Spectral Methods for Two-Point Boundary Value Problems

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1 Introduction and Problem Statement

Spectral methods are widely used to solve differential equations by approximating solutions with truncated series, such as Chebyshev polynomials. Derivatives are then computed using spectral differentiation matrices, which map function values at discrete nodes to their derivatives.

A major challenge arises as the number of nodes N increases: these differentiation matrices become ill-conditioned, amplifying roundoff errors and degrading the accuracy of computed derivatives, especially for higher-order derivatives.

In this study, we focus on linear two-point boundary value problems with constant coefficients:

$$Lu := u'' + \mu u' + \nu u = f(x), \quad x \in [-1, 1],$$

subject to Dirichlet boundary conditions $u(-1) = \alpha$ and $u(1) = \beta$. Standard spectral collocation methods can be unstable due to this ill-conditioning. To overcome this issue, the article proposes a **spectral integration method**, which reformulates the problem as an integral equation for u''. This approach allows for stable and spectrally accurate computation of both the solution and its derivatives. In the next section, we present the spectral integration formulation and discuss its numerical implementation.

2 Introduction and Problem Statement

Spectral methods approximate solutions of differential equations using truncated series, such as Chebyshev polynomials, with derivatives computed via spectral differentiation matrices. A well-known difficulty is that these matrices become ill-conditioned as the number of nodes N increases, leading to amplified roundoff errors and reduced accuracy, particularly for higher-order derivatives.

We study linear two-point boundary value problems of the form

$$Lu := u'' + \mu u' + \nu u = f(x), \qquad x \in [-1, 1],$$

with Dirichlet boundary conditions $u(-1) = \alpha$ and $u(1) = \beta$. Standard spectral collocation may become unstable due to the ill-conditioning of differentiation matrices.

To address this issue, the article adopts a **spectral integration method**, which reformulates the problem as an integral equation for $u''(x)$. This leads to a well-conditioned linear system for the Chebyshev coefficients and enables stable, spectrally accurate computation of the solution and its derivatives.

3 Method Explanation

3.1 Chebyshev Series Fundamentals

For a smooth function $f(x)$ on $[-1, 1]$, the Chebyshev expansion is:

$$f(x) = \sum_{k=0}^{\infty} a'_k T_k(x) \tag{1}$$

where $T_k(x) = \cos(k \arccos x)$ are Chebyshev polynomials and the prime notation indicates that a_0 is halved.

The coefficients are computed via:

$$a_k = \frac{2}{\pi} \int_{-1}^{1} f(x) T_k(x) (1 - x^2)^{-1/2} dx \tag{2}$$

3.2 Spectral Differentiation vs. Integration

For a Chebyshev expansion

$$f(x) = \sum_{k=0}^{N}{}' a_k T_k(x),$$

spectral differentiation and integration are defined term by term.

The derivative has the expansion

$$f'(x) = \sum_{k=0}^{N}{}' b_k T_k(x), \qquad b_k = \sum_{\substack{p=k+1 \\ p+k \text{ odd}}}^{N} p\, a_p,$$

which follows from $T'_p(x) = p\, U_{p-1}(x)$, where U_{p-1} is the Chebyshev polynomial of the second kind. Spectral differentiation is **ill-conditioned**, with error amplification $\sim O(N^2)$ in the ℓ_∞ norm.

For integration, let

$$I_n(x) = \int_{-1}^{x} T_n(t)\, dt.$$

Using the substitution $t = \cos \phi$ and $T_n(\cos \phi) = \cos(n\phi)$, one obtains

$$I_n(x) = \begin{cases} T_1(x) + T_0(x), & n = 0, \\ \dfrac{T_2(x) - T_0(x)}{4}, & n = 1, \\ \dfrac{1}{2} \left[\dfrac{T_{n+1}(x)}{n+1} - \dfrac{T_{n-1}(x)}{n-1} \right] + \mu_n, & n \geq 2, \end{cases}$$

where μ_n is a constant from the lower limit of integration.

Thus, the integral of $f(x)$ can be written as

$$\int_{-1}^{x} f(t)\, dt = \sum_{k=0}^{N}{}' d_k T_k(x),$$

with coefficients

$$d_k = \frac{a_{k-1} - a_{k+1}}{2k}, \quad k \geq 1, \qquad d_0 = \sum_{k=1}^{N} (-1)^{k+1} d_k.$$

Spectral integration is **well-conditioned** in the ℓ_∞ norm, in contrast to differentiation.

3.3 Integral Reformulation of Second-Order BVPs

Consider the one-dimensional Poisson equation

$$u''(x) = f(x), \qquad u(-1) = \alpha, \quad u(1) = \beta.$$

Rather than setting up recurrence relations as in other approaches, it is convenient to write a formal solution in integral form:

$$u(x) = \int_{-1}^{x} \int_{-1}^{t} f(\tau)\, d\tau dt + C_1 x + C_0,$$

where the constants C_1 and C_0 are chosen to satisfy the boundary conditions. The spectral integration matrix provides an efficient numerical means to compute this solution.

This approach can be generalized to a second-order boundary value problem:

$$u'' + \mu u' + \nu u = f(x), \quad u(-1) = \alpha, \quad u(1) = \beta.$$

Instead of solving for u directly, we define $\sigma(x) = u''(x)$ and write

$$u'(x) = \int_{-1}^{x} \sigma(t)\, dt + C_1, \qquad u(x) = \int_{-1}^{x} \int_{-1}^{t} \sigma(\tau)\, d\tau dt + C_1 x + C_0,$$

with C_0 and C_1 determined by the boundary conditions. Substituting these expressions into the original equation yields

$$\sigma(x) + \mu \left(\int_{-1}^{x} \sigma(t) dt + C_1 \right) + \nu \left(\int_{-1}^{x} \int_{-1}^{t} \sigma(\tau) d\tau dt + C_1 x + C_0 \right) = f(x). \tag{3}$$

3.4 Spectral Discretization

We expand

$$\sigma(x) = \sum_{k=0}^{N}{}' a_k T_k(x), \qquad f(x) = \sum_{k=0}^{N}{}' f_k T_k(x),$$

and apply the Chebyshev spectral integration formulas. The single integral becomes

$$\int_{-1}^{x} \sigma(t) dt = \sum_{k=1}^{N} d_k T_k(x) + C_1, \quad d_k = \frac{a_{k-1} - a_{k+1}}{2k}, \ k \geq 1,$$

and the double integral is

$$\int_{-1}^{x} \int_{-1}^{t} \sigma(\tau) d\tau dt = \sum_{k=2}^{N} u_k T_k(x) + \left(C_1 - \frac{d_2}{2} \right) x + C_0,$$

with

$$u_k = \frac{d_{k-1} - d_{k+1}}{2k} = \frac{1}{2k}\left(\frac{a_{k-2} - a_k}{2k-2} - \frac{a_k - a_{k+2}}{2k+2}\right), \quad k \geq 2.$$

Matching Chebyshev coefficients yields the linear system:

$$\begin{aligned} k = 0: &\quad a_0 + \mu C_1 + \nu C_0 = f_0, \\ k = 1: &\quad a_1 + \frac{\mu}{2}(a_0 - a_2) + \frac{\nu}{8}(8C_1 - a_1 + a_3) = f_1, \\ k \geq 2: &\quad a_k + \frac{\mu}{2k}(a_{k-1} - a_{k+1}) + \frac{\nu}{2k}\left(\frac{a_{k-2} - a_k}{2k-2} - \frac{a_k - a_{k+2}}{2k+2}\right) = f_k. \end{aligned}$$

The boundary conditions

$$u(-1) = \alpha, \qquad u(1) = \beta$$

add two dense equations to the system. Using $T_k(1) = 1$ and $T_k(-1) = (-1)^k$, the resulting $(N+3) \times (N+3)$ system consists of a pentadiagonal interior block augmented by two dense boundary rows, and can be solved in $O(N)$ work (about $10N$ flops). Using DCT and Clenshaw recursion, the overall computation scales as $O(N \log N)$.

4 Experiment

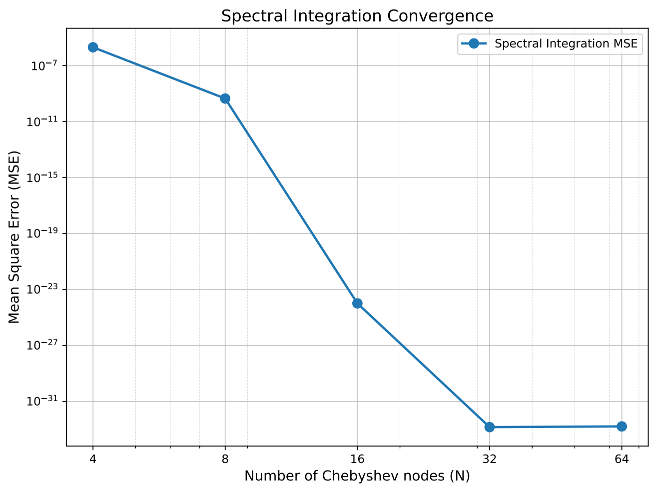
The following demonstrates the numerical solution of the Poisson equation using the Chebyshev spectral method:

$$u''(x) = \sin(\pi x), \quad x \in [-1, 1], \quad u(-1) = u(1) = 0,$$

whose exact solution is

$$u(x) = -\frac{\sin(\pi x)}{\pi^2}.$$

The approach first computes function values at Chebyshev nodes, then obtains the Chebyshev coefficients via the discrete Chebyshev transform (DCT-I). Using spectral integration, the coefficients are integrated twice to approximate $u(x)$, and Clenshaw recursion is applied to evaluate $u(x)$ at the nodes. Constants are then adjusted to satisfy the boundary conditions.



(a) Convergence plot for $u''(x) = \sin(\pi x)$.

N	CPU Time (s)	Mean Square Error
4	0.000999	2.051×10^{-6}
8	0.000997	4.446×10^{-10}
16	0.000585	1.018×10^{-24}
32	0.002010	1.461×10^{-33}
64	0.001885	1.637×10^{-33}

(b) CPU times and mean square errors.

5 Summary

The article presents a spectral integration method for solving linear two-point boundary value problems with constant coefficients. The method reformulates the differential equation as an integral equation and uses Chebyshev expansions with a spectral integration matrix, which provides stable and accurate computation of both the solution and its derivatives. This makes it particularly effective in problems where higher-order derivatives are needed and where standard spectral differentiation may be unstable.

The method works well under the assumption of smooth solutions and constant coefficients. Its efficiency relies on the use of Chebyshev nodes and the spectral integration matrix, allowing computations to scale as $O(N \log N)$.

Limitations arise when dealing with variable-coefficient problems. In such cases, the linear systems for the Chebyshev coefficients become dense, making direct solution computationally expensive. The performance of iterative methods depends on the complexity of the problem and may require many iterations for solutions with complicated behavior.

Potential improvements include developing efficient algorithms for variable-coefficient problems or extending the approach to nonlinear or higher-dimensional boundary value problems, while maintaining the stability and spectral accuracy observed for constant-coefficient cases.