Exercise (Slide 11)

$$P(X>1) = 1 - P(X \leq 1) =$$

$$= 1 - F(1) = 1 - (1 - e^{-1^2}) =$$

$$= 1 - 1 + e^{-1} = 0.368$$

Problem 7.1

$$F(0 \leq X \leq 1) = 1.0 \Rightarrow$$

$$\int_0^1 f(x)\,dx = 1.0 \Rightarrow \int_0^1 cx^3\,dx = 1.0 \Rightarrow$$

$$\left.\frac{cx^4}{4}\right|_0^1 = 1 \Rightarrow \frac{c}{4} - 0 = 1 \Rightarrow \boxed{c = 4}$$

$$P(0.4 < X < 0.5) = F(0.5) - F(0.4) =$$

$$\frac{\cancel{(4)}(0.5)^4}{\cancel{4}} - \frac{\cancel{(4)}(0.4)^4}{\cancel{4}} = 0.5^4 - 0.4^4 = 0.0369$$

Problem 7.2

$$P(X>1, Y<1) = \int_0^1 \int_1^\infty 2e^{-x} e^{-2y} \, dx \, dy$$

$$= \int_0^1 2e^{-2y} \left(-e^{-x} \Big|_1^\infty \right) dy =$$

$$= \int_0^1 2e^{-2y} \left(-0 + e^{-1} \right) dy =$$

$$= 2e^{-1} \int_0^1 e^{-2y} \, dy = 2e^{-1} \left(-\frac{e^{-2x}}{2} \Big|_0^1 \right) =$$

$$\boxed{= e^{-1} \left(-e^{-2} + 1 \right)}$$

$$P(X<Y) = \iint_{x<y} 2e^{-x} e^{-2y} \, dx \, dy =$$

$$= \int_0^\infty \int_0^y 2e^{-x} e^{-2y} \, dx \, dy = \int_0^\infty 2e^{-2y} \left(1 - e^{-y} \right) dy$$

$$= \int_0^\infty 2e^{-2y} \, dy - \int_0^\infty 2e^{-3y} \, dy =$$

$$= -\frac{2e^{-2y}}{2}\Big|_0^{\infty} - \left(-\frac{2e^{-3y}}{3}\Big|_0^{\infty} \right) =$$

$$= -0 + 1 - \left(-0 + \frac{2}{3} \right) = \frac{1}{3}$$

Problem 7.3

$$F_{X/Y}(a) = \iint_{X/Y \leq a} f(x,y)\, dx\, dy =$$

$$= \iint_{X/Y \leq a} e^{-x} e^{-y}\, dx\, dy = \int_0^{\infty} \int_0^{ay} e^{-x} e^{-y}\, dx\, dy$$

$$= \int_0^{\infty} \left(1 - e^{-ay}\right) e^{-y}\, dy =$$

$$= \left[-e^{-y} + \frac{e^{-(a+1)y}}{a+1} \right]_0^{\infty} = 1 - \frac{1}{a+1}$$

$$f_{X/Y}(a) = \frac{d F_{X/Y}(a)}{da} = \frac{1}{(a+1)^2}$$

Problem 7.4

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} =$$

$$= \frac{f(x,y)}{\int_{-\infty}^{\infty} f(x,y)\,dx} = \frac{\cancel{\frac{12}{5}}\, x\,(2-x-y)}{\int_0^1 \cancel{\frac{12}{5}}\, x\,(2-x-y)\,dx}$$

$$= \frac{x(2-x-y)}{(2-y)\int_0^1 x\,dx - \int_0^1 x^2\,dx} = \frac{x(2-x-y)}{\frac{(2-y)\,x^2}{2}\Big|_0^1 - \frac{x^3}{3}\Big|_0^1}$$

$$= \frac{x(2-x-y)}{\frac{2-y}{2} - \frac{1}{3}} = \frac{x(2-x-y)}{\frac{2}{3} - \frac{y}{2}} = \frac{6x(2-x-y)}{4-3y}$$