Last time:

$$\begin{tikzcd} IC_w \arrow[r] & H^*(IC_w) \cong B_w \in SBim \\ D^b_{mix}(B\backslash G/B) \arrow[u, phantom, "\ni"] & R = H^*_B(pt)\otimes H^*_B(pt)\text{-mod} \arrow[u, phantom, "\ni"] \end{tikzcd}$$

If $M \in R\text{-mod-}R$, then $HH_*(M) = M \overset{L}{\underset{R\otimes R^e}{\otimes}} \Delta(R)$

$M = H^*(B\backslash G/B, IC_w) = H^*_{B\times B}(IC_w)$ $IC_w \in Sh(G)$

$HH_*(M) = M \overset{L}{\underset{R\otimes R}{\otimes}} \Delta(R) = H^*_{\Delta(B)}(IC_w)$ $\Delta(B) \hookrightarrow B\times B \circlearrowright G$

$\Delta(B)$ acts on G by adjoint action

bigraded

$\boxed{Ex}$ $HH_*({}_RR_R) \cong H^*_{\Delta(B)}(IC_B) \cong H^*_{\Delta(B)}(B) \cong H^*_T(T)$

$T \overset{ad}{\circlearrowright} T$

$\Updownarrow$

trivial action

$\cong H^*_T(pt)\otimes H^*(T)$

diagonal grading

$\boxed{H^1(\mathbb{C}^*) \text{ has weight } 2}$

$$\begin{tikzcd} R \arrow[r, leftrightarrow] & IC_{B/B} \arrow[r, leftrightarrow] \arrow[d] & IC_B \arrow[d] \\ & G/B & G \end{tikzcd}$$

$HH_j({}_RR_R) \cong \bigoplus_i H^i_T(pt)\otimes H^{j-i}(T)$

weight i — $q^{\frac{i}{2}}$

weight $2(j-i)$ — q^{j-i}

$\boxed{\text{Compare}}$

$G = GL_2$

$n=2$

$R \cong \mathbb{R}[x_1,x_2]$

$HH_*({}_RR_R) \cong R(-4) \;\vdots\; (R\oplus R)(-2) \;\vdots\; R$

-2	-1	0
$\Updownarrow$	$\Updownarrow$	$\Updownarrow$
$H^2(T)\otimes H^*_T(pt)$	$H^1(T)\otimes H^*_T(pt)$	$H^0(T)\otimes H^*_T(pt)$

$T \subset GL_2$

$\cong \mathbb{C}^*\times\mathbb{C}^*$

$(1) \longleftrightarrow \langle 1\rangle = [1](\tfrac{1}{2})$

$[1] \longleftrightarrow [1]$

Computational tools for $H^*_{\Delta(B)}(IC_w)$

$T \curvearrowright X$ $\quad \mathcal{F} \in D^b_T(X)$ $\quad \mathcal{F}$ is called equivariantly formal if

T.F.A.E. ① $H^*_T(\mathcal{F})$ is a free module / $H^*_T(pt)$

② $\underset{E_2 \text{ page}}{\underset{\shortparallel}{H^*(\mathcal{F}) \otimes H^*_T(pt)}} \Longrightarrow H^*_T(\mathcal{F})$ degenerates at E_2 page

③ $\dim_{\mathbb{C}} H^*(\mathcal{F}) = \dim_{\mathbb{C}} H^*(X^T, \mathcal{F}|_{X^T})$

Ex If $T \curvearrowright X$ freely, then $\mathcal{F}$ is equivariantly formal $\Longleftrightarrow$ $H^*(\mathcal{F}) = 0$

Ex $T \curvearrowright X$ is $\underline{\text{GKM}}$ $\Longrightarrow$ can compute $H^*_T(\mathcal{F})$ using fixed points and one-dim'l orbits.

easy to verify if $\mathcal{F}$ is formal or not

ex $\mathcal{F} = IC_{\underline{\quad\quad}}$ ...
T-equiv. subvariety

Ex $X = \mathbb{P}^1 \circlearrowleft \mathbb{C}^*$ is GKM

$$\begin{tikzcd} \mathbb{C}^2\setminus\{0\} \arrow[d, "\pi"'] & (z_1, z_2) \arrow[d, mapsto] \\ \mathbb{P}^1 & {[z_1 : z_2]} \end{tikzcd}$$

$\mathbb{C}^* \curvearrowright z_2$

$\pi_* \underline{\mathbb{C}}_{\mathbb{C}^2\setminus\{0\}} \in D^b_T(\mathbb{P}^1)$

$H^*_T(\pi_* \underline{\mathbb{C}}_{\mathbb{C}^2\setminus\{0\}}) \cong H^*_T(\underline{\mathbb{C}}_{\mathbb{C}^2\setminus\{0\}})$

$\cong H^*(\mathbb{C}^2\setminus\{0\} / \mathbb{C}^*)$

finite dim'l

is a torsion module of $H^*_T(pt)$

$B \overset{Ad}{\curvearrowright} G$ is not GKM

Fact When $G = GL_n$ or SL_n

$IC_w \in D^b_{\Delta(B)}(G)$ are all equivariantly formal.

Pf $HH_*(B_{s_i})$ is a free module over R if we're in type A

$\Downarrow$

IC_{s_i} are equivariantly formal.

$$\Downarrow$$

$$\begin{tikzcd} IC_{s_{i_1}} \arrow[d] & * \; IC_{s_{i_2}} \arrow[d] & * \cdots * \; IC_{s_{i_m}} \arrow[d] \\ G & B\backslash G/B & \frac{G}{B_{ad}} \end{tikzcd}$$

are equiv. formal

decomposition thm.

$$= \bigoplus_{w\in W} IC_w[n_w]$$

formal as well.

$\boxed{\text{Prop}}$ $\mathcal{F} \in D^b_T(X)$ $\mathcal{F}$ equivariantly formal

$$\Updownarrow \; ? \qquad \Downarrow$$

$i : X^T \hookrightarrow X$

$$H^*_T(\mathcal{F}) \xhookrightarrow{i^*} H^*_T(X^T, \mathcal{F}|_T)$$

$$\cong$$

$$H^*(X^T, \mathcal{F}|_T) \otimes H^*_T(pt)$$

$T \overset{Ad}{\hookrightarrow} G$ then $G^T = T \subseteq G$ (G connected)

because T is a maximal torus.

$G = GL_n, SL_n \Rightarrow$

$$H^*_{\Delta(T)}(IC_w) \hookrightarrow H^*_{\Delta(T)}(T, IC_w|_T)$$

$$\begin{tikzcd} IC_w \arrow[d] \\ G \end{tikzcd}$$ is an injection.

"most complicated stalk"

$\boxed{\text{Ex}}$ $G = GL_2$ $HH_*(B_s)$ $B_s = R \otimes_{R^s} R(1)$

$$\updownarrow$$

$$\underline{\mathbb{C}}_{\mathbb{P}^1}[1](\tfrac{1}{2}) = IC_{\mathbb{P}^1}$$

$$=$$

$$H^*_{\Delta(T)}(G, IC_G)$$

$$=$$

$$H^*_T(pt) \otimes H^*(G)\,[\cdots] = H^*(S^1) \otimes H^*(S^3)$$

Relation to character sheaves :

$$\begin{tikzcd}
D^b_{mix}\left({}_B\backslash G/_B\right) \arrow[r, "\mathrm{For}^{B\times B}_{\Delta(B)}"] \arrow[rr, bend right, "CH"'] & D^b_{mix}\left(\frac{G}{\Delta(B)}\right) \arrow[r, "\mathrm{Ind}^{\Delta(G)}_{\Delta(B)}"] & D^b_{mix}\left(\frac{G_{ad}}{G}\right)
\end{tikzcd}$$

"coenter" "trace"

$$CH(\mathcal{F} * \mathcal{G}) \simeq CH(\mathcal{G} * \mathcal{F}) \qquad (tr(AB) = tr(BA))$$

$$\begin{tikzcd}
& \frac{G\times G/B}{G_{ad}} \arrow[dl, "\pi"'] \arrow[dr, "act"] & \\
\frac{G}{G_{ad}} & & {}_B\backslash G/_B = \frac{G/B\times G/B}{G}
\end{tikzcd}$$

Horocycle correspondence

$act(g, B) = (B, gBg^{-1})$

$$\boxed{CH \cong \bigoplus_i {}^p\mathcal{H}^i[-i]\ \pi_* act^! \langle -\dim_{\mathbb{C}} B \rangle}$$

$act: G\times G/B \to G/B\times G/B$ is smooth of relative dimension $\dim_{\mathbb{C}} B$

CH preserves pure $\boxed{\text{perverse}}$ sheaves of weight 0

$\boxed{\text{Ex}}$ If we want $HH_*({}_R R_R) \longleftrightarrow H^*_{\Delta(B)}(IC_B)$ $B\subset G$

$$\begin{tikzcd}
IC_B \arrow[r, leftrightarrow] \arrow[d] & IC_{\Delta(G/B)} \arrow[d] \\
{}_B\backslash G/_B & \frac{G/B\times G/B}{G}
\end{tikzcd}$$

$$\begin{tikzcd}
& \frac{G\times G/B}{G} \arrow[dl] \arrow[dr, "act"] & \\
\frac{G}{G} & \frac{\tilde{G}}{G} \arrow[dl] \arrow[dr] & \frac{G/B\times G/B}{G} \\
\frac{G}{G} & & \frac{\Delta(G/B)}{G} \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

$$\tilde{G} = act^{-1}(\Delta(G/B))$$

$$= \{(g, B) \mid gBg^{-1} = B\}$$

// Grothendieck–Springer alteration.

$$G\times^{B_{ad}} B$$

$$\frac{\tilde{G}}{G} = \frac{G\times^{B_{ad}} B}{G} = \frac{B}{B}{}_{ad}$$

$$HH_*({}_R R_R) \cong H^*_{\Delta(B)}(IC_B) \cong H^*_B(B) \cong H^*\left(\frac{\widetilde{G}}{G}\right)$$

$$\begin{tikzcd} H^*\left(\frac{\widetilde{G}}{G}\right) \arrow[d, equal] & \\ H^*(\pi_* \mathbb{C}_{\widetilde{G}/G}) \arrow[d] \arrow[dr, equal] & \\ \frac{G}{G} & CH(IC_B) \end{tikzcd}$$

Springer sheaf

Webster-Williamson