MATHEMATICS 121, FALL 2013
LINEAR ALGEBRA WITH APPLICATIONS

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Module #14, Proof:

Let $X \subset \mathbb{R}^2$ be an open set, and consider $\mathbf{f} : X \to \mathbb{R}^2$. Let $\mathbf{x_0}$ be a point in X. Prove that $\mathbf{f}$ is continuous at $\mathbf{x_0}$ if and only if for every sequence $\mathbf{x_i}$ converging to $\mathbf{x_0}$,

$$\lim_{i \to \infty} \mathbf{f}(\mathbf{x_i}) = \mathbf{f}(\mathbf{x_0}).$$

You may use the non-standard terms "good sequence" and "bad sequence," assuming that they have been defined as in this module.

Your proof will be valid for $f : \mathbb{R}^n \to \mathbb{R}^m$, but the use of $\mathbb{R}^2$ will let you draw meaningful diagrams.

This is an iff proof. First, we prove that if f is continuous, then every sequence $\boldsymbol{x_i}$ converging to $\boldsymbol{x_0}$ is a "good sequence".

Proof. Now, we know that our function is continuous, so $\forall \epsilon > 0, \exists \delta > 0$ such that $|\mathbf{x} - \mathbf{x_0}| < \delta \implies |\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x_0})| < \epsilon$. Consider any sequence $\mathbf{x_i}$ which converges to $\mathbf{x_0} \in X$. We must show that the sequence $\mathbf{f}(\mathbf{x_i})$ converges to $\mathbf{f}(\mathbf{x_0})$, or in other words, that $\forall \epsilon > 0, \exists N$ such that $\forall n > N, |\mathbf{f}(\mathbf{x_n}) - \mathbf{f}(\mathbf{x_0})| < \epsilon$. To find this N, we first find the δ such that $|\mathbf{x_n} - \mathbf{x_0}| < \delta \implies |\mathbf{f}(\mathbf{x_n}) - \mathbf{f}(\mathbf{x_0}))| < \epsilon$. But then note that from the definition of convergence, we have that $\forall \epsilon > 0, \exists M$ such that $\forall m > M, |\mathbf{x}_m - \mathbf{x_0}| < \epsilon$. Clearly, the M that works when $\epsilon = \delta$ is sufficient, so $N = M$ and every sequence is a "good sequence". $\square$

Now, we wish to prove that if every sequence $\mathbf{x_i}$ converging to $\mathbf{x_0}$ is a "good sequence', then f is continuous. We do this by proving the contrapositive: if f is discontinuous, then there exists a sequence $\mathbf{x_i}$ converging to $\mathbf{x_0}$ which is a "bad sequence".

Proof. Because $\mathbf{f}$ is discontinuous at $\mathbf{x_0}$, we know that $\exists \varepsilon > 0$ such that $\forall \delta > 0, |\mathbf{x} - \mathbf{x_0}| < \delta \implies |\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x_0})| \geq \epsilon$. Now construct the sequence $\mathbf{x_i}$ that converges to $\mathbf{x_0}$, but pick each x_m such that $|x_m - x_0| < \delta = \frac{1}{m}$ which implies that $|f(x_m) - f(x_0)| \geq \epsilon_0$ $\square$

Q.E.D.