Computation of $\chi_{a,q,t}$ CHHH ...; examples:

Easiest example is $T(1,0)$ ...

Recall: we have $B_{s_i} = R \otimes_{R^{s_i}} R(1) = R^e / \ldots$

$T_i = [B(-1) \xrightarrow{q} R]$

$T_{\underline{w}} = T_{s_{i_1}} \otimes \cdots \otimes T_{s_{i_k}}$ if $\underline{w} = s_{i_1} \cdots s_{i_k}$.

$R^e = R \otimes R^{op}$

$HHH^a(\underline{w}) = H^\bullet(\underbrace{\mathrm{Ext}^a_{R^e}(R, T_{\underline{w}})})$. ~~[Khova~~ [Khovanov]

Hochschild cohomology.

$R = \mathbb{C}[x]$, $R^e = \mathbb{C}[x, x']$. $W = \{id\}$. $R = R^e/(x-x')$

$HHH^a(T(1,0)) = \mathrm{Ext}^a_{R^e}(R, R)$

$R^e(-2) \xrightarrow{(x-x')} R^e \quad \xrightarrow{\mathrm{Hom}_{R^e}(-,R)}$

$T(2,0)$: $R^e \otimes \wedge^2 V^{(-4)} \longrightarrow R^e \otimes V^{(-2)} \longrightarrow R^e$ Koszul cpx

$(e_1 \wedge e_2) r \mapsto e_2(x-x')r - e_1(y-y')r.$

$e_1 r_1 + e_2 r_2 \mapsto (x-x')r_1 + (y-y')r_2$

$\mathrm{Hom}_{R^e}(-, R)$

$R(4) \xleftarrow{0} R \oplus R(-2) \xleftarrow{0} R$

$T(2,1)$

also unknot.

$$0 \to R \xrightarrow{(x-y') \hat{=} i} B(-1) \xrightarrow[\;]{\substack{(y-y') \\ \| \\ k=(x-x')}} B_{(-3)} \xrightarrow{q} R(-4) \to 0$$

$$\mathrm{Hom}_{R^e}\left(R^e/(x-x', y-y'),\, B\right) \overset{(x-y')}{=} \left(B/(x-x', y-y')\right) \cong R$$

$$R\mathrm{Hom}_{R^e}(R, B(-1)) = [B(-1) \xleftarrow{d_2'} V^* \otimes_k B(-1) \xleftarrow{d_1'} B(-1)].$$

$$d_1' : r \mapsto (x-x')e_1^* - (y-y')e_2^* = (x-x')(e_1^* + e_2^*)$$

$$d_2' : e_1^* r_1 + e_2^* r_2 \mapsto (y-y')r_1 + (x-x')r_2 = (x-x')(r_2 - r_1)$$

$$\Rightarrow \quad \mathrm{Hom}_{R^e}(R, B(-1)) = \ker d_1' = \ker k = \mathrm{Im}\, i = (x-y')R \cong R$$

$$\mathrm{Ext}^1_{R^e}(R, B(-1)) = \frac{\ker d_2'}{\mathrm{Im}\, d_1'} = \frac{\{(r_1, r_2) \mid r_2 - r_1 \in \ker k\}}{(x-x')(R \oplus R)} \,(2)$$

$$= \left(\frac{B}{\mathrm{Im}\, k} \oplus \underset{\substack{\| \\ \mathrm{Im}\, i}}{\ker k} \right)^{(2)} = (R \oplus R)(2)$$

$$\mathrm{Ext}^2_{R^e}(R, B(-1)) = \left(B/(x-x')B\right)(-4) = R(-4).$$

$$\mathrm{Hom}_{R^e}(R, R) \xleftarrow[x-y]{x-y'} \mathrm{Hom}_{R^e}(R, B(-1)) \quad \rightsquigarrow \quad HHH^0 = \mathbb{C}[x]$$

$$\mathrm{Ext}^1_{R^e}(R, R) \xleftarrow{id \oplus (x-y')} \mathrm{Ext}^1_{R^e}(R, B(-1)) \quad \rightsquigarrow \quad HHH^1 = \mathbb{C}[x](2)$$

$$\mathrm{Ext}^2_{R^e}(R, R) \xleftarrow{id} \mathrm{Ext}^2 \text{ --- } \qquad HHH^2 = 0$$

$K^0(SBim) \cong \mathcal{H}_W$

Hecke algebra

$\mathcal{H}_W = \mathbb{Z}[v^{\pm 1}] \langle T_s \mid s \in S \rangle$

- $(T_s - v^{-1})(T_s + v) = 0 \longrightarrow T_s^2 = (v^{-1} - v) T_s \cdot 1$, $\quad T_s^{-1} = T_s + v - v^{-1}$

$\underline{w} = s_{i_1} \cdots s_{i_k}$ reduced

- $\underbrace{T_s T_t T_s \cdots}_{\text{finite}} = \underbrace{T_t T_s T_t \cdots}_{\text{same \# as LHS}}$ $\qquad T_{\underline{w}} = \prod T_{s_{i_j}}$

$\iota : \mathcal{H}_W \to \mathcal{H}_W : v \mapsto v^{-1}, \ T_x \mapsto T_{x^{-1}}^{-1}$

$\iota(T_s + v) = T_s + v - v^{-1} + v^{-1} = T_s + v$

Define $\underline{\text{KL basis}} = \{C_x\}$ satisfying (labeled by W)

- $\iota(C_x) = C_x$
- $C_x = T_x + \sum_{y < x} h_{y,x} T_y \qquad h_{y,x} \in v\mathbb{Z}[v]$.

$y < x$: Bruhat order

Rmk: $v^{\ell(x) - \ell(y)} h_{x,y}$ K-L poly

$C_s = T_s + v$ ✓

Now back to $SBim \simeq QCoh(\mathfrak{h} \times \mathfrak{h})$

$\Delta_x := \{(xv, v) \mid v \in \mathfrak{h}\} \qquad x \in W$

$\Delta_A = \bigcup_{x \in A} \Delta_x \qquad$ write $R(A) := \mathbb{C}[\Delta_A]$

R

$\varepsilon : \mathcal{H}_W \longrightarrow K^0(SBim)$

- $\varepsilon\left(\sum_{w \le x} T_w v^{\ell(x) - \ell(w)}\right) \longmapsto R(\le x)(1)$

$\varepsilon(T_s + v) = R(\le s)(1) = [B_s]$ $\qquad \varepsilon(v) = R(1)$, $\quad \varepsilon(1) = [R]$

$$\mathcal{E}(T_s) = [B_s] - [R(1)]$$
$$= [(x-x')B_s] = [R_s(?)]$$

R-R bimodule structure is

R_s: twisted by γ_s on the right.

$\gamma_s: \begin{array}{c} R \to R \\ a \mapsto sa \end{array}$

$$\mathcal{E}(T_{\underline{w}}) = [R(\Delta_{\underline{w}} \,;\, ?)]$$

$$B_s \otimes_R B_s = B_i(1) \oplus B_i(-1) \quad \text{bc} \quad R = R^s \oplus R^s \alpha_s$$

$$C_s^2 = (T_s+v)(T_s+v) = T_s^2 + 2vT_s + v^2$$
$$= (v^{-1}-v)T_s + 1 + 2vT_s + v^2$$
$$= (v^{-1}+v)T_s + (v^{-1}+v)v = (v^{-1}+v)(T_s+v)$$
$$= (v^{-1}+v)C_s$$

Bott-Samelson bimodules:

$$B_{\underline{w}} = B_{s_{i_1}} \otimes_R \cdots \otimes_R B_{s_{i_k}}$$

Thm [Soergel; Elias; Williamson] $\mathcal{E}(C_{\underline{w}}) = B_{\underline{w}}$

Soergel $\downarrow$ geometric

Elias–Williamson $\downarrow$ Hodge theory of SBim

$\Rightarrow$ KL conj

Bott-Samelson varieties:

$$BS_{\underline{w}} = (P_{i_1} \times \cdots \times P_{i_k}) / B^k \quad \circlearrowleft T \text{ on the left}$$

$$(b_1, \dots, b_k)(p_1 \cdots p_k) = (p_1 b_1^{-1}, b_1 p_2 b_2^{-1}, \cdots, b_{k-1} p_k b_k^{-1})$$

$$BS_{\underline{w}} \xrightarrow{\text{resolution of singularity}} \overline{BwB}/B \subseteq G/B$$

~~Thm~~

$$H_T^\bullet(BS_{\underline{w}}) = B_{\underline{w}} = IH_T^\bullet(\overline{BwB/B})$$

$$H^\bullet(BS_{\underline{w}}) = B_{\underline{w}} \otimes_R \mathbb{C}$$

Example: sl_2: $BS_s = \mathbb{P}^1$ $\quad BS_{id} = pt$

$$H_T^\bullet(pt) = R = B_{id}$$

$$H_T^\bullet(\mathbb{P}^1) = R \otimes_{R^s} R = R \oplus R \otimes_{R^s} \alpha_s R^s = R \oplus \alpha_s R$$

$$\bullet \overset{\alpha_s}{\text{———}} \bullet$$

$(a,b) \in R \oplus R$

$\alpha_s \mid (a-b)$

$\langle (1,1), (0, \alpha_s) \rangle$

~~proof for the~~ proof is to use moment graph.

Example of) S_3 & $\underline{w} = s_1 s_2$

\# fixed pts $= 2^k$ if $\underline{w} = s_{i_1} \cdots s_{i_k}$

$(p_{i_1}, \dots, p_{i_k})$ $\quad p_{i_j} \in B$ or $s_{i_j} B$ $\quad$ bc $\quad t s_{j_1} s_{j_2} \cdots s_{j_k} = s_{j_1} \cdots s_{j_k} t'$ for some $t' \in T$

$$\begin{tikzcd}[column sep=small]
 & (s_1, s_2) \arrow[dl, dash, "\alpha_2"'] \arrow[dr, dash, "s_2\alpha_1 = (x_1 - x_3)"] & \\
(s_1, id) \arrow[dr, dash, "\alpha_{s_1}"'] & & (id, s_2) \arrow[dl, dash, "\alpha_{s_2}"] \\
 & (id, id) &
\end{tikzcd}$$

$s_1 = (1\ 2) \quad s_2 = (2\ 3)$

$\alpha_1 = x_1 - x_2 \quad \alpha_2 = x_2 - x_3$

Thm [GKM] $H_T^*(X) = \left\{ (a_x) \in \bigoplus_{x \in X^T} R_x \;\middle|\; x_E \text{ divides } \cdots a_x - a_y \text{ if } E \text{ connects } x, y \right\}$

$B_{\underline{s_1, s_2}} = R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} R \ni g \otimes f_1 \otimes f_2$

$$\downarrow$$

$$g\,(f_1 f_2,\ s_1(f_1) f_2,\ s_2(f_1 f_2),\ s_2(s_1(f_1) f_2))$$

since $R = R^s \oplus R^s \alpha_s$

$B_{\underline{w}}$ has a basis $\{ b_{\epsilon_1} \otimes \cdots \otimes b_{\epsilon_{i_k}} =: b_{\underline{\epsilon}} \}$

$b_{id} = 1 \otimes 1 \qquad b_s = \frac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s)$

In our example, $4\ b_{\underline{\epsilon}}$:

1) $1 \otimes 1 \otimes 1 \otimes 1 \longmapsto (1,1,1,1) \in H_T^*(X)$.

2) $\alpha_{s_1} \otimes 1 \otimes 1 \otimes 1 + 1 \otimes \alpha_{s_1} \otimes 1 \otimes 1 \longmapsto \alpha_1 (1,1,1) + (\alpha_1, s_1\alpha_1, s_2\alpha_s, s_2 s_1 \alpha_1)$

$\cdots$

$= \alpha_1 (1,1,1) + (\alpha_1, -\alpha_1, x_1 - x_3, x_3 - x_2)$

$= (\cancel{2\alpha_1}, 0, \quad 2x_1 - x_2 - x_3, x_3 - x_2)$.
$(2(x_1 - x_2))$

$\Uparrow$

$H_T^*(BS_{\underline{s_1 s_2}})$

3) $1 \otimes 1 \otimes \alpha_2 \otimes 1 + 1 \otimes 1 \otimes 1 \otimes \alpha_2$

$\cancel{\alpha_{s_1}\alpha_{s_2} \otimes 1 + 1 \otimes \alpha_{s_1}\alpha_{s_2} + \alpha_{s_1} \otimes \alpha_{s_2}}$

4) $(\alpha_{s_1} \otimes 1 + 1 \otimes \alpha_{s_1}) \otimes (\alpha_{s_2} \otimes 1 + 1 \otimes \alpha_{s_2})$