NOTES ON THE PROOF OF $P = W$ FOR $G = \mathrm{GL}_n$

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These are some notes compiled for the purpose of a series of seminar talks on $P = W$ given by the author based on the paper of Maulik and Shen. All mistakes are the fault of the author. In the following, unless otherwise noted, we work over $\mathbb{C}$, $G = \mathrm{GL}_n$ and C a smooth, projective curve of genus at least 2.

1. The Dolbeault Moduli Space and the Perverse Filtration

A Higgs bundle for $G = \mathrm{GL}_n$ is a pair (E, Φ) where E is a vector bundle of rank n on C and $\Phi\colon E \to E \otimes \Omega^1_C$ is a $\mathcal{O}_C$ linear section valued in 1-forms.

The Dolbeault moduli space M_{Dol} is the moduli space of stable Higgs bundles (E, Φ), where stability means the condition that all sub-Higgs bundles have strictly smaller slope

$$\mu(E) = \frac{\deg(E)}{\mathrm{r}k(E)}.$$

The space M_{Dol} admits a fibration, called the *Hitchin fibration*:

$$h\colon M_{Dol} \to A := \bigoplus_{i=1}^{n} H^0(C, \Omega^i_C), \quad \text{taking } (E, \Phi) \mapsto \text{char.poly}(\Phi)$$

where A is an affine space of half the dimension of M_{Dol}. The map h has several remarkable properties:

(1) h is surjective, proper, and Lagrangian (with symplectic structure on M_{Dol} induced by Nonabelian Hodge Correspondence).
(2) h is a *weak abelian fibration*, i.e. the generic fibers of h are abelian varieties.
(3) The pullback of the coordinate functions on A give a basis of Poisson commuting functions, giving the structure of a *completely integrable system*.

The fibration also defined the perverse filtration on cohomology $H^*(M_{Dol}, \mathbb{Q})$: We introduce this at the level of a proper map $f\colon X \to Y$, with $\dim(X) = a$ and $\dim(Y) = b$. Let $r = \dim X \times_Y X - \dim X$ be the defect to semismallness. [The fibers of h are equidimensional, so in the case of the Hitchin fibration,

$$r = \dim(M_{Dol}) - \dim(A) = \dim(A) = n^2(g-1) + 1$$

and $\dim(M_{Dol}) = 2n^2(g-1) + 2.$[1]] Then, the Perverse filtration is the increasing filtration on $H^m(X, \mathbb{Q})$ by

$$\begin{aligned} P_i H^m(X, \mathbb{Q}) &= \mathrm{Image}\left(H^{m-a+r}(Y, {}^p\tau_{\leq i} Rf_* \mathbb{Q}_X[a-r]) \to H^m(X, \mathbb{Q})\right) \\ &\overset{\text{our case}}{=} \mathrm{Image}\left(H^m(Y, {}^p\tau_{\leq i} Rf_* \mathbb{Q}_X) \to H^m(X, \mathbb{Q})\right) \end{aligned}$$

[1]These dimension computations follow as M_{Dol} is twice the dimension of A and $\dim A$ can be computed by Riemann-Roch.

By a 2010 paper of de Cataldo and Migliorini, this is quite explicit in our setting: Let $\Lambda^s \subset A$ be a general s dimensional linear section of A so that the Λ_s form a flag in A. Then

$$P_i H^m(X, \mathbb{Q}) = \ker\left(H^m(M_{Dol}, \mathbb{Q}) \to H^m(h^{-1}(\Lambda^{m-i-1}), \mathbb{Q})\right)$$

The latter filtration is called the “flag filtration” of h.

The perverse filtration is *not* multiplicative with respect to the cup product structure on $H^*(M_{Dol}, \mathbb{Q})$! This will come up later and will lead us to the notion of *strong perversity.*

2. The Betti Moduli Space (Character Variety) and the Weight Filtration

The Betti moduli space M_B is the character variety classifying isomorphism classes of irreducible local systems

$$\rho\colon \pi_1(C \setminus \{p\}) \to \mathrm{GL}_n(\mathbb{C})$$

sending a loop around $p \in C$ to $e^{2\pi i d/n} \cdot \mathrm{Id}_n$. Explicitly, for $g = g(C)$,

$$M_B = \left\{ a_k, b_k \in \mathrm{GL}_n, 1 \leq k \leq g\colon \prod_{j=1}^{g} [a_j, b_j] = e^{2\pi i d/n} \cdot \mathrm{Id}_n \right\} /\!\!/ \mathrm{GL}_n$$

Then M_B is affine, with nontrivial weight filtration $W_i H^*(M_{Dol}, \mathbb{Q})$ on its cohomology.

The weight filtration admits a “curious Hard Lefschetz” property due to Mellit: There exists a class $\alpha \in H^2(M_B)$ of type $(2,2)$ such that cup product with α induces isomorphisms

$$\alpha^l \cup -\colon \mathrm{Gr}^W_{\dim(M_B)-2l} H^*(M_B) \xrightarrow{\simeq} \mathrm{Gr}^W_{\dim(M_B)+2l} H^{*+2l}(M_B).$$

Intuition: This curious Hard Lefschetz comes from relative Hard Lefschetz for the Perverse filtration! Note in particular, that the shift by $2l$ is because the doubly indexed weight filtration is conjectured to agree with the singly indexed perverse filtration. Later, this Lefschetz type statement will be used to help prove the two filtrations agree.

3. The Nonabelian Hodge Correspondence

The nonabelian Hodge theorem relates these two dramatically different spaces M_{Dol} and M_B.

Theorem 3.1. *There is a canonical diffeomorphism $M_{Dol} \simeq M_B$, equipping both spaces with the structure of a hyperkähler manifold.*

This identification can be thought of as follows: A local system representing a point of M_B is equivalent to the data of a vector bundle E on C equipped with a flat connection ∇. The connection ∇ satisfies the Liebnitz rule

$$\nabla(f) = f \cdot \nabla + df$$

We may consider the family over $\mathbf{A}^1$ whose fiber over $z \in \mathbf{A}^1$ consists of pairs (E, ∇_z) where ∇_z is a z-connection, i.e. $\nabla(f) = f \cdot \nabla + z \cdot df$. Note that M_{Dol} is the fiber over 0 while all other fibers are isomorphic as complex varieties to M_B. The nonabelian Hodge theorem states this family is topologically trivial.

It was first considered by de Cataldo-Hausel-Migliorini what the meaning of the weight filtration was on $H^*(M_{Dol},\mathbb{Q})$ and vice versa. They came up with the following, proving it in the case of $n=2$ and $g\geq 2$ arbitrary:

Conjecture 3.2. ($P=W$) For any $k,m\geq 0$,

$$P_kH^m(M_{Dol},\mathbb{Q})=W_{2k}H^m(M_B,\mathbb{Q})=W_{2k+1}H^m(M_B,\mathbb{Q}).$$

4. A System of Generators for $H^*(M_B,\mathbb{Q})$ and a First Reduction

To avoid normalizations, we work with the Dolbeault moduli space for PGL_n. In degree d, we obtain this by the quotient of the of $M_{Dol,\deg=d,\mathrm{tr}=0}$ by the finite group $\Gamma=\mathrm{Pic}^0(C)[n]$. Denote the PGL_n Hitchin fibration by

$$\hat{h}\colon \hat{M}_{Dol}\to\hat{A}=\bigoplus_{i=2}^{n}H^0(C,\Omega_C^i)$$

There is a universal family $\mathcal{U}\to C\times\hat{M}_{Dol}$ which induces Chern characters

$$\mathrm{ch}_k(\mathcal{U})\in H^{2k}(C\times\hat{M}_{Dol},\mathbb{Q})$$

and therefore we get maps

$$c_k\colon H^*(C,\mathbb{Q})\to H^*(\hat{M}_{Dol},\mathbb{Q})$$

induced by the correspondence

$$\begin{tikzcd} & C\times\hat{M}_{Dol} \arrow[dl] \arrow[dr] & \\ C & & \hat{M}_{Dol} \end{tikzcd}$$

integrating with the kernel $\mathrm{ch}_k(\mathcal{U})$. Note that if Σ is a basis for $H^*(C,\mathbb{Q})$, then

$$\mathrm{ch}_k(\mathcal{U})=\sum_{\sigma\in\Sigma}\sigma^\vee\otimes c_k(\sigma)$$

The image of c_k gives us our genertors for cohomology as a $\mathbb{Q}$ algebra:

Theorem 4.1. *(Markman and Shende) The image $c_k(H^*(C,\mathbb{Q}))$ generates $H^*(\hat{M}_{Dol},\mathbb{Q})$ as a $\mathbb{Q}$ algebra, and moreover, for $\gamma\in H^m(C,\mathbb{Q})$,*

$$c_k(\gamma)\in W_{2k}H^m(M_B,\mathbb{Q})\cap F^kH^m(M_B,\mathbb{C})$$

Corollary 4.2. *The $P=W$ Conjecture is equivalent to the condition that $\prod_{i=1}^s c_{k_i}(\gamma_i)\in P_{\sum_i k_i}H^*(\hat{M}_{Dol},\mathbb{Q})$ for all products of tautological classes.*

Proof. The proof uses the fact that the $c_k(\gamma)$ generate under cup product together with Mellit's Curious Hard Lefschetz Theorem:

Indeed, suppose that the condition holds. Then, we see immediately that

$$W_{2k}(\hat{M}_B)\subset P_kH^*(\hat{M}_{Dol}).$$

Now, if $N=\dim\hat{M}_B$, then we know that $W_{2N}=P_N=H^*(M_B,\mathbb{Q})=:V$. By the inclusion

$$\dim W_0\leq\dim P_0\overset{\text{relative Lefschetz}}{=}\dim V/P_{N-1}\leq\dim V/W_{2(N-1)}$$

An application of Mellit's curious Hard Lefschetz proves that the first and last dimensions are equal, and hence the above inequalities are equalities. The rest of the terms are similar. $\square$

5. Nearby and Vanishing Cycles

We will use the algebraic tools of nearby and vanishing cycles to reduce the $P = W$ conjecture to a strong perversity condition on certain Chern classes. In this section, we review the motivation and construction of nearby and vanishing cycles.

There are two principal motivations for nearby cycles: One from measuring cohomology of nearby fibers and one from globalizing Milnor's fibration.

We begin by reviewing the Milnor fibration. For a map $f\colon \mathbb{C}^{n+1} \to \mathbb{C}$ with zero fiber $X_0 \subset X$ and $x \in X_0$, let $B_{\varepsilon,x}$ be a small ball centered at x with boundary $\partial B_{\varepsilon,x} = S_{\varepsilon,x}$. Milnor proved many results about the fibration which now bears his name. A few that are most relevant to us include:

Theorem 5.1. *(Milnor)*

(i) $B_{\varepsilon,x} \cap X_0$ is homeomorphic to the cone on K_x where $K_x = S_{\varepsilon,x} \cap X_0$ is the link at x.

(ii) The "Milnor fibration at x"

$$\frac{f}{|f|}\colon S_{\varepsilon,x} \setminus K_x \to S^1$$

is topologically trivial.

(iii) When (X_0, x) is an isolated hypersurface singularity germ, the fiber F_x of the Milnor fibration at x is homotopy equivalent to $\bigvee_{\mu_x} S^n$ where

$$\mu_x = \dim_{\mathbb{C}} \frac{\mathbb{C}[x_0, \ldots, x_n]}{(f_{x_0}, \ldots, f_{x_n})}$$

is the Milnor number of f at x.

(The spheres in part (iii) of Theorem 5.1 will correspond to the vanishing cycles we later construct.)

Example 5.2. Consider $f\colon \mathbb{C}^2 \to \mathbb{C}$ taking $f(x, y) = x^3 + y^2$. We take $x = (0, 0) \in X_0$. The link K_0 is the trefoil knot in S^3, and the Milnor fibration gives a topologically trivial fibration

$$S^3 \setminus (\text{trefoil knot}) \to S^1$$

whose fiber is homotopic to a wedge product $S^1 \vee S^1$.

Consider a proper family $f\colon X \to \mathbb{C}$ with fiber X_s a smooth hypersurface of $\mathbb{C}^N$ for $s \neq 0$ and X_0 singular. For s close enough to 0, $f^{-1}(D_\varepsilon(0))$ retracts onto X_0 and hence the inclusion $X_s \to f^{-1}(D)$ induces a specialization map on cohomology

$$sp^*\colon H^*(X_0, \mathbb{Z}) \to H^*(X_s, \mathbb{Z})$$

When the map is proper, we can upgrade this cohomological map to a non-holomorphic, but continuous, map

$$sp\colon X_s \to X_0$$

collapsing the vanishing cycles. [Constructed by McPherson.]

The nearby cycle functor will roughly measure $H^*(X_s)$, pushed forward under the specialization map. The vanishing cycles functor will measure the difference between $H^*(X_s)$ and $H^*(X_0)$, i.e. (dually) the cycles in $H_*(X_s)$ which collapse (or "vanish") as X_s degenerates to X_0.

Definition 5.3. Let D^* be the punctured disk and $\pi\colon \widetilde{D^*} \to D^*$ the universal cover of D^*. Consider the following diagram (with f proper)

$$\begin{tikzcd} X_0 \arrow[r, hook, "i"] \arrow[d] & X \arrow[d, "f"] & X^* \arrow[l, hook', "j"'] \arrow[d, "f^*"] & \widetilde{X^*} \arrow[l, "\hat{\pi}"'] \arrow[d] \\ \{0\} \arrow[r, hook] & D & D^* \arrow[l, hook'] & \widetilde{D^*} \arrow[l, "\pi"'] \end{tikzcd}$$

Then, the nearby cycles functor is

$$\psi_f \mathcal{F}^\bullet = i^* R(j \circ \hat{\pi})_* (j \circ \hat{\pi})^* \mathcal{F}^\bullet$$

It is immediate from the definition that, for any $x \in X_0$, the stalk of the sheaf cohomology of nearby cycles measures the cohomology of the Milnor fiber, i.e.

$$\mathcal{H}^k(\psi_f \mathcal{F}^\bullet)_x = H^k(F_x, \mathcal{F}^\bullet)$$

Moreover, when f is proper, we have a comparison result

Theorem 5.4. *When f is proper, $\psi_f \mathcal{F}^\bullet \simeq Rsp_*(\mathcal{F}^\bullet|_{X_s})$.*

The natural adjunction induces a canonical natural transformation

$$i^* \to \psi_f$$

Definition 5.5. The *vanishing cycles functor* φ_f is the cone of the map $i^* \to \psi_f$. It fits into an exact triangle

$$i^* \mathcal{F}^\bullet \to \psi_f \mathcal{F}^\bullet \to \varphi_f \mathcal{F}^\bullet \xrightarrow{[1]} \tag{5.1}$$

Note: It is not immediate from this definition that φ_f is functorial. One can prove that φ_f is the pullback of a representable functor along i.

Also note: In the literature, and in the rest of these notes, we will actually use a shift $\psi_f[-1]$ and $\varphi_f[-1]$ in place of ψ_f and φ_f. This is done so that nearby and vanishing cycles preserve perversity.

From the long exact sequence induced by the exact triangle, we have

$$\mathcal{H}^k(\varphi_f \mathcal{F}^\bullet)_x \simeq H^{k+1}(B_{\varepsilon,x}, B_{\varepsilon,x} \cap X_s, \mathcal{F}^\bullet) \quad \text{for } x \in X_0$$

When X is nonsingular, $B_{\varepsilon,x} \cap X_s$ is contractible and

$$\mathcal{H}^k(\varphi_f \underline{A}_X)_x \simeq \widetilde{H}^k(F_x, A)$$

is the reduced cohomology of the Milnor fiber F_x at $x \in X_0$. The stalk of the long exact sequence on cohomology assocaited to the exact triangle (5.1) looks like

$$\cdots \to H^k(B_{\varepsilon,x}, \mathcal{F}^\bullet) \to H^k(B_{\varepsilon,x} \cap X_s, \mathcal{F}^\bullet) \to H^{k+1}(B_{\varepsilon,x}, B_{\varepsilon,x} \cap X_s; \mathcal{F}^\bullet) \to \cdots$$

Corollary 5.6. *If X is nonsingular,* $\mathrm{Supp}(\varphi_f \underline{A}_X) \subset \mathrm{Sing}(X_0)$*, and if A is a field, e.g. $A = \mathbb{Q}$, then equality holds.*

Example 5.7. Consider the example of $f\colon \mathbb{C} \to \mathbb{C}$ taking $x \mapsto x^a$. The unique critical point is $X_0 = \{0\} \in \mathbb{C}$. We have $i^*\underline{\mathbb{Q}}_{\mathbb{C}} = \underline{\mathbb{Q}}_0$ while $\psi_f \underline{\mathbb{Q}}_{\mathbb{C}} = Rsp_*(\underline{\mathbb{Q}}_{\mu_a}) = \prod_{\mu_a} \underline{\mathbb{Q}}_0$. Hence, from the exact triangle,

$$\varphi_f \underline{\mathbb{Q}}_{\mathbb{C}} = (\underline{\mathbb{Q}}_0)^{a-1}.$$

So far, we have seen a construction that globalizes the Milnor fibration. The following tool is useful for computations:

Theorem 5.8. *(Analogue of Thom-Sebastiani) Let V denote the vanishing locus of a function and $k\colon V(f) \times V(g) \hookrightarrow V(f \boxtimes g)$ the natural inclusion map. Then, (with shifted nearby and vanishing cycle functors)*

$$k^*\Big(\varphi_{f\boxtimes g}(\mathcal{F}^\bullet \boxtimes^L \mathcal{G}^\bullet)\Big) \simeq \varphi_f \mathcal{F}^\bullet \boxtimes^L \varphi_g \mathcal{G}^\bullet$$

Example 5.9. Consider the case of a "Brieskorn singularity":

$$f\colon \mathbb{C}^n \to \mathbb{C}, \quad \text{by } (x_1, \dots, x_n) \mapsto \sum_i x_i^{a_i}$$

Then, by the Thom-Sebastiani Theorem combined with Example 5.7, we see that

$$\varphi_f(\underline{\mathbb{Q}}_{\mathbb{C}^n}) = k_*(\boxtimes_{i=1}^n \underline{\mathbb{Q}}_0^{a_i - 1}) = \underline{\mathbb{Q}}_0^{\sum_i (a_i - 1)}.$$

For us, the most important property of vanishing cycles will be its compatibility with the notion of "strong perversity".

Definition 5.10. An element $\gamma \in H^l(X, \mathbb{Q})$ has *strong perversity* c with respect to $f\colon X \to Y$ if the corresponding map $\gamma\colon Rf_*\mathbb{Q}_X \to Rf_*\mathbb{Q}_X[l]$ factors respects the perverse t structure by mapping

$$\begin{tikzcd}
{}^p\tau_{\leq i} Rf_*\mathbb{Q}_X \arrow[r, hook] \arrow[d] & Rf_*\mathbb{Q}_X \arrow[d, "\gamma"] \\
{}^p\tau_{\leq i+(c-l)} Rf_*\mathbb{Q}_X[l] \arrow[r, hook] & Rf_*\mathbb{Q}_X[l]
\end{tikzcd}$$

The benefit of strong perversity is that cup product with such classes respect the perverse filtration:

Lemma 5.11. *If $\gamma \in H^l(X, \mathbb{Q})$ has strong perversity c with respect to f, then $\gamma \cup -$ takes*

$$\gamma\colon P_i H^m(X, \mathbb{Q}) \to P_{i+c} H^{m+l}(X, \mathbb{Q})$$

Moreover, if $\gamma_i \in H^{l_i}(X, \mathbb{Q})$ has strong perversity c_i, then $\gamma_1 \cup \cdots \cup \gamma_s \in H^{l_1 + \cdots l_s}(X, \mathbb{Q})$ has strong perversity $\sum_i c_i$.

The key relationship with vanishing cycles can be formulated as follows: Consider a map $g\colon X \to \mathbf{A}^1$ where X is nonsingular and irreducible, and let $X' \subset X_0$ be the support of $\varphi_g := \varphi_g(IC_X)$.

Proposition 5.12. *With notation as above, suppose that X' is nonsingular and that the vanishing cycle sheaf is trivial on X', i.e. $\varphi_g \simeq IC_{X'} = \underline{\mathbb{Q}}_{X'}[\dim X']$. Further, suppose we have a commutative diagram*

$$\begin{tikzcd} X' \arrow[r, "i"] \arrow[d, "f'"'] & X \arrow[d, "f"'] \arrow[dr, "g"] & \\ Y' \arrow[r] & Y \arrow[r, "\nu"'] & \mathbf{A}^1 \end{tikzcd}$$

with f proper and g factoring through f. Then, if $\gamma \in H^l(X, \mathbb{Q})$ has strong perversity c with respect to f, then $i^*\gamma \in H^l(X', \mathbb{Q})$ has strong perversity c with respect to f'.

Proof. We apply φ_ν to the map

$$\gamma\colon Rf_*\mathbb{Q}_X \to Rf_*\mathbb{Q}_X[l]$$

and use a base change for vanishing cycles $Rf'_* \circ \varphi_g \simeq \varphi_\nu \circ Rf_*$ to get the map

$$\varphi_g(\gamma)\colon Rf'_*\varphi_g \to Rf'_*\varphi_g[l]$$

Since φ_ν commutes with the perverse t structure, this map takes ${}^p\tau_{\le i}$ to ${}^p\tau_{\le i+(c-l)}$. Then, since $\varphi_g \simeq \mathbb{Q}_{X'}[\dim X']$ is trivial, and $\varphi_g(\gamma) = i^*\gamma$, it follows that $i^*\gamma$ has strong perversity c with respect to f'. ☐

6. Reduction to Strong Perversity Condition on the Dolbeault Space

We now use the machinery of vanishing cycles to sheafify Corollary 4.2. We use upper script $\mathcal{L}$ to denote the twisted Hitchin system by the line bundle $\mathcal{L}$.

Maulik and Shen proved (in *Endoscopic decompositions and the Hausel–Thaddeus conjecture*), that for $p \in C$ there is a close relationship between $\hat{M}^{\mathcal{L}}_{Dol}$ and $\hat{M}^{\mathcal{L}(p)}_{Dol}$. Namely, we have a natural embedding

$$i\colon \hat{M}^{\mathcal{L}}_{Dol} \to \hat{M}^{\mathcal{L}(p)}_{Dol}$$

Proposition 6.1. *There is a function $g\colon \hat{M}^{\mathcal{L}(p)}_{Dol} \to \mathbf{A}^1$ factoring through $\hat{A}^{\mathcal{L}(p)}$ such that*

$$\varphi_g(IC_{\hat{M}^{\mathcal{L}(p)}_{Dol}}) = IC_{\hat{M}^{\mathcal{L}}_{Dol}} = \underline{\mathbb{Q}}_{\hat{M}^{\mathcal{L}}_{Dol}}[\dim \hat{M}^{\mathcal{L}}_{Dol}]$$

The main result is to prove the following:

Theorem 6.2. *For any effective line bundle $\mathcal{L}$, the class*

$$\mathrm{ch}_k(\mathcal{U}^{\mathcal{L}}) \in H^{2k}(C \times \hat{M}^{\mathcal{L}}_{Dol}, \mathbb{Q})$$

has strong perversity k with respect to $h^{\mathcal{L}}$.

Proof that Theorem 6.2 implies the hypotheses of Corollary 4.2. The main point is that we claim that if $\mathrm{ch}_k(\mathcal{U}^{\mathcal{L}(p)})$ has strong perversity k then $\mathrm{ch}_k(\mathcal{U}^{\mathcal{L}})$ has strong perversity k. For this, we rely on the key interaction Proposition 5.12 between vanishing cycles and strong perversity

Consider the commutative diagram

$$\begin{tikzcd} C \times \hat{M}^{\mathcal{L}}_{Dol} \arrow[r, hook, "i"] \arrow[d, "h^{\mathcal{L}}"] & C \times \hat{M}^{\mathcal{L}(p)}_{Dol} \arrow[d, "h^{\mathcal{L}(p)}"] \\ C \times \hat{A}^{\mathcal{L}} \arrow[r, hook] & C \times \hat{A}^{\mathcal{L}(p)} \end{tikzcd}$$

We have $i^*\mathrm{ch}_k(\mathcal{U}^{\mathcal{L}(p)}) = \mathrm{ch}_k(\mathcal{U}^{\mathcal{L}})$. Then, for the map $g\colon C \times \hat{M}^{\mathcal{L}(p)}_{Dol} \to \mathbf{A}^1$ from Proposition 6.1, φ_g is trivial, and so the claim follows from the Key Fact Prop 5.12.

The point of the claim is that, assuming Theorem 6.2, we can immediately conclude the same result for $\mathcal{L} = \mathcal{O}_C$ trivial. We have

$$P_k H^*(C \times \hat{M}_{Dol}, \mathbb{Q}) = H^*(C, \mathbb{Q}) \otimes P_k H^*(\hat{M}_{Dol}, \mathbb{Q}),$$

and if $\Sigma = \{\sigma_0, \sigma_1, \ldots, \sigma_{2g+1}, \sigma_{2g+2}\}$ is a homogeneous basis for $H^*(C, \mathbb{Q})$ with Poincaré dual basis $\{\sigma_0^\vee, \ldots, \sigma_{2g+2}^\vee\}$. Then,

$$\mathrm{ch}_k(\mathcal{U}) = \sum_{\sigma \in \Sigma} \sigma^\vee \otimes c_k(\sigma)$$

and so

$$\mathrm{ch}_k(\mathcal{U}) \cup (\sigma \otimes P_s H^*(\hat{M}_{Dol}, \mathbb{Q})) = \sigma^\vee(\sigma) \otimes \Big(c_k(\sigma) \cup P_s H^*(\hat{M}_{Dol}, \mathbb{Q})\Big) \subset H^*(C, \mathbb{Q}) \otimes P_{s+k} H^*(\hat{M}_{Dol}, \mathbb{Q})$$

Hence, $c_k(\sigma)$ sends P_s to P_{s+k}, and so also does $c_k(\gamma)$. The result follows. □

7. Yun's Global Springer Theory

An essential idea of the proof is to use Yun's Global Springer Theory to reduce strong perversity of the universal bundle on $\hat{M}^{par}$ to strong perversity of certain line bundles [related to the graded DAHA action] on the parabolic Hitchin fibration $\mathcal{M}^{par}$. For this section, all results work for general reductive groups G, though we will only make actions explicit for $G = \mathrm{GL}_n$.

7.1. **The (extended) affine Weyl group.** Reflecting the author of these notes' ignorance on the subject, we begin by reviewing affine root systems and the affine Weyl group. We will only review what is essential; for a better exposition, see Seth Shelley-Abrahamson, linked here.

Let R be an irreducible root system on a real vector space V with nondegenerate inner product $(-,-)$ and Weyl group W. Let P and Q denote the root and weight lattices, respectively.

The associated affine root system is

$$R^a = \{\alpha + n\delta\colon \alpha \in R,\, n \in \mathbb{Z}\}$$

The affine Weyl group associated to R^a is

$$W_a = \langle s_\alpha\colon \alpha \in R^a \rangle$$

where s_α is the orthogonal reflection around the affine hyperplane $\alpha^{-1}(0)$. The form $(-,-)$ on V induces a form, which we also call $(-,-)$, on affine linear functionals by forgetting the constant term and using the form on $V^\vee$. Let

$$t(v)(f) = f + (v, f)\delta$$

be the translation of f by v.

It is immediate from the definition that

$$W^a = W \rtimes t(P^\vee)$$

is the semidirect product of reflections in W with the translation by elements of the coroot lattice. The group W_a acts on connected components of $V \setminus \{\text{root hyperplanes}\}$ simply transitively; we call each connected component a "fundamental alcove." Then the group W_a is a Coxeter group with generators given by reflections over lines forming the boundary of a fundamental alcove.

We also have an action of the "extended affine Weyl group" $\widetilde{W} = W \rtimes t(Q^\vee)$ on the set of alcoves. This extended action is not faithful: For A a fundamental alcove, we set

$$\Omega = \{w \in \widetilde{W} : w \cdot A = A\}$$

This stabilizer subgroup Ω measures the discrepancy between the extended affine Weyl group and the affine Weyl group:

$$\widetilde{W} = W^a \rtimes \Omega$$

Moreover, Ω is a finite group with a simple description in terms of the coroot and coweight lattices of R. Namely, $\Omega = Q^\vee / P^\vee$.

8. The Parabolic Hitchin System

Recall the Grothendieck-Springer resolution

$$\widetilde{\mathfrak{g}} \to \mathfrak{g}.$$

This map is generically, over the regular, semisimple locus, a W cover, and Springer theory studies its fibers $\mathcal{B}_\gamma$. In particular, Springer theory realizes $H^*(\mathcal{B}_\gamma)$ as a W representation, in spite of the fact that W does not act on $\mathcal{B}_\gamma$ itself. This action, therefore, is very mysterious.

We will study a global analogue of this map ("global" in this context referring to working over a curve C). For a fixed line bundle D, we consider the Hitchin moduli *stack* $\mathcal{M}$ over $A \times C$; the S points of $\mathcal{M}$ are maps

$$S \times C \to [\mathfrak{g}/G]_D$$

The *parabolic Hitchin space* is the base change of the diagram

$$\begin{tikzcd} \mathcal{M}^{par} \arrow[r] \arrow[d] & {[\widetilde{\mathfrak{g}}/G]_D \simeq [\mathfrak{b}/B]_D} \arrow[d] \\ \mathcal{M} \arrow[r] & {[\mathfrak{g}/G]_D} \end{tikzcd}$$

Explicitly, we have that $\mathcal{M}^{par}$ is the moduli stack of data

$$(\mathcal{E}, \Phi, x, \mathcal{E}_x^B)$$

where $(\mathcal{E}, \Phi)$ is a Higgs bundle, $x \in C$, and $\mathcal{E}_x^B$ is a Borel reduction of the fiber of $\mathcal{E}$ at x. For the case of $G = \mathrm{GL}_n$, this consists of the data of a flag

$$\mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_n = \mathcal{E}_x$$

with $\dim \mathcal{E}_{i+1}/\mathcal{E}_i = 1$.

There is a map

$$h^{par} \colon \mathcal{M}^{par} \to A \times C$$

by composition with the Hitchin map for $\mathcal{M}$. Let $\widetilde{A} \to A$ denote the base change

$$\begin{tikzcd} \widetilde{A} \arrow[r] \arrow[d] & \mathfrak{t}_D \arrow[d] \\ A \arrow[r] & \mathfrak{g}/\!\!/G \simeq \mathfrak{t}/\!\!/W \end{tikzcd}$$

where $\mathfrak{t} \subset \mathfrak{g}$ is a choice of Cartan and $\mathfrak{g}/\!\!/G \simeq \mathfrak{t}/\!\!/W$ is the Chevalley restriction theorem. Considering the diagram

$$\begin{tikzcd} {[\widetilde{\mathfrak{g}}/G]} \arrow[r] \arrow[d] & \mathfrak{t} \arrow[d] \\ {[\mathfrak{g}/G]} \arrow[r] & \mathfrak{t}/\!\!/W \end{tikzcd}$$

obtained from the usual Grothendieck-Springer resolution, we deduce that the map h^{par} factors through the cameral cover

$$\widetilde{h} \colon \mathcal{M}^{par} \to \widetilde{A}$$

The Hitchin moduli space $\mathcal{M}^{par}$ is acted on by the symmetries of the Hitchin fibration $\mathcal{M}$. Namely, for the usual Hitchin system, there is a smooth, commutative group stack $\mathcal{P} \to A \times C$ with an action of $\mathcal{P}$ on $\mathcal{M}$ such that $\mathcal{M}$ is generically a torsor for the action of $\mathcal{P}$. While we will not describe these symmetries in general here, there is a concrete description of $\mathcal{P}$ for $G = \mathrm{GL}_n$ dating back to a construction of Hitchin. Namely, there is a universal "spectral curve"

$$\overline{A} \to A \times C$$

whose fiber over $a = (a_i) \in A$ is the zero set of the characteristic polynomial

$$\lambda^n + a_1 \lambda^{n-1} + \cdots + a_n$$

in the total space of the line bundle D, i.e. $\overline{A}$ fits into a Cartesian diagram

$$\begin{tikzcd} \overline{A} \arrow[r] \arrow[d] & \mathfrak{t}/\!\!/S_{n-1} \arrow[d] \\ A \arrow[r] & \mathfrak{t}/\!\!/S_n \end{tikzcd}$$

Definition 8.1. (For $G = \mathrm{GL}_n$.) The group of symmetries $\mathcal{P}$ is the Picard stack $\mathcal{P} := \mathrm{Pic}_{/A\times C}(\overline{A})$.

Let $A^{\heartsuit}$ denote the points in the Hitchin base whose image

$$C \to [\mathfrak{g}/\!\!/G]_D$$

is generically in the regular, semisimple locus, i.e. the spectral curve at a is integral, and $A^{\diamondsuit}$ the nonempty open locus where the spectral curve at a is smooth.

Proposition 8.2. *$\mathcal{M}$ is a torsor for the action of $\mathcal{P}$ over $\mathcal{A}^{\diamondsuit}$. Moreover, $\mathcal{M}$ is identified with the compactified relative Picard stack*

$$\mathcal{M} \simeq \overline{\mathrm{Pic}}_{/A^{\heartsuit}\times C}(\overline{A})$$

over $A^{\heartsuit}$. (Note: The compactified Picard stack of Y is the moduli stack classifying torsion-free, rank 1 $\mathcal{O}_Y$ modules.)

Ngô and Yun work over a slightly smaller locus of the Hitchin base, called the *anisotropic locus*. Namely, we define

$$A^{ani} := \{a \in A \colon \pi_0(\mathcal{P}_a) \text{ is finite}\}.$$

The anisotropic locus is open and is nonempty if and only if G is semisimple. We restrict to this case in the sequel, knowing that the reduction to this case can be done rather easily. The component group $\pi_0(\mathcal{P}_a)$ is an essential ingredient in the theory of endoscopy. In particular, the "jumps" in $\pi_0(\mathcal{P}_a)$ correspond to the endoscopic strata. For this reason, it is important to introduce the δ invariant.

Definition 8.3. The δ invariant on $\mathcal{A}^{\heartsuit}$ is given by

$$\delta \colon \mathcal{A}^{\heartsuit} \to \mathbb{Z}_{\geq 0}, \quad a \mapsto \dim(\text{affine part of } \mathcal{P}_a).$$

This is the sum of local δ invariants on $\mathcal{A}^{\heartsuit} \times C$ measuring the dimensions of the affine Springer fibers $\mathcal{M}_x(\gamma_{a,x})$.

The δ invariant is upper semicontinuous and hence determines a stratification on $A^{\heartsuit} \times C$.

Lemma 8.4. *Consider the map*

$$\mathcal{M}^{par}|_{A^{\heartsuit}} \to \mathcal{M} \times_{A^{\heartsuit}} \widetilde{A}^{\heartsuit}. \tag{8.1}$$

When $\deg(D) \geq 2g$, $(A^{\heartsuit} \times C)_0$ is the largest subset of $A^{\heartsuit} \times C$ over which this map is an isomorphism.

The proof of this fact uses the *product formula* to connect between the affine Springer fiber at a point (a, x) and the full curve.

Note that $\mathcal{A}^{\diamondsuit} \times C \subset (A^{\heartsuit} \times C)_0$. In particular, we immediately have:

Corollary 8.5. *There is an isomorphism*

$$\mathcal{M}^{par}|_{A^{\diamondsuit}} \simeq \mathcal{M} \times_{A^{\diamondsuit}} \widetilde{A}^{\diamondsuit}$$

8.1. **The Springer Action.** The first main result of Yun's thesis is the following action:

Theorem 8.6. *The extended affine Weyl group acts on the sheaf $Rh^{par}_*\mathbb{Q}_l$ over $A^{ani} \times C$.*

The construction of this action involves the Hecke correspondence for $\mathcal{M}^{par}$. Namely, let $\mathscr{H}ecke^{par}$ denote the ind-algebraic stack classifying tuples

$$(x, \mathcal{E}_i, \varphi_i, \mathcal{E}^B_{i,x}, \alpha),\ i = 1, 2$$

so that $(x, \mathcal{E}_i, \varphi_i, \mathcal{E}^B_{i,x}) \in \mathcal{M}^{par}$ and

$$\alpha\colon \mathcal{E}_1|_{(S\times C)\setminus\Gamma(x)} \xrightarrow{\sim} \mathcal{E}_2|_{(S\times C)\setminus\Gamma(x)}$$

is an isomorphism of parabolic Higgs bundles over the complement of the graph of x. (Note that for k points, the graph of x is just the point x in C.) There are obvious projections

$$\begin{tikzcd} & \mathscr{H}ecke^{par} \arrow[dl] \arrow[dr] & \\ \mathcal{M}^{par} & & \mathcal{M}^{par} \end{tikzcd}$$

The relation of this correspondence with the extended affine Weyl group comes from the following Proposition. To prepare, let $(A \times C)^{rs}$ denote the collection of points (a, x) such that $a(x) \in \mathfrak{g}^{rs}/\!\!/G$. The Hecke correspondence $\mathscr{H}ecke^{par}$ can be viewed as a stack over $A \times C$; we denote by $\mathscr{H}ecke^{par,rs}$ the restriction of this to $(A \times C)^{rs}$.

Proposition 8.7. *There is an action of $\widetilde{W}$ on the restriction $\mathcal{M}^{par,0}$ of $\mathcal{M}^{par}$ to $(A^\heartsuit \times C)_0$ such that the reduced structure of the restriction of $\mathscr{H}ecke$ to the regular semisimple locus decomposes as a disjoint union of graphs of $w \in \widetilde{W}$, i.e.*

$$(\mathscr{H}ecke^{par,rs})^{red} = \coprod_{w\in\widetilde{W}} \Gamma(w)$$

Construction of the action for $G = \mathrm{GL}_n$. Recall that for any $a \in A$, we have a spectral cover by pulling back

$$\begin{tikzcd} C_a \arrow[r] \arrow[d, "p_a"'] & \overline{A} \arrow[d] \\ C \arrow[r] & A \times C \end{tikzcd}$$

We note that $(a, x) \in (A^\heartsuit \times C)_0$ if and only if the spectral curve at a is smooth at the preimage of x. Then, the parabolic Hitchin fiber over (a, x) is described by

$$\mathcal{M}^{par}_{a,x} = \{\mathcal{F} \in \overline{\mathrm{Pic}}(C_a) \text{ together with an ordering of } p_a^{-1}(x)\}$$

Then, $W = S_n$ acts by permutation on the ordering of $p_a^{-1}(x)$ while $\lambda = (\lambda_j) \in \mathbb{Z}^n \simeq \mathbf{X}_*(T)$ acts by

$$\mathcal{F} \mapsto \mathcal{F} \otimes \mathcal{O}_{C_a}(\lambda_1 y_1 + \cdots \lambda_n y_n)$$

where $p_a^{-1}(a) = \{y_1, \ldots, y_n\}$ is the prescribed order on the preimage of x in C_a. $\square$

For every $w \in \widetilde{W}$, we define $\mathcal{H}_w$ to be the reduced closure of $\Gamma(w)|_{\mathscr{H}ecke^{par,rs}}$ in $\mathscr{H}ecke^{par}$. Convolution with $\mathcal{H}_w$ over the Hecke correspondence induces a map

$$[\mathcal{H}_w]_\#\colon h_*^{par}\overline{\mathbb{Q}}_l \to h_*^{par}\overline{\mathbb{Q}}_l.$$

Yun checks that this gives the desired action of $\widetilde{W}$ on $h_*^{par}\overline{\mathbb{Q}}_l$.

8.2. **Extending to the Graded DAHA.** In this section, we will discuss the geometric contributions that allow one to extend the action to an action of the graded (trigonometric) DAHA. As the author does not have a good understanding of DAHA, we will not give neither a definition nor an intuitive reason for considering this larger algebra or its action. Instead, we will only consider the trigonometric DAHA to be the extension of the group algebra $\mathbb{C}[\widetilde{W}]$ by the following pieces:

$$\mathbf{X}^*(T), \quad \mathbf{X}^*(\mathbb{G}_m^{rot}), \quad \mathbf{X}^*(\mathbb{G}_m^{cen}), \quad u$$

where the final u is simply an indeterminate. The action of each of these additional pieces is constructed as a cup product with a certain Chern class. The latter three are associated to standard sorts of bundles. However, the action of $\mathbf{X}^*(T)$ involves a very special type of line bundle on $\mathcal{M}^{par}$ which will play an important role in the proof of $P = W$. We focus on this case now.

Let Bun_G^{par} be the stack parametrizing $(x, \mathcal{E}, \mathcal{E}_x^B)$ where $\mathcal{E}$ is a G bundle, $x \in C$, and $\mathcal{E}_x^B$ is a B reduction of $\mathcal{E}$ at x. We have a tautological T torsor on Bun_G^{par} whose fiber at $(x, \mathcal{E}, \mathcal{E}_x^B)$ is $\mathcal{E}_x^B/N$ where N is the unipotent subgroup for B. Then any $\xi \in \mathbf{X}^*(T)$ induces a line bundle on Bun_G^{par} by twisting the tautological bundle by ξ. We denote by $\mathcal{L}(\xi)$ the pullback of this line bundle to $\mathcal{M}^{par}$ along the forgetful map. In Yun's global Springer theory, the action of $\xi \in \mathbf{X}^*(T)$ is by the cup product $- \cup c_1(\mathcal{L}(\xi))$.

The line bundles $\mathcal{L}(\xi)$ are essential building blocks for the $P = W$ theorem because they give the Chern roots of the universal bundle $\mathcal{U}$ on $\mathcal{M}$. Namely, for the parabolic Hitchin system,

$$c_1(\mathcal{U}^{par}) = \sum_{\xi \in \Sigma} c_1(\mathcal{L}(\xi))$$

where $\Sigma \subset \mathbf{X}^*(T)$ is a basis for the weight lattice. In this way, strong perversity of the universal bundle can be deduced from strong perversity of the line bundles $\mathcal{L}(\xi)$. For this reason, we now study the perversity of $\mathcal{L}(\xi)$.

In fact, in his paper "Langlands Duality and Global Springer Theory," Yun proves a first perversity result about these chern classes (see Lemma 8.8). In this paper, Yun was aiming to describe a relationship between his global Springer theory and Langlands duality. We sketch these results in Section 8.3 below, though this is not strictly necessary for Maulik and Shen's work.

We now state Yun's perversity result.

Lemma 8.8. *(Yun, Lemma 3.2.3) Over a Zariski dense open subset of $A \times C$, $c_1(\mathcal{L}(\xi))$ has strong perversity 1 for all $\xi \in \mathbf{X}^*(T)$.*

This will be the sole technical input from the Yun's analysis of the DAHA action that we will use to prove $P = W$.

Proof Sketch. To prove this result, we first reduce to the diamond locus, where the identification of $\mathcal{M}^{par}$ with the group scheme $\mathcal{P}$ gives an explicit description of the map. The reduction to the diamond locus is not automatic. It is given by the following stronger result.

Lemma 8.9. *Any simple constituent factor of $\widetilde{h}_*\overline{\mathbb{Q}}_\ell|_{A^{ani}}$ is the middle extension of its restriction to the regular semisimple locus $\widetilde{A}^{rs}$, i.e. the restriction of $\widetilde{A}$ to the regular semisimple locus*

$$(A \times C)^{rs} = \{(a, x)\colon a(x) \in \mathfrak{g}^{rs}/\!\!/G\}.$$

The proof of Lemma 8.9 uses Ngô's argument for the Support Theorem (see Section 9.1.) In particular, this is the key ingredient in Ngô's work that allows him to show that, over a finite field, δ regularity holds on a large enough open subset of the Hitchin base.

(Todo: To do: Insert computation now.) ∎

Showing $c_1(\mathcal{L}(\xi))$ has strong perversity 1 globally is equivalent to showing that the induced map

$${}^p\mathcal{H}^i(R\hat{h}^p_*\mathbb{Q}_{\hat{\mathcal{M}}^{par}}) \to^p \mathcal{H}^i(R\hat{h}^p_*\mathbb{Q}_{\hat{\mathcal{M}}^{par}}[2])$$

vanishes for all i. Since we have this result over a generic locus, extending our results over the entire Hitchin base can be done by decomposing the perverse cohomology groups into simple constituents and showing that each simple piece is of full support. We examine such a theorem in Section 9.

8.3. **Global Springer Theory and Langlands Duality.** There are important connections between the geometry of the Hitchin fibration for G and that of the Langlands dual group $G^\vee$. For example, there is an identification of Hitchin bases $A_G \simeq A_{G^\vee}$ and $\widetilde{A}_G \simeq \widetilde{A}_{G^\vee}$ given by a choice of Killing form, and so we get two torus fibrations

$$\begin{tikzcd} \mathcal{M}_G \arrow[dr, "\widetilde{h}"'] & & \mathcal{M}_{G^\vee} \arrow[dl, "\widetilde{h}^\vee"] \\ & \widetilde{A} & \end{tikzcd}$$

It was proved by Hausel-Thaddeus in the case $G = \mathrm{SL}_n, \mathrm{PGL}_n$, by Donagi-Pantev in the case of general G over $\mathbb{C}$, and by Chen-Zhu in general characteristic and for generl reductive G that $\widetilde{h}$ and $\widetilde{h}^\vee$ are generically dual torus fibrations, i.e. on $\widetilde{A}^\diamondsuit$, the groups $\mathcal{P}$ and $\mathcal{P}^\vee$ are dual "Beilinson 1-motives." This means that over $\widetilde{A}^\diamondsuit$, we have

$$\mathcal{P} = \mathrm{Hom}(\mathcal{P}^\vee, B\mathbb{G}_m)$$

and in particular, it means that if one ignores components and gerbe structure, the fibers of $\widetilde{h}$ and $\widetilde{h}^\vee$ are dual abelian varieties.

Likewise, Yun expected that the action of $\mathbf{X}^*(T)$ on $\widetilde{h}_*\overline{\mathbb{Q}}_\ell$ that he constructed using the Chern classes $c_1(\mathcal{L}(\lambda))$ could be studied on the dual side in terms of a different Springer action on $\widetilde{h}^\vee_*\overline{\mathbb{Q}}_\ell$. In particular, let us consider the pushforward $\widetilde{h}_*\overline{\mathbb{Q}}_\ell$. By the decomposition theorem, we have a direct sum decomposition of this into isotypic components

$$\widetilde{h}_*\overline{\mathbb{Q}}_\ell = \bigoplus_{\kappa\in\widehat{T}} (\widetilde{h}_*\overline{\mathbb{Q}}_\ell)_\kappa$$

for $\widehat{T} = \mathrm{Hom}(\mathbf{X}_*(T), \mathbb{G}_m)$. Let the *stable part* $(\widetilde{h}_*\overline{\mathbb{Q}}_\ell)_{st}$ be the isotypic component corresponding to the trivial κ. In general, there are two types of simple constituents:

- Those associated to a $\kappa \in Z\widehat{G}(\overline{\mathbb{Q}}_\ell)$, where $\widehat{G}$ is a split group over $\overline{\mathbb{Q}}_\ell$ which is Langlands dual to G. These are isomorphic to the stable part and have full support.
- Those with smaller support, which are associated to endoscopic groups.

Yun states results for the stable part, though well-phrased results hold for other isotypic pieces using the theory of endoscopy. If $d = \dim(\mathcal{M}_G^{par})$, we consider the sheaves

$$K = (\widetilde{h}_*\overline{\mathbb{Q}}_\ell)_{st}[d](d/2)$$

$$L = (\widetilde{h}^\vee_*\overline{\mathbb{Q}}_\ell)_{st}[d](d/2)$$

(Note: The point of these shifts and Tate twists is to center the pushforward with respect to Verdier and Langlands dualities.)

Let L^i and K^i denote the i-th perverse cohomology groups of L and K, respectively. For $\lambda \in \mathbf{X}^*(T^\vee)$, let $\mathrm{Ch}(\lambda)$ denote the map $L \to L[1](1)$ induced by cup product with $c_1(\mathcal{L}(\lambda))$. Accepting Yun's result that $\mathrm{Ch}(\lambda)$ induces the zero map $L^i \to L^{i+2}$, the nontrivial maps of interest become

$$\mathrm{Ch}^i(\lambda)\colon L^i \to L^{i+1}[1](1)$$

Likewise, we can view $\lambda \in \mathbf{X}_*(T)$ via the identification $\mathbf{X}^*(T^\vee) = \mathbf{X}_*(T)$. Doing so, we may consider a Springer map

$$\mathrm{Sp}^i(\lambda) =^p \mathcal{H}^i(\lambda - \mathrm{Id})\colon K^i \to K^{i+1}[1]$$

The main result is the following.

Theorem 8.10. *(Yun) There is an isomorphism $\mathbb{D}K^i \simeq L^i(i)$ where $\mathbb{D}$ denotes Verdier duality compatible with the Chern and Springer maps above in the sense that the following diagram commutes:*

$$\begin{tikzcd} K^{-i} \arrow[r, "\simeq"] \arrow[d, "\mathrm{Sp}^{-i}(\lambda)"] & \mathbb{D}K^i \arrow[r, "\simeq"] \arrow[d, "\mathbb{D}\mathrm{Sp}^{i+1}(\lambda)[1]"] & L^i(i) \arrow[d, "\mathrm{Ch}^i(\lambda)(i)"] \\ K^{-i-1} \arrow[r, "\simeq"] & \mathbb{D}K^{i+1}[1] \arrow[r, "\simeq"] & L^{i+1}[1](i+1) \end{tikzcd}$$

Proof. We will only sketch here the identification $\mathbb{D}K^i \simeq L^i(i)$, leaving the compatibility with the action maps for the interested reader to explore in Yun's paper.

The first major ingredient is a reduction to the diamond locus $\widetilde{A}^\diamondsuit$. This is immediate from Lemma 8.9.

Over the diamond locus, L^i and K^i are quite explicit. Namely, if $N = d - \dim(\widetilde{A})$ is the reltive dimension of $\widetilde{h}$, then we have

$$\begin{aligned} L^i_{\diamondsuit} &\simeq H^{i+N}(\mathcal{M}^{par}_{G^\vee}/\widetilde{A}^{\diamondsuit})_{st}(d/2) = \widetilde{p}^* H^{i+N}(\mathcal{M}_{G^\vee}/A^{\diamondsuit})_{st}(d/2) \\ &\simeq \widetilde{p}^* H^{i+N}(\mathcal{P}^\vee/A^{\diamondsuit})_{st}(d/2) \simeq \widetilde{p}^* \left(\bigwedge^{i+N} V_\ell(\mathcal{P}^{\vee,0}/A^{\diamondsuit})^* \right)(d/2) \end{aligned}$$

where $\widetilde{p} \colon \widetilde{A} \to A$ is the projection and $V_\ell(\mathcal{P}^{\vee,0}/A^{\diamondsuit})^* = \left(H^1(\widetilde{A}^{\diamondsuit}/A^{\diamondsuit}) \otimes_{\mathbb{Z}} \mathbf{X}^*(T^\vee)\right)_W$ is the relative ℓ-adic Tate module. Similarly, we can compute

$$\mathbb{D}K^i \simeq \widetilde{p}^* \left(\bigwedge^{i+N} V_\ell(\mathcal{P}^0/A^{\diamondsuit}) \right)(d/2 - N)$$

where $V_\ell(\mathcal{P}^0/A^{\diamondsuit}) \simeq \left(H_1(\widetilde{A}^{\diamondsuit}/A^{\diamondsuit}) \otimes_{\mathbb{Z}} \mathbf{X}_*(T)\right)^W$ is again a relative ℓ-adic Tate module. By a relative Poincaré duality, there is an identification $H_1(\widetilde{A}^{\diamondsuit}/A^{\diamondsuit}) \simeq H^1(\widetilde{A}^{\diamondsuit}/A^{\diamondsuit})(1)$ coming from the perfect pairing

$$H^1(\widetilde{A}^{\diamondsuit}/A^{\diamondsuit}) \otimes H^1(\widetilde{A}^{\diamondsuit}/A^{\diamondsuit}) \xrightarrow{\cup} H^2(\widetilde{A}^{\diamondsuit}/A^{\diamondsuit}) \simeq \overline{\mathbb{Q}}_\ell(-1).$$

This gives a duality isomorphism

$$\beta \colon V_\ell(\mathcal{P}^0/A^{\diamondsuit}) \simeq V_\ell(\mathcal{P}^{\vee,0}/A^{\diamondsuit})^*(1)$$

which gives the desired isomorphism. □

Note: A similar identity on ℓ-adic Tate modules serves as a crucial piece of Chen and Zhu's proof of Langlands duality in general characteristic.

9. Ngô's Support Theorem and a Parabolic Variant

Suppose that we have a proper map $f \colon X \to Y$ of smooth schemes. Then the powerful Decomposition Theorem says that the pushforward $Rf_*\mathbb{Q}_X$ decomposes into simple constituent pieces $IC(Z, \mathcal{L})$ where $\mathcal{L}$ is a local system on an open subset of Z. For the map $h^{ani} \colon \mathcal{M}^{ani} \to A^{ani}$, Ngô studied these supports, proving a powerful Support Theorem using ideas inspired by the work of Goresky and McPherson. This support theorem was the most important result leading to Ngô's proof of the Fundamental Lemma of Langlands, Shelstad, and Waldspulger. In type A, this was extended to the entire Hitchin base by work of Chaudinard-Laumon and by de Cataldo. In these notes, we will discuss these ideas in three subsections: the first discussing Ngô's Support Theorem and the ideas behind its proof, the second discussing the extension of this Support Theorem to the entire Hitchin base, and the third discussing Maulik and Shen's extension of Yun's parabolic support theorem, which serves as the critical ingredient in their proof of the $P = W$ conjecture in type A.

9.1. **Ngô's Support Theorem.** For general proper maps $f\colon X \to Y$, with X smooth, it can be quite difficult to determine the set of supports. If f is a smooth map, then the perverse cohomology sheaves coincide with the usual cohomology sheaves, which form a local system on Y. In particular, all supports are full. Similarly, if f is small, then all supports are full and if f is semismall, then the supports are given by the images of the maximal dimensional components of $X \times_Y X$ in Y. These cases apply to the Grothendieck-Springer resolution and Springer resolution, respectively. Beyond these cases, determining which supports appear can be subtle and very difficult. Ngô's contribution is extending support theorems to the case of δ-regular weak abelian fibrations, and then using these results to determine the supports for the Hitchin morphism. In this section, we review the structures involved and apply the results to the Hitchin fibration. The resulting statement is used in Ngô's proof of the Fundamental Lemma as well as Maulik and Shen's proof of the $P = W$ conjecture.

9.2. **Weak Abelian Fibrations and δ-Regularity.** Let $f\colon \mathcal{M} \to \mathcal{A}$ be a proper map with $\mathcal{M}$ smooth, and let $g\colon \mathcal{P} \to \mathcal{A}$ be a smooth, commutative group object over $\mathcal{A}$ which acts on $\mathcal{M}$ over $\mathcal{A}$

Definition 9.1. The morphisms f and g form a weak abelian fibration if the following are satisfied:

(1) (Relative Dimensions) The relative dimensions of f and g are the same.
(2) (Affine Stabilizers) If
$$1 \to R \to \mathcal{P} \to A \to 1$$
is the Chevalley decomposition of $\mathcal{P}$ with R the (fiberwise) maximal connected affine subgroup, then for every $a \in \mathcal{A}$ and $m \in \mathcal{M}_a$, we have
$$\mathrm{Stab}_{\mathcal{P}_a}(m) \subset R_a$$
(3) (Polarizable Tate Module) Let $H_1(\mathcal{P}/A) = H^{2g-1}(g_!\mathbb{Q}_\ell)$ denote the family whose fiber is the ℓ-adic Tate module $H_1(\mathcal{P}/A)_a = T_{\mathbb{Q}_\ell}(\mathcal{P}_a)$. Then, locally in the étale topology, there is an alternating form ψ on $H_1(\mathcal{P}/A)$ such that for all $a \in \mathcal{A}$, $\psi|_{T_{\mathbb{Q}_\ell}(R_a)} = 0$ and $\psi|_{T_{\mathbb{Q}_\ell}(A_s)}$ is non-degenerate.

The notion of weak abelian fibration is a good one in the sense that it is stable under arbitrary base change; however, it is poor in the sense that a weak abelian fibration need not have abelian varieties for fibers. The following notion of δ regularity is how we control (a) that the generic fibers are abelian varieties and (b) that the fibers degenerate in a way that is compatible with the Chevalley decomposition of $\mathcal{P}$.

To introduce δ regularity, we recall the invariant $\delta(a) = \dim(R_a)$ introduced in Definition 8.3. Let
$$\mathcal{A}_\delta = \{a \in \mathcal{A} \colon \delta(a) = \delta\}.$$
The function δ is always upper semicontinuous, and so the $\mathcal{A}_\delta$ give locally closed subsets stratifying the base $\mathcal{A}$. The condition of δ regularity is one that limits the codimension of these spaces and so limits the speed of degeneration.

Definition 9.2. A weak abelian fibration is δ regular if for all δ,

$$\mathrm{codim}_{\mathcal{A}}(\mathcal{A}_\delta) \leq \delta$$

In particular, $\mathcal{A}_0 \subset \mathcal{A}$ is dense, and over this locus, $\mathcal{P} = A$ is an abelian variety acting on $\mathcal{M}$ with finite stabilizers. In this case, the fibers of f can be identified with disjoint unions of abelian varieties.

Example 9.3. Let $X \to \mathcal{A}$ be a family of curves with reduced, irreducible fibers and with planar singularities. Let $\mathcal{P} = \mathrm{Jac}_{X/\mathcal{A}}$ be the relative Jacobian variety over $\mathcal{A}$ and let $\mathcal{M} = \overline{\mathrm{Jac}}_{X/\mathcal{A}}$ be the compactified relative Jacobian, i.e. the family whose fiber M_a is the moduli of torsion-free, rank 1 sheaves on X_a. The polarizability of the Tate module is given by the Weil pairing.

In this case, the δ invariant introduced above measures Serre's δ invariant, i.e. if X_a is the fiber of $X \to \mathcal{A}$ over a and if $c\colon \overline{X}_a \to X_a$ is the normalization map, then

$$\delta(a) = \dim H^0(X_a, c_*\mathcal{O}_{\overline{X}_a}/\mathcal{O}_{X_a}).$$

For smooth curves X_a, we have $\delta(a) = 0$. If X_a has an isolated singularity, then since we have planar singluarities we can find an affine patch $\mathrm{Spec}\left(\frac{k[x,y]}{f(x,y)}\right)$ with the isolated singularity at the origin. Suppose that $f(x,y) = x^n - y^m$ for coprime n and m. Then, $\delta(a) = \dim_k(k[t]/k[t^n, t^m])$. For such curves $x^n = y^m$, it is related to the Milnor number $\mu = \dim_k \Big(k[x,y]/(\partial_x f, \partial_y f)\Big) = (n-1)(m-1)$ by $\mu = 2\delta$. For general curves, we get the relationship

$$\mu = 2\delta - r + 1$$

where r is the number of locally irreducible branches at the singularity. [e.g. For a cuspidal cubic $r = 1$ while for a nodal cubic $r = 2$.]

It is not immediate that δ regularity should hold for any particular family $X \to A$ above. It is true for a versal deformation of a curve with planar singularities.

Example 9.4. Let $k = \mathbb{C}$ and f be an integrable system, i.e. f is of relative dimension $d = \dim(\mathcal{A})$, $\mathcal{M}$ carries a symplectic form and for any $a \in \mathcal{A}$ and $m \in \mathcal{M}_a$, $T_m\mathcal{M}_a$ is coisotropic with respect to the symplectic form on $\mathcal{M}$. If the weak abelian fibration is compatible with the integrable system structure in the sense that $\mathrm{Lie}(\mathcal{P}_a) \simeq T_a^*\mathcal{A}$ acts by the dual map

$$T_a^*\mathcal{A} \to T_m^*\mathcal{M} \simeq T_m\mathcal{M},$$

then this system is automatically δ regular. The δ regularity comes by identifying the image of $T_m\mathcal{M}$ in $T_a\mathcal{A}$ with the codimension $\delta(a)$ subspaces $\mathrm{Lie}(A_a)^* \subset \mathrm{Lie}(\mathcal{P}_a)^* \simeq T_aA$.

Note that this applies to the case of the anisotropic locus or semistable locus of the Hitchin system when $k = \mathbb{C}$ and $D = K_C$ is the canonical bundle.

Hitchin systems in characteristic zero do satisfy δ regularity, with the proof inspired by the work of Goresky, MacPherson, and Kottwitz. Over finite fields, one can still get the codimension estimate needed for δ regularity over an open dense subset of the anisotropic locus which is large enough to recover Waldspurger's version of the Fundamental Lemma.

For the purposes of Maulik and Shen's proof, we will only need the version over $k = \mathbb{C}$, but for general line bundles D of sufficiently high degree.

9.3. **Ngô's Support Theorem.** The main result is as follows:

Theorem 9.5. *Let $k = \overline{k}$ be algebraically closed. If $f\colon \mathcal{M} \to \mathcal{A}$ and $g\colon \mathcal{P} \to \mathcal{A}$ form a δ regular weak abelian fibration of relative dimension d with M smooth, then for any n and any simple perverse factor K of ${}^p\mathcal{H}^n(f_*\mathbb{Q}_\ell)$, there exists a simple factor of the top degree cohomology $R^{2d}f_*\mathbb{Q}_\ell$ restricted to some open set of $\mathcal{A}$ with the same support.*

The support theorem above follows from the refined version for not necessarily δ-regular fibrations:

Theorem 9.6. *Let f, g be a weak abelian fibration (not necessarily δ regular), and let*

$$\delta_Z = \min\{\delta(a)\colon a \in Z\}$$

be the minimal value of δ on the closed set Z. Then, for any simple factor K of ${}^p\mathcal{H}^n(f_*\mathbb{Q}_\ell)$, we have

$$\mathrm{codim}_{\mathcal{A}}(Z) \leq \delta_Z.$$

Moreover, when equality holds, the support Z appears as a support in the top degree cohomology $R^{2d}f_*\mathbb{Q}_\ell$ restricted to an open subset of $\mathcal{A}$.

We first address an important first case which serves as the inspiration for the proof: Let $f\colon X \to S$ be a proper family of curves with $\dim X = 2$, $\dim S = 1$, and the fibers of f being irreducible. Then, applying Poincaré duality to the decomposition

$$f_*\mathbb{Q}_\ell[2] = \bigoplus K[n]$$

implies that for every copy of $K[n]$ that appears, so also must $K^\vee[-n]$. Now, consider those K which are the IC sheaves of points. Then $K[n]$ contributes in ordinary cohomological degree $-n$ while $K^\vee[-n]$ contributes in degree n. Since $f_*\mathbb{Q}_\ell[2]$ is concentrated in ordinary cohomological degrees $-2, -1, 0$, we get $n = 0$, and K is a simple factor of $\mathcal{H}^0(f_*\mathbb{Q}_\ell[2]) = \mathcal{H}^2(f_*\mathbb{Q}_\ell)$, which is a constant sheaf of rank 1. Therefore, there can be no such factor supported at a point, and we conclude that all supports are full, i.e. $\mathrm{codim}_S(Z) < 1$.

With minimal modification of the above argument, Goreskey and MacPherson prove a more general "trivial" bound: If $f\colon X \to S$ is proper with all fibers having dimension d and X and S smooth over k, then for any support Z appearing in the decomposition of $f_*\mathbb{Q}_\ell$, we have

$$\mathrm{codim}_S(Z) \leq d \tag{9.1}$$

Moreover, if the fibers of f are all irreducible, then the inequality can be made strict.

The intuition for the rest of the proof of the result comes from a lifting argument. Though the assumptions that follow are *not* true for Hitchin systems, the false proof is nonetheless a model for the actual approach. We make the following (false) statements:

- For every $a \in Z$, there exists a lift of the abelian variety A_a to A_S for some étale neighborhood S of a in A.

- There exists a homomorphism $A_S \to P_S$ such that the composition with the projection $P_S \to A_S$ is an isogeny of A_S.

Assuming these two properties, we have an action of A_S on M_S with finite stabilizers, and so the quotient $[M_S/A_S]$ is smooth and proper over S with relative dimension δ_Z. By the Goresky-MacPherson inequality (equation (9.1)), it follows directly that $\mathrm{codim}_{\mathcal{A}}(Z) \leq \delta_Z$.

In general, these lifting assumptions do not hold, but the proof idea follows by mimicking these steps at the level of homology. Let $\Lambda_{\mathcal{P}} = g_! \mathbb{Q}_\ell[2d](d)$ be the complex concentrated in degrees $-2d, -2d+1, \ldots, -1, 0$. Concretely, for $0 \leq i \leq d$, we have the stalks $H^{-i}(\Lambda_{\mathcal{P}})_a = \bigwedge^i T_{\mathbb{Q}_\ell}(P_a)$ where $T_{\mathbb{Q}_\ell}$ again is the ℓ-adic Tate module. Then the cup product gives an algebra structure on $\Lambda_{\mathcal{P}}$, and we have a trace map

$$tr\colon \Lambda_{\mathcal{P}} \otimes f_! \mathbb{Q}_\ell \to f_! \mathbb{Q}_\ell$$

The proof will proceed with the lifting property being replaced with the condition that "$f_!\mathbb{Q}_\ell$ is a free module over the abelian part of $\Lambda_{\mathcal{P}}$". We will make this precise next.

Index the supports by Z_α for $\alpha \in \mathfrak{A}$, and let K_α^n denote the sum of all simple constituents of ${}^p\mathcal{H}^n(f_* \mathbb{Q}_\ell)$ with support Z_α. Put $K_\alpha = \bigoplus_n K_\alpha^n[-n]$.

For a fixed $\alpha \in \mathfrak{A}$, we have now associated K_α, with support Z_α. We now wish to introduce the action of the abelian variety. Let x_α be the generic point of Z_α, and consider the Chevalley sequence

$$1 \to R_\alpha \to \mathcal{P}_{x_\alpha} \to A_\alpha \to 1$$

For any homological lifting $T_{\mathbb{Q}_\ell}(A_\alpha) \to T_{\mathbb{Q}_\ell}(\mathcal{P}_{x_\alpha})$, we get an induced map $\Lambda_{A_\alpha} \to \Lambda_{\mathcal{P}_{x_\alpha}}$. In particular, every such lifting gives an action of Λ_{A_α} on K_α. We can now phrase our freeness lemma:

Lemma 9.7. *(Freeness) For any $\alpha \in \mathfrak{A}$ and for every lifting $T_{\mathbb{Q}_\ell}(A_\alpha) \to T_{\mathbb{Q}_\ell}(\mathcal{P}_{x_\alpha})$, we get that K_{α,x_α} is a free Λ_{A_α} module.*

We first show the proof of Theorem 9.6 assuming the Freeness Lemma:

Proof of Theorem 9.6. Fix $\alpha \in \mathfrak{A}$ and let $\delta_\alpha = \delta_{Z_\alpha} = \delta(x_\alpha)$. The Freeness Lemma implies that the degrees

$$\{n \in \mathbb{Z}\colon K^n_{\alpha,x_\alpha} \neq 0\}$$

are unions of intervals of length $2(d - \delta_\alpha)$. By Poincaré duality this set is symmetric over $\dim(M)$, and hence, there exists some degree $n \geq \dim(M) + d - \delta_\alpha$ such that $K^n_{\alpha,x_\alpha} \neq 0$. But also $K^n_{\alpha,x_\alpha}[-n]$ is in cohomological degree $n - \dim(Z_\alpha)$, so the trivial bound gives $n - \dim(Z_\alpha) \leq 2d$. Combining these gives

$$\dim(M) + d - \delta_\alpha \leq n \leq 2d + \dim(Z_\alpha)$$

which, when rearranged, gives $\mathrm{codim}_A(Z_\alpha) \leq \delta_\alpha$. $\square$

The proof of the Freeness Lemma uses a downward induction on $\dim(Z_\alpha)$ to reduce to the absolute case. We discuss this case now. Let $\mathcal{A} = \mathrm{Spec}(k)$ for $k = \overline{k}$. If $\mathcal{P} = A$ is an abelian variety, then the quotient $[\mathcal{M}/A]$ is proper over k and so the degeneration of the Leray spectral sequence proves that

$$H^*(\mathcal{M}) = H^*([\mathcal{M}/A]) \otimes \Lambda_A$$

is free over Λ_A. If $\mathcal{P}$ is instead defined over a finite field, then $\mathrm{Ext}^1(A, R)$ is annihilated by some integer N which is invertible in k, and so there is a pseudo-splitting $A \to \mathcal{P}$ such that the composite map $A \to \mathcal{P} \to A$ is multiplication by N. This induces a canonical lift $T_{\mathbb{Q}_\ell}(A) \to T_{\mathbb{Q}_\ell}(\mathcal{P})$ for which the Lemma can be reduced to the case of $\mathcal{P} = A$. For all other lifts, one can deduce the statement by a deformation argument, since the set of lifts satisfying the property can be thought of as an open, Frobenius-stable subset of the deformation space $\mathrm{Hom}(T_{\mathbb{Q}_\ell}A, T_{\mathbb{Q}_\ell}R)$. Finally, the case of $\mathcal{P}$ defined over an arbitrary field can be reduced to the case of k finite by a spreading out argument.

9.4. **Application to Hitchin Systems.** Next, we apply the Support Theorem to the Hitchin system. For simplicity, we will assume that G is semisimple. We have δ regularity over the anisotropic locus whenever the characteristic is zero, and hence, Theorem 9.5 applies. Of course, we'd like to apply it to fully understand the list of supports possible. We will see that these are closely related to the component group $\pi_0(\mathcal{P})$.

Recall that the anisotropic locus A^{ani} is the locus where $\pi_0(\mathcal{P}_a)$ is finite. Let $f^{ani}\colon \mathcal{M}^{ani} \to \mathcal{A}^{ani}$ be the Hitchin map. The action of $\mathcal{P}$ on the perverse cohomology groups ${}^p\mathcal{H}^n(f_*\mathbb{Q}_\ell)$ factors through the fiberwise component group $\pi_0(\mathcal{P})$, and moreover, for every $a \in A$, $\pi_0(\mathcal{P}_a)$ acts on $\pi_0(\mathcal{M}_a)$ simply transitively. We let the *stable locus* ${}^p\mathcal{H}^n(f_*\mathbb{Q}_\ell)_{st}$ denote the summand of ${}^p\mathcal{H}^n(f_*\mathbb{Q}_\ell)$ fixed by $\pi_0(\mathcal{P})$.

Corollary 9.8. *All simple constituents of ${}^p\mathcal{H}^n(f_*\mathbb{Q}_\ell)_{st}$ have full support.*

Proof. Any support appearing in ${}^p\mathcal{H}^n(f_*\mathbb{Q}_\ell)_{st}$ occurs in the top degree. Hence, it suffices to show $\mathcal{H}^{2d}(f_*\mathbb{Q}_\ell)_{st} \simeq \underline{\mathbb{Q}_\ell}$ is the constant sheaf. This follows from the fact that $\pi_0(\mathcal{P}_a)$ acts simply transitively on $\pi_0(\mathcal{M}_a)$. □

In general, however, there may be other supports appearing the in the top cohomology $\mathcal{H}^{2d}(f_*\mathbb{Q}_\ell)$. This is where the theory of endoscopy comes into play.

Corollary 9.9. *Any simple constituent of ${}^p\mathcal{H}^j(f_*\mathbb{Q}_\ell)_\kappa$ must have support A_H for H an unramified endoscopic group attached to κ.*

We will not define endoscopy here. (If there is interest, I can add this to a later version of these notes.)

9.5. **Extending to the Entire Hitchin Base.** (Todo:)

10. Conclusion of Result

The sketch of the proof is now as follows:

(1) Use vanishing cycles to reduce problem to strong perversity of the universal bundle on $\hat{M}$.

(2) Use the parabolic Hitchin system to split the universal bundle into Chern roots of certain line bundles $\mathcal{L}(\xi)$ on $\mathcal{M}^{par}$.
(3) Use a result of Yun to get strong perversity of $\mathcal{L}(\xi)$ generically.
(4) Use a parabolic support theorem to extend the strong perversity of $\mathcal{L}(\xi)$ over the entire Hitchin base.
(5) Use the Springer action to reduce strong perversity over $\mathcal{M}^{par}$ to strong perversity over $\hat{M}$.