MATHEMATICS 121, FALL 2013
LINEAR ALGEBRA WITH APPLICATIONS

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**Module #5, Proof:**
*Prove that in $\mathbb{R}^n$, a set of $n+1$ vectors cannot be linearly independent and a set of $n-1$ vectors cannot span.*

*A set of $n+1$ vectors in $\mathbb{R}^n$ cannot be linearly independent.* To see this, notice that if $n+1$ vectors are linearly independent, this means that there exists no non-trivial solutions for the following equation:

$$a_1\vec{\boldsymbol{v}}_1 + a_2\vec{\boldsymbol{v}}_2 + \cdots + a_n\vec{\boldsymbol{v}}_n + a_{n+1}\vec{\boldsymbol{v}}_{n+1} = 0$$

The above can be be transformed into the matrix equation $A\vec{\boldsymbol{x}} = \vec{\boldsymbol{0}}$ where the columns of $A$ consists of the $n+1$ vectors and the vector $\vec{\boldsymbol{x}}$ consists of $a_1, a_2, \cdots, a_n, a_{n+1}$. The matrix $A$ has $n+1$ columns and $n$ rows.
Since $A$ has more columns than rows, then $rref(A)$ must have at least one non-pivotal column (the number of pivotal columns is at most $n$). This indicates that there are free variables, and therefore an infinite number of solutions to the above system, contradicting the statement that the only solution is the trivial solution, and therefore our $n+1$ vectors cannot be linearly independent. $\square$

*A set of $n-1$ vectors in $\mathbb{R}^n$ cannot span $\mathbb{R}^n$.* To see this, notice that if $n-1$ vectors span $\mathbb{R}^n$, then there exists a solution to the following equation $\forall\vec{\boldsymbol{b}} \in \mathbb{R}^n$.

$$a_1\vec{\boldsymbol{v}}_1 + a_2\vec{\boldsymbol{v}}_2 + \cdots + a_{n-1}\vec{\boldsymbol{v}}_{n-1} = \vec{\boldsymbol{b}}$$

The above can be be transformed into the matrix equation $A\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}}$ where the columns of $A$ consists of the $n-1$ vectors and the vector $\vec{\boldsymbol{x}}$ consists of $a_1, a_2, \cdots, a_{n-1}$. The matrix $A$ has $n-1$ columns and $n$ rows.
Since $A$ has more rows than columns, then $rref(A)$ must have at least one row of zeros at the bottom (the number of non-zero rows is at most $n-1$). This indicates that system is not always consistent. In particular, let $E$ be the product of the elementary matrices necessary to row-reduce $A$. Then the vector $b \in \mathbb{R}^n = E^{-1}\vec{\boldsymbol{e}}_n$ will have a 1 in the $n$-th position when $[A \mid b]$ is row-reduced, and therefore the system will have no solution. Since $\vec{\boldsymbol{b}} \in \mathbb{R}^n$ but is not a linear combination of our $n-1$ vectors, the $n-1$ vectors cannot span $\mathbb{R}^n$. $\square$

Q.E.D.