MDS 6106 – Introduction to Optimization

Exercise Sheet Nr.: 01

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In the creation of this solution sheet, I worked together with:

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For correction:

Exercise						Σ
Grading						

A 1.1

Decision variable

x_1: the amount of beer mugs to produce,

x_2: the amount of champagne glasses to produce

y: work hour per day

maximize $f(x) = 200x_1 + 150x_2$

subject to
$$y \le 8$$
$$x_1 + x_2 \le 15y$$
$$9x_1 + 5.5x_2 \le 900$$
$$x_1, x_2 \in N^*$$

thus optimal solution is $X = (68. , 52)$

optimal value 21400

A 1.2

(a) Decision variable:

x_1 : the amount of Ale to produce

x_2 : the amount of Beer to produce

maximize $f(x) = 130x_1 + 230x_2$

subject to: $5x_1 + 15x_2 \le 480$

$4x_1 + 4x_2 \le 160$

$35x_1 + 20x_2 \le 1190$

$x_1, x_2 \in N^*$

(b) maximize: $f(x) = 130x_1 + 230x_2$

subject to: $5x_1 + 15x_2 \le 480$

$4x_1 + 4x_2 \le 160$

$35x_1 + 20x_2 \le 1190$

$1.5x_1 + 2.5x_2 \le 12 \times 7$

thus optimal solution is

$x^* = (6, 30)$

optimal value $f(x^*) = 7680$

A1.3

A_1 is symmetrica

A_1: $|\lambda I - A_1| = 0 \Rightarrow \begin{vmatrix} \lambda-1 & 0 & 0 \\ 0 & \lambda-2 & 0 \\ 0 & 0 & \lambda+1 \end{vmatrix} = 0 \Rightarrow \begin{cases} \lambda_1 = 1 \\ \lambda_2 = 2 \\ \lambda_3 = -1 \end{cases}$ thus A_1 is indefinite

A_2 is symmetrical

A_2: $|\lambda I - A_2| = 0 \Rightarrow \begin{vmatrix} \lambda-1 & 0 & -1 \\ 0 & \lambda-1 & -2 \\ -1 & -2 & \lambda-5 \end{vmatrix} = 0 \Rightarrow \begin{cases} \lambda_1 = 1 \\ \lambda_2 = 0 \\ \lambda_3 = 6 \end{cases}$ thus A_2 is positive semidefinite

A_3: $|\lambda I - A_3| = 0 \Rightarrow \begin{vmatrix} \lambda-1 & 0 & -1 \\ 0 & \lambda-1 & 1 \\ 0 & -2 & \lambda-4 \end{vmatrix} = 0 \Rightarrow \begin{cases} \lambda_1 = 1 \\ \lambda_2 = 2 \\ \lambda_3 = 3 \end{cases}$

$$x^T A_3 x = (x_1, x_2, x_3) \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ x_2 \\ x_3 \end{pmatrix} = (x_1, x_2+2x_3, x_1-x_2+4x_3) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1^2 + x_2^2 + 2x_2x_3 + x_1x_3 - x_2x_3 + 4x_3^2$$

$$= x_1^2 + x_2^2 + 4x_3^2 + x_2x_3 + x_1x_3$$

$$= \frac{1}{2}(x_2+x_3)^2 + \frac{1}{2}(x_1+x_3)^2 + \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + 3x_3^2 > 0$$ thus A_3 is positive definit

A_4: $|\lambda I - A_4| = 0 \Rightarrow \begin{vmatrix} \lambda-1 & 0 & -1 \\ 0 & \lambda+1 & 1 \\ 1 & -1 & \lambda \end{vmatrix} = 0 \Rightarrow \begin{cases} \lambda_1 = 0 \\ \lambda_2 = i \\ \lambda_3 = -i \end{cases}$

$$X^T A_4 X = (x_1, x_2, x_3) \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = (x_1+x_3, x_3-x_2, x_1-x_2) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1^2 + x_1x_3 + x_2x_3 - x_2^2 + x_1x_3 - x_2x_3$$

$$= x_1^2 - x_2^2 \begin{cases} = 0 & x_1^2 = x_2^2 \\ > 0 & x_1^2 > x_2^2 \\ < 0 & x_1^2 < x_2^2 \end{cases}$$ thus A_4 is indefinite

A 1.4

Decision variable:

r: radius of the circle

x: the center of the circle

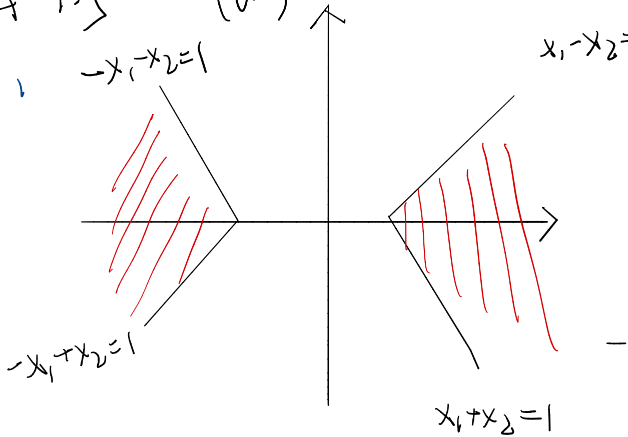
minimize: $f(r) = r$

subject to:
$$\|y^1 - x\| \le r$$
$$\|y^2 - x\| \le r$$
$$\|y^3 - x\| \le r$$
$$\vdots$$
$$\|y^k - x\| \le r$$

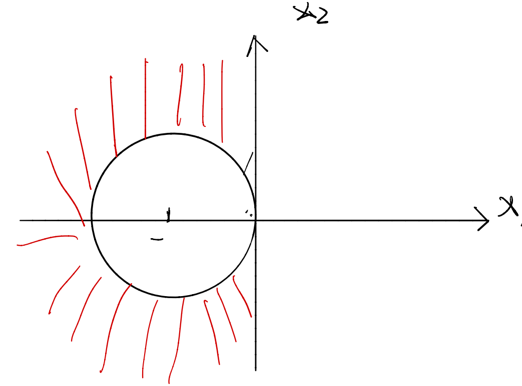
Classification: nonlinear . continuous . constrained

A 1.5 (a)

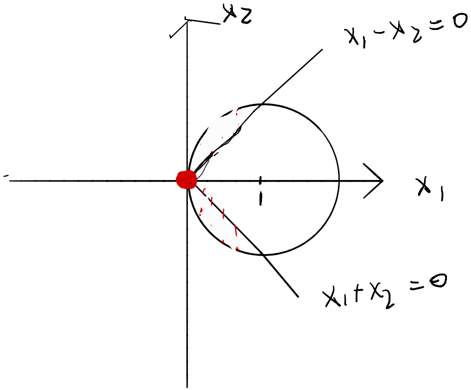
X_1



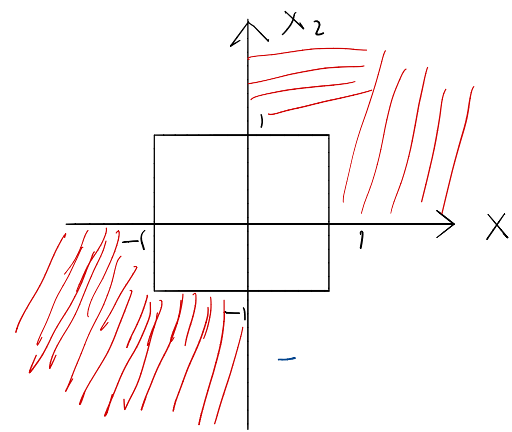
X_2



X_3

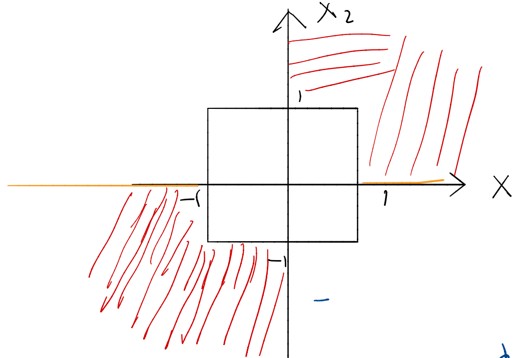


X_4



(b) X_3 is bounded, because there exists $B \in R$ with $\|x\| \le B$ for all $x \in X_3$

(c)

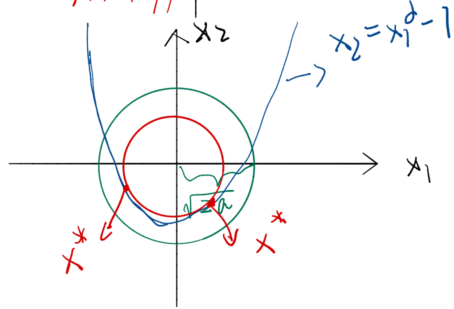


1° when $f(x) = x_1$, $X = \{(1, x_2) \mid 0 \le x_2 < 1\}$ $f(x)$ is local minimizer, and it's not strict. $f(x) = x_1$ doesn't exit global minimum

2° when $f(x) = \frac{1}{2}(x_1^2 + x_2^2)$

$\begin{cases} x_1 = 1 \\ x_2 = 0 \end{cases}$ $\begin{cases} x_1 = 0 \\ x_2 = 1 \end{cases}$ $\begin{cases} x_1 = -1 \\ x_2 = 0 \end{cases}$ $\begin{cases} x_1 = 0 \\ x_2 = -1 \end{cases}$, the $f(x)$ is local and global minimizer, but it's not strict

(d)



$$\min f(x) = \frac{1}{2}\|x\|^2 = \frac{1}{2}(x_1^2 + x_2^2)$$

$$1 - x_1^2 + x_2 = 0 \Rightarrow x_2 = x_1^2 - 1$$

thus
$$f(x) = \frac{1}{2}[x_1^2 + (x_1^2 - 1)^2]$$
$$= \frac{1}{2}[x_1^2 + x_1^4 - 2x_1^2 + 1]$$
$$= \frac{1}{2}(x_1^4 - x_1^2 + 1)$$
$$= \frac{1}{2}[(x_1^2 - \frac{1}{2})^2 + \frac{3}{4}]$$

when $x_1 = \pm\frac{\sqrt{2}}{2}$ $\min_{x \in R^2} f(x) = \frac{3}{8}$

thus x^* $(\frac{\sqrt{2}}{2}, -\frac{1}{2})$ and $(-\frac{\sqrt{2}}{2}, -\frac{1}{2})$

$$\nabla f(x^*) = \begin{pmatrix} \frac{\partial f(x)}{\partial x_1} \\ \frac{\partial f(x)}{\partial x_2} \end{pmatrix} = \begin{pmatrix} \pm\frac{\sqrt{2}}{2} \\ -\frac{1}{2} \end{pmatrix}$$

$$\nabla h(x^*) = \begin{pmatrix} \frac{\partial h(x)}{\partial x_1} \\ \frac{\partial h(x)}{\partial x_2} \end{pmatrix} = \begin{pmatrix} \mp\sqrt{2} \\ 1 \end{pmatrix}$$

thus $\nabla f(x^*)$ parallom $\nabla h(x^*)$

(e) $f(x) = -x^2$ $x \in [-1, +\infty)$

A 1.6

a) f_1: $\frac{\partial f_1(x)}{\partial x_1} = 4x_1(x_1^2 - x_2) + 2(x_1 - x_2^2) - 3(x_1-1)^2$ $\quad \frac{\partial f_1(x)}{\partial x_2} = -2(x_1^2 - x_2) - 4x_2(x_1 - x_2^2) + 3(x_2-1)^2$

$$H_{f_1}(x) = \begin{pmatrix} \frac{\partial f_1(x)}{\partial x_1 \partial x_1} & \frac{\partial f_1(x)}{\partial x_1 \partial x_2} \\ \frac{\partial f_1(x)}{\partial x_2 \partial x_1} & \frac{\partial f_1(x)}{\partial x_2 \partial x_2} \end{pmatrix} = \begin{pmatrix} 12x_1^2 - 4x_2 + 2 - 6(x_1-1), & -4x_1 - 4x_2 \\ -4x_1 - 4x_2 & 2 + 12x_2^2 - 4x_1 + 6(x_2-1) \end{pmatrix}$$

f_2: $\frac{\partial f_2(x)}{\partial x_1} = -\sin(x_1)\sin(x_2) - \frac{1}{1+x_2^2}$ $\quad \frac{\partial f_2(x)}{\partial x_2} = \cos(x_1)\cos(x_2) + \frac{2x_1x_2}{(1+x_2^2)^2}$

$$H_{f_2}(x) = \begin{pmatrix} -\cos(x_1)\sin(x_2) & -\sin(x_1)\cos(x_2) + \frac{2x_2}{(1+(x_2)^2)^2} \\ -\sin(x_1)\cos(x_2) + \frac{2x_2}{(1+x_2^2)^2} & -\cos(x_1)\sin(x_2) + \frac{2x_1 - 6x_1x_2^2}{(1+x_2^2)^3} \end{pmatrix}$$

f_3: $\frac{\partial f_3(x)}{\partial x_1} = 4x_1^3 + 6x_1^2 - 4x_2x_1$ $\quad \frac{\partial f_3(x)}{\partial x_2} = -2x_1^2 + 8x_2$

$$H_{f_3}(x) = \begin{pmatrix} 12x_1^2 + 12x_1 - 4x_2 & -4x_1 \\ -4x_1 & 8 \end{pmatrix}$$

(b) $\nabla f_3(x) = \begin{pmatrix} 4x_1^3+6x_1^2-4x_2x_1 \\ -2x_1^2+8x_2 \end{pmatrix} = 0 \Rightarrow \begin{cases} -2x_1^2+8x_2=0 \\ 4x_1^3+6x_1^2-4x_1x_2=0 \end{cases} \Rightarrow \begin{cases} x_2=\frac{1}{4}x_1^2 \\ 4x_1^3+6x_1^2-4x_1x_2=0 \end{cases}$

$\Rightarrow 4x_1^3+6x_1^2-x_1^3=0 \Rightarrow 3x_1^3+6x_1^2=0 \Rightarrow x_1^2(2+x_1)=0 \Rightarrow x_1=\begin{cases} -2 \\ 0 \end{cases}$

$x^*(-2,1)$ and $(0,0)$

so at this points $H_1f_3(x^*) = \begin{pmatrix} 20 & 8 \\ 8 & 8 \end{pmatrix}$ $H_2f_3(x^*) = \begin{pmatrix} 0 & 0 \\ 0 & 8 \end{pmatrix}$

due $H_1f_3(x^*)$ and $H_2f_3(x^*)$ all are symmrcal

$|\lambda I - H_1f_3(x^*)| = 0 \Rightarrow \begin{vmatrix} \lambda-20 & -8 \\ -8 & \lambda-8 \end{vmatrix} = 0 \Rightarrow (\lambda-20)(\lambda-8)-64=0$ $\lambda^2-28\lambda+96=0$

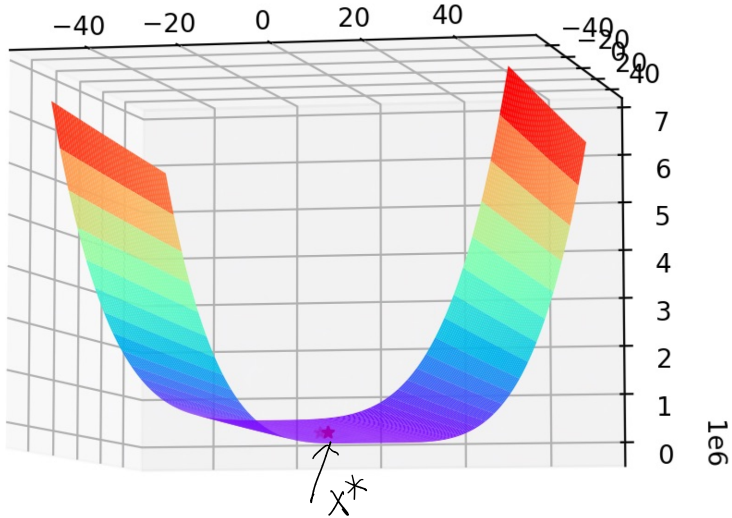
$\lambda=\begin{cases} 24 \\ 4 \end{cases}$

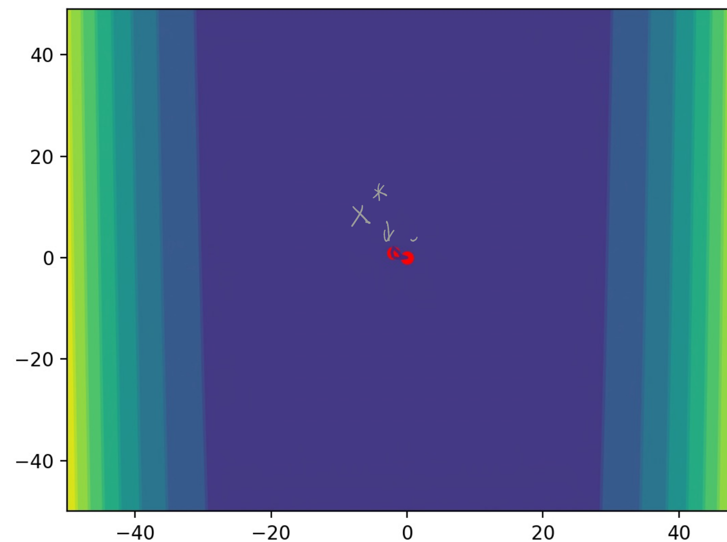
thus $H_1f_3(x^*)$ is positive definite

$|\lambda I - H_2f_3(x^*)| = 0 \Rightarrow \begin{vmatrix} \lambda & 0 \\ 0 & \lambda-8 \end{vmatrix} = 0 \Rightarrow \lambda=\begin{cases} 0 \\ 8 \end{cases}$

thus $H_2f_3(x^*)$ is positive semi-definite

(c)





Code:

Contour plot

```
import numpy as np
import matplotlib.pyplot as plt

#建立步长为0.01，即每隔0.01取一个点
step = 1
x = np.arange(-50,50,step)
y = np.arange(-50,50,step)
#也可以用x = np.linspace(-10,10,100)表示从-10到10，分100份
x1=[-2,0]
x2=[1,0]
#将原始数据变成网格数据形式
X,Y = np.meshgrid(x,y)
#写入函数，z是大写
Z = X**4+2*(X-Y)*X**2+4*Y**2
# 设置打开画布大小,长10，宽6
plt.figure(figsize=(10,6))
# 填充颜色，f即filled
plt.contourf(X,Y,Z)
# 画等高线
plt.contour(X,Y,Z)
plt.scatter(x1,x2,marker='o',c='red')

plt.show()
```

Surface 3D-plot

```
import matplotlib.pyplot as plt
import numpy as np

#建立步长为0.01，即每隔0.01取一个点
ax3=plt.axes(projection='3d')
step = 0.1
x = np.arange(-50,50,step)
y = np.arange(-50,50,step)
#也可以用x = np.linspace(-10,10,100)表示从-10到10，分100份
x1=[-2,0]
x2=[1,0]
X,Y = np.meshgrid(x,y)
Z = X**4+2*(X-Y)*X**2+4*Y**2
ax3.plot_surface(X,Y,Z,cmap='rainbow')
ax3.scatter(x1,x2,marker='*',c='red')
plt.show()
```

E1.7

a) T

b) F since strict local minimizer is unique

c) T when $f(x) = 1$

d) T when $f(x) = \cos(x)$

e) F since f just have one strict global minima

f) F since strict local minimum is unique