HW2

5. Prove that

$$(n-1)!h^{n-1}|(x-x_{n-1})(x-x_n)| \le |\omega_{n+1}(x)| \le n!h^{n-1}|(x-x_{n-1})(x-x_n)|,$$

where n is even, $-1 = x_0 < x_1 < \ldots < x_{n-1} < x_n = 1$, $x \in (x_{n-1}, x_n)$ and $h = 2/n$.
[*Hint* : let $N = n/2$ and show first that

$$\omega_{n+1}(x) = (x+Nh)(x+(N-1)h)\ldots(x+h)x \\ (x-h)\ldots(x-(N-1)h)(x-Nh). \qquad (8.74)$$

Then, take $x = rh$ with $N-1 < r < N$.]

$h = \frac{2}{n}$. let $N = \frac{n}{2}$, we have

$x_0 = -1 = -Nh$; $x_1 = -1+h = -Nh+h = -(N-1)h$

$\ldots$; $x_{N-1} = -h$; $x_N = 0$; $x_{N+1} = h$; $\ldots$; $x_{n-1} = -(\frac{n}{2}-n+1)h = (N-1)h$, $x_n = 1 = Nh$

$$\therefore\ \omega_{n+1}(x) = \prod_{i=0}^{n}(x-x_i) = \prod_{i=0}^{2N}[x-(-(N-i))h]$$

let $j = i-N$

$$= \prod_{j=-N}^{N}(x-jh) = (x-0)\cdot\prod_{j=-N}^{-1}(x-jh)\cdot\prod_{j=1}^{N}(x-jh)$$

$$= x\cdot\prod_{j=1}^{N}(x+jh)\cdot\prod_{j=1}^{N}(x-jh)$$

Then we proved (8.74)

$x \in (x_{n-1}, x_n) = ((N-1)h, Nh)$. Take $x = rh$ with $N-1 < r < N$

$$\therefore\ \omega_{n+1}(rh) = rh\cdot\prod_{j=1}^{N}[(r+j)h]\cdot\prod_{j=1}^{N}[(r-j)h]$$

$$= rh^{1+n}\cdot\prod_{j=1}^{N}(r+j)\cdot\prod_{j=1}^{N}(r-j)$$

$$|(x-x_{n-1})(x-x_n)| = |[rh-(N-1)h][rh-Nh]| = h^2|[r-(N-1)](r-N)|$$

$$\Rightarrow |\omega_{n+1}(x)| = h^{n+1}|r|\cdot\prod_{j=1}^{N}(r+j)\cdot\prod_{j=1}^{N}(r-j)$$

$$= |(x-x_{n-1})(x-x_n)|\cdot h^{n-1}\cdot|r|\prod_{j=1}^{N}(r+j)\cdot\prod_{j=1}^{N-2}(r-j) := P(r)$$

When $N-1 < r < N$,

① $r-(N-2) > (N-1)-(N-2) = 1$

$r-(N-3) > (N-1)-(N-3) = 2$

$\vdots$

$r-1 > (N-1)-1 = N-2$

$r > N-1$

$r+1 > (N-1)+1 = N$

$r+2 > (N-1)+2 = N+1$

$\vdots$

$r+N > (N-1)+N = 2N-1$

$\Rightarrow$ $P(r) > 1\cdot 2\cdots(N-2)(N-1)\cdot N\cdot(N+1)\cdots(2N-1)$

$= (2N-1)! = (n-1)!$

(U) $r-(N-2) < N-(N-2) = 2$

$r-(N-3) < N-(N-3) = 3 \quad \Rightarrow \quad P(r) < 2\cdot 3 \cdots (N-1)N(N+1)$

$\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (N+2)\cdots(2N)$

$r-1 < N-1 \qquad\qquad\qquad\qquad\qquad\qquad = (2N)! = n!$

$r < N$

$r+1 < N+1$

$r+2 < N+2$

$\vdots$

$r+N < 2N$

$\therefore \left|(x-x_{n-1})(x-x_n)\right| \cdot h^{n-1} \cdot (n-1)! \le \left|W_{n+1}(x)\right| \le \left|(x-x_{n-1})(x-x_n)\right| \cdot h^{n-1} \cdot n!$

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6. Under the assumptions of Exercise 5, show that $|\omega_{n+1}|$ is maximum if $x \in (x_{n-1}, x_n)$ (notice that $|\omega_{n+1}|$ is an even function).
[*Hint* : use (8.74) to prove that $|\omega_{n+1}(x+h)/\omega_{n+1}(x)| > 1$ for any $x \in (0, x_{n-1})$ with x not coinciding with any interpolation node.]

n: even

$$W_{n+1}(-x) = (-x) \cdot \prod_{j=1}^{N} (-x+jh) \cdot \prod_{j=1}^{N} (-x-jh) = (-1)^{1+n} \; W_{n+1}(x) = -W_{n+1}(x)$$

$\therefore$ W_{n+1} is even function.

$$\left| \frac{W_{n+1}(x+h)}{W_{n+1}(x)} \right| = \left| \frac{(x+h) \prod_{j=1}^{N} (x+h+jh) \cdot \prod_{j=1}^{N} (x+h-jh)}{x \cdot \prod_{j=1}^{N} (x+jh) \cdot \prod_{j=1}^{N} (x-jh)} \right|$$

$$= \left| \frac{\cancel{(x+h) \cdot [(x+2h) \cdots (x+Nh)}(x+(N+1)h)] \, [\cancel{x(x-h) \cdots (x+(2-N)h)} + \overset{(x-(N-1)h)}{\overset{\shortparallel}{(x+(1-N)h)}}]}{\cancel{x \cdot [(x+h)(x+2h) \cdots (x+Nh)] \, [(x-h)(x-2h) \cdots (x-(N-1)h)}) \cdot (x-Nh)]} \right|$$

$$= \left| \frac{x+(N+1)h}{x-Nh} \right|$$

$$(N+1)h < x + (N+1)h < (2N+1)h$$

If $x \in (0, x_n) = (0, Nh)$; $-Nh < x - Nh < 0$

$$\therefore \left| \frac{W_{n+1}(x+h)}{W_{n+1}(x)} \right| > \left| \frac{(N+1)h}{-Nh} \right| = \frac{N+1}{N} > 1$$

$\Rightarrow |W_{n+1}(x+h)| > |W_{n+1}(x)|$ if $x \in (0, x_n) = (0,1)$ — $(*)$

Consider $[0,1]$, $0 = x_N < x_{N+1} < \cdots < x_{2N} = 1$

define $I_k := (x_{N+k}, x_{N+k+1}) = (kh, (k+1)h)$, $k = 0, \cdots, N-1$

$m_k := \sup_{x \in I_k} |W_{n+1}(x)|$

If $x \in I_k$, then $x+h \in I_{k+1}$

By $(*)$, $|W_{n+1}(x+h)| > |W_{n+1}(x)|$, $\forall x \in I_k$

$$\therefore \underbrace{\sup_{y \in I_{k+1}} |W_{n+1}(y)|}_{m_{k+1}} = \sup_{x \in I_k} |W_{n+1}(x+h)| > \sup_{x \in I_k} |W_{n+1}(x)| = m_k$$

$\therefore$ $m_0 < m_1 < \cdots < m_{N-1}$ which implies $|W_{n+1}|$ is maximum if $x \in I_{N-1} = (x_{2N-1}, x_{2N}) = (x_{n-1}, x_n)$

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8. Determine an interpolating polynomial $Hf \in \mathbb{P}_n$ such that

$$(Hf)^{(k)}(x_0) = f^{(k)}(x_0), \qquad k = 0, \dots, n,$$

and check that

$$Hf(x) = \sum_{j=0}^{n} \frac{f^{(j)}(x_0)}{j!}(x - x_0)^j,$$

that is, the Hermite interpolating polynomial on one node coincides with the *Taylor polynomial*.

For a single node x_0, the Hermite polynomial can be written as $Hf(x) = \sum_{j=0}^{n} a_j (x-x_0)^j$, where the coefficients a_j are determined by the derivative conditions.

$$(Hf)^{(k)}(x) = \sum_{j=k}^{n} a_j \frac{j!}{(j-k)!}(x-x_0)^{j-k}$$

At $x = x_0$, $(x-x_0)^{j-k} = \begin{cases} 0, & \text{if } j > k \\ 1, & \text{if } j = k \end{cases}$

$\therefore (Hf)^{(k)}(x_0) = a_k \cdot \frac{k!}{0!} \cdot 1 = a_k \cdot k!$

From Hermite condition $(Hf)^{(k)}(x_0) = f^{(k)}(x_0)$, we get

$a_k \cdot k! = f^{(k)}(x_0) \Rightarrow a_k = \frac{f^{(k)}(x_0)}{k!}$, $k = 0, 1, \cdots, n$

, therefore

$$Hf(x) = \sum_{j=0}^{n} \frac{f^{(j)}(x_0)}{j!}(x-x_0)^j$$

this is exactly the Taylor polynomial of f at x_0. #

2. Show for $(n+1)$ Chebyshev points of the second kind, the barycentric weights are (after rescaling)

$$w_i = (-1)^i, \quad i=1, \cdots, n-1 \quad \text{and} \quad w_0 = \frac{1}{2}, \quad w_n = \frac{(-1)^n}{2}$$

<pf>

$$U_n(x) = \frac{\sin((n+1)\cos^{-1}x)}{\sin(\cos^{-1}x)}, \quad x \in [-1,1] \quad ; \quad U_n(\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}$$

Let $x_k = \cos\left(\frac{k\pi}{n}\right)$, $k = 0, 1, \cdots, n$

$$\frac{\sin(n+1)\theta}{\sin\theta} = \frac{\sin(n\theta)\cos\theta}{\sin\theta} + \frac{\cos(n\theta)\sin\theta}{\sin\theta}$$

$$\Rightarrow \quad U_n(\cos\theta) = U_{n-1}(\cos\theta)\cdot\cos\theta + T_n(\cos\theta)$$

$$\Rightarrow \quad \underset{\cos(n\cos^{-1}x)}{\underset{\shortparallel}{T_n(x)}} = U_n(x) - x\cdot U_{n-1}(x)$$

$$\therefore \quad T_n'(x) = -\sin(n\cos^{-1}x)\cdot\frac{-n}{\sqrt{1-x^2}} = \frac{n\sin(n\cos^{-1}x)}{\sqrt{1-x^2}}$$

Let $w_j = \dfrac{1}{\prod\limits_{\substack{k=0 \\ k\neq j}}^{n}(x_j - x_k)}$, $w_{n+1}(x) = \prod\limits_{k=0}^{n}(x - x_k)$

$x_1, \cdots, x_{n-1}$ be roots of $U_{n-1}(x)$, thus $\prod\limits_{k=1}^{n-1}(x-x_k) = \dfrac{U_{n-1}(x)}{2^{n-1}}$

$\Bigg($ $U_0(x) = 1$, $U_1(x) = 2x$

Suppose the leading coefficient of $U_k(x)$ is 2^k for $0 \le k \le n-1$

From $U_n(x) = 2x\cdot U_{n-1}(x) - U_{n-2}(x)$, we have the leading coefficient of U_n is $2\cdot 2^{n-1} = 2^n$; thus $U_{n-1}(x) = 2^{n-1}\cdot\prod\limits_{k=1}^{n-1}(x-x_k)$ $\Bigg)$

$$\therefore \quad w_{n+1}(x) = (x-1)(x+1)\prod_{k=1}^{n-1}(x-x_k) = (x^2-1)\frac{U_{n-1}(x)}{2^{n-1}}$$

$$w_{n+1}'(x) = \frac{1}{2^{n-1}}\left[2x\cdot U_{n-1}(x) + (x^2-1)\cdot U_{n-1}'(x)\right]$$

$$\therefore W_{n+1}'(1) = \frac{2n}{2^{n-1}} = \frac{n}{2^{n-2}} \Rightarrow W_0 = \frac{2^{n-2}}{n}$$

$$U_{n-1}(x) = \frac{\sin(n\cos^{-1}x)}{\sin(\cos^{-1}x)}, \quad U_{n-1}(1) = \frac{\sin(0)}{\sin(0)} \ \left(\frac{0}{0}\right)$$

$$\lim_{\theta \to 0} U_{n-1}(\cos\theta) = \lim_{\theta \to 0} \frac{\sin(n\theta)}{\sin\theta} = \lim_{\theta \to 0} \left(\frac{\sin(n\theta)}{n\theta} \cdot \frac{n\theta}{\sin\theta}\right) = n$$

$$W_{n+1}'(-1) = \frac{1}{2^{n-1}}(-2) \cdot U_{n-1}(-1) = \frac{-1}{2^{n-2}} \cdot (-1)^{n-1} n = \frac{(-1)^n}{2^{n-2}} \cdot n$$

$$\lim_{\theta \to \pi} U_{n-1}(\cos\theta) = \lim_{\theta \to \pi} \frac{\sin(n\theta)}{\sin\theta}$$

Let $\theta = \pi + \phi$

$\therefore \phi \to 0$ if $\theta \to \pi$

$$= \lim_{\phi \to 0} \frac{(-1)^n \sin(n\phi)}{-\sin\phi}$$

$\sin\theta = \sin(\pi + \phi) = -\sin\phi$

$\sin(n\theta) = \sin(n\pi + n\phi)$

$= \sin(n\pi)\cos(n\phi) + \cos(n\pi)\sin(n\phi)$

$= 0 + (-1)^n \sin(n\phi)$

$$= (-1)^{n-1} \lim_{\phi \to 0} \frac{\sin(n\phi)}{\sin\phi}$$

$$= (-1)^{n-1} \cdot n$$

$$\Rightarrow W_n = (-1)^n \cdot \frac{2^{n-2}}{n}$$

After scaling $\left(\times \frac{n}{2^{n-1}}\right)$, $W_0 = \frac{2^{n-2}}{n} \cdot \frac{n}{2^{n-1}} = \frac{1}{2}$

$$W_n = (-1)^n \cdot \frac{2^{n-2}}{n} \cdot \frac{n}{2^{n-1}} = \frac{(-1)^n}{2}$$

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