EE 263 Homework 3

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Fitting a model for hourly temperature

Solution:

(a) In order to find $a \in \mathbb{R}$ and $p \in \mathbb{R}^N$ (which is 24-periodic) that minimizes the RMS value of $y - \hat{y}$, we can rephrase our original predictor model as a linear system:

$$\hat{y} = Ax$$

where $\hat{y} \in \mathbb{R}^N$ and $x \in \mathbb{R}^{25}$, which represents our parameters (since p is 24-periodic). More precisely, we have:

$$x = \begin{bmatrix} p_{24} \\ p_{23} \\ \vdots \\ p_2 \\ p_1 \\ a \end{bmatrix} \in \mathbb{R}^{25}$$

As for A, we have:

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & N \\ 0 & 1 & 0 & \cdots & 0 & N-1 \\ 0 & 0 & 1 & \cdots & 0 & N-2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & N-23 \\ 1 & 0 & 0 & \cdots & 0 & N-24 \\ 0 & 1 & 0 & \cdots & 0 & N-25 \\ 0 & 0 & 1 & \cdots & 0 & N-26 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \end{bmatrix} \in \mathbb{R}^{N \times 25}$$

This means that A is a skinny and tall matrix, for an over-constrained system of equations. Finding the x that minimizes the RMS of $\hat{y} - y$ can be done by computing:

$$x = (A^T A)^{-1} A^T y$$

The x above gives us p as well as a.

(b) We now perform the process described in part (a). The trend parameter is:

$$a = -0.012075460503471858$$

A plot of the predictions as well as the observed values can be seen in Figure 1.

(c) The RMSE of the prediction error for tomorrow's temperatures is:

$$0.6521628280735887$$

A plot of the predicted and observed temperatures for tomorrow can be seen in Figure 2.

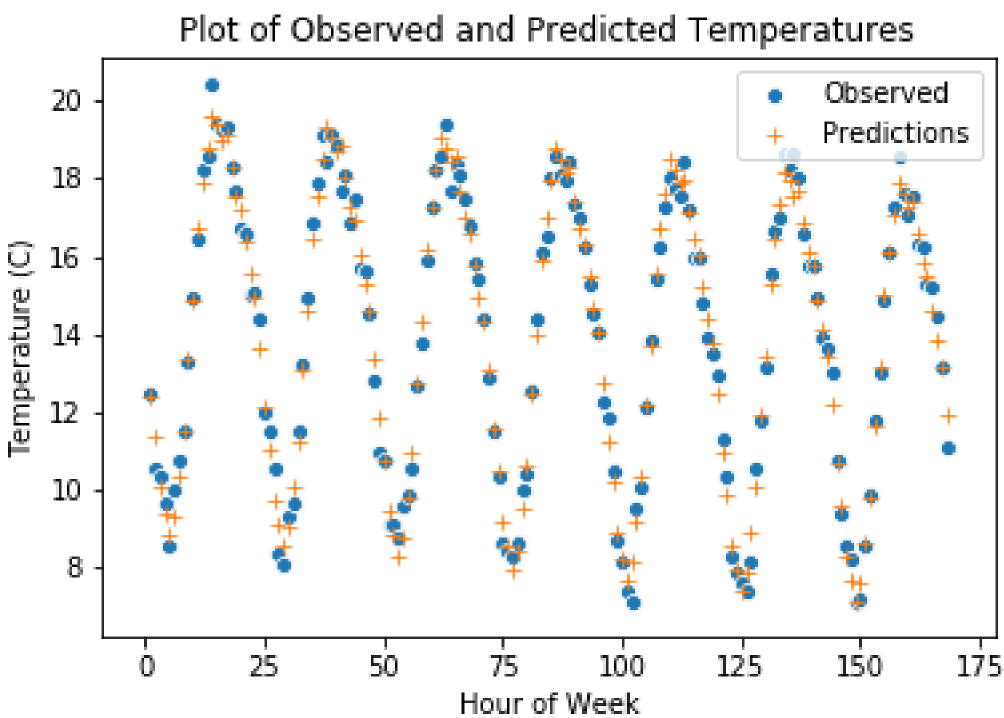

Figure 1: Plot of predicted (x) and observed (o) temperatures on training data.

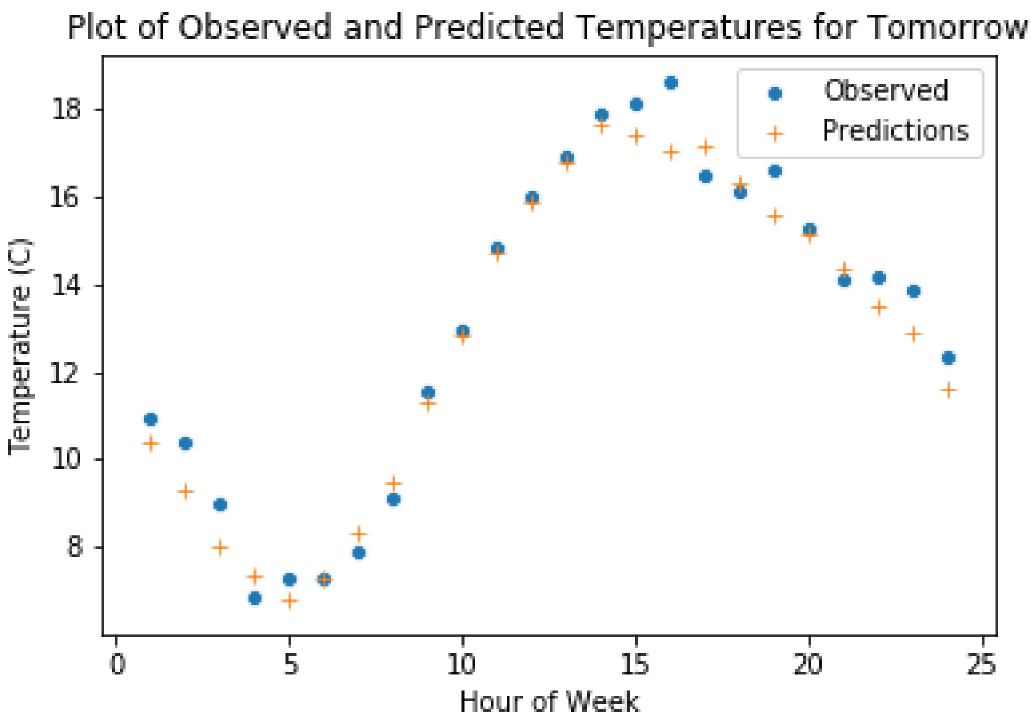

Figure 2: Plot of predicted (x) and observed (o) temperatures on test data.

Identifying a system from input/output data

Solution:

(a) We wish to find A such that:

$$J = \sum_{k=1}^{N} ||Ax^{(k)} - y^{(k)}||^2$$

is minimized. Taking the derivative wrt. A, we have:

$$\begin{aligned}
\frac{\partial J}{\partial A} &= \frac{\partial}{\partial A}\left[\sum_{k=1}^{N} ||Ax^{(k)} - y^{(k)}||^2\right] \\
&= \sum_{k=1}^{N} \frac{\partial}{\partial A}\left[||Ax^{(k)} - y^{(k)}||^2\right] \\
&= \sum_{k=1}^{N} \frac{\partial}{\partial A}\left[(Ax^{(k)} - y^{(k)})^T(Ax^{(k)} - y^{(k)})\right] \\
&= 2\sum_{k=1}^{N}(Ax^{(k)} - y^{(k)})x^{(k)T} \\
&= 2A\sum_{k=1}^{N} x^{(k)}x^{(k)T} - 2\sum_{k=1}^{N} y^{(k)}x^{(k)T}
\end{aligned}$$

So setting equal to 0 and solving for A, we have:

$$A = \left(\sum_{k=1}^{N} y^{(k)}x^{(k)T}\right)\left(\sum_{k=1}^{N} x^{(k)}x^{(k)T}\right)^{-1}$$

So we can find A if and only if the second term above is invertible.

(b) Implementing the method discussed above, we obtain the following A:

$$\hat{A} = \begin{bmatrix}
2.02992454 & 5.02077879 & 5.01040266 \\
0.0114300076 & 6.99991043 & 1.01061265 \\
7.04239020 & 0 & 6.94476335 \\
6.99765743 & 3.97592792 & 4.00242122 \\
9.01295285 & 1.04493868 & 6.99800225 \\
4.01187599 & 3.96488792 & 9.02674982 \\
4.98710794 & 6.97233996 & 8.03363399 \\
7.94249406 & 6.08754514 & 3.01735388 \\
0 & 8.97218370 & -0.0385465462 \\
1.06123427 & 8.02076138 & 7.02847693
\end{bmatrix}$$

This gives us a realtive error of 0.05814323689487761.

Robust regression using the Huber penalty function

Solution:

(a) This is a straight-forward application of weighed iterative least squares. In particular, our weight function is given by the below at iteration $k+1$:

$$w_i(a^{(k)}, b^{(k)}) = \frac{H_\delta(a^{(k)}t_i + b^{(k)} - x_i)}{(a^{(k)}t_i + b^{(k)} - x_i)^2}$$

We have to be a bit careful with the denominator to make sure it does not equal 0. Once we have the weights defined, we can use the standard update equation given by:

$$\begin{bmatrix} a^{(k+1)} \\ b^{(k+1)} \end{bmatrix} = (A^T W(a^{(k)}, b^{(k)})A)^{-1} A^T W(a^{(k)}, b^{(k)})x$$

where we have:

$$A = \begin{bmatrix} t_1 & 1 \\ \vdots & \vdots \\ t_N & 1 \end{bmatrix} \in \mathbb{R}^{N \times 2}$$

$$W = \begin{bmatrix} w_1(a,b) & 0 & \cdots & 0 \\ 0 & w_2(a,b) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & w_N(a,b) \end{bmatrix} \in \mathbb{R}^{N \times N}$$

(b) Applying the method above to "huber_penalty_function_data.m", we arrive at the following parameter values with the Huber loss:

$$a = -4.9920376, b = 23.5669247$$

for a total Huber loss of 0.40706964531961987.

As for ordinary least squares, we have the following parameters:

$$a = -3.94855818, b = 19.03697567$$

A visualization of the results in presented in Figure 3. We can see that the red line (Huber fit) matches the main data much more closely, since it is penalized less for the outliers. On the other hand, we see that the LS fit (green line) has to change slope and shift to try and accomodate the outlines, meaning the main datapints suffer.

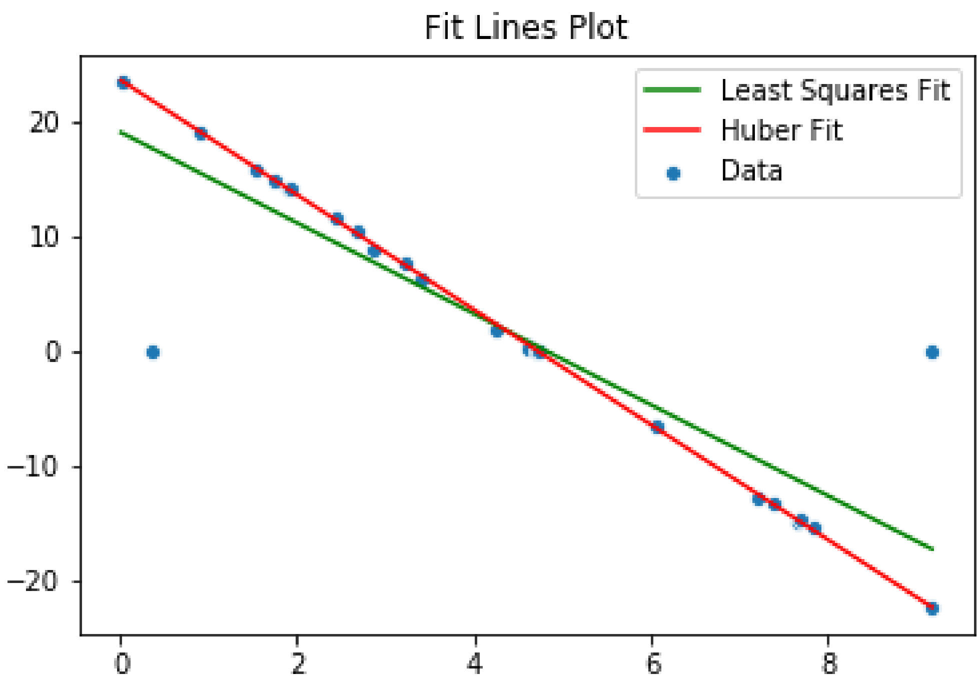

Figure 3: Plot of data as well as LS and Huber Fit

Estimating a signal with interference

Solution:

(a) We now formalize each of the methods described:

- In Nikola's process, the estimate $\hat{x}_{\text{Nikola}}$ is given by:

$$\hat{x}_{\text{Nikola}} = (A^T A)^{-1} A^T y$$

since it's just a straight-forward over-determined least squares problem. Note that this always has a solution if A is full-rank.

- Almir's method consists of estimating both $\hat{x}_{\text{Almir}}$ and $\hat{v}$. Basically, this involves rewriting the problem $y = Ax + Bv$ as a single matrix, like below:

$$\begin{aligned} y &= Ax + Bv \\ &= \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} \\ &= A'x' \end{aligned}$$

So now that we've presented it in this way, we have the solutions as:

$$\begin{bmatrix} \hat{x}_{\text{Almir}} \\ \hat{v} \end{bmatrix} = (A'^T A')^{-1} A'^T y$$

- This method follows the same set-up as the method above, so we re-use $A' \in \mathbb{R}^{m \times (n+p)}$ and $x' \in \mathbb{R}^{(n+p)}$. Using the same process as described, we find $\hat{x}_{\text{Almir}}$ and $\hat{v}$.

  We then form the pseudo-measurement $\hat{y} = y - B\hat{v}$. We now have the system:

  $$\hat{y} = Ax$$

  and now find our *true* estimate. Since this is just standard LS, we have:

  $$\hat{x}_{\text{Miki}} = (A^T A)^{-1} A^T \hat{y} = (A^T A)^{-1} A^T (y - B\hat{v})$$

(b) First, we show that Miki and Nikola's methods are *not* equivalent. In fact, let us consider the following system:

$$A = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in \mathbb{R}^{2 \times 1}$$
$$B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in \mathbb{R}^{2 \times 1}$$

and let us suppose that we make the following measurement:

$$y = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

By Nikola's method, we'll have the following $\hat{x}_{\text{Nikola}}$ estimate:

$$\begin{aligned} \hat{x}_{\text{Nikola}} &= (A^T A)^{-1} A^T y \\ &= \left(\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ &= 2 \end{aligned}$$

However, by Almir's method, we have:

$$\begin{aligned} \hat{x}_{\text{Almir}} &= \left(\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ &= \left(\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \right)^{-1} \begin{bmatrix} 2 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

From the above, we can clearly see that $\hat{x}_{\text{Almir}} \neq \hat{x}_{\text{Nikola}}$. As such, these two methods are not equivalent.

Next, we show that Miki's method is actually equivalent to Almir's method. That is, we show that $\hat{x}_{\text{Miki}} = \hat{x}_{\text{Almir}}$. First, we recall that $\hat{x}_{\text{Almir}}$ is is the unique solution satisfying the following system of equations:

$$\begin{aligned}
A'^T y &= A'^T A' \begin{bmatrix} \hat{x}_{\text{Almir}} \\ \hat{v} \end{bmatrix} \\
\implies \begin{bmatrix} A & B \end{bmatrix}^T y &= \begin{bmatrix} A & B \end{bmatrix}^T \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} \hat{x}_{\text{Almir}} \\ \hat{v} \end{bmatrix} && \text{(Definition of } A'\text{)} \\
\implies \begin{bmatrix} A^T y \\ B^T y \end{bmatrix} &= \begin{bmatrix} A^T A \hat{x}_{\text{Almir}} + A^T B \hat{v} \\ B^T A \hat{x}_{\text{Almir}} + B^T B \hat{v} \end{bmatrix} && \text{(Matrix Multiplication)} \\
\implies A^T B \hat{v} &= A^T y - A^T A \hat{x}_{\text{Almir}} && \text{(Taking just the top system)}
\end{aligned}$$

However, we note that following:

$$\begin{aligned}
\hat{x}_{\text{Miki}} &= (A^T A)^{-1} A^T (y - B\hat{v}) \\
&= (A^T A)^{-1} (A^T y - A^T B \hat{v}) && \text{(Distributing } A^T\text{)} \\
&= (A^T A)^{-1} (A^T y - (A^T y - A^T A \hat{x}_{\text{Almir}})) && \text{(Subtituting results from above)} \\
&= (A^T A)^{-1} (A^T A) \hat{x}_{\text{Almir}} && \text{(Simplifying)} \\
&= \hat{x}_{\text{Almir}}
\end{aligned}$$

As such, we can see that $\hat{x}_{\text{Miki}} = \hat{x}_{\text{Almir}}$ as expected.

(c) Almir's method works the best. It is equivalent to Miki's, but with less steps, and should provide better answers. Note that any solution $\hat{x}_{\text{Nikola}}$ will always lead to $\hat{y} \in \textbf{image}(A)$. However, with Almir's method, the only restriction is that $\hat{y} \in \textbf{image}(A) \bigcup \textbf{image}(B)$. As such, any observations y where $B^T y \neq 0$ will be better approximated by Almir's method (since it can accuraltey attribute some part of y to the output space of B).

However, it is interesting to note that Almir's method can be significantly more expensive than Nikola's, since it might require inverting a matrix that is twice as big.

HW4

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1 HW4

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1.1 Imports

[47]:
```
import numpy as np
from scipy import linalg
import seaborn as sns
import math
```

1.2 Problem 1: Fitting a model with hourly temperature

1.2.1 Part (b)

[5]:
```
"""Loading y vector of temperatures."""
y = [
    1.244822e+001,
    1.056210e+001,
    1.033174e+001,
    9.606498e+000,
    8.556261e+000,
    9.959871e+000,
    1.077175e+001,
    1.146578e+001,
    1.328712e+001,
    1.496854e+001,
    1.644381e+001,
    1.826723e+001,
    1.856764e+001,
    2.046171e+001,
    1.939858e+001,
    1.930111e+001,
    1.934969e+001,
    1.831423e+001,
    1.766321e+001,
```

```
    1.669083e+001,
    1.658678e+001,
    1.498528e+001,
    1.505830e+001,
    1.438545e+001,
    1.197421e+001,
    1.148069e+001,
    1.055407e+001,
    8.324995e+000,
    8.069414e+000,
    9.311587e+000,
    9.640021e+000,
    1.149359e+001,
    1.319649e+001,
    1.490358e+001,
    1.684989e+001,
    1.790743e+001,
    1.912722e+001,
    1.844008e+001,
    1.912928e+001,
    1.883938e+001,
    1.768906e+001,
    1.808924e+001,
    1.686000e+001,
    1.749298e+001,
    1.571663e+001,
    1.559855e+001,
    1.449279e+001,
    1.279592e+001,
    1.091916e+001,
    1.070769e+001,
    9.108555e+000,
    9.117091e+000,
    8.732967e+000,
    9.562628e+000,
    9.827205e+000,
    1.051960e+001,
    1.267284e+001,
    1.373827e+001,
    1.589141e+001,
    1.724662e+001,
    1.823062e+001,
    1.858238e+001,
    1.938788e+001,
    1.768315e+001,
    1.840879e+001,
    1.811321e+001,
```

```
    1.745971e+001,
    1.678153e+001,
    1.584773e+001,
    1.538232e+001,
    1.437838e+001,
    1.284104e+001,
    1.152896e+001,
    1.030374e+001,
    8.591905e+000,
    8.409658e+000,
    8.256319e+000,
    8.593017e+000,
    9.973718e+000,
    1.038751e+001,
    1.251709e+001,
    1.436635e+001,
    1.609681e+001,
    1.650109e+001,
    1.806422e+001,
    1.858905e+001,
    1.806650e+001,
    1.798032e+001,
    1.846825e+001,
    1.733324e+001,
    1.702398e+001,
    1.627260e+001,
    1.524704e+001,
    1.450239e+001,
    1.405536e+001,
    1.223471e+001,
    1.185029e+001,
    1.047937e+001,
    8.663015e+000,
    8.139456e+000,
    7.350008e+000,
    7.082453e+000,
    9.494333e+000,
    1.005408e+001,
    1.212076e+001,
    1.383665e+001,
    1.539110e+001,
    1.625036e+001,
    1.724103e+001,
    1.804050e+001,
    1.773624e+001,
    1.758339e+001,
    1.842454e+001,
```

```
    1.718926e+001,
    1.598398e+001,
    1.597279e+001,
    1.476660e+001,
    1.387474e+001,
    1.347138e+001,
    1.295600e+001,
    1.129482e+001,
    1.029512e+001,
    8.263162e+000,
    7.816357e+000,
    7.588277e+000,
    7.353837e+000,
    8.119445e+000,
    1.055820e+001,
    1.174310e+001,
    1.317865e+001,
    1.556603e+001,
    1.665215e+001,
    1.700492e+001,
    1.868391e+001,
    1.826153e+001,
    1.863960e+001,
    1.804732e+001,
    1.661052e+001,
    1.573907e+001,
    1.575054e+001,
    1.496028e+001,
    1.392356e+001,
    1.364802e+001,
    1.302191e+001,
    1.075686e+001,
    9.344309e+000,
    8.525904e+000,
    8.181372e+000,
    7.096798e+000,
    7.165355e+000,
    8.526029e+000,
    9.867947e+000,
    1.178932e+001,
    1.301388e+001,
    1.489993e+001,
    1.613176e+001,
    1.727414e+001,
    1.859689e+001,
    1.762994e+001,
    1.704406e+001,
```

```
    1.753179e+001,
    1.632080e+001,
    1.622418e+001,
    1.529564e+001,
    1.521687e+001,
    1.444662e+001,
    1.312329e+001,
    1.107770e+001,
]
N = 168
```

[6]:
```
def constructAMatrixFor1b(start:int, end:int):
    """We construct the matrix A as described in the homework assignment.

    Args:
        start: The starting timestep. Must be % 24 == 1, >= 1.
        end: The ending timestep. Must be a multiple of 24
        and must be >= 24.

    Returns:
        The matrix A.
    """
    assert start % 24 == 1 and start >= 1
    assert end % 24 == 0 and end >= 24
    identityMat = np.identity(24)
    onesPart = np.concatenate([identityMat.copy() for _ in range(start // 24,␣
 ↪end // 24)], axis=0)
    aColumn = np.flip(np.array(range(start, end + 1)))
    aColumn.shape = (aColumn.shape[0], 1)
    return np.append(onesPart, aColumn, axis=1)
```

[7]:
```
A = constructAMatrixFor1b(start=1, end=N)
```

[8]:
```
x = np.dot(np.linalg.inv(np.dot(A.T, A)), np.dot(A.T, np.flip(y)))
yhat = np.flip(np.dot(A, x))
```

[9]:
```
print("The trend parameter is a=%s" % (x[-1]))
```

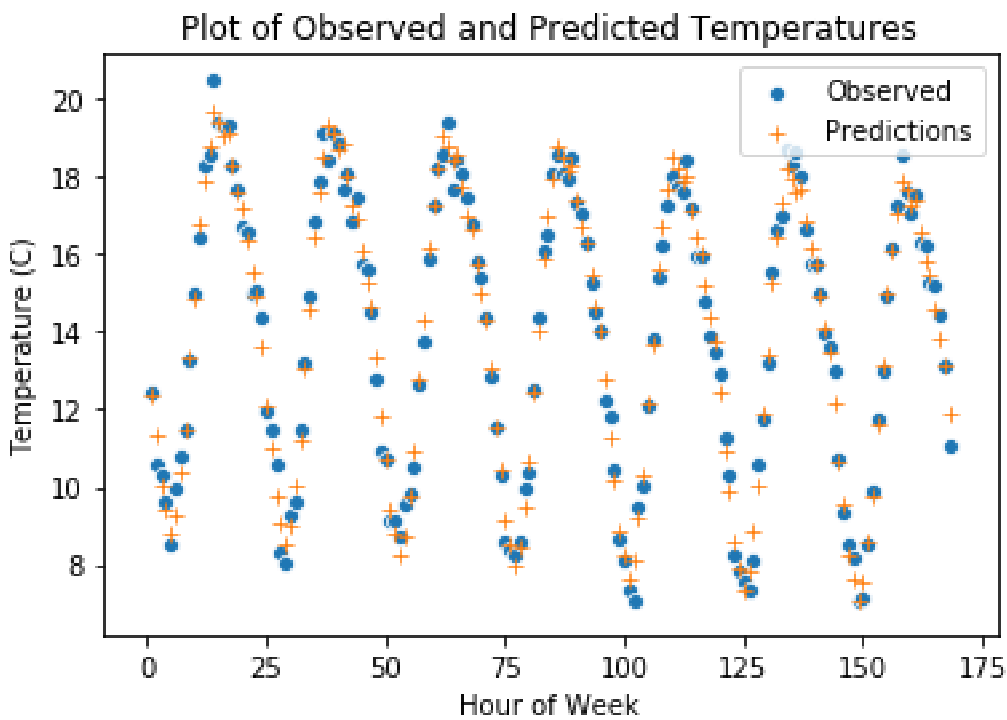
```
The trend parameter is a=-0.012075460503471858
```

[10]:
```
RMSE = np.sqrt(np.sum((y-yhat)**2) / len(yhat))
print("The RMSE on training data is: %s" % (RMSE))
```

```
The RMSE on training data is: 0.4050510038014836
```

[11]:
```
sns.scatterplot(x=range(1,N+1), y=y, label='Observed')
ax = sns.scatterplot(x=range(1,N+1), y=yhat, marker='+', label='Predictions')
ax.set_title('Plot of Observed and Predicted Temperatures')
```

```
ax.set(xlabel='Hour of Week', ylabel='Temperature (C)')
ax.get_figure().savefig("temps_on_train")
```



1.2.2 Part (c)

[12]:
```
# The measured values for the next 24 hours.
ytom = [
    1.092478e+001,
    1.039376e+001,
    8.989078e+000,
    6.891658e+000,
    7.317766e+000,
    7.259584e+000,
    7.909287e+000,
    9.120217e+000,
    1.152908e+001,
    1.293617e+001,
    1.484307e+001,
    1.600680e+001,
    1.692412e+001,
    1.785879e+001,
    1.814252e+001,
    1.862022e+001,
```

```
    1.646605e+001,
    1.611008e+001,
    1.657515e+001,
    1.526571e+001,
    1.411094e+001,
    1.418155e+001,
    1.385655e+001,
    1.231481e+001,
]
```

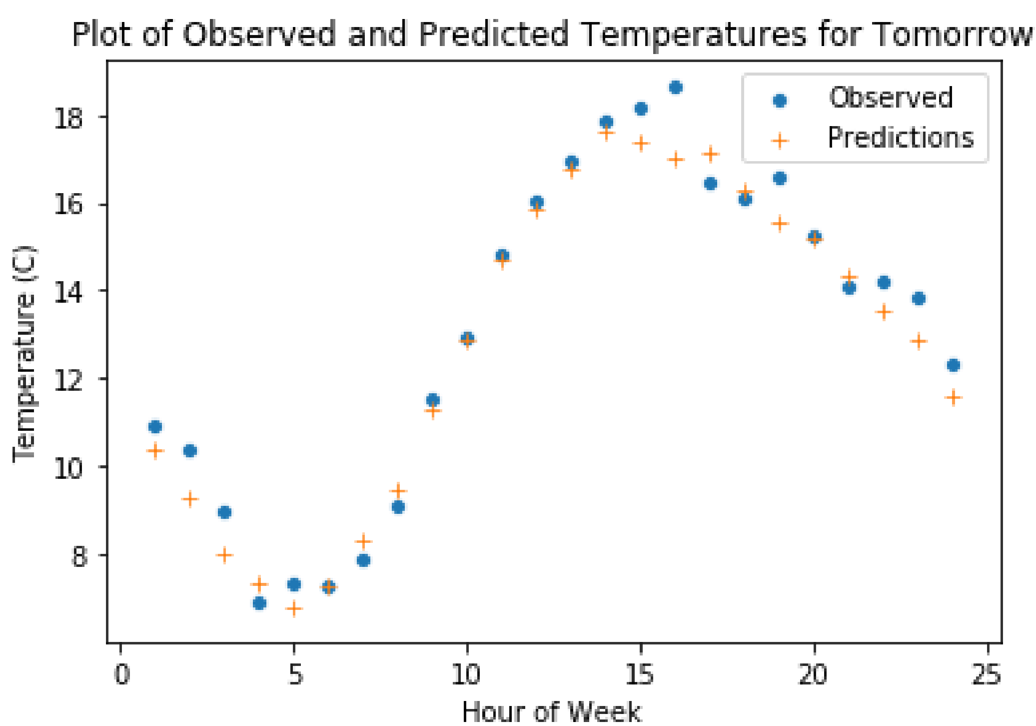
[13]:
```
APred = constructAMatrixFor1b(start=169, end=192)
```

[14]:
```
yhattom = np.flip(np.dot(APred, x))
```

[15]:
```
RMSETest = np.sqrt(np.sum((ytom-yhattom)**2) / len(ytom))
print("The RMSE of the prediction error for tomorrow's temperatures is: %s" %
 ↪(RMSETest))
```

```
The RMSE of the prediction error for tomorrow's temperatures is:
0.6521628280735887
```

[16]:
```
sns.scatterplot(x=range(1,25), y=ytom, label='Observed')
ax = sns.scatterplot(x=range(1,25), y=yhattom, marker='+', label='Predictions')
ax.set_title('Plot of Observed and Predicted Temperatures for Tomorrow')
ax.set(xlabel='Hour of Week', ylabel='Temperature (C)')
ax.get_figure().savefig("temps_on_test")
```



1.3 Problem 2: Identifying a system from input/output data

1.3.1 Part (b)

[17]:
```
# data for EE263 problem on system identification
n=3
m=10
N=100
X=np.array([
    5.8248,
    3.1342,
    0.3754,
    1.7580,
   -3.4826,
    8.4807,
    0.2953,
    8.9854,
    1.3203,
    4.3584,
   -7.2309,
   -3.5058,
    6.2299,
   -3.1949,
    2.8868,
   -1.8001,
   -0.6779,
   -6.7467,
   -6.3522,
    4.9229,
   -0.2244,
   -3.9947,
   -3.8259,
    4.3087,
   -0.2811,
    2.5674,
    1.9834,
    3.7811,
    2.0024,
   -6.7069,
    1.8752,
    5.6258,
    3.6432,
  -11.8873,
   -1.3689,
   -1.6147,
```

```
   1.5899,
  -2.5559,
  -0.0102,
   8.0326,
   4.2382,
   1.3405,
  -4.6174,
  -0.3525,
   0.7395,
  -2.7855,
  -1.6835,
   2.0761,
   7.7891,
 -12.2215,
  -5.4910,
   5.6132,
   2.9083,
  -1.3568,
   2.0710,
  -4.8891,
  -5.1073,
   1.5884,
   7.5805,
   3.7472,
  -2.5385,
   4.4265,
  -1.2405,
  -3.6312,
  -2.2252,
  -3.0646,
  -1.0457,
   2.8107,
  -5.3196,
   1.7579,
   5.6650,
   0.7500,
   3.5157,
  -0.2621,
  10.0925,
   4.6208,
  -9.0706,
   0.1749,
  -9.0393,
   5.1410,
   1.9730,
   3.1970,
   4.3711,
```

```
   8.7620,
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  -0.6871,
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```

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```

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   46.4896,
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```

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   -7.4045,
   10.7344,
  -17.5469,
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   11.0501,
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   79.5129,
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```

```
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   95.8613,
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  -57.7422,
   29.0651,
   11.7197,
   38.3672,
  -40.0050,
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   16.0110,
  -14.3773,
   10.8786,
   -7.5153,
  -33.1469,
  -38.3976,
  -65.6431,
  -42.4488,
   24.1370,
   44.9871,
    2.9564,
   36.7386,
   15.1112,
   14.6533,
   43.7402,
   54.1967,
   58.1558,
   41.7045])
Y=np.reshape(Y,(m,N), order='F')
```

[18]:
```
def solveForA(X, Y):
    """
    X is (n x N)
    Y is (m x N)

    We are looking to solve for A, which will be m x n.
    """
    (n, Nx) = X.shape
    (m, N) = Y.shape
    assert Nx == N
    # m x n -- formed from X and Y.
    xMatrix = np.zeros((n, n))
    yMatrix = np.zeros((m,n))
    for i in range(N):
        x = X[:, i]
        y = Y[:, i]
        x.shape = (n, 1)
        y.shape = (m, 1)
        xMatrix += np.dot(x, x.T)
        yMatrix += np.dot(y, x.T)
    assert len(linalg.null_space(xMatrix)[0]) == 0
    xInv = np.linalg.inv(xMatrix)
    return np.dot(yMatrix, xInv)
```

[19]:
```
minimizerA = solveForA(X, Y)
```

[20]:
```
minimizerA
```

[20]:
```
array([[ 2.02992454e+00,  5.02077879e+00,  5.01040266e+00],
       [ 1.14300076e-02,  6.99991043e+00,  1.01061265e+00],
       [ 7.04239020e+00, -2.54352520e-03,  6.94476335e+00],
       [ 6.99765743e+00,  3.97592792e+00,  4.00242122e+00],
       [ 9.01295285e+00,  1.04493868e+00,  6.99800225e+00],
       [ 4.01187599e+00,  3.96488792e+00,  9.02674982e+00],
       [ 4.98710794e+00,  6.97233996e+00,  8.03363399e+00],
       [ 7.94249406e+00,  6.08754514e+00,  3.01735388e+00],
       [ 9.43583486e-03,  8.97218370e+00, -3.85465462e-02],
       [ 1.06123427e+00,  8.02076138e+00,  7.02847693e+00]])
```

[21]:
```
def getRelApproxErr(mat, X, Y):
    (n, N) = X.shape
    error = 0.0
    for i in range(N):
        error += 1 / N * (linalg.norm(np.dot(mat, X[:, i]) - Y[:, i]) / linalg.
 ↪norm(Y[:, i]))
    return error
```

[22]:
```
print("The relative error is: %s" % (getRelApproxErr(minimizerA, X, Y)))
```

```
The relative error is: 0.05814323689487761
```

1.4 Proble 3: Robus regression using the Huber penalty function

1.4.1 Part (b)

[62]:
```
def huber(x, delta = 1):
    if abs(x) <= delta:
        return 0.5 * x**2
    else:
        return delta * (abs(x) - 0.5 * delta)
```

[63]:
```
N = 25
t = np.array([
        [0.0115],
        [0.3576],
        [0.9111],
        [1.5272],
        [1.7587],
        [1.8866],
        [1.9175],
        [2.4285],
        [2.6906],
        [2.8750],
        [3.2247],
        [3.4112],
        [4.2435],
        [4.6092],
        [4.6245],
        [4.7136],
        [4.7349],
        [6.0739],
        [7.2176],
        [7.3843],
        [7.6550],
        [7.7016],
        [7.8474],
        [9.1599],
        [9.1742],
])
x = np.array([
        [23.5187],
        [0.0000],
        [18.9909],
        [15.7897],
        [14.8507],
        [14.2145],
        [14.0918],
        [11.4933],
        [10.4921],
```

```
         [8.8340],
         [7.5857],
         [6.1800],
         [1.9019],
         [0.5615],
         [0.3277],
         [-0.1091],
         [-0.0124],
         [-6.5992],
         [-12.7377],
         [-13.2593],
         [-14.8588],
         [-14.6661],
         [-15.4040],
         [0.0000],
         [-22.2578],
])
```

[64]:
```
# Solve the sytem using least squares.
A = np.concatenate((t, np.ones((N,1))), axis=1)
```

[65]:
```
# Least squares solution is:
paramsLeastSquares = np.dot(np.linalg.inv(np.dot(A.T, A)), np.dot(A.T, x))
```

[131]:
```
def iterativeLeastSquares(A, x, startParams, delta=1,
                          maxIterations=1000,eps=1e-4):
    N, _ = A.shape
    prevParams = startParams
    huberV = np.vectorize(huber)
    for _ in range(maxIterations):
        W = huberV(np.diag((np.dot(A, prevParams) - x).flatten()))
        inverse = np.linalg.inv(np.dot(np.dot(A.T, W), A))
        newParams =  np.dot(inverse, np.dot(np.dot(A.T, W), x))
        huberLoss = np.sum(np.dot(A, newParams) - x)
        if linalg.norm(newParams - prevParams) < eps:
            return newParams, huberLoss
        prevParams = newParams
    return newParams, huberLoss
```

[132]:
```
huberParams, loss = iterativeLeastSquares(A, x, paramsLeastSquares)
```

[133]:
```
paramsLeastSquares
```

[133]:
```
array([[-3.94855818],
       [19.03697567]])
```

[134]:
```
 huberParams
```

[134]:
```
array([[-4.9920376],
       [23.5669247]])
```

[135]:
```
loss
```

[135]:
```
0.40706964531961987
```

[121]:
```
# Plot the data
sns.scatterplot(x=t.flatten(), y=x.flatten(), label="Data")
sns.lineplot(x=t.flatten(), y=np.dot(A, paramsLeastSquares).flatten(),␣
 ↪color='green', label='Least Squares Fit')
ax = sns.lineplot(x=t.flatten(), y=np.dot(A, huberParams).flatten(),␣
 ↪color='red', label='Huber Fit')
ax.set_title('Fit Lines Plot')
ax.get_figure().savefig('line_plots_for_huber_fit')
```