MATHEMATICS 121, FALL 2013
LINEAR ALGEBRA WITH APPLICATIONS

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Module #1, Proof:

1.1 Prove that the positive rational numbers are countably infinite but that the real numbers between 0 and 1 are uncountably infinite.

This is a two part proof. We beginning by proving that the positive rational numbers are finite.
In order to prove that the positive rational numbers are real, all we need to do is crate a bijection between them and $\mathbb{N}$. To do this, consider the following table:

$$\begin{array}{c|cccccccc}
 & 1 & & 2 & & 3 & & 4 & \cdots \\ \hline
1 & \frac{1}{1} & \rightarrow & \frac{2}{1} & & \frac{3}{1} & \rightarrow & \frac{4}{1} & \cdots \\
 & & \swarrow & & \nearrow & & \swarrow & & \\
2 & \frac{1}{2} & & \text{N/A} & & \frac{3}{2} & & \text{N/A} & \cdots \\
 & \downarrow & \nearrow & & \swarrow & & \nearrow & & \\
3 & \frac{1}{3} & & \frac{2}{3} & & \text{N/A} & & \frac{3}{4} & \cdots \\
 & & \swarrow & & \nearrow & & \swarrow & & \\
4 & \frac{1}{4} & & \text{N/A} & & \frac{3}{4} & & \text{N/A} & \cdots \\
 & \vdots & \nearrow & \vdots & \swarrow & \vdots & \nearrow & \vdots &
\end{array}$$

We can follow the arrows and assign each rational number a natural number depending on when we encounter it along our path. We skip over rationals which can be simplified. Therefore, since there is bijection between the set of positive rational numbers and natural numbers, we have proved that $\mathbb{Q}$ is countable.
Now we prove that the set of real numbers between 0 and 1 is uncountably infinite. Assume the set is countable and can therefore be listed as shown below:

$$A_1 = 0.\mathbf{1}453\cdots$$
$$A_2 = 0.5\mathbf{1}95\cdots$$
$$A_3 = 0.14\mathbf{5}1\cdots$$
$$A_4 = 0.988\mathbf{1}\cdots$$
$$\vdots$$

Now considered the decimal between 0 and 1 defined to the right of the decimal point with the bolded values changed in some arbitrary way (ie. add 1). This new number $A_k = 0.2262\cdots$ cannot possibly be in the list for it differs from the A_i decimal at the i-th decimal place. We therefore have a contradiction in our assumption that the set is countable. It must therefore be uncountable.
Q.E.D.