POSITROID VARIETIES

LIVIA XU

Following “Positroids, Knots, and q, t-Catalan Numbers” by Galashin and Lam, as well as “Postiroid Varieties: Juggling and Geometry” by Knutson, Lam, and Speyer.

Introduction: positroids as a finer stratification of Gr k, n than the Richardson varieties.

Contents

1. Notation and conventions

$$[n] = \{1, 2, \dots, n\}$$
$$\binom{[n]}{k} = \{S \subseteq [n] \mid |S| = k\}$$

Define a partial order on $\binom{[n]}{k}$ as follows: $I = \{i_1 < i_2 < \cdots < i_k\}$, $J = \{j_1 < j_2 < \cdots < j_k\}$. We write $I \leq J$ if $i_r \leq j_r$ for $r \in [k]$.

2. Schubert Cells, Richardson Varieties, and the Grassmannian

Fix $k, n \in \mathbb{N}$, $0 \leq k \leq n/$ For any $S \subseteq [n]$, let $\mathrm{Proj}_S \colon \mathbb{C}^n \to \mathbb{C}^k$ denote the projection onto the coordinates indexed by S. We use $e_1, \dots, e_n$ to denote the standard basis of $\mathbb{C}^n$.

Let $\mathrm{Fl}(n)$ denote the flag variety in $\mathbb{C}^n$. For $w \in S_n$, the *Schubert cell* $\mathring{X}_w$ and the *Schubert variety* X_w are

$$\mathring{X}_w = \{G_\bullet \in \mathrm{Fl}(n) \mid \dim(\mathrm{Proj}_{[j]}(G_i)) = |\{w([i]) \cap [j]\}| \text{ for all } i, j\},$$
$$X_w = \{G_\bullet \in \mathrm{Fl}(n) \mid \dim(\mathrm{Proj}_{[j]}(G_i)) \leq |\{w([i]) \cap [j]\}| \text{ for all } i, j\}.$$

Both have codimension $l(w)$ and they form a stratification of $\mathrm{Fl}(n)$: $X_w = \overline{\mathring{X}_w}$.

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Similarly, we can define the *opposite Schubert cell* $\mathring{X}^w$ and the *opposite Schubert variety* X^w to be

$$\mathring{X}^w = \{G_\bullet \in \mathrm{Fl}(n) \mid \dim(\mathrm{Proj}_{[n-j+1,n]}(G_i)) = |\{w([i]) \cap [n-j+1]\}| \text{ for all } i,j\},$$
$$X^w = \{G_\bullet \in \mathrm{Fl}(n) \mid \dim(\mathrm{Proj}_{[n-j+1,n]}(G_i)) \leq |\{w([i]) \cap [n-j+1]\}| \text{ for all } i,j\}.$$

Define the *Richardson varieties* as the transverse intersections of them:

$$X_u^w = X_u \cap X^w \text{ and } \mathring{X}_u^w = \mathring{X}_u \cap \mathring{X}^w.$$

The varieties X_u^w and $\mathring{X}_u^w$ are nonempty if and only if $v \leq w$. In this case each has codimension $l(w) - l(v)$. They also form a stratification of $\mathrm{Fl}(n)$.

If $E_\bullet$ denote the flag spanned by the standard basis, then $vE_\bullet$ is in X_u^w if and only if $u \leq v \leq w$.

Let $\mathrm{Gr}(k,n)$ denote the Grassmannian of k-planes in $\mathbb{C}^n$. We have the natural projection $\pi\colon \mathrm{Fl}(n) \to \mathrm{Gr}(k,n)$. We can give a partition of $\mathrm{Gr}(k,n)$ using Schubert cells labelled by $\binom{[n]}{k}$:

$$\mathring{X}_I = \{V \in \mathrm{Gr}(k,n) \mid \dim(\mathrm{Proj}_{[j]}(V)) = |\{I \cap [j]\}|\},$$
$$X_I = \{V \in \mathrm{Gr}(k,n) \mid \dim(\mathrm{Proj}_{[j]}(V)) = |\{I \cap [j]\}|\}.$$

Therefore,

$$\mathrm{Gr}(k,n) = \coprod_{I \in \binom{[n]}{k}} \mathring{X}_I \text{ and } X_J = \coprod_{I \geq J} \mathring{X}_I.$$

Similarly, define

$$\mathring{X}^I = \{V \in \mathrm{Gr}(k,n) \mid \dim(\mathrm{Proj}_{[n-j+1,n]}(V)) = |\{I \cap [n-j+1,n]\}|\},$$
$$X^I = \{V \in \mathrm{Gr}(k,n) \mid \dim(\mathrm{Proj}_{[n-j+1,n]}(V)) = |\{I \cap [n-j+1,n]\}|\}.$$

So for $J \in \binom{[n]}{k}$, the k-plane $\mathrm{Span}_{j \in J}\{e_j\}$ lies in X_I if and only if $I \leq J$, and lies in X^K if and only if $J \leq K$.

3. Affine Permutations and Juggling Patterns

Let $\widetilde{S}_n$ denote the set of bijections, called the *affine permutations*, $f\colon \mathbb{Z} \to \mathbb{Z}$ satisfying

$$f(i+n) = f(i) + n \text{ for all } i \in \mathbb{Z}.$$

Note that this group fits into an exact sequence

$$1 \longrightarrow \mathbb{Z}^n \xrightarrow{t} \widetilde{S}_n \longrightarrow S_n \longrightarrow 1,$$

where for $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n$, the corresponding element $t_\mu \in \widetilde{S}_n$ is given by $i \mapsto n\mu_i + i$ for $1 \leq i \leq n$. The squence splits since we can extend any $\sigma \in S_n$ to an affine permutation. So every $f \in \widetilde{S}_n$ can be uniquely factorized as $f = wt_\mu$, where $w \in S_n$ and $\mu \in \mathbb{Z}^n$. In this way,

$$f(i) = n\mu_{w^{-1}(i)} + i.$$

An affine permutation is *bounded* if for all $i \in \mathbb{Z}$, $i \leq f(i) \leq i + n$. In terms of the decomposition $f = wt_\mu$, this means that for $i \in [n]$, $0 \leq n\mu_i \leq n$, i.e., $\mu_i \in \{0,1\}$.

The *ball number* of an affine permutation f is the average

$$\mathrm{av}(f) = \frac{1}{n}\sum_{i=1}^{n}(f(i) - i).$$

This is always an integer. We use $\mathrm{Bound}(k,n)$ to denote the set of bounded affine permutation with ball number k. So if $f = wt_\mu \in \mathrm{Bound}(k,n)$, then exactly k elements of $\{\mu_i\}$ being 1.

For $f \in \mathrm{Bound}(k,n)$, define $\mathrm{st}(f,t) = \{f(i) - i \mid i \le t\}$. This is called the *juggling state of f at t*. We have that

$$\mathrm{st}(f,t) \cap [n] \in \binom{[n]}{k},$$

since we can replace t by $t \mod n$ and all we are looking at is

$$\{n\mu_{w^{-1}(i)} + i - t\}_{t-n<i\le t} \cap [n],$$

which is the same if we do not take the intersection (notice that $-n < i \le t - n$).

Jugglers treat these bounded affine permutations with ball number k as instructions on juggling k-balls. They say that

$$f(t) = \text{ the time that a ball thrown at time } t \text{ is next thrown.}$$

Then one can treat the number $f(t) - t \ge 0$ as how long the ball thrown at t is going to be in the air, i.e., the "height" of the throw at time t. If all balls are in the air at t, then there is nothing to be thrown and one should expect $f(t) = t$. One also interpret the juggling state

$$\mathrm{st}(f,t) \cap [n]$$

as where each ball is at time t.

A (k,n)-sequence of juggling states is a sequence

$$\mathcal{J} = (J_1, J_2, \dots, J_n) \in \binom{[n]}{k}^n$$

such that for each $i \in [n]$, we have that $J_{i+1} \supseteq (J_i \backslash \{1\}) - 1$, where -1 means to subtract 1 from each element and the indicies are cyclic modulo n. The collection of all such sequences is denoted $\mathrm{Jugg}(k,n)$.

Given $f \in \mathrm{Bound}(k,n)$, we can associate to it a juggling sequence

$$\mathcal{J}(f) = (\mathrm{st}(f,0) \cap [n], \mathrm{st}(f,1) \cap [n], \dots, \mathrm{st}(f,n) \cap [n]).$$

This is a bijection by realizing that both give the same *cyclic rank matrix*.

4. Cyclic Rank Matrices

Let $V \in \mathrm{Gr}(k,n)$. We define an infinite array $r_{\bullet\bullet}(V) = (r_{ij}(V))_{i,j\in\mathbb{Z}}$ as follows: For $i > j$, we set $r_{ij}(V) = j - i + 1$. For $i \le j$, we set

$$r_{ij}(V) = \dim(\mathrm{Proj}_{\{i,i+1,\dots,j\}}(V)),$$

where the indices are cyclic modulo n. Note that when $j \ge i + n - 1$, we project onto all of $[n]$. If V is the row span of a $k \times n$ matrix M, then r_{ij} is the rank of the submatrix of M consisting of columns $i, i+1, \dots, j$.

This matrix is a so-called *cyclic matrix of type* (k,n). These are $\infty \times \infty$ matrices $(r_{ij})_{i,j\in\mathbb{Z}}$ satisfying the following properties:

(C1') $r_{ij} = j - i + 1$ if $i > j$.

(C2') $r_{ij} = k$ if $j \ge i + n - 1$.

(C3) $r_{ij} - r_{(i+1)j} \in \{0,1\}$ and $r_{ij} - r_{(i-1)j} \in \{0,1\}$ for all i, j.

(C4) If $r_{(i+1)(j-1)} = r_{(i+1)j} = r_{i(j-1)}$, then $r_{ij} = r_{(i+1)(j-1)}$.

(C5) $r_{(i+n)(j+n)} = r_{ij}$.

Why they are cyclic rank matrices. Conditions (C1'), (C2') and (C5) are clear from the definitions. Let $M \in M_{k\times n}$ be such that its row span is V. (C3) says that adding a column to a matrix either preserves the rank of the matrix or increase it by one. Draw out the columns on the board, from i to j. The hypothesis condition of (C4) says that if $M_i = i$th column of M and M_j are in the span of $M_{i+1}, M_{i+2}, \ldots, M_{j-1}$, then

$$\dim \mathrm{Span}(M_i, M_{i+1}, \ldots, M_j) = \dim \mathrm{Span}(M_{i+1}, \ldots, M_{j-1}).$$

$\square$

5. Positroid Varieties

For any cyclic rank matrix r of type (k, n), let

$$\mathring{\Pi}_r = \{V \in \mathrm{Gr}(k, n) \mid r_{\bullet\bullet}(V) = r\}.$$

The following sets are all isomorphic as posets:

(1) $\mathrm{Bound}(k, n)$, Bounded affine permutations of type (k, n),

(2) $\mathcal{Q}(k, n)$, Equivalence classes of k-Bruhat intervals in S_n,

(3) $\mathrm{Jugg}(k, n)$, (k, n)-sequences of juggling states,

(4) Cyclic rank matrices of type (k, n),

(5) Positroids of rank k on $[n]$ (as a matroid).

They all index the *open positroid varieties* and the *positroid varieties*.

So under these correspondence, we can write $\mathring{\Pi}_f, \mathring{\Pi}_{\mathcal{J}}, \mathring{\Pi}_u^w$ where f is the bounded affine permutation, $\mathcal{J}$ the juggling pattern, or $\langle u, w\rangle$ the equivalence class of k-Bruhat interval corresponding to r.

If $u, w \in S_n$ and $u \le_k w$, then we can define $f_{u,w} \in \mathrm{Bound}(k, n)$ by

$$f_{u,w} = u t_{\omega_k} w^{-1},$$

where $\omega_k = (1, \ldots, 1, 0, \ldots, 0)$ has k 1s. This does not depend on the choice of representative of the equivalent class. A more explicit description is

$$f_{u,w}(i) = \begin{cases} u(w^{-1}(i)) & \text{if } w^{-1}(i) > k; \\ u(w^{-1}(i)) + n & \text{if } w^{-1}(i) \le k. \end{cases}$$

Proposition 5.1. *For $u, w \in S_n$, let $u \le_k w$ and $f_{u,w}$ be the corresponding affine permutation. We have $\mathring{\Pi}_f = \pi(\mathring{X}_u^w)$.*

Let $\chi = (12 \ldots n) \in S_n$.

Lemma 5.2. *For any $\mathcal{J} = (J_1, \ldots, J_n) \in \mathrm{Jugg}(k, n)$, we have*

$$\mathring{\Pi}_{\mathcal{J}} = \mathring{X}_{J_1} \cap \chi(\mathring{X}_{J_2}) \cap \cdots \cap \chi^{n-1}(\mathring{X}_{J_n}).$$

This can be done by unwrapping all these correspondences.

Therefore, we get

$$\mathrm{Gr}(k,n) = \coprod_{\mathcal{J}\in\mathrm{Jugg}(k,n)} \mathring{\Pi}_{\mathcal{J}}.$$

The sets $\mathring{\Pi}_{\mathcal{J}}$ are called the *open positroid varieties* and their closure $\Pi_{\mathcal{J}} = \overline{\mathring{\Pi}_{\mathcal{J}}}$ the *positroid varieties.*

It turns out that this decomposition of the Grassmannian coincide with a stratification given by Lusztig for partial flag varieties and we have a bunch of nice properties of these (open) positroid varieties.

Theorem 5.3. *Let $f = f_{u,v} \in \mathrm{Bound}(k,n)$. We already have $\mathring{\Pi}_f = \pi(\mathring{X}_u^w)$.*

(1) The varieties Π_f and $\mathring{\Pi}_f$ are irreducible of codimension $l(f)$, and $\mathring{\Pi}_f$ is smooth.

(2) For any Richardson variety X_u^w, whether or not $u \leq_k w$, the projection $\pi(X_u^w)$ is a closed positroid variety.

(3) Open positroid varieties form a stratification of the Grassmannian. So we have

$$\Pi_f = \coprod_{f'\geq f} \mathring{\Pi}_{f'} = X_{J_1} \cap \chi(X_{J_2}) \cap \cdots \cap \chi^{n-1}(X_{J_n}),$$

where $(J_1,\ldots,J_n) \in \mathrm{Jugg}(k,n)$ corresponds to f.

6. The Link Associated to an Open Positroid Variety

If $u \in S_n$, set $\beta(u)$ to be the corresponding braid, obtained by choosing a reduced word $u = s_{i_1}s_{i_2}\cdots s_{i_{l(u)}}$ and replacing each s_i with σ_i.

For $f = f_{v,w} \in \mathrm{Bound}(k,n)$, we define the *link associated to* f by

$$\beta_f \stackrel{\mathrm{def}}{=} \beta(w)\cdot\beta(v)^{-1},$$

and consider the closure $\hat{\beta}_f$.

By studying the torus action on $\mathring{\Pi}_f$ by $(\mathbb{C}^\times)^{n-1}$, one can show that $\hat{\beta}_f$ is a knot if and only if $\overline{f} \in S_n$, the reduction of $f \mod n$ (possible since f is a bounded affine permutation), contains only one cycle. In this case, the torus action is free.

7. Examples

The open positroid the paper is interested in is the top dimensional cell in $Gr(k,n)$ and is denoted as $\mathring{\Pi}_{k,n}$. The corresponding affine permutation is $f_{k,n}$. This top cell can be described using the Plüker coordinates:

$$\mathring{\Pi}_{k,n} = \{V \in \mathrm{Gr}\,k,n \mid \Delta_{1,2,\ldots,k}(V), \Delta_{2,3,\ldots,k+1}(V), \ldots, \Delta_{n,1,\ldots,k-1} \neq 0\}.$$

Example 7.1. The simplest example.

$n = 1, k = 1, w = v = (1)$. We have $f(i) = i + 1$. The corresponding juggling sequence is $(\{1\})$–throw a ball, catch it at the next unit, and throw it again. The corresponding braid is Id and the closure is the unknot.

In this case, $\mathrm{Gr}(1,1) = \{\star\}$ and nothing interesting is happening.

The hard part is to find this k.

Example 7.2. A more comlicated unknot. Look at $\mathrm{Fl}(2)$ and $\mathrm{Gr}(1,2)$ (since $\mathrm{Gr}(0,2)$ and $\mathrm{Gr}(2,2)$ are points). The interesting one is $n = 2, k = 1, v = (1), w = (12)$. The corresponding bounded affine permutation is $f(1) = 1$, $f(2) = 2 + 2 = 4$. To juggle this pattern, we perform a 2-throw on the ball at every even unit of time. The corresponding link is the unknot that looks like ∞.

If $k = 1, v = w = (1)$, then the bounded affine permutation is $f(1) = 2 + 1 = 3$, $f(2) = 2$. This looks like: 1-throw, catch the ball and wait, 1-throw. The corresponding link is two disjoint circles.

If $k = 1, v = w = (12)$, then the bounded affine permutation is the same as the previous case but the corresponding link is the unlink (one circle layed on the other).

The Richardson varieties in $\mathrm{Fl}(2) \cong \mathbb{P}^1$ are:

$$\mathrm{Fl}(2) = \mathring{X}_e^w \bigsqcup \mathring{X}_e^e \bigsqcup \mathring{X}_w^w,$$

where $\mathring{X}_e^w$ has dimension 1 and the other two are just $\{\star\}$.

To see this, we compute the Schubert cells:

$$\mathring{X}_e = \{v \mid \mathbb{C}v \text{ projected to } \mathbb{C}e_1 \text{ has dimension } 1\}$$
$$\mathring{X}_w = \{e_2\} = \{v \mid \mathbb{C}v \text{ projected to } \mathbb{C}e_1 \text{ has dimension } 0\}.$$

The first equality comes from $i = j = 1$, since in all other cases, we are either projecting a line to a plane or a plane to a line. The second also comes from $i = j = 1$. The opposite Schubert cells look like

$$\mathring{X}^w = \{v \mid \mathbb{C}v \text{ projected to } \mathbb{C}e_2 \text{ has dimension } 1\}$$
$$\mathring{X}^e = \{e_1\} = \{v \mid \mathbb{C}v \text{ projected to } \mathbb{C}e_2 \text{ has dimension } 0\}.$$

Therefore, we have

$$\mathring{X}_e^w = ((\mathbb{C}^2 \backslash \{0\})/\mathbb{C}^\times) \backslash \{e_1, e_2\}, \qquad \mathring{X}_e^e = \{e_1\}, \qquad \mathring{X}_w^w = \{e_2\}.$$

Example 7.3. The Trefoil knot. $n = 5$, $v = (1)$, $w = s_3 s_2 s_1 s_4 s_3 s_2 = (14253)$, $w^{-1} = (13524)$. This is $\mathring{\Pi}_{2,5}$. The corresponding affine permutation is:

$$1 \mapsto 3, \qquad 2 \mapsto 4, \qquad 3 \mapsto 5, \qquad 4 \mapsto 6, \qquad 5 \mapsto 7.$$

This is a boring juggling pattern, which is nothing but: 2-throw, 2-throw, 2-throw, etc.

draw picture