MATHEMATICS 121, FALL 2013
LINEAR ALGEBRA WITH APPLICATIONS

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Module #13, Proof:

Define a closed set A *by the property that its complement* $A^c = X - A$ *is an open set. Prove that if a sequence* p_n *of elements of a closed set A converges to p, then* $p \in A$*. Use only fundamental ideas of topology in your proof – do not assume that* $X = \mathbb{R}$*.*

Illustrate your conclusion by using the case where $X = \{1, 2, 3, 4\}$*, the topology is specified by the matrix*

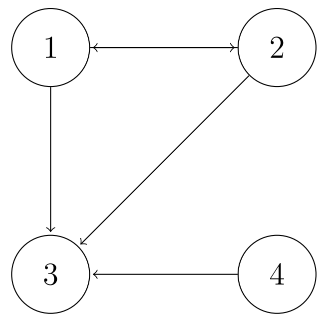
$$T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ and the sequence is } (4, 3, 2, 1, 3, 2, 1, 2, 1, 2, 1, \cdots).$$

Proof. By definition, if the sequence p_n converges to p, then for every open set S containing p, $\exists N$ such that $\forall n > N, a_n \in S$. Intuitively, this means that p_n gets inside S and stays there. We will proof that $p \in A$ by contradiction.
Assume that $p \notin A$. Then it must be the case that $p \in X - A$, which is an open set. By the definition of an open set, we know that there exists an open subset of $I \subset X - A$ containing p (ie, $p \in I$) which is contained entirely within $X - A$. But then our sequence must get inside and stay inside of this set, so it must contain elements which are not members of A, and we have a contradiction! Therefore our assumption that $p \notin A$ is false, and $p \in A$. □

Alternatively, we have the following proof as suggested in the small-group problems.

Proof. By definition, if the sequence p_n converges to p, then for every open set S containing p, $\exists N$ such that $\forall n > N, a_n \in S$. Intuitively, this means that p_n gets inside S and stays there.
Assume that $p \notin A$. Then it must be the case that $p \in X - A$. By definition of convergence, any open set containing p must also contain elements in A, therefore $p \in \overline{A}$. Then because $p \in X - A$, the set A cannot be closed because $\overline{A} \not\subseteq A$. Therefore, if $p \in X - A$ then A is not closed, and conversely, if A is closed then $p \notin X - A$, so $p \in A$. □

To illustrate, note the following diagram represents the topology specified by the matrix.



As we can see, the sequence eventually cycles between 1 and 2, but we should check all possible points of convergence.

1. The sequence converges to 1. Any open set that includes 1 must have $\{12\}$ as a subset. We can see that for $N = 5, n > N$ we have $a_n \in \{12\}$
2. The sequence converges to 2.Any open set that includes 2 must have $\{12\}$ as a subset. We can see that for $N = 5, n > N$ we have $a_n \in \{12\}$
3. The sequence converges to 3. Any open set that includes 3 must have $\{1234\}$ as a subset. We can see that for $N = 1, n > N$ we have $a_n \in \{123\}$.
4. The sequence does not converge to 4. The set $\{4\}$ is an open set containing 4, but for $N = 1, n > N$, we have $a_n \notin \{4\}$

This illustrated our proof. If we define $\{123\}$ as our closed set A (this means we ignore the first 4 v), we can see that every point to which the sequence converges is contained in this set.

Q.E.D.