EE 263 Homework 3

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Memory of a linear dynamical, time-invariant system

Solution:

(a) Let us first consider how we might check if a valid impose response of fixed size M exists. The first thing to note is that the covolution operator given in the problem statent can actually be written as a linear system:

$$\bar{y} = Ah$$

where $\bar{y} \in \mathbb{R}^{T-M}, h \in \mathbb{R}^M$ and $A \in \mathbb{R}^{(T-M)\times M}$. Note that we remove the first $\{y_1, \cdots, y_M\}$. In fact, we can use the matrix A as defined below:

$$A = \begin{bmatrix} u_M & u_{M-1} & u_{M-2} & \cdots & u_1 \\ u_{M+1} & u_M & u_{M-1} & \cdots & u_2 \\ u_{M+2} & u_{M+1} & u_M & \cdots & u_2 \\ \vdots & \vdots & \vdots & \ddots & \\ u_{T-2} & u_{T-3} & u_{T-4} & \cdots & u_{T-M-1} \\ u_{T-1} & u_{T-2} & u_{T-3} & \cdots & u_{T-M} \end{bmatrix}$$

$$y_{-1} = \begin{bmatrix} y_{M+1} \\ \vdots \\ y_T \end{bmatrix}$$

$$h = \begin{bmatrix} h_1 \\ \vdots \\ h_M \end{bmatrix}$$

We can see by inspection above that Ah performs the needed convolutions betweens u and h to obtain $\bar{y}$, by the properties of matrix multiplication (eg, to obtain y_i, we compute the dot product of the $i - M$-th row of A with h, which is exactly what our convolution dictates, and for $i \leq M$, the convolution is not fully defined so we ignore them).

In the problem statement, we're given the fact that $T > 2M$, so this means that there will always be at least $T - M > 2M - M = M$ rows in A, meaning it will always be a tall and skinny matrix.

As such, we can use the pseudo inverse to find the closests solution for h, computing:

$$\bar{h} = (A^T A)^{-1} A^T \bar{y}$$

Finally, once we've computed this h, we can re-compute that $\bar{y}$ given our inputs forming A, and see if this equals our what we started with. In other words, we have that:

$$||A\bar{h} - \bar{y}|| \leq \epsilon \implies M \text{ is a valid value}$$

(ϵ is needed to deal with floating point imprecision)

The above gives us a way to check if M is sufficient. To finallize our method, we simply iterate over the possible values of M from $M = 1, \cdots \frac{T}{2} - 1$ in order until we find a valid value.

(b) Applying the process we described above, we find that $M = 7$ is the smallest value that works. We also have:

$$h = \begin{bmatrix} 0.63 \\ 0.27 \\ 0.02 \\ 0.37 \\ 0.96 \\ 0.95 \\ 0.46 \end{bmatrix}$$