MATHEMATICS 121, FALL 2013
LINEAR ALGEBRA WITH APPLICATIONS

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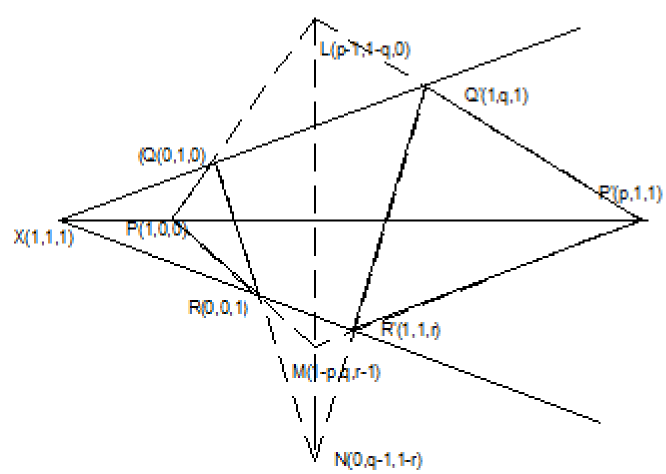
Module #10, Proof:

State and prove the theorem of Desargues in projective geometry. You may choose four points to have coordinates (1, 1, 1), (1, 0, 0), (0, 1, 0), and (0, 0, 1) respectively.

The Theorem of Desargues states that given three lines intersecting at a point X, three non-collinear points on the three lines labeld P, Q and R, and three different also non-collinear points on the lines labeled P', Q', R', then the points of intersection of PQ and $P'Q'$, PR and $P'R'$, and QR and $Q'R'$ (respectively, L,M, and N) are collinear. Wikipedia also states it more succinctly as follows: Two triangles are in perspective axially if and only if they are in perspective centrally.

While succinct, that definition uses terminology not used in class, though it is evident how it states the same idea.

This is clarified in the image below. The proof is a modified version of that appearing on Module 10, p9.



The proof involves nothing but cross products and is easy to do by hand if we introduce coordinates cleverly in a projective geometry.

- Choose $X = (1,1,1)$ and $P(1,0,0)$. Then P' is in the subspace spanned by X and P and must have coordinates of the form $(p,1,1)$ with $p \neq 1$.

- Choose $Q = (0,1,0)$. Then Q' is in the subspace spanned by X and Q and must have coordinates of the form $(1,q,1)$. $(q \neq 1)$.

- Choose $R(0,0,1)$. Then R' is in the subspace spanned by X and R and must have coordinates of the form $(1,1,r)$. $(r \neq 1)$.
- The cross product $L = (P' \times Q') \times (P \times Q) = (1-q, 1-p, pq-1) \times (0,0,1) = (1-p, q-1, 0)$.
- Similarly, $M = (R' \times P') \times (R \times P) = (1-r, rp-1, 1-p) \times (0,1,0) = (p-1, 0, 1-r)$
- Finally, $N = (Q' \times R') \times (Q \times R) = (qr-1, 1-r, 1-q) \times (1,0,0) = (0, 1-q, r-1)$.
- The sum of the vectors for L, M, and N is 0; so they are linearly dependent. As three-component vectors they lie in a plane (becase they are linearly dependent, but none are scalar multiples of anoter), which means that as projective points they are collinear.

Q.E.D.