1. Consider the boundary value problem (12.1)-(12.2) with $f(x) = 1/x$. Using (12.3) prove that $u(x) = -x\log(x)$. This shows that $u \in C^2(0,1)$ but $u(0)$ is not defined and u', u'' do not exist at $x = 0$ ($\Rightarrow$: if $f \in C^0(0,1)$, but not $f \in C^0([0,1])$, then u does not belong to $C^0([0,1])$).

For $x \in (0,1)$:

$$u(x) = \int_0^x s(1-x)\frac{1}{s}ds + \int_x^1 x(1-s)\frac{1}{s}ds$$

$$= \int_0^x (1-x)ds + x\int_x^1 (\frac{1}{s}-1)ds$$

$$= x(1-x) + x\left(\ln|s|-s\right)\Big|_{s=x}^{s=1}$$

$$= x - x^2 + x(-1-\ln x + x) = -x\ln x$$

$u(x) = -x\ln x \in C^2(0,1)$: $u'(x) = -\ln x - 1$, $u''(x) = -\frac{1}{x}$

As $x \to 0^+$: $u'(x) \longrightarrow \infty$, $u''(x) \to -\infty$.

So u, u' and u'' do not exist at $x = 0$.

4. Cerify the summation by parts formula

$$\sum_{j=0}^{n-1}(w_{j+1}-w_j)v_j = w_n v_n - w_0 v_0 - \sum_{j=0}^{n-1}(v_{j+1}-v_h)w_{j+1},$$

and show that, for $v_h \in V_h^0$,

$$(L_h v_h, v_h)_h = h^{-1}\sum_{j=0}^{n-1}(v_{j+1}-v_j)^2.$$

①.

$$\begin{aligned}\sum_{j=0}^{n-1}(W_{j+1}-W_j)V_j &= \sum_{j=0}^{n-1} W_{j+1}V_j - \sum_{j=0}^{n-1} W_j V_j \\ &= \sum_{j=1}^{n} W_j V_{j-1} - \sum_{j=0}^{n-1} W_j V_j \\ &= \sum_{j=1}^{n-1} W_j (V_{j-1}-V_j) + W_n V_{n-1} - W_0 V_0 \\ &= -\sum_{j=1}^{n-1} W_j (V_j - V_{j-1}) + W_n V_{n-1} - W_0 V_0 \\ &= -\sum_{j=0}^{n-1} W_{j+1}(V_{j+1}-V_j) + W_n(V_n - V_{n-1}) + W_n V_{n-1} - W_0 V_0 \\ &= W_n V_n - W_0 V_0 - \sum_{j=0}^{n-1}(V_{j+1}-V_j)W_{j+1}\end{aligned}$$

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②. L_h is defined by $(L_h V_h)_j = -\dfrac{V_{j+1}-2V_j+V_{j-1}}{h^2}$, $j=1,\cdots,n-1$

and $(U_h, V_h)_h = h\sum_{j=1}^{n-1} u_j V_j$

So $(L_h V_h, V_h)_h = h\sum_{j=1}^{n-1}\dfrac{-V_{j+1}+2V_j-V_{j-1}}{h^2}V_j = -\dfrac{1}{h}\sum_{j=1}^{n-1}(V_{j+1}-2V_j+V_{j-1})V_j$

$$\begin{aligned}-\sum_{j=1}^{n-1}(V_{j+1}-2V_j+V_{j-1})V_j &= -\sum_{j=1}^{n-1}(V_{j+1}-V_j)V_j + \sum_{j=1}^{n-1}(V_j - V_{j-1})V_j \\ &= -\sum_{j=1}^{n-1}(V_{j+1}-V_j)V_j + \sum_{j=0}^{n-2}(V_{j+1}-V_j)V_{j+1} \\ &= \sum_{j=0}^{n-1}(V_{j+1}-V_j)^2 + (V_1 - V_0)\underset{0}{\underset{\shortparallel}{V_0}} - (V_n - V_{n-1})\underset{0}{\underset{\shortparallel}{V_n}}\end{aligned}$$

Thus, $(L_h V_h, V_h)_h = \dfrac{1}{h}\sum_{j=0}^{n-1}(V_{j+1}-V_j)^2$

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6. Prove that $G^k(x_j) = hG(x_j, x_k)$, where G is Green's function introduced in (12.4) and G^k is its corresponding discrete counterpart solution of (12.4).

<u>Recall</u>: $G^k \in \overset{\circ}{V_h}$ as the solution to $L_h G^k = e^k$

, where $e^k(x_j) = \begin{cases} 1, & \text{if } k = j \\ 0, & \text{if } k \neq j \end{cases}$

$$G(x,s) = \begin{cases} s(1-x), & 0 \le s \le x \\ x(1-s), & x \le s \le 1 \end{cases} \quad ; \quad L_h(v)(x_j) = -\frac{v(x_{j+1}) - 2v(x_j) + v(x_{j-1})}{h^2}$$

<pf>

To show $L_h(hG(x_j, x_k)) = \delta_{jk}$ and satisfies boundary condition

① Boundary : $hG(\overset{\overset{0}{\shortparallel}}{x_0}, x_k) = 0$ & $hG(\overset{\overset{1}{\shortparallel}}{x_n}, x_k) = 0$

② $\boxed{j \neq k}$: Fix x_k , $G(x, x_k) = \begin{cases} x_k(1-x), & 0 \le x_k \le x \\ x(1-x_k), & x \le x_k \le 1 \end{cases}$

Since Green function is a straight line for $j \neq k$

$$, L_h(hG(x_j, x_k)) = -\frac{G(x_{j+1}, x_k) - 2G(x_j, x_k) + G(x_{j-1}, x_k)}{h} = 0$$

- $f(x) = ax + b$, $x_{j+1} = (j+1)h = jh + h = x_j + h$; $x_{j-1} = x_j - h$
- $f(x_{j+1}) - 2f(x_j) + f(x_{j-1}) = [a(x_j + h) + b] - 2(ax_j + b) + [a(x_j - h) + b] = 0$

$\boxed{j = k}$: $L_h(hG(x_k, x_k)) = -\dfrac{G(x_{k+1}, x_k) - 2G(x_k, x_k) + G(x_{k-1}, x_k)}{h}$

$$= -\frac{-h}{h} = 1$$

分子 $= x_k(1 - x_{k+1}) - 2x_k(1 - x_k) + x_{k-1}(1 - x_k)$

$= x_k(1 - x_k - h) - 2x_k + 2x_k^2 + (x_k - h)(1 - x_k)$

$= x_k - x_k^2 - x_k h - 2x_k + 2x_k^2 + x_k - x_k^2 - h + x_k h = -h$