7. Prove that the *gamma function*

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt, \qquad z \in \mathbb{C}, \quad \mathrm{Re} z > 0,$$

is the solution of the difference equation $\Gamma(z+1) = z\Gamma(z)$
[*Hint*: integrate by parts.]

$$\Gamma(z+1) = \int_0^{\infty} e^{-t} \cdot t^{z} dt = -t^{z} \cdot e^{-t} \Big|_0^{\infty} + \int_0^{\infty} e^{-t} \cdot z \cdot t^{z-1} dt$$

Let $u = t^{z}$, $dv = e^{-t} dt$
$du = z \cdot t^{z-1} dt$, $v = -e^{-t}$

$$= 0 + z \int_0^{\infty} e^{-t} \cdot t^{z-1} dt = z \cdot \Gamma(z) \quad \square$$

$$-t^{z} \cdot e^{-t} \Big|_0^{\infty} = \lim_{a \to \infty} -a^{z} \cdot e^{-a} + \lim_{b \to 0^+} b^{z} \cdot e^{-b}$$

$$= 0 + 0 = 0$$

9. Consider the following family of one-step methods depending on the real parameter α

$$u_{n+1} = u_n + h[(1 - \frac{\alpha}{2}) f(x_n, u_n) + \frac{\alpha}{2} f(x_{n+1}, u_{n+1})].$$

Study their consistency as a function of α; then, take $\alpha = 1$ and use the corresponding method to solve the Cauchy problem

$$\begin{cases} y'(x) = -10y(x), & x > 0, \\ y(0) = 1. \end{cases}$$

Determine the values of h in correspondance of which the method is absolutely stable.
[*Solution*: the family of methods is consistent for any value of α. The method of highest order (equal to two) is obtained for $\alpha = 1$ and coincides with the Crank-Nicolson method.]

For the exact solution $y(x)$, the local truncation error is:

$$\tau_{n+1} = \frac{y(x_{n+1}) - y(x_n)}{h} - \left[(1 - \frac{\alpha}{2}) f(x_n, y(x_n)) + \frac{\alpha}{2} f(x_{n+1}, y(x_{n+1}))\right]$$

Expand $y(x_{n+1})$ around x_n using Taylor series:

$$y(x_{n+1}) = y(x_n + h) = y(x_n) + h y'(x_n) + \frac{h^2}{2} y''(x_n) + \frac{h^3}{6} y'''(x_n) + O(h^4)$$

So $$\frac{y(x_{n+1}) - y(x_n)}{h} = y'(x_n) + \frac{h}{2} y''(x_n) + \frac{h^2}{6} y'''(x_n) + O(h^3)$$

Since $y'(x) = f(x, y(x))$ we have $y'(x_n) = f(x_n, y(x_n))$, $y'(x_{n+1}) = f(x_{n+1}, y(x_{n+1}))$

Expand $y'(x_{n+1})$ using Taylor series:

$$y'(x_{n+1}) = y'(x_n + h) = y'(x_n) + h y''(x_n) + \frac{h^2}{2} y'''(x_n) + O(h^3) = f(x_{n+1}, y(x_{n+1}))$$

$$(1-\tfrac{\alpha}{2}) f(x_n, y(x_n)) + \tfrac{\alpha}{2} f(x_{n+1}, y(x_{n+1}))$$

$$= (1-\tfrac{\alpha}{2}) f(x_n, y(x_n)) + \tfrac{\alpha}{2}\left[y'(x_n) + h y''(x_n) + \tfrac{h^2}{2} y'''(x_n) + O(h^3)\right]$$

$$= y'(x_n) + \tfrac{\alpha h}{2} y''(x_n) + \tfrac{\alpha h^2}{4} y'''(x_n) + O(h^3)$$

So $Z_{n+1} = y''(x_n)\left(\frac{h}{2} - \frac{\alpha h}{2}\right) + y'''(x_n)\left(\frac{h^2}{6} - \frac{\alpha h^2}{4}\right) + O(h^3)$

$$= \frac{h(1-\alpha)}{2} y''(x_n) + h^2\left(\frac{1}{6} - \frac{\alpha}{4}\right) y'''(x_n) + O(h^3)$$

The local truncation error is $Z_{n+1} = \frac{h(1-\alpha)}{2} y''(x_n) + O(h^2)$

Since $Z_{n+1} = O(h)$ for any value of α, the method is consistent for all α.

- For $\alpha \neq 1$, the method has order 1
- For $\alpha = 1$, the leading term vanishes, so $Z_{n+1} = O(h^2)$, giving second-order accuracy.

This confirms that $\alpha = 1$ gives the highest order method.

When $\alpha = 1$, the numerical scheme becomes:

$U_{n+1} = U_n + \frac{h}{2}[f(x_n, U_n) + f(x_{n+1}, U_{n+1})]$ (*)

For the differential equation $y'(x) = -10y(x)$, we have $f(x,y) = -10y$

Substitute into (*): $U_{n+1} = U_n + \frac{h}{2}(-10U_n - 10U_{n+1})$

$$= U_n - 5hU_n - 5hU_{n+1}$$

So $U_{n+1} = \frac{1-5h}{1+5h} U_n$

This is a linear recurrence relation, $U_{n+1} = rU_n$ with $r = \frac{1-5h}{1+5h}$

By induction, we can get $U_n = r^n U_0 = \left(\frac{1-5h}{1+5h}\right)^n U_0$

Given $y(0) = 1$, we have $U_0 = 1$. Therefore $U_n = \left(\frac{1-5h}{1+5h}\right)^n$

A numerical method is absolutely stable if $|U_n| \le |U_0|$, $\forall n$

So we need $|r^n| \le 1$, $\forall n$. This requires $|r| = \left|\frac{1-5h}{1+5h}\right| \le 1$

① . $0 < h < \frac{1}{5}$

Then $1-5h > 0$, and $|r| = \frac{1-5h}{1+5h}$

For stability, we need : $\frac{1-5h}{1+5h} \leq 1 \Rightarrow 1-5h \leq 1+5h$

$\Rightarrow -10h \leq 0$

This always true for $h > 0$

②. $h = \frac{1}{5}$

$|r| = \left|\frac{1-1}{1+1}\right| = 0 \leq 1$.

③. $h > \frac{1}{5}$

Then $1-5h < 0$, and $|r| = \frac{5h-1}{1+5h}$

For stability, we need : $5h-1 \leq 1+5h \Rightarrow -1 \leq 1$ always true.

Thus, $\alpha = 1$ is absolutely stable for all $h > 0$. $\square$