MATHEMATICS 121, FALL 2013
LINEAR ALGEBRA WITH APPLICATIONS

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Module #3, Proof:

For a 3×3 matrix A, define $\det(A)$ *in terms of the cross and dot products of the columns of the matrix. Then, using the definition of matrix multiplication and the linearity of the dot and cross products, prove that* $\det(AB) = \det(A)\det(B)$. We begin by making the following observation about

the first column of C, represented as $\vec{\boldsymbol{c}}_1$. Note that $b_{i,j}$ is the element in the i-th row and j-th column of B.

$$\vec{\boldsymbol{c}}_1 = A\vec{\boldsymbol{b}}_1 = A\sum_{i=1}^{3} b_{i,1}\vec{\boldsymbol{e}}_i = \sum_{i=1}^{3} b_{i,1}A\vec{\boldsymbol{e}}_i = \sum_{i=1}^{3} b_{i,1}\vec{\boldsymbol{a}}_i$$

The same is true for the second and third columns of C, so from the above, we have:

$$\det(C) = \vec{\boldsymbol{c}}_1 \times \vec{\boldsymbol{c}}_2 \cdot \vec{\boldsymbol{c}}_3 = (\sum_{i=1}^{3} b_{i,1}\vec{\boldsymbol{a}}_i) \times (\sum_{j=1}^{3} b_{j,2}\vec{\boldsymbol{a}}_j) \cdot (\sum_{i=1}^{3} b_{k,1}\vec{\boldsymbol{a}}_k)$$

We can then use the distributive law for the dot and cross products to arrive at:

$$\det(C) = (\sum_{i=1}^{3} b_{i,1})(\sum_{j=1}^{3} b_{j,2})(\sum_{i=1}^{3} b_{k,1})(\vec{\boldsymbol{a}}_i \times \vec{\boldsymbol{a}}_j \cdot \vec{\boldsymbol{a}}_k)$$

This sum consists of 27 terms, but all terms for which an index is repeated are zero, and therefore there are only six possibly non-zero terms which all involve $\vec{\boldsymbol{a}}_1 \times \vec{\boldsymbol{a}}_2 \cdot \vec{\boldsymbol{a}}_3$, thee with a plus sign and three with a negative sign. Therefore:

$$\det(C) = f(B)(\vec{\boldsymbol{a}}_1 \times \vec{\boldsymbol{a}}_2 \cdot \vec{\boldsymbol{a}}_3) = f(B)\det(A)$$

$f(B)$ is some is some function of the entries of B. The above formula is valid for all A, specifically $A = I$ for which $\det(A) = 1$, which means $\det(B) = f(B)\det(A) = f(B)$. Therefore, $\det(C) = \det(A)\det(B)$

Q.E.D.