MATHEMATICS 121, FALL 2013
LINEAR ALGEBRA WITH APPLICATIONS

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Luis Perez

Module #2, Proof:

Suppose that linear transformation $T : F^n \to F^m$ is represented by the $m \times n$ matrix $[T]$.

- a. Suppose that the matrix $[T]$ is invertible. Prove that the linear transformation T is one-to-one and onto, hence invertible.
- b. Suppose that linear transformation T is invertible. Prove that its inverse S is linear and that the matrix of S is $[S] = [T]^{-1}$

The linear transformation T is invertible if its matrix is invertible. Let $S : F^m \to F^n$ be the linear transformation for which $[S] = [T]^{-1}$ (ie, the matrix of S is the inverse of $[T]$).
Then $\forall \vec{\boldsymbol{x}}_1, \vec{\boldsymbol{x}}_2 \in F^n$, if $T(\vec{\boldsymbol{x}}_1) = T(\vec{\boldsymbol{x}}_2)$ we have:

$$\vec{\boldsymbol{x}}_1 = ([S][T])\vec{\boldsymbol{x}}_1 = S(T(\vec{\boldsymbol{x}}_1)) = S(T(\vec{\boldsymbol{x}}_{\boldsymbol{2}})) = ([S][T]) = \vec{\boldsymbol{x}}_2$$

This means $\vec{\boldsymbol{x}}_1 = \vec{\boldsymbol{x}}_2$ and therefore T is one-to-one.
On the other hand, let $\vec{\boldsymbol{y}} \in F^m$. Then we have:

$$\vec{\boldsymbol{y}} = ([T][S])\vec{\boldsymbol{y}} = T(S(\vec{\boldsymbol{y}}))$$

This means that $\forall \vec{\boldsymbol{y}} \in F^m$, $\exists \vec{\boldsymbol{x}} \in F^n$ such that $T(\vec{\boldsymbol{x}}) = \vec{\boldsymbol{y}}$, specifically $\vec{\boldsymbol{x}} = S(\vec{\boldsymbol{y}})$ and T is onto. Taken together, this means T is invertible.

$\square$

If T is invertible, then its inverse S is linear and the matrix of S is the inverse of the matrix of T.
First, we prove that S is linear. That is, we wish to prove that $S(a\vec{\boldsymbol{x}} + b\vec{\boldsymbol{y}}) = aS(\vec{x}) + bS(\vec{\boldsymbol{y}})$ for all $vectx, \vec{\boldsymbol{y}} \in F^m$. Observe the following:

$$aS(\vec{\boldsymbol{x}}) + bS(\vec{\boldsymbol{y}}) = S(T(aS(\vec{\boldsymbol{x}}) + bS(\vec{\boldsymbol{y}}))) = S(aT(S(\vec{\boldsymbol{x}})) + bT(S(\vec{\boldsymbol{y}}))) = S(a\vec{\boldsymbol{x}} + a\vec{\boldsymbol{y}})$$

Therefore, S is linear.
Now we wish to show that the matrix $[S] = [T]^{-1}$. We observe the following:
$[S][T] = [ST] = [id] = I_n$ and $[T][S] = [TS] = [id] = I_m$.
Thus, $[T]$ is invertible and its inverse, $[T]^{-1}$ is $[S]$. $\square$

Q.E.D.