Rational Cherednik algebras (RCA) & (torus) knot invariants

§1. Reps of RCA:

$n \geq 2$, $\mathfrak{h}$ $(n-1)$-dim vector space

standard rep of $S_n = W$

roots: $\alpha_{ij} = x_i - x_j \in \mathfrak{h}^*$, $i \neq j$

$\mathfrak{h}_{reg} = \mathfrak{h} \setminus \{\alpha_{ij} = 0\}$

$D(\mathfrak{h}_{reg})$: ring of diff ops on $\mathfrak{h}_{reg}$

$\langle (x_i - x_j)^{\pm}, \partial_{x_i} - \partial_{x_j} \rangle$

$\forall c \in \mathbb{C}$

Def: RCA H_c is the subalgebra of $D(\mathfrak{h}_{reg}) \rtimes W$

generated by W, $x_i - x_j$,

Dunkl operators $y_i = \partial_{x_i} - c \sum_{i \neq j} \frac{1 - s_{ij}}{x_i - x_j}$

Ex: $c=0$, $H_0 = D(\mathfrak{h}) \ltimes S_n$

Ex: $n=2$, $x_1 - x_2$, s_{12}

$$y_1 - y_2 = \partial_{x_1} - \partial_{x_2} - 2c \frac{1 - s_{12}}{x_1 - x_2}$$

$$(y_1 - y_2)(x_1 - x_2) = 2 - 4c$$

don't erase

$H_c \curvearrowright \mathbb{C}[\mathfrak{h}]$: polynomial rep.

$\Downarrow$

L_c

Thm (Berest - Etingof - Ginzburg 02')

Only when $c = \frac{m}{n}$, $m \in \mathbb{Z}$, $(m,n) = 1$,

H_c has nontrivial f.d rep.

When $c = \frac{m}{n}$, $m > 0$, $(m,n) = 1$, The only fd irep of L_c is L_c.

Def: Fourier transform:

$$\Phi_c: H_c \to H_c$$

$$x_i \mapsto y_i, \quad y_i \mapsto -x_i$$

$$s \mapsto s$$

- Dunkl bilinear form:

$$\mathbb{C}[\mathfrak{h}] \times \mathbb{C}[\mathfrak{h}] \to \mathbb{C}$$

$$(f, g) \mapsto (\Phi_c(f) g)\big|_{x=0}$$

$c = 0$, nondegenerate.

$I_c = \text{kernel}(-,-) \qquad L_c = \mathbb{C}[\mathfrak{h}]/I_c$

E.g.: $n=2$,

$$(y_1 - y_2)(x_1 - x_2) = 2 - 4c,$$

$$\begin{array}{ccc} \| & & \| \\ (x_1 - x_2,\ x_1 - x_2)_c & & 0 \end{array} \qquad c = \frac{1}{2}$$

$$c = \frac{1}{2}, \quad x_1 - x_2 \in \bar{I}_c \Rightarrow L_c \cong \mathbb{C}$$

$$(y_1 - y_2)(x_1 - x_2)^k = (2k - 4c)(x_1 - x_2)^k$$

$$= 0 \qquad c = \frac{k}{2}$$

$$\bar{I}_c = ((x_1 - x_2)^k), \quad \dim L_c = k$$

§2: relations with knot invariants:

Take $h = \sum x_i y_i + \sum y_i x_i$

under h-action, $L_c = \bigoplus L_c(\ell)$ wt spaces

q-character $\mathrm{tr}_q(L_c) \triangleq \sum \dim(L_c(\ell))\, q^{\ell}$

Thm [Gorsky-Oblomkov-Rasmussen-Shende '14]

$$a^{2u} \sum_{i=0}^{n-1} a^{2i}\, \mathrm{tr}_q(\mathrm{Hom}_{S_n}(\wedge^i \mathfrak{h}, L_c)) = P^{HOMFLY}_{a,q}(K_{m,n})$$

$u = \frac{(m-1)(n-1)}{2}$ genus of $\{x^m = y^n\}$

Conj: (GORS)

$$\cdots\cdots \mathrm{tr}_{q,t} \cdots\cdots = P^{KhR}_{a,q,t}(K_{m,n})$$

Filrations on L_c for t-grading

Geometry

F^{perv} [OY] $\quad F^{alg} \sim F^{ind}$ $\quad$ Comments

Thm [M]: $F^{alg} = F^{ind}$

Rmk: proof was inspired by "F^{Hodge}"

Evidence for the GORS-conj with F^{ind}:

$a=0$ part: $\quad c = k + \frac{1}{n} \quad k \geq 0$

$$P^{\#khR}_{a=0,q,t}(K_{m,n}) = q,t \text{ Catalan}$$

$\wr$ [Haiman] [GS]

$$ch_{q,t}(L^W_c).$$

$\Downarrow$

F^{ind}

§3: filtrations & proof strategy:

$\mathcal{C}[t_0]^{\geq m} = \mathcal{C}(\mathcal{C}[t_0]^{w}_{+})$ Let β_c be a --

Def: $F^{alg}_j L_c = \Phi_c\left[\underbrace{(m^{j+1})^{\perp_c}}_{(-,-)_c}\right]\underbrace{\beta_c}_{\text{hst wt vector in } L_c}$

auxillary (okzilary)

~~axillary~~

Def: When $c = \frac{1}{n}$,

$$0 = F^{ind}_{-1} L_{\frac{1}{n}} \subseteq F^{ind}_{0} L_{\frac{1}{n}} = L_{\frac{1}{n}}$$

When $m > 1$, F^{ind} is defined inductively s.t. the following isoms hold,

- (flip) $\quad eL_{\frac{m}{n}} \cong eL_{\frac{n}{m}}$

$e = \frac{1}{|W|}\sum_{w\in S_n} w \qquad e^2 = e$

$D(f_0)^{w}$

Σ

$eL_c = L_c^{w}, \rightsquigarrow$ rep of $eH_c e$.

$e_- = \frac{1}{|W|}\sum_{w\in S_n} (-1)^{\mathrm{sign}(w)} w \qquad e_-^2 = e_-$

$e_- L_c = L_c^{\mathrm{sign}}$

$\delta = \prod_{i<j} (x_i - x_j)$ Vandermonde.

- (shift)

$L_c \cong H_c e_- \delta \otimes eL_{c-1}$

$eH_{c-1}e$

$(\deg x = 0,\ \deg y = 1,\ \deg w = 0)$

Ex: $c = \frac{3}{2}$ $\quad L_c = \{ [x_1 - x_2] \} \quad \cancel{((x_1-x_2)^3)}$

$e L_{\frac{1}{2}}$ $\quad$ don't erase

$\exists !$

$e_- L_{\frac{3}{2}}$ $\quad$ $x_1 - x_2 = \delta$

$e L_{\frac{3}{2}}$. $\quad 1 \xleftarrow{y_1 - y_2}$ $\quad \xrightarrow{x_1 - x_2} (x_1 - x_2)^2$

$\deg(x_1 - x_2) = 0, \quad \deg(y_1 - y_2) = 1$

$$F_0^{ind} = \langle x_1 - x_2, (x_1 - x_2)^2 \rangle$$

$$F_1^{ind} = L_{\frac{3}{2}} = F_1^{alg} = L_{\frac{3}{2}}.$$

$$\cancel{F_0^{alg} = \Phi_c(\mathcal{H}_c)(x_1 - x_2)^2 = F_0^{ind}}$$

easier to show $F^{ind} \subseteq F^{alg}$.

As for "$F^{ind} \supseteq F^{alg}$":

$\underline{\underline{F_0}}$:

Classical : (c)

$$\mathbb{C}[\mathfrak{h}] = \mathcal{M}\textcircled{t}^{1}_{c}\mathcal{H}_c$$

$\mathcal{H}$:= space of harmonic polynomials

$= \{ h \mid \underline{\Phi}_{0_c}(\varphi)\, h = 0,\ \forall \varphi\ \mathbb{C}[\mathfrak{h}]^W_+ \}$.

$=$ $\mathbb{C}[\partial_1, \cdots, \partial_n]\,\delta$:

For general $c > 1$:

$$\mathbb{C}[\mathfrak{h}] = y \otimes^{t_c} \mathcal{H}_c$$

$\mathcal{H}_c$: space of c-harmonic polys.

$$= \{ h \mid \Phi_c(p)\, h = 0,\ \forall p \in \mathbb{C}[\mathfrak{h}]^{W}_{+} \}$$

$$= \mathbb{C}[y_1, \dots, y_n]\,\delta$$

$$F_0^{alg} = \Phi_c(y^{t_c})\,\beta_c$$

$$= \Phi_c(\mathcal{H}_c)\,\beta_c$$

~~Back to the exo~~

$$= \Phi_c(\mathbb{C}[y_1, \dots, y_n]\,\delta)\,\beta_c$$

$$= \underset{\sim}{\mathbb{C}[x_1, \cdots, x_n]}\, \underline{\Phi_c(\delta)\beta_c}$$

$$= \mathbb{C}[x_1, \cdots, x_n]\ \beta_{c-1}$$

$$\subseteq F_0^{ind} L_c.$$

$\underline{F_{\geq 1}:}$

Using the description of I_c & combinotorics of c-harmonic symmetric polynomials

based on Arc spaces

[Gorsky 12] [Dunkl ?].

$F_1^{alg} \subseteq F_1^{ind}$ can be reduced to a linear algebra problem:

$R := \mathbb{C}[2b]/\mathfrak{m}$: coinvariant algebra.

$$A = \begin{pmatrix} x_1^{n-1} & \cdots & x_1^{m-n} \\ \vdots & & \vdots \\ x_n^{n-1} & & x_n^{m-n} \end{pmatrix}_{n \times (n-m)} \qquad \begin{pmatrix} x_1^{m-n} & \cdots & x_n^{m-n} \\ \vdots & & \vdots \\ x_1^{n-1} & \cdots & x_n^{n-1} \end{pmatrix}_{(n-m)\times n} = B$$

$$R^n \xrightarrow{\quad A \quad} R^{n-m} \xrightarrow{\quad B \quad} R^n$$

$$BA = 0 \implies \mathrm{Im}A \subseteq \ker B$$

$$F_1^{ind} \supseteq F_1^{alg} \iff \mathrm{Im}A = \ker B.$$

$$\dim(\mathrm{Im}A) = \dim(\ker B)$$

$$\iff \cancel{\operatorname{rank} A = \operatorname{rank} B =}$$

$$= (\cancel{2}n-m)\frac{n!}{2}$$

$m = 2n-1,$

$$\operatorname{Im} A = (x_1^{n-1}, \dots, x_n^{n-1})$$

$$\ker B = \bigcap \ker(x_i^{n-1})$$

$$R / \operatorname{Im} A \cong H^*(Sp_{m,m})$$

$$\begin{pmatrix} 0 & 1 & & & \\ & 0 & 0 & & \\ & & 0 & \ddots & \\ & & & \ddots & \ddots \\ & & & & 0 \end{pmatrix}$$

[lusztig]

$$= \operatorname{Ind}_{S_2}^{S_n} \mathrm{triv}.$$

$\rightsquigarrow \dim R/I_m A = \frac{|S_n|}{|S_2|} = \frac{n!}{2}$

$\rightsquigarrow \dim I_m A \; \cancel{} = \frac{n!}{2}$

~~Eren~~ general m: proof idea comes from

"distributive lattices"

~~$|S|$~~

Q: A geometric interpretation

for general m?