1. Let $E_0(f)$ and $E_1(f)$ be the quadrature errors in (9.6) and (9.12). Prove that $|E_1(f)| \simeq 2|E_0(f)|$.

If $f \in C^2([a,b])$, the quadrature error is

$$E_0(f) = \frac{h^3}{3} f''(\xi),\ h = \frac{b-a}{2}, \qquad (9.6)$$

where ξ lies within the interval (a,b).

If $f \in C^2([a,b])$, the quadrature error is given by

$$E_1(f) = -\frac{h^3}{12} f''(\xi),\ h = b-a, \qquad (9.12)$$

where ξ is a point within the integration interval.

<pf>

$E_0(f) = \frac{(b-a)^3}{24} f''(\alpha)$, $E_1(f) = -\frac{(b-a)^3}{12} f''(\beta)$ where $\alpha, \beta \in (a,b)$

$$\frac{|E_1(f)|}{|E_0(f)|} = \left| \frac{\frac{(b-a)^3}{12} f''(\beta)}{\frac{(b-a)^3}{24} f''(\alpha)} \right| = 2 \cdot \frac{|f''(\beta)|}{|f''(\alpha)|}$$

assumption : $f \in C^2([a,b])$; $\exists m > 0$ s.t $|f''(x)| \geq m$, $\forall x \in [a,b]$

Claim: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t if $H = b-a < \delta$ then $\left| \frac{|E_1(f)|}{|E_0(f)|} - 2 \right| < \varepsilon$

By Claim, then we have $|E_1(f)| \approx 2|E_0(f)|$

<pf of claim>

Since f'' is continuous on a compact set $[a,b]$, then f'' is uniform continuous on $[a,b]$, i.e $\forall K > 0$, $\exists \delta > 0$ s.t if $|x-y| < \delta$, then $|f''(x) - f''(y)| < K$

Since $|\alpha - \beta| < b-a = H$, if $H < \delta$ then $|f''(\alpha) - f''(\beta)| < K$.

$$\left| \frac{|E_1(f)|}{|E_0(f)|} - 2 \right| = 2 \left| \frac{|f''(\beta)|}{|f''(\alpha)|} - 1 \right| = 2 \cdot \frac{\left| |f''(\beta)| - |f''(\alpha)| \right|}{|f''(\alpha)|} \leq 2 \frac{|f''(\beta) - f''(\alpha)|}{m}$$

Let $\varepsilon > 0$ be given. Take $K = \frac{m\varepsilon}{4}$. If $H < \delta$, then $\left| \frac{|E_1(f)|}{|E_0(f)|} - 2 \right| \leq 2 \cdot \frac{m\varepsilon}{4m} < \varepsilon$ $\square$

3. Let $I_n(f) = \sum_{k=0}^{n} \alpha_k f(x_k)$ be a Lagrange quadrature formula on $n+1$ nodes. Compute the degree of exactness r of the formulae:
(a) $I_2(f) = (2/3)[2f(-1/2) - f(0) + 2f(1/2)]$,
(b) $I_4(f) = (1/4)[f(-1) + 3f(-1/3) + 3f(1/3) + f(1)]$.
Which is the order of infinitesimal p for (a) and (b)?
[*Solution*: $r = 3$ and $p = 5$ for both $I_2(f)$ and $I_4(f)$.]

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sol. ① $f(x) = 1$

$I_2(1) = \frac{2}{3}[2 - 1 + 2] = \frac{2}{3} \cdot 3 = 2$

$I_4(1) = \frac{1}{4} \cdot 8 = 2$

$\int_{-1}^{1} 1\,dx = 2$ ✓ ✓

② $f(x) = x$

$I_2(x) = \frac{2}{3}[2 \cdot (-\frac{1}{2}) - 0 + 2 \cdot \frac{1}{2}] = \frac{2}{3} \cdot 0 = 0$

$I_4(x) = \frac{1}{4}[-1 + 3 \cdot (-\frac{1}{3}) + 3 \cdot (\frac{1}{3}) + 1] = 0$

$\int_{-1}^{1} x\,dx = 0$ ✓ ✓

③ $f(x) = x^2$, $I_2(x^2) = \frac{2}{3}[2 \cdot \frac{1}{4} - 0 + 2 \cdot \frac{1}{4}] = \frac{2}{3}$

$I_4(x^2) = \frac{1}{4}[1 + 3 \cdot \frac{1}{9} + 3 \cdot \frac{1}{9} + 1] = \frac{2}{3}$

$\int_{-1}^{1} x^2\,dx = \frac{1}{3}x^3 \Big|_{-1}^{1} = \frac{2}{3}$ ✓

④ $f(x) = x^3$, $I_2(x^3) = \frac{2}{3}[2(-\frac{1}{8}) - 0 + 2 \cdot \frac{1}{8}] = 0$

$\int_{-1}^{1} x^3\,dx = 0$ ✓

$I_4(x^3) = \frac{1}{4}[-1 + 3 \cdot (-\frac{1}{27}) + 3 \cdot \frac{1}{27} + 1] = 0$

⑤ $f(x) = x^4$, $I_2(x^4) = \frac{2}{3}[2 \cdot \frac{1}{16} + 0 + 2 \cdot \frac{1}{16}] = \frac{2}{3} \cdot \frac{1}{4} = \frac{1}{6}$

$I_4(x^4) = \frac{1}{4}[1 + 3 \cdot \frac{1}{3^4} + 3 \cdot \frac{1}{3^4} + 1] = \frac{1}{4}(2 + \frac{2}{27}) = \frac{1}{4} \cdot \frac{56}{27} = \frac{14}{27}$

$\int_{-1}^{1} x^4\,dx = \frac{1}{5}x^5 \Big|_{-1}^{1} = \frac{2}{5}$

Since $I_2(x^4) \neq \int_{-1}^{1} x^4\,dx$, $I_4(x^4) \neq \int_{-1}^{1} x^4\,dx$

, the formula are not exact for $f(x) = x^4$

Therefore, the degree of exactness r for $I_2(f)$ / $I_4(f)$ is 3.

The order of infinitesimal p is $3+2=5$

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5. Let $I_w(f) = \int_0^1 w(x)f(x)dx$ with $w(x) = \sqrt{x}$, and consider the quadrature formula $Q(f) = af(x_1)$. Find a and x_1 in such a way that Q has maximum degree of exactness r.
[*Solution*: $a = 2/3$, $x_1 = 3/5$ and $r = 1$.]

Exact for $f(x)=1$: $a \cdot 1 = \int_0^1 \sqrt{x} \cdot 1 dx = \frac{2}{3}x^{\frac{3}{2}} \Big|_0^1 = \frac{2}{3} \Rightarrow a = \frac{2}{3}$

Exact for $f(x)=x$: $a \cdot x_1 = \int_0^1 \sqrt{x} \cdot x dx = \int_0^1 x^{\frac{3}{2}} dx = \frac{2}{5}x^{\frac{5}{2}} \Big|_0^1 = \frac{2}{5} \Rightarrow x_1 = \frac{2}{5} \cdot \frac{3}{2} = \frac{3}{5}$

Verification accuracy :

$f(x) = x^2$, $Q(x^2) = \frac{2}{3} \cdot (\frac{3}{5})^2 = \frac{2}{3} \cdot \frac{9}{25} = \frac{6}{25}$

$I_w(x^2) = \int_0^1 \sqrt{x} \cdot x^2 dx = \int_0^1 x^{\frac{5}{2}} dx = \frac{2}{7} x^{\frac{7}{2}} \Big|_0^1 = \frac{2}{7}$

$\therefore$ the formula is not exact for $f(x) = x^2$

$\therefore$ $a = \frac{2}{3}$, $x_1 = \frac{3}{5}$, $r = 1$

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6. Let us consider the quadrature formula $Q(f) = \alpha_1 f(0) + \alpha_2 f(1) + \alpha_3 f'(0)$ for the approximation of $I(f) = \int_0^1 f(x)dx$, where $f \in C^1([0,1])$. Determine the coefficients α_j, for $j = 1, 2, 3$ in such a way that Q has degree of exactness $r = 2$.
[*Solution*: $\alpha_1 = 2/3$, $\alpha_2 = 1/3$ and $\alpha_3 = 1/6$.]

① $f(x) = 1$, $f'(x) = 0$
$Q(1) = I(1)$; $\alpha_1 + \alpha_2 = \int_0^1 1\,dx = 1$

② $f(x) = x$, $f'(x) = 1$
$Q(x) = I(x)$; $\alpha_2 + \alpha_3 = \int_0^1 x\,dx = \frac{1}{2}x^2\Big|_0^1 = \frac{1}{2}$

③ $f(x) = x^2$, $f'(x) = 2x$
$Q(x^2) = I(x^2)$; $\alpha_2 = \int_0^1 x^2\,dx = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3}$

$\therefore \alpha_3 = \frac{1}{2} - \alpha_2 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$, $\alpha_1 = 1 - \alpha_2 = 1 - \frac{1}{3} = \frac{2}{3}$

$\therefore \alpha_1 = \frac{2}{3}$, $\alpha_2 = \frac{1}{3}$, $\alpha_3 = \frac{1}{6}$ #