The Equations of Stellar Structure and Evolution in Neutron Stars

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December 6, 2013

Overview

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   - Non-Relativistic Regime: Newtownian Mechanics
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   - Relativistic Regime: The TOV Equations
2. The Equation of State
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   - Numerical Methods
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Motivation

Why study neutron stars?

- The equations of stellar structure and evolution connect an equation of state (EOS) to set of mass and radius points
- Can take observed data of mass and radius and obtain an EOS
- Can use an experimental EOS to produce a family of mass-radius data

Equations of Stellar Structure: Newtonian Picture

Determine the equations of stellar structure and evolution for single star without any significant contributing forces other than pressure and gravity acting upon the mass element.

$$P(r)dA - [P(r) + dP]dA - \rho(r)dAdrg(r) = 0 \tag{1}$$

- Assume hydrostatic equilibrium within the star

$$\frac{dP(r)}{dr} = -\rho(r)g(r) \tag{2}$$

- This implies that the pressure, P increases as the radial coordinate decreases.

Equations of Stellar Structure: Newtonian Picture

The Newtonian equations for stellar structure assuming hydrostatic equilibrium:

$$\frac{dP(r)}{dr} = -\frac{GM\rho}{r^2} \tag{3}$$

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho \tag{4}$$

- To fully solve eqs. (3, 4) for any point in the star an equation of state (EOS) is needed.

Equations of Stellar Structure: Neutron Stars

- Moving to more complex stellar structures eqs. (3, 4) need to be altered.
- The Newtonian gravitational theory is no longer valid for the case of neutron stars.
- Neutron stars are compact objects and General Relativitistic theory takes over.

Equations of Stellar Structure: Neutron Stars

- The idea of compact stars consisting of dense atomic nuclei was proposed by Landau in 1932, before the discovery of the neutron [Landau, 1932].
- Shortly after the discovery of the neutron, the prediction of such stars was confirmed while seeking an explanation for the origin of supernovae [Baade, 1934].
- The first models of neutron stars emerged in 1939 formulated by Tolman Oppenheimer and Volkoff, and would become known as the TOV equations [Oppenheimer, 1939].

Equations of Stellar Structure: General Relativistic Picture

Eqs. (4, 3) must be altered for the case of neutron stars.

- We begin with the Einstein equation

$$G^{\mu\nu} = 8\pi T^{\mu\nu}, \tag{5}$$

  where $G^{\mu\nu}$ is the Einstein tensor and $T^{\mu\nu}$ is the stress energy tensor.

We make the following assumptions

- Spherical Symmetry
- Perfect fluid
- Non-rotating body in hydrostatic equilibrium

Equations of Stellar Structure: General Relativistic Picture

Using the above information and employing known constaints on the Einstein tensors we arrive at the following equations

$$\frac{dP(r)}{dr} = -\frac{Gm\epsilon}{r^2}(1+\frac{P}{\epsilon c^2})(1+\frac{4\pi r^3 P}{mc^2})(1-\frac{2Gm}{rc^2})^{-1}. \tag{6}$$

$$\frac{dM(r)}{dr} = 4\pi\epsilon r^2, \tag{7}$$

These are known the equations known as the TOV equations [Oppenheimer, 1939].

Solving the TOV Equations: Numerical Methods

To solve the differential eqs. (7, 6) we will use the method of Runge Kutta integration.

To do this we combine several Euler-like steps assuming that we are given the IVP

$$\frac{dx}{dy} = f(x, y), \tag{8}$$

with initial values $y = y_0$ at $x = x_0$. Then the differential equation can be solved numerical expanded Euler steps, with some step-size δx.

Solving the TOV Equations: Numerical Methods

To perform one step in the integration of eq. (8) the following are calculated

$$k_1 = \delta x f(x_0, y_0), \tag{9}$$
$$k_2 = \delta x f(x_0 + 0.5\delta x, y_0 + 0.5k_1), \tag{10}$$
$$k_3 = \delta x f(x_0 + 0.5\delta x, y_0 + 0.5k_2), \tag{11}$$
$$k_4 = \delta x f(x_0 + \delta x, y_0 + k_3). \tag{12}$$

Thus the next step x_{n+1} is given by

$$x_{n+1} = x_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{13}$$

The Equation of State: Polytrope

To solve the equations of stellar structure for either the Newtonian or relativistic case we need a relationship between the pressure and density at points within the star.
The interior of a neutron star has varying densities in different regions.

- Core:
  In the core throughout computation a polytrope is used

$$P = K\epsilon^{\gamma} \tag{14}$$

- Crust:
  In the crust an interpolation function was used with data from Steiner et. al

$$\rho = \rho_1 + \frac{\rho_2 - \rho_1}{P_2 - P_1}(P - P_1) \tag{15}$$

Solving the TOV Equations: Code Structure

Solving eqs. (7, 6) to obtain a mass-radius curve

Boundary Conditions:

- $M(r = 0) = 0$
- $\epsilon(r = 0) = \epsilon_c$
- $P(r = 0) = P_c$

Solving the TOV Equations: Code Structure

Algorithm for solving the TOV equations numerically

- Using Runge Kutta advance one steps: M_{i+1} and P_{i+1}
- From P_{i+1} use defined EOS to get ϵ_{i+1}
- These values are the initial points for the next iteration
- If $P_{k+1} \leq P_{stop}$ then stop
  $\implies M_k = M(r_k) = M$ and $r_k = R$

Solving the TOV Equations: Results

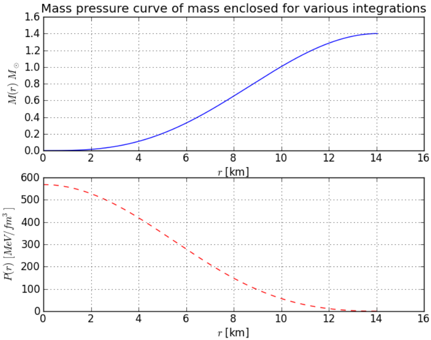

Figure: Mass enclosed curve for a $1.4M_\odot$ mass neutron star with a total radius of 14 km. Obtained for one integration of the TOV equations.

Solving the TOV Equations: Results

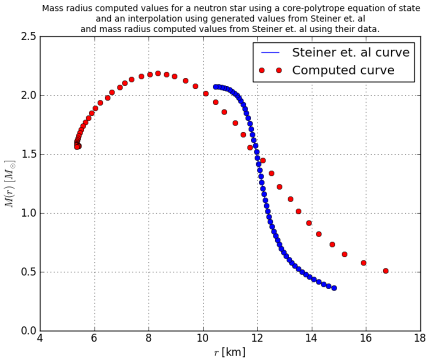

Figure: Mass-radius curve for computed solutions to TOV eqs. and Steiner et. al data values [Steiner et. al, 2010].

Conclusions: Future Considerations

For future work the following ideas would be implemented

- Implement a more realistic EOS in the core (dependent on baryon number, nuclear interactions, etc...)
- Use Monte Carlo methods to obtain constraints on the EOS based on MCMC fit results

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