MATHEMATICS 121, FALL 2013
LINEAR ALGEBRA WITH APPLICATIONS

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Module #11, Proof:

Prove that if $\vec{\mathbf{v}}_1, \cdots, \vec{\mathbf{v}}_n$ *are eigenvectors of* $A : \mathbb{R}^n \to \mathbb{R}^n$ *with distinct eigenvalues* $\lambda_1 \cdots \lambda_n$, *they are linearly independent. First do the proof by the "least number principle" (Theorem 2.7.4 in Hubbard), then reformulate the proof as a standard inductive argument.*

Following the instructions, we begin by proofing this with the least number principle.

Proof. Assume that there exists a smallest number j for which the vector $\vec{\boldsymbol{v}}_j$ is linearly dependent on the previous vectors $\vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2, \cdots, \vec{\boldsymbol{v}}_{j-1}$. It must then be the case that:

$$\vec{\boldsymbol{v}}_j = a_1\vec{\boldsymbol{v}}_1 + a_2\vec{\boldsymbol{v}}_2 + \cdots + a_{j-1}\vec{\boldsymbol{v}}_{j-1}$$

where not all a_i are zero because the zero vector cannot be an eigenvector. Then we have the following

$$\begin{aligned} \lambda_j\vec{\boldsymbol{v}}_j &= a_1\lambda_j\vec{\boldsymbol{v}}_1 + a_2\lambda_j\vec{\boldsymbol{v}}_2 + \cdots + a_{j-1}\lambda_j\vec{\boldsymbol{v}}_{j-1} && \text{(Multiply by } \lambda_j\text{)} \\ (-)\ \lambda_j\vec{\boldsymbol{v}}_j &= a_1\lambda_1\vec{\boldsymbol{v}}_1 + a_2\lambda_2\vec{\boldsymbol{v}}_2 + \cdots + a_{j-1}\lambda_{j-1}\vec{\boldsymbol{v}}_{j-1} && \text{(Apply } A \text{ to original)} \\ \hline 0 &= a_1(\lambda_j - \lambda_1)\vec{\boldsymbol{v}}_1 + a_2(\lambda_j - \lambda_2)\vec{\boldsymbol{v}}_2 + \cdots + a_{j-1}(\lambda_j - \lambda_{j-1})\vec{\boldsymbol{v}}_{j-1} \end{aligned}$$

Yet, because all λ_i are distinct and at least some of the a_i are non-zero, the above implies that the vectors $\vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2 \cdots \vec{\boldsymbol{v}}_{j-1}$ are linearly dependent! A contradiction, and therefore our assumption that there exists a least value j such that $\vec{\boldsymbol{v}}_j$ is a linear combination of the previous eigenvectors must be false. □

For the induction proof, we have the following:

Proof. Base case for $k = 1$: $\vec{\boldsymbol{v}}_1$ is clearly linearly independent since it is the only eigenvector, and eigenvectors cannot be 0.
Inductive step: $\vec{\boldsymbol{v}}_1, \cdots \vec{\boldsymbol{v}}_k$ are linearly independent. Then it must be the case that $\vec{\boldsymbol{v}}_{k+1}$ must also be linearly independent. To see this, assume that this is not the case. Then we have that

$$\vec{\boldsymbol{v}}_{k+1} = a_1\vec{\boldsymbol{v}}_1 + a_2\vec{\boldsymbol{v}}_2 + \cdots + a_k\vec{\boldsymbol{v}}_k$$

where not all a_i are zero because the zero vector cannot be an eigenvector. Then we have the following:

$$
\begin{array}{rll}
& \lambda_{k+1}\boldsymbol{\vec{v}}_{k+1} = a_1\lambda_k\boldsymbol{\vec{v}}_1 + a_2\lambda_k\boldsymbol{\vec{v}}_2 + \cdots + a_k\lambda_k\boldsymbol{\vec{v}}_k & \text{(Multiply by } \lambda_k\text{)} \\
(-) & \lambda_{k+1}\boldsymbol{\vec{v}}_{k+1} = a_1\lambda_1\boldsymbol{\vec{v}}_1 + a_2\lambda_2\boldsymbol{\vec{v}}_2 + \cdots + a_k\lambda_k\boldsymbol{\vec{v}}_k & \text{(Apply } A \text{ to original)} \\
\hline
& 0 = a_1(\lambda_{k+1} - \lambda_1)\boldsymbol{\vec{v}}_1 + a_2(\lambda_{k+1} - \lambda_2)\boldsymbol{\vec{v}}_2 + \cdots + a_k(\lambda_{k+1} - \lambda_k)\boldsymbol{\vec{v}}_k &
\end{array}
$$

This is a contradiction, because our inductive step guarantees the first k eigenvectors are linearly independent. Therefore, our assumption that $\boldsymbol{\vec{v}}_{k+1}$ is linearly independent must be false. $\square$

Q.E.D.