0.1 Computing KhR homology of $(2,1)$-torus knot

Apologies in advance for possible wrong shifts.

Fix notations:
$R = \mathbb{C}[x,y]$, $R^e := R \otimes_{\mathbb{C}} R^{op} = \mathbb{C}[x,y,x',y']$,
$W := S_2 = \langle \beta \rangle$, $B := R \otimes_{R^W} R(1)$.

Note that $B = R^e/(x+y-x'-y', xy-x'y')$ ($\deg(1) = -1$) and $R = R^e/(x-x', y-y')$. Therefore there is a graded quotient map $q : B(-1) \to R$ with $\mathrm{Ker}(q) = (x-x')$, equivalent to the diagonal closed embedding $\mathfrak{h} \to \mathfrak{h} \otimes_{\mathfrak{h}//W} \mathfrak{h}$.

On the other hand, since $(x-x')(x-y') = 0$ in B, we have a well-defined injective homomorphism $i : R \to B(1) : 1 \mapsto x - y' = x' - y$.

These two maps define the Rouquier complexes

$$T := [B(-1) \to R], \quad T^{-1} := [R \to B(1)].$$

Let us introduce the "bridge" homomorphism $k : B \to B(2) : 1 \mapsto x - x'$, whose kernel equals to $(x-y')$ (because $B/(x-y') = R$ is integral).

In conclusion, we have a graded exact sequence:

$$0 \to R \xrightarrow{i=(x-y')} B(-1) \xrightarrow{k=(x-x')} B(-3) \xrightarrow{q} R(-4) \to 0$$

Our goal is to compute $\mathrm{RHom}_{R^e}(R,T)$. First of all, R can be resolved by

$$\bigwedge^2 V \otimes_{\mathbb{C}} R^e \xrightarrow{d_2} V \otimes_{\mathbb{C}} R^e \xrightarrow{d_1} R^e$$

where $d_2 : (e_1 \bigwedge e_2)r \mapsto e_2(x-x')r - e_1(y-y')r$ and $d_1 : e_1 r_1 + e_2 r_2 \mapsto (x-x')r_1 + (y-y')r_2$.

As a result, $\mathrm{RHom}_{R^e}(R,R) = [R(-4) \xleftarrow{0} R \oplus R(-2) \xleftarrow{0} R]$ and

$$\mathrm{RHom}_{R^e}(R, B(-1)) = [B(-1) \xleftarrow{d_2'} V^* \otimes_{\mathbb{C}} B(-1) \xleftarrow{d_1'} B(-1)]$$

where $d_2' : e_1^* r_1 + e_2^* r_2 \to (y-y')r_1 + (x-x')r_2 = (x-x')(r_2 - r_1)$ and $d_1' : r \mapsto (x-x')e_1^* - (y-y')e_2^* = (x-x')(e_1^* + e_2^*)$ due to the relations defining B.

Therefore $\mathrm{Hom}_{R^e}(R, B(-1)) = \mathrm{Ker}(k) \cong R$. Moreover,

$$\mathrm{Ker}(d_1') = (e_1 + e_2)r + e_2(x-y')r'$$

hence

$R^1\mathrm{Hom}_{R^e}(R, B(-1)) \cong (e_1+e_2)B(-1)/(x-x') \oplus e_2(x-y')B(-1) = B(1)/\mathrm{Im}(k) \oplus \mathrm{Im}(i) \cong (R \oplus R)(-2)$.

Lastly $R^2\mathrm{Hom}_{R^e}(R, B(-1)) = B(-4)/(x-x') \cong R(-4)$.

It remains to compute the differentials between $R^i\mathrm{Hom}(R,B)$ and $R^i\mathrm{Hom}(R,B)$.

When $i = 2$, differential is simply the identity and so $\mathrm{HHH}^2(\beta) = 0$.

When $i = 0$, differential is the composition $i \circ q$ and hence $\mathrm{HHH}^0(\beta) = R/(x-y) \cong \mathbb{C}[x]$.

When $i = 1$, differential is $(id \oplus i \circ q)$ and we have $\mathrm{HHH}^1(\beta) = R(-2)/(x-y) \cong \mathbb{C}[x](-2)$.

As a result, we obtain $P_{a,t,q}(\beta) = \dfrac{1 + AQ^{-2}}{1 - Q^2} = \dfrac{1+a}{1-q}$ ($q = Q^2$ and $a = AQ^{-2}$)

0.2 $(2,k)$-torus link

First of all

$$T^k = [B(-1) \to R]^k \simeq [B(-2k+1) \to \cdots \to B(-1) \to R]$$

and hence

$$\mathrm{Hom}(R, T^k) = [R(-2k) \to \ldots R(-2) \to R]$$

where the differentials are $(x-y')$, 0, $\ldots$, from right to left.

As a result

$$P_{0,q,t}(T(2,2k+1)) = \frac{1}{1-Q^2} + \frac{T^{-2}Q^4}{1-Q^2} + \frac{T^{-4}Q^8}{1-Q^2} + \cdots = \frac{1 + qt^{-1} + q^2t^{-2} + \cdots + q^kt^{-k}}{1-q}$$