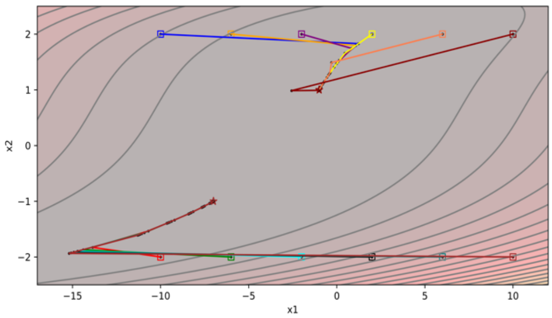
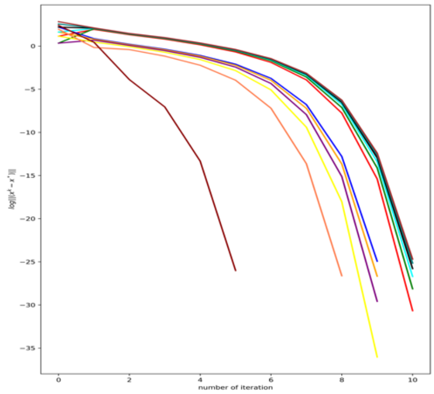
A4-1

a) We choose P=12 different initial points (-10, ±2), (-6, ±2), (-2, ±2), (2, ±2), (6,±2), (10, ±2). The figures which contains the paths is showing as Figure 1(a). The figure of $(\log ||x^k - x^*||)_k$ can be seen in Figure 1(b). As we can see from the figure, the convergence type is quadratic convergence.



(a) Paths of Newton method with different points



(b) $\log ||x^k - x^*||$ VS number of iteration

Figure 1 The plot of iteration process

By comparing the performance of the Newton method and gradient method tested in A3.3, we can derive Table 1. From the table, we can see that the Newton method is much better than the gradient method. The Newton method needs smallest iterations and time.

Method	tolerance	iteration	Time(s)
GM with backtracking	10^{-8}	2793	1.613
GM with diminishing	10^{-8}	85266	8.516
GM with exact line search	10^{-8}	695	1.007
Newton method	10^{-8}	9	0.035

Table 1 Performance of different methods

The python code is as follow

```
import numpy as np
import matplotlib.pyplot as plt
import time

def f1(x):
    x = x.reshape(x.size)
    return 3 + x[0] + ((1 - x[1]) * x[1] - 2) * x[1]
def f2(x):
    x = x.reshape(x.size)
    return 3 + x[0] + (x[1] - 3) * x[1]
def f(x):
    x = x.reshape(x.size)
```

```
    return f1(x) * f1(x) + f2(x) *f2(x)
def df(x):
    x = x.reshape(x.size)
    grad = np.zeros(2).reshape(2,1)
    grad[0] = 2 * f1(x) + 2 * f2(x)
    grad[1] = 2 * f1(x) * (2*x[1] - 3*(x[1]**2) - 2) + 2 * f2(x) * (2*x[1] - 3)
    return grad
def Hessian(x):
    x = x.reshape(x.size)
    hessian = np.zeros((2,2))
    hessian[0][0] = 4
    hessian[0][1] = 8*x[1]-6*(x[1]**2)-10
    hessian[1][0] = 8*x[1]-6*(x[1]**2)-10
    hessian[1][1] = 2*f1(x)*(-6*x[1]+2) + 2*(2*x[1]-3*(x[1]**2)-
2)**2+4*f2(x)+2*(2*x[1]-3)**2
    return hessian
def norm(x):
    x = x.reshape(x.size)
    return np.sqrt(x[0]**2 + x[1]**2)

color_list = ['red','blue', 'green', 'orange','cyan', 'purple', 'black','yellow',
'teal',
            'coral','brown', 'darkred']
Number_iterations = []

def plot_contour():
    X = np.arange(-17.5,12.5,0.05)
    Y = np.arange(-3, 3, 0.05)
    X,Y = np.meshgrid(X,Y)
    Z = np.zeros((X.shape[0], X.shape[1]))
    for i in range(X.shape[0]):
        for j in range(X.shape[1]):
            x = []
            x.append(X[i][j])
            x.append(Y[i][j])
            x = np.array(x)
            Z[i][j] = f(x)
    plt.contourf(X, Y, Z, 30, alpha=0.3, cmap=plt.cm.hot)
    plt.contour(X, Y, Z, 30, colors='grey')

def plot_line(xk_list, subfig_num):
    plt.figure(1)
    x = []
    y = []
```

```
    for i in range(xk_list.shape[0]):
        x.append(xk_list[i][0][0])
        y.append(xk_list[i][1][0])
    plt.plot(x,y, color = color_list[subfig_num-1], linewidth=1.5)
    plt.scatter(x, y, s=3, color='black')

def plot_convergence(y, subfig_num):
    plt.figure(2, figsize=(8,10))
    n = y.size
    x = np.arange(n)
    y = np.log(y)
    plt.plot(x, y, color = color_list[subfig_num-1], linewidth=2)
    plt.xlabel('number of iteration')
    plt.ylabel('$log||(x^k-x^*)||$')
    plt.tight_layout()

def check_dir(dk_tocheck, xk, gradient, hessian, beta1, beta2):
    dk_norm = norm(dk_tocheck)
    factor1 = np.dot(gradient.T, dk_tocheck)[0][0] < 0
    facotr2 = -(np.dot(gradient.T, dk_tocheck)[0][0]) >= beta1 * np.min([1,
dk_norm**beta2]) * dk_norm**2
    if factor1 == True and facotr2 == True:
        return True
    else:
        return False
def Global_Newton(initial, subfig_num):
    #paramaters
    s = 1
    sigma = 0.5
    gamma = 0.1
    beta1 = 1e-6
    beta2 = 0.1
    tol = 1e-8
    xk_list = []
    xk_xstar_list = []



    xk = initial
    num_iteration = 0
    xk_list.append(xk)



    gradient = df(xk)
    while norm(gradient) > tol:
```

```
        # deteriminate the direction
        hessian = Hessian(xk)
        dk_tocheck = np.linalg.solve(hessian, -gradient)
        good_dir = check_dir(dk_tocheck, xk, gradient, hessian, beta1, beta2)
        if(good_dir == False):
            dk = -gradient
        else:
            dk = dk_tocheck

        alphak = s
        while True:
            if f(xk + alphak*dk) - f(xk) <= gamma * alphak * (np.dot(gradient.T, 
dk)[0][0]):
                break
            alphak = alphak * sigma
        xk = xk + alphak * dk
        xk_list.append(xk)

        gradient = df(xk)
        num_iteration = num_iteration + 1

    Number_iterations.append(num_iteration)

    plt.figure(1)
    plt.scatter(xk_list[-1][0], xk_list[-1][1], s=60, marker='*',
                facecolors ='none', edgecolor= color_list[subfig_num-1])
    xk_list = np.array(xk_list)
    plot_line(xk_list, subfig_num)

    xstar_x1 = xk_list[-1][0][0]
    xstar_x2 = xk_list[-1][1][0]
    xstar = np.array([round(xstar_x1), xstar_x2]).reshape(2, 1)
    for i in range(xk_list.shape[0]):
        xk_xstar_list.append(norm(xk_list[i] - xstar))
    xk_xstar_list = np.array(xk_xstar_list)
    plot_convergence(xk_xstar_list, subfig_num)
# main begin

x1 = np.arange(-10, 11, 4)
x2 = np.arange(-2, 3, 4)

plt.figure(1, figsize=(10, 5))
plot_contour()
subfig_num = 1
```

```python
time_list = []
for i in range(6):
    for j in range(2):
        initial = np.zeros(2).reshape(2,1)
        initial[0][0] = x1[i]
        initial[1][0] = x2[j]
        plt.figure(1)
        plt.scatter(initial[0], initial[1], s=40, marker='s',
                    facecolors ='none', edgecolor= color_list[subfig_num-1])

        start = time.time()
        Global_Newton(initial, subfig_num)
        end = time.time()

        time_list.append(end - start)
        subfig_num = subfig_num + 1

plt.figure(1)
plt.xlabel('x1')
plt.ylabel('x2')
plt.xlim(-17, 12)
plt.ylim(-2.5, 2.5)
plt.savefig('A4_1_a', dpi=700)

plt.figure(2)
plt.savefig('A4_1_a_convergence', dpi=700)

print()
print('Number of iterations from different initial points:', Number_iterations)
print('Average number of iterations from different initial points:',
     sum(Number_iterations)/len(Number_iterations))

print('Calculating time from different initial points:', time_list)
print('Average calculating time from different initial points:',
     sum(time_list)/len(time_list))




"""
x1, x2 = symbols('x1 x2', real=True)
ans1 = diff(2*(3 + x1 + ((1 - x2) * x2 - 2) * x2) + 2*(3 + x1 + (x2 - 3) * x2),
x1).subs({x1:-7, x2:1})
ans2 = diff(2*(3 + x1 + ((1 - x2) * x2 - 2) * x2) + 2*(3 + x1 + (x2 - 3) * x2),
```

```
x2).subs({x1:-7, x2:1})
ans3 = diff(2*(3 + x1 + ((1 - x2) * x2 - 2) * x2)*(2*x2-3*x2**2-2) + 2*(3 + x1 +
(x2 - 3) * x2)*(2*x2-3), x1).subs({x1:-7, x2:1})
ans4 = diff(2*(3 + x1 + ((1 - x2) * x2 - 2) * x2)*(2*x2-3*x2**2-2) + 2*(3 + x1 +
(x2 - 3) * x2)*(2*x2-3), x2).subs({x1:-7, x2:1})
print(ans1)
print(ans2)
print(ans3)
print(ans4)
"""
```

b) By comparing the performance of the Newton method and gradient method with backtracking, we can derive Table 2. From the table, we can see that the Newton method is much better than the gradient method with backtracking. The Newton method always utilize the Newton direction. Newton method and gradient method both do not always use full step sizes αk = 1. The figures which contains the paths is showing as Figure 2. The figure of $(\log||x^k - x^*||)_k$ can be seen in Figure 3(a). By zooming in Figure 3(a), we get Figure 3(b). as we can see from the figure, the gradient method is linear convergence, and the Newton method is quadratic convergence.

Method	tolerance	iteration	Time(s)	Always use Newton direction	αk = 1
GM with backtracking	10^{-1}	54	0.05		False
	10^{-3}	5231	1.108		False
	10^{-5}	1096	2.303		False
Newton Method	10^{-1}	19	0.166	True	False
	10^{-3}	20	0.075	True	False
	10^{-5}	21	0.08	True	False

Table 2 performance of different methods on Rosenbrock function

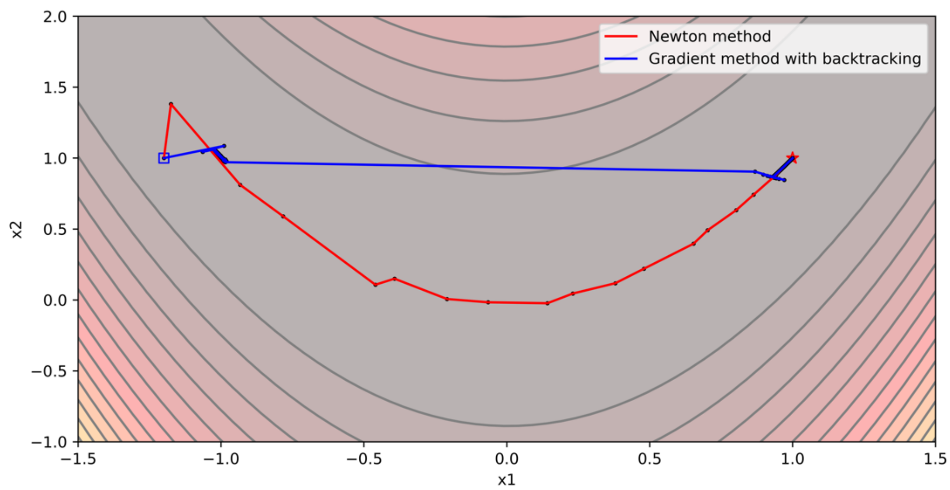


Figure 2 Path of Newton method and gradient mothod with backtracking

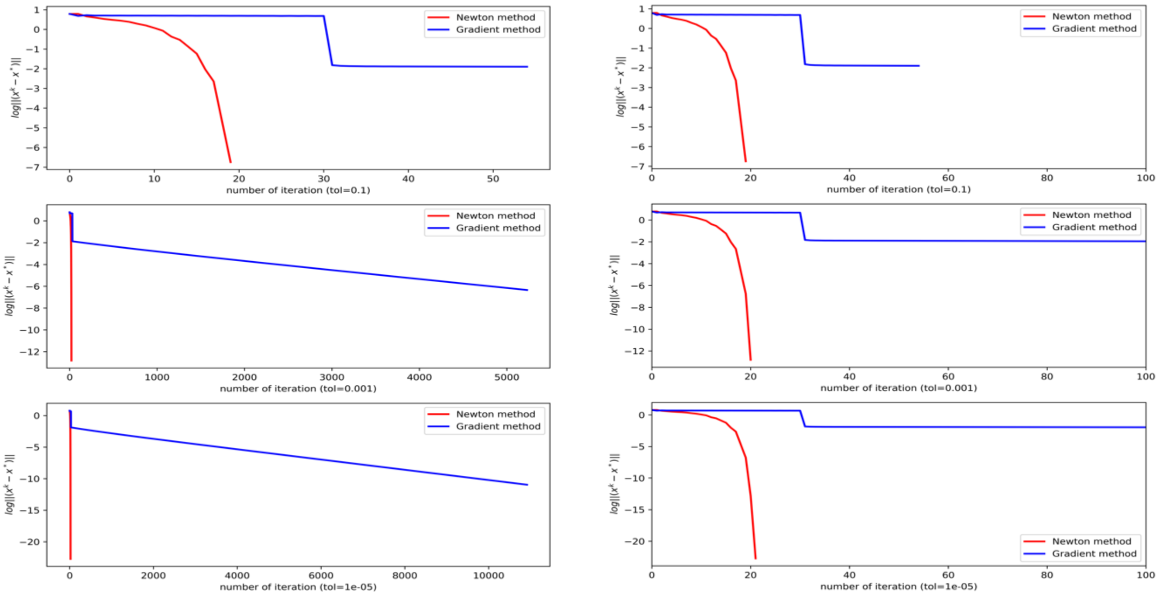


Figure 3 The plot of iteration of process

The python code to solve this problem is showing below.

```
import numpy as np
import matplotlib.pyplot as plt
import time

def f(x): # Rosenbrock function
    x = x.reshape(x.size)
    return 100*(x[1] - x[0]**2)**2 + (1-x[0])**2
def df(x):
    x = x.reshape(x.size)
    grad = np.zeros(2).reshape(2,1)
    grad[0] = -400 * x[0] * (x[1] - x[0]**2) + 2 * x[0] -2
    grad[1] = 200 * (x[1] - x[0]**2)
    return grad
def Hessian(x):
    x = x.reshape(x.size)
    hessian = np.zeros((2,2))
    hessian[0][0] = -400 * (x[1] - 3*x[0]**2) + 2
    hessian[0][1] = -400 * x[0]
    hessian[1][0] =  -400 * x[0]
    hessian[1][1] = 200
    return hessian
def norm(x):
    x = x.reshape(x.size)
    return np.sqrt(x[0]**2 + x[1]**2)
```

```
color_list = ['red','blue', 'green', 'orange','cyan', 'purple',
'black','yellow', 'teal',
            'coral','brown', 'darkred']
label_list = ['Newton method', 'Gradient method with backtracking']
tol_list = [1e-1, 1e-3, 1e-5]


def plot_contour():
    X = np.arange(-1.51,1.6,0.05)
    Y = np.arange(-1.55, 2.55, 0.05)
    X,Y = np.meshgrid(X,Y)
    Z = np.zeros((X.shape[0], X.shape[1]))
    for i in range(X.shape[0]):
        for j in range(X.shape[1]):
            x = []
            x.append(X[i][j])
            x.append(Y[i][j])
            x = np.array(x)
            Z[i][j] = f(x)
    plt.contourf(X, Y, Z, 20, alpha=0.3, cmap=plt.cm.hot)
    plt.contour(X, Y, Z, 20, colors='grey')

def plot_line(xk_list, subfig_num):
    plt.figure(1)
    x = []
    y = []
    for i in range(xk_list.shape[0]):
        x.append(xk_list[i][0][0])
        y.append(xk_list[i][1][0])
    plt.plot(x,y, color = color_list[subfig_num], linewidth=1.5,
label=label_list[subfig_num])
    plt.scatter(x, y, s=3, color='black')

def plot_convergence(y, method, subfig_num, tol_index):
    tol_index = tol_index + 1
    plt.figure(2, figsize=(8,10))
    plt.subplot(3,1,tol_index)
    n = y.size
    x = np.arange(n)
    y = np.log(y)
    plt.plot(x, y, label = method, color = color_list[subfig_num],
linewidth=2)
    plt.legend()
    plt.xlabel('number of iteration
```

```
(tol={})'.format(str(tol_list[tol_index-1])))
    plt.ylabel('$log||(x^k-x^*)||$')
    plt.tight_layout()
    #plt.xlim(0,100)

def check_dir(dk_tocheck, xk, gradient, hessian, beta1, beta2):
    dk_norm = norm(dk_tocheck)
    factor1 = np.dot(gradient.T, dk_tocheck)[0][0] < 0
    facotr2 = -(np.dot(gradient.T, dk_tocheck)[0][0]) >= beta1 * np.min([1,
dk_norm**beta2]) * dk_norm**2
    if factor1 == True and facotr2 == True:
        return True
    else:
        return False
def Global_Newton(initial, subfig_num, tol, tol_index):
    #paramaters

    xstar = np.array([1, 1]).reshape(2, 1)
    s = 1
    sigma = 0.5
    gamma = 1e-4
    beta1 = 1e-6
    beta2 = 0.1
    xk_list = []
    xk_xstar_list = []
    alphak_list_Newton = []
    Always_Use_Newton_Dir = True

    xk = initial
    num_iteration = 0
    xk_list.append(xk)
    xk_xstar_list.append((norm(xk-xstar)))

    gradient = df(xk)
    while norm(gradient) > tol:
        # deteriminate the direction
        hessian = Hessian(xk)
        dk_tocheck = np.linalg.solve(hessian, -gradient)
        good_dir = check_dir(dk_tocheck, xk, gradient, hessian, beta1,
beta2)
        if(good_dir == False):
            dk = -gradient
            Always_Use_Newton_Dir = False
        else:
```

```
            dk = dk_tocheck

        alphak = s
        alphak_list_Newton.append(alphak)

        while True:
            if f(xk + alphak*dk) - f(xk) <= gamma * alphak *
(np.dot(gradient.T, dk)[0][0]):
                break
            alphak = alphak * sigma
        alphak_list_Newton.append(alphak)
        xk = xk + alphak * dk
        xk_list.append(xk)
        xk_xstar_list.append((norm(xk-xstar)))

        gradient = df(xk)
        num_iteration = num_iteration + 1

    print('tolerance:', tol)
    print('Newton_num_iteration:', num_iteration)
    print('Always_Use_Newton_Dir:', Always_Use_Newton_Dir)
    print('alpha_k of Newton method', alphak_list_Newton)
    print()

    method = 'Newton method'
    xk_xstar_list = np.array(xk_xstar_list)
    plot_convergence(xk_xstar_list, method, subfig_num, tol_index)

    if tol == 1e-5:
        plt.figure(1)
        plt.scatter(xk_list[-1][0], xk_list[-1][1], s=60, marker='*',
                    facecolors ='none', edgecolor= 'r')
        xk_list = np.array(xk_list)
        plot_line(xk_list, subfig_num)

def gradient_method(initial, subfig_num, tol, tol_index):
    xstar = np.array([1, 1]).reshape(2, 1)
    s = 1
    sigma = 0.5
    gamma = 1e-4
    xk_list = []
    xk_xstar_list = []
    alphak_list_GM = []
```

```
    xk = initial
    gradient = df(xk)
    num_iteration = 0
    xk_list.append(xk)
    xk_xstar_list.append((norm(xk-xstar)))

    while norm(gradient) > tol:
        alphak = s
        alphak_list_GM.append(alphak)

        dk = -df(xk)
        while True:
            if f(xk + alphak*dk) - f(xk) <= gamma * alphak *
(np.dot(df(xk).T, dk)[0][0]):
                break
            alphak = alphak * sigma

        alphak_list_GM.append(alphak)
        xk = xk + alphak * dk
        xk_list.append(xk)
        xk_xstar_list.append((norm(xk-xstar)))

        gradient = df(xk)
        num_iteration = num_iteration + 1

    print('tolerance:', tol)
    print('Gradient_mothd_num_iteration:', num_iteration)
    print('alpha_k of GM method', alphak_list_GM)
    print()

    method = 'Gradient method'
    xk_xstar_list = np.array(xk_xstar_list)
    plot_convergence(xk_xstar_list, method, subfig_num, tol_index)

    if tol == 1e-5:
        plt.figure(1)
        plt.scatter(xk_list[-1][0], xk_list[-1][1], s=60, marker='*',
                facecolors ='none', edgecolor= 'r')
        xk_list = np.array(xk_list)
        plot_line(xk_list, subfig_num)

# main begin
```

```python
x1 = np.arange(-10, 11, 4)
x2 = np.arange(-2, 3, 4)

plt.figure(1, figsize=(10, 5))
plot_contour()

initial = np.array([[-1.2], [1]])
plt.figure(1)
plt.scatter(initial[0], initial[1], s=40, marker='s',
                 facecolors ='none', edgecolor= 'b')

print('----------Newton Method----------')
for index, tol in enumerate(tol_list):

    start = time.time()
    Global_Newton(initial, 0, tol, index)
    end = time.time()

    print('time:', end - start)
    print()

print('----------Gradient Method----------')
for index, tol in enumerate(tol_list):

    start = time.time()
    gradient_method(initial, 1, tol, index)
    end = time.time()

    print('time:', end - start)
    print()

plt.figure(1)
plt.xlabel('x1')
plt.ylabel('x2')
plt.xlim(-1.5, 1.5)
plt.ylim(-1, 2)
plt.legend()
# plt.show()
plt.savefig('A4_1_b', dpi=700)

plt.figure(2)
plt.savefig('convergence', dpi=700)
```

A 4-2

a) The code is as follows

```
import matplotlib.pyplot as plt
import time

def norm(x):
    x = x.reshape(x.size)
    return np.sqrt(np.sum(x**2))

def norm_square(x):
    x = x.reshape(x.size)
    return np.sum(x**2)

def huber(t):
    if np.abs(t) <= delta:
        return (t**2) / (2*delta)
    else:
        return np.abs(t) - delta/2

def d_huber(t):
    if np.abs(t) <= delta:
        return t/delta
    else:
        return t/np.abs(t)

def log_fun(t):
    return np.log(1 + t**2/v)

def d_log_fun(t):
    return (1/(1+t**2/v)) * (2*t/v)

def phi_1(x):
    return norm_square(x)

def d_phi_1(x):
    grad = 2*x
    return grad

def phi_2(x):
    x = x.reshape(x.size)
    Sum = 0
    for xi in x:
        Sum = Sum + huber(xi)
```

```
    return Sum

def d_phi_2(x):
    x = x.reshape(x.size)
    grad = np.zeros((x.size,1))
    for i in range(grad.size):
        grad[i] = d_huber(x[i])
    return grad



def phi_3(x):
    x = x.reshape(x.size)
    Sum = 0
    for xi in x:
        Sum = Sum + log_fun(xi)
    return Sum

def d_phi_3(x):
    x = x.reshape(x.size)
    grad = np.zeros((x.size,1))
    for i in range(grad.size):
        grad[i] = d_log_fun(x[i])
    return grad

color_list = ['blue', 'green', 'red', 'cyan', 'purple', 'yellow',
'orange', 'teal',
              'coral', 'darkred', 'brown', 'black']
def plot_convergence(y, method, subfig_num, tol):
    plt.figure(1)
    plt.subplot(2,1,subfig_num)
    n = y.size
    x = np.arange(n)
    y = np.log(y)
    plt.plot(x, y, label = method, color = color_list[subfig_num],
linewidth=2)
    plt.legend()
    plt.xlabel('number of iteration (tol={})'.format(tol))
    plt.ylabel('$log||gradient||$')
    plt.tight_layout()

def plt_compare_and_sparse(x_solution, subfig_num):
    plt.figure(subfig_num+1, figsize=(8,20))
    plt.subplot(2,1,1)
    plt.scatter(x_solution, x_star, color=color_list[0], s=2)
```

```
    xmin = x_solution.min()
    xmax = x_solution.max()

    plt.plot([x_star.min(), x_star.max()], [x_star.min(), x_star.max()], '-
-', color='red', linewidth=1, label='diagonal line')
    plt.xlim(xmin-0.02, xmax+0.01)
    plt.xlabel('Solution')
    plt.ylabel('$x^*$')
    plt.legend()

    plt.subplot(2,1,2)
    n = x_solution.size
    plt.plot([0,n], [0,0], '--', color='red', linewidth=1)
    plt.scatter(np.arange(n)+1, x_solution, color=color_list[1], s=2)
    plt.xlabel('$i$ (the $i^{th}$ unit of solution)')
    plt.ylabel('The value of $i^{th}$ unit in solution')

    plt.tight_layout()
    plt.savefig('compare_f' +str(subfig_num), dpi=700)


def f1(x):
    return 1/2 * norm_square(np.dot(A,x) - b) + mu * phi_1(x)

def f2(x):
    return 1/2 * norm_square(np.dot(A,x) - b) + mu * phi_2(x)

def f3(x):
    return 1/2 * norm_square(np.dot(A,x) - b) + mu * phi_3(x)

def df1(x):
    return np.dot(A.T, np.dot(A, x)-b) + mu * d_phi_1(x)

def df2(x):
    return np.dot(A.T, np.dot(A, x)-b) + mu * d_phi_2(x)

def df3(x):
    return np.dot(A.T, np.dot(A, x)-b) + mu * d_phi_3(x)

def AGM(initial, smooth_func_type):
    if smooth_func_type == 1:
        df = df1
        alpha_k = 1 / L1
        method = 'AGM method on $f_1$'
```

```
    elif smooth_func_type == 2:
        df = df2
        alpha_k = 1 / L2
        method = 'AGM method on $f_2$'

    tol = 1e-4  # vary
    x_minus = initial
    xk = initial

    tk_minus = 1
    tk = 1

    xk_list = []
    xk_list.append(xk)
    norm_gradient_list = []
    num_iteration = 0

    gradient = df(xk)
    norm_gradient_list.append(norm(gradient))

    while norm(gradient) > tol:
        beta_k = (tk_minus - 1)/tk
        y = xk + beta_k * (xk - x_minus)

        x_minus = xk
        xk = y - alpha_k * df(y)
        xk_list.append(xk)

        tk_minus = tk
        tk = 1/2 * (1 + np.sqrt(1+4*tk**2))

        gradient = df(xk)
        norm_gradient_list.append(norm(gradient))
        # print(norm(gradient))
        num_iteration  = num_iteration + 1

    xk_list = np.array(xk_list)
    print(xk_list.shape)
    norm_gradient_list = np.array(norm_gradient_list)
    plot_convergence(norm_gradient_list, method, smooth_func_type, tol)

    x_solution = xk_list[-1]
    print('norm of (xk-x_star):', norm(x_solution - x_star))
    plt_compare_and_sparse(x_solution, smooth_func_type)
```

```
# main begin
#parameters

np.random.seed(2222)
n = 3000
m = 300
s = 30
mu = 1
delta = 1e-3
v = 1e-5

A = np.random.randn(m, n)
mask = np.random.choice(np.arange(1,n+1), s, replace=False)
x_star = np.zeros((n,1))
for i in range(n):
    if i+1 in mask:
        x_star[i][0] = np.random.randn(1)[0]

b = np.dot(A, x_star) + 0.01 * np.random.randn(m, 1)
L1 = 2*mu + np.linalg.norm(np.dot(A.T, A), ord = 2)
L2 = mu*(1/delta) + np.linalg.norm(np.dot(A.T, A), ord = 2)

initial = np.zeros((n,1))

plt.figure(1, figsize=(8,12))

print('----------f1----------')
start = time.clock()
AGM(initial, 1)
end = time.clock()
print('time:', end - start)
print()

print('----------f2----------')
start = time.clock()
AGM(initial, 2)
end = time.clock()
print('time:', end - start)
print()
```

```
plt.figure(1)
plt.savefig('4_2_a_convergence.png', dpi=700)
```

b) The code is as follow

```
import numpy as np
import matplotlib.pyplot as plt
import time


def norm(x):
    x = x.reshape(x.size)
    return np.sqrt(np.sum(x**2))

def norm_square(x):
    x = x.reshape(x.size)
    return np.sum(x**2)

def huber(t):
    if np.abs(t) <= delta:
        return (t**2) / (2*delta)
    else:
        return np.abs(t) - delta/2

def d_huber(t):
    if np.abs(t) <= delta:
        return t/delta
    else:
        return t/np.abs(t)

def log_fun(t):
    return np.log(1 + t**2/v)

def d_log_fun(t):
    return (1/(1+t**2/v)) * (2*t/v)

def phi_1(x):
    return norm_square(x)

def d_phi_1(x):
    grad = 2*x
    return grad
```

```
def phi_2(x):
    x = x.reshape(x.size)
    Sum = 0
    for xi in x:
        Sum = Sum + huber(xi)
    return Sum

def d_phi_2(x):
    x = x.reshape(x.size)
    grad = np.zeros((x.size,1))
    for i in range(grad.size):
        grad[i] = d_huber(x[i])
    return grad


def phi_3(x):
    x = x.reshape(x.size)
    Sum = 0
    for xi in x:
        Sum = Sum + log_fun(xi)
    return Sum

def d_phi_3(x):
    x = x.reshape(x.size)
    grad = np.zeros((x.size,1))
    for i in range(grad.size):
        grad[i] = d_log_fun(x[i])
    return grad
def f1(x):
    return 1/2 * norm_square(np.dot(A,x) - b) + mu * phi_1(x)

def f2(x):
    return 1/2 * norm_square(np.dot(A,x) - b) + mu * phi_2(x)

def f3(x):
    return 1/2 * norm_square(np.dot(A,x) - b) + mu * phi_3(x)

def df1(x):
    return np.dot(A.T, np.dot(A, x)-b) + mu * d_phi_1(x)

def df2(x):
    return np.dot(A.T, np.dot(A, x)-b) + mu * d_phi_2(x)

def df3(x):
```

```
    return np.dot(A.T, np.dot(A, x)-b) + mu * d_phi_3(x)

color_list = ['brown', 'green', 'red', 'blue', 'cyan', 'purple', 'yellow',
'orange', 'teal',
              'coral', 'darkred', 'black']
def plot_convergence(y, method, subfig_num, tol, knownL):
    plt.figure(2-knownL)

    if knownL == True:
        sum_fig = 2
    else:
        sum_fig = 3
    plt.subplot(sum_fig, 1, subfig_num)
    n = y.size
    x = np.arange(n)
    y = np.log(y)
    plt.plot(x, y, label = method, color = color_list[subfig_num],
linewidth=2)
    plt.legend()
    plt.xlabel('number of iteration (tol={})'.format(tol))
    plt.ylabel('$log||gradient||$')
    plt.tight_layout()

def plt_compare_and_sparse(x_solution, subfig_num, knownL):
    if knownL == True:
        Type = 'knwonL'
        fignum_begin = 3
    else:
        Type = 'UnknownL'
        fignum_begin = 5

    plt.figure(fignum_begin + subfig_num - 1, figsize=(8,20))
    plt.subplot(2,1,1)
    plt.scatter(x_solution, x_star, color=color_list[3], s=2)
    xmin = x_solution.min()
    xmax = x_solution.max()

    plt.plot([x_star.min(), x_star.max()], [x_star.min(), x_star.max()], '-
-', color='red', linewidth=1, label='diagonal line')
    plt.xlim(xmin-0.02, xmax+0.01)
    plt.xlabel('Solution')
    plt.ylabel('$x^*$')
    plt.legend()
```

```
    plt.subplot(2,1,2)
    n = x_solution.size
    plt.plot([0,n], [0,0], '--', color='red', linewidth=1)
    plt.scatter(np.arange(n)+1, x_solution, color=color_list[1], s=2)
    plt.xlabel('$i$ (the $i^{th}$ unit of solution)')
    plt.ylabel('The value of $i^{th}$ unit in solution')

    plt.tight_layout()
    plt.savefig('compare_f' +str(subfig_num)+'_'+Type, dpi=700)

def IGM_Known_L(inital, smooth_func_type):
    if smooth_func_type == 1:
        df = df1
        L = L1
        method = 'IGM method on $f_1$ with known L'
    elif smooth_func_type == 2:
        df = df2
        L = L2
        method = 'IGM method on $f_2$ with known L'

    tol = 1e-4
    beta = 0.5
    alpha = 1.99 * (1-beta) / L

    x_minus = initial
    xk = initial
    xk_list = []
    xk_list.append(xk)
    norm_gradient_list = []
    num_iteration = 0

    gradient = df(xk)
    norm_gradient_list.append(norm(gradient))
    while norm(gradient) > tol:
        y = xk + beta * (xk - x_minus)
        x_minus = xk
        xk = y - alpha * df(xk)

        xk_list.append(xk)

        gradient = df(xk)
        norm_gradient_list.append(norm(gradient))

        num_iteration  = num_iteration + 1
```

```
    xk_list = np.array(xk_list)
    print(xk_list.shape)
    x_solution = xk_list[-1]

    print('norm of (xk-x_star):', norm(x_solution - x_star))
    norm_gradient_list = np.array(norm_gradient_list)
    knownL = True
    plot_convergence(norm_gradient_list, method, smooth_func_type, tol,
knownL)
    plt_compare_and_sparse(x_solution, smooth_func_type, knownL)

def IGM_Unknown_L(initial, smooth_func_type):
    if smooth_func_type == 1:
        df = df1
        f = f1
        method = 'IGM method on $f_1$ with unknown L'
    elif smooth_func_type == 2:
        df = df2
        f = f2
        method = 'IGM method on $f_2$ with unknown L'
    else:
        df = df3
        f = f3
        method = 'IGM method on $f_3$ with unknown L'

    tol = 1e-4  # vary
    beta = 0.5  # vary
    l = 1  #vary
    alpha = 1.99 * (1-beta) / l

    x_minus = initial
    xk = initial

    xk_list = []
    xk_list.append(xk)

    num_iteration = 0
    norm_gradient_list = []
    gradient = df(xk)
    norm_gradient_list.append(norm(gradient))

    while norm(gradient) > tol:
```

```
        y = xk + beta * (xk - x_minus)
        xk_bar = y - alpha * df(xk)
        while f(xk_bar) - f(xk) > np.dot(df(xk).T, xk_bar-xk) + l/2 * norm_square(xk_bar-xk):
            l = 2 * l
            alpha = 1.99 * (1-beta) / l
            xk_bar = y - alpha * df(xk)

        x_minus = xk
        xk = xk_bar
        xk_list.append(xk)

        gradient = df(xk)
        norm_gradient_list.append(norm(gradient))

        num_iteration  = num_iteration + 1

    xk_list = np.array(xk_list)
    print(xk_list.shape)
    x_solution = xk_list[-1]

    print('norm of (xk-x_star):', norm(x_solution - x_star))
    norm_gradient_list = np.array(norm_gradient_list)
    knownL = False
    plot_convergence(norm_gradient_list, method, smooth_func_type, tol, knownL)
    plt_compare_and_sparse(x_solution, smooth_func_type, knownL)



# main begin
#parameters

np.random.seed(2222)
n = 3000
m = 300
s = 30
delta = 1e-3
v = 1e-4

mask = np.random.choice(np.arange(1,n+1), s, replace=False)
x_star = np.zeros((n,1))
for i in range(n):
    if i+1 in mask:
        x_star[i][0] = np.random.randn(1)[0]
```

```
A = np.random.randn(m, n)
c = 0.01 * np.random.randn(m, 1)
b = np.dot(A, x_star) + c



initial = np.zeros((n,1))

print('----------Known L----------')

# for known L: f1 and f2
plt.figure(1, figsize=(8, 12))
mu = 1
L1 = 2*mu + np.linalg.norm(np.dot(A.T, A), ord = 2)
start = time.clock()
IGM_Known_L(initial, 1)
end = time.clock()
print('f1 time:', end-start)

mu = 1
L2 = mu*(1/delta) + np.linalg.norm(np.dot(A.T, A), ord = 2)
start = time.clock()
IGM_Known_L(initial, 2)
end = time.clock()
print('f2 time:', end-start)

plt.figure(1)
plt.savefig('KnownL.png', dpi=700)


print('----------Unknown L----------')

# for unknown L: f1, f2, f3
plt.figure(2, figsize=(8, 18))

mu = 1
start = time.clock()
IGM_Unknown_L(initial, 1)
end = time.clock()
print('f1 time:', end-start)

mu = 1
start = time.clock()
IGM_Unknown_L(initial, 2)
```

```
end = time.clock()
print('f2 time:', end-start)

mu = 0.1
start = time.clock()
IGM_Unknown_L(initial, 3)
end = time.clock()
print('f3 time:', end-start)

plt.figure(2)
plt.savefig('UnknownL.png', dpi=700)
```

(1) AGM

For AGM method, we applied it into f_1 and f_2. For f_1 we choose the parameters as: μ=1. For f_2, we choose the as: μ=1， δ = 10 . We select x = 0 as the initial point and tol= 10 The performance is measured by comparing $\|\nabla f(x^k)\|$. The result is showed in Figure 4. Then, we reconstruct solution and compate them with x^*, the result is showed as Figure 5.

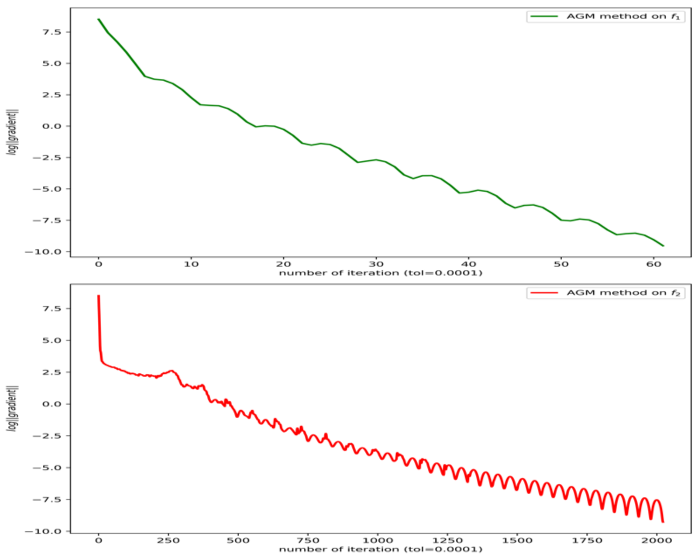
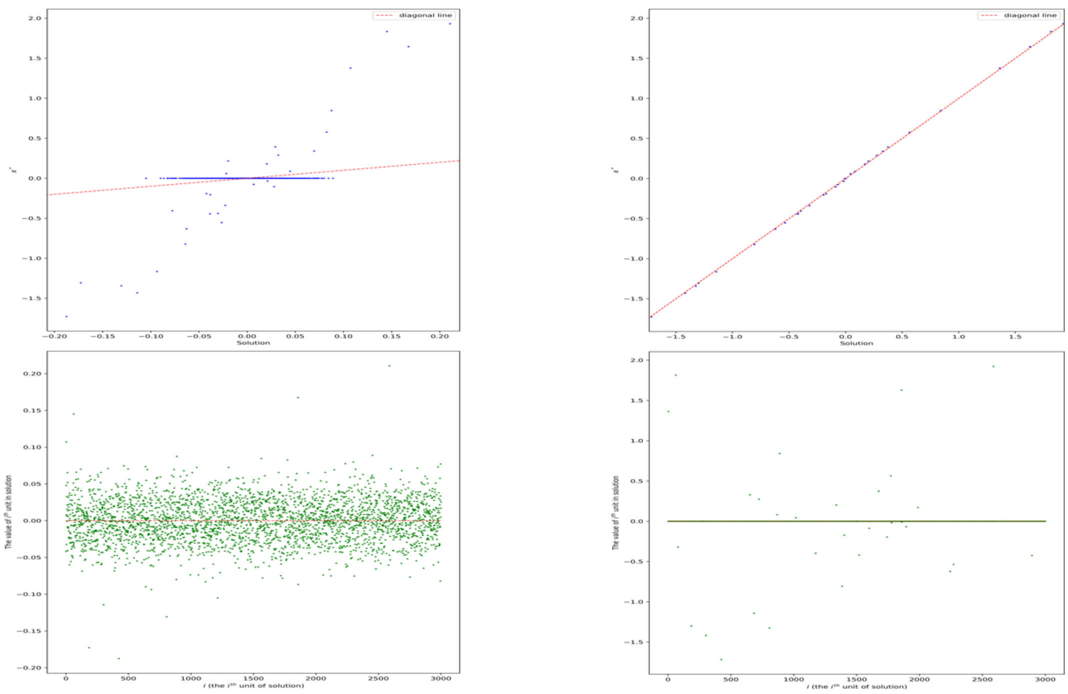


Figure 4 The plot of iteration by AGM



(a) AGM on f_1 (b) AGM on f_2

Figure 5 the comparison between solutions and x^*, and check the sparse of solutions

As we can see from Figure 4. When we apply AGM to f_1, the gradient can converge in 62 times, which is much smaller than that in f_2 (2023 times). As we can see from Figure 5(a), When using AGM, f_1 is not a good model. This is because the solution is far away from x^*, which means the solution is not what we want. Besides, the solution is not sparse, there are small number of 0 in the solution's units actually. On the contrary, as we can see from Figure 5(b), when using AGM, f_2 is really a good model. The solution is very close to x^*, because they are almost totally on the diagonal line. Besides, the solution is also sparse, there are just small number of units in the solution are not 0, but most units of them are 0, which means the solution is sparse.

(2)IGM

For IGM method, we applied it into f_1, f_2 and f_3. We seperated the experiments into two parts: "Know Lipschitz constant" and "Not know Lipschitz constant".

a): Know Lopschitz constant

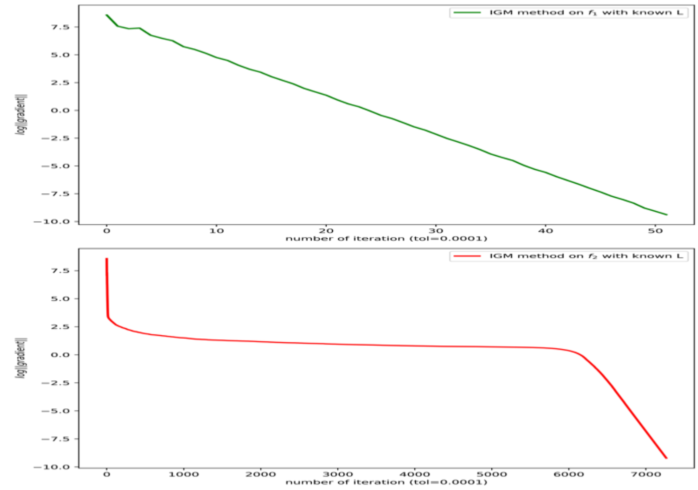


Figure 6 The plot of iteration process by IGM with known L

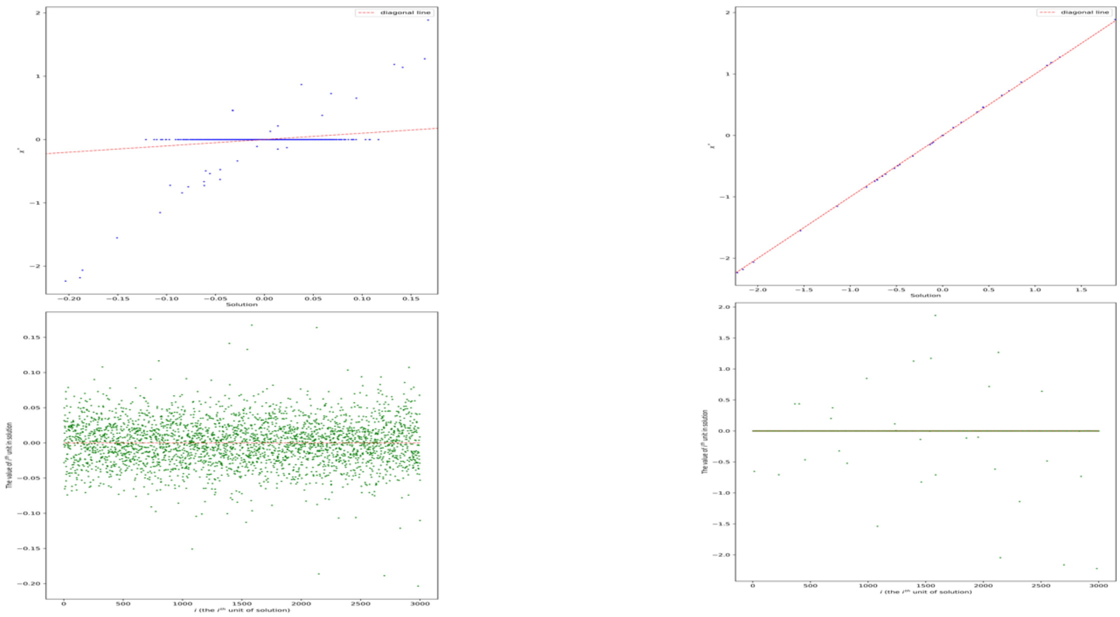


Figure 7 The comparison between solutions and X*, and check the sparse of solutions

In this part, we applied IGM into f_1 and f_2, we computed L_1 and L_2 explicitly. For f_1, we choose the parameters as: $\mu = 1$. For f_2, we choose the parameters as: $\mu = 1, \delta = 10^{-3}$. We select $x^0 = 0$ as the initial point and tol= 10 . The performance is measured by comparing $\|\nabla f(x)\|$. The result is showed in Figure 6. Then, we reconstruct solution and compate them with x^*, the result is showed as Figure 7.

As we can see from Figure 6, when we apply IGM with known L to f_1, the gradient can converge in 52 times, which is much smaller than that in f_2 (7267 times). And we can see that there is a plateau period in the convergence process of the gradient of f_2.

As we can see from Figure 7(a), When using IGM with known L, f_1 is not a good model. This is because the solution is far away from x^*, which means the solution is not what we want. Besides, the solution is not sparse, there are small number of 0 in the solution's units actually. On the contrary, as we can see from Figure 7(b), when using IGM with known L, f_2 is really a good model. The solution is very close to x^*, because they are almost totally on the diagonal line. Besides, the solution is also sparse, there are just small number of units in the solution are not 0, but most units of them are 0, which means the solution is sparse.

• (b): Not know Lipschitz constant In this part, we applied IGM into f_1, f_2 and f_3, we do not compute L_1, L_2 and L_3 explicitly. On the contrary, we treat the Lipschitz constant as unknown, and we use Algorithm 1 (mentioned before) as a variant to do IGM. For f_1, we choose the parameters as: $\mu = 1$. For f_2, we choose -3 -3 The parametersas: μ=1,δ=10 . Forf3,wechoosetheparametersas: μ=1,δ=10 ,ν= 10^{-4}. We select x = 0 as the initial point and tol= 10^{-4}. The performance is measured by comparing $\|\nabla f(x^k)\|$. The result is showed in Figure 8. Then, we reconstruct solution and compate them with x^*, the result is showed as Figure 9.

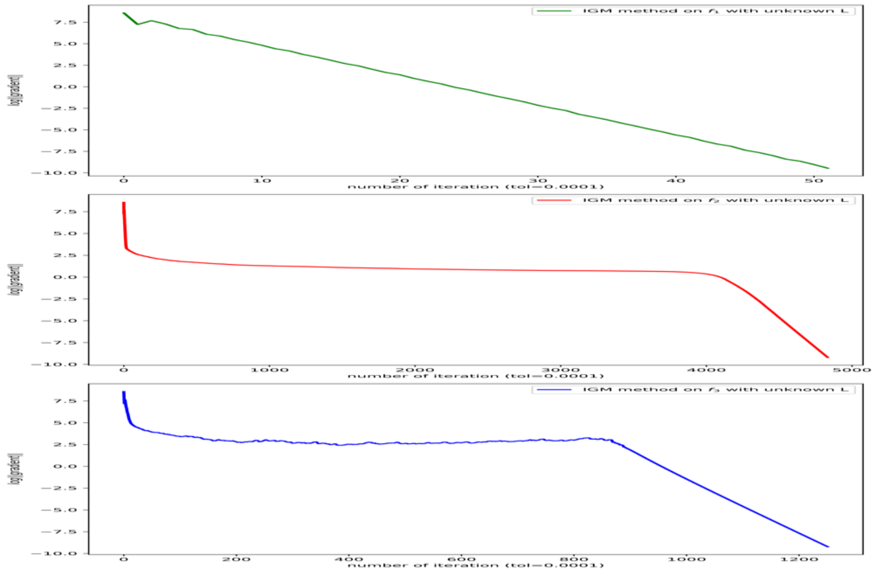


Figure 8 The plot of iteration process by IGM with unknown L

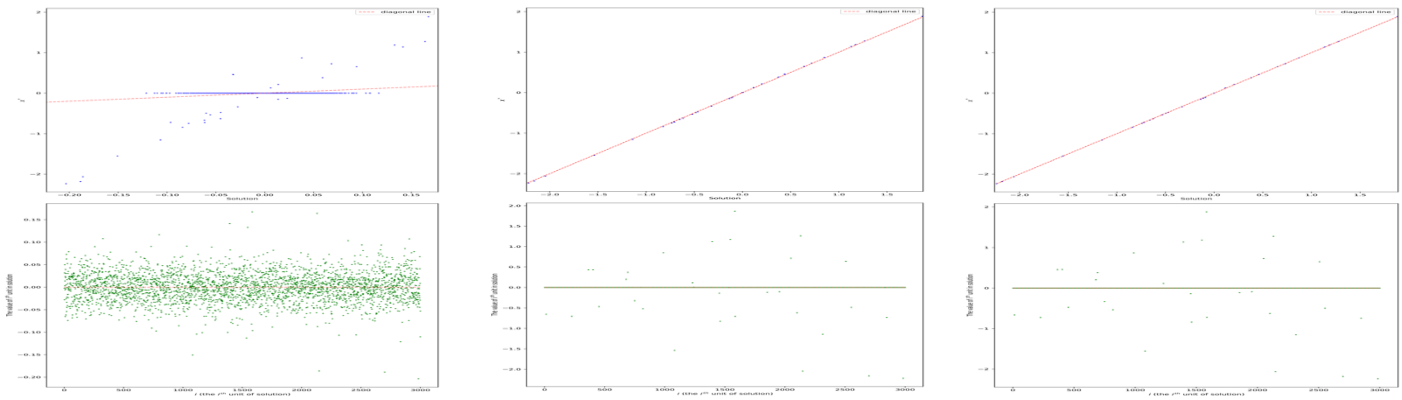


Figure 9 The comparison between solutions and X*, and check the sparse of solutions

As we can see from Figure 8, when we apply IGM with unknown L, the gradient of f_1 can converge in 52 times, the gradient of f_2 converges in 4835 times, and the gradient of f_3 converges in 1252 times. And we can see that there is a plateau period in the convergence process of the gradient of f_2 and the gradient of f_3.

As we can see from Figure 9(a), When using IGM with unknown L, f_1 is not a good model. This is because the solution is far away from x^*, which means the solution is not what we want. Besides, the solution is not sparse, there are small number of 0 in the solution's units actually. On the contrary, as we can see from Figure 9(b) and Figure 9(c), when using IGM with unknown L, f_2 and f_3 are both good models. The solution is very close to x^*, because they are almost totally on the diagonal line. Besides, the solution is also sparse, there are just small number of units in the solution are not 0, but most units of them are 0, which means the solution is sparse.

(3) Summary

Above all, we have applied AGM and IGM (with known L and unknown L) method to solve f_1, f_2 and f_3 models respectively. The iteration times is concluded as Table 3.

Method	Model	tolerance	iteration	Time(s)	$\|\|x^k - x^*\|\|$
AGM	f_1	10^{-4}	62	0.978	4.775
	f_2	10^{-4}	2023	115.113	0.085
IGM(know L)	f_1	10^{-4}	52	0.406	5.392
	f_2	10^{-4}	7276	417.809	0.086
IGM(not know L)	f_1	10^{-4}	52	0.488	5.392
	f_2	10^{-4}	4835	626.245	0.086
	f_3	10^{-4}	1252	149.43	0.036

As we can see from Table 3, model f_3 is the best in these tasks. When and do not compute L explicitly, the iteration times is 1252, which is smaller than model f_2. And the solution of model f_3 is the most close to x^*, as we can see the value $\|x^k - x^*\|$ equals 0.036, which is very small.