Relation with q,t - Catalan:

Branches & Puiseux expansions:

$\tilde{C} \longrightarrow C$ normalization

$$q \longmapsto \underset{}{\overset{\cup}{p}}$$

At q: $x = t^k + \cdots$

$y = c\,t^l + \cdots$

Puiseux: $f = c\,t^{\frac{l}{k}} + \cdots$

Ex: $y^2 = x^3 + x^2$ $\quad t \mapsto (t^2-1,\ t^3-t)$

$y = \pm(x + \frac{1}{2}x^2 + \cdots)$

$y^2 = x^3$ $\quad t \mapsto (t^2, t^3)$

$y = x^{\frac{3}{2}}$

Results:

- ~~irreducibility from embedding dims~~
- planar singularity $\Rightarrow$ $\overline{JC} \setminus JC$ can be described
- Rational unibranched $\Rightarrow$ $\overline{J}C = \prod \overline{J}C_p$

Jacobi factor.
analytic type.

- Jacobi factor w/ one Puiseux pair (p,q)

$\Rightarrow$ affine cells labeled by

$\Gamma^{p,q}$ - semimodules Δ; $\dim = \dim \Delta$

$$= \sum_{j=0}^{p-1} \left| [a_j, a_j+q) \setminus \Delta \right|$$

Piontkowski

$$0 \in \Delta \ \& \ 0 + \Gamma^{p,q} \subseteq \Delta$$

$$\Gamma^{p,q} = \{ap+bq,\ a,b \in \mathbb{Z}_{\geq 0}\}$$

From $\overline{JC}$ to Sp_γ

Beauville: (generalized by Ngô into product formula of Hitchin fibres).

$\pi : X_p \longrightarrow \overline{JC}$ is a homeomorphism
(example: $y^2 = x^3$)

$X_p :=$ $\mathbb{C}[[t^p, t^q]]$-submod in $\mathbb{C}((t))$ $\exists N$ s.t.

① $t^N \mathbb{C}[[t]] \subset M \subset t^{-N}\mathbb{C}[[t]]$

② $M / t^N\mathbb{C}[[t]]$ is proj of $rk = rk \dfrac{\mathbb{C}[[t^p, t^q]]}{t^N\mathbb{C}[[t]]}$

$t^{-N}\mathbb{C}[[t]]/M$ $\qquad$ $\mathbb{C}[[t]] / \mathbb{C}[[t^p, t^q]]$

In the case of $\{x^p = y^q\}$

$$M \longleftrightarrow \Gamma_M = \{n \in \mathbb{Z} \mid t^n \in M\}$$

is a $\langle p, q\rangle$-module

② $\Rightarrow \#(\mathbb{Z}_{\geq 0} - M)$

$= \#(\mathbb{Z}_{\geq 0} - \Gamma)$

$\Leftrightarrow \quad 0 \in \Delta, \qquad \Delta \subseteq \mathbb{Z}_{\geq 0}$

$\langle p, q \rangle + \Delta \subseteq \Delta$

$\Rightarrow \langle p, q \rangle \subseteq \Delta$

$$Sp_\gamma = \left\{ \Lambda \subseteq \mathbb{C}((x))^p \;\middle|\; \begin{array}{l} \gamma\Lambda \subseteq \Lambda \\ \dim \Lambda / \Lambda \cap \mathcal{O}^p = \dim \mathcal{O}^p / \mathcal{O}^p \cap \Lambda \end{array} \right\}$$

$\mathbb{C}((x))^p \;\|\; \mathbb{C}((x))[y]/(y^p - x^q)$

$\cong$

X_p

0: fix the vol.

q,t - Catalan

$pq - px - qy$

-3	-6	-9	-12
1	-2	-5	-8
5	2	-1	-4

$\Gamma^{3,4} = \{3, 4, 6, 7, 8, \cdots\}$

missing ones $1, 2, 5$

$\# \{\triangle\} = \#$ dyck paths $=$ Catalan #

$$C_{p,q}(q,t) = \sum q^{\mathrm{dinv}(\Delta)} t^{\mathrm{area}(\Delta)}$$

$$\mathrm{dinv}(\Delta) = \left\{ x \in \Delta \,\middle|\, \frac{a(x)}{l(x)+1} \leq \frac{p}{q} \leq \frac{a(c)+1}{l(c)} \right\}$$

$$q^3 + q^2 t + q t^2 + t^3 + qt = C_{3,4}(q,t)$$

$\Gamma_{3,4}$	δ 3	$\mathbb{A}^3$
$+\langle t^5 \rangle$	2	$\mathbb{A}^2$
$+\langle t^2, t^5 \rangle$	1	$\mathbb{A}^2$
$+\langle t, t^5 \rangle$	1	$\mathbb{A}^1$
$\mathbb{C}[[t]]$	0	pt

$$P^{\text{hikita}}_{\bar{J}}(q,t) = \sum q^{\delta(M)} (qt^2)^{\dim[M]}$$

$$= 1 + q^2t^2 + q^3t^4 + q^4t^4 + q^6t^6$$

only proved for $nk+1$

$$\sum \dim gr^P_j H^{2i}(\bar{J})\, q^j t^{2i} \triangleq P^{\text{Perv}}_{\bar{J}}(q,t)$$

$$= (qt^2)^{\mu}\, C\left(q, \frac{1}{qt^2}\right)$$

q,t symmetry

Know:

$$e_2^{\delta-j}: \quad gr^P_j H^{2i}(\bar{J}) \xrightarrow{\sim} gr^P_{2\mu-j} H^{2(\mu-j+i)}(\bar{J})$$

$$C(q,t) = t^{\mu} P(q, (qt)^{-\frac{1}{2}}) \quad ①$$

$$C(t,q) = q^{\mu} P(t, (qt)^{-\frac{1}{2}}) \quad ②$$

① $gr_{k+u-\ell} H^{2(u-\ell)}(\bar{J})\, q^k t^\ell$

$\downarrow e_2^{\ell-k}$

② $gr_{\ell+u-\ell} H^{2(u-k)}(\bar{J})\, q^k t^\ell$ $\quad 2(u-k) = 2(u-(k+u-\ell)+u-\ell)$

$k+u-\ell+2(\ell-k)$

q,t symmetry proved by [Mellit, shuffle conj]

a different proof: Curious Lefschetz for weight filtration

Hikita filtration: on $H_*^{BM}(\bar{J}) \cong H^*(\bar{J})$

$h_{\leq d} H_*^{BM}(\bar{J}) = \mathrm{Im}(H_*^{BM}(\bar{J}_{\leq d}) \longrightarrow H_*^{BM}(\bar{J}))$

$\perp$ $P^{hikita}_{\bar{J}}(q,t) \overset{y^p = x^q}{=} \sum q^j (q^{\frac{1}{2}} t)^i gr_i^h(\bar{J})$

$\tilde{h}^{\geq d} = \ker(H^*(\bar{J}) \to H^*(\bar{J}_{\leq d}))$ why not just naively counting cells ??

$$\bar{J}_{\leq d} = \bigcup_{\ell \leq d} \bar{J}_\ell$$

$$\bar{J}_e = \{ M \mid \delta(M) = e \} \quad \delta\text{-stratificati[on]}$$

Conj ([KT]) (by Yun in unibranch case)

when $\operatorname{rk} \Lambda = 1$.

$\rightsquigarrow \bar{J}$ in unibranch case

$$H^{2j}(\bar{P}/\Lambda)$$

$$\shortparallel$$

$$P_{\leq j+k} H^{2j}(\bar{P}/\Lambda) \oplus \tilde{h}^{>k} H^{2j}(\bar{P}/\Lambda)$$

$\forall j, k$.

Relation with Hikita's work:

[Haiman]

$$\mathrm{ch}(DR_n, z; q, t) = \nabla e_n$$

Shuffle || conj

$$\sum_{\substack{\Delta:\text{Dyck path}\\ \text{with label } x}} \sum t^{\mathrm{area}(\Delta)} q^{\dim(\Delta)} z^{x}$$

? [Hikita]

$$\mathcal{F}\left(\mathrm{gr}^{h}_{\cdot} H_*^{BM}(Sp_\nu^\Delta), z, q^{-1}, t\right).$$

Here

$$\nu = \begin{pmatrix} t^2 & t & & \\ & & \ddots & \\ & & & t \\ & & & t^2 \end{pmatrix}$$

$$\begin{array}{ccc} \widetilde{Sp}_\nu^\Delta / T & \cong & \hat{B}_\nu \subseteq G(K)/I \\ & & \downarrow \\ & & \cancel{X} \subseteq G(K)/G(O) \end{array}$$

$$\downarrow$$
$$\amalg S^{\Delta}_{p_\gamma} \;/\!/\; S_{p_\gamma}$$

Last remark: (will be talked about by Linus)

(3,4)-case can also be computed

using braid variety

$\cong$

E_6 cluster variety.

$\{x^3 = y^4\}$ is type E_6 curve singularity...