Compactified Jacobians & Arc spaces.

$(p,q)=1$, $p,q\geq 1$, $\overline{C_{\frac{q}{p}} = \{x^p = y^q\}}$

$\mathcal{O}_{\frac{q}{p}} = \mathbb{C}[\mathrm{Maps}_1^{rig}(\mathbb{P}^1, C_{\frac{q}{p}})]$
$\infty \mapsto \infty$ $\circlearrowleft \varphi$

rigidified: $\Rightarrow$ <u>rigidification at ∞</u> ?

see the following:

$\varphi^*: \quad \mathbb{C}[t^p, t^q] \longrightarrow \mathbb{C}[t]$

$\forall f$ of deg n, "$\deg(\varphi^* f - f) \leq n-2$"
(${SL}_n$ feature)

Explicit description of $\mathcal{O}_{\frac{q}{p}}$

$$\varphi^*(t^p) = t^p + \sum_{i=2}^{p} e_i t^{p-i}$$

$$\varphi^*(t^q) = t^q + \sum_{i=2}^{q} f_j t^{q-j}$$

ring homomorphism $\Rightarrow$

$$\left(t^p+\sum_{i=2}^{p} e_i t^{p-i}\right)^{q} = \left(t^q+\sum_{i=2}^{q} f_j t^{q-j}\right)^{p} \qquad (*)$$

$$\mathcal{O}_{\frac{q}{p}} = \frac{\mathbb{C}[e_2,\dots,e_p,f_2,\dots,f_q]}{\text{conditions defined by } (*)}$$

$$= \frac{\mathbb{C}[e_2,\dots,e_p]}{(f_{q+1}(e_2,\dots,e_p),\dots)}$$

$$= \frac{\mathbb{C}[e_2,\dots,e_p]}{(f_{q+1},\dots,f_{q+p-1})}$$

$$\cong eL_{\frac{q}{p}}$$

$$\dim \mathcal{O}_{\frac{q}{p}} = \binom{p+q}{p}\frac{1}{p+q}$$

$$\Vert$$

$$\dim H^*\left(J_{\frac{q}{p}}\right)$$

$J_{\frac{q}{p}}$: compactified Jacobian of $C_{\frac{q}{p}}$ (sing at ∞)

$\curvearrowright$

$\mathbb{G}_m$

$$\#\left(J_{\frac{q}{p}}^{\mathbb{G}_m}\right) = \text{Catalan } \#$$

Slogan: $\mathrm{gr}^P H^*(J_{\frac{a}{p}}) \circlearrowleft eH_{\frac{a}{p}}e$.

$\cong$

$\mathcal{O}_{\frac{a}{p}}$.

defined by [Yun] & [GK].

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grading: degrees of poly

filtration: $\mathfrak{m} = (e_2 \cdots e_p) \subseteq \mathcal{O}_{\frac{a}{p}}$

<u>Conj</u> [OY17] $F_i^{\mathfrak{m}} \mathcal{O}_{\frac{a}{p}} = \mathfrak{m}^i$

$$\mathrm{gr}_j^P H^{2i}(J_{\frac{a}{p}}) \cong \mathrm{gr}_{i-j}^{\mathfrak{m}} \mathcal{O}_{\frac{a}{p}}[j]$$

<u>$q = pk+1$</u> follows from

my result.

1

$$\sum \dim H^{2i}(J_{\frac{q}{p}})\, t^{i} = \sum t^{|D|}$$

2

proved when $q = pk+1$

by G-Mazin

$$ch_t(\Gamma(\mathrm{Hilb}_0, \mathcal{O}(k)))$$
$$\|$$
$$ch_t(eL_c)$$

OY

$\Longrightarrow$ Conj for $q = pk+1$

$\underline{\underline{J_{\frac{a}{p}} \text{ living in } \mathcal{M}}}$ $-1-\frac{1}{p}$

$$\mathcal{M} := \mathcal{M}^{\text{stable}}\left(\mathbb{P}(p,1),\ \mathcal{O}\left(\frac{q}{p}\right)\right)$$

$$p,\ \mathcal{O}\left(\frac{-q(p-1)}{2}\right)\ ?$$

$$\mathbb{P}(p,1) = \mathbb{C}^2 \setminus 0 \big/ t(x,y) = (t^p x, ty)$$

$$\mathrm{Aut}([1,0]) = \mu_p$$

- $$\deg(\mathcal{L}) = \sum_x \frac{1}{|\mathrm{Aut}(x)|} \deg_x(f)$$

$$\mathrm{proj} \bigoplus_{i=1}^{\infty} \mathbb{C}[\xi, y]_i \qquad \deg \xi = p \quad \deg y = 1$$

$$1, \dots, p-1 \quad p, \dots, 2p-1$$

dim 1   dim 2.

$\mathrm{Pic} \cong \frac{1}{p}\mathbb{Z}$

Theory built by Neitzeiz for DM-curve

$\mathbb{G}_m$-actions on $M \to A$

$\mathbb{G}_m^{dil}$: scaling the Higgs field

$\mathbb{G}_m^{rot} \curvearrowright X$: $t(x,y) = (t^{-1}x, y)$.

$$\Gamma(X, \mathcal{O}(\tfrac{q}{p})) = \mathbb{C}[\xi, \eta]_q$$

$$A = \bigoplus_{i=1}^{p} \Gamma(X, \wedge^i \mathcal{O}(\tfrac{q}{p}))$$

$$= \bigoplus_{i=1}^{p} \mathbb{C}[\xi, \eta]_{iq}$$

$$G_m^{dil} : \quad t(p_0, \cdots, p_p)$$

$$= (t\, p_0, \cdots, t^p p_p)$$

$$G_m^{rot} : \quad t(p_i(x, y))$$

$$= p_i(t^{-1} x, y)$$

$$G_m^{\frac{q}{p}} \hookrightarrow G_m^{rot} \times G_m^{dil}$$

$$t \longmapsto (t^p, t^q)$$

$$t\, P_i(x, y)$$

$$= t^{iq} P_i(t^{-p}x, y)$$

$a \in \mathbb{C}$

$$\mathbb{A}^{\mathbb{G}_m^{\frac{q}{p}}} = (0, \dots, 0, a \cdot \xi^q) \cong \mathbb{A}^1$$

because: $t \cdot \xi^q = t^{pq} \cdot (t^{-p}\xi)^q$

$$= \xi^q$$

$\mathbb{G}_m$ Contrats.

$$\begin{tikzcd}
\mathcal{M}^{ell} \arrow[r] & \mathbb{A}^{ell}. \\
\mathcal{M}_{\frac{a}{p}} \arrow[u, phantom, "\cup" rotate=90] \arrow[r] & \mathbb{A}_{\frac{a}{p}} \setminus \{0\} \arrow[u, phantom, "\cup" rotate=90] \\
\mathcal{M}_1 \arrow[u, hook] \arrow[r] & (0, \cdots, 0, \zeta^{a}) \arrow[u, hook]
\end{tikzcd}$$

$\mathcal{M}_{\frac{a}{p}} \cong \mathcal{M}_1 \times \mathbb{G}_m$

$$\mathcal{M}_1 \cong J_{\frac{a}{p}}$$

$$H^*(\mathcal{M}^{ell}) = H^*(\mathcal{M}|_{A_{\frac{a}{p}} \setminus 0}).$$

$$\overline{J}_{\frac{a}{p}} \times \mathbb{G}_m$$

Claim: $H^*(\mathcal{M}^{ell})_{pure} \cong H^*(\overline{J}_{\frac{a}{p}})$.

$$H^*(\overline{J}_{\frac{a}{p}}) \otimes [\mathbb{G}_m]$$

$$H^*(\overline{J}_{\frac{a}{p}})_{pure} = H^*(\overline{J}_{\frac{a}{p}})$$

Thm [H-T, M] $\mathcal{E}^{univ}_{can}$

descend from M to the coarse moduli space, $H^*(M^{ell})_{pure}$ is

generated by β_i that

show up in

$$c_j(\mathcal{E}^{univ}) = \sum \alpha_i \otimes \underline{\beta_i}$$

$$\in H^*(X \times M^{ell})$$

$$c_2, \ldots, c_p, \rightsquigarrow \beta_2, \ldots, \beta_p.$$

$\rightsquigarrow H^*(\bar{J}_{\frac{a}{p}})$ generated by

$\beta_2, \ldots, \beta_p$.

$R_{s,\varepsilon}$: $s=0,\ \varepsilon=1 \rightsquigarrow G_{\frac{a}{p}}$

$s=1,\ \varepsilon=0, \rightsquigarrow \bar{J}_{\frac{a}{p}}$

$\deg s = (1,0)$ $\deg(e_i) \ldots$

$\deg(\varepsilon) = (0,1)$ $\deg(f_i) \ldots$

$\rightsquigarrow \underbrace{R_{s,\varepsilon} = \bigoplus R_{a,b}}$

From this bigrading, one can buid F^{ε} on $H^*(\bar{J}\frac{a}{p})$ and show

$$F^{\varepsilon} = C^{[0 \leftarrow 1 \leftarrow b]} = P$$

$$\underbrace{C_k H^*(\bar{J}\frac{a}{p})}_{\varepsilon = 0} = \text{" monomials in } e_i \text{ of deg } \leq k \text{ "}$$

$(\deg e_i = i)$

$$H^*_{\mathbb{G}_m}\left(\overline{J}_{\frac{a}{p}}\right) = R_{s=1}$$

$$\begin{tikzcd} & H^*_{\mathbb{G}_m}\left(\overline{J}_{\frac{a}{p}}\right) \arrow[dl, "\varepsilon = 1"'] \arrow[dr, "\varepsilon = 0"] & \\ H^*_{\varepsilon=1}\left(\overline{J}_{\frac{a}{p}}\right) & & H^*\left(\overline{J}_{\frac{a}{p}}\right) \end{tikzcd}$$

" $F^{\varepsilon}\, \mathcal{O}_{\frac{a}{p}} \supseteq F^{my}\, \mathcal{O}_{\frac{a}{p}}$ "

"=" if R is flat over ε, s

$$\begin{array}{c} \mathrm{QMap}(\mathbb{P}^1, \mathcal{N}/G) \\ \Big\downarrow h \\ \underbrace{A} \end{array} \quad \left\{ \begin{array}{l} \mathfrak{g}/G \\ (\mathfrak{g} \times S_t)/G \end{array} \right.$$

product formula $\dfrac{h^{-1}(a)}{P_a} = \text{GASF}$

$h \rightsquigarrow$ Perverse filtration

Analogue of H-T, M for OMap

$\rightsquigarrow$ Chern filtration