MATHEMATICS 121, FALL 2013
LINEAR ALGEBRA WITH APPLICATIONS

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Module #8, Proof:

Suppose that you have a cubic polynomial equation with three distinct roots x_1, x_2, x_3. Explain what is a "symmetric polynomial" and an "elementary symmetric polynomial" in these roots, and show that each elementary symmetric polynomial is equal in magnitude to a coefficient in the equation. Prove that the symmetric polynomials of degree k form an abstract vector space whose dimension is equal to the number of distinct ways of writing k as a sum of 3 or fewer integers and that this space has a basis in which each basis vector is a product of powers of elementary symmetric polynomials. Thereby prove that any symmetric polynomial, elementary or not, is a function of the coefficients in the equation.

A "symmetric polynomial" is simply a polynomial that is a function of all three roots x_1, x_2, x_3, constructed in such a manner that switching subscripts won't affect the overall behavior of the polynomial. An "elementary symmetric polynomial" is a symmetric polynomial such that none of the roots are raised to a power greater than one.

To see that each elementary symmetric polynomial is equal in magnitude to a coefficient in the equation, note that we can write a cubic polynomial as $p(x) = x^3 + Ax^2 + Bx + C$. Furthermore, we can write this same polynomial in factored form as $p(x) = (x - x_1)(x - x_2)(x - x_3)$. Multiplying out these terms can never generate a term with a root raised to a power greater than one because each root occurs only once. Furthermore, the roots can be written in any order, not affecting the resulting coefficients, and so the coefficients in the equation are each elementary symmetric polynomials. We can very this by multiplying out $p(x)$.

$$p(x) = x^3 - (x_1 + x_2 + x_3)x^2 + (x_1x_2 + x_2x_3 + x_1x_3)x - (x_1x_2x_3) = x^3 + Ax^2 + Bx + C$$

Evidently, each elementary symmetric polynomial, $S_1 = x_1 + x_2 + x_3, S_2 = x_1x_2 + x_1x_2 + x_2x_3$, and $S_3 = x_1x_2x_3$ is equal in magnitude to a coefficient in the equation.

Next, we show that symmetric polynomials of degree k form an abstract vector space. First, note that it is a known fact that polynomials of degree k form an abstract vector space, P_k, and therefore the symmetric polynomials of degree k satisfy the vector space axioms if they form an abstract subspace of P_k. Note that the space of symmetric polynomials is closed under scalar multiplication since multiplying by a scalar does not affect the roots, degree, nor the overall behavior of the polynomial. Additionally, it is closed under vector addition since adding two symmetric polynomials leads to a polynomial that is also symmetric (it is still a function of the roots, and you can switch indexes without affecting the polynomial), and of the same degree (except for 0, but this is trivially also a member of the subspace).

Next, we show that degree of this abstract vector space is equal to the number of distinct ways of writing k as the sum of three or fewer integers. To do this, we create a bijection between the symmetric polynomials of degree k and the number of ways of writing k as a sum of three non-negative integers. Notice that a symmetric polynomial of degree k is uniquely determined by it's first term $x_1^a x_2^b x_3^c$ where $a + b + c = k$. This is because each symmetric polynomial must be of the form $p(x) = x_1^a x_2^b x_3^c + x_1^c x_2^a x_3^b + x_1^b x_2^c x_3^a$ in order for it to *be* symmetric. So looking at just the first term, we can uniquely determine the entire polynomial. For degree 5, for example, we have 5 symmetric polynomials uniquely determined by their first term, which can be mapped bijectively to the five unique set of non-negative integers which can sum to 5 as follows:

$$\begin{aligned}
p_1 &= x_1^5 x_2^0 x_3^0 + \cdots + x_1^0 x_2^0 x_3^5 \Leftrightarrow 5+0+0=5 \\
p_2 &= x_1^4 x_2^1 x_3^0 + \cdots + x_1^1 x_2^0 x_3^4 \Leftrightarrow 4+1+0=5 \\
p_3 &= x_1^3 x_2^1 x_3^1 + \cdots + x_1^1 x_2^1 x_3^3 \Leftrightarrow 3+1+1=5 \\
p_4 &= x_1^3 x_2^2 x_3^0 + \cdots + x_1^2 x_2^0 x_3^3 \Leftrightarrow 3+2+0=5 \\
p_5 &= x_1^2 x_2^2 x_3^1 + \cdots + x_1^2 x_2^1 x_3^2 \Leftrightarrow 2+2+1=5
\end{aligned}$$

The above generalizes as explained.

Now we wish to show that this vector space also has a basis consisting of the product of powers of elementary symmetric polynomials. Denote the elementary symmetric polynomials as $S_1 = x_1 + x_2 + x_3, S_2 = x_1x_2 + x_2x_3 + x_1x_3$, and $S_3 = x_1x_2x_3$. We can therefore write any polynomial consisting of powers of elementary symmetric polynomials as $p_{l,m,n} = (S_1)^l(S_2)^m(S_3)^n$. These will be the vectors in the new basis, and our task is now to construct enough of them such that they form a basis of our vector space. Let us denote the vectors in the old basis as $x_1^a x_2^b x_3^c + \cdots x_1^c x_2^a x^3 b$, since as noted above these form a basis and are uniquely determined by the first term. Then notice that $(S_1)^l(S_2)^m(S_3)^n = (x_1+x_2+x_3)^l + (x_1x_2+x_2x_3+x_1x_3)^m(x_1x_2x_3)^n = x_1^{l+m+n} x_2^{m+n} x_3^n + \cdots$, so the new basis vectors can be expressed in terms of the old basis vectors when multiplied out. Furthermore, if for each old basis vector (uniquely determined by the set of numbers a, b, c), we pick a new vector $p_{l,m,n}$ such that $n = c, m = b - c, l = a - b$ to be in our new basis, then we will have constructed an equivalent basis of the powers of elementary symmetric polynomials, each of which can be represented as a linear combination of our old basis.

Therefore, any symmetric polynomial can be expressed as a linear combination of powers of elementary symmetric polynomials, and vice versa (this is just a change of basis problem since we now have two equivalent basis as constructed above). This means that every symmetric polynomials is a function of the coefficients in the equation (because we can represent any symmetric polynomial as the linear combination of powers of elementary symmetric polynomials).

For a concrete example, let us continual with our basis of degree 5 symmetric polynomials. With the above outlined construction, we have:

$$\begin{aligned}
\text{Old Basis} &\Leftrightarrow \text{New Basis}\\
x_1^5x_2^0x_3^0+\cdots &\Leftrightarrow (x_1+x_2+x_3)^5\\
x_1^4x_2^1x_3^0+\cdots &\Leftrightarrow (x_1+x_2+x_3)^3(x_1x_2+x_1x_3+x_2x_3)\\
x_1^3x_2^2x_3^0+\cdots &\Leftrightarrow (x_1+x_2+x_3)(x_1x_2+x_1x_3+x_2x_3)^2\\
x_1^3x_2^1x_3^1+\cdots &\Leftrightarrow (x_1+x_2+x_3)^2(x_1x_2x_3)\\
x_1^2x_2^2x_3^1+\cdots &\Leftrightarrow (x_1x_2+x_1x_3+x_2x_3)(x_1x_2x_3)
\end{aligned}$$

Q.E.D.