Algebraic Topology

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Contents

Lecture 1: Introduction

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- First part: fundamental groups. We follow Munkres:
  - Chap 9 'Fundamental group'
  - Chap 11 'The Seifert-Van Kampen thm' and
  - Chap 12/13 'Classification covering spaces'
- Second part: Homology groups, via cudi, chapter of a book.
- 10 problems (solve them during the semester) No feedback during the semester, but asking questions is allowed. Working together is allowed.
- Exam (completely open book) First 4 questions: 1h30 prep. Last one 30 min, no prep
  1. Theoretical question (open book, explain the proof, ...) /4
  2. New problem (comparable to one of the 10 problems) /4
  3. Explain your solution n-th problem solved at home /4
  4. Explain your solution m-th problem solved at home /4
  5. 4 small questions /4

  After the exam, hand in your solutions of the other problems. After a quick look, the points of 3. and 4. can be ± 1 (in extreme cases ± 2)
- No exercise classes for this course!

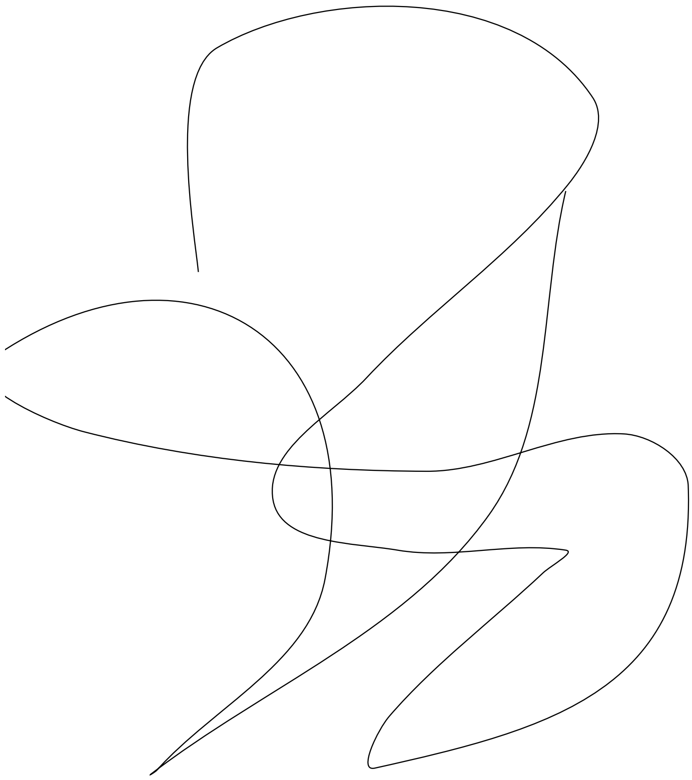

Figure 1: kTest inserting figures generated by inkscape

Chapter 0

Introduction

0.1 What is algebraic topology?

Functor from category of topological spaces to the category of groups.

- Category: set of spaces and morphisms.
- Functor: $X \rightsquigarrow G_X$ and $f : X \to Y \rightsquigarrow f_* : G_X \to G_Y$ such that
  - $(f \circ g)_* = f_* \circ g_*$
  - $(1_X)_* = 1_{G_X}$

  Two systems we'll discuss:

- fundamental groups
- homology groups

Example. Suppose we have a functor. If $G_X \not\cong G_Y$, then X and Y are not homeomorphic. If 'shadows' are different, then objects themselves are different too.

Proof. Suppose X and Y are homeomorphic. Then $\exists f : X \to Y$ and $g : Y \to X$, maps (maps are always continuous in this course), such that $g \circ f = 1_X$ and $f \circ g = 1_Y$. Then $f_* : G_X \to G_Y$ and $g_* : G_Y \to G_X$ such that $(g \circ f)_* = (1_X)_*$ and $(f \circ g)_* = (1_Y)_*$. Using the rules discussed previously, we get

$$g_* \circ f_* = 1_{G_X} \quad f_* \circ g_* = 1_{G_Y},$$

which means that $f_* : G_X \to G_Y$ is an isomorphism.

0.2 Fundamental group

Pick a base point x_0 and consider it fixed. (The fundamental group will not depend on it. We assume all spaces are path connected) $X \rightsquigarrow \pi(X)$.

- A loop based at $x_0 \in X$ is a map $f : I = [0,1] \to X$, $f(0) = f(1) = x_0$.
- Loops are equivalent if one can be deformed in the other in a continuous way, with the base point fixed.
- The fundamental group consists of equivalent classes of loops.

Example. Let $X = B^2$ (2 dimensional disk). Then $\pi(B^2) = 1$, because every loop is equivalent to the 'constant' loop.

Example. Let $X = S^1$ and pick x_0 on the circle. Two options:

- The loop is trivial equivalent to the constant loop
- The loop goes around the circle.
- The loop goes around the circle, twice.
- The loop goes around the circle, clockwise, once
- ...

$\pi(S^1) \cong \mathbb{Z}$ (proof will follow).

The composition of loops is simply pasting them. In the case of the circle, the loop $-1 \circ$ the loop 2 is the loop 1.

Suppose $\alpha : I \to X$ and $f : X \to Y$. Then we define

$$f_*[\alpha] = [f \circ \alpha].$$

Theorem 1 (Fixed point theorem of Brouwer)**.** Any continuous map from a rectangle to itself has at least one fixed point.

Proof. Suppose there is no fixed point, so $f(x) \neq x$ for all $x \in B^2$. Then we can construct map $r : B^2 \to S_1$ as follows: take the intersection of the boundary and half ray between $f(x)$ and x.

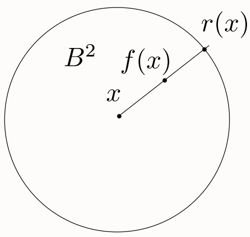

Figure 1: Proof of the Brouwer fixed point theorem

If x lies on the boundary, we have the identity map. This map is continuous. Then we have $S^1 \to B^2 \to S^1$, via the inclusion and r. Looking at the fundamental groups:

$$\pi(S^1) = \mathbb{Z} \to \pi(B^2) = 1 \to \pi(S^1) = \mathbb{Z}.$$

The map from $\pi(S^1) \to \pi(S^1)$ is the identity map, but the first map maps everything on 1. $\lightning$

Chapter 9

Fundamental group

9.51 Homotopy of paths

Definition 1 (Homotopy). Let $f, g : X \to Y$ be maps (so continuous). Then a homotopy between f and g is a continuous map $H : X \times I \to Y$ such that

- $H(x, 0) = f(x)$, $H(x, 1) = g(x)$
- For all $t \in I$, define $f_t : X \to Y : x \mapsto H(x, t)$

We say that f is homotopic with g, we write $f \simeq g$. If g is a constant map, we say that f is null homotopic.

Definition 2 (Path homotopy). Let $f, g : I \to X$ be two paths such that $f(0) = g(0) = x_0$ and $f(1) = g(1) = x_1$. Then $H : I \times I \to X$ is a path homotopy between f and g, if and only if

- $H(s, 0) = f(s)$ and $H(s, 1) = g(s)$ (homotopy between maps)
- $H(0, t) = x_0$ and $H(1, t) = x_1$ (start and end points fixed)

Notation: $f \simeq_p g$.

Lemma 1. $\simeq$ and $\simeq_p$ are equivalence relations.

Proof.

- Reflective: $F(x, t) = f(x)$
- Symmetric: $G(x, t) = H(x, 1 - t)$
- Transitive: Suppose $f \simeq g$ and $g \simeq h$, with H_1, H_2 resp.

$$H(x,t) = \begin{cases} H_1(x, 2t) & 0 \le t \le \frac{1}{2} \\ H_2(x, 2t-1) & \frac{1}{2} \le t \le 1 \end{cases}. \qquad \square$$

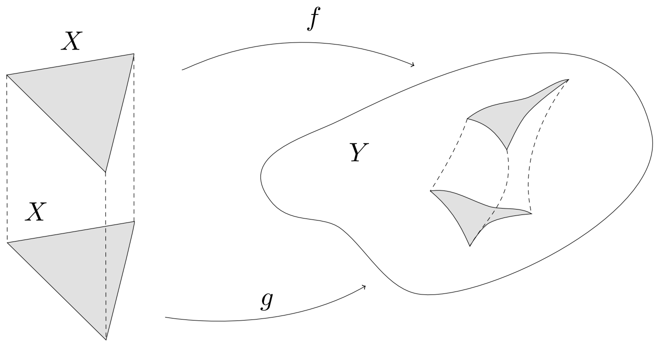

Figure 9.1: General example of a homotopy.

Example (Trivial, but important). Let $C \subset \mathbb{R}^n$ be a convex subset.

- Any two maps $f, g : X \to C$ are homotopic.
- Any two paths $f, g : I \to C$ with $f(0) = g(0)$ and $g(1) = f(1)$ are path homotopic.

Choose $H : X \times I \to C : (x, t) \to H(x, t) = (1 - t)f(x) + tg(x)$.

Product of paths

Let $f : I \to X$, $g : I \to X$ be paths, $f(1) = g(0)$. Define

$$f * g : I \to X : s \mapsto \begin{cases} f(2s) & 0 \leq s \leq \frac{1}{2} \\ g(2s-1) & \frac{1}{2} \leq s \leq 1. \end{cases}$$

Remark. If f is path homotopic to f' and g path homotopic to g' (which means that $f(1) = f'(1) = g(0) = g'(0)$), then $f * g \simeq_p f' * g'$.

So we can define $[f] * [g] := [f * g]$ with $[f] := \{g : I \to X \mid g \simeq_p f\}$

Theorem 2.

1. $[f] * ([g] * [h])$ is defined iff $([f] * [g]) * [h]$ is defined and in that case, they are equal.
2. Let e_x denote the constant path $e_x : I \to X : s \mapsto x$, $x \in X$. If $f(0) = x_0$ and $f(1) = x_1$ then $[e_{x_0}] * [f] = [f]$ and $[f] * [e_{x_1}] = [f]$.
3. Let $\overline{f} : I \to X : s \mapsto f(1-s)$. Then $[f]*[\overline{f}] = [e_{x_0}]$ and $[\overline{f}]*[f] = [e_{x_1}]$

Proof. First, two observations

- Suppose $f \simeq_p g$ via homotopy H, $f, g : I \to X$. Let $k : X \to Y$.

Then $k \circ f \simeq_p k \circ g$ using $k \circ H$.

- If $f * g$ (not necessarily path homotopic). Then $k \circ (f * g) = (k \circ f) * (k \circ g)$.

Now, the proof

2. Take $e_0 : I \to I : s \mapsto 0$. Take $i : I \to I : s \mapsto s$ Then $e_0 * i$ is a path from 0 to $1 \in I$. The path i is also such a path. Because I is a convex subset, $e_0 * i$ and i are path homotopic, $e_0 * i \simeq_p i$. Using one of our observations, we find that

$$\begin{aligned} f \circ (e_0 * i) &\simeq_p f \circ i \\ (f \circ e_0) * (f \circ i) &\simeq_p f \\ e_{x_0} * f &\simeq_p f \\ [e_{x_0}] * [f] &= [f]. \end{aligned}$$

3. Note that $i * \bar{i} \simeq_p e_0$. Now, applying the same rules, we get

$$\begin{aligned} f \circ (i * \bar{i}) &\simeq_p f \circ e_0 \\ f * \overline{f} &\simeq_p e_{x_0} \\ [f] * [\overline{f}] &= [e_{x_0}]. \end{aligned}$$

1. Remark: Only defined if $f(1) = g(0), g(1) = h(0)$. Note that $f * (g * h) \neq (f * g) * h$. The trajectory is the same, but the speed is not.

   Assume the product is defined. Suppose $[a, b]$, $[c, d]$ are intervals in $\mathbb{R}$. Then there is a unique positive (positive slope), linear map from $[a, b] \to [c, d]$. For any $a, b \in [0, 1)$ with $0 < a < b < 1$, we define a path

$$\begin{aligned} k_{a,b} : [0,1] &\longrightarrow X \\ [0, a] &\xrightarrow{\text{lin.}} [0,1] \xrightarrow{f} X \\ [a, b] &\xrightarrow{\text{lin.}} [0,1] \xrightarrow{g} X \\ [b, 0] &\xrightarrow{\text{lin.}} [0,1] \xrightarrow{h} X \end{aligned}$$

   Then $f * (g * h) = k_{\frac{1}{2}, \frac{3}{4}}$ and $(f * g) * h = k_{\frac{1}{4}, \frac{1}{2}}$

   Let γ be that path $\gamma : I \to I$ with the following graphs:

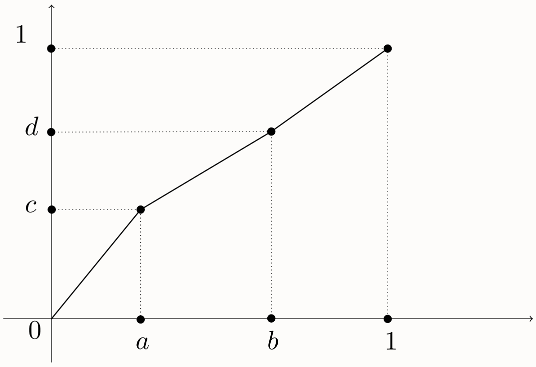

Figure 9.2: proof path

Note that $\gamma \simeq_p i$. Now, using the fact that composition of positive linear maps is positive linear.

$$k_{c,d} \circ \gamma \simeq_p k_{c,d} \circ i$$
$$k_{a,b} \simeq_p k_{c,d},$$

which is what we wanted to show.

□

9.52 Fundamental group

Definition 3. Let X be a space and $x_0 \in X$, then the fundamental group of X based at x_0 is

$$\pi(X, x_0) = \{[f] \mid f : I \to X, f(0) = f(1) = x_0\}.$$

(Also $\pi_1(X, x_0)$ is used, first homotopy group of X based at x_0)

For $[f], [g] \in \pi(X, x_0)$, $[f] * [g]$ is always defined, $[e_{x_0}]$ is an identity element, $*$ is associative and $[f]^{-1} = [\overline{f}]$. This makes $(\pi(X, x_0), *)$ a group.

Example. If $C \subset \mathbb{R}^n$, convex then $\pi(X, x_0) = 1$. E.g. $\pi(B^2, x_0) = 1$.

Remark. All groups are a fundamental group of some space.

Question: how does the group depend on the base point?

Theorem 3 (52.1)**.** Let X be a space, $x_0, x_1 \in X$ and $\alpha : I \to X$ a path from x_0 to x_1. Then

$$\begin{aligned}\hat{\alpha} : \pi(X, x_0) &\longrightarrow \pi(x, x_1)\\ [f] &\longmapsto [\overline{\alpha}] * [f] * [\alpha].\end{aligned}$$

is an isomorphisms of groups. Note however that this isomorphism depends on α.

Proof. Let $[f], [g] \in \pi_1(X, x_0)$. Then

$$\begin{aligned}\hat{\alpha}([f] * [g]) &= [\overline{\alpha}] * [f] * [g] * [\alpha]\\ &= [\overline{\alpha}] * [f] * [\alpha] * [\overline{\alpha}] * [g] * [\alpha]\\ &= \hat{\alpha}[f] * \hat{\alpha}[g].\end{aligned}$$

We can also construct the inverse, proving that these groups are isomorphic. □

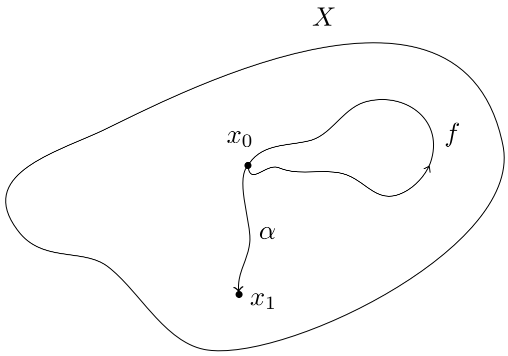

Figure 9.3: Construction of the group homomorphism

Remark. If $f : (x, x_0) \to (Y, y_0)$ is a map of pointed topology spaces ($f : X \to Y$ continuous and $f(x_0) = y_0$). Then

$$f_* : \pi(X, x_0) \to \pi(Y, y_0) : [\gamma] \mapsto [f \circ \gamma]$$

is a morphism of groups, because of the two 'rules' discussed previously, with

$$(f \circ g)_* = f_* \circ g_* \qquad (1_X)_* = 1_{\pi(X, x_0)}.$$