MATHEMATICS 121, FALL 2013
LINEAR ALGEBRA WITH APPLICATIONS

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Module #15, Proof: Prove that a continuous real-valued function f defined on a compact subset $C \subset \mathbb{R}^n$ has a supremum M and that there is a point $\mathbf{a} \in C$ (a maximum) where $f(\mathbf{a}) = M$.

First, we prove that a continuous real-valued function $\mathbf{f}$ defined on a compact subset $C \subset \mathbb{R}^n$ has a supremum M.

Proof. Assume that $\mathbf{f}$ is unbounded. Then we can construct a sequence $\mathbf{f}(\mathbf{a_1}) > 1, \mathbf{f}(\mathbf{a_2}) > 2, \cdots \mathbf{f}(\mathbf{a_n}) > N$. Because C is compact, by the Bolzano-Weierstrass Theorem, we can extract a convergent subsequence $\mathbf{a_i}$ which converges to $a \in C$. Because $\mathbf{f}$ is continuous, we know that $\forall \epsilon > 0, \exists \delta > 0$ such that $|\mathbf{x} - a| < \delta \implies |\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a})| < \epsilon$. Using the triangle inequality, we have

$$\begin{aligned} |\mathbf{f}(\mathbf{x})| &= |\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a}) + \mathbf{f}(\mathbf{a})| \\ &\leq |\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a})| + |\mathbf{f}(\mathbf{a})| & < \epsilon + \mathbf{f}(\mathbf{a}) \end{aligned}$$

Because $\mathbf{a_i}$ converges to $\mathbf{a}$, we have $|\mathbf{a_i} - \mathbf{a}| < \delta$ for sufficiently large i. But as soon $i > \mathbf{f}(\mathbf{a}) + \epsilon$, we have from our definition of $\mathbf{f}$ that $\mathbf{f}(\mathbf{a_i}) > i > \mathbf{f}(\mathbf{a})$, a contradiction in our definition of continuity. Therefore, our assumption that $\mathbf{f}$ is unbounded is wrong, and therefore $\mathbf{f}$ must have a supremum, M. □

Next, we show that there exists a point $\mathbf{a} \in C$ such that $\mathbf{f}(\mathbf{a}) = M$.

Proof. Using the above information, we know that there is a sequence $\mathbf{x_i}$ such that as $i \to \infty$, $\mathbf{f}(\mathbf{x_i}) = M$. Using Bolzano-Weierstrass again, we can extract a convergent subsequence $\mathbf{a_i}$ which converges to some point $\mathbf{a} inC$. Then it is clear that as $i \to \infty$, $\mathbf{a_i} \to \mathbf{a}$ and $\mathbf{f}(\mathbf{a}) = \mathbf{f}(\mathbf{a_i}) = M$. □

Q.E.D.