5. Prove the estimate (12.23).

$$\|\tau_h\|_h^2 \le 3 \left(\|f\|_h^2 + \|f\|_{L^2(0,1)}^2 \right) \qquad\qquad (12.23)$$

[*Hint*: for each internal node x_j, $j = 1, \ldots, n-1$, integrate by parts (12.21) to get

$$\tau_h(x_j)$$

$$= -u''(x_j) - \frac{1}{h^2} \left[\int_{x_j - h}^{x_j} u''(t)(x_j - h - t)^2 \, dt - \int_{x_j}^{x_j + h} u''(t)(x_j + h - t)^2 \, dt \right].$$

Then, pass to the squares and sum $\tau_h(x_j)^2$ for $j = 1, \ldots, n-1$. On noting that $(a + b + c)^2 \le 3(a^2 + b^2 + c^2)$, for any real numbers a, b, c, and applying the Cauchy-Schwarz inequality yields the desired result.]

Let $A_j = -u''(x_j)$, $B_j = -\frac{1}{h^2} \int_{x_j-h}^{x_j} u''(t)(x_j - h - t)^2 \, dt$

$$C_j = \frac{1}{h^2} \int_{x_j}^{x_j + h} u''(t)(x_j + h - t)^2 \, dt$$

using $(a + b + c)^2 \le 3(a^2 + b^2 + c^2)$, we get

$$\tau_h(x_j)^2 = (A_j + B_j + C_j)^2 \le 3(A_j^2 + B_j^2 + C_j^2)$$

Sum over $j = 1, \ldots, n-1$ and multiply by h to form the discrete L^2 norm

$$\|\tau_h\|_h^2 = h \sum_{j=1}^{n-1} \tau_h(x_j)^2 \le 3h \sum_{j=1}^{n-1} A_j^2 + 3h \sum_{j=1}^{n-1} B_j^2 + 3h \sum_{j=1}^{n-1} C_j^2$$

Now estimate the last two sums by Cauchy-Schwarz applied to each integral. For the first integral in B_j,

$$\int_{x_j-h}^{x_j} u''(t)(x_j - h - t)^2 \, dt \le \int_{x_j-h}^{x_j} (u''(t))^2 \, dt \cdot \int_{x_j-h}^{x_j} (x_j - h - t)^4 \, dt$$

Change variable $s = t - (x_j - h) \in [0, h]$ in the last integral to get

$$\int_{x_j-h}^{x_j} (x_j - h - t)^4 \, dt = \int_0^h s^4 \, ds = \frac{h^5}{5}$$

Thus,

$$B_j^2 \le \frac{1}{h^4} \cdot \frac{h^5}{5} \int_{x_j-h}^{x_j} (u''(t))^2 \, dt = \frac{h}{5} \int_{x_j-h}^{x_j} (u''(t))^2 \, dt$$

An identical calculation for C_j yields $\quad C_j^2 \le \frac{h}{5} \int_{x_j}^{x_j+h} (u''(t))^2 \, dt$

Insert these bounds into the sum:

$$\|z_h\|_h^2 \leq 3h \sum_{j=1}^{n-1} (u''(x_j))^2 + 3h \sum_{j=1}^{n-1} \frac{h}{5} \int_{x_j-h}^{x_j} (u''(t))^2 dt + 3h \sum_{j=1}^{n-1} \frac{h}{5} \int_{x_j}^{x_j+h} (u''(t))^2 dt$$

$$\because \quad h \sum_{j=1}^{n-1} (u''(x_j))^2 = \|u''\|_h^2$$

$$\sum_{j=1}^{n-1} \int_{x_j-h}^{x_j} (u''(t))^2 dt + \sum_{j=1}^{n-1} \int_{x_j}^{x_j+h} (u''(t))^2 dt \leq 2 \int_0^1 (u''(t))^2 dt = 2 \|u''\|_{L^2(0,1)}^2$$

$$\therefore \quad \|z_h\|_h^2 \leq 3 \|u''\|_h^2 + \frac{3}{5} h^2 \cdot 2 \|u''\|_{L^2(0,1)}^2 = 3 \|u''\|_h^2 + \frac{6h^2}{5} \|u''\|_{L^2(0,1)}^2$$

Finally use the PDE relation $-u'' = f$, so $u'' = -f$.
Hence $\|u''\|_h = \|f\|_h$ and $\|u''\|_{L^2} = \|f\|_{L^2}$. Also $h \leq 1$.
Thus,

$$\|z_h\|_h^2 \leq 3 \left(\|f\|_h^2 + \|f\|_{L^2(0,1)}^2 \right) \quad \square$$

7. Let $g = 1$ and prove that $T_h g(x_j) = \frac{1}{2}x_j(1-x_j)$.

[*Solution*: use the definition (12.25) with $g(x_k) = 1$, $k = 1,\ldots,n-1$ and recall that $G^k(x_j) = hG(x_j, x_k)$ from the exercise above. Then

$$T_h g(x_j) = h\left[\sum_{k=1}^{j} x_k(1-x_j) + \sum_{k=j+1}^{n-1} x_j(1-x_k)\right]$$

from which, after straightforward computations, one gets the desired result.]

$$w_h = T_h g, \qquad w_h = \sum_{k=1}^{n-1} g(x_k)G^k. \qquad (12.25)$$

$$T_h g(x_j) = h\left[\sum_{k=1}^{j} x_k(1-x_j) + \sum_{k=j+1}^{h-1} x_j(1-x_k)\right]$$

$$= h\sum_{k=1}^{j} kh(1-x_j) + h\sum_{k=j+1}^{h-1} x_j(1-kh) \qquad \sum_{k=j+1}^{h-1}k = \sum_{k=1}^{h-1}k - \sum_{k=1}^{j}k$$

$$= h^2(1-x_j)\cdot\frac{j(j+1)}{2} + hx_j\left[(h-1)\cdot(j+1)+1 - h\cdot\left(\frac{(h-1)h}{2} - \frac{j(j+1)}{2}\right)\right]$$

$$\left(h-j-1 - \frac{h-1}{2} + \frac{j(j+1)}{2h} \overset{!!}{=} \frac{h-1}{2} - j + \frac{j(j+1)}{2h}\right)$$

$h = \frac{1}{h}$

$x_j = jh = \frac{j}{h}$

$$= \frac{1}{h^2}\left(1-\frac{j}{h}\right)\cdot\frac{j(j+1)}{2} + \frac{j}{h^2}\left[\frac{h-1}{2} - j + \frac{j(j+1)}{2h}\right]$$

$$= \left(\frac{1}{h^2} - \frac{j}{h^3}\right)\cdot\frac{j(j+1)}{2} + \frac{j(h-1)}{2h^2} - \frac{j^2}{h^2} + \frac{j^2(j+1)}{2h^3}$$

$$= \frac{j^2 + jh}{2h^2} - \frac{j^2}{h^2} = \frac{1}{2}\cdot x_j^2 + \frac{1}{2}\cdot x_j - x_j^2 = \frac{1}{2}(x_j - x_j^2)$$

$$= \frac{1}{2}x_j(1-x_j) \qquad \square$$

8. Prove Young's inequality (12.40).

$$ab \le \varepsilon a^2 + \frac{1}{4\varepsilon}b^2, \qquad \forall a, b \in \mathbb{R}, \quad \forall \varepsilon > 0. \qquad\qquad (12.40)$$

Fix any $\varepsilon > 0$, and for $a, b \in \mathbb{R}$,

$$\left(\sqrt{\varepsilon}a - \frac{b}{2\sqrt{\varepsilon}}\right)^2 \ge 0 \qquad \text{or} \qquad \varepsilon a^2 - 2\sqrt{\varepsilon}a \cdot \frac{b}{2\sqrt{\varepsilon}} + \frac{b^2}{4\varepsilon} \ge 0$$

Thus,

$$ab \le \varepsilon a^2 + \frac{b^2}{4\varepsilon}, \qquad \forall a, b \in \mathbb{R}, \quad \forall \varepsilon > 0 \qquad \square$$

9. Show that $\|v_h\|_h \le \|v_h\|_{h,\infty} \ \forall v_h \in V_h$.

Let the mesh be $x_j = jh$ for $j = 0, 1, \dots, N$ with $Nh = 1$ and define

$$\|v_h\|_h := \left(h \sum_{j=1}^{N-1} v_j^2\right)^{\frac{1}{2}}, \quad \|v_h\|_{h,\infty} := \max_{1 \le j \le N-1} |v_j|$$

For every index j we have $|v_j| \le \|v_h\|_{h,\infty}$, hence

$$\sum_{j=1}^{N-1} v_j^2 \le \sum_{j=1}^{N-1} \|v_h\|_{h,\infty}^2 = (N-1)\|v_h\|_{h,\infty}^2$$

Multiplying by h and taking square roots gives

$$\|v_h\|_h = \left(h \sum_{j=1}^{N-1} v_j^2\right)^{\frac{1}{2}} \le \sqrt{h(N-1)} \, \|v_h\|_{h,\infty}$$

Since $h(N-1) = 1 - h \le 1$, we obtain $\sqrt{h(N-1)} \le 1$, and therefore

$$\|v_h\|_h \le \|v_h\|_{h,\infty} \qquad \square$$

11. Discretize the fourth-order differential operator $Lu(x) = -u^{(iv)}(x)$ using centered finite differences.

[*Solution*: apply twice the second order centered finite difference operator L_h defined in (12.9).]

$$(L_h W_h)(x_j) = -\frac{W_{j+1} - 2W_j + W_{j-1}}{h^2}$$

$$(L_h^2 W)_j = L_h(L_h W)_j = -\frac{(L_h W)_{j+1} - 2(L_h W)_j + (L_h W)_{j-1}}{h^2}$$

$$= -\frac{1}{h^2}\left[-\frac{W_{j+2} - 2W_{j+1} + W_j}{h^2} + 2\frac{W_{j+1} - 2W_j + W_{j-1}}{h^2} - \frac{W_j - 2W_{j-1} + W_{j-2}}{h^2} \right]$$

$$= \frac{1}{h^4}\left(W_{j+2} - 2W_{j+1} + W_j - 2W_{j+1} + 4W_j - 2W_{j-1} + W_j - 2W_{j-1} + W_{j-2} \right)$$

$$= \frac{W_{j+2} - 4W_{j+1} + 6W_j - 4W_{j-1} + W_{j-2}}{h^4} \qquad \square$$