Homework 6

EE 263 Stanford University Summer 2019

Due: August 7, 2019

Question **8**, and **9** are **optional** questions (they will not be graded) and you do not need to submit them to gradescope. However, doing them will help improve your understanding and grasp of the material.

1. **Optimal operation of a two-state chemical reactor.** Consider a chemical reactor containing n compounds, labeled $1, \ldots, n$. Let $x_i(t)$ be the amount of compound i in the reactor at time t. The chemical reactor has two modes of operation, labeled 1 and 2. (For example, the first mode may be operating the reactor at a low temperature, and the second mode may be operating the reactor at a high temperature.) For simplicity we assume that the mode of operation can be changed instantaneously. When we operate the reactor in mode j, the vector of compound amounts evolves according to the equation

$$\dot{x}(t) = A_j x(t).$$

We are given the vector $x(0) \in \mathbb{R}^n$ of initial compound amounts, and the dynamics matrices A_1 and A_2. Our objective is to maximize the amount of compound k at time T, where $k \in \{1, \ldots, n\}$ and $T > 0$ are given.

a) Suppose the reactor operates in mode 1 for $0 \leq t \leq T_0$, and mode 2 for $T_0 < t \leq T$. Explain how to choose the time T_0 in order to maximize the amount of compound k at time T. Your answer only needs to be accurate to two decimal digits.

b) Apply your method to the data given in `chemical_reactor_data.m`. Report the optimal value of T_0 and the corresponding amount of compound k at time T; submit a plot showing all of the components of $x(t)$ as functions of time on a single set of axes.

c) Suppose the reactor operates in mode 1 for $0 \leq t \leq T_1$ and $T_2 < t \leq T$, and mode 2 for $T_1 < t \leq T_2$. Explain how to choose the times T_1 and T_2 in order to maximize the amount of compound k at time T. Your answers for T_1 and T_2 only need to be accurate to two decimal digits.

d) Apply your method to the data given in `chemical_reactor_data.m`. Report the optimal values of T_1 and T_2 and the corresponding amount of compound k at time T; submit a plot showing all of the components of $x(t)$ as functions of time on a single set of axes.

2. Harmonic oscillator. The system $\dot{x} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} x$ is called a *harmonic oscillator.*

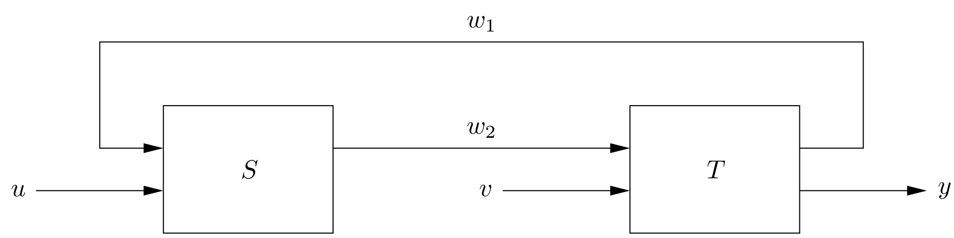
a) Find the eigenvalues, resolvent, and state transition matrix for the harmonic oscillator. Express $x(t)$ in terms of $x(0)$.

b) Sketch the vector field of the harmonic oscillator.

c) The state trajectories describe circular orbits, *i.e.*, $\|x(t)\|$ is constant. Verify this fact using the solution from part (a).

d) You may remember that circular motion (in a plane) is characterized by the velocity vector being orthogonal to the position vector. Verify that this holds for any trajectory of the harmonic oscillator. Use only the differential equation; do not use the explicit solution you found in part (a).

3. Interconnection of linear systems. Often a linear system is described in terms of a block diagram showing the interconnections between components or subsystems, which are themselves linear systems. In this problem you consider the specific interconnection shown below:



Here, there are two subsystems S and T. Subsystem S is characterized by

$$\dot{x} = Ax + B_1 u + B_2 w_1, \qquad w_2 = Cx + D_1 u + D_2 w_1,$$

and subsystem T is characterized by

$$\dot{z} = Fz + G_1 v + G_2 w_2, \qquad w_1 = H_1 z, \qquad y = H_2 z + J w_2.$$

We don't specify the dimensions of the signals (which can be vectors) or matrices here. You can assume all the matrices are the correct (*i.e.*, compatible) dimensions. Note that the subscripts in the matrices above, as in B_1 and B_2, refer to different matrices. Now the problem. Express the overall system as a single linear dynamical system with input, state, and output given by

$$\begin{bmatrix} u \\ v \end{bmatrix}, \qquad \begin{bmatrix} x \\ z \end{bmatrix}, \qquad y,$$

respectively. Be sure to explicitly give the input, dynamics, output, and feedthrough matrices of the overall system. If you need to make any assumptions about the rank or invertibility of any matrix you encounter in your derivations, go ahead. But be sure to let us know what assumptions you are making.

4. **Analysis of investment allocation strategies.** Each year or period (denoted $t = 0, 1, \ldots$) an investor buys certain amounts of one-, two-, and three-year certificates of deposit (CDs) with interest rates 5%, 6%, and 7%, respectively. (We ignore minimum purchase requirements, and assume they can be bought in any amount.)

   - $B_1(t)$ denotes the amount of one-year CDs bought at period t.
   - $B_2(t)$ denotes the amount of two-year CDs bought at period t.
   - $B_3(t)$ denotes the amount of three-year CDs bought at period t.

   We assume that $B_1(0) + B_2(0) + B_3(0) = 1$, *i.e.*, a total of 1 is to be invested at $t = 0$. (You can take $B_j(t)$ to be zero for $t < 0$.) The total payout to the investor, $p(t)$, at period t is a sum of six terms:

   - $1.05B_1(t-1)$, *i.e.*, principle plus 5% interest on the amount of one-year CDs bought one year ago.
   - $1.06B_2(t-2)$, *i.e.*, principle plus 6% interest on the amount of two-year CDs bought two years ago.
   - $1.07B_3(t-3)$, *i.e.*, principle plus 7% interest on the amount of three-year CDs bought three years ago.
   - $0.06B_2(t-1)$, *i.e.*, 6% interest on the amount of two-year CDs bought one year ago.
   - $0.07B_3(t-1)$, *i.e.*, 7% interest on the amount of three-year CDs bought one year ago.
   - $0.07B_3(t-2)$, *i.e.*, 7% interest on the amount of three-year CDs bought two years ago.

   The total wealth held by the investor at period t is given by

   $$w(t) = B_1(t) + B_2(t) + B_2(t-1) + B_3(t) + B_3(t-1) + B_3(t-2).$$

   Two re-investment allocation strategies are suggested.

   - *The 35-35-30 strategy.* The total payout is re-invested 35% in one-year CDs, 35% in two-year CDs, and 30% in three-year CDs. The initial investment allocation is the same: $B_1(0) = 0.35$, $B_2(0) = 0.35$, and $B_3(0) = 0.30$.
   - *The 60-20-20 strategy.* The total payout is re-invested 60% in one-year CDs, 20% in two-year CDs, and 20% in three-year CDs. The initial investment allocation is $B_1(0) = 0.60$, $B_2(0) = 0.20$, and $B_3(0) = 0.20$.

   a) Describe the investments over time as a linear dynamical system $x(t+1) = Ax(t)$, $y(t) = Cx(t)$ with $y(t)$ equal to the total wealth at time t. Be *very clear* about what the state $x(t)$ is, and what the matrices A and C are. You will have two such linear systems: one for the 35-35-30 strategy and one for the 60-20-20 strategy.

   b) *Asymptotic wealth growth rate.* For each of the two strategies described above, determine the asymptotic growth rate, defined as $\lim_{t\to\infty} w(t+1)/w(t)$. (If this limit doesn't exist, say so.) *Note:* simple numerical simulation of the strategies (*e.g.*, plotting $w(t+1)/w(t)$ versus t to guess its limit) is *not* acceptable. (You can, of course, simulate the strategies to *check* your answer.)

c) *Asymptotic liquidity.* The total wealth at time t can be divided into three parts:

- $B_1(t) + B_2(t-1) + B_3(t-2)$ is the amount that matures in one year (*i.e.*, the amount of principle we could get back next year)
- $B_2(t) + B_3(t-1)$ is the amount that matures in two years
- $B_3(t)$ is the amount that matures in three years (*i.e.*, is least liquid)

We define liquidity ratios as the ratio of these amounts to the total wealth:

$$\begin{aligned}
L_1(t) &= (B_1(t) + B_2(t-1) + B_3(t-2))/w(t), \\
L_2(t) &= (B_2(t) + B_3(t-1))/w(t), \\
L_3(t) &= B_3(t)/w(t).
\end{aligned}$$

For the two strategies above, do the liquidity ratios converge as $t \to \infty$? If so, to what values? *Note:* as above, simple numerical simulation alone is *not* acceptable.

d) Suppose you could change the *initial* investment allocation for the 35-35-30 strategy, *i.e.*, choose some other nonnegative values for $B_1(0)$, $B_2(0)$, and $B_3(0)$ that satisfy $B_1(0) + B_2(0) + B_3(0) = 1$. What allocation would you pick, and how would it be better than the $(0.35,\ 0.35,\ 0.30)$ initial allocation? (For example, would the asymptotic growth rate be larger?) How much better is your choice of initial investment allocations? *Hint for part d:* think *very carefully* about this one. *Hint for whole problem:* watch out for nondiagonalizable, or nearly nondiagonalizable, matrices. Don't just blindly type in matlab commands; check to make sure you're computing what you think you're computing.

5. Some basic properties of eigenvalues. Show the following:

a) The eigenvalues of A and A^T are the same.

b) A is invertible if and only if A does not have a zero eigenvalue.

c) If the eigenvalues of A are $\lambda_1, \ldots, \lambda_n$ and A is invertible, then the eigenvalues of A^{-1} are $1/\lambda_1, \ldots, 1/\lambda_n$.

d) The eigenvalues of A and $T^{-1}AT$ are the same.

Hint: you'll need to use the facts that $\det A = \det(A^\mathsf{T})$, $\det(AB) = \det A \det B$, and, if A is invertible, $\det A^{-1} = 1/\det A$.

6. Optimal espresso cup pre-heating. At time $t = 0$ boiling water, at 100°C, is poured into an espresso cup; after P seconds (the 'pre-heating time'), the water is poured out, and espresso, with initial temperature 95°C, is poured in. (You can assume this operation occurs instantaneously.) The espresso is then consumed exactly 15 seconds later (yes, instantaneously). The problem is to choose the pre-heating time P so as to maximize the temperature of the espresso when it is consumed.

We now give the thermal model used. We take the temperature of the liquid in the cup (water or espresso) as one state; for the cup we use an n-state finite element model. The vector $x(t) \in \mathbb{R}^{n+1}$ gives the temperature distribution at time t: $x_1(t)$ is the liquid (water or espresso) temperature at time t, and $x_2(t), \ldots, x_{n+1}(t)$ are the temperatures of the elements in the cup. All of these are in degrees C, with t in seconds. The dynamics are

$$\frac{d}{dt}(x(t) - 20 \cdot \mathbf{1}) = A(x(t) - 20 \cdot \mathbf{1}),$$

where $A \in \mathbb{R}^{(n+1)\times(n+1)}$. (The vector $20 \cdot \mathbf{1}$, with all components 20, represents the ambient temperature.) The initial temperature distribution is

$$x(0) = \begin{bmatrix} 100 \\ 20 \\ \vdots \\ 20 \end{bmatrix}.$$

At $t = P$, the liquid temperature changes instantly from whatever value it has, to 95; the other states do not change. Note that the dynamics of the system are the same before and after pre-heating (because we assume that water and espresso behave in the same way, thermally speaking).

We have *very generously* derived the matrix A for you. You will find it in `espressodata.m`. In addition to `A`, the file also defines `n`, and, respectively, the ambient, espresso and preheat water temperatures `Ta` (which is 20), `Te` (95), and `Tl` (100).

Explain your method, submit your code, and give final answers, which must include the optimal value of P and the resulting optimal espresso temperature when it is consumed. Give both to an accuracy of one decimal place, as in

> '$P = 23.5$ s, which gives an espresso temperature at consumption of 62.3°C.'

(This is not the correct answer, of course.)

7. **Real modal form.** Generate a matrix A in $\mathbb{R}^{10\times 10}$ using `A=randn(10)`. (The entries of A will be drawn from a unit normal distribution.) Find the eigenvalues of A. If by chance they are all real, please generate a new instance of A. Find the real modal form of A, *i.e.*, a matrix S such that $S^{-1}AS$ has the real modal form given in lecture 11. Your solution should include a clear explanation of how you will find S, the source code that you use to find S, and some code that checks the results (*i.e.*, computes $S^{-1}AS$ to verify it has the required form).

8. **Jordan form of a block matrix.** We consider the block 2×2 matrix

$$C = \begin{bmatrix} A & I \\ 0 & A \end{bmatrix}.$$

Here $A \in \mathbb{R}^{n\times n}$, and is diagonalizable, with real, distinct eigenvalues $\lambda_1, \ldots, \lambda_n$. We'll let $v_1, \ldots, v_n$ denote (independent) eigenvectors of A associated with $\lambda_1, \ldots, \lambda_n$.

a) Find the Jordan form J of C. Be sure to explicitly describe its block sizes.

b) Find a matrix T such that $J = T^{-1}CT$.

9. Affine dynamical systems. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is called *affine* if it is a linear function plus a constant, *i.e.*, of the form $f(x) = Ax + b$. Affine functions are more general than linear functions, which result when $b = 0$. We can generalize linear dynamical systems to *affine dynamical systems*, which have the form

$$\dot{x} = Ax + Bu + f, \quad y = Cx + Du + g.$$

Fortunately we don't need a whole new theory for (or course on) affine systems; a simple shift of coordinates converts it to a linear dynamical system. Assuming A is invertible, define $\tilde{x} = x + A^{-1}f$ and $\tilde{y} = y - g + CA^{-1}f$. Show that $\tilde{x}$, u, and $\tilde{y}$ are the state, input, and output of a linear dynamical system.