MATHEMATICS 121, FALL 2013
LINEAR ALGEBRA WITH APPLICATIONS

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Module #12, Proof:

- For real $n \times n$ matrix A, prove that if all the polynomials $p_i(t)$ are simple and have real roots, then there exists a basis for $\mathbb{R}^n$ consisting of eigenvectors of A.
- Prove that if there exists a basis for $\mathbb{R}^n$ consisting of eigenvectors of A, then all the polynomials $p_i(t)$ are simple and have real roots.

Assume that for each $\vec{\boldsymbol{w}} = \vec{\boldsymbol{e}}_i$ we choose, the corresponding polynomial $p_i(t)$ has simple, real roots. Then consider the subspace E_i which is the image of the matrix with columns $\vec{\boldsymbol{e}}_i, A\vec{\boldsymbol{e}}_i, A^2\vec{\boldsymbol{e}}_i \cdots A^{m_i}\vec{\boldsymbol{e}}_i$. We assert that there exists a basis of eigenvectors for E_i. To see this, notice that because $p_i(t)$ only has simple, real roots, then:

$$p_i(t) = (t - \lambda_1 I)(t - \lambda_2 I) \cdots (t - \lambda_{m_i} I) = 0$$

Plugging in A, and factoring out any of the terms, we have that $(A - \lambda_j I)q_j(A)\vec{\boldsymbol{e}}_i = 0$ where $q_j(t)$ is the polynomial left behind after factoring $(t - \lambda_j I)$, and therefore $q_j(t)\vec{\boldsymbol{e}}_i$ is an eigenvector of A. For example, we have

1. Leaving out the first term of $p(A), (A - \lambda_1 I)$, we know that $q_1(A) = (A - \lambda_2 I) \cdots (A - \lambda_{m_i} I)\vec{\boldsymbol{e}}_{\boldsymbol{i}}$ is an eigenvector with eigenvalue λ_1.
2. Leaving out the second term of $p(A), (A - \lambda_2 I)$, we know that $q_2(A) = (A - \lambda_1 I) \cdots (A - \lambda_{m_i} I)\vec{\boldsymbol{e}}_{\boldsymbol{i}}$ is an eigenvector with eigenvalue λ_2.
3. We can continue this process until, leaving out the last term of $p(A), (A - \lambda_{m_i} I)$, we have that $q_{m_i}(A) = (A - \lambda_1 I)(A - \lambda_2 I) \cdots (A - \lambda_{m_i - 1} I)\vec{\boldsymbol{e}}_{\boldsymbol{i}}$ is an eigenvector with eigenvalue λ_{m_i}.

We have therefore constructed m_i eigenvectors with distinct eigenvalues. By Module 11, the eigenvectors must be linearly independent. Furthermore, each eigenvector is a linear combination of the columns of the matrix whose image is E_i (you just have to multiply out $q_j(A)$ to see this). Additionally, the matrix is constructed so that you stop as soon as you encounter one linearly dependent vector, therefore even though A has $m_i + 1$ columns, the dim(Im A) is only m_i, and our set of m_i eigenvectors therefore forms a basis of E_i.
Now consider the space E which is the image of the matrix $\left[E_1 \mid E_2 \mid \cdots \mid E_n\right]$. Because each E_i contains the basis vector $\vec{\boldsymbol{e}}_i$, then the image of this matrix is $\mathbb{R}^n$. Furthermore, as described earlier each E_i has a basis consisting only of

eigenvectors, and therefore the image of our matrix also has a basis consisting of eigenvectors. More precisely, $\vec{\boldsymbol{e}}_i \in E_i$ which is spanned by eigenvectors only, so removing all e_i from our new matrix does not affect the image of the matrix. Then it is the case that if for a real $n \times n$ matrix A all the polynomials $p_i(t)$ are simple and have real roots, then there exists a basis for of R^n consisting only of eigenvectors of A.
Now we prove that if there exists a basis for R^n consisting of eigenvectors of A, then all the polynomials $p_i(t)$ are simple and have real roots. To see this, note that because we have a basis of eigenvectors then for any basis vector $\vec{\boldsymbol{e}}_i$, we can write $\vec{\boldsymbol{e}}_i = a_1\vec{\boldsymbol{v}}_1 + a_2\vec{\boldsymbol{v}}_2 = \cdots a_n\vec{\boldsymbol{v}}_n$. Note that our basis need not consists of eigenvectors with distinct eigenvalues; they only need to form a basis. Then consider the polynomial $p_i(t) = (t - \lambda_1 I)(t - \lambda_2 I)\cdots(t - \lambda_k I$ where $k \leq n$ (depending on whether we have repeated eigenvalues or not). Then applying $p_i(A)$ to $\vec{\boldsymbol{e}}_i$ yields 0 because we have included every possible eigenvector in the polynomial. If we remove any factor, then we are no longer guaranteed to obtain 0. Therefore all $p_i(t)$ (ie, polynomials for which $\vec{\boldsymbol{e}}_i$ is a linear combination of eigenvectors, as occurs when we construct our matrix $\left[\vec{\boldsymbol{e}}_i \mid A\vec{\boldsymbol{e}}_{\boldsymbol{i}} \mid \cdots \mid A^{m_i}\vec{\boldsymbol{e}}_i\right]$ with the smallest possible m_i) are simple and have real roots.

Q.E.D.