MATHEMATICS 121, FALL 2013
LINEAR ALGEBRA WITH APPLICATIONS

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Module #6, Proof:

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Prove that Ker T and Img T are subspaces of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively and that dim(Ker T) + dim(Img T) = n.

Ker T is a subspace of $\mathbb{R}^n$. By definition, Ker T is in the domain of T, therefore it is a subset of $\mathbb{R}^n$.
First, notice $T(\vec{0}) = 0$ and therefore $\vec{0} \in$ Ker T and Ker T is non-empty. Let $\vec{\boldsymbol{x}}_1, \vec{\boldsymbol{x}}_2 \in$ Ker T. This means $T(\vec{\boldsymbol{x}}_1) = 0$ and $T(\vec{\boldsymbol{x}}_2) = 0$.
Then $T(a\vec{\boldsymbol{x}}_1 + b\vec{\boldsymbol{x}}_2) = aT(\vec{\boldsymbol{x}}_1) + bT(\vec{\boldsymbol{x}}_2) = 0$, and therefore $(a\vec{\boldsymbol{x}}_1 + b\vec{\boldsymbol{x}}_2) \in$ Ker T meaning Ker T is closed under addition and scalar multiplication and is therefore a subspace. $\square$

Img T is a subspace of $\mathbb{R}^m$. By definition, Img T is in the codomain of T, therefore it is a subset of $\mathbb{R}^m$.
First, notice $T(\vec{0}) = 0$ and therefore $\vec{0} \in$ Img T and Img T is non-empty. Let $\vec{\boldsymbol{b}}_1, \vec{\boldsymbol{b}}_2 \in$ Img T. This means $T(\vec{\boldsymbol{x}}_1) = \vec{\boldsymbol{b}}_1$ and $T(\vec{\boldsymbol{x}}_2) = \vec{\boldsymbol{b}}_2$ for some $\vec{\boldsymbol{x}}_1, \vec{\boldsymbol{x}}_2 \in \mathbb{R}^n$.
Then $T(a\vec{\boldsymbol{x}}_1 + c\vec{\boldsymbol{x}}_2) = aT(\vec{\boldsymbol{x}}_1) + cT(\vec{\boldsymbol{x}}_2) = a\vec{\boldsymbol{b}}_1 + c\vec{\boldsymbol{b}}_2$, and therefore $(a\vec{\boldsymbol{b}}_1 + c\vec{\boldsymbol{b}}_2) \in$ Im T meaning Im T is closed under addition and scalar multiplication and is therefore a subspace of $\mathbb{R}^m$. $\square$

dim(Ker T) + dim(Img T) = n. The liner transformation can be represented by an $n \times m$ matrix A. Each column of the row-reduced form of A will be either pivotal or non-pivotal. Therefore pivotal columns + non-pivotal columns = n. But the pivotal columns in $rref(A)$ correspond (bijectively) to the set of linearly independent columns in A, and since the Im A is the span of its columns, dim(Im T) = dim(Im A) = number of pivotal columns in $rref(A)$. Similarly, for each of the non-pivotal columns in $rref(A)$ we can construct $\vec{\boldsymbol{v}}_i \in R^n$ with zeros in the positions corresponding to all other non-pivotal columns and a 1 in the position corresponding to the pivotal column from which $\vec{\boldsymbol{v}}$ is being constructed. By construction, this set of vectors is linearly independent. Therefore dim(Ker T) = dim(Ker A) = number of non-pivotal columns.
Taken together, this means dim(Ker T) + dim(Img T) = n. $\square$

Q.E.D.