Introduction to Optimization: Homework #4

Due on December 12, 2020

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Assignment A4.1

Implement the globalized Newton method (Lecture L-08, slide 26) with backtracking that was presented in the lecture as a function newton_glob in MATLAB or Python.
The following input functions and parameters should be considered:

- obj, grad, hess - function handles that calculate and return the objective function $f(x)$, the gradient $\nabla f(x)$, and the Hessian $\nabla^2 f(x)$ at an input vector $x \in \mathbb{R}^n$. You can treat these handles as functions or fields of a class or structure f or use them directly as input. (For example, your function can have the form newton_glob(obj, grad,hess, ...)).
- x^0- the initial point.
- tol - a tolerance parameter. The method should stop whenever the current iterate x^k satisfies the criterion $\left\|\nabla f\left(x^k\right)\right\| \leq$ tol.
- $\beta_1, \beta_2 > 0-$ parameters for the Newton condition.
- $s > 0, \sigma, \gamma \in (0,1)-$ parameters for backtracking and the Armijo condition.

You can again organize the latter parameters in an appropriate options class or structure. You can use the backslash operator $A\backslash b$ in MATLAB or numpy . linalg.solve (A, b) to solve the linear system of equations $Ax = b$. If the computed Newton step $d^k = -\nabla^2 f\left(x^k\right)^{-1} \nabla f\left(x^k\right)$ is a descent direction and satisfies

$$-\nabla f\left(x^k\right)^\top d^k \geq \beta_1 \min\left\{1, \left\|d^k\right\|^{\beta_2}\right\} \left\|d^k\right\|^2$$

we accept it as next direction. Otherwise, the gradient direction $d^k = -\nabla f\left(x^k\right)$ is chosen. The method should return the final iterate x^k that satisfies the stopping criterion.

a) Test your implementation on the following problem:

$$\min_{x \in \mathbb{R}^2} f(x) = f_1(x)^2 + f_2(x)^2$$

where $f_1 : \mathbb{R}^2 \to \mathbb{R}$ and $f_2 : \mathbb{R}^2 \to \mathbb{R}$ are given by:

$$f_1(x) := 3 + x_1 + ((1 - x_2)\, x_2 - 2)\, x_2$$
$$f_2(x) := 3 + x_1 + (x_2 - 3)\, x_2$$

(This problem was discussed in Assignment A3.3).

– Generate a plot of the solution paths of Newton's method for a variety of initial points similar to Assignment A3.3c. Let us again define the set

$$\mathcal{X}^0 := \left\{x \in \mathbb{R}^2 : x_1 \in \{-10, 10\}, x_2 \in [-2, 2]\right\} \cup \left\{x \in \mathbb{R}^2 : x_1 \in [-10, 10], x_2 \in \{-2, 2\}\right\}.$$

Run the globalized Newton method with parameters $(s, \sigma, \gamma) = (1, 0.5, 0.1)$ and $(\beta_1, \beta_2) = \left(10^{-6}, 0.1\right)$ to solve the problem $\min_x f(x)$ with p different initial points selected from the set $\mathcal{X}^0$.

The initial points should uniformly cover the different parts of the set $\mathcal{X}^0$ and you can use the tolerance tol $= 10^{-8}$ and $p \in [10, 20] \cap \mathbb{N}$. Create a single figure that contains all of the solution paths generated for the different initial points. The initial points and limit points should be clearly visible. Add a contour plot of the function f in the background of your figure.

Report the behavior and performance of the Newton method and compare it to the convergence of the gradient method tested in A3.3. In particular, discuss the number of required iterations (on average). Which type of convergence can typically be observed?

Assignment A4.1 continued on next page…

b) Test your approach on the Rosenbrock function $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x) = 100\left(x_2 - x_1^2\right)^2 + (1 - x_1)^2$$

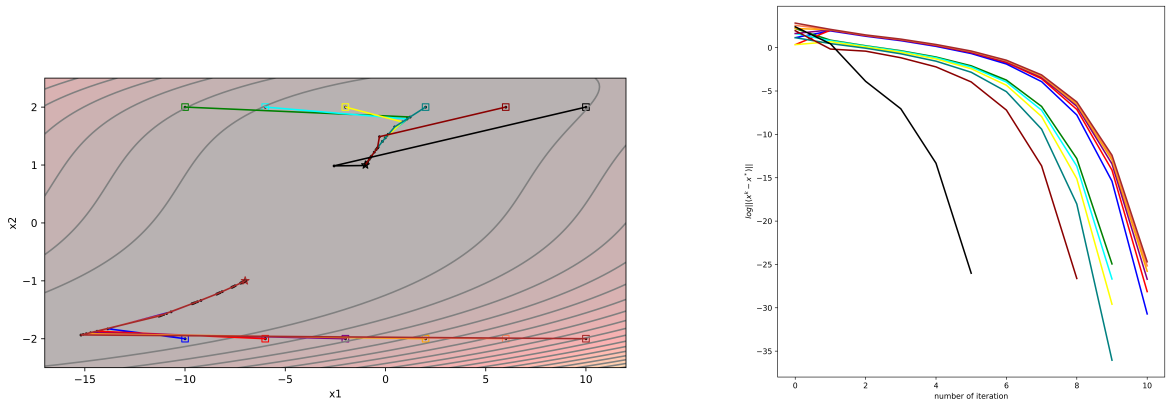
with initial point $x^0 = (-1.2, 1)^\top$ and parameter $(s, \sigma, \gamma) = \left(1, 0.5, 10^{-4}\right)$ and $(\beta_1, \beta_2) = \left(10^{-6}, 0.1\right)$.($\gamma$ is smaller here). Besides the globalized Newton method also run the gradient method with backtracking $\left((s, \sigma, \gamma) = \left(1, 0.5, 10^{-4}\right)\right)$ on this problem and compare the performance of the two approaches for different tolerances tol $\in \left\{10^{-1}, 10^{-3}, 10^{-5}\right\}$.

Does the Newton method always utilize the Newton direction? Which type of convergence can be observed? Does the method always use full step sizes $\alpha_k = 1$? In contrast, what is the typical behavior of the gradient method?

Solution

Subproblem (a)

We choose p =12 different initial points as (-10, ±2), (-6, ±2), (-2, ±2), (2, ±2), (6, ±2), (10, ±2). The figures which contains the paths is showing as Figure 1(a). The figure of $\left(\log \left\|x^k - x^*\right\|\right)_k$ can be seen in Figure 1(b). As we can see from the figure, the convergence type is quadratic convergence.



(a) Paths of Newton method with different initial points

(b) $\log \|x^k - x^*\|$ VS number of iteration

Figure 1 The plot of iteration process

By comparing the performance of the Newton method and gradient method tested in A3.3, we can derive Table 1. From the table, we can see that the Newton method is much better than the gradient method. The Newton method needs smallest iterations and time.

Table 1 Performance of different methods

Method	tolerance	iteration	time(s)
GM with backtracking	10^{-8}	2793	1.613
GM with diminishing	10^{-8}	85266	8.516
GM with exact line search	10^{-8}	695	1.007
Newton method	10^{-8}	9	0.035

The python code to solve this problem is showing below.

```
import numpy as np
import matplotlib.pyplot as plt
import time

def f1(x):
    x = x.reshape(x.size)
    return 3 + x[0] + ((1 - x[1]) * x[1] - 2) * x[1]
def f2(x):
    x = x.reshape(x.size)
    return 3 + x[0] + (x[1] - 3) * x[1]
def f(x):
    x = x.reshape(x.size)
    return f1(x) * f1(x) + f2(x) *f2(x)
def df(x):
    x = x.reshape(x.size)
    grad = np.zeros(2).reshape(2,1)
    grad[0] = 2 * f1(x) + 2 * f2(x)
    grad[1] = 2 * f1(x) * (2*x[1] - 3*(x[1]**2) - 2) + 2 * f2(x) * (2*x[1] - 3)
    return grad
def Hessian(x):
    x = x.reshape(x.size)
    hessian = np.zeros((2,2))
    hessian[0][0] = 4
    hessian[0][1] = 8*x[1]-6*(x[1]**2)-10
    hessian[1][0] = 8*x[1]-6*(x[1]**2)-10
    hessian[1][1] = 2*f1(x)*(-6*x[1]+2) + 2*(2*x[1]-3*(x[1]**2)-2)**2+4*f2(x)+2*(2*x[1]-3)**2
    return hessian
def norm(x):
    x = x.reshape(x.size)
    return np.sqrt(x[0]**2 + x[1]**2)

color_list = ['blue', 'green', 'red', 'cyan', 'purple', 'yellow', 'orange', 'teal',
              'coral', 'darkred', 'brown', 'black']
Number_iterations = []

def plot_contour():
    X = np.arange(-17.5,12.5,0.05)
    Y = np.arange(-3, 3, 0.05)
    X,Y = np.meshgrid(X,Y)
    Z = np.zeros((X.shape[0], X.shape[1]))
    for i in range(X.shape[0]):
        for j in range(X.shape[1]):
            x = []
            x.append(X[i][j])
            x.append(Y[i][j])
            x = np.array(x)
            Z[i][j] = f(x)
    plt.contourf(X, Y, Z, 30, alpha=0.3, cmap=plt.cm.hot)
    plt.contour(X, Y, Z, 30, colors='grey')

def plot_line(xk_list, subfig_num):
    plt.figure(1)
    x = []
    y = []
    for i in range(xk_list.shape[0]):
        x.append(xk_list[i][0][0])
        y.append(xk_list[i][1][0])
    plt.plot(x,y, color = color_list[subfig_num-1], linewidth=1.5)
    plt.scatter(x, y, s=3, color='black')

def plot_convergence(y, subfig_num):
```

Assignment A4.1 continued on next page...

```
    plt.figure(2, figsize=(8,10))
    n = y.size
    x = np.arange(n)
    y = np.log(y)
    plt.plot(x, y, color = color_list[subfig_num-1], linewidth=2)
    plt.xlabel('number of iteration')
    plt.ylabel('$log||(x^k-x^*)||$')
    plt.tight_layout()

def check_dir(dk_tocheck, xk, gradient, hessian, beta1, beta2):
    dk_norm = norm(dk_tocheck)
    factor1 = np.dot(gradient.T, dk_tocheck)[0][0] < 0
    facotr2 = -(np.dot(gradient.T, dk_tocheck)[0][0]) >= beta1 * np.min([1, dk_norm**beta2]) *
                                                        dk_norm**2
    if factor1 == True and facotr2 == True:
        return True
    else:
        return False
def Global_Newton(initial, subfig_num):
    #paramaters
    s = 1
    sigma = 0.5
    gamma = 0.1
    beta1 = 1e-6
    beta2 = 0.1
    tol = 1e-8
    xk_list = []
    xk_xstar_list = []


    xk = initial
    num_iteration = 0
    xk_list.append(xk)


    gradient = df(xk)
    while norm(gradient) > tol:
        # deteriminate the direction
        hessian = Hessian(xk)
        dk_tocheck = np.linalg.solve(hessian, -gradient)
        good_dir = check_dir(dk_tocheck, xk, gradient, hessian, beta1, beta2)
        if(good_dir == False):
            dk = -gradient
        else:
            dk = dk_tocheck

        alphak = s
        while True:
            if f(xk + alphak*dk) - f(xk) <= gamma * alphak * (np.dot(gradient.T, dk)[0][0]):
                break
            alphak = alphak * sigma
        xk = xk + alphak * dk
        xk_list.append(xk)

        gradient = df(xk)
        num_iteration = num_iteration + 1

    Number_iterations.append(num_iteration)

    plt.figure(1)
    plt.scatter(xk_list[-1][0], xk_list[-1][1], s=60, marker='*',
                facecolors ='none', edgecolor= color_list[subfig_num-1])
```

```
    xk_list = np.array(xk_list)
    plot_line(xk_list, subfig_num)

    xstar_x1 = xk_list[-1][0][0]
    xstar_x2 = xk_list[-1][1][0]
    xstar = np.array([round(xstar_x1), xstar_x2]).reshape(2, 1)
    for i in range(xk_list.shape[0]):
        xk_xstar_list.append(norm(xk_list[i] - xstar))
    xk_xstar_list = np.array(xk_xstar_list)
    plot_convergence(xk_xstar_list, subfig_num)
# main begin

x1 = np.arange(-10, 11, 4)
x2 = np.arange(-2, 3, 4)

plt.figure(1, figsize=(10, 5))
plot_contour()
subfig_num = 1

time_list = []
for i in range(6):
    for j in range(2):
        initial = np.zeros(2).reshape(2,1)
        initial[0][0] = x1[i]
        initial[1][0] = x2[j]
        plt.figure(1)
        plt.scatter(initial[0], initial[1], s=40, marker='s',
                    facecolors ='none', edgecolor= color_list[subfig_num-1])

        start = time.clock()
        Global_Newton(initial, subfig_num)
        end = time.clock()

        time_list.append(end - start)
        subfig_num = subfig_num + 1

plt.figure(1)
plt.xlabel('x1')
plt.ylabel('x2')
plt.xlim(-17, 12)
plt.ylim(-2.5, 2.5)
plt.savefig('A4_1_a', dpi=700)

plt.figure(2)
plt.savefig('A4_1_a_convergence', dpi=700)

print()
print('Number of iterations from different initial points:', Number_iterations)
print('Average number of iterations from different initial points:',
      sum(Number_iterations)/len(Number_iterations))

print('Calculating time from different initial points:', time_list)
print('Average calculating time from different initial points:',
      sum(time_list)/len(time_list))
```

Subproblem (b)

By comparing the performance of the Newton method and gradient method with backtracking, we can derive Table 2. From the table, we can see that the Newton method is much better than the gradient method with backtracking. The Newton method always utilize the Newton direction. Newton method and gradient method both do not always use full step sizes $\alpha_k = 1$. The figures which contains the paths is showing as

Figure 2. The figure of $\left(\log \left\|x^k - x^*\right\|\right)_k$ can be seen in Figure 3(a). By zooming in Figure 3(a), we get Figure 3(b). As we can see from the figure, the gradient method is linear convergence, and the Newton method is quadratic convergence.

Table 2 Performance of different methods on Rosenbrock function

Method	tolerance	iteration	time(s)	always use Newton direction	α_k always $= 1$
GM with backtracking	10^{-1}	54	0.050		False
	10^{-3}	5231	1.108		False
	10^{-5}	10916	2.303		False
Newton Method	10^{-1}	19	0.166	True	False
	10^{-3}	20	0.075	True	False
	10^{-5}	21	0.080	True	False

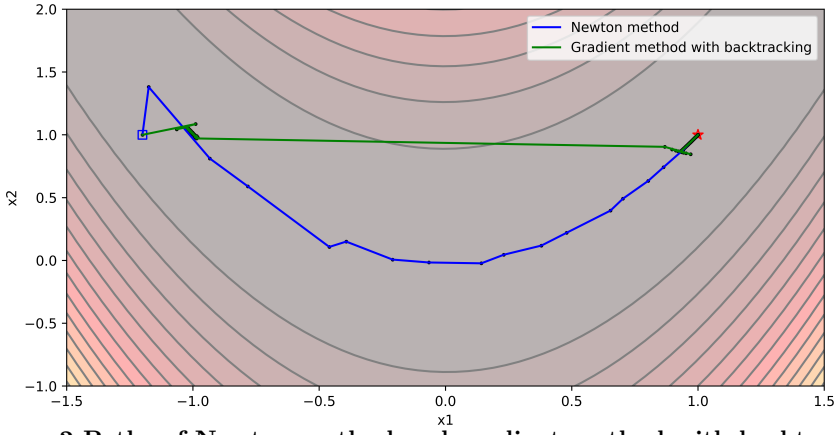
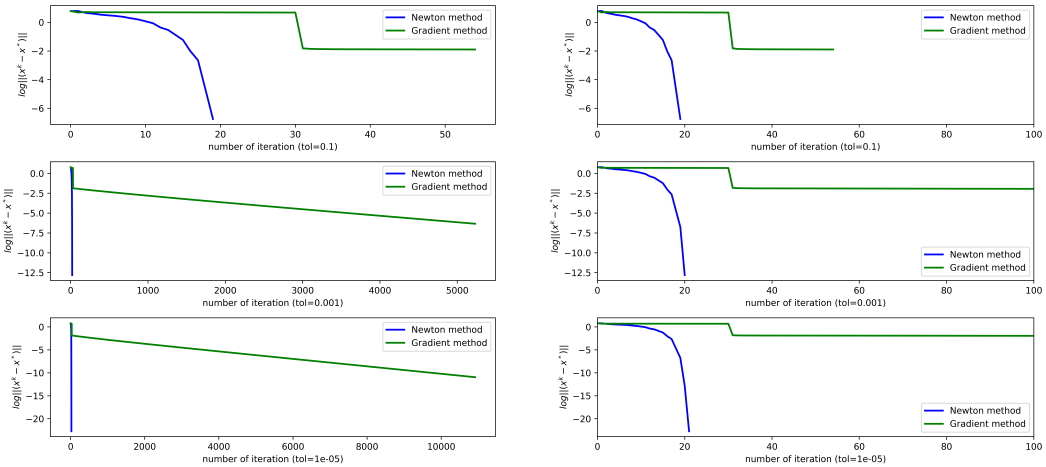


Figure 2 Paths of Newton method and gradient mothod with backtracking



(a) $\log \|x^k - x^*\|$ VS number of iteration (b) $\log \|x^k - x^*\|$ VS number of iteration (zoom in)

Figure 3 The plot of iteration process

The python code to solve this problem is showing below.

```
import numpy as np
import matplotlib.pyplot as plt
import time

def f(x): # Rosenbrock function
    x = x.reshape(x.size)
    return 100*(x[1] - x[0]**2)**2 + (1-x[0])**2
def df(x):
    x = x.reshape(x.size)
    grad = np.zeros(2).reshape(2,1)
    grad[0] = -400 * x[0] * (x[1] - x[0]**2) + 2 * x[0] -2
    grad[1] = 200 * (x[1] - x[0]**2)
    return grad
def Hessian(x):
    x = x.reshape(x.size)
    hessian = np.zeros((2,2))
    hessian[0][0] = -400 * (x[1] - 3*x[0]**2) + 2
    hessian[0][1] = -400 * x[0]
    hessian[1][0] =  -400 * x[0]
    hessian[1][1] = 200
    return hessian
def norm(x):
    x = x.reshape(x.size)
    return np.sqrt(x[0]**2 + x[1]**2)

color_list = ['blue', 'green', 'red', 'cyan', 'purple', 'yellow', 'orange', 'teal',
              'coral', 'darkred', 'brown', 'black']
label_list = ['Newton method', 'Gradient method with backtracking']
tol_list = [1e-1, 1e-3, 1e-5]


def plot_contour():
    X = np.arange(-1.51,1.6,0.05)
    Y = np.arange(-1.55, 2.55, 0.05)
    X,Y = np.meshgrid(X,Y)
    Z = np.zeros((X.shape[0], X.shape[1]))
    for i in range(X.shape[0]):
        for j in range(X.shape[1]):
            x = []
            x.append(X[i][j])
            x.append(Y[i][j])
            x = np.array(x)
            Z[i][j] = f(x)
    plt.contourf(X, Y, Z, 20, alpha=0.3, cmap=plt.cm.hot)
    plt.contour(X, Y, Z, 20, colors='grey')

def plot_line(xk_list, subfig_num):
    plt.figure(1)
    x = []
    y = []
    for i in range(xk_list.shape[0]):
        x.append(xk_list[i][0][0])
        y.append(xk_list[i][1][0])
    plt.plot(x,y, color = color_list[subfig_num], linewidth=1.5, label=label_list[subfig_num])
    plt.scatter(x, y, s=3, color='black')

def plot_convergence(y, method, subfig_num, tol_index):
    tol_index = tol_index + 1
    plt.figure(2, figsize=(8,10))
    plt.subplot(3,1,tol_index)
```

```
    n = y.size
    x = np.arange(n)
    y = np.log(y)
    plt.plot(x, y, label = method, color = color_list[subfig_num], linewidth=2)
    plt.legend()
    plt.xlabel('number of iteration (tol={})'.format(str(tol_list[tol_index-1])))
    plt.ylabel('$log||(x^k-x^*)||$')
    plt.tight_layout()
    #plt.xlim(0,100)

def check_dir(dk_tocheck, xk, gradient, hessian, beta1, beta2):
    dk_norm = norm(dk_tocheck)
    factor1 = np.dot(gradient.T, dk_tocheck)[0][0] < 0
    facotr2 = -(np.dot(gradient.T, dk_tocheck)[0][0]) >= beta1 * np.min([1, dk_norm**beta2]) *
                                                         dk_norm**2
    if factor1 == True and facotr2 == True:
        return True
    else:
        return False
def Global_Newton(initial, subfig_num, tol, tol_index):
    #paramaters

    xstar = np.array([1, 1]).reshape(2, 1)
    s = 1
    sigma = 0.5
    gamma = 1e-4
    beta1 = 1e-6
    beta2 = 0.1
    xk_list = []
    xk_xstar_list = []
    alphak_list_Newton = []
    Always_Use_Newton_Dir = True

    xk = initial
    num_iteration = 0
    xk_list.append(xk)
    xk_xstar_list.append((norm(xk-xstar)))

    gradient = df(xk)
    while norm(gradient) > tol:
        # deteriminate the direction
        hessian = Hessian(xk)
        dk_tocheck = np.linalg.solve(hessian, -gradient)
        good_dir = check_dir(dk_tocheck, xk, gradient, hessian, beta1, beta2)
        if(good_dir == False):
            dk = -gradient
            Always_Use_Newton_Dir = False
        else:
            dk = dk_tocheck

        alphak = s
        alphak_list_Newton.append(alphak)

        while True:
            if f(xk + alphak*dk) - f(xk) <= gamma * alphak * (np.dot(gradient.T, dk)[0][0]):
                break
            alphak = alphak * sigma
        alphak_list_Newton.append(alphak)
        xk = xk + alphak * dk
        xk_list.append(xk)
        xk_xstar_list.append((norm(xk-xstar)))
```

```
        gradient = df(xk)
        num_iteration = num_iteration + 1

    print('tolerance:', tol)
    print('Newton_num_iteration:', num_iteration)
    print('Always_Use_Newton_Dir:', Always_Use_Newton_Dir)
    print('alpha_k of Newton method', alphak_list_Newton)
    print()

    method = 'Newton method'
    xk_xstar_list = np.array(xk_xstar_list)
    plot_convergence(xk_xstar_list, method, subfig_num, tol_index)

    if tol == 1e-5:
        plt.figure(1)
        plt.scatter(xk_list[-1][0], xk_list[-1][1], s=60, marker='*',
                    facecolors ='none', edgecolor= 'r')
        xk_list = np.array(xk_list)
        plot_line(xk_list, subfig_num)

def gradient_method(initial, subfig_num, tol, tol_index):
    xstar = np.array([1, 1]).reshape(2, 1)
    s = 1
    sigma = 0.5
    gamma = 1e-4
    xk_list = []
    xk_xstar_list = []
    alphak_list_GM = []


    xk = initial
    gradient = df(xk)
    num_iteration = 0
    xk_list.append(xk)
    xk_xstar_list.append((norm(xk-xstar)))

    while norm(gradient) > tol:
        alphak = s
        alphak_list_GM.append(alphak)

        dk = -df(xk)
        while True:
            if f(xk + alphak*dk) - f(xk) <= gamma * alphak * (np.dot(df(xk).T, dk)[0][0]):
                break
            alphak = alphak * sigma

        alphak_list_GM.append(alphak)
        xk = xk + alphak * dk
        xk_list.append(xk)
        xk_xstar_list.append((norm(xk-xstar)))

        gradient = df(xk)
        num_iteration = num_iteration + 1

    print('tolerance:', tol)
    print('Gradient_mothd_num_iteration:', num_iteration)
    print('alpha_k of GM method', alphak_list_GM)
    print()

    method = 'Gradient method'
    xk_xstar_list = np.array(xk_xstar_list)
    plot_convergence(xk_xstar_list, method, subfig_num, tol_index)
```

```
    if tol == 1e-5:
        plt.figure(1)
        plt.scatter(xk_list[-1][0], xk_list[-1][1], s=60, marker='*',
                facecolors ='none', edgecolor= 'r')
        xk_list = np.array(xk_list)
        plot_line(xk_list, subfig_num)

# main begin

x1 = np.arange(-10, 11, 4)
x2 = np.arange(-2, 3, 4)

plt.figure(1, figsize=(10, 5))
plot_contour()

initial = np.array([[-1.2], [1]])
plt.figure(1)
plt.scatter(initial[0], initial[1], s=40, marker='s',
                    facecolors ='none', edgecolor= 'b')

print('----------Newton Method----------')
for index, tol in enumerate(tol_list):

    start = time.clock()
    Global_Newton(initial, 0, tol, index)
    end = time.clock()

    print('time:', end - start)
    print()

print('----------Gradient Method----------')
for index, tol in enumerate(tol_list):

    start = time.clock()
    gradient_method(initial, 1, tol, index)
    end = time.clock()

    print('time:', end - start)
    print()

plt.figure(1)
plt.xlabel('x1')
plt.ylabel('x2')
plt.xlim(-1.5, 1.5)
plt.ylim(-1, 2)
plt.legend()
plt.show()
plt.savefig('A4_1_b', dpi=700)

plt.figure(2)
plt.savefig('convergence', dpi=700)
```

Assignment A4.2

In this exercise, we consider the ℓ_1 -optimization problem

$$\min_x f(x) = \frac{1}{2}\|Ax - b\|^2 + \mu\|x\|_1$$

where $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m$ and $\mu > 0$ are given. In order to handle the nonsmoothness of the ℓ_1 -norm, we substitute $\varphi(x) = \|x\|_1$ in (1) with one of the following smooth options:

$$\varphi_1(x) = \|x\|_2^2, \quad \varphi_2(x) = \sum_{i=1}^{n} \varphi_{\text{hub}}(x_i), \quad \text{and} \quad \varphi_3(x) = \sum_{i=1}^{n} \varphi_{\log}(x_i)$$

where

$$\varphi_{\text{hub}}(t) := \begin{cases} \frac{1}{2\delta}t^2 & \text{if } |t| \le \delta \\ |t| - \frac{1}{2}\delta & \text{if } |t| > \delta \end{cases} \quad \text{and} \quad \varphi_{\log}(t) := \log\left(1 + t^2/\nu\right), \quad \delta, \nu > 0$$

The data A and b is generated as follows: choose $n, m, s \in \mathbb{N}$ with $s \le m \le n$ and create a mask mask = randperm (n, s).(The vector mask contains s different integers from 1 to n). We then construct a sparse signal $x^* \in \mathbb{R}^n$ via

$$x^* = \text{zeros(n, 1)} \quad x^*(\text{mask}) = \text{randn(s, 1)}$$

Hence, x^* is an n -dimensional vector with s -nonzero (randomly selected) entries that follow a Normal distribution. We choose $A = \text{randn(m, n)}$ and generate the partial measurements b via $b = Ax^* + 0.01 \cdot \text{randn(m, 1)}$. The goal is to reconstruct the original sparse signal x^* from the much smaller measurements b via solving the problem (1) and its smooth variants.

a) Implement the accelerated gradient method (Lecture L-09, slide 29) for problem (1) using φ_1 and φ_2. The extrapolation parameter and step size should be chosen as follows:

$$\alpha_k = \frac{1}{L}, \quad \beta_k = \frac{t_{k-1} - 1}{t_k}, \quad t_k = \frac{1}{2}\left(1 + \sqrt{1 + 4t_{k-1}^2}\right), \quad t_{-1} = t_0 = 1$$

Here, L denotes the Lipschitz constants of the gradient mappings $\nabla f_1(x) = A^\top(Ax - b) + \nabla\varphi_1(x)$ and $\nabla f_2(x) = A^\top(Ax - b) + \nabla\varphi_2(x)$, respectively. It can be shown that the two constants are given by

$$L_1 = 2\mu + \left\|A^\top A\right\| \text{ (for } \nabla f_1) \quad \text{and} \quad L_2 = \mu\delta^{-1} + \left\|A^\top A\right\| \text{ (for } \nabla f_2)$$

b) Implement the inertial gradient method (Lecture L – 09, slide 42) for problem (1) using φ_1, φ_2, and φ_3. Here, you can implement the following variant for unknown Lipschitz constant:

Algorithm 1: The Inertial Gradient Method for Unknown Lipschitz Constants

1 Initialization: Choose an initial point $x^0 \in \mathbb{R}^n$, set $x^{-1} = x^0, \beta \in [0, 1), \ell > 0$, and calculate $\alpha = 1.99(1 - \beta)/\ell$
for $k = 0, 1, 2, \ldots$ **do**
2 Compute $y^{k+1} = x^k + \beta\left(x^k - x^{k-1}\right)$ and set $\bar{x}^{k+1} = y^{k+1} - \alpha\nabla f\left(x^k\right)$
3 **while** $f\left(\bar{x}^{k+1}\right) - f\left(x^k\right) > \nabla f\left(x^k\right)^\top\left(\bar{x}^{k+1} - x^k\right) + \frac{\ell}{2}\left\|\bar{x}^{k+1} - x^k\right\|^2$ **do**
Set $\ell = 2 \cdot \ell$ and $\alpha = 1.99(1 - \beta)/\ell$ and recompute $\bar{x}^{k+1} = y^{k+1} - \alpha\nabla f\left(x^k\right)$
4 Set $x^{k+1} = \bar{x}^{k+1}$

c) Test your implementations for $m = 300, n = 3000$, and $s = 30$. Report and compare the performance of the methods using the different sparse models. For f_1 and f_2 you can choose $\mu \in [0.1, 10]$ and $\delta \in \left[10^{-5}, 10^{-3}\right]$; for f_3 you can set $\mu \in \left[10^{-3}, 10^{-1}\right]$ and $\nu \in \left[10^{-5}, 10^{-4}\right]$. You can select $x^0 = 0$ as initial point and tol $= 10^{-4}$. (Performance can be measured by comparing $\left\|\nabla f\left(x^k\right)\right\|$ or $\left|f\left(x^k\right) - f^*\right| / \max\{1, |f^*|\}$ where f^* is an approximation of the optimal objective function value).

Plot the reconstructed solutions and compare them with the true solution x^*. Check whether your solutions are truly sparse - which of the models and algorithms performed best in these tasks?

Solution

Subproblem (a)

The python code to solve this problem is showing below.

```
import numpy as np
import matplotlib.pyplot as plt
import time

def norm(x):
    x = x.reshape(x.size)
    return np.sqrt(np.sum(x**2))

def norm_square(x):
    x = x.reshape(x.size)
    return np.sum(x**2)

def huber(t):
    if np.abs(t) <= delta:
        return (t**2) / (2*delta)
    else:
        return np.abs(t) - delta/2

def d_huber(t):
    if np.abs(t) <= delta:
        return t/delta
    else:
        return t/np.abs(t)

def log_fun(t):
    return np.log(1 + t**2/v)

def d_log_fun(t):
    return (1/(1+t**2/v)) * (2*t/v)

def phi_1(x):
    return norm_square(x)

def d_phi_1(x):
    grad = 2*x
    return grad

def phi_2(x):
    x = x.reshape(x.size)
    Sum = 0
    for xi in x:
        Sum = Sum + huber(xi)
    return Sum

def d_phi_2(x):
    x = x.reshape(x.size)
    grad = np.zeros((x.size,1))
    for i in range(grad.size):
        grad[i] = d_huber(x[i])
    return grad


def phi_3(x):
    x = x.reshape(x.size)
    Sum = 0
    for xi in x:
        Sum = Sum + log_fun(xi)
```

```
    return Sum

def d_phi_3(x):
    x = x.reshape(x.size)
    grad = np.zeros((x.size,1))
    for i in range(grad.size):
        grad[i] = d_log_fun(x[i])
    return grad

color_list = ['blue', 'green', 'red', 'cyan', 'purple', 'yellow', 'orange', 'teal',
              'coral', 'darkred', 'brown', 'black']
def plot_convergence(y, method, subfig_num, tol):
    plt.figure(1)
    plt.subplot(2,1,subfig_num)
    n = y.size
    x = np.arange(n)
    y = np.log(y)
    plt.plot(x, y, label = method, color = color_list[subfig_num], linewidth=2)
    plt.legend()
    plt.xlabel('number of iteration (tol={})'.format(tol))
    plt.ylabel('$log||gradient||$')
    plt.tight_layout()

def plt_compare_and_sparse(x_solution, subfig_num):
    plt.figure(subfig_num+1, figsize=(8,20))
    plt.subplot(2,1,1)
    plt.scatter(x_solution, x_star, color=color_list[0], s=2)
    xmin = x_solution.min()
    xmax = x_solution.max()

    plt.plot([x_star.min(), x_star.max()], [x_star.min(), x_star.max()], '--', color='red',
                                                    linewidth=1, label='diagonal line')
    plt.xlim(xmin-0.02, xmax+0.01)
    plt.xlabel('Solution')
    plt.ylabel('$x^*$')
    plt.legend()

    plt.subplot(2,1,2)
    n = x_solution.size
    plt.plot([0,n], [0,0], '--', color='red', linewidth=1)
    plt.scatter(np.arange(n)+1, x_solution, color=color_list[1], s=2)
    plt.xlabel('$i$ (the $i^{th}$ unit of solution)')
    plt.ylabel('The value of $i^{th}$ unit in solution')

    plt.tight_layout()
    plt.savefig('compare_f' +str(subfig_num), dpi=700)


def f1(x):
    return 1/2 * norm_square(np.dot(A,x) - b) + mu * phi_1(x)

def f2(x):
    return 1/2 * norm_square(np.dot(A,x) - b) + mu * phi_2(x)

def f3(x):
    return 1/2 * norm_square(np.dot(A,x) - b) + mu * phi_3(x)

def df1(x):
    return np.dot(A.T, np.dot(A, x)-b) + mu * d_phi_1(x)

def df2(x):
    return np.dot(A.T, np.dot(A, x)-b) + mu * d_phi_2(x)
```

```
def df3(x):
    return np.dot(A.T, np.dot(A, x)-b) + mu * d_phi_3(x)

def AGM(initial, smooth_func_type):
    if smooth_func_type == 1:
        df = df1
        alpha_k = 1 / L1
        method = 'AGM method on $f_1$'
    elif smooth_func_type == 2:
        df = df2
        alpha_k = 1 / L2
        method = 'AGM method on $f_2$'

    tol = 1e-4  # vary
    x_minus = initial
    xk = initial

    tk_minus = 1
    tk = 1

    xk_list = []
    xk_list.append(xk)
    norm_gradient_list = []
    num_iteration = 0

    gradient = df(xk)
    norm_gradient_list.append(norm(gradient))

    while norm(gradient) > tol:
        beta_k = (tk_minus - 1)/tk
        y = xk + beta_k * (xk - x_minus)

        x_minus = xk
        xk = y - alpha_k * df(y)
        xk_list.append(xk)

        tk_minus = tk
        tk = 1/2 * (1 + np.sqrt(1+4*tk**2))

        gradient = df(xk)
        norm_gradient_list.append(norm(gradient))
        # print(norm(gradient))
        num_iteration  = num_iteration + 1

    xk_list = np.array(xk_list)
    print(xk_list.shape)
    norm_gradient_list = np.array(norm_gradient_list)
    plot_convergence(norm_gradient_list, method, smooth_func_type, tol)

    x_solution = xk_list[-1]
    print('norm of (xk-x_star):', norm(x_solution - x_star))
    plt_compare_and_sparse(x_solution, smooth_func_type)


# main begin
#parameters

np.random.seed(2222)
n = 3000
m = 300
s = 30
```

```
mu = 1
delta = 1e-3
v = 1e-5

A = np.random.randn(m, n)
mask = np.random.choice(np.arange(1,n+1), s, replace=False)
x_star = np.zeros((n,1))
for i in range(n):
    if i+1 in mask:
        x_star[i][0] = np.random.randn(1)[0]

b = np.dot(A, x_star) + 0.01 * np.random.randn(m, 1)
L1 = 2*mu + np.linalg.norm(np.dot(A.T, A), ord = 2)
L2 = mu*(1/delta) + np.linalg.norm(np.dot(A.T, A), ord = 2)

initial = np.zeros((n,1))

plt.figure(1, figsize=(8,12))

print('----------f1----------')
start = time.clock()
AGM(initial, 1)
end = time.clock()
print('time:', end - start)
print()

print('----------f2----------')
start = time.clock()
AGM(initial, 2)
end = time.clock()
print('time:', end - start)
print()

plt.figure(1)
plt.savefig('4_2_a_convergence.png', dpi=700)
```

Subproblem (b)

The python code to solve this problem is showing below.

```
import numpy as np
import matplotlib.pyplot as plt
import time

def norm(x):
    x = x.reshape(x.size)
    return np.sqrt(np.sum(x**2))

def norm_square(x):
    x = x.reshape(x.size)
    return np.sum(x**2)

def huber(t):
    if np.abs(t) <= delta:
        return (t**2) / (2*delta)
    else:
        return np.abs(t) - delta/2

def d_huber(t):
    if np.abs(t) <= delta:
        return t/delta
    else:
        return t/np.abs(t)
```

```
def log_fun(t):
    return np.log(1 + t**2/v)

def d_log_fun(t):
    return (1/(1+t**2/v)) * (2*t/v)

def phi_1(x):
    return norm_square(x)

def d_phi_1(x):
    grad = 2*x
    return grad

def phi_2(x):
    x = x.reshape(x.size)
    Sum = 0
    for xi in x:
        Sum = Sum + huber(xi)
    return Sum

def d_phi_2(x):
    x = x.reshape(x.size)
    grad = np.zeros((x.size,1))
    for i in range(grad.size):
        grad[i] = d_huber(x[i])
    return grad

def phi_3(x):
    x = x.reshape(x.size)
    Sum = 0
    for xi in x:
        Sum = Sum + log_fun(xi)
    return Sum

def d_phi_3(x):
    x = x.reshape(x.size)
    grad = np.zeros((x.size,1))
    for i in range(grad.size):
        grad[i] = d_log_fun(x[i])
    return grad
def f1(x):
    return 1/2 * norm_square(np.dot(A,x) - b) + mu * phi_1(x)

def f2(x):
    return 1/2 * norm_square(np.dot(A,x) - b) + mu * phi_2(x)

def f3(x):
    return 1/2 * norm_square(np.dot(A,x) - b) + mu * phi_3(x)

def df1(x):
    return np.dot(A.T, np.dot(A, x)-b) + mu * d_phi_1(x)

def df2(x):
    return np.dot(A.T, np.dot(A, x)-b) + mu * d_phi_2(x)

def df3(x):
    return np.dot(A.T, np.dot(A, x)-b) + mu * d_phi_3(x)

color_list = ['brown', 'green', 'red', 'blue', 'cyan', 'purple', 'yellow', 'orange', 'teal',
              'coral', 'darkred', 'black']
def plot_convergence(y, method, subfig_num, tol, knownL):
```

```
    plt.figure(2-knownL)

    if knownL == True:
        sum_fig = 2
    else:
        sum_fig = 3
    plt.subplot(sum_fig, 1, subfig_num)
    n = y.size
    x = np.arange(n)
    y = np.log(y)
    plt.plot(x, y, label = method, color = color_list[subfig_num], linewidth=2)
    plt.legend()
    plt.xlabel('number of iteration (tol={})'.format(tol))
    plt.ylabel('$log||gradient||$')
    plt.tight_layout()

def plt_compare_and_sparse(x_solution, subfig_num, knownL):
    if knownL == True:
        Type = 'knwonL'
        fignum_begin = 3
    else:
        Type = 'UnknownL'
        fignum_begin = 5

    plt.figure(fignum_begin + subfig_num - 1, figsize=(8,20))
    plt.subplot(2,1,1)
    plt.scatter(x_solution, x_star, color=color_list[3], s=2)
    xmin = x_solution.min()
    xmax = x_solution.max()

    plt.plot([x_star.min(), x_star.max()], [x_star.min(), x_star.max()], '--', color='red',
                                                    linewidth=1, label='diagonal line')
    plt.xlim(xmin-0.02, xmax+0.01)
    plt.xlabel('Solution')
    plt.ylabel('$x^*$')
    plt.legend()

    plt.subplot(2,1,2)
    n = x_solution.size
    plt.plot([0,n], [0,0], '--', color='red', linewidth=1)
    plt.scatter(np.arange(n)+1, x_solution, color=color_list[1], s=2)
    plt.xlabel('$i$ (the $i^{th}$ unit of solution)')
    plt.ylabel('The value of $i^{th}$ unit in solution')

    plt.tight_layout()
    plt.savefig('compare_f' +str(subfig_num)+'_'+Type, dpi=700)

def IGM_Known_L(inital, smooth_func_type):
    if smooth_func_type == 1:
        df = df1
        L = L1
        method = 'IGM method on $f_1$ with known L'
    elif smooth_func_type == 2:
        df = df2
        L = L2
        method = 'IGM method on $f_2$ with known L'

    tol = 1e-4
    beta = 0.5
    alpha = 1.99 * (1-beta) / L

    x_minus = initial
```

```
    xk = initial
    xk_list = []
    xk_list.append(xk)
    norm_gradient_list = []
    num_iteration = 0

    gradient = df(xk)
    norm_gradient_list.append(norm(gradient))
    while norm(gradient) > tol:
        y = xk + beta * (xk - x_minus)
        x_minus = xk
        xk = y - alpha * df(xk)

        xk_list.append(xk)

        gradient = df(xk)
        norm_gradient_list.append(norm(gradient))

        num_iteration  = num_iteration + 1

    xk_list = np.array(xk_list)
    print(xk_list.shape)
    x_solution = xk_list[-1]

    print('norm of (xk-x_star):', norm(x_solution - x_star))
    norm_gradient_list = np.array(norm_gradient_list)
    knownL = True
    plot_convergence(norm_gradient_list, method, smooth_func_type, tol, knownL)
    plt_compare_and_sparse(x_solution, smooth_func_type, knownL)

def IGM_Unknown_L(initial, smooth_func_type):
    if smooth_func_type == 1:
        df = df1
        f = f1
        method = 'IGM method on $f_1$ with unknown L'
    elif smooth_func_type == 2:
        df = df2
        f = f2
        method = 'IGM method on $f_2$ with unknown L'
    else:
        df = df3
        f = f3
        method = 'IGM method on $f_3$ with unknown L'

    tol = 1e-4  # vary
    beta = 0.5  # vary
    l = 1  #vary
    alpha = 1.99 * (1-beta) / l

    x_minus = initial
    xk = initial

    xk_list = []
    xk_list.append(xk)

    num_iteration = 0
    norm_gradient_list = []
    gradient = df(xk)
    norm_gradient_list.append(norm(gradient))

    while norm(gradient) > tol:
        y = xk + beta * (xk - x_minus)
```

```
        xk_bar = y - alpha * df(xk)
        while f(xk_bar) - f(xk) > np.dot(df(xk).T, xk_bar-xk) + l/2 * norm_square(xk_bar-xk):
            l = 2 * l
            alpha = 1.99 * (1-beta) / l
            xk_bar = y - alpha * df(xk)

        x_minus = xk
        xk = xk_bar
        xk_list.append(xk)

        gradient = df(xk)
        norm_gradient_list.append(norm(gradient))

        num_iteration  = num_iteration + 1

    xk_list = np.array(xk_list)
    print(xk_list.shape)
    x_solution = xk_list[-1]

    print('norm of (xk-x_star):', norm(x_solution - x_star))
    norm_gradient_list = np.array(norm_gradient_list)
    knownL = False
    plot_convergence(norm_gradient_list, method, smooth_func_type, tol, knownL)
    plt_compare_and_sparse(x_solution, smooth_func_type, knownL)

# main begin
#parameters

np.random.seed(2222)
n = 3000
m = 300
s = 30
delta = 1e-3
v = 1e-4

mask = np.random.choice(np.arange(1,n+1), s, replace=False)
x_star = np.zeros((n,1))
for i in range(n):
    if i+1 in mask:
        x_star[i][0] = np.random.randn(1)[0]
A = np.random.randn(m, n)
c = 0.01 * np.random.randn(m, 1)
b = np.dot(A, x_star) + c

initial = np.zeros((n,1))

print('----------Known L----------')

# for known L: f1 and f2
plt.figure(1, figsize=(8, 12))
mu = 1
L1 = 2*mu + np.linalg.norm(np.dot(A.T, A), ord = 2)
start = time.clock()
IGM_Known_L(initial, 1)
end = time.clock()
print('f1 time:', end-start)

mu = 1
L2 = mu*(1/delta) + np.linalg.norm(np.dot(A.T, A), ord = 2)
start = time.clock()
IGM_Known_L(initial, 2)
end = time.clock()
```

```
print('f2 time:', end-start)

plt.figure(1)
plt.savefig('KnownL.png', dpi=700)

print('----------Unknown L----------')
# for unknown L: f1, f2, f3
plt.figure(2, figsize=(8, 18))

mu = 1
start = time.clock()
IGM_Unknown_L(initial, 1)
end = time.clock()
print('f1 time:', end-start)

mu = 1
start = time.clock()
IGM_Unknown_L(initial, 2)
end = time.clock()
print('f2 time:', end-start)

mu = 0.1
start = time.clock()
IGM_Unknown_L(initial, 3)
end = time.clock()
print('f3 time:', end-start)

plt.figure(2)
plt.savefig('UnknownL.png', dpi=700)
```

Subproblem (c)

(1) AGM

For AGM method, we applied it into f_1 and f_2. For f_1, we choose the parameters as: $\mu = 1$. For f_2, we choose the parameters as: $\mu = 1, \delta = 10^{-3}$. We select $x^0 = 0$ as the initial point and tol$= 10^{-4}$. The performance is measured by comparing $\|\nabla f(x^k)\|$. The result is showed in Figure 4. Then, we reconstruct solution and compate them with x^*, the result is showed as Figure 5.

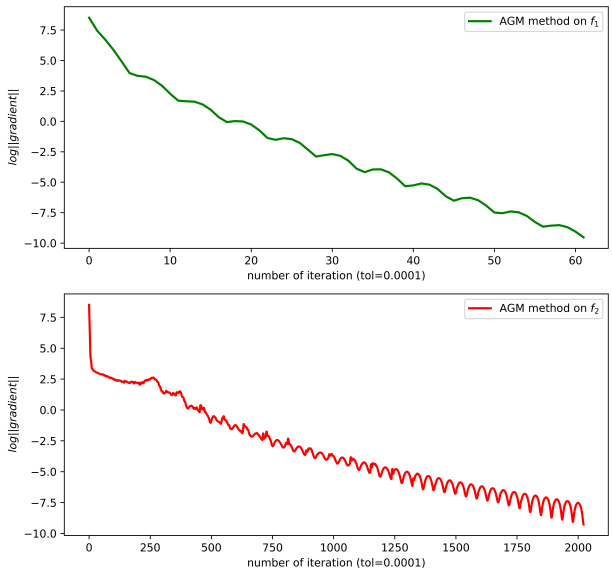
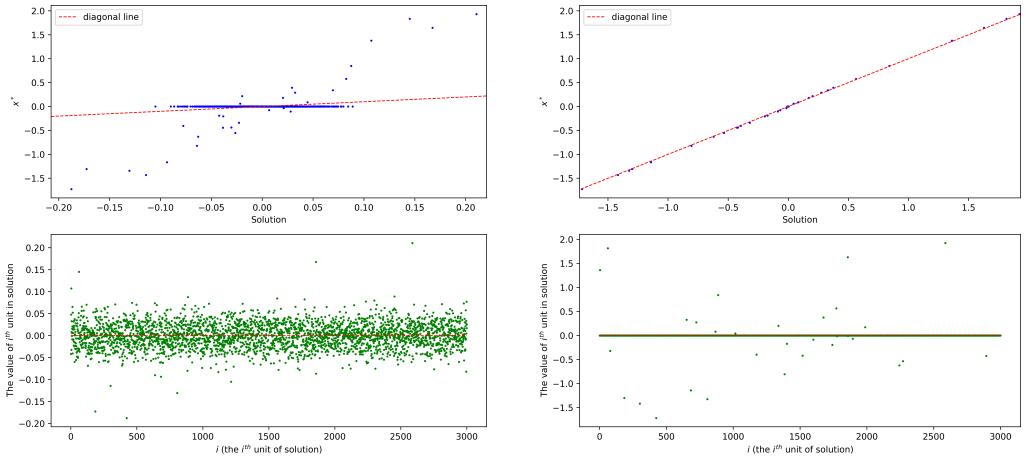


Figure 4 The plot of iteration process by AGM



(a) AGM on f_1 (b) AGM on f_2

Figure 5 The comparison between solutions and x^*, and check the sparse of solutions

As we can see from Figure 4. When we apply AGM to f_1, the gradient can converge in 62 times, which is much smaller than that in f_2 (2023 times). As we can see from Figure 5(a), When using AGM, f_1 is not a good model. This is because the solution is far away from x^*, which means the solution is not what we want. Besides, the solution is not sparse, there are small number of 0 in the solution's units actually. On the contrary, as we can see from Figure 5(b), when using AGM, f_2 is really a good model. The solution is very close to x^*, because they are almost totally on the diagonal line. Besides, the solution is also sparse, there are just small number of units in the solution are not 0, but most units of them are 0, which means the solution is sparse.

(2) IGM

For IGM method, we applied it into f_1, f_2 and f_3. We seperated the experiments into two parts: "Know Lipschitz constant" and "Not know Lipschitz constant".

- (a): Know Lipschitz constant

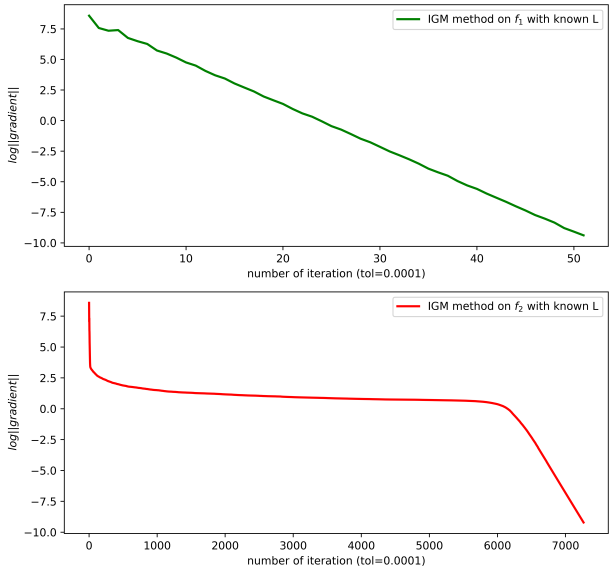
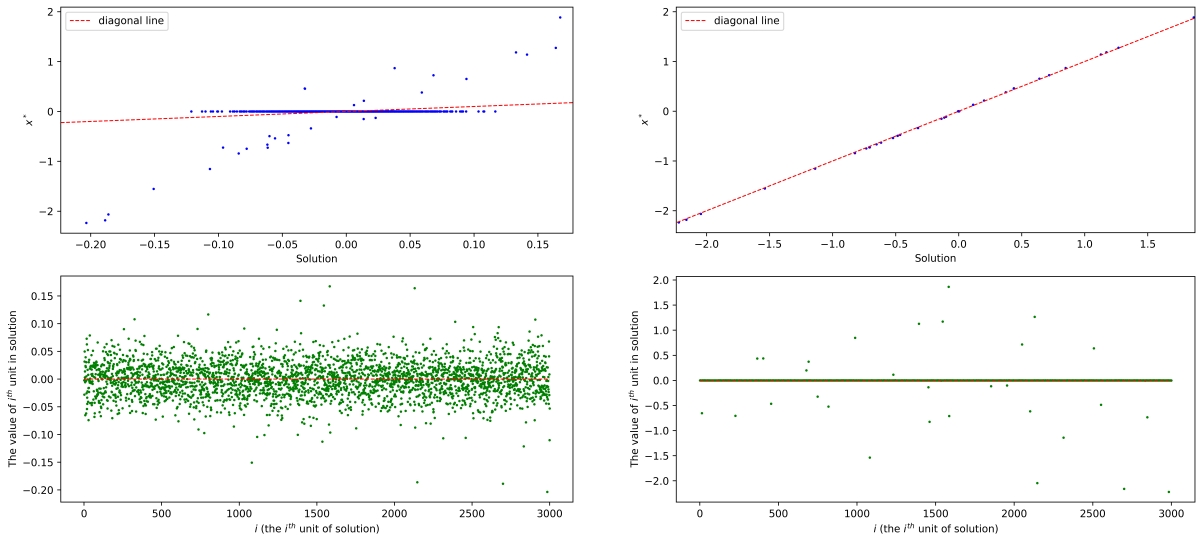


Figure 6 The plot of iteration process by IGM with known L



(a) IGM on f_1 with known L

(b) IGM on f_2 with known L

Figure 7 The comparison between solutions and x^*, and check the sparse of solutions)

In this part, we applied IGM into f_1 and f_2, we computed L_1 and L_2 explicitly. For f_1, we choose the parameters as: $\mu = 1$. For f_2, we choose the parameters as: $\mu = 1, \delta = 10^{-3}$. We select $x^0 = 0$ as the initial point and tol$= 10^{-4}$. The performance is measured by comparing $\|\nabla f(x^k)\|$. The result is showed in Figure 6. Then, we reconstruct solution and compate them with x^*, the result is showed as Figure 7.

As we can see from Figure 6, when we apply IGM with known L to f_1, the gradient can converge in 52 times, which is much smaller than that in f_2 (7267 times). And we can see that there is a plateau period in the convergence process of the gradient of f_2.

As we can see from Figure 7(a), When using IGM with known L, f_1 is not a good model. This is because the solution is far away from x^*, which means the solution is not what we want. Besides, the solution is not sparse, there are small number of 0 in the solution's units actually. On the contrary, as we can see from Figure 7(b), when using IGM with known L, f_2 is really a good model. The solution is very close to x^*, because they are almost totally on the diagonal line. Besides, the solution is also sparse, there are just small number of units in the solution are not 0, but most units of them are 0, which means the solution is sparse.

- (b): Not know Lipschitz constant

  In this part, we applied IGM into f_1, f_2 and f_3, we do not compute L_1, L_2 and L_3 explicitly. On the contrary, we treat the Lipschitz constant as unknown, and we use Algorithm 1 (mentioned before) as a variant to do IGM. For f_1, we choose the parameters as: $\mu = 1$. For f_2, we choose the parameters as: $\mu = 1, \delta = 10^{-3}$. For f_3, we choose the parameters as: $\mu = 1, \delta = 10^{-3}, \nu = 10^{-4}$.We select $x^0 = 0$ as the initial point and tol$= 10^{-4}$. The performance is measured by comparing $\|\nabla f(x^k)\|$. The result is showed in Figure 8. Then, we reconstruct solution and compate them with x^*, the result is showed as Figure 9.

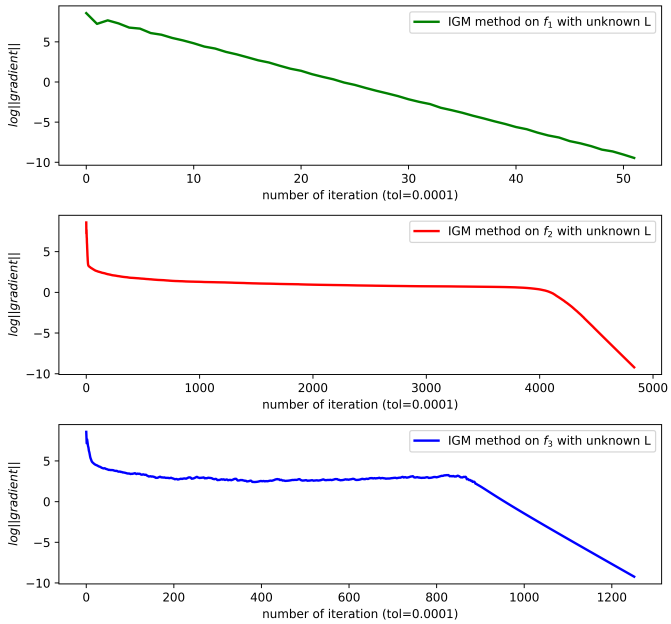
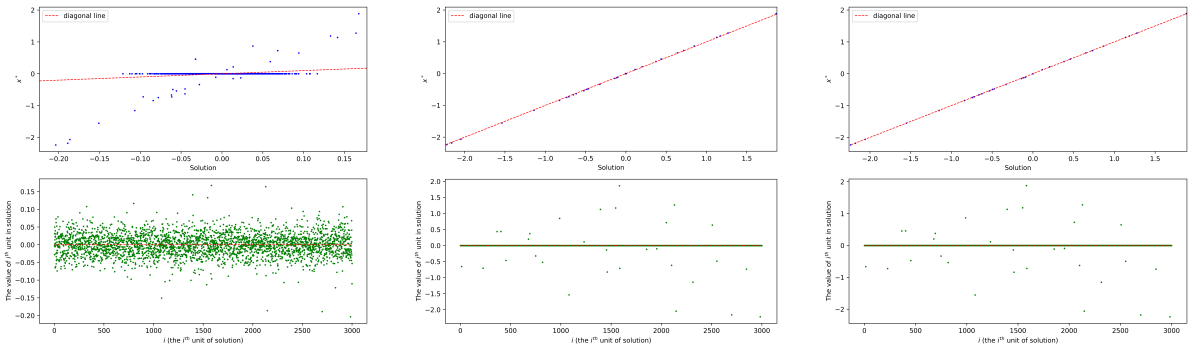


Figure 8 The plot of iteration process by IGM with unknown L



(a) IGM on f_1 with unknown L (b) IGM on f_2 with unknown L (c) IGM on f_3 with unknown L

Figure 9 The comparison between solutions and x^*, and check the sparse of solutions

As we can see from Figure 8, when we apply IGM with unknown L, the gradient of f_1 can converge in 52 times, the gradient of f_2 converges in 4835 times, and the gradient of f_3 converges in 1252 times. And we can see that there is a plateau period in the convergence process of the gradient of f_2 and the gradient of f_3.

As we can see from Figure 9(a), When using IGM with unknown L, f_1 is not a good model. This is because the solution is far away from x^*, which means the solution is not what we want. Besides, the solution is not sparse, there are small number of 0 in the solution's units actually. On the contrary, as we can see from Figure 9(b) and Figure 9(c), when using IGM with unknown L, f_2 and f_3 are both good models. The solution is very close to x^*, because they are almost totally on the diagonal line. Besides, the solution is also sparse, there are just small number of units in the solution are not 0, but most units of them are 0, which means the solution is sparse.

(3) Summary

Above all, we have applied AGM and IGM (with known L and unknown L) method to solve f_1, f_2 and f_3 models respectively. The iteration times is concluded as Table 3.

Table 3 Performance of different methods on different models

Method	model	tolerance	iteration	time(s)	$\|x^k - x^*\|$
AGM	f_1	10^{-4}	62	0.978	4.775
	f_1	10^{-4}	2023	115.113	0.085
IGM (know L)	f_1	10^{-4}	52	0.406	5.392
	f_2	10^{-4}	7276	417.809	0.086
IGM (not know L)	f_1	10^{-4}	52	0.488	5.392
	f_2	10^{-4}	4835	626.245	0.086
	f_3	10^{-4}	1252	149.43	0.036

As we can see from Table 3, model f_3 is the best in these tasks. When applied model f_3 with IGM and do not compute L explicitly, the iteration times is 1252, which is smaller than model f_2. And the solution of model f_3 is the most close to x^*, as we can see the value $\|x^k - x^*\|$ equals 0.036, which is very small.

The python code to solve this problem is showed in subproblem(a) and subproblem(b) sections.