§ Links + Braids

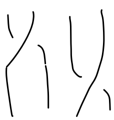
Def Braid group $Br_n = \langle \sigma_1, \ldots, \sigma_n : \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i-j| \geq 2, \ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \rangle$.

Think of σ_i : 

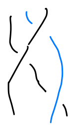
"up to isotopy"

Composition: "Vertically stack the braids"

$\sigma_1 \sigma_3 = \sigma_3 \sigma_1$

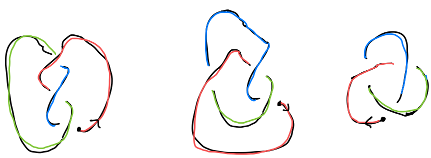
 

$\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2$

Given $\beta \in Br_n$, can form braid closure $\hat{\beta}$. It's a link.

Ex $\beta = \sigma_1^3 \in Br_2$.

$\hat{\beta} =$ 

Trefoil

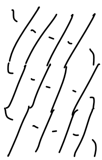
Ex $\beta = \sigma_1^5 \in Br_2$

$\hat{\beta} =$ Cinquefoil 

and similarly for σ_1^{2n+1}.

Defn. $T(m,n) = (m,n)$-torus link is the closure of

$(\sigma_1 \cdots \sigma_{m-1})^n \in Br_m$.

 $= (\sigma_1 \cdots \sigma_4)^3 \rightsquigarrow T(5,3)$

Prop.
- $T(m,n) \cong T(n,m)$
- # components $= \gcd(m,n)$
- Each component is a $T\left(\frac{m}{\gcd}, \frac{n}{\gcd}\right)$ torus knot.
- Linking number between any two components is constant.

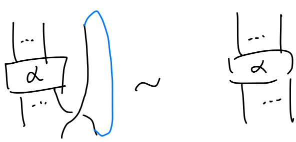
Thm. Any link is the closure of some braid.

Thm. $\hat{\beta}_1 = \hat{\beta}_2$ iff β_1 and β_2 are related by a sequence of the following moves:

$\alpha, \beta \in Br_n$: $\quad \alpha\beta \sim \beta\alpha$ "conjugation"

$\alpha\sigma_n \sim \alpha$ "positive stabilization"

in Br_{n+1} $\quad \alpha\sigma_n^{-1} \sim \alpha$ "negative stabilization"



§ KR Homology

Construction of KR homology of a link L:

- Write $L = \hat{\beta}$ for some $\beta = \sigma_{i_1}^{\varepsilon_1} \cdots \sigma_{i_\ell}^{\varepsilon_\ell}$ $\varepsilon_i = \pm 1$
- Define $HHH(\sigma_{i_1}^{\varepsilon_1}, \sigma_{i_2}^{\varepsilon_2}, \cdots, \sigma_{i_\ell}^{\varepsilon_\ell})$
- Show it depends only on L, ie. invariant under braid relations + Markov moves.

  ⚠ Negative stabilization $\rightsquigarrow$ grading shift.

$R = \mathbb{C}[x_1, \text{—}, x_n]$. graded: $\deg(x_i) = 2$.

$R^{s_i} = \mathbb{C}[x_1, \text{—}, x_{i-1}, x_i + x_{i+1}, x_i x_{i+1}, x_{i+2}, \text{—}, x_n]$.

$(R,R)\text{-bimod} \longleftrightarrow \mathbb{C}[x_1, -, x_n, x_1', -, x_n']\text{-mod}$

For $M = \bigoplus_i M_i$ graded (R,R)-bimod, $M(1)$ has $M(1)_i = M_{i+1}$.

Def. Category of Soergel bimods is smallest full subcat. of graded (R,R)-bimods containing R,

$$B_i := R \otimes_{R^{s_i}} R(1) = \frac{\mathbb{C}[x_1, \text{—}, x_n, x_1', \text{—}, x_n']}{\left\langle \begin{matrix} x_i + x_{i+1} = x_i' + x_{i+1}' \\ x_i x_{i+1} = x_i' x_{i+1}' \end{matrix}, \; x_j = x_j' \; (j \neq i, i+1) \right\rangle},$$

closed under $\oplus$, $\ominus$, $\otimes$, (1). Denoted $SBim_n$.

direct summand

Rmk. This category is additive but not abelian. Hence $D^b(SBim_n)$ is not well behaved.

Instead we will be in homotopy category $K^b(SBim_n)$.

(up to chain homotopy, rather than up to quasi iso.)

Lem. There are natural maps $B_i(-1) \longrightarrow R$ and $R \longrightarrow B_i(1)$

$1 \longmapsto 1$. $\qquad 1 \longmapsto x_i - x_{i+1}'$

Def. $T_i := [B_i(-1) \to \underline{R}]$ in $K^b(SBim_n)$.

$T_i^{-1} = [\underline{R} \to B_i(1)]$

Thm T_i satisfy braid relations up to chain homotopy.

$\rightsquigarrow$ Rouquier complex $T_\beta = T_{i_1}^{\varepsilon_1} \otimes \cdots \otimes T_{i_\ell}^{\varepsilon_\ell}$

Defn. i-th Hochschild cohomology of $M_\bullet \in K^b(SBim_n)$ is

$$HH^i(M_\bullet) = [\cdots \to \mathrm{Ext}^i_{(R,R)}(M_k) \xrightarrow{d_M} \mathrm{Ext}^i_{(R,R)}(M_{k+1}) \to \cdots] \in K^b(R\text{-mod})$$

graded

Defn Khovanov - Rozansky homology is

grading: Q.

$$HHH^{A,Q,T}(\beta) = H^T(HH^A(T_\beta))$$

$$= \frac{\ker(d_M : \mathrm{Ext}^A_{(R,R)}(M_T) \to \mathrm{Ext}^A_{(R,R)}(M_{T+1}))}{\mathrm{img}(d_M : \mathrm{Ext}^A_{(R,R)}(M_{T-1}) \to \mathrm{Ext}^A_{(R,R)}(M_T))}$$

Thm. HHH is invariant under conj. and pos. stab., and neg. stab. raises it up by one A-degree.

§ Links + Braids 2

Given $w = s_{i_1} \cdots s_{i_\ell} \in S_n$ reduced expression

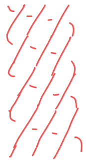
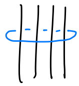
$\rightsquigarrow \beta(w) = \sigma_{i_1} \cdots \sigma_{i_\ell} \in Br_n$ "minimal braid lift of w"

[CGGS, pg 7] "Alg. weaves and braid varieties"

Lem $\beta(w)$ independent of choice of expression.

Defn • $HT_n \in Br_n$ "half twist" is $\beta(w_0)$ longest permutation.

• FT_n "full twist" is $FT_n = HT_n^2$

• $FT_n =$  Shorthand:  or 

• $Z(Br_n) = \langle FT_n \rangle$.

Ex $\beta = FT_2 \, FT_3$  $\rightsquigarrow \hat{\beta} =$ 

Not a torus link, but is a positroid link.

§ Recursions.

Hogancamp + friends found many recursions which compute $HHH(M_\bullet)$ for various $M_\bullet \in K^b(SBim_n)$. (None of them supsets any other, I think)

[HogMel] "Torus Link Homology": $HHH(T_\beta)$ $\forall$ torus links

[Hog] from abstract: Computes e.g. $HHH(T_{FT_{n-1}^a FT_n^b})$

These recursions involve $HHH(M_\bullet)$ for $M_\bullet \neq T_\beta$.