Example (Slide 3)

$p(1) = p(2) = \dots = p(6) = \frac{1}{6}$

$E[X] = 1\left(\frac{1}{6}\right) + 2\left(\frac{1}{6}\right) + \dots + 6\left(\frac{1}{6}\right) = \frac{7}{2}$

Example (Slide 5)

$E[X] = \int_0^{1.5} x \frac{1}{1.5}\, dx = \frac{x^2}{3}\Big|_0^{1.5} = 0.75$

Example (Slide 7)

Y: cost

$E[Y] = \int y \overset{?}{f_Y(y)}\, dy$

$F_Y(a) = P(Y \leq a) = P(X^3 \leq a) = P(X \leq a^{1/3})$

$= \int_0^{a^{1/3}} dx = a^{1/3} - 0 = a^{1/3}$

$$f_Y(a) = \frac{dF_Y}{da} = \frac{1}{3} a^{-2/3}$$

$$E[Y] = \int_0^1 a f_Y(a)\, da =$$

$$= \int_0^1 a \frac{1}{3} a^{-2/3}\, da = \frac{1}{3}\int_0^1 a^{1/3}\, da$$

$$= \frac{1}{\cancel{3}} \frac{\cancel{3}}{4} a^{4/3} \Big|_0^1 = \frac{1}{4}$$

<u>Proof of $E[aX+b]$</u>

$$E[aX+b] = \sum_x (ax+b)\, p(x) =$$

$$= \sum_x ax\, px + \sum_x b\, p(x) =$$

$$= a\sum_x x\, p(x) + b\sum_x p(x) =$$

$$= a\, E[x] + b\,(1) = a\, E[x] + b$$

Problem 8.1

$$\int_0^1 f_X(x)\,dx = 1 \Rightarrow \int_0^1 (a+bx^2)\,dx = 1 \Rightarrow$$

$$ax\Big|_0^1 + b\frac{x^3}{3}\Big|_0^1 = 1 \Rightarrow a + \frac{b}{3} = 1$$

$$E[X] = \int_0^1 x\, f_X(x)\,dx = \frac{3}{5} \Rightarrow$$

$$\int_0^1 (ax + bx^3)\,dx = \frac{3}{5} \Rightarrow$$

$$\int_0^1 \left(x - \frac{bx}{3} + bx^3\right)dx = \frac{3}{5} \Rightarrow$$

$$\frac{x^2}{2}\Big|_0^1 - \frac{bx^2}{6}\Big|_0^1 + \frac{bx^4}{4}\Big|_0^1 = \frac{3}{5} \Rightarrow$$

$$\frac{1}{2} - \frac{b}{6} + \frac{b}{4} = \frac{3}{5} \Rightarrow \frac{5-6}{10} = \frac{2b-3b}{12}$$

$$\Rightarrow -\frac{1}{10} = -\frac{b}{12} \Rightarrow b = \frac{12}{10} = \frac{6}{5}$$

$$a = \frac{3}{5}$$

Alternative formula for Var(X)

$$Var(X) = E[(X-\mu)^2] =$$

$$E[X^2 - 2\mu X + \mu^2] =$$

$$= E[X^2] - E[2\mu X] + E[\mu^2] =$$

$$= E[X^2] - 2\mu E[X] + \mu^2 =$$

$$= E[X^2] - 2\mu^2 + \mu^2 = E[X^2] - \mu^2$$

$$Var(X) = E[X^2] - (E[X])^2$$

Var(aX+b)

$$= E[(aX+b)^2] - (E[aX+b])^2 =$$

$$= E[a^2X^2 + 2abX + b^2] - (aE[X]+b)^2 =$$

$$= a^2 E[X^2] + \cancel{2abE[X]} + \cancel{b^2} - a^2(E[X])^2 - \cancel{2abE[X]}$$

$$\cancel{-b^2}$$

$$= a^2 E[x^2] - a^2(E[x])^2 = a^2 Var(x)$$

Problem 8.2

$$X_i = \begin{cases} 1 \\ 0 \end{cases}$$

$$E[X] = \sum x_i p_i = \sum \left[(1)p + (0)(1-p) \right]$$

$$= \sum p = np$$

$$Var(x) = E[x^2] - (E[x])^2 = np - (np)^2$$

$$= np(1-np)$$