

From Cherednik algebras to knot homology via cuspidal \mathcal{D} -modules

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Abstract

We show that the triply-graded Khovanov-Rozansky homology of the torus knot $T_{m,n}$ can be recovered from the finite-dimensional representation $L_{\frac{m}{n}}$ of the rational Cherednik algebra at slope $\frac{m}{n}$, endowed with the Hodge filtration coming from the cuspidal character \mathcal{D} -module $\mathbf{N}_{\frac{m}{n}}$. Our approach involves expressing the associated graded of $\mathbf{N}_{\frac{m}{n}}$ in terms of a dg module closely related to the action of the shuffle algebra on the equivariant K-theory of the Hilbert scheme of points on \mathbb{C}^2 , thereby proving the rational master conjecture. As a corollary, we identify the Hodge filtration with the inductive and algebraic filtrations on $L_{\frac{m}{n}}$.

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1 Introduction

1.1 Rational Cherednik algebras and link homology

Recent years have seen extraordinary connections between seemingly unrelated mathematical objects across different fields, indexed by pairs of coprime natural numbers m, n . Topologically, there is the (m, n) -torus knot $T_{m,n}$, which winds m times around a circle in the interior of the torus, and n times around its axis of rotational symmetry. Surprisingly, the Khovanov-Rozansky homology ([Kho07]) of $T_{m,n}$ can be captured by the following representation-theoretic object.

Write $c = \frac{m}{n}$. Let \mathfrak{h} denote the $(n-1)$ -dimensional standard representation of the symmetric group S_n . The rational Cherednik algebra H_c (also called rational DAHA) is a deformation of $\mathcal{D}(\mathfrak{h}) \ltimes S_n$ at the parameter c where $\mathcal{D}(\mathfrak{h})$ is the ring of differential operators on \mathfrak{h} . The algebra H_c has a unique finite-dimensional irreducible representation which we denote by L_c . Under the action of the Euler field h_c , L_c decomposes into a direct sum of eigenspaces $\bigoplus_{\ell \in \mathbb{Z}} L_c(\ell)$.

Conjecture 1.1. ([GORS14, Conjecture 1.2]) *There exists a filtration F on L_c compatible with the order filtration on H_c and the h_c -grading such that there is an isomorphism*

$$\mathrm{HHH}(T_{m,n}) \cong \mathrm{Hom}_{S_n}(\wedge^\bullet \mathfrak{h}, \mathrm{gr}_\bullet^F(\bigoplus L_c(\bullet))).$$

of triply graded vector spaces such that the following gradings are matched

$$\begin{aligned} \text{internal } q\text{-grading} &\leftrightarrow h_c\text{-grading} \\ \text{Hochschild homological } a\text{-grading} &\leftrightarrow \text{wedge degree of } \wedge^\bullet \mathfrak{h} \\ \text{usual homological } t\text{-grading} &\leftrightarrow \text{filtration on } L_c \end{aligned}$$

In this paper for any coprime (m, n) , $m > n$ we will define a Hodge filtration F^H on L_c and show that

Theorem 1.2. (Corollary 5.15) *Conjecture 1.1 holds with respect to the Hodge filtration F^H when $m > n$ such that $(m, n) = 1$.*

The Hodge filtration is defined on $L_c^{S_n}$ for all $m > 0$, in which cases we also have that Conjecture 1.1 holds for the degree of $a = 0$.

Building upon the framework established by Calaque-Enrique-Etingof [CEE09], we define F^H on L_c by viewing $L_c^{S_n}$ as the $\mathrm{Sym}^m \mathbb{C}^n$ - \mathfrak{sl}_n -isotypic component in the cuspidal character $\mathcal{D}(\mathfrak{sl}_n)$ -module \mathbf{N}_c following Saito's theory. Using the Springer resolution and Laumon's result on pushforward of filtered \mathcal{D} -modules, we are able to write the associated graded of \mathbf{N}_c in terms of an explicit dg module, which we call the cuspidal dg module (for further discussion, see 1.3). The cuspidal dg module is closely related to the nilpotent commuting variety, which has the punctual Hilbert scheme of points in \mathbb{C}^2 as a GIT quotient. In this sense, the appearance of the equivariant K-theory of the Hilbert scheme of points on \mathbb{C}^2 that we will discuss below is natural.

1.2 Hilbert schemes and the rational master conjecture

Another avatar of the theme of two coprime integers comes from the spherical DAHA. In [BS12], Burban and Schiffmann show that the spherical DAHA can be generated by elements $P_{m,n}$ labeled by pairs of integers. Using the generator $P_{m,n}$, Cherednik [Che13] reconstructs the refined Chern-Simons knot invariant in the sense of Aganagic-Shakirov ([AS12]) of the m, n -torus knot in the form of the three-variable torus knot superpolynomial.

The generator $P_{m,n}$ is also closely related to the finite-dimensional representation $L_{\frac{m}{n}}$ via the geometry of the Hilbert scheme. In [SV13], Schiffmann and Vasserot define an action of the spherical DAHA on $K^{\mathbb{C}^* \times \mathbb{C}^*}(\mathrm{Hilb}^n(\mathbb{C}^2))$, the equivariant K-theory of the Hilbert scheme of points on \mathbb{C}^2 . In [GN15], Gorsky and Neguț propose that the bigraded Frobenius character of

L_c can be read off from the action of $P_{m,n}$ on $K^{\mathbb{C}^* \times \mathbb{C}^*}(\text{Hilb}^n(\mathbb{C}^2))$. In this paper, we settle their conjecture:

Theorem 1.3. (Theorem 5.13) *The bigraded Frobenius character of L_c with respect to the Hodge filtration and the Euler field is given by*

$$\text{ch}_{S_n \times \mathbb{C}^* \times \mathbb{C}^*}(L_c) = (P_{m,n} \cdot 1)(q, q^{-1}t)$$

Theorem 1.3 is related to Theorem 1.2 as follows. The computation of the Khovanov-Rozansky homology for torus links had long been a challenging open problem, which was finally addressed by Elias, Hogancamp, and others ([Hog17, EH19, Hog18]) through recursive methods. The culmination of this work is a shuffle conjecture style formula [Mel22]. From this formula, it follows that the Euler characteristic of $\text{HHH}(T_{m,n})$ equals the knot superpolynomial of $T_{m,n}$, interpreted as a certain matrix coefficient of the action of $P_{m,n}$ on $K^{\mathbb{C}^* \times \mathbb{C}^*}(\text{Hilb}^n(\mathbb{C}^2))$, as observed in [GN15]. Therefore, Theorem 1.3 implies Theorem 1.2.

It is worth noting that in the case of $m = n + 1$, L_{n+1} is isomorphic to the space of diagonal harmonics [Gor03] and $P_{n+1,n} \cdot 1$ can be identified with ∇e_n [GN15, Corollary 6.5], where ∇ is the nabla operator ([BG98]) and e_n is the elementary symmetric polynomial of degree n . Therefore, Theorem 1.3 may be viewed as a generalization of the master conjecture proved by Haiman [Hai03, Theorem 4.2.5].

1.3 DG punctual flag Hilbert schemes and the work of Ginzburg

Notably, as observed by Neguț in [Neg15a], the action of the shuffle generators $P_{m,n}$ on $K^{\mathbb{C}^* \times \mathbb{C}^*}(\text{Hilb}^n(\mathbb{C}^2))$ is not visible using the classical Nakajima correspondence, and the notion of a flag Hilbert scheme turns out to be necessary. Indeed, there has been some intriguing open conjectures relating the coherent category of the flag Hilbert scheme and the homotopy category of Soergel bimodules ([GNR21]). However, it is well-known that the naive flag Hilbert scheme is highly singular. A remedy is to rather use a dg scheme: the dg flag Hilbert scheme $\text{FHilb}_{\text{dg}}^n(\mathbb{C}^2)$ is defined independently in [Gin12] and [GNR21]. In spite of its derived nature, $\text{FHilb}_{\text{dg}}^n(\mathbb{C}^2)$ is by definition a local complete intersection.

In [Gin12], Ginzburg studies the isospectral commuting variety by expressing the associated graded of the Harish-Chandra \mathcal{D} -module with respect to the Hodge filtration in terms of the dg flag commuting variety. In type A , the isospectral commuting variety has the isospectral Hilbert scheme as a GIT quotient and the dg flag commuting variety has the dg flag Hilbert scheme as a GIT quotient. Therefore, the result in *loc. cit.* implies that the pushforward of the structure sheaf of $\text{FHilb}_{\text{dg}}^n(\mathbb{C}^2)$ to $\text{Hilb}^n(\mathbb{C}^2)$ is the Procesi bundle. Using the main result of *loc. cit.* plus examination into the singular support of the Harish-Chandra \mathcal{D} -module, Gordon gives an alternative proof of the Macdonald positivity conjecture ([Gor12]).

We adapt a very similar approach as [Gin12] in the setting of nontrivial monodromy and naturally obtain what we call the cuspidal dg module. We also define a Catalan dg module, whose GIT quotient specializes to the dg punctual flag Hilbert scheme defined in [GNR21]. We show the pushforwards of these two dg modules to the commuting variety correspond to the same equivariant K-theory classes and we state the sheaf-theoretic identification as a conjecture

(Conjecture 5.3). While this paper was under editing, Gorsky and Neguț generously shared a copy of their preprint [GN24]. Their Conjecture 2.2 is a non-derived version of our Conjecture 5.3.

1.4 Filtrations and future directions

Regarding Conjecture 1.1, the authors of [GORS14] have proposed several filtrations: the algebraic(Chern) filtration, the inductive filtration and the geometric(perverse) filtration, which are shown to coincide ([OY16], [Ma24]). As a corollary of Theorem 1.2, we show that

Proposition 1.4. *(Proposition 5.18) The Hodge filtration equals the inductive and algebraic filtrations on L_c when $m > n$ for coprime m, n and on $L_c^{S_n}$ for all $m > 0$ coprime to n .*

The inductive filtration is also only defined on L_c for $m > n$, but the algebraic filtration is well-defined on L_c for all $m > 0$ with $(m, n) = 1$. As a corollary of Proposition 1.4, we also have that

Proposition 1.5. *(Proposition 5.20) For all integers $m > 0$ coprime to n , with respect to the algebraic filtration, Conjecture 1.1 holds.*

There are two natural generalizations of our setting: replacing $\text{Hilb}^n(\mathbb{C}^2)$ by Gieseker varieties and allowing m, n to be non-coprime. The former is related to the study of representations of a quantized Gieseker moduli algebra ([EKLS21]) and the study of higher Catalan numbers and a finite Shuffle conjecture ([GSV23]). The latter is related to the study of rational DAHA representations of minimal support and torus link homology ([EGL15]). Despite the current absence of definitions of the inductive, algebraic and geometric filtrations in these settings, we believe that a notion of Hodge filtration is still available, and via a similar method, bigraded Frobenius characters can be computed and related to the corresponding link invariants. On the other hand, it is conjectured that the stable envelopes on $\text{Hilb}^n(\mathbb{C}^2)$ are closely related to the DAHA Verma modules [GN17, Conjecture 6.5]. In particular, suitable filtrations on Verma modules, satisfying properties such as compatibility with the parabolic induction and restriction functors, are necessary for such a program. We believe the Hodge filtration might be the correct candidate; however, it is neither clear nor has it been explored which \mathcal{D} -modules correspond to the Verma modules. We plan to investigate these directions in future work.

Acknowledgement: I would like to express my deepest gratitude to my advisor, Victor Ginzburg, for suggesting this problem. His previous work forms the foundation of almost all the arguments presented here, and this paper would not have been possible without his guidance and inspiration. I am also very grateful to Andrei Neguț, especially for pointing out the reference [Neg22, (2.34) and (2.35)], which plays an important role in the proofs in Section 5.2. Additionally, I would like to thank Eugene Gorsky for valuable discussions and for giving me the opportunities to present this work while it was still in progress. My genuine thanks also go to Oscar Kivinen and Minh-Tam Trinh for helpful discussions in the early stage of this project.

2 Quantum Hamiltonian reductions and the \mathbb{Z} -algebra construction

Fix an integer $n \geq 2$. Let $\overline{G} = \mathrm{GL}_n$ with Lie algebra $\overline{\mathfrak{g}} := \mathfrak{gl}_n$. Let $G = \mathrm{SL}_n$, $\mathfrak{g} = \mathfrak{sl}_n$. Let $W = S_n$ be the Weyl group.

2.1 Representations of Rational Cherednik algebras

Fix the maximal torus $T \subset G$ with Lie algebra $\mathfrak{h} \subset \mathfrak{g}$, consisting of diagonal matrices. For any $c \in \mathbb{C}$, we define the rational Cherednik algebra $H_c := H_c(\mathfrak{h})$ to be the \mathbb{C} -algebra generated by \mathfrak{h} , \mathfrak{h}^* and W with relations

$$\begin{aligned} [x, x'] &= [y, y'] = 0, \quad wxw^{-1} = w(x), \quad wyw^{-1} = w(y) \\ [y, x] &= x(y) - \sum_{s \in S} c \langle \alpha_s, y \rangle \langle \alpha_s^\vee, x \rangle s \end{aligned}$$

where $x, x' \in \mathfrak{h}^*$, $y, y' \in \mathfrak{h}$, $w \in W$, $S \subset W$ is the set of simple reflections and α_s , resp. α_s^\vee , is the root, resp. coroot, associated to s . In particular $H_0 = \mathcal{D}(\mathfrak{h}) \ltimes S_n$ (for a definition of $\mathcal{D}(\mathfrak{h})$, see Section 2.2.1).

For any W -representation τ , we may regard it as a $\mathbb{C}[\mathfrak{h}^*] \ltimes W$ -representation by requiring $\mathbb{C}[\mathfrak{h}^*]$ to act trivially and define the Verma module

$$M_c(\tau) = H_c \otimes_{\mathbb{C}[\mathfrak{h}^*] \ltimes W} \tau.$$

The Verma module $M_c(\tau)$ has a maximal proper submodule with an irreducible quotient $L_c(\tau)$.

Theorem 2.1. ([BEG03, Theorem 1.2]) *When $c = \frac{m}{n}$ for positive integer m coprime to n , the only irreducible finite-dimensional representation of H_c is $L_c(\mathrm{triv})$. Moreover, only when $c = \frac{m}{n}$ for integer m coprime to n does H_c have finite-dimensional representations.*

Below, we will simply write $L_c := L_c(\mathrm{triv})$. Let $\mathcal{O}(H_c)$ denote the BGG category \mathcal{O} of the rational Cherednik algebra, which is a full subcategory of $H_c\text{-mod}$ whose objects are finitely generated over H_c such that the $S(\mathfrak{h})$ action is locally nilpotent ([GGOR03]). Then as τ varies over irreducible W -representations, the modules $L_c(\tau)$ give a complete list of irreducible objects in $\mathcal{O}(H_c)$.

Let $e := \frac{1}{n!} \sum_{w \in W} w$ be the symmetrizing idempotent in H_c . Then inside H_c , $e\mathbb{C}[\mathfrak{h}] \cong \mathbb{C}[\mathfrak{h}]^W$ and $eS(\mathfrak{h}) \cong S(\mathfrak{h})^W$. The spherical Cherednik algebra is defined by $A_c := eH_ce$. By [BE09, Theorem 4.1], for all c satisfying

$$c \notin \left\{ \frac{a}{b} \in (-1, 0) \mid a, b \in \mathbb{Z}, 2 \leq b \leq n \right\} \quad (1)$$

there is a Morita equivalence

$$H_c\text{-mod} \rightarrow A_c\text{-mod}, \quad M \mapsto eM.$$

Denote the BGG category \mathcal{O} of the spherical Cherednik algebra by $\mathcal{O}(A_c)$.

Let R^+ be a chosen set of positive roots. Write $\delta = \prod_{\alpha \in R^+} \alpha$. Let $\mathfrak{h}_r = \{\delta \neq 0\}$ denote the regular locus of \mathfrak{h} , i.e., when the diagonals are pairwise distinct. The action of H_c on $\mathbb{C}[\mathfrak{h}] \cong H_c \otimes_{\mathbb{C}[\mathfrak{h}^*] \ltimes W} \mathbb{C}$ gives the Dunkl embedding of H_c into $\mathcal{D}(\mathfrak{h}_r) \ltimes W$. In particular, assume $\{x_i\}$ is a basis of \mathfrak{h}^* and $\{y_i\}$ is its dual basis of \mathfrak{h} . Then the Dunkl operator associated to y_i is $D_{y_i, c} := \partial_{x_i} - c \sum_{s \in S} \frac{\alpha_s(x_i)}{\alpha_s} (1 - s)$.

Localized at $\delta := \prod_{\alpha \in \Delta^+} \alpha$, we have that $(H_c)_\delta \cong \mathcal{D}(\mathfrak{h}_r)_\delta \ltimes W$. Define a symmetry on $(H_c)_\delta$ by sending

$$x \mapsto x, \quad D_{y_i, c} \mapsto D_{y_i, -c}, \quad w \mapsto \text{sign}(w)w$$

which restricts to an isomorphism $A_d \rightarrow \delta^{-1} A_{-d-1} \delta$ ([GGS09, 5.6, 5.8]). This induces an equivalence of categories, which we will use later in the paper:

$$\Omega_d = \Omega_{-d-1}^{-1} : A_d\text{-mod} \cong A_{-d-1}\text{-mod}.$$

2.2 Quantum Hamiltonian reduction functors

2.2.1 Notations and conventions on \mathcal{D} -modules

For any smooth algebraic variety X , denote the sheaf of differential operators on X by \mathcal{D}_X and write $\mathcal{D}(X) := \Gamma(X, \mathcal{D}_X)$. Suppose $\{x_i\}$ is a local coordinate system of X in an affine open subset U . Then $\mathcal{D}_X(U)$ is the algebra generated by x_i, ∂_{x_i} where ∂_{x_i} is the partial derivative along the direction of x_i , satisfying the relations that $[\partial_{x_i}, x_j] = \delta_{i,j}$.

We define the order filtration F^{ord} on \mathcal{D}_X by setting

$$(F_k^{\text{ord}} \mathcal{D}_X)(U) = \sum_{|\alpha| \leq k} \mathcal{O}(U) \partial_x^\alpha$$

i.e., it is the filtration induced by setting $\deg(x) = 0$ and $\deg(\partial_x) = 1$.

Let $\pi : T^*X \rightarrow X$ be the projection map. Then with respect to the order filtration, we have that

$$\text{gr}^{\text{ord}} \mathcal{D}_X \cong \pi_* \mathcal{O}_{T^*X}.$$

For any coherent \mathcal{D}_X -module M with an increasing filtration F , we say (M, F) is a filtered \mathcal{D}_X -modules if $F_i M = 0$ for $i \ll 0$ and $F_i \mathcal{D}_X \cdot F_j M \subset F_{i+j} M$. With the latter condition, we are able to define an associated coherent sheaf on \mathcal{O}_{T^*X} by

$$\tilde{\text{gr}}^F(M) := \pi^{-1} \text{gr}^F M \otimes_{\pi^{-1} \pi_* \mathcal{O}_{T^*X}} \mathcal{O}_{T^*X}.$$

We say the filtration F of M is good if $\tilde{\text{gr}}^F(M)$ is a coherent sheaf on T^*X .

The support of $\tilde{\text{gr}}^F(M)$ in T^*X is independent of the choice of a good filtration, which we will call the singular support of M and denote by $SS(M)$.

2.2.2 Mirabolic \mathcal{D} -modules and Cherednik algebras

Let $V = \mathbb{C}^n$ be the standard representation of \overline{G} and $\mathfrak{G} = \mathfrak{g} \ltimes V$. Let $\tau : \overline{\mathfrak{g}} \rightarrow \mathcal{D}(\mathfrak{G})$ denote the embedding induced by differentiating the diagonal action of \overline{G} on \mathfrak{G} . Our convention is that

for any $f \in \mathcal{O}(\mathfrak{G})$ and $g \in \overline{G}$, $(g \cdot f)(x) = f(g^{-1} \cdot x)$. In particular, let $\mathbf{1} \in \overline{\mathfrak{g}}$ denote the identity matrix. Then $\tau(\mathbf{1}) = -\sum_{i=1}^n v_i \partial_{v_i}$.

Define the shifted embedding τ_d by a constant $d \in \mathbb{C}$ by (later we will take d to be $-c$)

$$\tau_d : \overline{\mathfrak{g}} \rightarrow \mathcal{D}(\mathfrak{G}), \quad x \rightarrow \tau(x) - d \cdot \text{tr}(x).$$

Define

$$\Lambda := \{(X, Y, i, j) \in \mathfrak{g} \times \mathcal{N} \times V \times V^* \mid [X, Y] + ij = 0\},$$

which is a Lagrangian subvariety of $T^*\mathfrak{G}$. Following [BG15], we call a finitely generated $\mathcal{D}_{\mathfrak{G}}$ -module mirabolic if it is locally finite as a $\tau(\overline{\mathfrak{g}})$ -module and its singular support is contained in Λ . Denote the category of mirabolic $\mathcal{D}_{\mathfrak{G}}$ -modules by $\mathcal{C}(\mathfrak{G})$.

Fix a nonzero element $\text{vol} \in \bigwedge^n V^*$. Following [BFG06, (5.3.2)], we define a regular function on \mathfrak{G} by

$$s(x, v) = \langle \text{vol}, v \wedge xv \wedge \cdots \wedge x^{n-1}v \rangle. \quad (2)$$

Denote $\mathfrak{G}_{\text{cyc}} := \{(x, v) \in \mathfrak{G} \mid s(x, v) \neq 0\}$, which is equivalently the locus when v is a cyclic vector for x , i.e., $\mathbb{C}[x]v = V$. It is well-known that the composition of the projection $\mathfrak{G}_{\text{cyc}} \rightarrow \mathfrak{g}$ and the Chevalley map $\mathfrak{g} \rightarrow \mathfrak{h}/W$ is a principal \overline{G} -bundle, such that the diagonal \overline{G} -action on $\mathfrak{G}_{\text{cyc}}$ acts freely on the fibers. Therefore we have an isomorphism

$$\iota : \mathbb{C}[\mathfrak{G}_{\text{cyc}}]^{\overline{G}} \cong \mathbb{C}[\mathfrak{h}]^W. \quad (3)$$

Note that $s^{-d} \in \mathbb{C}[\mathfrak{G}_{\text{cyc}}]^{\tau_d(\overline{\mathfrak{g}})}$. In fact $\mathbb{C}[\mathfrak{G}_{\text{cyc}}]^{\tau_d(\overline{\mathfrak{g}})} = \mathbb{C}[\mathfrak{G}_{\text{cyc}}]^G s^{-d}$. Moreover,

Theorem 2.2. ([GG06, Theorem 1.3.1], [GGS09, Theorem 8.1]) *The radial part map*

$$\mathcal{D}(\mathfrak{G})^{\overline{G}} \rightarrow \mathcal{D}(\mathfrak{h}/W), \quad u \mapsto \left[\mathbb{C}[\mathfrak{h}/W] \ni f \mapsto \iota(s^d u(s^{-d} \iota^{-1}(f))) \right]$$

induces an isomorphism

$$\mathcal{H}_d : (\mathcal{D}(\mathfrak{G})/\mathcal{D}(\mathfrak{G})_{\tau_d(\overline{\mathfrak{g}})})^{\overline{G}} \cong A_{d-1}, \quad (4)$$

which induces the quantum Hamiltonian reduction functor

$$\mathbb{H}_d : \mathcal{C}(\mathfrak{G}) \rightarrow \mathcal{O}(A_{d-1}), \quad \mathbf{M} \mapsto \Gamma(\mathfrak{G}, \mathbf{M})^{\tau_d(\overline{\mathfrak{g}})}$$

such that

$$\mathcal{C}(\mathfrak{G})/\text{Ker}(\mathbb{H}_d) \cong \mathcal{O}(A_{d-1}).$$

Remark 2.3. *The “ -1 ” factor in the subscript of A_{d-1} is consistent with the classical result of Harish-Chandra that*

$$\mathcal{D}(\mathfrak{g})^G \rightarrow \mathcal{D}(\mathfrak{h}/W), \quad u \mapsto \left[\mathbb{C}[\mathfrak{h}/W] \ni f \mapsto \delta(u|_{\mathbb{C}[\mathfrak{h}/W]})(\delta^{-1}f) \right]$$

induces an isomorphism $(\mathcal{D}(\mathfrak{g})/\mathcal{D}(\mathfrak{g})\mathrm{ad}(\mathfrak{g}))^G \cong D(\mathfrak{h})^W$. Here δ is the product of all positive roots and the restriction $u|_{\mathbb{C}[\mathfrak{h}/W]}$ is taken with respect to the Chevalley isomorphism $C[\mathfrak{g}]^G \cong \mathbb{C}[\mathfrak{h}]^W$.

2.3 Cuspidal mirabolic \mathcal{D} -modules

2.3.1 Functors on \mathcal{D} -modules

Let $f : X \rightarrow Y$ be a morphism of smooth algebraic varieties. Let $\Omega_{X/Y} = \Omega_X \otimes f^*\omega_Y^{-1}$ be the relative canonical bundle.

Define $\mathcal{D}_{X \rightarrow Y} := f^*\mathcal{D}_Y$ and $\mathcal{D}_{Y \leftarrow X} := \mathcal{D}_{X \rightarrow Y} \otimes_{\mathcal{O}_X} \Omega_{X/Y}$. There are a $(\mathcal{D}_X, f^{-1}\mathcal{D}_Y)$ -bimodule structure on $\mathcal{D}_{X \rightarrow Y}$ and a $(f^{-1}\mathcal{D}_Y, \mathcal{D}_X)$ -bimodule structure on $\mathcal{D}_{Y \leftarrow X}$ ([HTT08, 1.3]).

Suppose M is a \mathcal{D}_X -module and N is a \mathcal{D}_Y -module. We define the \mathcal{D} -module pullback of N to be $f^!N := \mathcal{D}_{X \rightarrow Y} \otimes_{f^{-1}\mathcal{D}_Y}^L f^{-1}N$ and the \mathcal{D} -module pushforward of M to be $f_!M = Rf_*(\mathcal{D}_{Y \leftarrow X} \otimes_{\mathcal{D}_X}^L M)$.

One can similarly define the proper pushforward $f_!$ of a \mathcal{D} -module using the Verdier duality. When i is a locally closed embedding, we define the minimal extension functor $i_{!*}$ to be the image under the canonical morphism $i_! \rightarrow i_*$.

Definition 2.4. *We say a local system on a locally closed subset $U \xrightarrow{i} X$ is clean if its minimal extension coincides with the extensions using $i_!$ or i_* .*

2.3.2 Cuspidal local systems

Let \mathcal{N} be the subvariety in \mathfrak{g} consisting of nilpotent matrices. Then the 0 fiber of the fibration $\mathfrak{G}_{\mathrm{cyc}} \rightarrow \mathfrak{h}/W$ is

$$U := \{(x, v) \in \mathcal{N} \times V \mid s(x, v) \neq 0\}.$$

The diagonal G action on U is transitive and free. Hence $\pi_1(U) = \mathbb{Z}$. We have that $g \cdot s = \det(g)^{-1}s$ for any $g \in G$ and thus every simple local system on U of finite order monodromy is given by the multi-valued section s^a for some rational number a . We will denote the \mathcal{D}_U -module corresponding to such a local system by \mathcal{E}_a .

Let \mathcal{N}_r denote the unique regular G -orbit in \mathcal{N} , fix an element $x \in \mathcal{N}_r$ and write $\mathrm{Stab}_G(x)$ for the stabilizer of x in SL_n . Then the fibration

$$\begin{array}{ccc} \mathrm{Stab}_G(x) & \longrightarrow & G \\ & & \downarrow \\ & & \mathcal{N}_r \end{array}$$

induces exact sequence

$$1 = \pi_1(G) \rightarrow \pi_1(\mathcal{N}_r) \rightarrow \pi_0(\mathrm{Stab}_G(x)) \rightarrow 1$$

which implies $\pi_1(\mathcal{N}_r) \cong \pi_0(\mathrm{Stab}_G(x)) = \pi_0(Z(G)) = \mathbb{Z}/n\mathbb{Z}$. Moreover, every simple local system on \mathcal{N}_r of finite order monodromy is defined by the representation of $\mathbb{Z}/n\mathbb{Z}$ given by $e^{2\pi i b}$ for

some $b \in \frac{1}{n}\mathbb{Z}$. We will denote the $\mathcal{D}_{\mathcal{N}_r}$ -module corresponding to such a local system by \mathcal{F}_b . Both \mathcal{E}_a and \mathcal{F}_b are G -equivariant.

The projection $U \rightarrow \mathcal{N}$ has image inside \mathcal{N}_r and the fibers of this projection are isomorphic to $\mathfrak{F} := \mathbb{C}^{n-1} \times \mathbb{C}^*$. The fibration

$$\begin{array}{ccc} \mathfrak{F} & \longrightarrow & U \\ & & \downarrow \\ & & \mathcal{N}_r \end{array}$$

induces an exact sequence

$$1 \rightarrow (\mathbb{Z} = \pi_1(\mathfrak{F})) \xrightarrow{n} (\mathbb{Z} = \pi_1(U)) \rightarrow (\pi_1(\mathcal{N}_r) = \mathbb{Z}/n\mathbb{Z}) \rightarrow 1.$$

It follows that \mathcal{E}_a is the pullback of the local system \mathcal{F}_a on \mathcal{N}_r for $a \in \frac{\mathbb{Z}}{n}$.

2.3.3 Cuspidal mirabolic \mathcal{D} -modules and DAHA representations

Let $c = \frac{m}{n}$ for positive integer m coprime to n .

Definition 2.5. • The cuspidal character $\mathcal{D}_{\mathfrak{g}}$ -module with parameter c is the minimal extension of the local system \mathcal{F}_c to \mathfrak{g} , which we denote by \mathbf{N}_c .

- The cuspidal mirabolic $\mathcal{D}_{\mathfrak{g}}$ -module of parameter c is defined to be the minimal extension, i.e. $!*$ -extension, of \mathcal{E}_c to \mathfrak{g} , which we denote by $\overline{\mathbf{N}}_c$.

By [Lus86], \mathcal{F}_c is clean. On the other hand, \mathcal{E}_c is not clean, but we have the following:

Lemma 2.6. We have $\overline{\mathbf{N}}_c \cong \mathbf{N}_c \boxtimes \mathcal{O}_V$ as SL_n -equivariant $\mathcal{D}_{\mathfrak{g}}$ -modules. Moreover

$$SS(\overline{\mathbf{N}}_c) = \{(x, y) \in \mathcal{N} \times \mathcal{N} \mid [x, y] = 0\} \times V. \quad (5)$$

Proof. The function s has degree n along the direction of V . Since $c \cdot n = m$, the local system \mathcal{E}_c , defined by the multi-valued function s^c , has no monodromy along the V direction. Therefore the minimal extension of \mathcal{E}_c to $\mathcal{N}_r \times V$ is $\mathcal{F}_c \boxtimes \mathcal{O}_V$ and the first statement follows from the cleanness of \mathcal{F}_c . On the other hand, it is known that the singular support of a cuspidal character \mathcal{D} -module equals $\{(x, y) \in \mathcal{N} \times \mathcal{N} \mid [x, y] = 0\}$ and hence the second statement of the lemma follows. \square

Theorem 2.7. ([CEE09, Theorem 9.19])

- There is a A_{-c-1} -action on $(\Gamma(\mathfrak{g}, \mathbf{N}_c) \otimes \mathrm{Sym}^m V)^{\mathrm{SL}_n}$.
- Under this action, $(\Gamma(\mathfrak{g}, \mathbf{N}_c) \otimes \mathrm{Sym}^m V)^{\mathrm{SL}_n}$ is isomorphic to $\Omega_c(\mathrm{eL}_c)$.

Corollary 2.8. We have $\mathbb{H}_{-c}(\overline{\mathbf{N}}_c) \cong \Omega_c(\mathrm{eL}_c)$ as A_{-c-1} -modules.

Proof. Follows from Lemma 2.6, Theorem 2.7 and the fact that $\mathbb{C}[V]^{\tau-c(1)} = \mathrm{Sym}^m V$. \square

2.4 Hilbert schemes of points

Let $\text{Hilb}^n := \text{Hilb}^n(\mathbb{C}^2)$ be the moduli space of ideals of colength n in $\mathbb{C}[x, y]$. It is a smooth and quasi-projective variety of dimension $2n$. The Hilbert-Chow map $\text{Hilb}^n \rightarrow (\mathbb{C}^2)^n/S_n$, defined by sending a colength n ideal I to the subvariety defined by the quotient $\mathbb{C}[x, y]/I$, is a resolution of singularity.

Define

$$\widetilde{\text{Hilb}}^n := \{(X, Y, v) \in \mathfrak{g} \times \mathfrak{g} \times V \mid [X, Y] = 0, \mathbb{C}[X, Y]v = V\}_{\text{red}}$$

where the subscript $_{\text{red}}$ refers to taking the reduced structure. The diagonal G -action on $\widetilde{\text{Hilb}}$ is free and the resulting GIT quotient is $\widetilde{\text{Hilb}}^n // G = \text{Hilb}^n$ ([Nak99]).

We also define

$$\widetilde{\text{Hilb}}_0^n = (\mathcal{N} \times \mathcal{N} \times V) \cap \widetilde{\text{Hilb}}^n$$

which is a open subvariety of the singular support (5). The GIT quotient $\text{Hilb}_0^n := \widetilde{\text{Hilb}}_0^n // G$ is the punctual Hilbert scheme which is the zero fiber of the Hilbert-Chow map and is irreducible of dimension $n - 1$.

Let \mathcal{V} be the tautological vector bundle on Hilb^n of rank n characterized by $\mathcal{V}|_I = \mathbb{C}[x, y]/I$ for any $I \in \text{Hilb}^n$. Define $\mathcal{O}_{\text{Hilb}^n}(1) = \wedge^n \mathcal{V}$. Denote by \mathcal{V}_{st} the direct summand of \mathcal{V} such that $\mathcal{V} := \mathcal{O}_{\text{Hilb}^n} \oplus \mathcal{V}_{\text{st}}$.

2.5 The Gordon-Stafford functor

In [GS05] Gordon and Stafford define a functor from the category of A_c -modules equipped with a good filtration (i.e., filtrations for which the associated graded is finitely generated as a $\mathbb{C}[\mathfrak{h} \times \mathfrak{h}]^W$ -module) to the category of coherent sheaves on $\text{Hilb}^n(\mathbb{C}^2)$, motivated by the following diagram.

$$\begin{array}{ccc} ? & \xrightarrow[\text{gr}]{} & \mathcal{O}_{\text{Hilb}^n} \\ \uparrow & & \uparrow \text{Hilbert-Chow} \\ A_c & \xrightarrow[\text{gr}]{} & \mathbb{C}[\mathfrak{h} \times \mathfrak{h}]^{S_n} \end{array}$$

The definition of their functor, which we denote by GS is based on the Proj construction of the Hilbert scheme due to Haiman [Hai98, Proposition 2.6]. Let $J_0 = \mathbb{C}[\mathfrak{h}]^W$ and for each $k \geq 1$ let $J_k = (\mathbb{C}[\mathfrak{h}]^{\text{sign}})^k$ be the product of k copies of J_1 in $\mathbb{C}[\mathfrak{h}]$. Put $J = \bigoplus_{k \geq 0} J_k$. Then $\text{Hilb}^n = \text{Proj}(J)$. Therefore, any finitely generated graded J -module gives a coherent sheaf on Hilb^n .

We define two subspaces in $(H_d)_\delta$:

$${}_{d+1}P_d := eH_d\delta e_-, \quad {}_dQ_{d+1} := e_-\delta^{-1}H_{d+1}e.$$

Both ${}_{d+1}P_d$ and ${}_dQ_{d+1}$ inherit from $(H_d)_\delta$ the order filtration such that $\deg(x) = 0$, $\deg(y) = 1$.

For any $d \in \mathbb{C}$, the isomorphism

$$eH_d e \cong e\delta^{-1}H_{d+1}\delta e$$

gives a (A_{d+1}, A_d) -module structure on ${}_{d+1}P_d$ and a (A_d, A_{d+1}) -module structure on ${}_dQ_{d+1}$. Thus we can inductively define for any $k \in \mathbb{Z}_{>0}$,

$${}_{d+k}P_d := {}_{d+k}P_{d+k-1} \otimes_{A_{d+k-1}} {}_{d+k-1}P_d,$$

and similarly for ${}_{d-k}Q_d$. We equip ${}_{d+k}P_d$ and ${}_{d-k}Q_d$ with the tensor product filtrations.

For any A_{-d-1} -module L , one has ([GGS09, Proposition 5.8])

$${}_{d+k}P_d \otimes_{A_d} \Omega_{-d-1}L \cong \Omega_{-(d+k)-1}(-{(d+k)-1}Q_{-d-1} \otimes_{A_{-d-1}} L). \quad (6)$$

Moreover, ${}_{c+k}P_c$ is a (A_{c+k}, A_c) -module and hence defines a shift functor

$$S_{c,k} : A_c\text{-mod} \rightarrow A_{c+k}\text{-mod}, \quad M \mapsto {}_{c+k}P_c \otimes_{A_c} M.$$

If M is equipped with a filtration, one equips $S_{c,k}(M)$ with the tensor product filtration.

Now for an A_c -module M equipped with a good filtration F , the Gordon-Stafford functor associates to (M, F) a coherent sheaf on Hilb^n defined by

$$GS(M, F) = Sh(\text{gr} \bigoplus_{k \geq 0} S_{c,k}M)$$

where we take associated graded with respect to the tensor product filtration and Sh denotes the sheaf associated to the graded module.

2.6 Relating the functors

The parameter c in this subsection can be any complex number.

2.6.1 Compatibility of filtrations

Let $\mathbf{M} \in \mathcal{C}(\mathfrak{G})$ be a mirabolic \mathcal{D} -module endowed with a good filtration F . Write $(M, F) = \Gamma(\mathfrak{G}, (\mathbf{M}, F))$. The filtration F restricts to a filtration on $M^{\tau_{-c-k}(\bar{\mathfrak{g}})}$ for any $k \in \mathbb{Z}_{\geq 0}$ and hence also on $L := \Omega_{-c-1} \circ \mathbb{H}_{-c}(\mathbf{M})$ as Ω_{-c-1} preserves the order filtration.

For any G -module E , write $E^{\det^{-k}} := \{f \in E \mid g \cdot f = \det(g)^{-k}f, \forall g \in G\}$. Also, denote $\mathcal{D}_d(\mathfrak{G}) := \mathcal{D}(\mathfrak{G})/\mathcal{D}_d(\mathfrak{G})\tau_d(\bar{\mathfrak{g}})$. Then there is a homomorphism

$$\phi_M^k : \mathcal{D}_{-c}(\mathfrak{G})^{\det^{-k}} \otimes_{A_{-c-1}} M^{\tau_{-c}(\bar{\mathfrak{g}})} \rightarrow M^{\tau_{-c-k}(\bar{\mathfrak{g}})} \quad (7)$$

given by the left multiplication of $\mathcal{D}(\mathfrak{G})$ on M . By [BG15, Theorem 1.3.5], ϕ_M^k is an isomorphism of A_{-c-k} -modules.

Moreover, by [GGS09, Theorem 5.3(1)], when each of the rational numbers $-c-1, -c-2, \dots, -c-k-1$ satisfies the condition (1), there is an isomorphism

$$\mathcal{D}_{-c}(\mathfrak{G})^{\det^{-k}} \cong {}_{-c-k-1}Q_{-c-1} \quad (8)$$

of filtered modules with respect to the order filtrations on $\mathcal{D}(\mathfrak{G})$ and $(H_c)_\delta$.

Using (6), we have

$$\Omega_{c+k}(c+kP_c \otimes_{A_c} L) = {}_{-(c+k)-1}Q_{-c-1} \otimes_{A_{-c-1}} \Omega_c(L).$$

The tensor product filtration on the right hand side is induced from the filtration on L and the order filtrations on A_{-c-1} and ${}_{-(c+k)-1}Q_{-c-1}$. By (8), we have a filtered isomorphism

$$\Omega_{c+k}(c+kP_c \otimes_{A_c} L) \cong \mathcal{D}_{-c}(\mathfrak{G})^{\det^{-k}} \otimes_{A_{-c-1}} M^{\tau-c}(\bar{\mathfrak{g}}). \quad (9)$$

Consider the tensor product filtration F^T on $\mathcal{D}_{-c}(\mathfrak{G})^{\det^{-k}} \otimes_{A_{-c-1}} M^{\tau-c}(\bar{\mathfrak{g}})$ and the sub-filtration on $M^{\tau-c-k}(\bar{\mathfrak{g}})$. By definition of a filtered $\mathcal{D}(\mathfrak{G})$ -module, ϕ_M^k is a filtered homomorphism, i.e., for any $i \geq 0$,

$$\phi_M^k F_i^T \left(\mathcal{D}_{-c}(\mathfrak{G})^{\det^{-k}} \otimes_{A_{-c-1}} M^{\tau-c}(\bar{\mathfrak{g}}) \right) \subset F_i(M^{\tau-c-k}(\bar{\mathfrak{g}})).$$

This is not an equality in general, i.e., ϕ_M^k is not necessarily a filtered isomorphism. Given (9), we see that ϕ_M^k is a filtered isomorphism if and only if

$$\mathrm{gr}^{F^T} S_{c,k} L \cong \mathrm{gr}^F \Omega_{-c-k-1} M^{\tau-c-k}(\bar{\mathfrak{g}}). \quad (10)$$

2.6.2 Commutativity of the diagram

Let $F\mathcal{C}^G(\mathfrak{G})$ be the abelian category whose objects are pairs (\mathbf{M}, F) , where \mathbf{M} is a G -equivariant mirabolic \mathcal{D} -module and F is a good filtration on \mathbf{M} . We define a descent functor

$$\begin{aligned} \Psi_c : F\mathcal{C}^G(\mathfrak{G}) &\rightarrow \mathrm{Coh}(\mathrm{Hilb}^n) \\ (\mathbf{M}, F) &\mapsto \mathrm{desc}_c(\widetilde{\mathrm{gr}}^F \mathbf{M}|_{\widetilde{\mathrm{Hilb}}}) := Sh \bigoplus_{\ell \geq 0} \Gamma(\widetilde{\mathrm{Hilb}}, \mathrm{gr}^F \mathbf{M})^{\tau-c-\ell}(\bar{\mathfrak{g}}) \end{aligned} \quad (11)$$

whose essential image lands in $\mathrm{Coh}(\mathrm{Hilb}_1^n)$ where $\mathrm{Hilb}_1^n \subset \mathrm{Hilb}^n$ is the preimage of $\{(x, 0) \in (\mathbb{C}^n)^2\}/S_n$ under the Hilbert-Chow map. Later we will write $\mathrm{desc} := \mathrm{desc}_0$.

Remark 2.9. *Since \mathfrak{sl}_n is simply connected, any \mathfrak{sl}_n -action on a coherent sheaf can always be integrated into an SL_n -action. However, whether a \mathfrak{gl}_n -action on a coherent sheaf can be integrated into a GL_n -action depends on whether the center acts by integer eigenvalues. In (11), we consider shifted descent with respect to a \mathfrak{gl}_n -action that may have non-integer eigenvalues. When the eigenvalues lie within $-c + \mathbb{Z}$, taking desc_c is equivalent to first defining an integrable \mathfrak{gl}_n -action by precomposing the original action with τ_c and then taking GL_n -invariants.*

Let $F\mathcal{O}(A_c)$ be the abelian category of well-filtered (i.e., equipped with a good filtration) A_c -modules whose essential image under the forgetful functor lies in $\mathcal{O}(A_c)$.

Let $x = (x_{ij})$ be the standard coordinates of \mathfrak{gl}_n and $(\partial) = (\partial_{x_{ij}})_{1 \leq i, j \leq n}$. Let $\{x_i\}$ and $\{y_i\}$ be dual bases of \mathfrak{h}^* and \mathfrak{h} .

Denote $h := \frac{1}{2} \sum_{1 \leq i, j \leq n} (x_{ij} \partial_{x_{ij}} + x_{ij} \partial_{x_{ij}})$ and $h_c := \Omega_{-c-1}(\mathcal{H}_{-c}(h)) = \frac{1}{2} \sum_{i=1}^n (x_i y_i + y_i x_i)$. (The homomorphism \mathcal{H}_{-c} is defined in (4).)

For any $\mathbf{M} \in F\mathcal{C}^G(\mathfrak{G})$, resp. $L \in F\mathcal{O}(A_c)$, the action of h , resp. h_c , is semisimple and induces a \mathbb{Z} -grading on \mathbf{M} , resp. L . This grading together with the filtration induces a $\mathbb{C}^* \times \mathbb{C}^*$ -equivariant structure on $\Psi_c(\mathbf{M})$, resp. $GS(L)$. On the other hand, the torus $\mathbb{C}^* \times \mathbb{C}^*$ acts on \mathbb{C}^2 by the scalar action on coordinates and hence acts on Hilb^n .

The grading on Hilb^n induced by the filtration corresponds to the embedding $\mathbb{C}^* \hookrightarrow \mathbb{C}^* \times \mathbb{C}^*$, $t \mapsto (1, t)$, while the action of h , resp. h_c , corresponds to the anti-diagonal embedding of $\mathbb{C}^* \hookrightarrow \mathbb{C}^* \times \mathbb{C}^*$: $t \mapsto (t, t^{-1})$ on Hilb^n .

Let $\text{Coh}^{\mathbb{C}^* \times \mathbb{C}^*}(\text{Hilb}^n)$ denote the category of $\mathbb{C}^* \times \mathbb{C}^*$ -equivariant coherent sheaves on Hilb^n . Consider the following diagram:

$$\begin{array}{ccc} & F\mathcal{C}^G(\mathfrak{G}) & \\ \Omega_{-c-1} \circ \mathbb{H}_{-c} \swarrow & & \searrow \Psi_c \\ F\mathcal{O}(A_c) & \xrightarrow{GS} & \text{Coh}^{\mathbb{C}^* \times \mathbb{C}^*}(\text{Hilb}^n) \end{array} \quad (12)$$

Proposition 2.10. *Let $(\mathbf{M}, F) \in F\mathcal{C}^G(\mathfrak{G})$ and $(M, F) = \Gamma(\mathfrak{G}, (\mathbf{M}, F))$. Suppose that $-c - k$ satisfies (1) for all $k \in \mathbb{N}$. Then the following are equivalent*

- (a) $\Psi_c(\mathbf{M}) \cong GS \circ \Omega_{-c-1} \circ \mathbb{H}_{-c}(\mathbf{M})$. That is, the diagram (12) commutes.
- (b) ϕ_M^ℓ (defined in (7)) is a filtered isomorphism for all $\ell \gg 0$.
- (c) ϕ_M^ℓ is a filtered isomorphism for all $\ell \geq 0$.

Proof. Let $L = \Omega_{-c-1} \circ \mathbb{H}_c(M)$. By definition

$$GS(L) = Sh\left(\bigoplus_{\ell \geq 0} \text{gr}^{F^T} S_{c,\ell} L\right)$$

where F^T denotes the tensor product filtration induced by the filtration on L .

On the other hand,

$$\Psi_c(\mathbf{M}) = Sh\left(\bigoplus_{\ell \geq 0} \Gamma(\widetilde{\text{Hilb}}, \text{gr}^F \mathbf{M})^{\tau_{-c-\ell}(\bar{\mathfrak{g}})}\right)$$

Therefore, the equality $GS(L) = \Psi_c(\mathbf{M})$ is equivalent to

$$\Gamma(\widetilde{\text{Hilb}}, \text{gr}^F \mathbf{M})^{\tau_{-c-\ell}(\bar{\mathfrak{g}})} \cong \text{gr}^{F^T} S_{c,\ell} L$$

when $\ell \gg 0$.

By [GGS09, Proposition 7.4]

$$\Gamma(\widetilde{\text{Hilb}}, \text{gr}^F \mathbf{M})^{\tau_{-c-\ell}(\bar{\mathfrak{g}})} = \Gamma(T^* \mathfrak{G}, \text{gr}^F \mathbf{M})^{\tau_{-c-\ell}(\bar{\mathfrak{g}})} = \text{gr}^F \Gamma(\mathfrak{G}, \mathbf{M})^{\tau_{-c-\ell}(\bar{\mathfrak{g}})}$$

for $\ell \gg 0$, because $\widetilde{\text{Hilb}}$ is the semistable locus with respect to \det .

Since $\Omega_{-c-\ell-1}$ preserves filtration, i.e., $\text{gr}^F \Omega_{-c-\ell-1} M^{\tau_{-c-k}(\bar{\mathfrak{g}})} \cong \text{gr}^F M^{\tau_{-c-\ell}(\bar{\mathfrak{g}})}$, given (10) we conclude that (a) is equivalent to (b).

As for the implication (b) \Rightarrow (c), for any $k \geq 0$, take $\ell \gg 0$ and consider

$$\begin{array}{ccc}
\mathcal{D}_{-c}(\mathfrak{G})^{\det^{-\ell}} \otimes_{A_{-c-k-1}} \mathcal{D}_{-c}(\mathfrak{G})^{\det^{-k}} \otimes_{A_{-c-1}} M^{\tau-c}(\bar{\mathfrak{g}}) & \xrightarrow{id \otimes \phi_M^k} & \mathcal{D}_{-c}(\mathfrak{G})^{\det^{-\ell}} \otimes_{A_{-c-k-1}} M^{\tau-c-k}(\bar{\mathfrak{g}}) \\
\downarrow \text{mul} \otimes id & & \downarrow \phi_M^\ell \\
\mathcal{D}_{-c}(\mathfrak{G})^{\det^{-\ell-k}} \otimes_{A_{-c-1}} M^{\tau-c}(\bar{\mathfrak{g}}) & \xrightarrow{\phi_M^{\ell+k}} & M^{\tau-c-\ell-k}(\bar{\mathfrak{g}})
\end{array}$$

Here the map $\text{mul} : \mathcal{D}_{-c}(\mathfrak{G})^{\det^{-\ell}} \otimes_{A_{-c-k-1}} \mathcal{D}_{-c}(\mathfrak{G})^{\det^{-k}} \rightarrow \mathcal{D}_{-c}(\mathfrak{G})^{\det^{-\ell-k}}$ is defined by multiplication. By [GGS09, Lemma 5.2(2)], mul is a filtered isomorphism. Also, by assumption, $\phi_M^{\ell+k}$, ϕ_M^ℓ are filtered isomorphisms. As a result, ϕ_M^k is also a filtered isomorphism. This concludes the proof of the proposition. \square

3 Hodge filtration on the cuspidal mirabolic \mathcal{D} -module

From now on, $c = \frac{m}{n}$ for a positive integer m coprime to n .

Fix the Borel subgroup $B \subset G$ with Lie algebra $\mathfrak{b} \subset \mathfrak{g}$ of upper triangular matrices.

3.1 Defining the Hodge filtration

Write $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ the nilpotent radical and $\mathfrak{n}_r = \mathfrak{n} \cap \mathcal{N}_r$. We give \mathfrak{n} coordinates by

$$\begin{pmatrix} 0 & x_1 & * & * & * & * \\ 0 & 0 & x_2 & * & * & * \\ & & & \cdots & * & * \\ 0 & 0 & 0 & \cdots & 0 & x_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \quad (13)$$

Then $\mathfrak{n}_r = \mathfrak{n} \setminus D$ where D is a simple normal crossing (SNC) divisor defined by $D := \{x_1 x_2 \cdots x_{n-1} = 0\}$.

The restriction of \mathcal{E}_c to $U_0 := \{(x, v) \in \mathfrak{n} \times V \mid s(x, v) \neq 0\}$ is a local system given by the multi-valued function

$$s^c|_{U_0} = x_c x_2^{2c} \cdots x_{n-1}^{(n-1)c} v_n^m. \quad (14)$$

Similarly, $\mathbb{L}_0 := \mathcal{F}_c|_{\mathfrak{n}_r}$ is defined by the multi-valued function

$$s_0^c := x_1^c x_2^{2c} \cdots x_{n-1}^{(n-1)c}. \quad (15)$$

Let F_0 be the order filtration on \mathbb{L}_0 . Then (\mathbb{L}_0, F_0) defines a variation of Hodge structure of rank 1.

Write $i_0 : \mathfrak{n}_r \hookrightarrow \mathfrak{n}$. Saito's theory [Sai90] (as stated in [Pop18, Theorem 4.3.5]) implies that there exists a unique Hodge module structure on $\mathbb{L}_D := (i_0)_! \mathbb{L}_0$. Following [Pop18, 4.4], Hodge filtrations across SNC divisors can be described explicitly as follows.

First of all, we have that

$$\Gamma(\mathfrak{n}, \mathbb{L}_D) = \mathcal{D}(\mathfrak{n}) / \left(\sum_{i=1}^{n-1} \mathcal{D}(\mathfrak{n})(x_i \partial_{x_i} + ic) + \mathcal{D}(\mathfrak{n})S([\mathfrak{n}, \mathfrak{n}]) \right).$$

The $\mathcal{D}_{\mathfrak{n}}$ -module \mathbb{L}_D is a regular meromorphic extension of \mathbb{L}_0 across the SNC divisor D . Inside \mathbb{L}_D , we have Deligne's canonical extension $\mathbb{L}_0^{>-1}$ ([Del70]), which is a lattice extending \mathbb{L}_0 such that the residues ([HTT08, 5.2.2]) of the meromorphic connection under this lattice along all the components of D lie in $(-1, 0]$. By lattice, we mean $\mathbb{L}_0^{>-1}$ is a rank 1 free $\mathcal{O}_{\mathfrak{n}}$ -module such that

$$\mathbb{L}_D = \mathbb{L}_0^{>-1} \otimes_{\mathcal{O}_{\mathfrak{n}}} \mathcal{O}_{\mathfrak{n}}[D] = \mathcal{D}_{\mathfrak{n}} \cdot \mathbb{L}_0^{>-1}$$

Here $\mathcal{O}_{\mathfrak{n}}[D]$ is the sheaf of rational functions on \mathfrak{n} that are regular on U_0 .

In our case, for

$$[s_0^c] := x_1^{-[c]} x_2^{-[2c]} \dots x_{n-1}^{-[(n-1)c]}, \quad (16)$$

we can compute that

$$s_0^c [s_0^c]^{-1} = x_1^{[c]-c} x_2^{[2c]-2c} \dots x_{n-1}^{[(n-1)c]-(n-1)c}$$

and

$$x_i \partial_{x_i} (s_0^c [s_0^c]^{-1}) = [ic] - ic \in (-1, 0], \quad i = 1, \dots, n-1.$$

Hence $\mathbb{L}_0^{>-1} = \mathcal{O}_{\mathfrak{n}}[s_0^c]^{-1} s_0^c \subset \mathbb{L}_0$.

On $\mathbb{L}_0^{>-1}$ we have the filtration

$$F_k^0 \mathbb{L}_0^{>-1} = \mathbb{L}_0^{>-1} \cap (i_0)_* F_k^0 \mathbb{L}_0 = \mathcal{O}_{\mathfrak{n}}[s_0^c]^{-1}$$

for all $k \geq 0$ and 0 otherwise. Hence the induced filtration on $\mathcal{D}_{\mathfrak{n}} \cdot \mathbb{L}_0^{>-1}$ is

$$F_k^D (\mathcal{D}_{\mathfrak{n}} \cdot \mathbb{L}_0^{>-1}) = \sum F_i^{\text{ord}} \mathcal{D}_{\mathfrak{n}} F_0^{k-i} \mathbb{L}_0^{>-1} = (F_k^{\text{ord}} \mathcal{D}_{\mathfrak{n}}) \cdot [s_0^c]^{-1}$$

for all $k \geq 0$ and 0 otherwise.

The pushforward of the Hodge module (\mathbb{L}_D, F_0) along $i_{\mathfrak{n}}$ has underlying $\mathcal{D}_{\mathfrak{g}}$ -module $\mathbb{L} = (i_{\mathfrak{n}})_\dagger \mathbb{L}_D$ with the Hodge filtration filtration $F_{\mathbb{L}}$ on \mathbb{L} defined by the formula ([Pop18, 1.5])

$$\Gamma(F_k^{\mathbb{L}}((i_{\mathfrak{n}})_\dagger(\mathbb{L}_D, F^D))) = \text{Im} \left(\left(\sum_q \Gamma(F_q^{\text{ord}} \mathcal{D}_{\mathfrak{g} \leftarrow \mathfrak{n}}) \otimes \Gamma(F_{k-q}^D(\mathbb{L}_D)) \right) \rightarrow \Gamma((i_{\mathfrak{n}})_\dagger(\mathbb{L}_D)) \right)$$

3.1.1 Associated graded of \mathbb{L}

Recall that $U_0 := \{(x, v) \in \mathfrak{n} \times V \mid s(x, v) \neq 0\}$. Let $\overline{\mathbb{L}}$ be the minimal extension of $\mathcal{E}_c|_{U_0}$ to \mathfrak{G} . One may run the same procedure to define a Hodge module structure on $\overline{\mathbb{L}}$, which we denote by $(\overline{\mathbb{L}}, F^{\overline{\mathbb{L}}}, \bar{i}_*(\overline{\mathbb{L}}_0)_{\mathbb{Q}})$. Here $\bar{i}: U_0 \rightarrow \mathfrak{G}$ and $(\overline{\mathbb{L}}_0)_{\mathbb{Q}}$ is the locally constant sheaf underlying $\overline{\mathbb{L}}_0 := \mathcal{E}_c|_{U_0}$.

Define

$$\mu_{\lceil c \rceil} = (\mu_{\lceil c \rceil}(1), \dots, \mu_{\lceil c \rceil}(n)) := (\lceil c \rceil, \lceil 2c \rceil - \lceil c \rceil, \dots, \lceil nc \rceil - \lceil (n-1)c \rceil). \quad (17)$$

and

$$\mu_{\lfloor c \rfloor} = (\mu_{\lfloor c \rfloor}(1), \dots, \mu_{\lfloor c \rfloor}(n)) := (\lfloor c \rfloor, \lfloor 2c \rfloor - \lfloor c \rfloor, \dots, \lfloor nc \rfloor - \lfloor (n-1)c \rfloor). \quad (18)$$

Then $(\mu_{\lceil c \rceil}(1), \dots, \mu_{\lceil c \rceil}(n)) = (\mu_{\lfloor c \rfloor}(n), \dots, \mu_{\lfloor c \rfloor}(1))$.

Write down the standard \mathfrak{sl}_2 -triple of SL_n :

$$E = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \quad F = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ n-1 & 0 & \cdots & 0 & 0 \\ 0 & 2(n-2) & \cdots & 0 & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & 1-n & 0 \end{pmatrix} \quad H = [E, F] \quad (19)$$

Write $\mathfrak{X} = \overline{B \cdot (F + \mathrm{stab}_{\mathfrak{g}}(E))}$, consisting of matrices with 0 entries below the lower subdiagonal.

Under the coordinates (13) and

$$\begin{pmatrix} * & * & \cdots & * & * & * \\ y_1 & * & \cdots & * & * & * \\ 0 & y_2 & \cdots & * & * & * \\ & & \cdots & & & \\ 0 & 0 & \cdots & y_{n-2} & * & * \\ 0 & 0 & \cdots & 0 & y_{n-1} & * \end{pmatrix} \quad (20)$$

we define

$$\mathfrak{Y}_0 := \{(x, y) \in \mathfrak{n} \times \mathfrak{X} \mid x \in \mathfrak{n}, y \in \mathfrak{X}, x_i y_i = 0, 1 \leq i \leq n-1\}.$$

Let α be the weight associated to the relative canonical bundle $\Omega_{\mathfrak{n}/\mathfrak{g}}$ and $\gamma_n = (0, 0, \dots, 0, 1)$.

Let $i_V : V \hookrightarrow T^*V$ be the zero section. Endow $\mathcal{O}_{\mathfrak{Y}_0}$, resp. $\mathcal{O}_{\mathfrak{Y}_0} \boxtimes (i_V)_* \mathcal{O}_V$ with the trivial B , resp. \overline{B} -equivariant structure. Let \mathbb{C}_λ be the 1-dimensional representation of \overline{B} associated to the character λ . Denote the embedding $\mathfrak{Y}_0 \rightarrow \mathfrak{g} \times \mathfrak{g}$ by $i_{\mathfrak{Y}_0}$.

As in Remark 2.9, we make $\widetilde{\mathrm{gr}}^{\mathrm{H}} \overline{\mathbb{L}}$ a \overline{B} -equivariant sheaf by precomposing the original $\overline{\mathfrak{b}}$ -action with τ_c so that it is integrable.

Lemma 3.1. *As a \overline{B} -equivariant $\mathcal{O}_{T^*\mathfrak{g}}$ -module, $\widetilde{\mathrm{gr}}^{\mathrm{H}} \overline{\mathbb{L}} = (i_{\mathfrak{Y}_0})_* \mathcal{O}_{\mathfrak{Y}_0} \otimes \mathbb{C}_{\mu_{\lceil c \rceil} + \alpha - m\gamma_0}$.*

Proof. As a result of the identification:

$$\Gamma(\mathfrak{g}, \mathbb{L}) \cong \mathcal{D}(\mathfrak{g}) / (\mathcal{D}(\mathfrak{g}) \cdot \mathcal{O}(\mathfrak{b}_-) + \sum_{i=1}^{n-1} \mathcal{D}(\mathfrak{g})(x_i \partial_{x_i} + ic) + \mathcal{D}(\mathfrak{g}) \cdot S([\mathfrak{n}, \mathfrak{n}])$$

we have that $\widetilde{\mathrm{gr}}^{\mathrm{H}} \overline{\mathbb{L}} \cong (i_{\mathfrak{Y}_0})_* \mathcal{O}_{\mathfrak{Y}_0}$ as a $\mathcal{O}_{T^*\mathfrak{g}}$ -module. Furthermore, the first non-vanishing degree of $\mathrm{gr}^{\mathrm{H}} \overline{\mathbb{L}}$ equals $\Omega_{\mathfrak{n}/\mathfrak{g}} \otimes \mathbb{C} \cdot \lceil s_0^c \rceil^{-1}$.

We conclude the lemma by observing that the \overline{B} -action on $[s_0^c]^{-1}$ (see (16)) is exactly by the character $\mu_{[c]} - m\omega_0$. \square

We similarly have that

Lemma 3.2. *As a \overline{B} -equivariant $\mathcal{O}_{T^*\mathfrak{g}}$ -module,*

$$\widetilde{\mathrm{gr}}^H \overline{\mathbb{L}} = ((i_{\mathfrak{y}_0})_* \mathcal{O}_{\mathfrak{y}_0} \boxtimes (i_V)_* \mathcal{O}_V) \otimes \mathbb{C}_{\mu_{[c]} + \alpha}.$$

Proof. Comparing Lemmas 3.1 and 3.2, the extra factor of $m\gamma_n$ comes from the difference between $s^c|_{U_0}$ (eq. (14)) and s_0^c (eq. (15)). \square

3.2 Functors on equivariant Hodge modules

In this section, we state two important results about Hodge modules that will play crucial roles in our study on the cuspidal character \mathcal{D} -module later.

Let G be an arbitrary affine algebraic group with a closed subgroup H . For any smooth variety X with a G -action, consider the diagram

$$\begin{array}{ccc} G \times X & \xrightarrow{\pi} & G/H \times X \\ \downarrow pr & & \downarrow \overline{pr} \\ X & & X \end{array}$$

where pr is the second projection, $\pi : (g, x) \mapsto (\overline{g}, g \cdot x)$ and $\overline{pr} : (\overline{g}, x) \mapsto x$.

For any H -equivariant \mathcal{D}_X -module \mathcal{F} , there is a unique G -equivariant $\mathcal{D}_{G/H \times X}$ -module \mathcal{E} such that $pr^* \mathcal{F} \cong \pi^* \mathcal{E}$. We denote $\mathcal{E} = \mathrm{Ind}_H^G \mathcal{F}$ and $\widetilde{\mathrm{Ind}}_H^G := \overline{pr}_* \circ \mathrm{Ind}_H^G$.

Similarly, for any H -equivariant coherent \mathcal{O}_X -module \mathcal{F} , we write $\mathrm{ind}_H^G \mathcal{F}$ to denote the associated G -equivariant $\mathcal{O}_{G/H \times X}$ -module.

Denote the abelian category of G -equivariant Hodge modules on a smooth variety X by $\mathrm{HM}^G(X)$ (for a definition, see [Ach, Chapter 5]). We also recall that for any morphism $f : X \rightarrow Y$, there are two associated maps ([HTT08, 2.4]), which form the so-called Lagrangian correspondence

$$T^*X \xleftarrow{\rho_f} X \times_Y T^*Y \xrightarrow{\varpi_f} T^*Y.$$

Combining the cases when $f = \pi$ or $f = pr$, we obtain a commutative diagram

$$\begin{array}{ccccc} T^*(G \times X) & \xleftarrow{\rho_{pr}} & G \times T^*X & \xrightarrow{\varpi_{pr}} & T^*X \\ \uparrow \rho_\pi & & & \searrow q & \\ T^*(G/H \times X) \times_{G/H \times X} (G \times X) & \xrightarrow{\varpi_\pi} & T^*(G/H \times X) & \xleftarrow{s} & G/H \times T^*X \end{array} \quad (21)$$

Here q is the natural quotient map but the embedding s does not simply come from the zero section. Instead, it is the composition map $\varpi_\pi \circ \rho_\pi^{-1} \circ \rho_{pr} \circ q^{-1}$, which is clearly well-defined.

Proposition 3.3. *For any $\mathbf{M} \in \mathrm{HM}^G(G/H \times X)$, $SS(\mathbf{M}) \subset s(G/H \times T^*X)$ and the following diagram commutes, where $\tilde{\mathrm{gr}}$ is taken with respect to Hodge filtration.*

$$\begin{array}{ccc} \mathrm{HM}^H(X) & \xrightarrow[\sim]{\mathrm{Ind}_H^G} & \mathrm{HM}^G(G/H \times X) \\ \downarrow \tilde{\mathrm{gr}} & & \downarrow \tilde{\mathrm{gr}} \\ \mathrm{Coh}^H(T^*X) & \xrightarrow[\sim]{\mathrm{ind}_H^G} & \mathrm{Coh}^G(s(G/H \times T^*X)) \end{array}$$

Proof. That the functor Ind_H^G is an isomorphism in the level of Hodge modules can be found in [Ach, Theorem 6.2]. It remains to show the essential image of $\mathrm{HM}^G(G/H \times \mathfrak{G})$ under $\tilde{\mathrm{gr}}$ is as desired and the diagram is commutative.

Suppose $\mathbf{M} = \mathrm{Ind}_H^G \mathbb{L}$ for $\mathbb{L} \in \mathrm{HM}^H(\mathfrak{G})$. By definition, $\pi^\dagger \mathbf{M} = pr^\dagger \mathbb{L}$ as G -equivariant $\mathcal{D}_{G \times X}$ -modules. Taking associated graded of both sides and using [Kas03, Theorem 4.7], we obtain an isomorphism in $\mathrm{Coh}^G(T^*(G \times X))$

$$(\rho_\pi)_* \varpi_\pi^* \tilde{\mathrm{gr}}^\bullet \mathbf{M} \cong (\rho_{pr})_* \varpi_{pr}^* \tilde{\mathrm{gr}}^\bullet \mathbb{L}. \quad (22)$$

In particular, we see $SS(\mathbf{M}) \subset \varpi_\pi \rho_\pi^{-1}(G \times T^*X) \subset s(G/H \times T^*X)$.

Given the diagram (21), the identity (22) implies $\varpi_{pr}^* \tilde{\mathrm{gr}}^\bullet \mathbb{L} = q^* \tilde{\mathrm{gr}}^\bullet \mathbf{M}$. Therefore $\mathrm{ind}_B^G \tilde{\mathrm{gr}}^\bullet \mathbb{L} = \tilde{\mathrm{gr}}^\bullet \mathbf{M}$ and the diagram in the statement commutes. \square

We will also need the following result on direct images of Hodge modules. Let $p : X \rightarrow Y$ be a projective morphism between two smooth varieties. For the functor $p_\dagger : \mathrm{HM}(X) \rightarrow \mathrm{D}^b \mathrm{HM}(Y)$, see [Sch, 23]. Let $\widetilde{\Omega_{X/Y}}$ denote the pullback of the relative canonical bundle $\Omega_{X/Y}$ to $X \times_Y T^*Y$.

Theorem 3.4. ([Lau83, 2.3.2], [Sch, 28]) *The diagram below commutes*

$$\begin{array}{ccc} \mathrm{HM}^G(X) & \xrightarrow{p_\dagger} & \mathrm{D}^b \mathrm{HM}^G(Y) \\ \downarrow \tilde{\mathrm{gr}} & & \downarrow \tilde{\mathrm{gr}} \\ \mathrm{Coh}^G(T^*X) & \xrightarrow{R(\varpi_p)_*(\widetilde{\Omega_{X/Y}} \otimes^L L\rho_p^*(-))} & \mathrm{D}^b \mathrm{Coh}^G(T^*Y) \end{array}$$

where $\tilde{\mathrm{gr}}$ is taken with respect to the Hodge filtration.

3.3 Back to the cuspidal setting

We apply the two results in the last section to \mathbf{N}_c and $\overline{\mathbf{N}}_c$.

Since every regular nilpotent x is contained in a unique Borel subalgebra, there are embeddings of \mathcal{N}_r into $\mathcal{B} \times \mathfrak{g}$ and of U into $\mathcal{B} \times \mathfrak{G}$:

$$\begin{array}{ccc} \mathcal{N}_r & \hookrightarrow & \mathfrak{g} \\ & \searrow & \uparrow p \\ & & \mathcal{B} \times \mathfrak{g} \end{array} \qquad \begin{array}{ccc} U & \hookrightarrow & \mathfrak{G} \\ & \searrow \tilde{i} & \uparrow \\ & & \mathcal{B} \times \mathfrak{G} \end{array}$$

Define the Springer cuspidal character $\mathcal{D}_{\mathfrak{g}}$ -module \mathbf{M}_c to be the minimal extension of \mathcal{F}_c to $\mathcal{B} \times \mathfrak{g}$ and the Springer cuspidal mirabolic $\mathcal{D}_{\mathfrak{g}}$ -module $\overline{\mathbf{M}}_c$ to be the minimal extension of \mathcal{E}_c to $\mathcal{B} \times \mathfrak{g}$. The local system \mathcal{F}_c is also clean with respect to the inclusion \tilde{i} , i.e., $\mathbf{M}_c = \tilde{i}_+ \mathcal{F}_c$. Therefore by functoriality $p_+ \mathbf{M}_c = \mathbf{N}_c$. Similar to Lemma 2.6, we have that $\overline{\mathbf{M}}_c = \mathbf{M}_c \boxtimes \mathcal{O}_V$.

Lemma 3.5. (1) The G -equivariant \mathcal{D} -module underlying $\mathrm{Ind}_B^G(\mathbb{L}, F^{\mathbb{L}})$ is \mathbf{M}_c .

(2) The G -equivariant \mathcal{D} -module underlying $\mathrm{Ind}_B^G(\overline{\mathbb{L}}, F^{\overline{\mathbb{L}}})$ is $\overline{\mathbf{M}}_c$.

Proof. We only show (1) and the same argument applies to (2) since $\overline{\mathbb{L}} \cong \mathbb{L} \boxtimes \mathcal{O}_V$. Consider the following cartesian diagram

$$\begin{array}{ccccc} \mathfrak{n}_r & \xleftarrow{p_0} & G \times \mathfrak{n}_r & \xrightarrow{\pi_0} & G \times^B \mathfrak{n}_r & \xlongequal{\quad} & \mathcal{N}_r \\ \downarrow i_{0,r} & & \downarrow i_r & & & & \downarrow i \\ \mathfrak{g} & \xleftarrow{p} & G \times \mathfrak{g} & \xrightarrow{\quad \pi \quad} & \mathcal{B} \times \mathfrak{g} & & \end{array}$$

Since $\mathcal{E}_c = \mathrm{Ind}_B^G \mathbb{L}_0$, using base change twice, we have

$$\pi^\dagger i_+ (\mathcal{E}_c) = (i_r)_+ \pi_0^\dagger (\mathcal{E}_c) = (i_r)_+ p_0^\dagger (\mathcal{F}_c) = p^\dagger (i_{0,r})_+ (\mathcal{F}_c)$$

which proves the lemma. \square

We let $\mathfrak{Y} = G \times^B \mathfrak{Y}_0$ with an embedding $i_{\mathfrak{Y}} : \mathfrak{Y} \rightarrow T^*(\mathcal{B} \times \mathfrak{g})$ defined by $\overline{(g, x, y)} \mapsto (\overline{g}, [x, y], g \cdot x, g \cdot y)$. Let \mathcal{L}_λ be the \overline{G} -equivariant line bundle on \mathcal{B} associated to the weight λ . Write $\tilde{\mathcal{L}}_\lambda := (\pi_{T^*(\mathcal{B} \times \mathfrak{g}) \rightarrow \mathcal{B}})^* \mathcal{L}_\lambda$. Endow $(i_{\mathfrak{Y}})_* \mathcal{O}_{\mathfrak{Y}}$, resp. $(i_{\mathfrak{Y}})_* \mathcal{O}_{\mathfrak{Y}} \boxtimes \mathcal{O}_V$, with the trivial G , resp. \overline{G} -equivariant structure. As in Remark 2.9, we make $\tilde{\mathrm{gr}}^H \mathbf{M}_c$ a \overline{G} -equivariant sheaf by precomposing the original $\overline{\mathfrak{g}}$ -action with τ_c so that it is integrable.

Corollary 3.6. As \overline{G} -equivariant \mathcal{O} -modules,

- $\tilde{\mathrm{gr}}^H \mathbf{M}_c \cong \tilde{\mathcal{L}}_{\mu_{[c]} + \alpha - m\gamma_n} \otimes (i_{\mathfrak{Y}})_* \mathcal{O}_{\mathfrak{Y}}$.
- $\tilde{\mathrm{gr}}^H \overline{\mathbf{M}}_c \cong (\tilde{\mathcal{L}}_{\mu_{[c]} + \alpha} \otimes (i_{\mathfrak{Y}})_* \mathcal{O}_{\mathfrak{Y}}) \boxtimes (i_V)_* \mathcal{O}_V$.

Proof. By Proposition 3.3 and Lemma 3.5, we have $\tilde{\mathrm{gr}}^H \mathbf{M}_c \cong \mathrm{ind}_B^G \tilde{\mathrm{gr}}^{\mathbb{L}} \mathbb{L}$ and $\tilde{\mathrm{gr}}^H \overline{\mathbf{M}}_c \cong \mathrm{ind}_B^G \tilde{\mathrm{gr}}^{\overline{\mathbb{L}}} \overline{\mathbb{L}}$. It remains to use Lemma 3.1, Lemma 3.2 and the fact that $\mathrm{ind}_B^G \mathbb{C}_\lambda = \mathcal{L}_\lambda$. \square

4 The cuspidal dg module and bigraded characters

4.1 The cuspidal dg module

Recall the coordinates from (13) and (20). When $x \in \mathfrak{n}$ and $y \in \mathfrak{X}$, we have $[x, y] \in \mathfrak{b}$. Moreover, the diagonals of $[x, y]$ equal

$$x_1 y_1, x_2 y_2 - x_1 y_1, \dots, -x_{n-1} y_{n-1}.$$

As a result, $x_i y_i = 0$, $1 \leq i \leq n-1$ if and only if $[x, y] = 0 \bmod \mathfrak{n}$. That is to say, the following diagram is Cartesian:

$$\begin{array}{ccc} \mathfrak{Y} & \xrightarrow{q_{\mathfrak{n}}} & G \times^B \mathfrak{n} \\ \downarrow & & \downarrow \\ G \times^B (\mathfrak{n} \times \mathfrak{X}) & \xrightarrow{q_{\mathfrak{b}}} & G \times^B \mathfrak{b} \end{array} \quad (23)$$

with $q_{\mathfrak{b}} : G \times^B (\mathfrak{n} \times \mathfrak{X}) \rightarrow G \times^B \mathfrak{b}$ given by $(g, x, y) \mapsto (g, [x, y])$ and $q_{\mathfrak{n}}$ is the restriction of $q_{\mathfrak{b}}$ to \mathfrak{Y} .

On \mathcal{B} we have the vector bundle $\underline{\mathfrak{b}}^*$ (resp. $\underline{\mathfrak{b}}$) whose total space equals $G \times^B \mathfrak{b}^*$ (resp. $G \times^B \mathfrak{b}$). Let $\pi_{\mathfrak{b}} : G \times^B \mathfrak{b} \rightarrow \mathcal{B}$ be the projection and $\iota_{\mathfrak{b}} : \mathcal{B} \rightarrow G \times^B \mathfrak{b}$ be the zero section.

The Koszul complex $(\wedge^\bullet \pi_{\mathfrak{b}}^* \underline{\mathfrak{b}}^*, \partial_{\mathfrak{b}})$, with differential given by contraction with the canonical section of $\pi_{\mathfrak{b}}^* \underline{\mathfrak{b}}$, is quasi-isomorphic to $(\iota_{\mathfrak{b}})_* \mathcal{O}_{\mathcal{B}}$.

One can similarly define $\pi_{\mathfrak{n}}$, $\iota_{\mathfrak{n}}$, $\underline{\mathfrak{n}}^*$, etc., such that $(\wedge^\bullet \pi_{\mathfrak{n}}^* \underline{\mathfrak{n}}^*, \partial_{\mathfrak{n}})$ is quasi-isomorphic to $(\iota_{\mathfrak{n}})_* \mathcal{O}_{\mathcal{B}}$. We define a dg algebra by

$$\mathcal{A}'' := ((\wedge^\bullet (\pi_{\mathfrak{n}} \circ q_{\mathfrak{n}})^* \underline{\mathfrak{n}}^*, q_{\mathfrak{n}}^* \partial_{\mathfrak{n}})).$$

By definition, the associated dg scheme $\text{Spec}(\mathcal{A}'')$ makes the following diagram Cartesian.

$$\begin{array}{ccc} \text{Spec}(\mathcal{A}'') & \longrightarrow & \mathcal{B} \\ \downarrow & & \downarrow \iota_{\mathfrak{n}} \\ \mathfrak{Y} & \xrightarrow{q_{\mathfrak{n}}} & G \times^B \mathfrak{n} \end{array} \quad (24)$$

Given the Cartesian diagram (23), we have that $(i_{\mathfrak{Y} \rightarrow G \times^B (\mathfrak{n} \times \mathfrak{X})})_* \mathcal{A}''$ is quasi-isomorphic to

$$\mathcal{A} := ((\wedge^\bullet q_{\mathfrak{b}}^* \pi_{\mathfrak{b}}^* \underline{\mathfrak{b}}^*, q_{\mathfrak{b}}^* \partial_{\mathfrak{b}})).$$

Because (24) and the diagram on the left of (25) are Cartesian, diagram on the right of (25) is also Cartesian.

$$\begin{array}{ccc} G \times^B T^* \mathfrak{g} & \longrightarrow & \mathcal{B} \\ \downarrow & & \downarrow \\ T^*(\mathcal{B} \times \mathfrak{g}) & \longrightarrow & G \times^B \mathfrak{n} \end{array} \quad \begin{array}{ccc} \text{Spec}(\mathcal{A}'') & \longrightarrow & \mathcal{B} \times T^* \mathfrak{g} \\ \downarrow & & \downarrow \\ \mathfrak{Y} & \longrightarrow & T^*(\mathcal{B} \times \mathfrak{g}) \end{array} \quad (25)$$

Diagrams (23), (24) and (25) can be combined into Figure 4.1.

Next, we consider the following maps

$$\begin{array}{ccccc} \mathfrak{Y} & \xrightarrow{i_{\mathfrak{Y}}} & T^*(\mathcal{B} \times \mathfrak{g}) & \xleftarrow{i_{\mathcal{B} \times T^* \mathfrak{g}}} & \mathcal{B} \times T^* \mathfrak{g} \\ & \searrow p_{\mathfrak{Y}} & \downarrow p_{T^*(\mathcal{B} \times \mathfrak{g})} & \swarrow p_{\mathcal{B} \times T^* \mathfrak{g}} & \\ & & T^* \mathfrak{g} & & \end{array}$$

Applying this lemma to (27) in the setting of $Y = \mathcal{B} \times T^*\mathfrak{g}$ and $Z = \mathfrak{Y}$, we obtain

$$\begin{aligned}
& R(p_{\mathcal{B} \times T^*\mathfrak{g}})_* \left(L i_{\mathcal{B} \times T^*\mathfrak{g}}^* (\tilde{\mathcal{L}}_{\mu_{[c]} + \alpha} \otimes (i_{\mathfrak{Y}})_* \mathcal{O}_{\mathfrak{Y}}) \otimes \pi_{T^*\mathfrak{g}}^* \Omega_{\mathcal{B}} \right) \\
&= R(p_{T^*(\mathcal{B} \times \mathfrak{g})})_* \left(\tilde{\mathcal{L}}_{\mu_{[c]}} \otimes (i_{\mathfrak{Y}})_* \mathcal{O}_{\mathfrak{Y}} \otimes^L (i_{\mathcal{B} \times T^*\mathfrak{g}})_* \mathcal{O}_{\mathcal{B} \times T^*\mathfrak{g}} \right) \\
&= R(p_{T^*(\mathcal{B} \times \mathfrak{g})})_* R(i_{\mathfrak{Y}})_* L(i_{\mathfrak{Y}})^* \left(\tilde{\mathcal{L}}_{\mu_{[c]}} \otimes (i_{\mathcal{B} \times T^*\mathfrak{g}})_* \mathcal{O}_{\mathcal{B} \times T^*\mathfrak{g}} \right) \\
&= R(p_{\mathcal{B} \times T^*\mathfrak{g}})_* L(i_{\mathfrak{Y}})^* \left(\tilde{\mathcal{L}}_{\mu_{[c]}} \otimes (i_{\mathcal{B} \times T^*\mathfrak{g}})_* \mathcal{O}_{\mathcal{B} \times T^*\mathfrak{g}} \right). \tag{28}
\end{aligned}$$

Since the diagram on the right of (25) is Cartesian, we have

$$L(i_{\mathfrak{Y}})_* \left(\tilde{\mathcal{L}}_{\mu_{[c]}} \otimes (i_{\mathcal{B} \times T^*\mathfrak{g}})_* \mathcal{O}_{\mathcal{B} \times T^*\mathfrak{g}} \right) = R(i_{\mathfrak{Y}})_* \mathcal{A}_c''.$$

Therefore, (28) equals

$$R(p_{\mathcal{B} \times T^*\mathfrak{g}})_* R(i_{\mathfrak{Y}})_* \mathcal{A}_c'' = R(p_{\mathfrak{Y}})_* \mathcal{A}_c'' = R p_* \mathcal{A}_c.$$

Therefore, \overline{G} -equivariantly $\tilde{\mathfrak{g}}^{\mathrm{H}} \overline{\mathbf{N}}_c = R p_* \mathcal{A}_c \boxtimes (i_V)_* \mathcal{O}_V$. \square

4.2 The equivariant K -theory of the Hilbert scheme

Write $A := \mathbb{C}^* \times \mathbb{C}^*$. Let $K^A(\mathrm{Hilb}^n)$ denote the equivariant K -theory group, which is a module over $K^A(pt) = \mathbb{C}[q^{\pm}, t^{\pm}]$.

Define the isospectral Hilbert scheme IHilb^n to be the reduced fibered product of the following diagram

$$\begin{array}{ccc}
\mathrm{IHilb}^n & \xrightarrow{\beta} & \mathbb{C}^{2n} \\
\downarrow \alpha & & \downarrow \\
\mathrm{Hilb}^n & \longrightarrow & \mathbb{C}^{2n}/S_n
\end{array}$$

A deep result of Haiman states that $\mathcal{P} := \alpha_* \mathcal{O}_{\mathrm{IHilb}^n}$ is a vector bundle of rank $n!$, which is known since as the Procesi bundle.

Moreover, the generalized McKay correspondence of Bridgeland-King-Reid implies an isomorphism

$$\beta_* \alpha^* : K^A(\mathrm{Hilb}^n) \cong K^{S_n \times A}(\mathbb{C}^{2n}). \tag{29}$$

The Grothendieck group $K^{S_n \times A}(\mathbb{C}^{2n})$ is freely generated by $V_{\lambda} \otimes \mathbb{C}[\mathbb{C}^{2n}]$ over $\mathbb{C}[q^{\pm}, t^{\pm}]$ where V_{λ} is the irreducible representation of S_n associated to $\lambda \vdash n$. Let s_{λ} be the Schur function associated to λ . Then the bigraded Frobenius character of $V_{\lambda} \otimes \mathbb{C}[\mathbb{C}^{2n}]$ is

$$\mathrm{ch}_{S_n \times \mathbb{C}^* \times \mathbb{C}^*}(V_{\lambda} \otimes \mathbb{C}[\mathbb{C}^{2n}]) = s_{\lambda} \left(\frac{z}{(1-q)(1-t)} \right).$$

Therefore, composing (29) with $\text{ch}_{S_n \times \mathbb{C}^* \times \mathbb{C}^*}$ establishes an isomorphism

$$\kappa : K^A(\text{Hilb}^n) \cong \{f \in \mathbb{C}(q, t)[z_1, \dots, z_n]^{S_n} \mid f((1-q)(1-t)z) \text{ has coefficients in } \mathbb{C}[q^\pm, t^\pm]\}. \quad (30)$$

such that $\kappa(\mathcal{V}_\lambda) = s_\lambda(\frac{z}{(1-q)(1-t)})$ where $\mathcal{V}_\lambda = \text{Hom}_{S_n}(V_\lambda, \mathcal{P})$.

Let λ^t denote the transpose of the partition $\lambda \vdash n$. The modified Macdonald polynomials $\tilde{H}_\lambda(z; q, t)$ are the unique symmetric polynomials satisfying

$$\begin{aligned} \tilde{H}_\lambda((1-q)z; q, t) &\in \mathbb{C}(q, t)\{s_\mu \mid \mu \geq \lambda\}; \\ \tilde{H}_\lambda((1-t)z; q, t) &\in \mathbb{C}(q, t)\{s_\mu \mid \mu \geq \lambda^t\}; \\ (\tilde{H}_\lambda(z; q, t), s_{(n)}) &= 1. \end{aligned}$$

The A -fixed points in Hilb^n are in bijection with partitions of n . For any $\lambda \vdash n$, let I_λ be the associated fixed point and $[I_\lambda]$ be the K-theory class corresponding to the skyscraper sheaf supported on I_λ .

Proposition 4.3. ([Hai03, Theorem 4.1.5 and Proposition 5.4.1]) *The image of $[I_\lambda]$ under (30) is \tilde{H}_λ .*

Throughout, we may not distinguish between partitions and Young diagrams.

For a box x inside a Young diagram σ , let a, ℓ , resp. a', ℓ' , denote its arm and leg, resp. coarm and coleg (demonstrated in Figure (4.2)). For a Young tableau, define the weight of the box x labeled i by $\chi_i = q^{a'(x)}t^{\ell'(x)}$.

Define

$$g_\lambda = \prod_{x \in \sigma} (1 - q^{a(x)+1}t^{-\ell(x)})(1 - q^{-a(x)}t^{\ell(x)+1})$$

which is the bigraded character of the cotangent space at I_λ in $\text{Hilb}^n(\mathbb{C}^2)$.

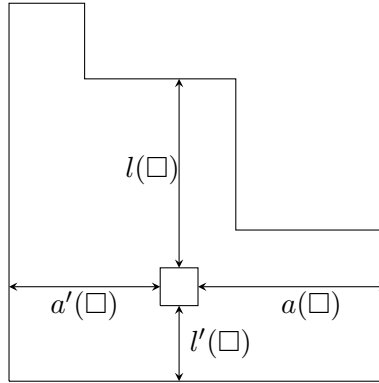


Figure 4.2: ([GN15, Fig.1]) Arm, leg, co-arm and co-leg

Let $\iota_\lambda : \{I_\lambda\} \hookrightarrow \text{Hilb}^n$ denote the embedding. For any $[\mathcal{F}] \in K^A(\text{Hilb}^n)$, under the isomor-

phism (30) we have the localization formula ([CG10, Proposition 5.10.3]):

$$\kappa([\mathcal{F}]) = \sum_{\lambda \vdash n} \frac{\tilde{H}_\lambda}{g_\lambda} \text{ch}_A(L_{\ell_\lambda^*} \mathcal{F}). \quad (31)$$

4.3 Bigraded character of L_c

4.3.1 Principal nilpotent pairs

In [Gin00], a pair of commuting elements (x_1, x_2) in $\mathfrak{g} \times \mathfrak{g}$ is called a principal nilpotent pair if

- (x_1, x_2) is regular, i.e., the joint centralizer of x_1 and x_2 is of minimal dimension.
- For any $(t_1, t_2) \in \mathbb{C}^* \times \mathbb{C}^*$, there exists some $g \in G$ such that $(t_1 x_1, t_2 x_2) = (\text{Ad}(g)x_1, \text{Ad}(g)x_2)$.

It is shown in [Gin00, Theorem 1.2] that for every principal nilpotent pair (e_1, e_2) , there exists an associated semisimple pair $\mathbf{h} = (h_1, h_2)$ such that \mathbf{h} is regular and $[h_i, e_j] = \delta_{ij} e_j$ for $i, j = 1, 2$.

The adjoint action of (h_1, h_2) decomposes \mathfrak{g} into weight spaces $\mathfrak{g} = \bigoplus_{a,b \in \mathbb{Z}^2} \mathfrak{g}_{a,b}$ such that $\text{ad}(h_1)x = ax$ and $\text{ad}(h_2)x = bx$ for all $x \in \mathfrak{g}_{a,b}$.

For every fixed principal nilpotent pair \mathbf{e} with associated semisimple pair \mathbf{h} , let $\rho : \mathbb{C}^* \times \mathbb{C}^* \rightarrow G$ be the 2-parameter subgroup with differential at the identity being $\mathbb{C}^2 \rightarrow \mathfrak{g}$: $(1, 0) \mapsto h_1$, $(0, 1) \mapsto h_2$. We define a $\mathbb{C}^* \times \mathbb{C}^*$ action on \mathfrak{g} by $\text{Ad}(\rho)$ and a $\mathbb{C}^* \times \mathbb{C}^*$ action $\mathfrak{g} \times \mathfrak{g}$ by

$$(t_1, t_2)(x, y) = (t_1^{-1} \text{Ad}(\rho(t_1, t_2))x, t_2^{-1} \text{Ad}(\rho(t_1, t_2))y) \quad (32)$$

such that \mathbf{e} is a fixed point under this action. Note that under these actions, $(\mathfrak{g} \oplus \mathfrak{g})_{a,b} = \mathfrak{g}_{a+1,b} \oplus \mathfrak{g}_{a,b+1}$.

4.3.2 Stalks of the cuspidal dg module

In the case of \mathfrak{sl}_n , principal nilpotent pairs up to conjugation are in bijection with partitions of n . Indeed, for a partition λ , let e_1 be the associated Jordan normal form and e_2 be the Jordan normal form associated to the transpose λ^t . Then it is easy to check that (e_1, e_2) defines a principal nilpotent pair and all principal nilpotent pairs can be constructed in this way. For $\lambda \vdash n$, let \mathbf{e}_λ denote the corresponding principal nilpotent pair up to conjugacy.

The $\mathbb{C}^* \times \mathbb{C}^*$ -action defined in Section 4.3.1 induces a $\mathbb{C}^* \times \mathbb{C}^*$ -action on $(Rp_* \mathcal{A}_c)|_{\mathbf{e}_\lambda}$. Consider this action versus the bigrading on $GS(\text{eL}_c)|_{I_\lambda}$ induced by F^H and the Euler field $h_c \in H_c$. Because the $\text{Ad}(\rho)$ -action is lost when taking descent, one has that

Lemma 4.4. *There is a bigraded isomorphism between vector spaces $(Rp_* \mathcal{A}_c)|_{\mathbf{e}_\lambda} \cong GS(\text{eL}_c)|_{I_\lambda}$.*

Definition 4.5. *We call a Young tableau an almost standard Young tableau (ASYT) if the labels increase rightwards on rows and upwards on columns, with the exception that the labels are allowed to decrease up to 1 going up.*

For an example, see Appendix A.2.2 for a full list of all ASYT of three boxes. Recall that if the labels increase rightwards on rows and upwards on columns, we obtain a standard Young tableau. Let ASYT_λ , resp. SYT_λ , be the set of almost standard Young tableaux, resp. standard Young tableaux, of shape λ .

Remark 4.6. *Almost standard Young tableaux appear in the discussion of the “eccentric correspondence” in [Neg15b, 4.5] and [GN24, 2.3], which is invented to study the shuffle generators defined by eq. (42). When specialized, the eccentric correspondence captures the geometry of the cuspidal dg algebra \mathcal{A} at homological degree 0.*

Since \mathbf{h} is regular, the Borel subalgebras containing both h_1 and h_2 are in bijection with the Weyl group. We fix such a bijection $w \leftrightarrow \mathbf{b}_w$. Write $\mathfrak{Z} := G \times^B (\mathfrak{n} \times \mathfrak{X})$ and $\mathbf{n}_w = [\mathbf{b}_w, \mathbf{b}_w]$ for $w \in W$. We use the subscript r to indicate the regular locus, i.e., when the stablizer of the G -action is of the minimal possible dimension. Then $\mathfrak{Z}_r^A = \sqcup_{w \in W} \mathfrak{Z}_r^{w,A}$ with

$$\mathfrak{Z}_r^{w,A} := \{\mathbf{b}_w\} \times (\mathbf{n}_w \oplus \mathfrak{X}_w) \cap (\mathfrak{g} \oplus \mathfrak{g})_r^A.$$

Similar to [BG13, Lemma 4.4.1], we have that

Lemma 4.7. *For $\mathbf{e} = (e_1, e_2)$ with associated principal semisimple pair (h_1, h_2) , the set of Borel subalgebras that contains e_1, e_2, h_1, h_2 are in bijection with ASYT_λ .*

We will call such Borel subalgebras almost adapted.

Though $[\mathbf{n}_w, \mathfrak{X}_w] = \mathbf{b}_w$, we will insist on writing $[\mathbf{n}_w, \mathfrak{X}_w]$ to emphasize the A -action on it is induced by composing (32) with $[-, -]$. Note that $\mathfrak{g}_{1,1} = [\mathbf{n}_w, \mathfrak{X}_w]^A$. We fix an A -stable subspace $\mathfrak{R}_w \subset [\mathbf{n}_w, \mathfrak{X}_w]$ such that $[\mathbf{n}_w, \mathfrak{X}_w] = \mathfrak{R}_w \oplus \mathfrak{g}_{1,1}$.

For any bigraded vector space V , denote $\lambda(V) = \sum_{i=0}^{\dim(V)} (-1)^i \text{ch}_A(\wedge^i V)$. Following [BG13, 4.3], we adopt the Ω -notation by setting

$$\Omega\left(\sum_{i,j} a_{i,j} q^i t^j\right) = \prod (1 - q^i t^j)^{a_{i,j}} \quad (33)$$

and $\Omega^0(F) = \Omega(F - a_{0,0})$. Also, we write

$$\omega(x) = \frac{(1-x)(1-qt x)}{(1-qx)(1-tx)}$$

Analogous to [BG13, Theorem 4.5.1], we have

Proposition 4.8. *The bigraded character of the stalk $(Rp_* \mathcal{A}_c)|_{\mathbf{e}_\lambda}$ is*

$$\text{ch}_A((Rp_* \mathcal{A}_c)|_{\mathbf{e}_\lambda}) = g_\lambda \frac{(1-qt)^{n-1}}{(1-t)^{n-1}(-t)^{n-1}} \sum_{\sigma \in \text{ASYT}_\mathbf{e}} \frac{\Xi_\sigma \prod_{i=1}^n \chi_{n-i+1}^{\mu_{[c]}(i)}}{\hat{\prod}_{i=1}^{n-1} (1 - \frac{\chi_i}{t\chi_{i+1}})} \quad (34)$$

where

$$\Xi_\sigma := \hat{\prod}_i \frac{1}{(1 - \chi_i^{-1})} \hat{\prod}_{1 \leq i < j \leq n} \omega\left(\frac{\chi_i}{\chi_j}\right) \quad (35)$$

The “restricted” product $\hat{\prod}$ means we ignore all the zero linear denominators.

Proof. Denote the commuting variety $\{(x, y) \in \mathfrak{g} \times \mathfrak{g} \mid [x, y] = 0\}_{\text{red}}$ by \mathfrak{C} . Let \mathfrak{C}_r denote the regular locus in \mathfrak{C} .

By [BG13, Proposition 3.8.6],

$$\text{ch}_A((Rp_*\mathcal{A}_c)|_{\mathbf{e}_\lambda}) = \lambda((T_{\mathfrak{C}_r^A}^*\mathfrak{C}_r)|_{\mathbf{e}_\lambda}) \cdot \sum_{\mathfrak{b}_w \text{ almost adapted}} \lambda((T_{\mathfrak{Z}_r^{w,A}}^*\mathfrak{Z}_r)|_{\mathbf{e}_\lambda})^{-1} \lambda(\mathfrak{R}_w^*) \prod_{i=1}^n \chi_{n-i+1}^{\mu_{[c]}(i)}.$$

One has that $\lambda(\mathfrak{R}_w) = \lambda(([\mathfrak{n}_w, \mathfrak{X}_w]/\mathfrak{g}_{1,1})^*)$. Moreover, by [BG13, Lemma 3.9.1],

$$\begin{aligned} \lambda((T_{\mathfrak{C}_r^A}^*\mathfrak{C}_r)|_{\mathbf{e}_\lambda}) &= g_\lambda \cdot \lambda((\mathfrak{g}/(\text{Stab}(e) \oplus \mathfrak{h}))^*) \\ \lambda((T_{\mathfrak{Z}_r^{w,A}}^*\mathfrak{Z}_r)|_{\mathbf{e}_\lambda}) &= \lambda(\mathfrak{n}_w) \oplus \lambda((\mathfrak{n}_w \oplus \mathfrak{X}_w)/(\mathfrak{g} \oplus \mathfrak{g})^A \cap (\mathfrak{n}_w \oplus \mathfrak{X}_w))^* \end{aligned}$$

Let R , resp. R^+ , denote the set of all roots, resp. positive roots with respect to \mathfrak{b}_w . Also recall that the weight of the box x labeled i is defined by $\chi_i = q^{a'(x)}t^{l'(x)}$. We have that

$$\begin{aligned} \text{ch}_A(\mathfrak{h}^*) &= n - 1 \\ \text{ch}_A(\mathfrak{g}^*) &= n - 1 + \sum_{\alpha_1, \alpha_2 \in R} q^{\alpha_1(h_1)}t^{\alpha_2(h_2)} = n - 1 + \sum_{1 \leq i \neq j \leq n} \chi_i \chi_j^{-1} \\ \text{ch}_A(\text{Stab}(e)^*) &= \sum_{(a,b) \in YT_\lambda} q^{-a}t^{-b} = \sum_{i=1}^n \chi_i^{-1} \\ \text{ch}_A(\mathfrak{n}_w) &= \sum_{\alpha \in R^+} q^{-\alpha(h_1)}t^{-\alpha(h_2)} = \sum_{1 \leq i < j \leq n} \chi_i^{-1} \chi_j \\ \text{ch}_A(\mathfrak{n}_w^* \oplus \{0\}) &= \sum_{\alpha \in R^+} q^{1+\alpha(h_1)}t^{\alpha(h_2)} = q \sum_{1 \leq i < j \leq n} \chi_i \chi_j^{-1} \\ \text{ch}_A(\{0\} \oplus \mathfrak{X}_w^*) &= \sum_{\alpha \in R^+} q^{\alpha(h_1)}t^{1+\alpha(h_2)} + (n-1)t + \sum_{\alpha \in R^+, \text{ simple}} q^{-\alpha(h_1)}t^{1-\alpha(h_2)} \\ &= (n-1)t + t \sum_{1 \leq i < j \leq n} \chi_i \chi_j^{-1} + t \sum_{i=1}^{n-1} \chi_{i+1} \chi_i^{-1} \\ \chi([\mathfrak{n}_w, \mathfrak{X}]) &= (n-1)qt + \sum_{\alpha \in R^+} q^{1+\alpha(h_1)}t^{1+\alpha(h_2)} = (n-1)qt + qt \sum_{1 \leq i < j \leq n} \chi_i \chi_j^{-1} \end{aligned}$$

Since $\lambda(V) = \Omega(\text{ch}_A(V))$ and $\lambda(V/V^A) = \Omega^0(\text{ch}_A(V))$, we further deduce that

$$\lambda((\mathfrak{g}/(\text{Stab}(e) \oplus \mathfrak{h}))^*) \cdot \left(\lambda(\mathfrak{n}_w) \oplus \lambda((\mathfrak{n}_w \oplus \mathfrak{X}_w)/(\mathfrak{g} \oplus \mathfrak{g})^A \cap (\mathfrak{n}_w \oplus \mathfrak{X}_w))^* \right)^{-1}$$

$$\begin{aligned}
&= \Omega\left(\sum_{1 \leq i \neq j \leq n} \chi_i \chi_j^{-1}\right) \Omega((n-1)qt + qt \sum_{1 \leq i < j \leq n} \chi_i \chi_j^{-1}) \\
&\quad \left(\Omega\left(\sum_{i=1}^n \chi_i^{-1}\right) \Omega\left(\sum_{1 \leq i < j \leq n} \chi_i^{-1} \chi_j\right) \Omega^0\left(q \sum_{1 \leq i < j \leq n} \chi_i \chi_j^{-1}\right)\right)^{-1} \\
&\quad \left(\Omega((n-1)t) \Omega^0\left(t \sum_{1 \leq i < j \leq n} \chi_i \chi_j^{-1}\right) \Omega^0\left(t \sum_{i=1}^{n-1} \chi_{i+1} \chi_i^{-1}\right)\right)^{-1} \\
&= \left(\frac{1-qt}{1-t}\right)^{n-1} \Omega^0((1+qt-q-t) \sum_{1 \leq i < j \leq n} \chi_i \chi_j^{-1}) \Omega\left(\sum_{i=1}^n \chi_i^{-1}\right)^{-1} \Omega^0\left(t \sum_{i=1}^{n-1} \chi_{i+1} \chi_i^{-1}\right)^{-1} \\
&= \left(\frac{1-qt}{1-t}\right)^{n-1} \sum_{\sigma \in \text{ASYT}_e} \frac{\hat{\Pi}_{1 \leq i < j \leq n} \omega\left(\frac{\chi_i}{\chi_j}\right)}{\hat{\Pi}_i (1 - \chi_i^{-1}) \hat{\Pi}_{i=1}^{n-1} (1 - t \frac{\chi_{i+1}}{\chi_i})} \\
&= \frac{(1-qt)^{n-1}}{(1-t)^{n-1} (-t)^{n-1}} \sum_{\sigma \in \text{ASYT}_e} \frac{\chi_1 / \chi_n \Xi_\sigma}{\hat{\Pi}_{i=1}^{n-1} (1 - \frac{\chi_i}{t \chi_{i+1}})}
\end{aligned}$$

The proposition now follows from the equality

$$(\mu_{\lceil c \rceil}(1), \dots, \mu_{\lceil c \rceil}(n)) = (\mu_{\lfloor c \rfloor}(1), \dots, \mu_{\lfloor c \rfloor}(n)) + (1, 0, \dots, 0, -1). \quad \square$$

5 The cuspidal and Catalan dg modules and shuffle generators

5.1 The Catalan dg module

Following Section 4.1 closely, we define an analogue of the cuspidal dg module. On \mathcal{B} we also have the vector bundle $[\mathbf{n}, \mathbf{n}]^*$ (resp. $[\mathbf{n}, \mathbf{n}]$) whose total space equals $G \times^B [\mathbf{n}, \mathbf{n}]^*$ (resp. $G \times^B [\mathbf{n}, \mathbf{n}]$). Let $\pi_{[\mathbf{n}, \mathbf{n}]} : G \times^B [\mathbf{n}, \mathbf{n}] \rightarrow \mathcal{B}$ be the projection and $\iota_{[\mathbf{n}, \mathbf{n}]} : \mathcal{B} \rightarrow G \times^B [\mathbf{n}, \mathbf{n}]$ be the zero section. The Koszul complex $(\wedge^\bullet \pi_{[\mathbf{n}, \mathbf{n}]}^* [\mathbf{n}, \mathbf{n}]^*, \partial_{[\mathbf{n}, \mathbf{n}]})$, with differential given by contraction with the canonical section of $\pi_{[\mathbf{n}, \mathbf{n}]}^* [\mathbf{n}, \mathbf{n}]$, is quasi-isomorphic to $(\iota_{[\mathbf{n}, \mathbf{n}]})_* \mathcal{O}_{\mathcal{B}}$. Put $q_{[\mathbf{n}, \mathbf{n}]} : G \times^B (\mathbf{n} \times \mathbf{n}) \rightarrow G \times^B [\mathbf{n}, \mathbf{n}]$ by $(g, x, y) \mapsto (g, [x, y])$.

We define

$$\mathcal{A}'_c := (\pi_{G \times^B (\mathbf{n} \times \mathbf{n}) \rightarrow \mathcal{B}})^* \mathcal{L}_{\mu_{\lfloor c \rfloor}} \otimes (\wedge^\bullet (\pi_{[\mathbf{n}, \mathbf{n}]} \circ q_{[\mathbf{n}, \mathbf{n}]})^* ([\mathbf{n}, \mathbf{n}])^*, q_{[\mathbf{n}, \mathbf{n}]}^* \partial_{[\mathbf{n}, \mathbf{n}]}) \quad (36)$$

and call it the Catalan dg module at slope c .

Warning: Note that the definitions (26) and (36) use different line bundles.

Write $p' : G \times^B (\mathbf{n} \times \mathbf{n}) \rightarrow \mathfrak{g} \times \mathfrak{g} : (g, x, y) \mapsto (g \cdot x, g \cdot y)$. The $\mathbb{C}^* \times \mathbb{C}^*$ -actions in Section 4.3.1 induce a $\mathbb{C}^* \times \mathbb{C}^*$ -action on $(Rp'_* \mathcal{A}'_c)|_{\mathbf{e}}$. Similar to Proposition 4.8, we have that

Proposition 5.1. *The bigraded character of the stalk $R(p_{G \times^B (\mathbf{n} \times \mathbf{n})})_* \mathcal{A}'_c|_{\mathbf{e}}$ is given by*

$$\text{ch}_A((Rp'_* \mathcal{A}'_c)|_{\mathbf{e}_\lambda}) = g_\lambda \sum_{\sigma \in \text{SYT}_\lambda} \frac{\Xi_\sigma \prod_{i=1}^n \chi_i^{\mu_c(n-i+1)}}{\hat{\Pi}_{i=1}^{n-1} (1 - qt \frac{\chi_i}{\chi_{i+1}})} \quad (37)$$

where Ξ_σ is as defined in (35).

Remark 5.2. [BG13, Theorem 4.5.1] states that the q, t -character of the stalk of the rank $n!$ vector bundle on the commuting variety at \mathbf{e}_λ is

$$\frac{g_\lambda}{(1-q)^n(1-t)^n} \sum_{\sigma \in \text{SYT}_\lambda} \Xi_\sigma. \quad (38)$$

As a comparison, the dg algebra in [Gin12] is defined by pulling $(\wedge^\bullet \pi_{\mathbf{n}}^* \underline{\mathbf{n}}^*, \partial_{\mathbf{n}})$ back to $G \times^B (\mathfrak{b} \times \mathfrak{b})$ and the Catalan dg algebra is defined by pulling $(\wedge^\bullet \pi_{[\mathbf{n}, \mathbf{n}]}^* [\mathbf{n}, \mathbf{n}]^*, \partial_{[\mathbf{n}, \mathbf{n}]})$ back to $G \times^B (\mathbf{n} \times \mathbf{n})$.

Compare (38) and (37) when $\prod_{i=1}^n \chi_i^{\mu_c(n-i+1)} = 1$. The differences lie in the terms $(1-q)^n(1-t)^n$ and $\prod_{i=1}^{n-1} (1 - qt \frac{\chi_i}{\chi_{i+1}})$.

The term $(1-q)^n(1-t)^n$ arises from the distinction between the support being $\mathfrak{b} \times \mathfrak{b}$ versus $\mathbf{n} \times \mathbf{n}$ (in \mathfrak{gl}_n). The term $\prod_{i=1}^{n-1} (1 - qt \frac{\chi_i}{\chi_{i+1}})$ results from the complex being defined by \mathbf{n}^* versus $[\mathbf{n}, \mathbf{n}]^*$.

For a $\mathbb{C}^* \times \mathbb{C}^*$ -module \mathcal{F} , we use the notation $q^a t^b \mathcal{F}$ to shift the original action by the weight (a, b) . We will prove in the next sections that the descents of $Rp'_* \mathcal{A}'_c$ and $q^{1-n} Rp_* \mathcal{A}_c$ correspond to the same equivariant K -theory classes on Hilb^n . We conjecture that

Conjecture 5.3. *There exists a $\text{GL}_n \times \mathbb{C}^* \times \mathbb{C}^*$ -equivariant isomorphism:*

$$Rp'_* \mathcal{A}'_c \cong q^{1-n} Rp_* \mathcal{A}_c.$$

One should note that a priori it is not clear whether $Rp'_* \mathcal{A}'_c$ is concentrated in one degree. In contrast, the sheaf $Rp_* \mathcal{A}_c$ is automatically concentrated in one degree as it is the associated graded of \mathbf{N}_c .

5.2 Cuspidal vs Catalan

5.2.1 Shuffle algebras

Define

$$K = \mathbb{C}(q, t)(z_1, z_2, \dots)^{S_\infty}.$$

We endow K with a $\mathbb{C}(q, t)$ -algebra structure via the shuffle product

$$f(z_1, \dots, z_k) * g(z_1, \dots, z_\ell) = \frac{1}{k! \ell!} \text{Sym} [f(z_1, \dots, z_k) g(z_{k+1}, \dots, z_{k+\ell}) \prod_{i=1}^k \prod_{j=k+1}^{k+\ell} \omega(\frac{z_i}{z_j})].$$

Here Sym denotes symmetrization.

Definition 5.4. *The shuffle algebra \mathfrak{A} is defined as the subspace of K consisting of rational functions in the form of*

$$F(z_1, \dots, z_k) = \frac{f(z_1, \dots, z_k) \prod_{1 \leq i < j \leq k} (z_i - z_j)^2}{\prod_{1 \leq i \neq j \leq k} (z_i - qz_j)(z_i - tz_j)}$$

such that f is a symmetric Laurent series satisfying the wheel conditions:

$$f(z_1, z_2, z_3, \dots) = 0 \text{ if } \left\{ \frac{z_1}{z_2}, \frac{z_2}{z_3}, \frac{z_3}{z_1} \right\} = \left\{ q, t, \frac{1}{qt} \right\}.$$

It is shown in [SV13] that there is an isomorphism between \mathfrak{A} and the positive half of the elliptic Hall algebra. Moreover,

Theorem 5.5. ([FT11, SV13]) *There exists a geometric action of the algebra \mathfrak{A} on the vector space $\bigoplus_{n \geq 0} K^A(\text{Hilb}^n) \otimes_{\mathbb{C}[q^\pm, t^\pm]} \mathbb{C}(q, t)$.*

5.2.2 Shuffle generators

Following [Neg22], we define¹

$$P_{n,m} = \text{Sym} \left(\frac{\prod_{i=1}^n z_{n-i+1}^{[ic] - \lfloor (i-1)c \rfloor}}{\prod_{i=1}^{n-1} (1 - qt \frac{z_i}{z_{i+1}})} \prod_{1 \leq i < j \leq n} \omega \left(\frac{z_i}{z_j} \right) \right) \quad (39)$$

According to [BS12], $P_{n,m}$ with $n \geq 1$, $m \in \mathbb{Z}$ generate the shuffle algebra \mathfrak{A} .

By [Neg22, (2.34) and (2.35)], (39) equals

$$P_{n,m} = \left(\frac{(1-qt)}{(1-t)(-qt)} \right)^{n-1} \text{Sym} \left(\frac{\prod_{i=1}^n z_{n-i+1}^{[ic] - \lfloor (i-1)c \rfloor}}{\prod_{i=1}^{n-1} (1 - \frac{z_i}{tz_{i+1}})} \prod_{1 \leq i < j \leq n} \omega \left(\frac{z_i}{z_j} \right) \right). \quad (40)$$

Proposition 5.6. ([Neg15a, Proposition 5.5], [GN15, (49)]) *Under the action in Theorem 5.5,*

$$P_{n,m} \cdot 1 = \left(\frac{(1-q)(1-t)}{(1-qt)} \right)^n \sum_{\lambda \vdash n} \frac{\tilde{H}_\lambda}{g_\lambda} \sum_{\sigma \in \text{SYT}_\lambda} \frac{\prod_{i=1}^n \chi_{n-i+1}^{[ic] - \lfloor (i-1)c \rfloor}}{\prod_{i=1}^{n-1} (1 - qt \frac{\chi_i}{\chi_{i+1}})} \Theta_\sigma \quad (41)$$

$$= \frac{(1-t)(1-q)^{n-1}}{(1-qt)(-qt)^{n-1}} \sum_{\lambda \vdash n} \frac{\tilde{H}_\lambda}{g_\lambda} \sum_{\sigma \in \text{ASYT}_\lambda} \frac{\prod_{i=1}^n \chi_{n-i+1}^{[ic] - \lfloor (i-1)c \rfloor}}{\prod_{i=1}^{n-1} (1 - \frac{\chi_i}{t\chi_{i+1}})} \Theta_\sigma \quad (42)$$

where

$$\Theta_\sigma := \prod_{i=1}^n (1 - qt\chi_i) \prod_{1 \leq i < j \leq n} \omega^{-1} \left(\frac{\chi_j}{\chi_i} \right).$$

Proof. The first identity is exactly [Neg15a, Proposition 5.5] (see also [GN15, (49)]). We show the second identity closely following the proof of *loc. cit.*

As in the proof of [Neg15a, Proposition 5.5] (see also [GN15, (42)]), the localization formula

¹Note that the $P_{m,n}$ here and in [Neg22] is denoted by $\tilde{P}_{m,n}$ in [GN15], as a certain modification of the $P_{m,n}$ in [Neg14].

(31) and [Neg15a, Theorem 4.7] imply that $P_{n,m} \cdot 1$ equals

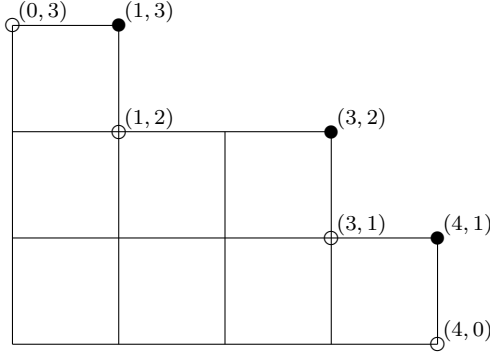
$$\gamma^{n-1} \sum_{\mu \vdash n} \frac{[I_\mu]}{g_\mu} \int \frac{\prod_{i=1}^n z_{n-i+1}^{\lfloor ic \rfloor - \lfloor (i-1)c \rfloor}}{\prod_{i=1}^{n-1} (1 - \frac{z_i}{tz_{i+1}})} \prod_{1 \leq i < j \leq n} \omega\left(\frac{z_i}{z_j}\right) \prod_{\square \in \lambda} \prod_{i=1}^n \left(\omega^{-1}\left(\frac{z_i}{\chi(\square)}\right) (1 - qtz_i) \frac{dz_i}{2\pi i z_i} \right) \quad (43)$$

where $\gamma = \frac{(1-qt)}{(1-t)(-qt)}$ and the integral is taken along contours separating the poles of the function to be integrated.

For a partition $\lambda \vdash n$, [Neg15a, (5.4)] (proved in detail in [BG13, Lemma 4.8.5])

$$\prod_{\square \in \lambda} \left(\omega^{-1}\left(\frac{z}{\chi(\square)}\right) (1 - qtz) \right) = \frac{\prod_{\square \text{ inner corner of } \mu} (1 - \frac{qtz}{\chi(\square)})}{\prod_{\square \text{ outer corner of } \mu} (1 - \frac{qtz}{\chi(\square)})}.$$

where the notion of inner/outer corners is illustrated below, with hollow circles indicating the inner corners of the partition and the solid circles indicating the outer corners.



[Neg15a, Figure 5.1]

When we integrate over z_n , we pick up a residue whenever qtz_n equals to the weight of some outer corner of the partition λ . Label the box adjacent to this corner by n . Then by change of variables the residue we pick up is the weight of the box labeled by $n-1$ and the integral in (43) becomes

$$\chi_n \int \left[\frac{\prod_{i=1}^n z_{n-i+1}^{\lfloor ic \rfloor - \lfloor (i-1)c \rfloor}}{\prod_{i=1}^{n-1} (1 - \frac{z_i}{tz_{i+1}})} \prod_{1 \leq i < j \leq n} \omega\left(\frac{z_i}{z_j}\right) \prod_{\square \in \lambda} \prod_{i=1}^n \left(\omega^{-1}\left(\frac{z_i}{\chi(\square)}\right) (1 - qtz_i) \frac{dz_i}{2\pi i z_i} \right) \right] \Big|_{z_n = \chi_n}$$

Next, when we integrate over z_{n-1} , we pick up a residue whenever

- qtz_{n-1} equals the weight of some outer corner of λ . (z_{n-1} equals to the weight of some inner corner of λ)
- $q \frac{z_{n-1}}{\chi_n} = 1$ or $t \frac{z_{n-1}}{\chi_n} = 1$. (z_{n-1} equals to the weight of some box to the left or below the box labeled by n in the last step.)
- $\frac{z_{n-1}}{t\chi_n} = 1$. (z_{n-1} equals to the weight of some box above the box labeled by n in the last step.)

These three conditions together with the condition for z_n exactly define all the possible relative positions between the box labeled by n and the box labeled by $n-1$ in an ASYT such that the box labeled by n is in a corner. One can argue similarly starting from any label on a corner.

Repeating this procedure, we conclude that the integral in (43) equals

$$\begin{aligned} & \sum_{\text{ASYT}} \prod_{i=1}^n \chi_i \left[\frac{\prod_{i=1}^n z_{n-i+1}^{[ic] - [(i-1)c]}}{\prod_{i=1}^{n-1} (1 - \frac{z_i}{tz_{i+1}})} \hat{\prod}_{1 \leq i < j \leq n} \omega\left(\frac{z_i}{z_j}\right) \hat{\prod}_{\square \in \lambda} \hat{\prod}_{i=1}^n \left(\omega^{-1}\left(\frac{z_i}{\chi(\square)}\right) (1 - qt z_i) \frac{1}{z_i} \right) \right]_{z_i = \chi_i} \\ &= \left(\frac{(1-q)(1-t)}{(1-qt)} \right)^n \sum_{\text{ASYT}_\lambda} \left[\frac{\prod_{i=1}^n \chi_{n-i+1}^{[ic] - [(i-1)c]}}{\prod_{i=1}^{n-1} (1 - \frac{\chi_i}{t\chi_{i+1}})} \hat{\prod}_{1 \leq i < j \leq n} \omega^{-1}\left(\frac{\chi_j}{\chi_i}\right) \prod_{i=1}^n (1 - qt \chi_i) \right]. \end{aligned}$$

The factor $\left(\frac{(1-q)(1-t)}{(1-qt)} \right)^n$ comes from $\hat{\prod}_{i=1}^n \omega^{-1}\left(\frac{\chi_i}{\chi_i}\right)$. □

5.3 Cuspidal and Catalan dg modules vs shuffle generators

Proposition 5.7.

$$\begin{aligned} (P_{n,m} \cdot 1) &= \sum_{\lambda \vdash n} \text{ch}_{q,t}((\text{Rp}_* \mathcal{A}'_c)|_{\mathbf{e}_\lambda}) \frac{\tilde{H}_\lambda}{g_\lambda} \\ &= q^{1-n} \sum_{\lambda \vdash n} \text{ch}_{q,t}((\text{Rp}_* \mathcal{A}_c)|_{\mathbf{e}_\lambda}) \frac{\tilde{H}_\lambda}{g_\lambda} \end{aligned}$$

Proof. Recall the following expression from formulae (41) and (42) of $P_{m,n} \cdot 1$:

$$\Theta_\sigma = \Omega(qt \sum_{i=1}^n \chi_i - (1-q)(1-t) \sum_{1 \leq i < j \leq n} \frac{\chi_j}{\chi_i})$$

For a Young tableau σ with n boxes and positive integer $k \leq n$, we let $\sigma(k)$ denote the Young sub-tableau consisting of the first k labels in σ . Then $\Theta_\sigma = \prod_{i=1}^n \Theta(\sigma(j))$ with

$$\Theta(\sigma(j)) := \Omega(qt \chi_j - (1-q)(1-t) \sum_{1 \leq i < j} \frac{\chi_j}{\chi_i}). \quad (44)$$

Also recall the following expression from formulae (34) and (37) for \mathcal{A}_c and \mathcal{A}'_c :

$$\Xi_\sigma = \Omega\left(-\sum_{i=1}^n \chi_j^{-1} + (1-q)(1-t) \sum_{1 \leq i < j \leq n} \frac{\chi_i}{\chi_j}\right) = \prod_{i=1}^n \Xi(\sigma(j))$$

with

$$\Xi(\sigma(j)) := \Omega^0(-\chi_j^{-1} + (1-q)(1-t) \sum_{1 \leq i < j} \frac{\chi_i}{\chi_j}).$$

In view of Propositions 4.8, 5.1 and 5.6, it suffices to show that for all $i = 1, \dots, n$,

$$\Theta(\sigma(i)) = \Xi(\sigma(i)) \frac{g_{\sigma(i)}}{g_{\sigma(i-1)}} \frac{1-qt}{(1-q)(1-t)}. \quad (45)$$

We consider the case $i = n$; the proof in the general case is similar.

We let R_n (resp. C_n) denote the set of boxes of σ in the same row (resp. same column) as $\sigma \setminus \sigma(n-1)$ (excluding $\sigma \setminus \sigma(n-1)$). For $x \in \sigma$ we write $a_k(x), \ell_k(x)$ to indicate the arm and leg of x in $\sigma(k)$. Then we have that

$$\frac{g_{\sigma(n)}}{g_{\sigma(n-1)}} = (1-q)(1-t) \prod_{x \in C_n \cup R_n} \frac{1 - q^{1+a_n(x)} t^{-\ell_n(x)}}{1 - q^{1+a_{n-1}(x)} t^{-\ell_{n-1}(x)}} \cdot \frac{1 - q^{-a_n(x)} t^{\ell_n(x)+1}}{1 - q^{-a_{n-1}(x)} t^{\ell_{n-1}(x)+1}}.$$

Moreover, it is shown in [BG13, 4.10] that

$$\Xi(\sigma(n)) = \prod_{x \in C_n} \frac{1 - q^{1+a_{n-1}(x)} t^{-\ell_{n-1}(x)}}{1 - q^{1+a_n(x)} t^{-\ell_n(x)}} \prod_{x \in R_n} \frac{1 - q^{-a_{n-1}(x)} t^{\ell_{n-1}(x)+1}}{1 - q^{-a_n(x)} t^{\ell_n(x)+1}}.$$

Therefore the right hand side of (45) equals

$$(1-qt) \prod_{x \in R_n} \frac{1 - q^{1+a_n(x)} t^{-\ell_n(x)}}{1 - q^{1+a_{n-1}(x)} t^{-\ell_{n-1}(x)}} \prod_{x \in C_n} \frac{1 - q^{-a_n(x)} t^{\ell_n(x)+1}}{1 - q^{-a_{n-1}(x)} t^{\ell_{n-1}(x)+1}}$$

which can be also written as Ω applied to

$$qt + \sum_{x \in R_n} (q^{1+a_n(x)} t^{-\ell_n(x)} - q^{1+a_{n-1}(x)} t^{-\ell_{n-1}(x)}) + \sum_{x \in C_n} (q^{-a_n(x)} t^{\ell_n(x)+1} - q^{-a_{n-1}(x)} t^{\ell_{n-1}(x)+1}) \quad (46)$$

Assume the box labeled by n is at (c, r) and the partition is $(\lambda_1, \dots, \lambda_{\ell(\lambda)})$ with $\lambda_1 \geq \dots \geq \lambda_{\ell(\lambda)}$. By the proof of [BG13, Lemma 4.10.2], we have that (46) equals

$$\begin{aligned} & qt + \sum_{i=1}^{c-1} (q^{c-i+1} - q^{c-i}) t^{r+1-\lambda_{i+1}} + \sum_{j=0}^{r-1} q^{-\lambda_{j+1}+c+1} (t^{r-j+1} - t^{r-j}) \\ & = qt + q^{c+1} t^{r+1} \left(\sum_{i=1}^c (q^{-i+1} - q^{-i}) t^{-\lambda_i} + \sum_{j=1}^r q^{-\lambda_j} (t^{-j+1} - t^{-j}) \right). \end{aligned} \quad (47)$$

By [BG13, Lemma 4.8.5]

$$\sum_{i=1}^c (q^{-i+1} - q^{-i}) t^{-\lambda_i} + \sum_{j=1}^r q^{-\lambda_j} (t^{-j+1} - t^{-j}) = -(1-q^{-1})(1-t^{-1}) B_{\sigma \setminus \{(c,r)\}}(q^{-1}, t^{-1}) + 1 - q^c t^{-r}.$$

Here $B_\mu = \sum_{(\alpha, \beta) \in \mu} q^\alpha t^\beta$ for any Young diagram μ .

Therefore, (47) equals

$$-q^c t^r (1-q)(1-t) B_{\sigma(n-1)}(q^{-1}, t^{-1}) + q^{c+1} t^{r+1},$$

which is exactly $qt\chi_n - (1-q)(1-t) \sum_{1 \leq i < n} \frac{\chi_n}{\chi_i}$. From the expression (44), we obtain the identity (45) and the proposition follows. \square

Remark 5.8. A similar formula appears in [KT22, Lemma 5.13]

We will express the formula (41) as $P_{m,n} \cdot 1 = \sum_{\lambda \vdash n} c_{m,n}^\lambda \tilde{H}_\lambda$, with

$$c_{m,n}^\lambda = \left(\frac{(1-q)(1-t)}{(1-qt)} \right)^n \sum_{\sigma \in \text{SYT}_\lambda} \frac{\prod_{i=1}^n \chi_{n-i+1}^{[ic] - [(i-1)c]}}{g_\lambda \prod_{i=1}^{n-1} (1 - qt \frac{\chi_i}{\chi_{i+1}})} \Theta_\sigma.$$

According to [GN15, Conjecture 6.1] which is implied by [Mel21, Theorem 5.8],

$$\sum_{\lambda \vdash n} c_{m,n}^\lambda = \sum_D q^{\mu - \text{area}(D)} t^{\text{dinv}(D)} = \sum_D q^{\text{dinv}(D)} t^{\mu - \text{area}(D)} \quad (48)$$

is the q, t -Catalan number. Here $\mu = \frac{(m-1)(n-1)}{2}$. The sums on the middle and right are taken over all $\frac{m}{n}$ -Dyck paths D . The area and dinv are two combinatorial statistics associated to each dyck path with nonnegative integer values. In particular, when $\text{area}(D) = 0$, $\text{dinv}(D) = \mu$ and when $\text{dinv}(D) = 0$, $\text{area}(D) = \mu$. Interested readers can refer to [GN15, 6.2] for definitions.

Corollary 5.9. With respect to the natural $\mathbb{C}^* \times \mathbb{C}^*$ -action on \mathcal{A}_c ,

- $\text{ch}_{A \times S_n} \Gamma(\text{Hilb}^n, \mathcal{P} \otimes \text{desc}(Rp_* \mathcal{A}_c \boxtimes \mathcal{O}_V)|_{\widetilde{\text{Hilb}^n}}) = q^{n-1} P_{m,n} \cdot 1.$
- $\text{ch}_A \Gamma(\text{Hilb}^n, \text{desc}(Rp_* \mathcal{A}_c \boxtimes \mathcal{O}_V)|_{\widetilde{\text{Hilb}^n}}) = q^{n-1} \sum_{\lambda \vdash n} c_{m,n}^\lambda.$

Remark 5.10. Let $\tilde{\mathcal{P}}$ be the rank $n!$ vector bundle on \mathfrak{C} as defined in [Gin12]. Then one similarly has that

- $\text{ch}_{A \times S_n} \Gamma(\mathfrak{C}_r, \tilde{\mathcal{P}} \otimes Rp_* \mathcal{A}_c)^{\bar{G}} = q^{n-1} P_{m,n} \cdot 1.$
- $\text{ch}_A \Gamma(\mathfrak{C}_r, Rp_* \mathcal{A}_c)^{\bar{G}} = q^{n-1} \sum_{\lambda \vdash n} c_{m,n}^\lambda.$

Lemma 5.11. $\Gamma(T^* \mathfrak{G}, \tilde{\text{gr}}^{\text{H}} \overline{\mathbf{N}}_c)^{\tau-c(\mathfrak{g})} = \Gamma(\widetilde{\text{Hilb}^n}, \tilde{\text{gr}}^{\text{H}} \overline{\mathbf{N}}_c)^{\tau-c(\mathfrak{g})}.$

Proof. Clearly we have the inclusion “ \subset ”. Thus we only need to count the dimensions. By Corollary 2.8, the left hand side equals $\tilde{\text{gr}}^{\text{H}} \text{eL}_c$, which is known to have dimension $\frac{(m+n-1)!}{m!n!}$, the Catalan number. Moreover, by Corollary 5.9 and Proposition 4.1, the dimension of the right hand side equals to $\sum_{\lambda \vdash n} c_{m,n}^\lambda (q=1, t=1).$

By (48), $\sum_{\lambda \vdash n} c_{m,n}^\lambda (q=1, t=1)$ equals the number of $\frac{m}{n}$ -dyck paths, which is known to be $\frac{(m+n-1)!}{m!n!}$. \square

Proposition 5.12. *Hodge filtrations are compatible with shift functors, i.e., when $c > 1$ the following isomorphism is filtered with respect to the Hodge filtration*

$$\mathrm{eL}_c \cong \mathrm{eH}_c \delta \mathrm{e}_- \otimes_{\mathrm{eH}_{c-1}} \mathrm{eL}_{c-1}$$

where the filtration on the right hand side is the tensor product filtration with H_c filtered by $\deg y = 1$ and $\deg x = \deg w = 0$.

Proof. Recall the homomorphism $\phi_{\overline{\mathbf{N}}_c}^k$ from (7). Given the equivalence (b) \Leftrightarrow (c) in Proposition 2.10, it suffices to show that $\mathrm{gr}(\phi_{\overline{\mathbf{N}}_c}^k)$ is an isomorphism for $k \gg 0$.

We claim that when $k \gg 0$, $\Gamma(T^*\mathfrak{G}, \widetilde{\mathrm{gr}}^H(\mathcal{D}_{-c}(\mathfrak{G})^{\det^{-k}})) = \Gamma(\widetilde{\mathrm{Hilb}}^n, \widetilde{\mathrm{gr}}^H(\mathcal{D}_{-c}(\mathfrak{G})^{\det^{-k}})) = \Gamma(\mathcal{O}_{\mathrm{Hilb}}(k))$.

The first equality is [GGS09, Proposition 7.4]. The second equality follows from the \overline{G} -equivariant isomorphism: $(\mathcal{O}_{\mathfrak{g} \times \mathfrak{g}} \boxtimes \pi_V^* \mathcal{O}_{\mathbb{P}^{n-1}}(k))|_{\widetilde{\mathrm{Hilb}}^n} \cong \mathcal{O}_{\widetilde{\mathrm{Hilb}}^n}(k)$. Here $\pi_V : V \setminus \{0\} \rightarrow \mathbb{P}^{n-1}$ is the quotient map.

As a result of the claim, Lemma 5.11 and Proposition 4.1, when $k \gg 0$ we have that

$$\mathrm{gr}(\mathcal{D}_{-c}(\mathfrak{G})^{\det^{-k}} \otimes_{\mathrm{A}_c} \Gamma(\mathfrak{G}, \mathbf{N}_c)^{\tau-c(\bar{\mathfrak{g}})}) \cong \Gamma(\mathrm{Hilb}_n, \mathcal{O}_{\mathrm{Hilb}^n}(k) \otimes \mathrm{desc}((Rp_* \mathcal{A}_c \boxtimes (i_V)_* \mathcal{O}_V)|_{\widetilde{\mathrm{Hilb}}^n}))$$

which is isomorphic to

$$\Gamma(\widetilde{\mathrm{Hilb}}^n, (Rp_*(\mathcal{A}_c \otimes (\pi_{\mathfrak{Y} \rightarrow \mathcal{B}})^* \mathcal{L}_{(k, \dots, k)})) \boxtimes (i_V)_* \mathcal{O}_V) \cong \Gamma(\widetilde{\mathrm{Hilb}}^n, Rp_* \mathcal{A}_{c+k} \boxtimes (i_V)_* \mathcal{O}_V).$$

Using Lemma 5.11 and Proposition 4.1 again, we see that

$$\Gamma(\widetilde{\mathrm{Hilb}}^n, Rp_* \mathcal{A}_{c+k} \boxtimes (i_V)_* \mathcal{O}_V) \cong \mathrm{gr}^H \Gamma(\mathfrak{G}, \overline{\mathbf{N}}_c)^{\bar{\mathfrak{g}}-c-k}$$

and the proposition follows. \square

Corollary 5.12 allows us to extend F^H from eL_c to L_c for all $c = \frac{m}{n} > 1$ by defining a tensor product filtration on

$$\mathrm{L}_c \cong \mathrm{H}_c \delta \mathrm{e}_- \otimes_{\mathrm{A}_{c-1}} \mathrm{eL}_{c-1}. \quad (49)$$

Theorem 5.13. (i) *The bigraded Frobenius character of L_c with respect to the Hodge filtration and the Euler field \mathfrak{h}_c is*

$$\mathrm{ch}_{S_n \times \mathbb{C}^* \times \mathbb{C}^*} \mathrm{gr}^H \mathrm{L}_c = (P_{m,n} \cdot 1)(q, q^{-1}t).$$

(ii) *In $\mathrm{Coh}^{\mathbb{C}^* \times \mathbb{C}^*}(\mathrm{Hilb}^n)$, we have that*

$$GS(\mathrm{eL}_c) \cong q^{1-n} \mathrm{desc}((Rp_* \mathcal{A}_c \boxtimes (i_V)_* \mathcal{O}_V)|_{\widetilde{\mathrm{Hilb}}^n}).$$

Proof. Because of Proposition 5.12 and Proposition 2.10, we have $GS(\mathrm{eL}_c) = \Psi_c(\overline{\mathbf{N}}_c)$.

By [GS05, Theorem 4.5], $GS(\mathrm{e}H_c) = \mathcal{P}$. This plus (49) implies that as $\mathbb{C}^* \times \mathbb{C}^*$ -modules, $\mathrm{gr}^H L_c = \Gamma(\mathrm{Hilb}^n, \mathcal{P} \otimes \Psi(\overline{N}_c))$.

By Proposition 4.1,

$$\mathcal{P} \otimes \Psi(\overline{N}_c) = q^a t^b (\mathcal{P} \otimes \mathrm{desc}((Rp_* \mathcal{A}_c \boxtimes (i_V)_* \mathcal{O}_V)|_{\widetilde{\mathrm{Hilb}^n}))$$

whose space of global sections has bigraded Frobenius character $q^{a+n-1} t^b P_{m,n} \cdot 1$ according to Corollary 5.9, for some integers a, b .

As a consequence, $\mathrm{ch}_A(\mathrm{gr}^H \mathrm{e}L_c) = q^{a+n-1} t^b \sum_{\lambda \vdash n} c_{m,n}^\lambda(q, q^{-1}t)$. The change of variable $(q, q^{-1}t)$ comes from the fact that the Euler field h_c acts by weight $(1, -1)$.

It remains to show $q^{a+n-1} t^b = 1$. However, the highest, resp. lowest weight of $\mathrm{e}L_c$ under the action of h_c is μ , resp. $-\mu$. Given (48) we see that $q^{a+n-1} t^b = 1$ and the theorem follows. \square

5.4 Link homology and filtrations

We can now conclude Theorem 1.2 from the introduction.

In $\bigoplus_{n \geq 0} K^A(\mathrm{Hilb}^n) \otimes_{\mathbb{C}[q^\pm, t^\pm]} \mathbb{C}(a, q, t)$, define $\Lambda(\mathcal{V}_{\mathrm{st}}, a) = \bigoplus_{i=0}^{n-1} a^i (\wedge^i \mathcal{V}_{\mathrm{st}})$.

Theorem 5.14. ([Mel22, Corollary 3.4]) *Up to a constant factor, the triply graded Euler characteristic $\mathrm{ch}_{a,q,t}(\mathrm{HHH}(T_{m,n}))$ equals the matrix coefficient $\langle \Lambda(\mathcal{V}_{\mathrm{st}}, a) | P_{m,n} | 1 \rangle$.*

As a corollary of Theorem 5.14 and Theorem 5.13, we have

Corollary 5.15. *For $m > n$ and $(m, n) = 1$, there is a triply graded isomorphism when $m > n$:*

$$\mathrm{HHH}(T_{m,n}) \cong \mathrm{Hom}_{S_n}(\wedge^\bullet \mathfrak{h}, \mathrm{gr}_\bullet^H(\bigoplus L_c(\bullet))). \quad (50)$$

Since the left hand side of (50) is m, n -symmetric, we see that

Corollary 5.16. *Hodge filtrations on $\mathrm{e}L_{\frac{m}{n}}$ and $\mathrm{e}L_{\frac{n}{m}}$ are compatible with the isomorphism $\mathrm{e}L_{\frac{m}{n}} \cong \mathrm{e}L_{\frac{n}{m}}$.*

Consider a partial order on the positive rational numbers in the following way: for coprime pairs (m, n) , $\frac{m}{n} \prec \frac{m+n}{n}$; if $n < m$, then $\frac{m}{n} \prec \frac{n}{m}$.

One can always go from $c = \frac{m}{n}$ to $\frac{1}{n'}$ for some integer $n' > 1$ through a chain of rational numbers decreasing under the order $(\mathbb{Q}_{>0}, \prec)$.

Definition 5.17. [GORS14, Theorem 4.1] *We define a filtration F^{ind} inductively as follows:*

$$0 = F_{-1}^{\mathrm{ind}} \mathrm{e}L_{\frac{1}{n}} \subset F_0^{\mathrm{ind}} \mathrm{e}L_{\frac{1}{n}} = \mathrm{e}L_{\frac{1}{n}} = L_{\frac{1}{n}}.$$

Next, F^{ind} is defined inductively under the order $(\mathbb{Q}_{>0}, \prec)$ using the isomorphisms:

$$\text{when } m, n > 1, \quad \mathrm{e}L_{\frac{m}{n}} \cong \mathrm{e}L_{\frac{n}{m}} \quad ([CEE09, 8.2])$$

$$\text{when } c > 1, \quad L_c \cong H_c \mathrm{e}_- \otimes_{\mathrm{e}H_{c-1} \mathrm{e}} \mathrm{e}L_{c-1} \quad ([GS05, Theorem 1.6])$$

Combining Proposition 5.12 and Corollary 5.16, we obtain that

Proposition 5.18. $F_j^H L_c = F_j^{\text{ind}} L_c$ when $c > 1$ and $F_j^H eL_c = F_j^{\text{ind}} eL_c$ when $c > 0$.

On H_c we have the Fourier transform defined by

$$\Phi_c(x_i) = y_i, \quad \Phi_c(y_i) = -x_i, \quad \Phi_c(w) = w \quad (51)$$

which defines the Dunkl bilinear form

$$(-, -)_c : \mathbb{C}[\mathfrak{h}] \times \mathbb{C}[\mathfrak{h}] \rightarrow \mathbb{C}, \quad (f, g)_c = [\Phi_c(f)g]|_{x_i=0}.$$

When $c = \frac{m}{n} > 0$, with m, n coprime, $(-, -)_c$ has a nonzero kernel I_c and the resulting quotient $\mathbb{C}[\mathfrak{h}]/I_c$ is exactly isomorphic to L_c ([DO03, Proposition 2.34]). Inside $\mathbb{C}[\mathfrak{h}]$, take the ideal $\mathfrak{a} := (\mathbb{C}[\mathfrak{h}]_+^W)$. Let β_c be a highest weight vector in L_c and let ${}^{\perp_c}$ denote orthogonal complement with respect to $(-, -)_c$.

Definition 5.19. The algebraic filtration² is defined by

$$F_i^{\mathfrak{a}}(L_c) = \Phi_c[(\mathfrak{a}^{i+1})^{\perp_c}] \beta_c.$$

Proposition 5.20. For $m > 0$ and $(m, n) = 1$, there is a triply graded isomorphism:

$$\text{HHH}(T_{m,n}) \cong \text{Hom}_{S_n}(\wedge^{\bullet} \mathfrak{h}, \text{gr}_{\bullet}^{\mathfrak{a}}(\oplus L_c(\bullet))). \quad (52)$$

Proof. It is shown in [Ma24] that $F^{\text{ind}} = F^{\mathfrak{a}}$. This plus Proposition 5.18 and Corollary 5.15 implies that Proposition 5.20 holds when $m > n$. Denote the right hand side of (52) by $\mathfrak{H}_{m,n}$, it remains to show a triply graded isomorphism $\mathfrak{H}_{m,n} \cong \mathfrak{H}_{n,m}$.

By [Gor13, Corollary 1.1], for all $k \geq 0$ there is an isomorphism

$$\text{Hom}_{S_n}(\wedge^k \mathbb{C}^{n-1}, L_{\frac{m}{n}}) \cong \text{Hom}_{S_m}(\wedge^k \mathbb{C}^{m-1}, L_{\frac{n}{m}}). \quad (53)$$

via identifications with spaces of differential forms on a zero-dimensional moduli space associated with the plane curve singularity $x^m = y^n$. By [GORS14, Proposition 1.5], (53) is a bigraded isomorphism with respect to the algebraic filtration and the Euler field h_c . This finishes the proof of the proposition. \square

6 Fourier transforms of cuspidal character \mathcal{D} -modules

In this section, we prove an auxiliary result about the Fourier transform of the cuspidal character \mathcal{D} -module. Although unrelated to the main results of the paper, this may be of independent interest.

²In [Ma24] (and [GORS14]) the filtration $F^{\mathfrak{a}}$ is called the power filtration, while the algebraic filtration is its associated Kazhdan filtration.

6.1 The cuspidal character \mathcal{D} -module is stable under the Fourier transform

The map $\mathfrak{g} \times \mathfrak{g}^* \rightarrow \mathfrak{g}^* \times \mathfrak{g}$ sending $(x, x^*) \mapsto (x^*, -x)$ induces an isomorphism $\sigma_1 : \mathcal{D}(\mathfrak{g}) \cong \mathcal{D}(\mathfrak{g}^*)$.

Further identifying $\sigma_2 : \mathcal{D}(\mathfrak{g}^*) \cong \mathcal{D}(\mathfrak{g})$ via a non-degenerate bilinear form $\mathfrak{g} \cong \mathfrak{g}^*$, we have obtain the Fourier transform induced by $\sigma := \sigma_2 \circ \sigma_1$:

$$\mathbb{F} : \mathcal{D}_{\mathfrak{g}}\text{-mod} \rightarrow \mathcal{D}_{\mathfrak{g}}\text{-mod}.$$

Proposition 6.1. *As $\mathcal{D}_{\mathfrak{g}}$ -modules, $\mathbb{F}\mathbf{N}_c \cong \mathbf{N}_c$.*

Proof. Put $\iota : T^*\mathfrak{g} \rightarrow T^*\mathfrak{g}$, $(x, x^*) \mapsto (x^*, -x)$. Then

$$SS(\mathbb{F}(\mathbf{N}_c)) = \iota(SS(\mathbf{N}_c)) \subset \mathcal{N} \times \mathcal{N}.$$

By [Lus87], $\mathbb{F}(\mathbf{N}_c)$ is again a cuspidal character \mathcal{D} -module and is determined by the monodromy of its restriction to \mathcal{N}_r . Therefore the proposition follows from the following claim:

Claim: There is an isomorphism $\iota^\dagger(\mathbb{F}(\mathbf{N}_c)) \cong \mathcal{F}_c$ where $\iota : \mathcal{N}_r \hookrightarrow \mathfrak{g}$.

Recall that the cuspidal character \mathcal{D} -module \mathbf{N}_c can be expressed as $\widetilde{\text{Ind}}_B^G \mathbb{L}$. Therefore $\mathbb{F}(\mathbf{N}_c) = \widetilde{\text{Ind}}_B^G \mathbb{F}(\mathbb{L})$.

Since

$$\Gamma(\mathfrak{g}, \mathbb{L}) = \mathcal{D}(\mathfrak{g}) / (\mathcal{D}(\mathfrak{g}) \cdot O(\mathfrak{b}_-) + \sum_i \mathcal{D}(\mathfrak{g})(x_i \partial_{x_i} - ic) + \mathcal{D}(\mathfrak{g}) \cdot S([\mathfrak{n}, \mathfrak{n}])) \quad (54)$$

we see that

$$\Gamma(\mathfrak{g}, \mathbb{F}(\mathbb{L})) = \mathcal{D}(\mathfrak{g}) / (\mathcal{D}(\mathfrak{g}) \cdot S(\mathfrak{b}) + \sum_i \mathcal{D}(\mathfrak{g})(\partial_{y_i} y_i + ic) + \mathcal{D}(\mathfrak{g}) \cdot O([\mathfrak{n}_-, \mathfrak{n}_-])). \quad (55)$$

Consider the standard \mathfrak{sl}_2 -triple E, F, H (defined by (19)). Then (54) and (55) tells us that \mathbb{L} is the minimal extension of a local system supported on $B \cdot E$ and $\mathbb{F}(\mathbb{L})$ is the minimal extension of a local system supported on $B \cdot F$ defined by a multi-valued function

$$y_1^{-c} y_2^{-2c} \cdots y_{n-1}^{-(n-1)c}. \quad (56)$$

Moreover, $\mathbb{L}|_{T \cdot E}$ is a local system on $T \cdot E$ such that

$$\mathcal{F}_c = \widetilde{\text{Ind}}_T^G i_{\dagger}^E (i^E)^\dagger \iota^\dagger \mathbb{L}$$

with $i^E : T \cdot E \rightarrow \mathcal{N}_r$. Similarly, $\mathbb{F}(\mathbf{N}_c)|_{T \cdot F}$ is a local system on $T \cdot F$ such that

$$(\mathbb{F}(\mathbf{N}_c))|_{\mathcal{N}_r} = \widetilde{\text{Ind}}_T^G i_{\dagger}^F (i^F)^\dagger \iota^\dagger \mathbb{F}(\mathbb{L})$$

with $i^F : T \cdot F \rightarrow \mathcal{N}_r$.

It suffices to show that $(i^F)^\dagger \mathcal{F}_c = (i^F)^\dagger \iota^\dagger \mathbb{F}(\mathbb{L})$. Recall that the pullback of \mathcal{F}_c along the

fibration $q : U \rightarrow \mathcal{N}_r$ is \mathcal{E}_c and \mathcal{E}_c is defined by the multi-valued function s^c (eq. (2)).

$$s^c|_{T \cdot F} = v_1^{nc} y_1^{(n-1)c} y_2^{(n-2)c} \cdots y_{n-1}^c.$$

Therefore, $(i^F)^\dagger \mathcal{F}_c$ is defined by the multi-valued function

$$y_1^{(n-1)c} y_2^{(n-2)c} \cdots y_{n-1}^c.$$

Finally, the lemma follows from the observation that the functions $y_1^{(n-1)c} y_2^{(n-2)c} \cdots y_{n-1}^c$ and $y_1^{-c-1} y_2^{-2c-1} \cdots y_{n-1}^{-(n-1)c-1}$ (eq. (56)) define the same local system on $T \cdot F$ as

$$((n-1)c, (n-2)c, \dots, c) - (m, m, \dots, m) = (-c, -2c, \dots, -(n-1)c). \quad \square$$

6.2 An explicit description of the Fourier transform

Let $x = (x_{ij})$ be the standard coordinates of \mathfrak{gl}_n and $(\partial) = (\partial_{x_{ij}})_{1 \leq i, j \leq n}$. Take

$$e := \frac{1}{2} \text{tr}(x^2), \quad f := -\frac{1}{2} \text{tr}(\partial^2), \quad h = \sum_{1 \leq i, j \leq n} x_{ij} \partial_{x_{ij}} + (n^2 - 1)/2. \quad (57)$$

Clearly, $[e, f] = h$, $[h, e] = 2e$ and $[h, f] = -2f$. Notice that $[e, -]$, $[f, -]$, $[h, -]$ all preserve the homogeneous components of $\mathcal{D}(\mathfrak{g})$ (with $\deg(x_{ij}) = \deg(\partial_{x_{ij}}) = 1$). In other words, this \mathfrak{sl}_2 -action on $\mathcal{D}(\mathfrak{g})$ is locally finite and thus integrable. Moreover, $e - f$ acts on $\mathcal{D}(\mathfrak{g})$ via

$$[e - f, x_{ij}] = [-f, x_{ij}] = \partial_{x_{ij}}, \quad [e - f, \partial_{x_{ij}}] = [e, \partial_{x_{ij}}] = -x_{ij}. \quad (58)$$

Hence its exponential $\text{Ad } e^{\frac{i\pi}{2}(e-f)}$ gives exactly the Fourier transform σ .

On the other hand, any $\mathcal{D}(\mathfrak{g})$ -module inherits such an \mathfrak{sl}_2 -action. By [CEE09, example 63], the action of $\{e, f, h\}$ on \mathbf{N}_c is locally finite and thus also integrable. Denote the action of $e^{\frac{i\pi}{2}(e-f)}$ on \mathbf{N}_c by Φ . Then (58) implies that

$$\Phi(x_{ij}a) = \partial_{x_{ij}} \Phi(a), \quad \Phi(\partial_{x_{ij}}a) = -x_{ij} \Phi(a), \quad \forall a \in \mathbf{N}_c, 1 \leq i, j \leq n.$$

Therefore, Φ gives an explicit isomorphism between \mathbf{N}_c and $\mathbb{F}(\mathbf{N}_c)$.

A Examples

We compile some computations of $\sum_{\lambda \vdash n} \frac{\text{ch}_{q,t}((Rp_*\mathcal{A}_c)|_{\mathbf{e}_\lambda})}{g_\lambda}$ and $\sum_{\lambda \vdash n} \frac{\text{ch}_{q,t}((Rp_*\mathcal{A}'_c)|_{\mathbf{e}_\lambda})}{g_\lambda}$. In view of [KT22], setting $q \rightarrow 1$ these statistics are also Shalika germs in the corresponding cases.

A.1 $n = 2$

A.1.1 Catalan $\mathcal{A}'_{k+\frac{1}{2}}$

$$\begin{array}{c} \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} \\ \underbrace{\hspace{1.5cm}} \\ q^k \\ \hline 1 - \frac{t}{q} \end{array} + \begin{array}{c} \begin{array}{|c|} \hline 2 \\ \hline 1 \\ \hline \end{array} \\ \underbrace{\hspace{1.5cm}} \\ t^k \\ \hline 1 - \frac{q}{t} \end{array}$$

$$= \sum_{i=0}^k q^i t^{k-i}$$

A.1.2 Cuspidal $\mathcal{A}_{k+\frac{1}{2}}$

$$\begin{array}{c} \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} \\ \underbrace{\hspace{1.5cm}} \\ q^{k+1} \\ \hline 1 - \frac{t}{q} \end{array} + \begin{array}{c} \begin{array}{|c|} \hline 2 \\ \hline 1 \\ \hline \end{array} \\ \underbrace{\hspace{1.5cm}} \\ t^{k+1}(1-q)(1-qt) \\ \hline (1-t)(1-t^2)(1-\frac{q}{t}) \end{array} + \begin{array}{c} \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} \\ \underbrace{\hspace{1.5cm}} \\ t^k(1-qt^2) \\ \hline (1-t^2)(1-\frac{1}{t}) \end{array}$$

$$= q \left(\sum_{i=0}^k q^i t^{k-i} \right)$$

A.2 $n = 3$

A.2.1 Catalan $\mathcal{A}'_{\frac{2}{3}}$

$$\begin{aligned}
& \frac{\overbrace{\begin{array}{|c|c|} \hline 2 & \\ \hline 1 & 3 \\ \hline \end{array}}^q}{\left(1 - \frac{t}{q}\right) \left(1 - \frac{q^2}{t}\right)} + \frac{\overbrace{\begin{array}{|c|c|} \hline 3 & \\ \hline 1 & 2 \\ \hline \end{array}}^t}{\left(1 - \frac{q}{t}\right) \left(1 - \frac{t^2}{q}\right)} + \frac{\overbrace{\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline \end{array}}^q}{\left(1 - \frac{t}{q}\right) \left(1 - \frac{t}{q^2}\right)} + \frac{\overbrace{\begin{array}{|c|} \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array}}^t}{\left(1 - \frac{q}{t}\right) \left(1 - \frac{q}{t^2}\right)} \\
& = q + t
\end{aligned}$$

A.2.2 Cuspidal $\mathcal{A}_{\frac{2}{3}}$

$$\begin{aligned}
& \frac{\overbrace{\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline \end{array}}^{q^3}}{\left(1 - \frac{t}{q}\right) \left(1 - \frac{t}{q^2}\right)} \\
& + \frac{\overbrace{\begin{array}{|c|c|} \hline 3 & \\ \hline 1 & 2 \\ \hline \end{array}}^{q(1-q^2)t(1-qt)}}{\left(1-t\right) \left(1 - \frac{q^2}{t}\right) \left(1 - \frac{t}{q}\right) \left(1 - \frac{t^2}{q}\right)} + \frac{\overbrace{\begin{array}{|c|c|} \hline 1 & \\ \hline 2 & 3 \\ \hline \end{array}}^{qt(1-qt)^2}}{\left(1-t\right)^2 \left(1 - \frac{q}{t}\right) \left(1 - \frac{t^2}{q}\right)} + \frac{\overbrace{\begin{array}{|c|c|} \hline 2 & \\ \hline 1 & 3 \\ \hline \end{array}}^{q(1-qt^2)}}{\left(1 - \frac{1}{t}\right) (1-t) \left(1 - \frac{t^2}{q}\right)} \\
& + \frac{\overbrace{\begin{array}{|c|} \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array}}^{(1-q)^2 t^3 (1-qt)^2}}{\left(1-t\right)^2 \left(1-t^2\right)^2 \left(1 - \frac{q}{t^2}\right) \left(1 - \frac{q}{t}\right)} + \frac{\overbrace{\begin{array}{|c|} \hline 3 \\ \hline 1 \\ \hline 2 \\ \hline \end{array}}^{(1-q)t^3(1-qt)(1-qt^2)}}{\left(1 - \frac{1}{t}\right) (1-t) (1-t^2) (1-t^3) \left(1 - \frac{q}{t^2}\right)} \\
& + \frac{\overbrace{\begin{array}{|c|} \hline 2 \\ \hline 3 \\ \hline 1 \\ \hline \end{array}}^{(1-q)t^2(1-qt)(1-qt^2)}}{\left(1 - \frac{1}{t}\right) (1-t) (1-t^2) (1-t^3) \left(1 - \frac{q}{t^2}\right)} + \frac{\overbrace{\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}}^{t(1-qt^2)(1-qt^3)}}{\left(1 - \frac{1}{t^2}\right) \left(1 - \frac{1}{t}\right) (1-t^2) (1-t^3)} \\
& = q^2(q+t)
\end{aligned}$$

A.3 $n = 4$

A.3.1 Catalan $\mathcal{A}'_{\frac{3}{4}}$

$$\begin{aligned}
& \frac{\overbrace{\begin{array}{|c|c|c|} \hline 3 & & \\ \hline 1 & 2 & 4 \\ \hline \end{array}}^{qt(1-t)}}{\left(1 - \frac{q^2}{t}\right) \left(1 - \frac{t}{q}\right)^2 \left(1 - \frac{t^2}{q^2}\right)} + \frac{\overbrace{\begin{array}{|c|c|c|} \hline 2 & & \\ \hline 1 & 3 & 4 \\ \hline \end{array}}^{qt}}{\left(1 - \frac{q}{t}\right) \left(1 - \frac{t}{q}\right) \left(1 - \frac{t^2}{q^2}\right)} \\
& + \frac{\overbrace{\begin{array}{|c|c|} \hline 3 & 4 \\ \hline 1 & 2 \\ \hline \end{array}}^{(1-q)qt}}{\left(1 - \frac{q}{t}\right) \left(1 - \frac{q^2}{t}\right) \left(1 - \frac{t}{q}\right)^2} + \frac{\overbrace{\begin{array}{|c|c|c|} \hline 4 & & \\ \hline 1 & 2 & 3 \\ \hline \end{array}}^{q^3}}{\left(1 - \frac{t}{q^3}\right) \left(1 - \frac{t}{q^2}\right) \left(1 - \frac{t}{q}\right)} + \frac{\overbrace{\begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \end{array}}^{q^3}}{\left(1 - \frac{q^3}{t}\right) \left(1 - \frac{t}{q^2}\right) \left(1 - \frac{t}{q}\right)} \\
& + \frac{\overbrace{\begin{array}{|c|} \hline 4 \\ \hline 2 \\ \hline 1 & 3 \\ \hline \end{array}}^{qt(1-t)}}{\left(1 - \frac{q^2}{t^2}\right) \left(1 - \frac{q}{t}\right)^2 \left(1 - \frac{t^2}{q}\right)} + \frac{\overbrace{\begin{array}{|c|} \hline 4 \\ \hline 3 \\ \hline 1 & 2 \\ \hline \end{array}}^{qt}}{\left(1 - \frac{q^2}{t^2}\right) \left(1 - \frac{q}{t}\right) \left(1 - \frac{t}{q}\right)} \\
& + \frac{\overbrace{\begin{array}{|c|c|} \hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array}}^{qt(1-t)}}{\left(1 - \frac{q}{t}\right)^2 \left(1 - \frac{t}{q}\right) \left(1 - \frac{t^2}{q}\right)} + \frac{\overbrace{\begin{array}{|c|} \hline 3 \\ \hline 2 \\ \hline 1 & 4 \\ \hline \end{array}}^{t^3}}{\left(1 - \frac{q}{t^2}\right) \left(1 - \frac{q}{t}\right) \left(1 - \frac{q}{t^3}\right)} + \frac{\overbrace{\begin{array}{|c|} \hline 4 \\ \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array}}^{t^3}}{\left(1 - \frac{q}{t^3}\right) \left(1 - \frac{q}{t^2}\right) \left(1 - \frac{q}{t}\right)} \\
& = q^3 + q^2t + qt + qt^2 + t^3
\end{aligned}$$

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