix} \cdots & y_1^T & \cdots \\ \cdots & y_2^T & \cdots \\ \cdots & \vdots & \cdots \\ \cdots & y_T^T & \cdots \end{bmatrix} \in \mathbb{R}^{T \times n}$$

## Simultaneously estimating student ability and exercise difficulty

**Solution:** Our problem is to find  $\hat{G}$  which approximates  $G$ . In fact, what we wish to do is find the matrix  $\hat{G}$  of the form:

$$\hat{G} = \begin{bmatrix} \frac{1}{d_1} \\ \frac{1}{d_2} \\ \vdots \\ \frac{1}{d_m} \end{bmatrix} \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$$

which minimizes:

$$J = \sqrt{\frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n (G_{ij} - \hat{G}_{ij})^2}$$

Note that since  $\sqrt{\cdot}$  is monotonic and  $mn$  are constants, minimizing  $J$  is equivalent to minimizing:

$$\begin{aligned} \sqrt{mn}J &= \sqrt{\sum_{i=1}^m \sum_{j=1}^n (G_{ij} - \hat{G}_{ij})^2} \\ &= \|G - \hat{G}\|_F \end{aligned}$$

As such, we can rephrase our problem as:

$$\begin{array}{ll} \text{minimize} & \|G - \hat{G}\|_F \\ \text{subject to} & \text{Rank}(\hat{G}) = 1 \end{array}$$

From lecture, we know that this problem has the solution of:

$$\hat{G} = \sigma_1 u_1 v_1^T$$

where the SVD decomposition of  $G$  is given by:

$$G = \sum_{i=1}^r \sigma_i u_i v_i^T$$

where  $\sigma_1 \geq \sigma_2 \geq \cdots \sigma_r > 0$ . This approximation is unique if  $\sigma_1 > \sigma_2$ , otherwise multiple approximations achieve the same loss.

From the above model, we basically have what we want, except for the fact that we must

make sure that the exam question difficulties are normalized. We therefore have:

$$\begin{aligned}
 d_i &= \frac{1}{u_{1i}} \left( \frac{1}{m} \sum_{j=1}^m \frac{1}{u_{1j}} \right)^{-1} \\
 &= \frac{m}{u_{1i} \sum_{j=1}^m \frac{1}{u_{1j}}} \\
 \Rightarrow a_i &= \frac{m\sigma_1 v_{1i}}{\sum_{j=1}^m \frac{1}{u_{1j}}}
 \end{aligned}$$

Where the implication above comes from the fact that  $\frac{a_i}{d_i} = \sigma_1 u_{1i} v_{1i}$ . The above gives us our solution, ignoring the constraints that  $a_j$  be non-negative and  $d_i$  be positive.

Carrying out the proposed method above, we obtain the normalized difficulties as the vector:

$$d = \begin{bmatrix} 0.943 \\ 1.278 \\ 0.902 \\ 0.920 \\ 0.773 \\ 1.042 \\ 1.143 \end{bmatrix}$$

for which the optimal value achieved is  $J_{\text{opt}} = 5.675923069899214$ . The ratio of  $J_{\text{opt}}$  and the RMSE of  $G$  is 0.35742617468116206.



## Square matrices and the SVD

**Solution:**

- (a) False.
- (b) False.
- (c) False.
- (d) True.
- (e) False.
- (f) True.

## Principal-components analysis of decathlon data

### Solution:

- (a) A plot showing  $\sigma_j$  versus  $j$  is shown in Figure 3. A plot showing  $p_j$  versus  $j$  is shown in Figure 4. We also see that:

$$p_2 = 0.490183068840158$$

- (b) The requested plot of the first two principal components can be seen in Figure 5. It's clear from the plot that some events with similarity appear to be close to each other. For example, shot put and discus are both throwing related events. Similarly, the pole vault and javelin both require similar physical attributes.
- (c) The plot generated by the given command can be seen in Figure 6. It's clear from the plot that  $v_1$  most closely appears to represent  $r$ , since the intensity varies as the points move along the axis corresponding to  $v_1$ .
- (d) The requested plots are presented in Figure 7 and Figure 8.

From the plots, we can derive an intuitive interpretation for the first two left singular vectors of  $X$ .  $u_1$  appears to correlate with the athletes' total standardized scores,  $t$ .  $u_2$  on the other hand appears to correlate most closely with  $\delta$ , which is a measure of how much better an athlete is at running/leg events (100m, long jump, 100m hurdles) compared to throwing events (shot put, discus, javelin).

The code used for the above problems is now presented:

```

1 % load data
2 decathlon_pca_data
3
4 X = zscore(scores);
5 [U,S,V] = svd(X);
6 D = diag(S);
7
8 % part a
9 figure
10 plot(1:n, D);
11 title('\sigma_j vs j');
12 xlabel('j');
13 ylabel('\sigma_j');
14 saveas(gcf, 'sigma-vs-j.jpg');
15
16 p = cumsum(D.^2) / sum(D.^2);

```

```
17 figure
18 plot(1:n, p);
19 title('p_j vs j');
20 xlabel('j')
21 ylabel('p_j');
22 saveas(gcf, 'power-vs-j.jpg');
23
24 p2 = p(2);
25
26 % part b
27 v1 = V(:,1);
28 v2 = V(:,2);
29
30 figure
31 scatter(v1, v2);
32 text(v1, v2, events);
33 title('Top 2 components of event data')
34 xlabel('v_1')
35 ylabel('v_2')
36 saveas(gcf, 'principal_components.jpg')
37
38
39 % part c
40 t = sum(X,2);
41 r = t' * X;
42 spatial_plot(v1, v2, r, 3);
43 title('Spatial plot with r for intensity (color)')
44 xlabel('v_1')
45 ylabel('v_2')
46 saveas(gcf, 'intensity-plot.jpg')
47
48 % part d
49 u1 = U(:, 1);
50 u2 = U(:, 2);
51 delta = (X(:, 1) + X(:,5) + X(:,6)) - (X(:,3) + X(:,7) + X(:,9));
52
53 spatial_plot(u1, u2, t, length(names));
54 title('Spatial plot with t (athlete standardized scores) for
    intensity (color)')
55 xlabel('u_1')
56 ylabel('u_2')
57 saveas(gcf, 'spatial-t.jpg')
58
```

```
59 spatial_plot(u1 , u2 , delta , length(names));  
60 title('Spatial plot with r for intensity (color)')  
61 xlabel('u_1')  
62 ylabel('u_2')  
63 saveas(gcf, 'spatial_delta.jpg')
```

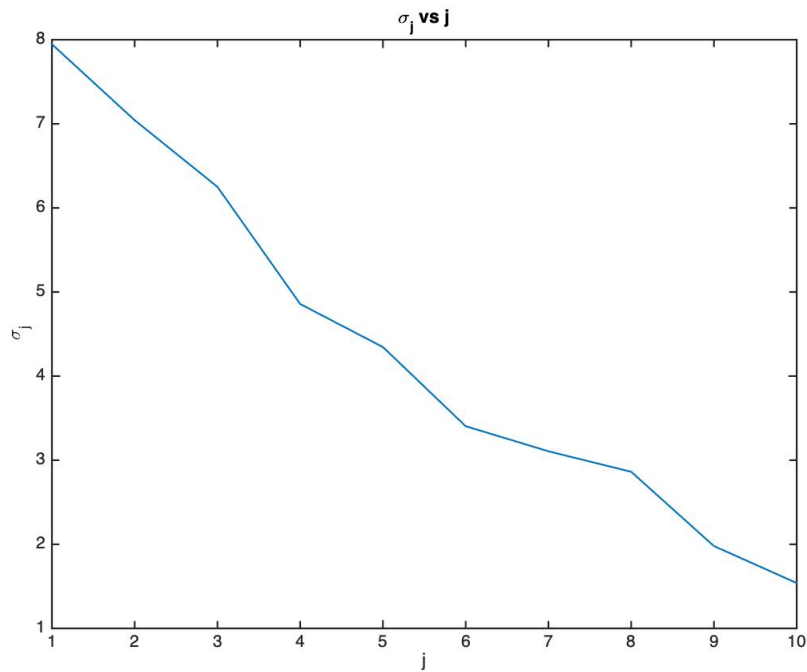


Figure 3: Plot of total power,  $\sigma_j$ , versus  $j$

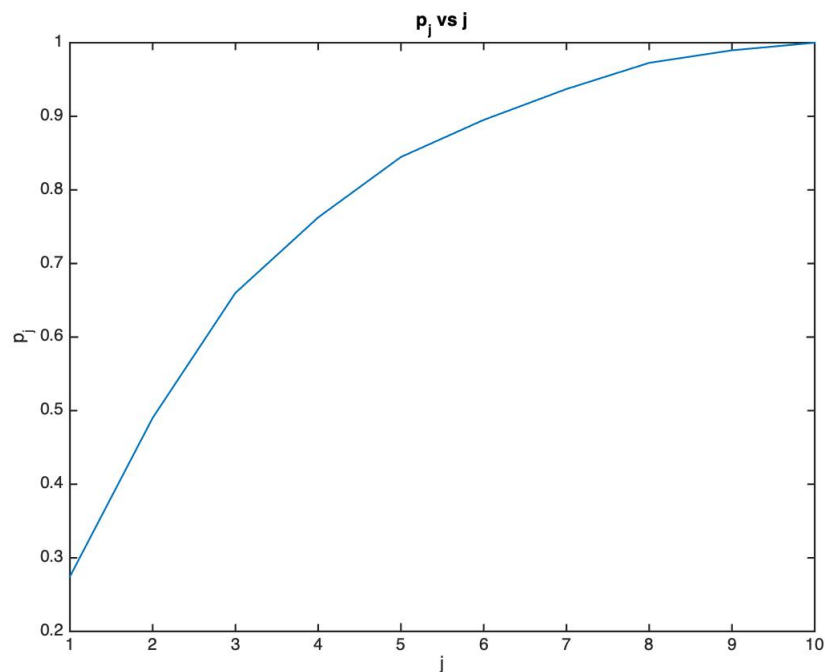


Figure 4: Plot of fraction of total power,  $p_j$ , versus  $j$

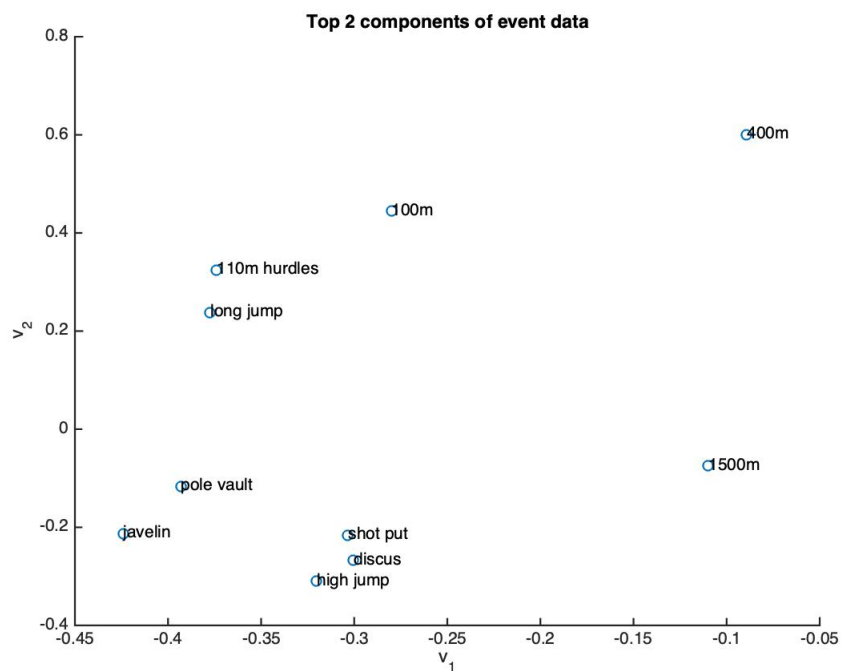


Figure 5: Plot of events by the first two principal components

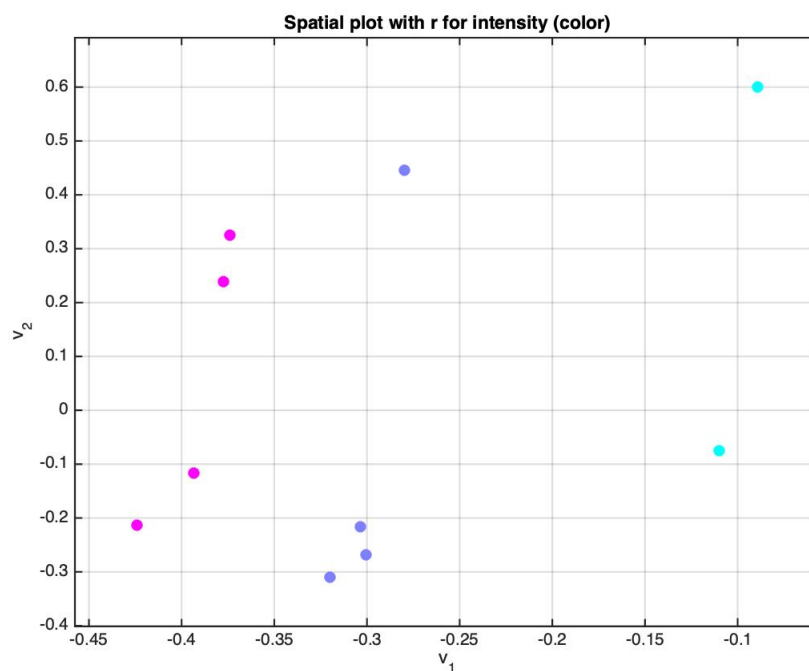


Figure 6: Plot of events colored by correlation between event and athlete's standardized test scores.

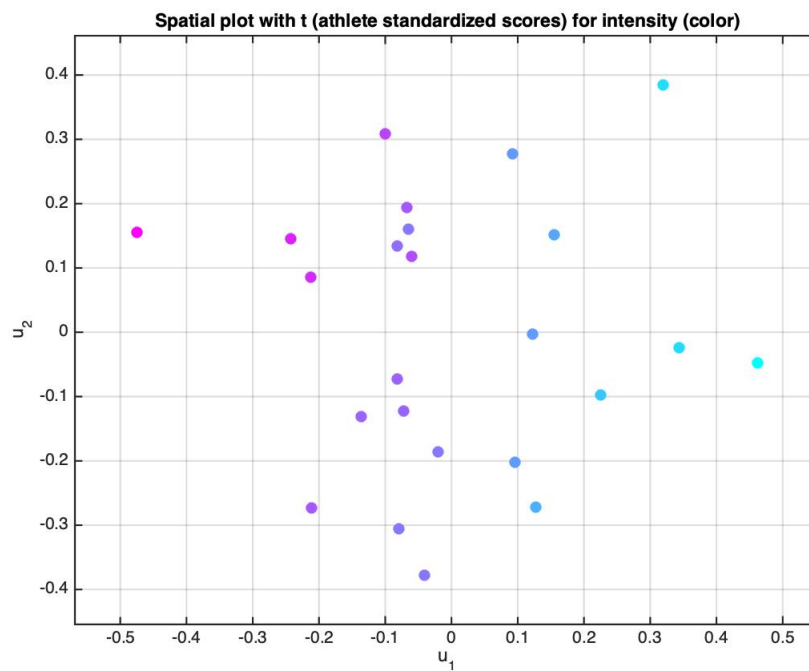


Figure 7: Plot of events colored by correlation athlete's standardized test scores.

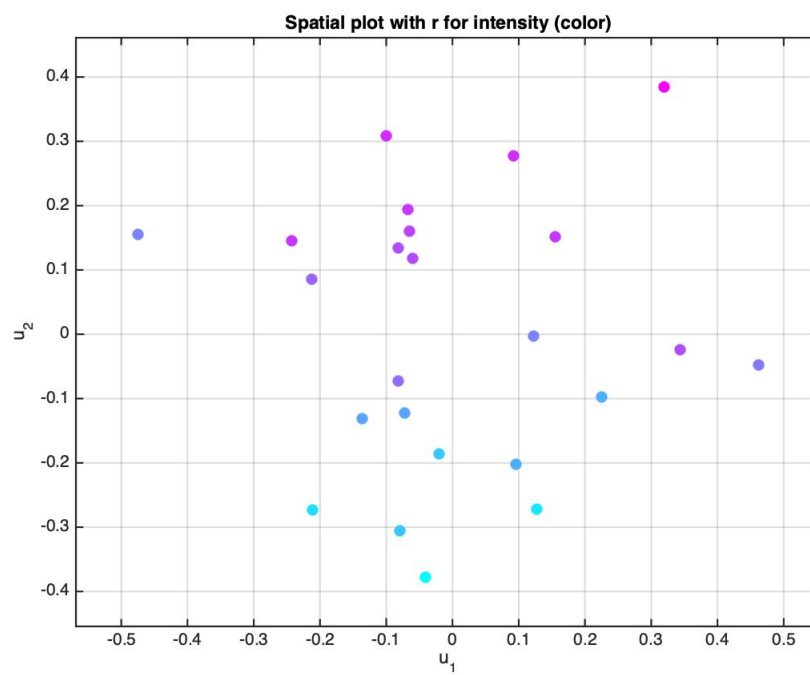


Figure 8: Plot of events colored by delta in scores between (100m, long jump, 100m hurdles) and (shot put, discus, javelin).