9. Given the following set of data

(*) $\{f_0 = f(-1) = 1,\ f_1 = f'(-1) = 1,\ f_2 = f'(1) = 2,\ f_3 = f(2) = 1\}$,

prove that the Hermite-Birkoff interpolating polynomial H_3 does not exist for them.
[*Solution* : letting $H_3(x) = a_3x^3 + a_2x^2 + a_1x + a_0$, one must check that the matrix of the linear system $H_3(x_i) = f_i$ for $i = 0, \ldots, 3$ is singular.]

Let $H_3(x) = a_3 x^3 + a_2 x^2 + a_1 x + a_0$, $H_3'(x) = 3a_3 x^2 + 2a_2 x + a_1$

From condition (*):
$$\begin{cases} -a_3 + a_2 - a_1 + a_0 = 1 \\ 3a_3 - 2a_2 + a_1 = 1 \\ 3a_3 + 2a_2 + a_1 = 2 \\ 8a_3 + 4a_2 + 2a_1 + a_0 = 1 \end{cases}$$

Write in matrix form :
$$\underbrace{\begin{pmatrix} -1 & 1 & -1 & 1 \\ 3 & -2 & 1 & 0 \\ 3 & 2 & 1 & 0 \\ 8 & 4 & 2 & 1 \end{pmatrix}}_{\text{Let } A} \begin{pmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}$$

$$\det(A) = \begin{vmatrix} -1 & 1 & -1 & 1 \\ 3 & -2 & 1 & 0 \\ 3 & 2 & 1 & 0 \\ 9 & 3 & 3 & 0 \end{vmatrix} = - \begin{vmatrix} 3 & -2 & 1 \\ 3 & 2 & 1 \\ 9 & 3 & 3 \end{vmatrix} \times(-1)$$

$$= - \begin{vmatrix} 3 & -2 & 1 \\ 0 & 4 & 0 \\ 9 & 3 & 3 \end{vmatrix} = -4 \cdot \begin{vmatrix} 3 & 1 \\ 9 & 3 \end{vmatrix} = -4(9-9) = 0$$

Since the coefficient matrix A is singular ($\det(A) = 0$), the linear system has no solution.
Thus, the Hermite - Birkhoff interpolating polynomial H_3 does not exist for the given data. □

12. Let $f(x) = \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$; then, consider the following rational approximation

$$r(x) = \frac{a_0 + a_2x^2 + a_4x^4}{1 + b_2x^2}, \tag{8.75}$$

called the *Padé approximation*. Determine the coefficients of r in such a way that

$$f(x) - r(x) = \gamma_8 x^8 + \gamma_{10} x^{10} + \ldots.$$

[*Solution*: $a_0 = 1$, $a_2 = -7/15$, $a_4 = 1/40$, $b_2 = 1/30$.]

We want $f(x) - r(x) = \cos x - \dfrac{a_0 + a_2x^2 + a_4x^4}{1 + b_2x^2} = \gamma_8 x^8 + \gamma_{10} x^{10} + \ldots$

iff $(1 + b_2x^2) f(x) - (a_0 + a_2x^2 + a_4x^4) = O(x^8)$

Expand $(1 + b_2x^2) f(x) = (1 + b_2x^2)\cos x$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots\right) + \left(b_2x^2 - \frac{b_2x^4}{2!} + \frac{b_2x^6}{4!} - \frac{b_2x^8}{6!} + \frac{b_2x^{10}}{8!} - \ldots\right)$$

$$= 1 - \left(\frac{1}{2!} - b_2\right)x^2 + \left(\frac{1}{4!} - \frac{b_2}{2!}\right)x^4 - \left(\frac{1}{6!} - \frac{b_2}{4!}\right)x^6 + \left(\frac{1}{8!} - \frac{b_2}{6!}\right)x^8 - \ldots$$

Collect terms by powers of x, and set up the coefficient equations

$$(1 + b_2x^2) f(x) - (a_0 + a_2x^2 + a_4x^4)$$
$$= (1 - a_0) - \left(\frac{1}{2} - b_2 + a_2\right)x^2 + \left(\frac{1}{24} - \frac{b_2}{2} - a_4\right)x^4 - \left(\frac{1}{720} - \frac{b_2}{24}\right)x^6 + \left(\frac{1}{8!} - \frac{b_2}{6!}\right)x^8 - \ldots$$
$$= O(x^8)$$

For this to hold, the coefficients of x^0, x^2, x^4, x^6 must be zero:

$$\begin{cases} 1 - a_0 = 0 \\ \frac{1}{2} - b_2 + a_2 = 0 \\ \frac{1}{24} - \frac{b_2}{2} - a_4 = 0 \\ \frac{1}{720} - \frac{b_2}{24} = 0 \end{cases} \Rightarrow \begin{cases} a_0 = 1 \\ b_2 = \frac{24}{720} = \frac{1}{30} \\ a_2 = b_2 - \frac{1}{2} = \frac{-14}{30} = -\frac{7}{15} \\ a_4 = \frac{1}{24} - \frac{b_2}{2} = \frac{1}{24} - \frac{1}{60} = \frac{5-2}{120} = \frac{1}{40} \end{cases}$$

Therefore, the Padé approximation is: $r(x) = \dfrac{1 - \frac{7}{15}x^2 + \frac{1}{40}x^4}{1 + \frac{1}{30}x^2}$ $\square$

Verification:

coefficient of $x^8 = \dfrac{1}{8!} - \dfrac{\frac{1}{30}}{6!} \neq 0$