- X smooth projective variety $/\mathbb{C}$ of complex dimension n, then the Hodge decomposition theorem says that $H^k(X;\mathbb{C}) \cong \bigoplus_{p+q=k} H^p(X, \Omega_X^q)$

  This decomposition comes from the resolution of sheaves:

$$\mathbb{C}_X \longrightarrow \left[\Omega_X^0 \xrightarrow{\partial} \Omega_X^1 \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Omega_X^n \to 0\right]$$

  $\mathbb{C}_X$: constant sheaf

  Ω_X^i: sheaf of holomorphic i-forms

  We can further resolve Ω_X^i:

$$\Omega_X^i \longrightarrow \left[\Omega_X^{i,0} \xrightarrow{\bar\partial} \Omega_X^{i,1} \xrightarrow{\bar\partial} \Omega_X^{i,2} \xrightarrow{\bar\partial} \cdots\right]$$

  $\Omega_X^{i,j}$ = sheaf of type (i,j) differential forms (with i dz's and j $d\bar z$'s)

  - $H^p(X, \Omega_X^i)$ are the cohomologies of the $\bar\partial$ complex above.

- The Hodge filtration $F^p\Omega^* = [\cdots 0 \to \Omega^p \xrightarrow{\partial} \Omega^{p+1} \xrightarrow{\partial} \cdots]$ on $\Omega^* = [\Omega^0 \xrightarrow{\partial} \Omega^1 \xrightarrow{\partial} \cdots]$ induces the Hodge-to-de Rham spectral sequence, whose E_1-page $E_1^{p,q} \cong H^p(X, \Omega^q)$ it converges to the E_∞-page $H^{p+q}(X;\mathbb{C})$.

- The Hodge decomposition theorem states that for X compact Kähler, the spectral sequence degenerates at E_1 page, so $\bigoplus_{p+q=k} H^p(X, \Omega^q) \xrightarrow{\cong} H^k(X;\mathbb{C})$

- From the examples above, we can define: A rational Hodge structure of pure weight w on a $\mathbb{C}$-vector space V is the following data:
  - A $\mathbb{Q}$-structure on V: a $\mathbb{Q}$-vector space $V_\mathbb{Q}$ s.t. $V_\mathbb{Q} \otimes_\mathbb{Q} \mathbb{C} \xrightarrow{\cong} V$
  - A decomposition $V = \bigoplus_{p+q=w} V^{p,q}$ s.t. $V^{q,p} = \overline{V^{p,q}}$

    This information is equivalent to a Hodge filtration $F^pV = \bigoplus_{i \geq p} V^{i,j}$ s.t. $F^pV \oplus \overline{F}^{w+1-p} = V$ for any p $\quad V^{p,q} = F^pV \cap \overline{F^qV}$

- Once we understand compact Kähler manifolds, we can seek to understand closed embeddings of compact Kähler manifolds and the open complement.

  $\boxed{\text{Ex}}$ X complex projective, $Y \subset X$ nonsingular divisor, hence also complex projective. the Hodge decomposition is preserved by the pull-back $i^*: H^*(X) \to H^*(Y)$ induced by $i: Y \subset X$

  We get the long exact sequence: $\quad U = X \setminus Y$

$$\cdots \to H^m(X) \xrightarrow{i^*} H^m(Y) \xrightarrow{\partial} H_c^{m+1}(U) \longrightarrow H^{m+1}(X) \xrightarrow{i^*} H^{m+1}(Y) \xrightarrow{\partial} \cdots$$

Since $H^m(Y)$ has a pure Hodge structure of weight m, $H^{m+1}(X)$ has a pure weight $m+1$,
$H_c^{m+1}(U)$ should have a mixed Hodge structure of weights $m, m+1$.

$\boxed{\text{Def}}$ A mixed Hodge structure $(V_{\mathbb{Q}}, W_\bullet, F^\bullet)$ is:

- A finite dimensional $\mathbb{Q}$-v.s. $V_{\mathbb{Q}}$
- $W_\ell \subseteq W_{\ell+1}$ increasing filtration of $V_{\mathbb{Q}}$
- $F^p \supseteq F^{p+1}$ decreasing filtration of $V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$

s.t. $F^\bullet$ induces a pure Hodge structure of weight ℓ on $gr_\ell^W V_{\mathbb{Q}}$

$\boxed{\text{Ex}}$ on $H_c^{m+1}(U;\mathbb{Q})$, the weight filtration is given by $W_m = \mathrm{Im}(\partial : H^m(Y;\mathbb{Q}) \to H_c^{m+1}(U;\mathbb{Q}))$

$W_{m+1} = H_c^{m+1}(U;\mathbb{Q})$

This LES is induced by the triangle $j_! j^! \mathbb{Q}_X \to \mathbb{Q}_X \to i_* i^* \mathbb{Q}_X$

$i : Y \hookrightarrow X \quad j : \underset{U}{\underset{\shortparallel}{X \setminus Y}} \hookrightarrow X$

We can also consider the LES induced by the triangle
$i_! i^! \mathbb{Q}_X \to \mathbb{Q}_X \to j_* j^* \mathbb{Q}_X$, and get the Gysin exact sequence:

$$H^{m-2}(Y;\mathbb{Q}) \to H^m(X;\mathbb{Q}) \xrightarrow{j^*} H^m(U;\mathbb{Q}) \xrightarrow{res} H^{m-1}(Y;\mathbb{Q})$$

taking the residue along divisor Y

To keep track of the Hodge structure, we'll need to introduce the Tate twist:

$\mathbb{Q}(1) :=$ weight -2 pure Hodge structure on $\mathbb{Q} \cong 2\pi i \mathbb{Q} \subseteq \mathbb{C}$

Then the LES preserving the Hodge structure is:

$$H^{m-2}(Y;\mathbb{Q})(-1) \to H^m(X;\mathbb{Q}) \to H^m(U;\mathbb{Q}) \xrightarrow{res} H^{m-1}(Y;\mathbb{Q})(-1)$$

pure wt m — mixed wts $m, m+1$ — pure wt $m-1+2 = m+1$

$\boxed{\text{Explanation of } 2\pi i}$

Consider $\mathbb{G}_m$ defined over $\mathbb{Q}$ $\quad \mathbb{G}_m = \mathrm{Spec}\, \mathbb{Q}[x, x^{-1}]$, then $H^*_{dR}(\mathbb{G}_m, \mathbb{Q}) \xrightarrow{\cong} H^*(\mathbb{C}^\times, \mathbb{Q})$ by the algebraic de Rham theorem.

Consider $Y \subset X$ smooth divisor, then we have

$$0 \to \Omega_X^\bullet \to \Omega_X^\bullet(\log Y) \xrightarrow{res} \Omega_Y^{\bullet-1} \to 0 \qquad H^*(\Omega_X^\bullet(\log Y)) \xrightarrow{\cong} H^*(X \setminus Y)$$

Take $Y = 0 \subset \mathbb{A}^1 = X$

$$\begin{array}{lccccccccc} 1 & 0 \to & \mathbb{Q}[x]dx & \to & \mathbb{Q}[x]\frac{dx}{x} & \xrightarrow{res} & \mathbb{Q} & \to & 0 \\ & & \uparrow & & \uparrow d & & \uparrow & & \\ 0 & 0 \to & \mathbb{Q}[x] & \to & \mathbb{Q}[x] & \to & 0 & \to & 0 \end{array}$$

$\Rightarrow H^1(\mathbb{G}_m)$ has a Hodge structure pure of weight 2, the generator is $\frac{dx}{x}$
since $\int_{S^1} \frac{dx}{x} = 2\pi i$, if the natural basis

in $H_1(\mathbb{G}_m)$ is $[S^1]$, then $\frac{dx}{x} \longleftrightarrow 2\pi i$ in $\mathbb{C}$

$\boxed{\text{Theorem}}$ $\overline{X}$ smooth projective, $X \subset \overline{X}$, $D = \overline{X} \setminus X$ is a normal crossing divisor. $D = \bigcup_i D_i$ then there's a spectral sequence

$$E_1^{-p,q} = H^{q-2p}(D^{(p)}) \Rightarrow gr_W^q H^{q-p}(X) \qquad D^{(p)} = \text{locus of } p\text{-fold intersection}$$

This spectral sequence degenerates at the E_2 page.

$$\begin{array}{ll} H^0(D^{(2)}) \to H^2(D^{(1)}) \to H^4(\overline{X}) & \\ \quad H^1(D^{(1)}) \to H^3(\overline{X}) & \\ \quad H^0(D^{(1)}) \to H^2(\overline{X}) & \\ \quad 0 \quad H^1(\overline{X}) & \\ \quad 0 \quad H^0(\overline{X}) & \end{array}$$

$\Downarrow$ converges to

$$\begin{array}{ll} gr_W^2 H^1 & gr_W^2 H^2 \\ & gr_W^1 H^1 \\ & gr_W^0 H^0 \end{array}$$

$\boxed{\text{Ex}}$ $X = \mathbb{G}_m$, $\overline{X} = \mathbb{P}^1$. $D = \{0, \infty\}$

$D^{(1)} = D \qquad D^{(2)} = \emptyset$

$$\begin{array}{l} H^0(\{0,\infty\}) \to H^2(\mathbb{P}^1) \\ \qquad H^1(\mathbb{P}^1) \\ \qquad H^0(\mathbb{P}^1) \end{array} \cong \begin{array}{l} \mathbb{C}^2 \to \mathbb{C} \\ \qquad 0 \\ \qquad \mathbb{C} \end{array}$$

$\Downarrow$

$$\begin{array}{ll} \mathbb{C} & 0 \\ 0 & 0 \\ 0 & \mathbb{C} \end{array}$$

$\Rightarrow gr_W^0 H^0(\mathbb{G}_m) \cong \mathbb{C}$

$gr_W^2 H^1(\mathbb{G}_m) \cong \mathbb{C}$

We also saw above that $H^1(\mathbb{G}_m)$ is pure of weight 2.

$\boxed{\text{Braid varieties}}$

$$B_i(z) = \begin{array}{c} \hspace{2em} i \;\; i+1 \\ \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & \begin{pmatrix} 0 & 1 \\ 1 & z \end{pmatrix} & \\ & & & \ddots \\ & & & & 1 \end{pmatrix} \end{array}$$

Given $\beta = \sigma_{i_1} \cdots \sigma_{i_k}$. $B_\beta(z_1, \ldots, z_k) = \prod_{j=1}^{k} B_{i_j}(z_j)$

$X(\beta) := \{(z_1, \ldots, z_k) \mid B_\beta \text{ is upper triangular}\}$

A numerical way of seeing the weight filtration: $\mathbb{F}_q$ point count.

Mixed Hodge numbers $\quad h^{p,q;j}(X) := \dim_{\mathbb{C}}\left(Gr_p^F \, Gr_{p+q}^W \, H^j(X) \right)$

$$h_c^{p,q;j}(X) := \dim_{\mathbb{C}}\left(Gr_p^F \, Gr_{p+q}^W \, H_c^j(X) \right)$$

Mixed Hodge polynomials $\quad H(X,x,y,t) := \sum h^{p,q;j}(X)\, x^p y^q t^j$

$$H_c(X,x,y,t) := \sum h_c^{p,q;j}(X)\, x^p y^q t^j$$

Poincaré Duality: For a smooth connected variety X of complex dimension d:

$$H_c(X,x,y,t) = (xyt^2)^d \, H\left(X, \tfrac{1}{x}, \tfrac{1}{y}, \tfrac{1}{t}\right)$$

$\boxed{\text{Thm}}$ (Katz) Assume X has polynomial $\mathbb{F}_q$ point count $|X(\mathbb{F}_q)| = P_X(q)$

then $P_X(xy) = H_c(X,x,y,-1)$

$\boxed{\text{Ex}}$ $X = \mathbb{C}^*$ $\qquad j : X \hookrightarrow \bar{X} = \mathbb{P}^1 \hookleftarrow \{0,\infty\} : i$

$$H_c^0(\mathbb{C}^*) \to H^0(\mathbb{P}^1) \to H^0(\{0,\infty\}) \to H_c^1(\mathbb{C}^*) \to H^1(\mathbb{P}^1) \to \overset{\overset{0}{\shortparallel}}{H^1(\{0,\infty\})} \to H_c^2(\mathbb{C}^*) \to H^2(\mathbb{P}^1)$$

$\Rightarrow \; gr_W^0 H_c^1(\mathbb{C}^*) \cong \mathbb{C} \qquad gr_W^2 H_c^2(\mathbb{C}^*) \cong \mathbb{C}$

$\quad gr_0^F gr_W^0 H_c^1(\mathbb{C}^*) \cong \mathbb{C} \qquad gr_1^F gr_W^2 H_c^2(\mathbb{C}^*)$

$H_c(\mathbb{C}^*, x,y,t) = t + xyt^2 \qquad t=-1 \Rightarrow xy - 1$

$$\Updownarrow$$

$$|\mathbb{G}_m(\mathbb{F}_q)| = q-1$$

$\boxed{\text{Ex}}$ $X = \mathbb{P}^n$, $\quad \underset{\underset{H_c(\mathbb{P}^n,x,y,t)}{\shortparallel}}{H(\mathbb{P}^n,x,y,t)} = 1 + xyt^2 + (xyt^2)^2 + \dots + (xyt^2)^n$

$$|\mathbb{P}^n(\mathbb{F}_q)| = 1 + q + q^2 + \dots + q^n$$

$\boxed{\text{Ex}}$ This polynomial is a lot less informative for $X =$ elliptic curve E

$H^0(E) \cong \mathbb{C}$: pure Hodge str of wt 0 $\qquad gr_0^F gr_W^0(H^0(E)) \cong \mathbb{C}$

$H^1(E) \cong \mathbb{C}^2$ pure of wt 1 : $\quad gr_0^F gr_W^1(H^1(E)) \cong \mathbb{C} \qquad gr_1^F gr_W^1(H^1(E)) \cong \mathbb{C}$

$H^2(E) \cong \mathbb{C}$ pure of wt 2 : $\quad gr_1^F gr_W^2(H^2(E)) \cong \mathbb{C}$ $\qquad \uparrow$ because $H^1(E)$ contains dz and $d\bar{z}$

$$H(E, x, y, t) = 1 + (x+y)t + xyt^2$$
$$\parallel$$
$$H_c(E, x, y, t)$$

$H_c(E, x, y, -1) = (1-x)(1-y)$ is not a polynomial in xy

$$\Updownarrow$$

$|E(\mathbb{F}_q)|$ is not a polynomial of q

$(\ |E(\mathbb{F}_p)| = p+1-a \quad \leftrightarrow \quad \mathrm{Frob}^2 - a\,\mathrm{Frob} + p = 0 \ )$

$\mathrm{Frob} \curvearrowright H^1_{ét}(E, \overline{\mathbb{Q}}_\ell) \quad \ell \neq p$

Hecke algebra :

$\boxed{\text{Thm}}$ The Hecke algebra of S_n is the same as the convolution algebra

$$\left(\mathbb{C}\left[B(\mathbb{F}_q) \backslash GL_n(\mathbb{F}_q) / B(\mathbb{F}_q) \right], * \right)$$

$\boxed{\text{Ex}}$ $n=2$: $GL(2)/B \simeq \mathbb{P}^1$ identity 1 in the Hecke algebra:

the indicator function on $B(\mathbb{F}_q) \backslash B(\mathbb{F}_q) / B(\mathbb{F}_q)$

$T_s = (\sqrt{q})^{-1} \cdot$ indicator function on $B(\mathbb{F}_q) \backslash B(\mathbb{F}_q) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} B(\mathbb{F}_q) / B(\mathbb{F}_q)$

relations: constant function on $\mathbb{P}^1 = 1 + \sqrt{q}\, T_s$

constant function $*$ constant function $=$ constant function $\cdot \# \mathbb{P}^1(\mathbb{F}_q)$

$= (q+1)$ constant function

$$\Rightarrow (1 + \sqrt{q}\, T_s)^2 = (q+1)(1 + \sqrt{q}\, T_s)$$

$$\Rightarrow q T_s^2 + 2\sqrt{q}\, T_s + 1 = q+1 + (q+1)\sqrt{q}\, T_s$$

$$T_s^2 - \left(\sqrt{q} + \tfrac{1}{\sqrt{q}}\right) T_s - 1 = 0 \qquad \left(T_s - \tfrac{1}{\sqrt{q}}\right)\left(T_s + \sqrt{q}\right) = 0 \qquad \sqrt{q} = v^{-1}$$

Language of sheaves :

Hecke algebra categorifies to $D^b_{mix}(B \backslash G / B)$

Soergel bimodules $\leftrightarrow$ cohomology of intersection cohomology sheaves on $B \backslash G / B$.

3 miraculous theorems:

$\boxed{\text{Thm}}$ (Ginzburg) $\quad \mathrm{Hom}^*(IC_{w_1}, IC_{w_2}) \xrightarrow{\cong} \mathrm{Hom}^*_{H^*({}_B\backslash G/_B)}(H^*(IC_{w_1}), H^*(IC_{w_2}))$

Homological grading $\longleftrightarrow$ internal grading

$\boxed{\text{Thm}}$ $\quad K^b(IC) \xrightarrow{\cong} D^b_{mix}({}_B\backslash G/_B)$ $\quad$ ("projective resolution result")

Via Ginzburg's thm, $K^b(IC) \xrightarrow{\cong} K^b(SBim)$

Hence complexes in $K^b(SBim)$ can be viewed as sheaves with a mixed Hodge structure (mixed Hodge modules) in $D^b_{mix}({}_B\backslash G/_B)$

$\boxed{\text{Thm}}$ The Hodge structure on $\mathrm{Hom}^*(IC_{w_1}, IC_{w_2})$ is "diagonal"

$\boxed{\text{Dictionary for } gl_2}$ $\quad R = \mathbb{C}[x_1, x_2] \quad |x_1| = |x_2| = 2$

Diagonal bimodule $R \quad \longleftrightarrow \quad$ skyscraper sheaf at $0 \in \mathbb{P}^1$
$\underline{\mathbb{C}}_0$

$B_s = R \otimes_{R^s} R(1) \quad \longleftrightarrow \quad$ constant sheaf on $\mathbb{P}^1$ with cohomological and tate twists
$\underline{\mathbb{C}}_{\mathbb{P}^1}[1](\tfrac{1}{2})$

$$T_s = \begin{array}{c} \begin{matrix} 0 & & 1 \end{matrix} \\ \left[\underbrace{R \otimes_{R^s} R(1)}_{\substack{\| \\ B_s}} \xrightarrow{m} R(1) \right] \end{array} \quad \longleftrightarrow \quad j_! \underline{\mathbb{C}}_{\mathbb{A}^1}[1](\tfrac{1}{2}) \; : \; j_! \underline{\mathbb{C}}_{\mathbb{A}^1}[1](\tfrac{1}{2}) \to \left[\underline{\mathbb{C}}_{\mathbb{P}^1}[1](\tfrac{1}{2}) \to \underline{\mathbb{C}}_0[1](\tfrac{1}{2}) \right]$$

$$T_s^{-1} : \begin{array}{c} \begin{matrix} -1 & & 0 \end{matrix} \\ \left[R(-1) \to \underbrace{R \otimes_{R^s} R(1)}_{\substack{\| \\ B_s}} \right] \end{array} \quad \longleftrightarrow \quad j_* \underline{\mathbb{C}}_{\mathbb{A}^1}[1](\tfrac{1}{2}) \qquad \underline{\underline{\mathbb{C}}_0[-1](-\tfrac{1}{2}) \to \underline{\mathbb{C}}_{\mathbb{P}^1}[1](\tfrac{1}{2})} \to j_* \underline{\mathbb{C}}_{\mathbb{A}^1}[1](\tfrac{1}{2})$$

$1 \longmapsto 1 \otimes \alpha_s + \alpha_s \otimes 1$

$\to$