

EE 263 Homework 6

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Optimal operation of a two-state chemical reactor

Solution:

- (a) We have the dynamics equation $\dot{x}(t) = A_j x(t)$. This can be solved directly as the following:

$$x(t) = e^{tA_j} x(0)$$

If we operate the first reactor for T_0 time followed by the second reactor for $T - T_0$ time, our final state will be:

$$x(T) = e^{(T-T_0)A_2} x(T_1) = e^{(T-T_0)A_2} e^{T_0 A_1} x(0)$$

With the above, we can now write a function $C_k(T_0)$ which will give us the amount of compounds k at time T . We have:

$$C_k(T_0) = e_k^T e^{(T-T_0)A_2} e^{T_0 A_1} x(0)$$

where e_k is the k -th unit vector. Since our answer only needs to be accurate to two decimal places, computing the optimal value of T_0 that maximizes the above can be done through a simple search. We simply compute $C_k(t)$ for all $0 < t \leq T$ at intervals of size 0.001, and report the value t_{\max} which achieves the maximum C_k .

- (b) We perform the method described above. See Figure 1 for plot. We conclude that: The optimal value is $T_0 = 0.61$ with 0.37 units of compound 2 at time $T = 1$.
- (c) We follow a similar set-up as before. In this case, the final state of the system will be given by:

$$x(T) = e^{(T-T_2)A_1} e^{(T_2-T_1)A_2} e^{T_1 A_1} x(0)$$

We can similarly define a function for the k -th chemical as:

$$C_k(T_1, T_2) = e_k^T e^{(T-T_2)A_1} e^{(T_2-T_1)A_2} e^{T_1 A_1} x(0)$$

Since we're once again only interested in a solution accurate up to 2 decimal places, we can just try suitable values of T_1 and T_2 in intervals of size 0.001.

- (d) We perform the method described above. See Figure 2 for plot. We conclude that: The optimal value is $T_1 = 0.49$ and $T_2 = 0.87$ with 0.38 units of compound 2 at time $T = 1$.

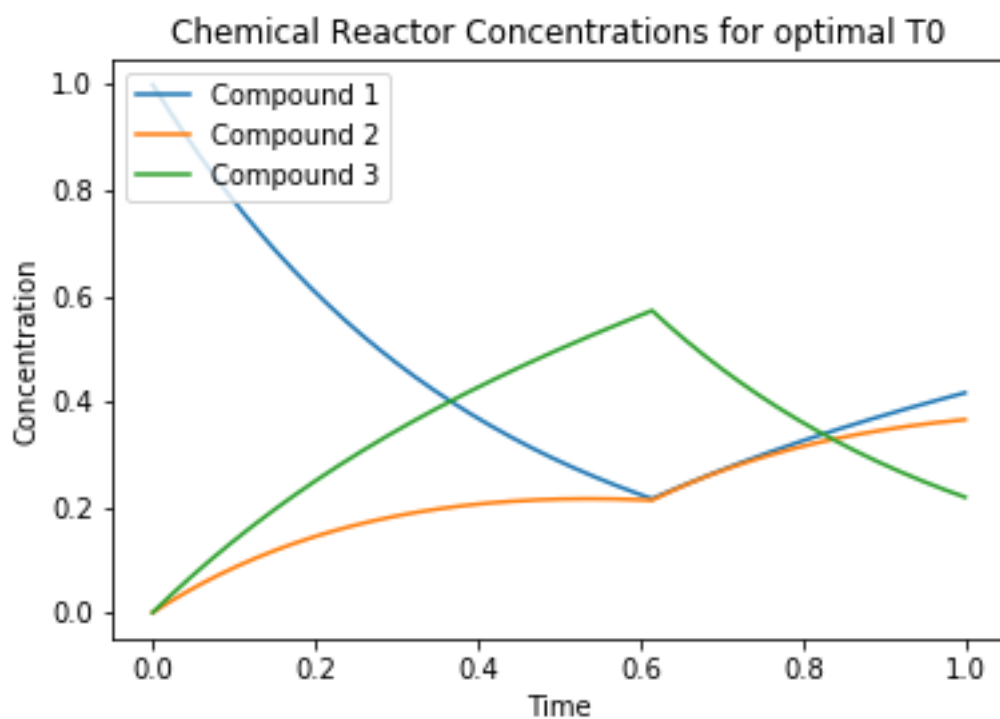


Figure 1: Chemical concentrations for $T_0 = 0.61$ (part b)

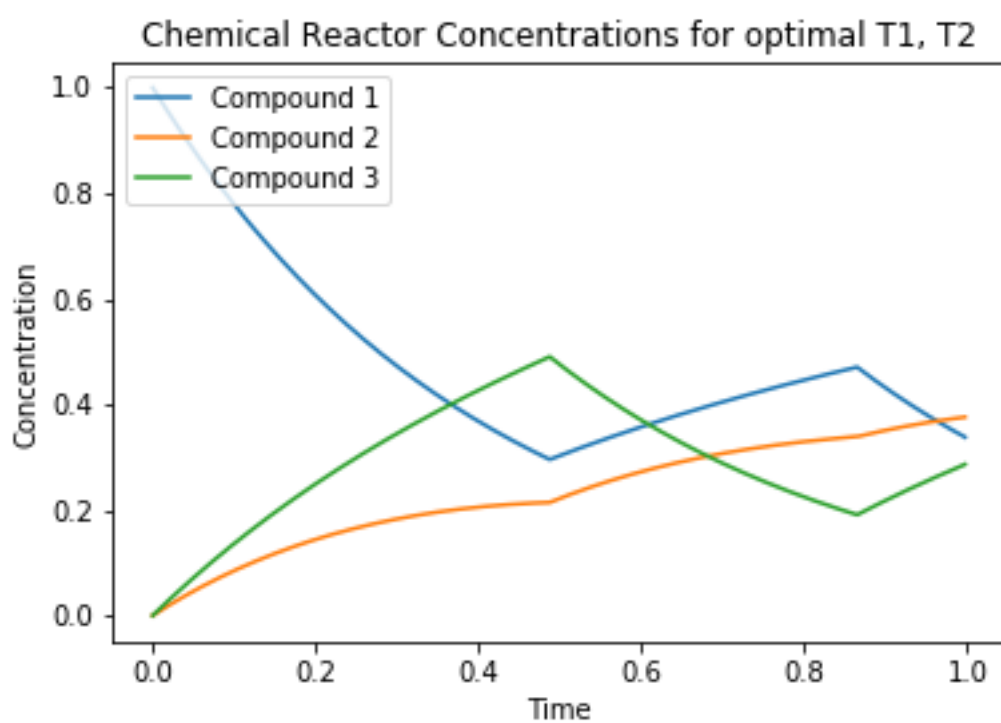


Figure 2: Chemical concentrations for $T_1 = 0.49$ and $T_2 = 0.87$ (part d)

Harmonic Oscillator

Solution:

- (a) We find the eigenvalues, resolvent, and state transition matrix for the matrix:

$$A = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}$$

The resolvent in our case is given by:

$$(sI - A)^{-1} = \begin{bmatrix} s & -w \\ w & s \end{bmatrix}^{-1} \\ \frac{1}{s^2 + w^2} \begin{bmatrix} s & w \\ -w & s \end{bmatrix}$$

We can also derive the eigenvalues by simply solving:

$$\det(A) = s^2 + w^2 = 0 \\ \implies s = \pm iw \quad (\text{Where } i = \sqrt{-1})$$

From the above, we can compute the state transition matrix as:

$$\Phi = \begin{bmatrix} \cos \omega t & \sin \omega t \\ -\sin \omega t & \cos \omega t \end{bmatrix}$$

Finally, we can express $x(t)$ as:

$$x(t) = \Phi x(0)$$

- (b) The vector field is sketched out in Figure 3.

- (c) We wish to verify that $\|x(t)\|$ is constant. Let's do this directly as:

$$\begin{aligned} \|x(t)\| &= \sqrt{x(t)^T x(t)} \\ &= \sqrt{x(0)^T \Phi^T \Phi x(0)} \\ &= \sqrt{x(0)^T \begin{bmatrix} \cos^2 \omega t + \sin^2 \omega t & -\cos \omega t \sin \omega t + \sin \omega t \cos \omega t \\ \sin \omega t \cos \omega t - \cos \omega t \sin \omega t & \sin^2 \omega t + \cos^2 \omega t \end{bmatrix} x(0)} \\ &= \sqrt{x(0)^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(0)} \\ &= \sqrt{x(0)^T x(0)} \\ &= \|x(0)\| \end{aligned}$$

From the above, we can conclude that $\|x(t)\|$ is constant for all t .

- (d) We verify directly that the velocity vector is always orthogonal to the position vector. Take $x \in \mathbb{R}^2$ to be some arbitrary position. Then we have:

$$\begin{aligned}
 x^T \dot{x} &= x^T \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} x && \text{(Given dynamics)} \\
 &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
 &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} \omega x_2 \\ -\omega x_1 \end{bmatrix} \\
 &= \omega x_1 x_2 - \omega x_1 x_2 \\
 &= 0
 \end{aligned}$$

As such, we conclude our proof.

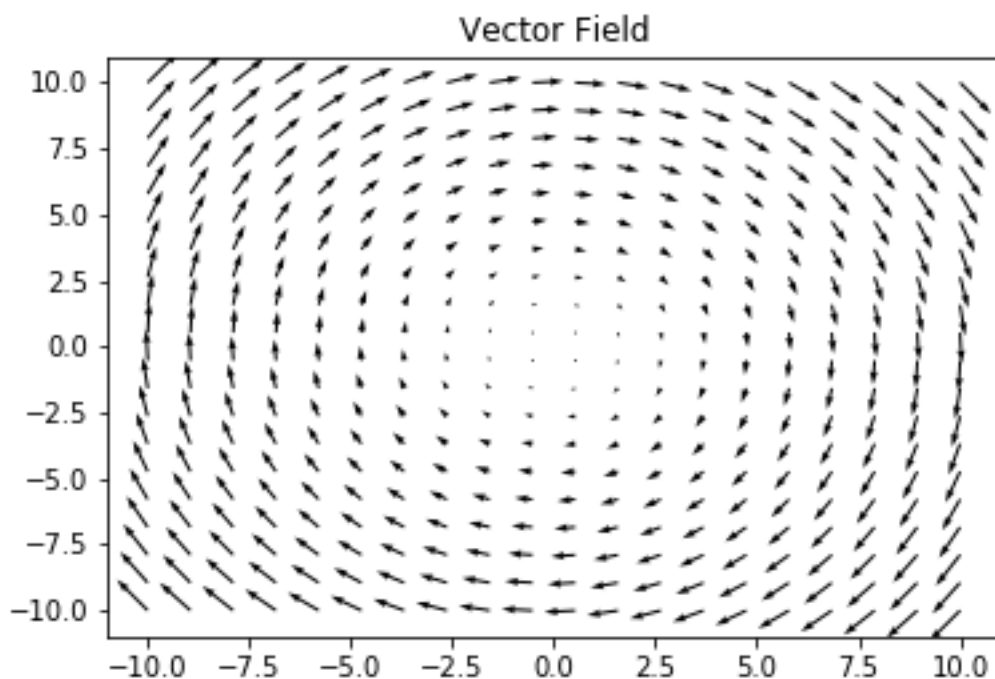


Figure 3: Vector Field Sketch for the Harmonic Oscillator

Interconnection of linear systems

Solution: Our task is to express the overall system (consisting of subsystems S and T) as a single linear dynamical system with input, state, and output given by:

$$\hat{u} = \begin{bmatrix} u \\ v \end{bmatrix}, \hat{x} = \begin{bmatrix} x \\ z \end{bmatrix}, \hat{y} = y$$

This is actually relatively straight-forward. First, we need to express each of the equations of each sub-system as a linear combination of only u, v, x, z and y . We begin with the dynamics equations:

$$\begin{aligned} \dot{x} &= Ax + B_1u + B_2w_1 && \text{(Given dynamics of subsystem } S) \\ &= Ax + B_1u + B_2H_1z && \text{(Using the fact that } w_1 = H_1z) \\ \dot{z} &= Fz + G_1v + G_2w_2 && \text{(Given dynamics of subsystem } T) \\ &= Fz + G_1v + G_2(Cx + D_1u + D_2w_1) && \text{(Using given formula for } w_2) \\ &= Fz + G_1v + G_2(Cx + D_1u + D_2H_1z) && \text{(Using formula for } w_1) \\ &= (F + G_2D_2H_1)z + G_2Cx + G_1v + G_2D_1u && \text{(Grouping like-terms)} \end{aligned}$$

Note that we can express the equations above in matrix form as:

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A & B_2H_1 \\ G_2C & F + G_2D_2H_1 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B_1 & 0 \\ G_2D_1 & G_1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

We can immediately tell from the above our dynamics and input matrix for the single linear dynamical system.

Next, we tackle the output equation. We focus on expression y as a function of x, z, u and v .

$$\begin{aligned} y &= H_2z + Jw_2 && \text{(Given output equation)} \\ &= H_2z + J(Cx + D_1u + D_2w_1) && \text{(Equation for } w_2) \\ &= H_2z + J(Cx + D_1u + D_2H_1z) && \text{(Equation for } w_1) \\ &= (H_2 + JD_2H_1)z + JCx + JD_1u && \text{(Grouping like terms)} \end{aligned}$$

We can rewrite the above in matrix form as:

$$y = [JC \quad H_2 + JD_2H_1] \begin{bmatrix} x \\ z \end{bmatrix} + [JD_1 \quad 0] \begin{bmatrix} u \\ v \end{bmatrix}$$

From the above, we can immediately read off our output and feed-through matrices. We did not need to make any assumptions.

However, to be extremely explicit, we have now written our system in the form:

$$\begin{aligned} \dot{\hat{x}} &= \hat{A}\hat{x} + \hat{B}\hat{u} \\ \hat{y} &= \hat{C}\hat{x} + \hat{D}\hat{u} \end{aligned}$$

where:

$$\hat{x} = \begin{bmatrix} x \\ z \end{bmatrix}$$

$$\hat{u} = \begin{bmatrix} u \\ v \end{bmatrix}$$

$$\hat{y} = y$$

$$\hat{A} = \begin{bmatrix} A & B_2 H_1 \\ G_2 C & F + G_2 D_2 H_1 \end{bmatrix}$$

$$\hat{B} = \begin{bmatrix} B_1 & 0 \\ G_2 D_1 & G_1 \end{bmatrix}$$

$$\hat{C} = [JC \quad H_2 + JD_2 H_1]$$

$$\hat{D} = [JD_1 \quad 0]$$

Analysis of investment allocation strategies

Solution:

- (a) It turns out that both investment strategies can be described using essentially the same framework. However, for simplicity, we start with the 35 – 35 – 30 strategy. We define our input state to be:

$$x(t) = \begin{bmatrix} B_1(t) \\ B_2(t) \\ B_3(t) \\ B_2(t-1) \\ B_3(t-1) \\ B_3(t-2) \end{bmatrix} \in \mathbb{R}^6$$

Note that $B_i(t) = 0$ for $t < 0$. Our initial state is therefore:

$$x(0) = \begin{bmatrix} 0.35 \\ 0.35 \\ 0.30 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^6$$

We can immediately compute our total wealth at time t as:

$$y(t) = Cx(t) = w(t) = B_1(t) + B_2(t) + B_3(t) + B_2(t-1) + B_3(t-1) + B_3(t-2)$$

where

$$C = [1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1] \in \mathbb{R}^{1 \times 6}$$

For the dynamics matrix, we have:

$$x(t+1) = Ax(t)$$

where $A \in \mathbb{R}^{6 \times 6}$ is given by:

$$\begin{aligned}
 A &= \underbrace{\begin{bmatrix} 0.35 \\ 0.35 \\ 0.30 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1.05 & 0.06 & 0.07 & 1.06 & 0.07 & 1.07 \end{bmatrix}}_{\text{Total payout is re-invested by purchasing new bonds}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}}_{\text{Update state for one time-step}} \\
 &= \begin{bmatrix} 0.3675 & 0.021 & 0.0245 & 0.371 & 0.0245 & 0.3735 \\ 0.3675 & 0.021 & 0.0245 & 0.371 & 0.0245 & 0.3735 \\ 0.315 & 0.018 & 0.021 & 0.318 & 0.021 & 0.321 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}
 \end{aligned}$$

The easiest way to explain is in parts. The first summand when multiplied by $x(t)$ serves to compute the total payout at time t , and then re-distributes this payout by purchasing new bonds at $t+1$, $B_1(t+1)$, $B_2(t+1)$, $B_3(t+1)$. The second summand simply updates our state, since $B_i(t)$ is the “the amount of i -year CDs bought at period t ”, and since one-time period is passing, these take the place of $B_i(t-1)$ in our state vector.

With the above complete, we move on to the 60 – 20 – 20 strategy. We use the same framework as above, with just a few minor modifications. With this strategy, we have:

$$x(0) = \begin{bmatrix} 0.6 \\ 0.2 \\ 0.2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and we have:

$$\begin{aligned}
 A &= \underbrace{\begin{bmatrix} 0.60 \\ 0.20 \\ 0.20 \\ 0 \\ 0 \\ 0 \end{bmatrix} [1.05 \quad 0.06 \quad 0.07 \quad 1.06 \quad 0.07 \quad 1.07]}_{\text{Total payout is re-invested by purchasing new bonds}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}}_{\text{Update state for one time-step}} \\
 &= \begin{bmatrix} 0.63 & 0.036 & 0.042 & 0.636 & 0.042 & 0.642 \\ 0.21 & 0.012 & 0.014 & 0.212 & 0.014 & 0.214 \\ 0.21 & 0.012 & 0.014 & 0.212 & 0.014 & 0.214 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}
 \end{aligned}$$

The C matrix remains the same.

(b) We want to understand the limit:

$$\lim_{t \rightarrow \infty} \frac{w(t+1)}{w(t)}$$

From the above, we know that:

$$w(t) = Cx(t) = CA^t x(0)$$

As such, we need to understand the limiting behavior of $A^t x(0)$. We recall from lecture the following formulation, for diagonalizable A (we check that this is true for the 35-35-30 strategy):

$$x(t) = A^t x(0) = \sum_{i=1}^6 \lambda_i^t (w_i^T x(0) v_i) \quad (A \text{ is diagonalizable for 35-35-30 strategy})$$

where λ_i, w_i, v_i are the eigenvalues, right, and left eigenvectors. As such, to understand the limiting behavior of $x(t)$ as $t \rightarrow \infty$, we need to compute these values for A .

Doing this for the 35 – 35 – 30 strategy, we have the absolute of the eigenvalues:

$$\begin{aligned}
 \lambda_1 &= 1.06265258 \\
 \lambda_2 &= -0.32657629 + 0.44206583j \\
 \lambda_3 &= -0.32657629 - 0.44206583j \\
 \lambda_4 &= \lambda_5 = \lambda_6 = 0
 \end{aligned}$$

As such, we can see that λ_1 will dominate as $t \rightarrow \infty$ (it has the largest absolute value). With this, we can simplify our ratio:

$$\begin{aligned}
 \lim_{t \rightarrow \infty} \frac{w(t+1)}{w(t)} &= \lim_{t \rightarrow \infty} \frac{CA^{t+1}x(0)}{CA^t x(0)} \\
 &= \lim_{t \rightarrow \infty} \frac{C \sum_{i=1}^6 \lambda_i^{t+1} (w_i^T x(0) v_i)}{C \sum_{i=1}^6 \lambda_i^t (w_i^T x(0) v_i)} && \text{(Substituting } A^t x(0)) \\
 &= \lim_{t \rightarrow \infty} \frac{\lambda_1^{t+1} C w_1^T x(0) v_1}{\lambda_1^t C w_1^T x(0) v_1} && \text{(As } t \rightarrow \infty, \text{ only } \lambda_1 \text{ survives)} \\
 &= \lim_{t \rightarrow \infty} \frac{\lambda_1^{t+1}}{\lambda_1^t} && \text{(Factor out and cancel values not dependent on } t) \\
 &= \lambda_1
 \end{aligned}$$

As such, we have that the wealth ration is simply given by the dominant eigenvalue. In the case of the 35 – 35 – 30 we have:

$$\lim_{t \rightarrow \infty} \frac{w(t+1)}{w(t)} = \lambda_1 = 1.06265258$$

We would like to carry out a similar analysis for the 60-20-20 strategy. However, it turns out that the state matrix for this strategy is not diagonalizable. As such, we must use the generalized Jordan Canonical form. In this case, we have:

$$x(t) = A^t x(0) = \sum_{i=1}^4 T_i J_i^t (S_i^T x(0)) \quad (A \text{ is not-diagonalize for 60-20-20 strategy})$$

By a similar analysis for the case of the 60 – 20 – 20 strategy, we have the eigenvalues:

$$\begin{aligned}
 \lambda_1 &= 1.05978649 \\
 \lambda_2 &= -0.20189324 + 0.40145558i \\
 \lambda_3 &= -0.20189324 - 0.40145558i \\
 \lambda_4 &= \lambda_5 = \lambda_6 = 0
 \end{aligned}$$

It turns out that the eigenvectors associated with the 0 eigenvalue only span a space of one-dimension, and as such, this is the reason why the matrix is not diagonalizable. However, our limiting analysis doesn't actually change a whole lot, given that λ_1 will

still overpower all other terms. In fact, we have:

$$\begin{aligned}
 \lim_{t \rightarrow \infty} \frac{w(t+1)}{w(t)} &= \lim_{t \rightarrow \infty} \frac{CA^{t+1}x(0)}{CA^t x(0)} \\
 &= \lim_{t \rightarrow \infty} \frac{C(\sum_{i=1}^3 \lambda_i^{t+1}(w_i^T x(0)v_i) + T_4 J_4^{t+1}(S_4^T x(0)))}{C(\sum_{i=1}^3 \lambda_i^t(w_i^T x(0)v_i) + T_4 J_4^{t+1}(S_4^T x(0)))} \\
 &\quad \text{(Substituting } A^t x(0)) \\
 &= \lim_{t \rightarrow \infty} \frac{\lambda_1^{t+1} C w_1^T x(0) v_1}{\lambda_1^t C w_1^T x(0) v_1} \\
 &\quad \text{(As } t \rightarrow \infty, \text{ only } \lambda_1 \text{ survives since } J_4^t = \lambda_4^t \hat{J}_4 = \hat{J}_4) \\
 &= \lim_{t \rightarrow \infty} \frac{\lambda_1^{t+1}}{\lambda_1^t} \quad \text{(Factor out and cancel values not dependent on } t) \\
 &= \lambda_1
 \end{aligned}$$

which means that we have:

$$\lim_{t \rightarrow \infty} \frac{w(t+1)}{w(t)} = \lambda_1 = 1.05978649$$

- (c) We follow a similar strategy as above. We know that for both of our matrices, we have a dominant eigenvalue λ_1 with corresponding eigenvectors w_1, v_1 . Keeping this

in mind, we have:

$$\begin{aligned}
\lim_{t \rightarrow \infty} \frac{B_1(t) + B_2(t-1) + B_3(t-2)}{w(t)} &= \lim_{t \rightarrow \infty} \frac{\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} x(t)}{Cx(t)} \\
&\quad \text{(Expressing all values as functions of } x(t)) \\
&= \lim_{t \rightarrow \infty} \frac{\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} A^t x(0)}{CA^t x(0)} \\
&\quad \text{(Using the fact that } x(t) = A^t x(0)) \\
&= \lim_{t \rightarrow \infty} \frac{\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \sum_{i=1}^6 \lambda_i^t (w_i^T x(0) v_i)}{C \sum_{i=1}^6 \lambda_i^t (w_i^T x(0) v_i)} \\
&\quad \text{(Expand into combination of modes)} \\
&= \lim_{t \rightarrow \infty} \frac{\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \lambda_1^t (w_1^T x(0) v_1)}{C \lambda_1^t (w_1^T x(0) v_1)} \\
&\quad \text{(Only dominant eigenvalue remains as } t \rightarrow \infty) \\
&= \frac{\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} (w_1^T x(0) v_1)}{C (w_1^T x(0) v_1)} \lim_{t \rightarrow \infty} \frac{\lambda^t}{\lambda^t} \\
&\quad \text{(Factor out constants)} \\
&= \frac{\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} (w_1^T x(0) v_1)}{C (w_1^T x(0) v_1)} \quad \text{(Limit is 1)} \\
&= \frac{\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} v_1}{C v_1} \\
&\quad \text{(Get rid of scalar } w_1^T x(0))
\end{aligned}$$

A very similar argument applies, even for the 60-20-20 matrix which is not diagonalizable since we still have λ_1 as dominant.

By very similar arguments (which we don't repeat for succinctness), we have:

$$\begin{aligned}
\lim_{t \rightarrow \infty} L_2(t) &= \frac{\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} v_1}{C v_1} \\
\lim_{t \rightarrow \infty} L_3(t) &= \frac{\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} v_1}{C v_1}
\end{aligned}$$

As such, we can see that for both strategies, the liquidity ratios do converge as $t \rightarrow \infty$ (since both strategy matrices have one dominant eigenvalue λ_1 with $|\lambda_1| > 1$ and for the non-diagonalizable case, the duplicitious eigenvalue is $\lambda_4 = 0$, which means it also dies off). In fact, they converge to the formulas presented above.

Using code to compute the actual results, we have the following. The liquidity ratios for 35 – 35 – 30 strategy are:

$$L_1 = 0.5033876854426759$$

$$L_2 = 0.33681212927259585$$

$$L_3 = 0.15980018528472825$$

The liquidity ratios for 60 – 20 – 20 strategy are:

$$L_1 = 0.6215267348660467$$

$$L_2 = 0.24989770184766186$$

$$L_3 = 0.12857556328629144$$

- (d) The *initial* investment allocation has no effect on the asymptotic growth rate (as long as there is *some* allocation). In the long-term, the 35-35-30 strategy achieves a steady state defined by the v_1 eigenvector, regardless of what the initial investment allocation is. Similarly, the liquidity ratios are not affected, and similarly achieve the same ratios in the long-term.

As such, the only possible *initial* investment allocations that we'd consider picking would be given by:

$$x(0) = f(v_1)$$

where f sets the last 3 entries to 0 and normalize the vector to sum to 1. The only difference with this allocation is that we would expect it to achieve the steady-state more quickly, but asymptotically, there is no difference.

Some basic properties of eigenvalues

Solution:

- (a) Proving the the eigenvalues of A and A^T are the same boils down to show that their characteristic polynomials, $\chi_A(s)$ and $\chi_{A^T}(s)$ are the same. We have:

$$\begin{aligned}
 \chi_{A^T}(s) &= \det(sI - A^T) \\
 &= \det([sI - A]^T) \\
 (sI - A^T &= (sI - A)^T \text{ since the transpose leaves the diagonals unchanged and } A \text{ is square}) \\
 &= \det(sI - A) \quad (\det(X) = \det(X^T) \text{ by the hint provided}) \\
 &= \chi_A(s)
 \end{aligned}$$

As such, we have the the characteristic polynomials of both A and A^T are the same. Therefore, their roots (eigenvalues) must also be the same.

- (b) We must prove both directions. For the first direction, suppose A is invertible, but A has a zero eigenvalue. Then this implies that the characteristic polynomial is 0 when evaluated with $s = 0$. More correctly, we must have:

$$\begin{aligned}
 \chi_A(0) &= \det(0I - A) \\
 &= \det(-A) \\
 &= \det(A) \\
 &= 0
 \end{aligned}$$

which implies A is not invertible. This is a contradiction. As such, we must have that if A is invertible, does not have a zero eigenvalue.

For the other direction, suppose A has a zero eigenvalue, but it is not invertible. Then we have:

$$\begin{aligned}
 \det(A) &= \det(-A) \\
 &= \det 0I - A \\
 &= \chi_A(0) = 0
 \end{aligned}$$

which means that 0 must be an eigenvalue of A , a contradiction. As such, we conclude that if 0 is not an eigenvalue of A , A must be invertible.

- (c) Let us focus on one eigenvalue, λ_i , with corresponding eigenvector v_i . Since A is invertible, we know that $\lambda_i \neq 0$. As such, we have:

$$\begin{aligned}
 A^{-1}v_i &= A^{-1}\frac{\lambda_i v_i}{\lambda_i} && \text{(Multiplying by 1, we know that } \lambda_i \neq 0) \\
 &= \frac{1}{\lambda_i} A^{-1} A v_i && \text{(Using the fact that } \lambda_i v_i = A v_i) \\
 &= \frac{1}{\lambda_i} v_i
 \end{aligned}$$

As such, we have that $\frac{1}{\lambda_i}$ is an eigenvector of A^{-1} .

- (d) Similarly to the first sub-problem, this boils down to showing that the characteristic polynomials of both matrices are the same.

$$\begin{aligned}
 \chi_{T^{-1}AT}(s) &= \det(sI - T^{-1}AT) \\
 &= \det(sT^{-1}T - T^{-1}AT) && (T^{-1}T = I) \\
 &= \det(T^{-1}(sI)T - T^{-1}AT) && (\text{Insert identity and move scalar}) \\
 &= \det(T^{-1}[sI - A]T) && (\text{Factoring}) \\
 &= \det(T^{-1}) \det(T) \det(sI - A) && (\text{Properties of determinant}) \\
 &= \frac{1}{\det(T)} \det(T) \det(sI - A) && (\text{More properties of determinant}) \\
 &= \det(sI - A) \\
 &= \chi_A(s)
 \end{aligned}$$

As such, we have the the characteristic polynomials of the matrices are the same. Therefore their eigenvalues are the same.

Optimal espresso cup pre-heating

Solution: We begin by developing a model for the temperatures at a given time. We have the matrix $A \in \mathbb{R}^{(n+1) \times (n+1)}$ and initial state $x(0) \in \mathbb{R}^{n+1}$. The dynamics of the system, at all times, are given by:

$$\frac{d}{dt}(x(t) - 20 \cdot \mathbf{1}) = A(x(t) - 20 \cdot \mathbf{1})$$

From the above, we immediately have that after P seconds of pre-heating, the state will be:

$$x(P) = e^{PA}(x(0) - 20 \cdot \mathbf{1}) + 20 \cdot \mathbf{20}$$

We define a modified version of $x(P)$, call it $\tilde{x}(P)$, where $x_i(P) = \tilde{x}_i(P)$ for $i \neq 1$, and $\tilde{x}_1(P) = 95$. Then the temperature distribution at the time of drinking the espresso will be:

$$x(P + 15) = e^{15A}(\tilde{x}(P) - 20 \cdot \mathbf{1}) + 20 \cdot \mathbf{1}$$

Then the temperature of espresso consumptions is given by $T(P) = x_1(P + 15)$. We can solve this by just searching for P directly, sampling enough values of P and computing $T(P)$. With this method, we obtain the following answer:

The optimal value of P is $P = 11.11$ s which gives an espresso temperature at consumption of 87.60C.

Real modal form

Solution: Constructing S is somewhat straight-forward. Let us assume that $A \in \mathbb{R}^{n \times n}$ is a diagonalizable matrix with at least one non-real eigenvalue. We can write $A = T^{-1}\Lambda T$. We summarize a few facts covered in lecture. Let $v_j \in \mathbb{R}^n$ be an eigenvector with corresponding eigenvalue $\lambda_j = a_j + ib_j$ where $b_j \neq 0$. This means that we have:

$$Av_j = \lambda_j v_j$$

Expanding out the above into real and imaginary parts, we have:

$$\begin{aligned} Av_j &= A(\mathcal{R}(v_j) + i\mathcal{I}(v_j)) \\ &= A\mathcal{R}(v_j) + iA\mathcal{I}(v_j) \\ &= (a_j + ib_j)(\mathcal{R}(v_j) + i\mathcal{I}(v_j)) \\ &= a_j\mathcal{R}(v_j) - b_j\mathcal{I}(v_j) + i(b_j\mathcal{R}(v_j) + a_j\mathcal{I}(v_j)) \\ \implies A\mathcal{R}(v_j) &= a_j\mathcal{R}(v_j) - b_j\mathcal{I}(v_j) \\ A\mathcal{I}(v_j) &= b_j\mathcal{R}(v_j) + a_j\mathcal{I}(v_j) \end{aligned}$$

The last two equations can be written in matrix form as:

$$A \begin{bmatrix} \mathcal{R}(v_j) & \mathcal{I}(v_j) \end{bmatrix} = \begin{bmatrix} \mathcal{R}(v_j) & \mathcal{I}(v_j) \end{bmatrix} \begin{bmatrix} a_j & b_j \\ -b_j & a_j \end{bmatrix}$$

If we follow a similar process for the conjugate eigenvector, \bar{v}_j , which has corresponding eigenvalue $\bar{\lambda}_j$, we'll arrive at the same set of equations as above. First, let us show that \bar{v}_j is also an eigenvector of A . We have:

$$\begin{aligned} A\bar{v}_j &= \bar{A}v_j && (A \text{ is real}) \\ &= \bar{\lambda}_j v_j && (Av_j = \lambda_j v_j) \\ &= \bar{\lambda}_j \bar{v}_j && (\text{Shows } \bar{v}_j \text{ is also eigenvector}) \end{aligned}$$

Following a similar process to above, we have:

$$\begin{aligned} A\bar{v}_j &= A(\mathcal{R}(v_j) - i\mathcal{I}(v_j)) \\ &= A\mathcal{R}(v_j) - iA\mathcal{I}(v_j) \\ &= (a_j - ib_j)(\mathcal{R}(v_j) - i\mathcal{I}(v_j)) \\ &= a_j\mathcal{R}(v_j) - b_j\mathcal{I}(v_j) - i(b_j\mathcal{R}(v_j) + a_j\mathcal{I}(v_j)) \\ \implies A\mathcal{R}(v_j) &= a_j\mathcal{R}(v_j) - b_j\mathcal{I}(v_j) \\ A\mathcal{I}(v_j) &= b_j\mathcal{R}(v_j) + a_j\mathcal{I}(v_j) \end{aligned}$$

These are the same set of equations implied by the conjugate.

A similar argument also applies to all conjugate pairs of complex eigenvalues. As such, let take v_1, \dots, v_r to be the eigenvectors with real eigenvalues. Let us then take v_{r+1}, \dots, v_n to

the the eigenvectors with complex eigenvalues, ordered such that eigenvectors are followed by their conjugate, if not already present. Then, we can construct the matrix S as follows:

$$S = \begin{bmatrix} v_1 & \cdots & v_r & v_{r+1} & v_{r+3} & \cdots & v_{n-1} \end{bmatrix} \in \mathbb{R}^{n \times n}$$

From the arguments made above, we must have:

$$\begin{aligned} AS &= S \mathbf{diag} \left(\lambda_1, \dots, \lambda_r, \begin{bmatrix} a_{r+1} & b_{r+1} \\ -b_{r+1} & a_{r+1} \end{bmatrix}, \begin{bmatrix} a_{r+3} & b_{r+3} \\ -b_{r+3} & a_{r+3} \end{bmatrix}, \dots, \begin{bmatrix} a_{n-1} & b_{n-1} \\ -b_{n-1} & a_{n-1} \end{bmatrix} \right) \\ \Rightarrow S^{-1}AS &= \mathbf{diag} \left(\lambda_1, \dots, \lambda_r, \begin{bmatrix} a_{r+1} & b_{r+1} \\ -b_{r+1} & a_{r+1} \end{bmatrix}, \begin{bmatrix} a_{r+3} & b_{r+3} \\ -b_{r+3} & a_{r+3} \end{bmatrix}, \dots, \begin{bmatrix} a_{n-1} & b_{n-1} \\ -b_{n-1} & a_{n-1} \end{bmatrix} \right) \end{aligned}$$

We now have the real-modal form we were interested in. For verification, see the attached code.