MATHEMATICS 121, FALL 2013
LINEAR ALGEBRA WITH APPLICATIONS

September 16, 2013
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Module #4, Proof:

Let R be a 3×3 orthogonal matrix, for which $R^T R = I$. Prove that
$R(\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}) = \det(R)(R\vec{\boldsymbol{u}} \times R\vec{\boldsymbol{v}})$

Construct the matrix $B = \left[R\vec{\boldsymbol{u}} \mid R\vec{\boldsymbol{v}} \mid \vec{\boldsymbol{w}}\right]$. Then:

$$\begin{aligned} R^T \left[R\vec{\boldsymbol{u}} \mid R\vec{\boldsymbol{v}} \mid \vec{\boldsymbol{w}}\right] &= \left[R^T R\vec{\boldsymbol{u}} \mid R^T R\vec{\boldsymbol{v}} \mid R^T\vec{\boldsymbol{w}}\right] \\ &= \left[\vec{\boldsymbol{u}} \mid \vec{\boldsymbol{v}} \mid R^T\vec{\boldsymbol{w}}\right] \end{aligned}$$

Taking the determinant of both sides, we have:

$$\begin{aligned} (\det R^T)(\det \left[R\vec{\boldsymbol{u}} \mid R\vec{\boldsymbol{v}} \mid \vec{\boldsymbol{w}}\right]) &= \det \left[\vec{\boldsymbol{u}} \mid \vec{\boldsymbol{v}} \mid R^T\vec{\boldsymbol{w}}\right] \\ \det(R)(R\vec{\boldsymbol{u}} \times R\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{w}}) &= \vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}} \cdot R^T\vec{\boldsymbol{w}} \end{aligned}$$

But we know that $\vec{\boldsymbol{a}} \cdot A\vec{\boldsymbol{b}} = A^T\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}}$, so taking $\vec{\boldsymbol{a}} = \vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{b}} = \vec{\boldsymbol{w}}$, we arrive at:

$$\det(R)(R\vec{\boldsymbol{u}} \times R\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{w}}) = R(\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}) \cdot \vec{\boldsymbol{w}}$$

We know this must hold true $\forall\vec{\boldsymbol{w}}$, in particular $\vec{\boldsymbol{e_1}}, \vec{\boldsymbol{e_2}}$, and $\vec{\boldsymbol{e_3}}$. Therefore:
$\det(R)(R\vec{\boldsymbol{u}} \times R\vec{\boldsymbol{v}}) = R(\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}})$

Q.E.D.