Last time: complex geometry $\longleftrightarrow$ $\mathbb{F}_q$ point counts

This time $pt_0 = \operatorname{Spec} \mathbb{F}_q$ $\quad pt = \operatorname{Spec} \overline{\mathbb{F}_q}$ $\quad \begin{array}{c} pt \\ \downarrow \\ pt_0 \end{array}$

$\pi_1^{ét}(pt_0) = \operatorname{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q) \ni \operatorname{Frob} : a \mapsto a^q$

(Analogy: $\operatorname{Spec} \mathbb{F}_q \leftrightarrow S^1$

$\operatorname{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q) \leftrightarrow \pi_1(S^1) \cong \mathbb{Z}$
$\overset{\shortparallel}{\hat{\mathbb{Z}}}$

$$\begin{array}{ccc} \operatorname{Spec} \overline{\mathbb{F}_q} & \longleftrightarrow & \mathbb{R} \\ \downarrow & & \downarrow \\ \operatorname{Spec} \mathbb{F}_q & \longleftrightarrow & S^1 \end{array}$$)

étale locsystems on $\operatorname{Spec} \mathbb{F}_q$ with coefficients $\overline{\mathbb{Q}}_\ell$ $\quad (\ell, q) = 1$

$\updownarrow$

a representation of $\pi_1^{ét}(\operatorname{Spec} \mathbb{F}_q) \circlearrowright M$ $\quad M$ is a $\overline{\mathbb{Q}}_\ell$ v.s.

the action is continuous.

$(\operatorname{locsys}(S^1) \leftrightarrow \pi_1(S^1)\text{-rep} \leftrightarrow \mathbb{Z}\text{-rep} \leftrightarrow F^n \circlearrowright M)$

$\Rightarrow$ get an extra "grading" from eigenvalues of Frobenius.

$$\begin{array}{ccc} X_0 & & X = X_0 \times_{\operatorname{Spec} \mathbb{F}_q} \operatorname{Spec} \overline{\mathbb{F}_q} \\ \downarrow & & \downarrow \\ \operatorname{Spec} \mathbb{F}_q & & \operatorname{Spec} \overline{\mathbb{F}_q} \end{array}$$

sheaves: complexes of étale sheaves.

on X: $D^b_{ét}(X, \overline{\mathbb{Q}}_\ell)$

on X_0: $D^b_{mix}(X_0, \overline{\mathbb{Q}}_\ell)$ $\qquad \underline{D_{mix}}(X_0, \overline{\mathbb{Q}}_\ell)$

$D^b_{mix}(X_0, \overline{\mathbb{Q}}_\ell) \to D^b_{ét}(X, \overline{\mathbb{Q}}_\ell)$

(to get rid of the grading)

- $\operatorname{Hom}(\mathcal{F}, \mathcal{G})$ are Homs compatible with Frob action (no Frob. actions here)

$\underline{\operatorname{Hom}}(\mathcal{F}, \mathcal{G})$ are all the Homs, remember Frob action

$$\operatorname{Hom}^*(\mathcal{F}, \mathcal{G}) = (\underline{\operatorname{Hom}}^*(\mathcal{F}, \mathcal{G}))^{\operatorname{Frob}}$$

- $0 \to \underline{\operatorname{Hom}}^{n-1}(\mathcal{F}_1, \mathcal{F}_2)_{\operatorname{Frob}} \to \operatorname{Hom}^n(\mathcal{F}_1, \mathcal{F}_2) \to \underline{\operatorname{Hom}}^n(\mathcal{F}_1, \mathcal{F}_2)^{\operatorname{Frob}} \to 0$

(If Frob is semi-simple, we can simplify)

- Frob $\curvearrowright M/_{\overline{\mathbb{Q}}_\ell}$ (v.s.) + constraints on the eigenvalues λ

$$|\lambda| = q^{\frac{1}{2}w} \qquad \overline{\mathbb{Q}} \subseteq \mathbb{C}, \quad \overline{\mathbb{Q}} \subseteq \overline{\mathbb{Q}}_\ell$$

w: weight $\boxed{w \in \mathbb{Z}}$

$$W_{\leq w}(M) = \bigoplus_{\substack{\lambda \\ |\lambda| = q^{\frac{1}{2}k} \\ k \leq w}} \text{generalized eigenspaces of } \lambda.$$

- $\boxed{\text{Ex}}$ $\mathbb{F}_q$ point counts: $\mathrm{Frob}_q \curvearrowright X$ Fixed points: $X(\mathbb{F}_q)$

Grothendieck $|X(\mathbb{F}_q)| = \sum_i (-1)^i \left(\mathrm{tr}\, H^i_{\acute{e}t,c}(X), \mathrm{Frob}\right).$

$\boxed{\text{Ex}}$ $\mathbb{P}^n$: $H^0(\mathbb{P}^n)$ $H^2(\mathbb{P}^n)$ . . . $H^{2n}(\mathbb{P}^n)$

Frob $\circlearrowright$ 1, Frob $\circlearrowright$ q, q^n

$|\mathbb{P}^n(\mathbb{F}_q)| = 1 + q + \cdots + q^n$ has a polynomial point count.

$\boxed{\text{Ex}}$ E = elliptic curves $H^0(E)$ $H^1(E)$ $H^{1,1}(E)$

1, dz $d\bar{z}$, $dz \wedge d\bar{z}$

$H^1 = H^{1,0} \oplus H^{0,1}$

Frob: 1 $\mathrm{Frob}^2 - a\,\mathrm{Frob} + q = 0$ q $\leftarrow$ wt 2

$\mathrm{tr} = 1 - a + q$

$= |E(\mathbb{F}_q)|$

wt 0

eigenvalues: $\dfrac{a \pm \sqrt{4q - a^2}\, i}{2}$

(Hasse: $a^2 \leq 4q$)

norms: $q^{\frac{1}{2}}$

wt 1

Achar–Riche If X_0 has an affine paving, and

$$X_0 = \coprod_{s \in S} X_s \qquad IC_{\overline{X_s}} = \text{middle extension of } \overline{\mathbb{Q}}_{\ell, X_s}[\ell(s)]\left(\tfrac{\ell(s)}{2}\right)$$

$$X_s \simeq \mathbb{A}^{\ell(s)}$$

$$\mathcal{H}^i(IC_{\overline{X_s}}|_{X_t}) = \begin{cases} 0 & \text{if } i \not\equiv \dim X_s \pmod 2 \\ \overline{\mathbb{Q}}_\ell(\frac{-i}{2})^{\oplus r} & \text{else} \end{cases}$$

$\boxed{\text{Ex}}$ $\mathbb{P}^1 = 0 \sqcup \mathbb{A}^1_\infty$ $\quad i: 0 \hookrightarrow \mathbb{P}^1$

$IC_{\mathbb{P}^1} = \overline{\mathbb{Q}}_{\ell,\mathbb{P}^1}[1](\frac{1}{2})$

$i^* IC_{\mathbb{P}^1} \cong \underline{\overline{\mathbb{Q}}_\ell}_0 [1](\frac{1}{2})$ $\quad \mathcal{H}^0(i^* IC_{\mathbb{P}^1}) = 0$

$\mathcal{H}^{-1}(i^* IC_{\mathbb{P}^1}) \cong \overline{\mathbb{Q}}_\ell(\frac{1}{2})$

This condition can be checked when $X = G/P$ $\quad G$: Kac Moody group

P: parabolic

$B \curvearrowright G/P$

affine paving by B orbits

can verify conditions above

by decomposition thm

$\boxed{\text{Thm}}$ If X_0 satisfies the conditions above,

then $\text{Frob} \circlearrowright \underline{\text{Hom}}^n(F, G)$ is semisimple.

$F, G \in \underline{D}^b_{mix}(X_0)$

<u>Hecke category</u>: $\quad {}_B\backslash G/B$ $\quad G$ reductive group

<u>Ginzburg</u>: $\quad IC_w \in D^b_{mix}({}_B\backslash G/B)$ $\quad IC_w$ supported on ${}_B\backslash\overline{BwB}/B \subseteq {}_B\backslash G/B$

$\downarrow$

$j_{w!*}\, \underline{\overline{\mathbb{Q}}_\ell}[\ell(w)](\frac{\ell(w)}{2})$ $\quad IC_w$ are pure of weight 0.

$\langle 1 \rangle = [1](\frac{1}{2})$ $\quad IC_w\langle n \rangle$ pure of wt 0.

X: projective variety: smooth $\quad H^i_{et}(X)$ has weight i

$\updownarrow$

$\overline{\mathbb{Q}}_{\ell X}$ is pure of weight 0

$\updownarrow$

favourite shift: $\quad \overline{\mathbb{Q}}_{\ell X}\langle n \rangle$ is pure of wt 0. for any n

$IC_X = \overline{\mathbb{Q}}_{\ell X}\langle \dim X \rangle$

Additive cats:

$$\{ IC_w\langle n\rangle \}_\oplus \subseteq D^b_{mix}(B\backslash G/B) \longrightarrow \{B_w(n)\}_\oplus = SBim$$

$$IC_w \longmapsto H^*(B\backslash G/B, IC_w).$$

linear over $H^*(B\backslash pt/B)$

$$\cap\!\!| \quad H^*_B(pt)\otimes H^*_B(pt) = R\otimes R$$

$$Ext^*(IC_{w_1}, IC_{w_2}) \xrightarrow{\cong} Hom(B_{w_1}, B_{w_2})$$

naive Hom between B_{w_1} and B_{w_2}

$$Hom(B_{w_1}, B_{w_2}) \cong \bigoplus_n grHom(B_{w_1}, B_{w_2}(n))$$

parity

$$\bigoplus_n Ext^0(IC_{w_1}, IC_{w_2}\langle n\rangle) \longrightarrow \bigoplus_n grHom(B_{w_1}, B_{w_2}(n))$$

bigraded, diagonally graded.

$\mathbb{Z}$-graded.

$R=\mathbb{C}[x_1\cdots x_n]$ $\boxed{|x_i|=2}$

$$H^*_B(pt) = H^*_T(pt)$$

$$H^*_{\mathbb{G}_m}(pt) \cong H^*(\mathbb{P}^\infty) = H^0\oplus H^2\oplus H^4\oplus\cdots$$

wt: $0 \quad 2 \quad 4 \quad \cdots$

$$= \overline{\mathbb{Q}}_\ell \oplus \overline{\mathbb{Q}}_\ell\langle 2\rangle \oplus \overline{\mathbb{Q}}_\ell\langle 4\rangle \oplus \cdots$$

$\boxed{Ex}$ $G=GL(2)$

① diagonal bimodule: $IC_e = IC_{B\backslash B/B} \longleftrightarrow {}_R R_R$

$$IC_{B\backslash B/B} \in D^b(B\backslash G/B)$$

$$H^*(IC_{B\backslash B/B}) \cong H^*(pt/B) \cong R$$

② $B_s = IC_{B\backslash GL(2)/B} = \overline{\mathbb{Q}}_{\ell,\mathbb{P}^1}[1](\tfrac{1}{2})$

$$H^*(B\backslash GL(2)/B) = H^*_B(GL(2)/B) \cong H^*_T(\mathbb{P}^1)$$

<u>Equiv. formality</u> : $G \curvearrowright X \rightsquigarrow H^*_G(X) = H^*(X \overset{G}{\times} EG)$

$$\begin{array}{ccc} X \hookrightarrow X \overset{G}{\times} EG & & \\ \downarrow & \xrightarrow[\text{sequence}]{\text{Serre spectral}} & E_2^{p,q} = H^p(X) \otimes H^q(BG) \Longrightarrow E_\infty = H^*_G(X) \\ BG & & \end{array}$$

If X is GKM, then the spectral sequence degenerate at the E_2 page

G/P $\Rightarrow H^*_G(X) \cong H^*(X) \otimes H^*(BG)$ as $H^*(BG)$-modules

<u>Equiv. localization</u> : $T \curvearrowright X$

$$H^*_T(X) \longrightarrow H^*_T(X^T) \cong H^*(X^T) \otimes H^*_T(pt)$$

ring homomorphism. ring isom.

• when $T \curvearrowright X$ is equivariantly formal, then <u>this is an injection</u> !

$\boxed{\text{Ex}}$ $T = (\mathbb{C}^*)^2 \curvearrowright \mathbb{P}^1$ $(\mathbb{P}^1)^T = \{0, \infty\}$

$$H^*_T(\mathbb{P}^1) \underset{i^*}{\hookrightarrow} H^*(0 \cup \infty) \otimes \underbrace{H^*_T(pt)}_{\substack{\| \\ R \\ \| \\ \mathbb{C}[x_1, x_2]}}$$

i^* also preserves the weight grading.

$|x_i| = (2,2)$

$$\| \\ R \oplus R$$

$$\begin{array}{rcl} 1 & \hookrightarrow & (1, 1) \\ x_1 & \hookrightarrow & (x_1, x_1) \\ x_2 & \hookrightarrow & (x_2, x_2) \\ y = c_1(\mathcal{O}(-1)) & \hookrightarrow & (x_1, x_2) \end{array}$$

$\mathcal{O}(-1)$: tautological line bundle

$\mathbb{C}[x_1, x_2, y] \big/ (y - x_1)(y - x_2) = 0$

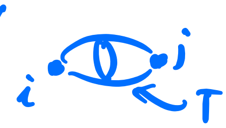
$|x_i| = |y| = (2,2)$

<u>GKM</u> : The image in $R^{\oplus k \text{ fixed pts}}$ $f_i \in R$

is $(f_1, \dots f_k)$ s.t. for all 1-dim'l orbits of T

$$f_i\big|_{\ker(T \to \mathbb{C}^*)} = f_j\big|_{\ker(T \to \mathbb{C}^*)}$$

$\mathbb{P}^1$ $\|$



$\boxed{T \to \mathbb{C}^*}$