MATHEMATICS 121, FALL 2013
LINEAR ALGEBRA WITH APPLICATIONS

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Module #9, Proof:

Using the axioms for an affine plane, which include all the axioms for a two-dimensional abstract vector space, prove the following:

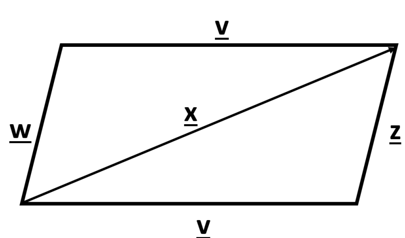
- Parallel lines have proportional direction vectors.
- Opposite sides of a parallelogram are equal vectors.

First, we want to show that parallel lines have proportional direction vectors. Notice that if two lines have proportional direction vectors, then they are parallel by definition. Show we only proof the forward direction. Begin by letting $L = p + s\underline{\boldsymbol{v}}$ and $L' = p' + s'\underline{\boldsymbol{v}}'$. Because they are parallel, we know that $\forall s, s' \in \mathbb{R}$, $p + s\underline{\boldsymbol{v}} \neq p' + s'\underline{\boldsymbol{v}}'$. Rearranging, we have that

$$\forall s, s' \in \mathbb{R}, p - p' \neq s'\underline{\boldsymbol{v}}' - s\underline{\boldsymbol{v}}'$$

Therefore, $\underline{\boldsymbol{v}}$ and $\underline{\boldsymbol{v}}'$ cannot span our two-dimensional space V because their linear combinations do not include $p - p' \in V$. This means they are linearly dependent, and therefore have proportional direction vectors.

Next, we wish to show that opposite sides of a parallelogram are equal vectors. We begin by proving that if a quadrilateral has two opposite sides that are equal vectors, then it is a parallelogram. The below diagram will be helpful:

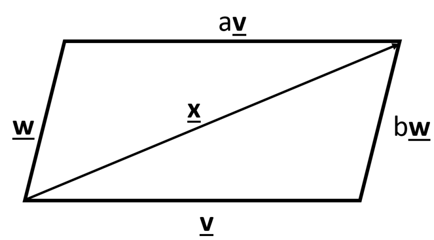


We can see that $\underline{\boldsymbol{x}} = \underline{\boldsymbol{v}} + \underline{\boldsymbol{z}}$ and that it also equals $\underline{\boldsymbol{w}} + \underline{\boldsymbol{v}}$. Setting the two expressions for $\underline{\boldsymbol{x}}$ equal, we have:

$$\underline{\boldsymbol{v}} + \underline{\boldsymbol{z}} = \underline{\boldsymbol{w}} + \underline{\boldsymbol{v}}\underline{\boldsymbol{z}} = \underline{\boldsymbol{w}}$$

Therefore, our quadrilateral is a parallelogram.

From the definition of parallelogram, we know that opposite sides are proportional. Therefore we have the following schematic:



We can see that once again $\underline{\boldsymbol{x}}$ can be represented in two ways. We have:

$$\underline{\boldsymbol{x}} = \underline{\boldsymbol{v}} + b\underline{\boldsymbol{w}}$$

and

$$\underline{\boldsymbol{x}} = a\underline{\boldsymbol{v}} + \underline{\boldsymbol{w}}$$

Equating the two sides, we have:

$$\underline{\boldsymbol{v}} + b\underline{\boldsymbol{w}} = a\underline{\boldsymbol{v}} + \underline{\boldsymbol{w}}$$

and

$$(b-1)\underline{\boldsymbol{w}} = (a-1)\underline{\boldsymbol{v}}$$

If $a \neq 1$, then $\underline{\boldsymbol{v}} = \frac{b-1}{a-1}\underline{\boldsymbol{v}}$ and the two cannot span the entire plane contradicting the fact that $\underline{\boldsymbol{v}}$ and $\underline{\boldsymbol{w}}$ are linearly independent. A similar argument in the case of b shows that both a and b must be 1, and therefore opposite sides of a parallelogram are equal vectors.

Q.E.D.