MATHEMATICS 121, FALL 2013
LINEAR ALGEBRA WITH APPLICATIONS

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**Module #7, Proof:**

*Start with a minimal set of axioms for an abstract vector space over a field $F$, assuming only that each vector has a right inverse, that there exists a right additive identity, that the associative and distributive laws hold, and that the multiplicative identity 1 in $F$ is also a multiplicative identity for vectors. Prove that each right inverse is also a left inverse, that the right identity is also a left identity, and that vector addition is commutative.*

We begin with the following axioms over the abstract vector space $V$ over and arbitrary field $F$:

1. $\forall \underline{\mathbf{v}} \in V$, $\exists (-\underline{\mathbf{v}}) \in V$ such that $\underline{\mathbf{v}} + (-\underline{\mathbf{v}}) = \underline{\mathbf{0}}$
2. $\exists \underline{\mathbf{0}} \in V$ such that $\forall \underline{\mathbf{v}} \in V$, $\underline{\mathbf{v}} + \underline{\mathbf{0}} = \underline{\mathbf{v}}$
3. Vector addition is associative
4. Scalar and vector addition is distributive
5. The multiplicative identity 1 in $F$ is also the multiplicative identity for vectors

We wish to proof that each right inverse is also a left inverse (ie. $\forall \underline{\mathbf{v}} \in V$, $\underline{\mathbf{v}} + (-\underline{\mathbf{v}}) = (-\underline{\mathbf{v}}) + \underline{\mathbf{v}}$). First, we note that by Axiom 1: $\exists \underline{\mathbf{w}} \in V$ such that $(-\underline{\mathbf{v}}) + \underline{\mathbf{w}} = 0$.

$$
\begin{aligned}
\underline{\mathbf{v}} + (-\underline{\mathbf{v}}) &= (-\underline{\mathbf{v}}) + \underline{\mathbf{w}} && \text{(Both equations equal } \underline{\mathbf{0}}\text{)}\\
\underline{\mathbf{v}} + (-\underline{\mathbf{v}}) &= (-\underline{\mathbf{v}}) + \underline{\mathbf{0}} + \underline{\mathbf{w}} && \text{(Axiom 2)}\\
\underline{\mathbf{v}} + (-\underline{\mathbf{v}}) &= (-\underline{\mathbf{v}}) + \underline{\mathbf{v}} + (-\underline{\mathbf{v}}) + \underline{\mathbf{w}} && \text{(Axiom 1)}\\
\underline{\mathbf{v}} + (-\underline{\mathbf{v}}) &= (-\underline{\mathbf{v}}) + \underline{\mathbf{v}} + 0 && \text{(Axiom 1 and Axiom 3)}\\
\underline{\mathbf{v}} + (-\underline{\mathbf{v}}) &= (-\underline{\mathbf{v}}) + \underline{\mathbf{v}} && \text{(Axiom 2)}
\end{aligned}
$$

Therefore the right inverse is also the left inverse. We modify axiom 1 to reflect this. We have Axiom 1: $\forall \underline{\mathbf{v}} \in V$, $\exists (-\underline{\mathbf{v}}) \in V$ such that $\underline{\mathbf{v}} + (-\underline{\mathbf{v}}) = (-\underline{\mathbf{v}}) + \underline{\mathbf{v}} = \underline{\mathbf{0}}$. Now we wish to show that the right identity is also the left identity.

That is $\forall \underline{\mathbf{v}} \in V$, $\underline{\mathbf{v}} + \underline{\mathbf{0}} = \underline{\mathbf{0}} + \underline{\mathbf{v}}$.

$$
\begin{aligned}
\underline{\mathbf{v}} + \underline{\mathbf{0}} &= \underline{\mathbf{v}} + \underline{\mathbf{0}} && \text{(Setting equal to itself)}\\
\underline{\mathbf{v}} + \underline{\mathbf{0}} &= \underline{\mathbf{v}} + (-\underline{\mathbf{v}}) + \underline{\mathbf{v}} && \text{(Axiom 1)}\\
\underline{\mathbf{v}} + \underline{\mathbf{0}} &= \underline{\mathbf{0}} + \underline{\mathbf{v}} && \text{(Axiom 1 and Axiom 3)}
\end{aligned}
$$

Therefore, the right additive identity is also the left additive identity. We modify Axiom 2 to reflect this. We have Axiom 2: $\exists \underline{\mathbf{0}} \in V$ such that $\forall \underline{\mathbf{v}} \in V$, $\underline{\mathbf{v}} + \underline{\mathbf{0}} = \underline{\mathbf{0}} + \underline{\mathbf{v}} = \underline{\mathbf{v}}$
Now we wish to show that vector addition is commutative.
That is $\forall \underline{\mathbf{a}}, \underline{\mathbf{b}} \in V$, $\underline{\mathbf{a}} + \underline{\mathbf{b}} = \underline{\mathbf{b}} + \underline{\mathbf{a}}$.

$$\begin{aligned}
\underline{\mathbf{a}} + \underline{\mathbf{b}} &= \underline{\mathbf{a}} + \underline{\mathbf{b}} && \text{(Equal to itself.)} \\
(1+1)(\underline{\mathbf{a}} + \underline{\mathbf{b}}) &= (1+1)(\underline{\mathbf{a}} + \underline{\mathbf{b}}) && \text{(Multiply on left by (1+1))} \\
(1+1)\underline{\mathbf{a}} + (1+1)\underline{\mathbf{b}} &= 1(\underline{\mathbf{a}} + \underline{\mathbf{b}}) + 1(\underline{\mathbf{a}} + \underline{\mathbf{b}}) && \text{(Axiom 4)} \\
1\underline{\mathbf{a}} + 1\underline{\mathbf{a}} + 1\underline{\mathbf{b}} + 1\underline{\mathbf{b}} &= 1\underline{\mathbf{a}} + 1\underline{\mathbf{b}} + 1\underline{\mathbf{a}} + 1\underline{\mathbf{b}} && \text{(Axiom 4)} \\
\underline{\mathbf{a}} + \underline{\mathbf{a}} + \underline{\mathbf{b}} + \underline{\mathbf{b}} &= \underline{\mathbf{a}} + \underline{\mathbf{b}} + \underline{\mathbf{a}} + \underline{\mathbf{b}} && \text{(Axiom 5)} \\
(-\underline{\mathbf{a}}) + \underline{\mathbf{a}} + \underline{\mathbf{a}} + \underline{\mathbf{b}} + \underline{\mathbf{b}} + (-\underline{\mathbf{b}}) &= (-\underline{\mathbf{a}}) + \underline{\mathbf{a}} + \underline{\mathbf{b}} + \underline{\mathbf{a}} + \underline{\mathbf{b}} + (-\underline{\mathbf{b}}) \\
& && \text{(Adding same to left and right.)} \\
\underline{\mathbf{0}} + \underline{\mathbf{a}} + \underline{\mathbf{b}} + \underline{\mathbf{0}} &= \underline{\mathbf{0}} + \underline{\mathbf{b}} + \underline{\mathbf{a}} + \underline{\mathbf{0}} && \text{(Axiom 1)} \\
\underline{\mathbf{a}} + \underline{\mathbf{b}} + &= \underline{\mathbf{b}} + \underline{\mathbf{a}} && \text{(Axiom 2)}
\end{aligned}$$

Q.E.D.