

# Homework 7

EE 263 Stanford University Summer 2019

Due: Aug. 14, 2019

Question 7, and 8 are **optional** questions (they will not be graded) and you do not need to submit them to gradescope. However, doing them will help improve your understanding and grasp of the material. They relate to the controllability and observability of states.

**1. Properties of symmetric matrices.** In this problem  $P$  and  $Q$  are symmetric matrices. For each statement below, either give a proof or a specific counterexample.

- a) If  $P \geq 0$  then  $P + Q \geq Q$ .
- b) If  $P \geq Q$  then  $-P \leq -Q$ .
- c) If  $P > 0$  then  $P^{-1} > 0$ .
- d) If  $P \geq Q > 0$  then  $P^{-1} \leq Q^{-1}$ .
- e) If  $P \geq Q$  then  $P^2 \geq Q^2$ .

*Hint:* you might find it useful for part (d) to prove  $Z \geq I$  implies  $Z^{-1} \leq I$ .

**2. Drawing a representation of a graph.** Consider an undirected graph with  $n$  nodes, and  $m$  edges. We want to draw a representation of this graph in the plane, which means assigning coordinates  $(x_i, y_i) \in \mathbb{R}^2$  to node  $i$  for  $i = 1, \dots, n$ . Let  $x \in \mathbb{R}^n$  be the vector of  $x$ -coordinates of the nodes, and  $y \in \mathbb{R}^n$  be the vector of  $y$ -coordinates of the nodes. One desirable property of our representation is that nodes that are connected by an edge should not be too far apart. We can try to find a representation that has this property by minimizing the objective

$$J = \sum_{(i,j) \in E} ((x_i - x_j)^2 + (y_i - y_j)^2),$$

where  $E$  is the edge set of the graph. Note that  $J$  is the sum of the squares of the lengths of all of the edges in the graph. Just minimizing  $J$  does not usually yield a sensible solution, so we need to introduce some additional constraints. Since the objective  $J$  is not affected by shifting all of the coordinates by some fixed amount, we will assume that the coordinates are centered: that is,

$$\sum_{i=1}^n x_i = 0, \quad \text{and} \quad \sum_{i=1}^n y_i = 0.$$

Another problem with just minimizing  $J$  is that there is a trivial solution: set  $x = y = 0$ . To force the nodes to spread out, we impose the constraints

$$\sum_{i=1}^n x_i^2 = 1, \quad \sum_{i=1}^n y_i^2 = 1, \quad \text{and} \quad \sum_{i=1}^n x_i y_i = 0.$$

The first constraint says that the  $x$ -coordinates have unit variance, while the second constraint says that the  $y$ -coordinates have unit variance; the third constraint says that the  $x$ -coordinates and  $y$ -coordinates are uncorrelated. Even with all of these additional constraints, there is still not a unique set of coordinates that minimize  $J$ . For example, if  $x$  and  $y$  are some set of coordinates satisfying the constraints, and  $Q \in \mathbb{R}^{2 \times 2}$  is orthogonal, then

$$\begin{bmatrix} \tilde{x}_i \\ \tilde{y}_i \end{bmatrix} = Q \begin{bmatrix} x_i \\ y_i \end{bmatrix}, \quad i = 1, \dots, n$$

is another set of coordinates that also satisfies the constraints, and achieves the same value of  $J$ . Intuitively, we can rotate or reflect any set of coordinates to obtain another set of coordinates that is just as good. We will live with this ambiguity.

- a) Explain how to find coordinates  $x$  and  $y$  that minimize  $J$  subject to the centering and spreading constraints.
- b) The file `draw_graph_representation_data.m` defines the following variables.
  - `A`, the adjacency matrix of the graph
  - `x_circ`, a vector of  $x$ -coordinates obtained using a simple method that places the nodes at equally spaced points on a circle (the radius of the circle is chosen to satisfy the spreading constraints)
  - `y_circ`, a vector of  $y$ -coordinates corresponding to `x_circ`

Apply your method to the graph described by `A`. Report the optimal value of  $J$ , and draw a corresponding representation of the graph. For comparison, compute the value of  $J$  associated with `x_circ` and `y_circ`, and draw the corresponding representation of the graph.

**3. Blind signal detection.** A binary signal  $s_1, \dots, s_T$ , with  $s_t \in \{-1, 1\}$  is transmitted to a receiver, which receives the (vector) signal  $y_t = a s_t + v_t \in \mathbb{R}^n$ ,  $t = 1, \dots, T$ , where  $a \in \mathbb{R}^n$  and  $v_t \in \mathbb{R}^n$  is a noise signal. We'll assume that  $a \neq 0$ , and that the noise signal is centered around zero, but is otherwise unknown. (This last statement is vague, but it will not matter.)

The receiver will form an approximation of the transmitted signal as

$$\hat{s}_t = w^T y_t, \quad t = 1, \dots, T,$$

where  $w \in \mathbb{R}^n$  is a weight vector. Your job is to choose the weight vector  $w$  so that  $\hat{s}_t \approx s_t$ . If you knew the vector  $a$ , then a reasonable choice for  $w$  would be  $w = a^\dagger = a/\|a\|^2$ . This choice is the smallest (in norm) vector  $w$  for which  $w^T a = 1$ .

Here's the catch: You don't know the vector  $a$ . Estimating the transmitted signal, given the received signal, when you don't know the mapping from transmitted to received signal (in this case, the vector  $a$ ) is called *blind signal estimation* or *blind signal detection*.

Here is one approach. Ignoring the noise signal, and assuming that we have chosen  $w$  so that  $w^\top y_t \approx s_t$ , we would have

$$(1/T) \sum_{t=1}^T (w^\top y_t)^2 \approx 1.$$

Since  $w^\top v_t$  gives the noise contribution to  $\hat{s}_t$ , we want  $w$  to be as small as possible. This leads us to choose  $w$  to minimize  $\|w\|$  subject to  $(1/T) \sum_{t=1}^T (w^\top y_t)^2 = 1$ . This doesn't determine  $w$  uniquely; we can multiply it by  $-1$  and it still minimizes  $\|w\|$  subject to  $(1/T) \sum_{t=1}^T (w^\top y_t)^2 = 1$ . So we can only hope to recover either an approximation of  $s_t$  or of  $-s_t$ ; if we don't know  $a$  we really can't do any better. (In practice we'd use other methods to determine whether we have recovered  $s_t$  or  $-s_t$ .)

- a) Explain how to find  $w$ , given the received vector signal  $y_1, \dots, y_T$ , using concepts from the class.
- b) Apply the method to the signal in the file `bs_det_data.m`, which contains a matrix  $Y$ , whose columns are  $y_t$ . Give the weight vector  $w$  that you find. Plot a histogram of the values of  $w^\top y_t$  using `hist(w'*Y,50)`. You'll know you're doing well if the result has two peaks, one negative and one positive. Once you've chosen  $w$ , a reasonable guess of  $s_t$  (or, possibly, its negative  $-s_t$ ) is given by

$$\tilde{s}_t = \text{sign}(w^\top y_t), \quad t = 1, \dots, T,$$

where  $\text{sign}(u)$  is  $+1$  for  $u \geq 0$  and  $-1$  for  $u < 0$ . The file `bs_det_data.m` contains the original signal, as a row vector  $\mathbf{s}$ . Give your error rate, *i.e.*, the fraction of times for which  $\tilde{s}_t \neq s_t$ . (If this is more than 50%, you are welcome to flip the sign on  $w$ .)

- 4. Simultaneously estimating student ability and exercise difficulty..** Each of  $n$  students takes an exam that contains  $m$  questions. Student  $j$  receives a grade  $G_{ij} \geq 0$  on question  $i$ . One simple model for predicting grades is to estimate

$$G_{ij} \approx \hat{G}_{ij} = \frac{a_j}{d_i},$$

where  $a_j \geq 0$  is the ability of student  $j$ , and  $d_i > 0$  is the difficulty of question  $i$ . To ensure a unique model, we normalize the exam question difficulties so that the mean exam question difficulty across the  $m$  questions is 1; otherwise,  $a_j$  and  $d_i$  would not be uniquely determined because we can scale them both by any positive constant without changing our estimates.

In this problem you are given the matrix  $G \in \mathbb{R}^{m \times n}$  of grades. Your task is to find a set of nonnegative student abilities, and a set of positive, normalized question difficulties such that  $G_{ij} \approx \hat{G}_{ij}$ . In particular, choose your model in order to minimize the root mean squared error:

$$J = \sqrt{\frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n (G_{ij} - \hat{G}_{ij})^2}.$$

- a) Explain how to solve this problem. You can ignore the constraints that the  $a_j$  be nonnegative, and the  $d_i$  be positive.
- b) Carry out your method on the data in `ability_difficulty_data.m`. Report the optimal value of  $J$ , the ratio of the optimal value of  $J$  and the root mean squared value of  $G_{ij}$ ; give the difficulties of the exam questions.

**5. Square matrices and the SVD..** Let  $A$  be an  $n \times n$  real matrix. State whether each of the following statements is true or false. Do not give any explanation or show any work.

- a) If  $x$  is an eigenvector of  $A$ , then  $x$  is either a left or right singular vector of  $A$
- b) If  $\lambda$  is an eigenvalue of  $A$ , then  $|\lambda|$  is a singular value
- c) If  $A$  is symmetric, then every singular value of  $A$  is also an eigenvalue of  $A$
- d) If  $A$  is symmetric, then every singular vector of  $A$  is also an eigenvector of  $A$
- e) If  $A$  is symmetric with the following singular value decomposition

$$A = U\Sigma V^T$$

then  $U = V$

- f) If  $A$  is invertible, then

$$\sigma_i \neq 0 \quad \text{for all } i = 1, \dots, n$$

**6. Principal-components analysis of decathlon data.** In the decathlon athletes compete in  $n = 10$  different track-and-field events: 100 m run, long jump, shot put, high jump, 400 m run, 110 m hurdle, discus, pole vault, javelin, and 1500 m run. The file `decathlon_pca_data.m` contains the results for the  $m = 24$  athletes who completed all of the events in the decathlon in the 2008 Olympics; in particular, it defines the following variables.

- `m` and `n`, the numbers of athletes and events, respectively
- `names`, a cell array containing the last names and countries of the athletes
- `events`, a cell array containing the names of the events
- `scores`, a matrix containing the score of each athlete on each event; each row corresponds to an athlete, and each column corresponds to an event; the orderings of athletes and events are the same as in `names` and `events`, respectively

First, we standardize the data by forming the matrix  $X \in \mathbb{R}^{m \times n}$  such that

$$X_{ij} = \frac{\text{scores}(i, j) - m_j}{s_j},$$

where

$$m_j = \frac{1}{m} \sum_{i=1}^m \text{scores}(i, j) \quad \text{and} \quad s_j = \sqrt{\frac{1}{m-1} \sum_{i=1}^m (\text{scores}(i, j) - m_j)^2}$$

are, respectively, the mean and standard deviation of the scores for the  $j$ th event.

- a) Let  $X = \sum_{i=1}^m \sigma_i u_i v_i^T$  be the singular-value decomposition of  $X$ . Submit a plot showing  $\sigma_j$  versus  $j$ . The fraction of the total power that is contained in the first  $j$  dyads of the singular-value decomposition is defined to be

$$p_j = \frac{\sum_{j=1}^j \sigma_j^2}{\sum_{j=1}^n \sigma_j^2}.$$

Submit a plot of  $p_j$  versus  $j$ , and report  $p_2$ , the fraction of the total power that is contained in the first two dyads.

- b) Use the command `text` to create a plot in which the point  $((v_1)_j, (v_2)_j)$  is labeled with the name of the  $j$ th event. Do similar events appear to be close to each other on your plot?
- c) Let  $t_i = \sum_{j=1}^n X_{ij}$  be the total standardized score of the  $i$ th athlete. Then, we have that

$$r_j = \sum_{i=1}^m t_i X_{ij}$$

is proportional to the correlation between an athlete's total standardized score and his standardized score on the  $j$ th event. Download the function `spatial_plot` from the course website. Let  $v_1$  and  $v_2$  be the first two right singular vectors of  $X$ . Submit a copy of the plot generated using the following command.

```
spatial_plot(v1 , v2 , r , 3)
```

Which right singular vector seems to represent  $r$ ?

- d) Let  $u_1$  and  $u_2$  be the first two left singular vectors of  $X$ . Define the vector  $\delta \in \mathbb{R}^m$

$$\delta_i = (X_{i1} + X_{i5} + X_{i6}) - (X_{i3} + X_{i7} + X_{i9}),$$

and let  $t = (t_1, \dots, t_m)$  be the vector of the total standardized scores of the athletes. Submit copies of the plots generated using the following commands.

```
spatial_plot(u1 , u2 , t , length(names));
spatial_plot(u1 , u2 , delta , length(names));
```

Give intuitive interpretations of the first two left singular vectors of  $X$ .

**7. Minimum energy required to steer the state to zero.** Consider a controllable discrete-time system  $x(t+1) = Ax(t) + Bu(t)$ ,  $x(0) = x_0$ . Let  $E(x_0)$  denote the minimum energy required to drive the state to zero, *i.e.*

$$E(x_0) = \min \left\{ \sum_{\tau=0}^{t-1} \|u(\tau)\|^2 \mid x(t) = 0 \right\}.$$

An engineer argues as follows:

This problem is like the minimum energy reachability problem, but ‘turned backwards in time’ since here we steer the state from a given state to zero, and in the reachability problem we steer the state from zero to a given state. The system  $z(t+1) = A^{-1}z(t) - A^{-1}Bv(t)$  is the same as the given one, except time is running backwards. Therefore  $E(x_0)$  is the same as the minimum energy required for  $z$  to reach  $x_0$  (a formula for which can be found in the lecture notes).

Either justify or refute the engineer's statement. You can assume that  $A$  is invertible.

**8. Sensor selection and observer design.** Consider the system  $\dot{x} = Ax$ ,  $y = Cx$ , with

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(This problem concerns observer design so we've simplified things by not even including an input.) (The matrix  $A$  is the same as in problem ??, just to save you typing; there is no other connection between the problems.) We consider observers that (exactly and instantaneously) reconstruct the state from the output and its derivatives. Such observers have the form

$$x(t) = F_0 y(t) + F_1 \frac{dy}{dt}(t) + \cdots + F_k \frac{d^k y}{dt^k}(t),$$

where  $F_0, \dots, F_k$  are matrices that specify the observer. (Of course we require this formula to hold for any trajectory of the system and any  $t$ , *i.e.*, the observer has to work!) Consider an observer defined by  $F_0, \dots, F_k$ . We say the *degree* of the observer is the largest  $j$  such that  $F_j \neq 0$ . The degree gives the highest derivative of  $y$  used to reconstruct the state. If the  $i$ th columns of  $F_0, \dots, F_k$  are all zero, then the observer doesn't use the  $i$ th sensor signal  $y_i(t)$  to reconstruct the state. We say the observer *uses* or *requires* the sensor  $i$  if at least one of the  $i$ th columns of  $F_0, \dots, F_k$  is nonzero.

- a) What is the minimum number of sensors required for such an observer? List all combinations (*i.e.*, sets) of sensors, of this minimum number, for which there is an observer using only these sensors.
- b) What is the minimum degree observer? List all combinations of sensors for which an observer of this minimum degree can be found.