NOTES ON THE PROOF OF $P = W$ FOR $G = \mathrm{GL}_n$

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These are some notes compiled for the purpose of a series of seminar talks on $P = W$ given by the author base on the paper of Maulik and Shen. All mistakes are the fault of the author. In the following, unless otherwise noted, we work over $\mathbb{C}$, $G = \mathrm{GL}_n$ and C is a smooth, projective curve of genus at least 2.

1. The Dolbeault Moduli Space and the Perverse Filtration

A Higgs bundle for $G = \mathrm{GL}_n$ is a pair (E, Φ) where E is a vector bundle of rank n on C and $\Phi\colon E \to E \otimes \Omega^1_C$ is a $\mathcal{O}_C$ linear section valued in 1-forms.

The Dolbeault moduli space M_{Dol} is the moduli space of stable Higgs bundles (E, Φ), where stability means the condition that all sub-Higgs bundles have strictly smaller slope

$$\mu(E) = \frac{\deg(E)}{\mathrm{r}k(E)}.$$

The space M_{Dol} admits a fibration, called the *Hitchin fibration*:

$$h\colon M_{Dol} \to A := \bigoplus_{i=1}^{n} H^0(C, \Omega^i_C), \quad \text{taking } (E, \Phi) \mapsto \text{char.poly}(\Phi)$$

where A is an affine space of half the dimension of M_{Dol}. The map h has several remarkable properties:

(1) h is surjective, proper, and Lagrangian (with symplectic structure on M_{Dol} induced by Nonabelian Hodge Correspondence).
(2) h is a *weak abelian fibration*, i.e. the generic fibers of h are abelian varieties.
(3) The pullback of the coordinate functions on A give a basis of Poisson commuting functions, giving the structure of a *completely integrable system*.

The fibration also defined the perverse filtration on cohomology $H^*(M_{Dol}, \mathbb{Q})$: We introduce this at the level of a proper map $f\colon X \to Y$, with $\dim(X) = a$ and $\dim(Y) = b$. Let $r = \dim X \times_Y X - \dim X$ be the defect to semismallness. [The fibers of h are equidimensional, so in the case of the Hitchin fibration,

$$r = \dim(M_{Dol}) - \dim(A) = \dim(A) = n^2(g-1) + 1$$

and $\dim(M_{Dol}) = 2n^2(g-1) + 2.$[1]] Then, the Perverse filtration is the increasing filtration on $H^m(X, \mathbb{Q})$ by

$$P_i H^m(X, \mathbb{Q}) = \mathrm{Image}\left(H^{m-a+r}(Y, {}^p\tau_{\leq i} Rf_* \mathbb{Q}_X[a-r]) \to H^m(X, \mathbb{Q})\right)$$
$$\overset{\text{our case}}{=} \mathrm{Image}\left(H^m(Y, {}^p\tau_{\leq i} Rf_* \mathbb{Q}_X) \to H^m(X, \mathbb{Q})\right)$$

[1]These dimension computations follow as M_{Dol} is twice the dimension of A and $\dim A$ can be computed by Riemann-Roch.

By a 2010 paper of de Cataldo and Migliorini, this is quite explicit in our setting: Let $\Lambda^s \subset A$ be a general s dimensional linear section of A so that the Λ_s form a flag in A. Then

$$P_i H^m(X, \mathbb{Q}) = \ker\left(H^m(M_{Dol}, \mathbb{Q}) \to H^m(h^{-1}(\Lambda^{m-i-1}), \mathbb{Q})\right)$$

The latter filtration is called the "flag filtration" of h.

The perverse filtration is *not* multiplicative with respect to the cup product structure on $H^*(M_{Dol}, \mathbb{Q})$! This will come up later and will lead us to the notion of *strong perversity.*

2. The Betti Moduli Space (Character Variety) and the Weight Filtration

The Betti moduli space M_B is the character variety classifying isomorphism classes of irreducible local systems

$$\rho\colon \pi_1(C \setminus \{p\}) \to \mathrm{GL}_n(\mathbb{C})$$

sending a loop around $p \in C$ to $e^{2\pi i d/n} \cdot \mathrm{Id}_n$. Explicitly, for $g = g(C)$,

$$M_B = \left\{ a_k, b_k \in \mathrm{GL}_n, 1 \leq k \leq g\colon \prod_{j=1}^{g} [a_j, b_j] = e^{2\pi i d/n} \cdot \mathrm{Id}_n \right\} /\!\!/ \mathrm{GL}_n$$

Then M_B is affine, with nontrivial weight filtration $W_i H^*(M_{Dol}, \mathbb{Q})$ on its cohomology.

The weight filtration admits a "curious Hard Lefschetz" property due to Mellit: There exists a class $\alpha \in H^2(M_B)$ of type $(2,2)$ such that cup product with α induces isomorphisms

$$\alpha^l \cup -\colon \mathrm{Gr}^W_{\dim(M_B)-2l} H^*(M_B) \xrightarrow{\simeq} \mathrm{Gr}^W_{\dim(M_B)+2l} H^{*+2l}(M_B).$$

Intuition: This curious Hard Lefschetz comes from relative Hard Lefschetz for the Perverse filtration! Note in particular, that the shift by $2l$ is because the doubly indexed weight filtration is conjectured to agree with the singly indexed perverse filtration. Later, this Lefschetz type statement will be used to help prove the two filtrations agree.

3. The Nonabelian Hodge Correspondence

The nonabelian Hodge theorem relates these two dramatically different spaces M_{Dol} and M_B.

Theorem 3.1. *There is a canonical diffeomorphism $M_{Dol} \simeq M_B$, equipping both spaces with the structure of a hyperkähler manifold.*

This identification can be thought of as follows: A local system representing a point of M_B is equivalent to the data of a vector bundle E on C equipped with a flat connection ∇. The connection ∇ satisfies the Liebnitz rule

$$\nabla(f) = f \cdot \nabla + df$$

We may consider the family over $\mathbf{A}^1$ whose fiber over $z \in \mathbf{A}^1$ consists of pairs (E, ∇_z) where ∇_z is a z-connection, i.e. $\nabla(f) = f \cdot \nabla + z \cdot df$. Note that M_{Dol} is the fiber over 0 while all other fibers are isomorphic as complex varieties to M_B. The nonabelian Hodge theorem states this family is topologically trivial.

It was first considered by de Cataldo-Hausel-Migliorini what the meaning of the weight filtration was on $H^*(M_{Dol}, \mathbb{Q})$ and vice versa. They came up with the following, proving it in the case of $n = 2$ and $g \geq 2$ arbitrary:

Conjecture 3.2. ($P = W$) For any $k, m \geq 0$,

$$P_k H^m(M_{Dol}, \mathbb{Q}) = W_{2k} H^m(M_B, \mathbb{Q}) = W_{2k+1} H^m(M_B, \mathbb{Q}).$$

4. A System of Generators for $H^*(M_B, \mathbb{Q})$ and a First Reduction

To avoid normalizations, we work with the Dolbeault moduli space for PGL_n. In degree d, we obtain this by the quotient of the of $M_{Dol,\deg=d,\mathrm{tr}=0}$ by the finite group $\Gamma = \mathrm{Pic}^0(C)[n]$. Denote the PGL_n Hitchin fibration by

$$\hat{h} \colon \hat{M}_{Dol} \to \hat{A} = \bigoplus_{i=2}^{n} H^0(C, \Omega_C^i)$$

There is a universal family $\mathcal{U} \to C \times \hat{M}_{Dol}$ which induces Chern characters

$$\mathrm{ch}_k(\mathcal{U}) \in H^{2k}(C \times \hat{M}_{Dol}, \mathbb{Q})$$

and therefore we get maps

$$c_k \colon H^*(C, \mathbb{Q}) \to H^*(\hat{M}_{Dol}, \mathbb{Q})$$

induced by the correspondence

$$\begin{tikzcd} & C \times \hat{M}_{Dol} \arrow[dl] \arrow[dr] & \\ C & & \hat{M}_{Dol} \end{tikzcd}$$

integrating with the kernel $\mathrm{ch}_k(\mathcal{U})$. Note that if Σ is a basis for $H^*(C, \mathbb{Q})$, then

$$\mathrm{ch}_k(\mathcal{U}) = \sum_{\sigma \in \Sigma} \sigma^\vee \otimes c_k(\sigma)$$

The image of c_k gives us our genertors for cohomology as a $\mathbb{Q}$ algebra:

Theorem 4.1. *(Markman and Shende) The image $c_k(H^*(C, \mathbb{Q}))$ generates $H^*(\hat{M}_{Dol}, \mathbb{Q})$ as a $\mathbb{Q}$ algebra, and moreover, for $\gamma \in H^m(C, \mathbb{Q})$,*

$$c_k(\gamma) \in W_{2k} H^m(M_B, \mathbb{Q}) \cap F^k H^m(M_B, \mathbb{C})$$

Corollary 4.2. *The $P = W$ Conjecture is equivalent to the condition that $\prod_{i=1}^s c_{k_i}(\gamma_i) \in P_{\sum_i k_i} H^*(\hat{M}_{Dol}, \mathbb{Q})$ for all products of tautological classes.*

Proof. The proof uses the fact that the $c_k(\gamma)$ generate under cup product together with Mellit's Curious Hard Lefschetz Theorem:

Indeed, suppose that the condition holds. Then, we see immediately that

$$W_{2k}(\hat{M}_B) \subset P_k H^*(\hat{M}_{Dol}).$$

Now, if $N = \dim \hat{M}_B$, then we know that $W_{2N} = P_N = H^*(M_B, \mathbb{Q}) =: V$. By the inclusion

$$\dim W_0 \leq \dim P_0 \overset{\text{relative Lefschetz}}{=} \dim V/P_{N-1} \leq \dim V/W_{2(N-1)}$$

An application of Mellit's curious Hard Lefschetz proves that the first and last dimensions are equal, and hence the above inequalities are equalities. The rest of the terms are similar. $\square$

5. Nearby and Vanishing Cycles

We will use the algebraic tools of nearby and vanishing cycles to reduce the $P = W$ conjecture to a strong perversity condition on certain Chern classes. In this section, we review the motivation and construction of nearby and vanishing cycles.

There are two principal motivations for nearby cycles: One from measuring cohomology of nearby fibers and one from globalizing Milnor's fibration.

We begin by reviewing the Milnor fibration. For a map $f\colon \mathbb{C}^{n+1} \to \mathbb{C}$ with zero fiber $X_0 \subset X$ and $x \in X_0$, let $B_{\varepsilon,x}$ be a small ball centered at x with boundary $\partial B_{\varepsilon,x} = S_{\varepsilon,x}$. Milnor proved many results about the fibration which now bears his name. A few that are most relevant to us include:

Theorem 5.1. *(Milnor)*

(i) $B_{\varepsilon,x} \cap X_0$ is homeomorphic to the cone on K_x where $K_x = S_{\varepsilon,x} \cap X_0$ is the link at x.

(ii) The "Milnor fibration at x"

$$\frac{f}{|f|}\colon S_{\varepsilon,x} \setminus K_x \to S^1$$

is topologically trivial.

(iii) When (X_0, x) is an isolated hypersurface singularity germ, the fiber F_x of the Milnor fibration at x is homotopy equivalent to $\bigvee_{\mu_x} S^n$ where

$$\mu_x = \dim_{\mathbb{C}} \frac{\mathbb{C}[x_0, \dots, x_n]}{(f_{x_0}, \dots, f_{x_n})}$$

is the Milnor number of f at x.

(The spheres in part (iii) of Theorem 5.1 will correspond to the vanishing cycles we later construct.)

Example 5.2. Consider $f\colon \mathbb{C}^2 \to \mathbb{C}$ taking $f(x, y) = x^3 + y^2$. We take $x = (0, 0) \in X_0$. The link K_0 is the trefoil knot in S^3, and the Milnor fibration gives a topologically trivial fibration

$$S^3 \setminus (\text{trefoil knot}) \to S^1$$

whose fiber is homotopic to a wedge product $S^1 \vee S^1$.

Consider a proper family $f\colon X \to \mathbb{C}$ with fiber X_s a smooth hypersurface of $\mathbb{C}^N$ for $s \neq 0$ and X_0 singular. For s close enough to 0, $f^{-1}(D_\varepsilon(0))$ retracts onto X_0 and hence the inclusion $X_s \to f^{-1}(D)$ induces a specialization map on cohomology

$$sp^*\colon H^*(X_0, \mathbb{Z}) \to H^*(X_s, \mathbb{Z})$$

When the map is proper, we can upgrade this cohomological map to a non-holomorphic, but continuous, map

$$sp\colon X_s \to X_0$$

collapsing the vanishing cycles. [Constructed by McPherson.]

The nearby cycle functor will roughly measure $H^*(X_s)$, pushed forward under the specialization map. The vanishing cycles functor will measure the difference between $H^*(X_s)$ and $H^*(X_0)$, i.e. (dually) the cycles in $H_*(X_s)$ which collapse (or "vanish") as X_s degenerates to X_0.

Definition 5.3. Let D^* be the punctured disk and $\pi\colon \widetilde{D^*} \to D^*$ the universal cover of D^*. Consider the following diagram (with f proper)

$$\begin{tikzcd}
X_0 \arrow[r, hook, "i"] \arrow[d] & X \arrow[d, "f"] & X^* \arrow[l, hook', "j"'] \arrow[d, "f^*"] & \widetilde{X^*} \arrow[l, "\hat{\pi}"'] \arrow[d] \\
\{0\} \arrow[r, hook] & D & D^* \arrow[l, hook'] & \widetilde{D^*} \arrow[l, "\pi"']
\end{tikzcd}$$

Then, the nearby cycles functor is

$$\psi_f \mathcal{F}^\bullet = i^* R(j \circ \hat{\pi})_* (j \circ \hat{\pi})^* \mathcal{F}^\bullet$$

It is immediate from the definition that, for any $x \in X_0$, the stalk of the sheaf cohomology of nearby cycles measures the cohomology of the Milnor fiber, i.e.

$$\mathcal{H}^k(\psi_f \mathcal{F}^\bullet)_x = H^k(F_x, \mathcal{F}^\bullet)$$

Moreover, when f is proper, we have a comparison result

Theorem 5.4. *When f is proper, $\psi_f \mathcal{F}^\bullet \simeq Rsp_*(\mathcal{F}^\bullet|_{X_s})$.*

The natural adjunction induces a canonical natural transformation

$$i^* \to \psi_f$$

Definition 5.5. The *vanishing cycles functor* φ_f is the cone of the map $i^* \to \psi_f$. It fits into an exact triangle

$$i^* \mathcal{F}^\bullet \to \psi_f \mathcal{F}^\bullet \to \varphi_f \mathcal{F}^\bullet \xrightarrow{[1]} \tag{5.1}$$

Note: It is not immediate from this definition that φ_f is functorial. One can prove that φ_f is the pullback of a representable functor along i.

Also note: In the literature, and in the rest of these notes, we will actually use a shift $\psi_f[-1]$ and $\varphi_f[-1]$ in place of ψ_f and φ_f. This is done so that nearby and vanishing cycles preserve perversity.

From the long exact sequence induced by the exact triangle, we have

$$\mathcal{H}^k(\varphi_f \mathcal{F}^\bullet)_x \simeq H^{k+1}(B_{\varepsilon,x}, B_{\varepsilon,x} \cap X_s, \mathcal{F}^\bullet) \quad \text{for } x \in X_0$$

When X is nonsingular, $B_{\varepsilon,x} \cap X_s$ is contractible and

$$\mathcal{H}^k(\varphi_f \underline{A}_X)_x \simeq \widetilde{H}^k(F_x, A)$$

is the reduced cohomology of the Milnor fiber F_x at $x \in X_0$. The stalk of the long exact sequence on cohomology assocaited to the exact triangle (5.1) looks like

$$\cdots \to H^k(B_{\varepsilon,x}, \mathcal{F}^\bullet) \to H^k(B_{\varepsilon,x} \cap X_s, \mathcal{F}^\bullet) \to H^{k+1}(B_{\varepsilon,x}, B_{\varepsilon,x} \cap X_s; \mathcal{F}^\bullet) \to \cdots$$

Corollary 5.6. *If X is nonsingular,* $\mathrm{Supp}(\varphi_f \underline{A}_X) \subset \mathrm{Sing}(X_0)$*, and if A is a field, e.g. $A = \mathbb{Q}$, then equality holds.*

Example 5.7. Consider the example of $f\colon \mathbb{C} \to \mathbb{C}$ taking $x \mapsto x^a$. The unique critical point is $X_0 = \{0\} \in \mathbb{C}$. We have $i^*\underline{\mathbb{Q}}_{\mathbb{C}} = \underline{\mathbb{Q}}_0$ while $\psi_f \underline{\mathbb{Q}}_{\mathbb{C}} = Rsp_*(\underline{\mathbb{Q}}_{\mu_a}) = \prod_{\mu_a} \underline{\mathbb{Q}}_0$. Hence, from the exact triangle,

$$\varphi_f \underline{\mathbb{Q}}_{\mathbb{C}} = (\underline{\mathbb{Q}}_0)^{a-1}.$$

So far, we have seen a construction that globalizes the Milnor fibration. The following tool is useful for computations:

Theorem 5.8. *(Analogue of Thom-Sebastiani) Let V denote the vanishing locus of a function and $k\colon V(f) \times V(g) \hookrightarrow V(f \boxtimes g)$ the natural inclusion map. Then, (with shifted nearby and vanishing cycle functors)*

$$k^*\Big(\varphi_{f\boxtimes g}(\mathcal{F}^\bullet \boxtimes^L \mathcal{G}^\bullet)\Big) \simeq \varphi_f \mathcal{F}^\bullet \boxtimes^L \varphi_g \mathcal{G}^\bullet$$

Example 5.9. Consider the case of a "Brieskorn singularity":

$$f\colon \mathbb{C}^n \to \mathbb{C}, \quad \text{by } (x_1, \ldots, x_n) \mapsto \sum_i x_i^{a_i}$$

Then, by the Thom-Sebastiani Theorem combined with Example 5.7, we see that

$$\varphi_f(\underline{\mathbb{Q}}_{\mathbb{C}^n}) = k_*(\boxtimes_{i=1}^n \underline{\mathbb{Q}}_0^{a_i-1}) = \underline{\mathbb{Q}}_0^{\sum_i (a_i-1)}.$$

For us, the most important property of vanishing cycles will be its compatibility with the notion of "strong pervesity".

Definition 5.10. An element $\gamma \in H^l(X, \mathbb{Q})$ has *strong perversity* c with respect to $f\colon X \to Y$ if the corresponding map $\gamma\colon Rf_*\mathbb{Q}_X \to Rf_*\mathbb{Q}_X[l]$ factors respects the perverse t structure by mapping

$$\begin{tikzcd}
{}^p\tau_{\le i} Rf_*\mathbb{Q}_X \arrow[r, hook] \arrow[d] & Rf_*\mathbb{Q}_X \arrow[d, "\gamma"] \\
{}^p\tau_{\le i+(c-l)} Rf_*\mathbb{Q}_X[l] \arrow[r, hook] & Rf_*\mathbb{Q}_X[l]
\end{tikzcd}$$

The benefit of strong perversity is that cup product with such classes respect the perverse filtration:

Lemma 5.11. *If $\gamma \in H^l(X, \mathbb{Q})$ has strong perversity c with respect to f, then $\gamma \cup -$ takes*

$$\gamma\colon P_i H^m(X, \mathbb{Q}) \to P_{i+c} H^{m+l}(X, \mathbb{Q})$$

Moreover, if $\gamma_i \in H^{l_i}(X, \mathbb{Q})$ has strong perversity c_i, then $\gamma_1 \cup \cdots \cup \gamma_s \in H^{l_1 + \cdots l_s}(X, \mathbb{Q})$ has strong perversity $\sum_i c_i$.

The key relationship with vanishing cycles can be formulated as follows: Consider a map $g\colon X \to \mathbf{A}^1$ where X is nonsingular and irreducible, and let $X' \subset X_0$ be the support of $\varphi_g := \varphi_g(IC_X)$.

Proposition 5.12. *With notation as above, suppose that X' is nonsingular and that the vanishing cycle sheaf is trivial on X', i.e. $\varphi_g \simeq IC_{X'} = \underline{\mathbb{Q}}_{X'}[\dim X']$. Further, suppose we have a commutative diagram*

$$\begin{tikzcd} X' \arrow[r, "i"] \arrow[d, "f'"'] & X \arrow[d, "f"'] \arrow[dr, "g"] & \\ Y' \arrow[r] & Y \arrow[r, "\nu"'] & \mathbf{A}^1 \end{tikzcd}$$

with f proper and g factoring through f. Then, if $\gamma \in H^l(X, \mathbb{Q})$ has strong perversity c with respect to f, then $i^*\gamma \in H^l(X', \mathbb{Q})$ has strong perversity c with respect to f'.

Proof. We apply φ_ν to the map

$$\gamma \colon Rf_*\mathbb{Q}_X \to Rf_*\mathbb{Q}_X[l]$$

and use a base change for vanishing cycles $Rf'_* \circ \varphi_g \simeq \varphi_\nu \circ Rf_*$ to get the map

$$\varphi_g(\gamma) \colon Rf'_*\varphi_g \to Rf'_*\varphi_g[l]$$

Since φ_ν commutes with the perverse t structure, this map takes ${}^p\tau_{\leq i}$ to ${}^p\tau_{\leq i+(c-l)}$. Then, since $\varphi_g \simeq \mathbb{Q}_{X'}[\dim X']$ is trivial, and $\varphi_g(\gamma) = i^*\gamma$, it follows that $i^*\gamma$ has strong perversity c with respect to f'. $\square$

6. Reduction to Strong Perversity Condition on the Dolbeault Space

We now use the machinery of vanishing cycles to sheafify Corollary 4.2. We use upper script $\mathcal{L}$ to denote the twisted Hitchin system by the line bundle $\mathcal{L}$.

Maulik and Shen proved (in *Endoscopic decompositions and the Hausel–Thaddeus conjecture*), that for $p \in C$ there is a close relationship between $\hat{M}^{\mathcal{L}}_{Dol}$ and $\hat{M}^{\mathcal{L}(p)}_{Dol}$. Namely, we have a natural embedding

$$i \colon \hat{M}^{\mathcal{L}}_{Dol} \to \hat{M}^{\mathcal{L}(p)}_{Dol}$$

Proposition 6.1. *There is a function $g \colon \hat{M}^{\mathcal{L}(p)}_{Dol} \to \mathbf{A}^1$ factoring through $\hat{A}^{\mathcal{L}(p)}$ such that*

$$\varphi_g(IC_{\hat{M}^{\mathcal{L}(p)}_{Dol}}) = IC_{\hat{M}^{\mathcal{L}}_{Dol}} = \underline{\mathbb{Q}}_{\hat{M}^{\mathcal{L}}_{Dol}}[\dim \hat{M}^{\mathcal{L}}_{Dol}]$$

The main result is to prove the following:

Theorem 6.2. *For any effective line bundle $\mathcal{L}$, the class*

$$\mathrm{ch}_k(\mathcal{U}^{\mathcal{L}}) \in H^{2k}(C \times \hat{M}^{\mathcal{L}}_{Dol}, \mathbb{Q})$$

has strong perversity with respect to $h^{\mathcal{L}}$.

Proof that Theorem 6.2 implies the hypotheses of Corollary 4.2. The main point is that we claim that if $\mathrm{ch}_k(\mathcal{U}^{\mathcal{L}(p)})$ has strong perversity k then $\mathrm{ch}_k(\mathcal{U}^{\mathcal{L}})$ has strong perversity k. For this, we rely on the key interaction Proposition 5.12 between vanishing cycles and strong perversity

Consider the commutative diagram

$$\begin{tikzcd} C \times \hat{M}_{Dol}^{\mathcal{L}} \arrow[r, hook, "i"] \arrow[d, "h^{\mathcal{L}}"] & C \times \hat{M}_{Dol}^{\mathcal{L}(p)} \arrow[d, "h^{\mathcal{L}(p)}"] \\ C \times \hat{A}^{\mathcal{L}} \arrow[r, hook] & C \times \hat{A}^{\mathcal{L}(p)} \end{tikzcd}$$

We have $i^*\mathrm{ch}_k(\mathcal{U}^{\mathcal{L}(p)}) = \mathrm{ch}_k(\mathcal{U}^{\mathcal{L}})$. Then, for the map $g\colon C \times \hat{M}_{Dol}^{\mathcal{L}(p)} \to \mathbf{A}^1$ from Proposition 6.1, φ_g is trivial, and so the claim follows from the Key Fact Prop 5.12.

The point of the claim is that, assuming Theorem 6.2, we can immediately conclude the same result for $\mathcal{L} = \mathcal{O}_C$ trivial. We have

$$P_k H^*(C \times \hat{M}_{Dol}, \mathbb{Q}) = H^*(C, \mathbb{Q}) \otimes P_k H^*(\hat{M}_{Dol}, \mathbb{Q}),$$

and if $\Sigma = \{\sigma_0, \sigma_1, \ldots, \sigma_{2g+1}, \sigma_{2g+2}\}$ is a homogeneous basis for $H^*(C, \mathbb{Q})$ with Poincaré dual basis $\{\sigma_0^\vee, \ldots, \sigma_{2g+2}^\vee\}$. Then,

$$\mathrm{ch}_k(\mathcal{U}) = \sum_{\sigma \in \Sigma} \sigma^\vee \otimes c_k(\sigma)$$

and so

$$\mathrm{ch}_k(\mathcal{U}) \cup (\sigma \otimes P_s H^*(\hat{M}_{Dol}, \mathbb{Q})) = \sigma^\vee(\sigma) \otimes \Big(c_k(\sigma) \cup P_s H^*(\hat{M}_{Dol}, \mathbb{Q})\Big) \subset H^*(C, \mathbb{Q}) \otimes P_{s+k} H^*(\hat{M}_{Dol}, \mathbb{Q})$$

Hence, $c_k(\sigma)$ sends P_s to P_{s+k}, and so also does $c_k(\gamma)$. The result follows. □

7. Yun's Global Springer Theory

8. Ngô's Support Theorem and a Parabolic Variant

9. Conclusion of Result