1. Prove that Heun's method has order 2 with respect to h.
[*Hint*: notice that $h\tau_{n+1} = y_{n+1} - y_n - h\Phi(t_n, y_n; h) = E_1 + E_2$, where

$$E_1 = \int_{t_n}^{t_{n+1}} f(s, y(s))ds - \frac{h}{2}[f(t_n, y_n) + f(t_{n+1}, y_{n+1})]$$

and

$$E_2 = \frac{h}{2}\{[f(t_{n+1}, y_{n+1}) - f(t_{n+1}, y_n + hf(t_n, y_n))]\},$$

where E_1 is the error due to numerical integration with the trapezoidal method and E_2 can be bounded by the error due to using the forward Euler method.]

<pf>

$$y_{n+1} = y_n + \frac{h}{2}\left[f_n + f(t_{n+1}, u_n + hf_n)\right]$$

Let $\Phi(t_n, y_n; h) = \frac{1}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_n + hf(t_n, y_n))\right]$

$$\tau_{n+1} = \frac{y_{n+1} - y_n}{h} - \Phi(t_n, y_n; h) = \frac{1}{h}\int_{t_n}^{t_{n+1}} f(s, y(s))ds - \Phi(t_n, y_n; h)$$

$$E_1 = \int_{t_n}^{t_{n+1}} f(s, y(s))ds - \frac{h}{2}[f(t_n, y_n) + f(t_{n+1}, y_{n+1})]$$ $\rightarrow$ Trapezoidal approximation error, $O(h^3)$

$$E_2 = \frac{h}{2}\{[f(t_{n+1}, y_{n+1}) - f(t_{n+1}, y_n + hf(t_n, y_n))]\}$$

$y(t_{n+1}) = y(t_n) + h y'(t_n) + \frac{h^2}{2} y''(c)$, $c \in (t_n, t_{n-1})$

$\Rightarrow y_{n+1} = y_n + h \cdot f(t_n, y_n) + O(h^2)$

$\Rightarrow f(t_{n+1}, y_{n+1}) = f(t_{n+1}, y_n + hf(t_n, y_n)) + O(h^2) \Rightarrow E_2 = \frac{h}{2} \cdot O(h^2) = O(h^3)$

So $\tau_{n+1} = \dfrac{E_1 + E_2}{h} = \dfrac{O(h^3)}{h} = O(h^2)$ $\square$

2. Prove that the Crank-Nicoloson method has order 2 with respect to h.
[*Solution*: using (9.12) we get, for a suitable ξ_n in (t_n, t_{n+1})

$$y_{n+1} = y_n + \frac{h}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right] - \frac{h^3}{12} f''(\xi_n, y(\xi_n))$$

or, equivalently,

$$\frac{y_{n+1} - y_n}{h} = \frac{1}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right] - \frac{h^2}{12} f''(\xi_n, y(\xi_n)). \qquad (11.90)$$

Therefore, relation (11.9) coincides with (11.90) up to an infinitesimal of order 2 with respect to h, provided that $f \in C^2(I)$.]

<pf>

Let $y(t)$ be the exact solution of the ODE $y' = f(t, y)$, and set $g(t) := f(t, y(t))$

On the time interval $[t_n, t_{n+1}]$ with $h = t_{n+1} - t_n$, apply the trapezoidal formula g:

$$\int_{t_n}^{t_{n+1}} g(t)\,dt = \frac{h}{2}\left(g(t_n) + g(t_{n+1})\right) - \frac{h^3}{12} g''(\xi_n)$$

for some $\xi_n \in (t_n, t_{n+1})$

Because $y_{n+1} - y_n = \int_{t_n}^{t_{n+1}} g(t)\,dt$, we obtain

$$y_{n+1} - y_n = \frac{h}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right] - \frac{h^3}{12} g''(\xi_n)$$

Let $$z_{n+1} = \frac{y_{n+1} - y_n}{h} - \frac{1}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right]$$

Then $z_{n+1} = O(h^2)$ $\square$