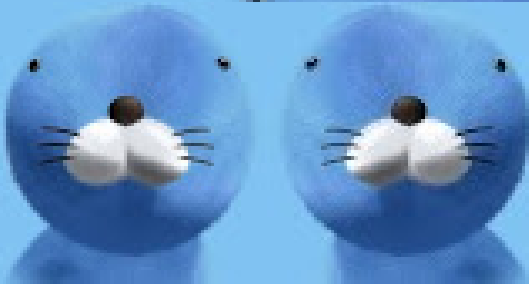


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# Notes on Kinematic Fit

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February 28, 2024

# Introduction

Assume that we have measured a momentum of two particles, which decay from a mother particle with an exact momentum  $P_0$ . In the real world, every measurement inherently carries some errors. Consequently, the measured momentum may not satisfy momentum conservation:

$$P_0 = P_1 + P_2; \quad P_{1,meas} + P_{2,meas} \neq P_0 \quad (1)$$

However, since these measurements did not incorporate our prior knowledge from physics, we can make a more informed estimate of the measured parameters. If the momentum resolution of particles 1 and 2 is well-known, then we can express  $\chi^2$  as

$$\chi^2 = \frac{(P_1 - P_{1,meas})^2}{\sigma_1^2} + \frac{(P_2 - P_{2,meas})^2}{\sigma_2^2} \quad (2)$$



# Introduction

By incorporating the *Kinematic Constraints*, specifically *momentum conservation*, into our example, we introduce additional terms known as *Lagrange Multiplier* to Equation (3):

$$\chi^2 = \frac{(P_1 - P_{1,meas})^2}{\sigma_1^2} + \frac{(P_2 - P_{2,meas})^2}{\sigma_2^2} + 2\lambda(P_{1,meas} + P_{2,meas} - P_0) \quad (3)$$

We then proceed to evaluate the conditions for local minima, i.e. setting the partial derivatives equal to zero:

$$\frac{1}{2} \frac{\partial \chi^2}{\partial P_1} = \frac{(P_1 - P_{1,meas})}{\sigma_1^2} + \lambda = 0 \quad (4)$$

$$\frac{1}{2} \frac{\partial \chi^2}{\partial P_2} = \frac{(P_2 - P_{2,meas})}{\sigma_2^2} + \lambda = 0 \quad (5)$$

$$\frac{1}{2} \frac{\partial \chi^2}{\partial \lambda} = (P_1 + P_2 - P_0) = 0 \quad (6)$$



# Introduction

By solving the equations 4,5,6, we obtain the following expressions:

$$P_1 = \frac{\sigma_2^2 P_{1,meas} - \sigma_1^2 P_{2,meas} + \sigma_1^2 P_0}{\sigma_1^2 + \sigma_2^2} \quad (7)$$

$$P_2 = \frac{\sigma_1^2 P_{2,meas} - \sigma_2^2 P_{1,meas} + \sigma_2^2 P_0}{\sigma_1^2 + \sigma_2^2} \quad (8)$$

$$\lambda = \frac{P_{1,meas} + P_{2,meas} - P_0}{\sigma_1^2 + \sigma_2^2}. \quad (9)$$

Now, we have obtained the 'corrected' measurements with minimized  $\chi^2$ , which incorporates momentum conservation. Let us delve into the interpretation of these equations.

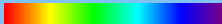


# Introduction

In a straightforward interpretation,  $\lambda$  can be viewed as a kind of 'normalized variance' of the kinematic constraint. It quantifies the error in momentum conservation( $P_{1,meas} + P_{2,meas} - P_0$ ) relative to the overall resolution( $\sigma_1^2 + \sigma_2^2$ ). Equation (4) implies that

$$P_1 = P_{1,meas} + \sigma_1^2 \lambda, \quad (10)$$

suggesting that the corrected momentum( $P_1$ ) is essentially the measured momentum( $P_{1,meas}$ ) augmented by a term proportional to the detector resolution and the normalized error of the kinematic constraint. Thus, we can assert that we have applied a statistically fair correction to the momentum, taking into account both the detector resolution and kinematic constraints.



# Fitting in General

Assume that you have a set of measurements,  $\mathbf{m} = \{m_1, m_2 \dots m_N\}$ , and some unmeasured data,  $\mathbf{u} = \{u_1, u_2 \dots u_J\}$  to be estimated. Kinematic constraints can be represented by sets of equations  $\mathbf{f} = \{f_1(m_1, m_2, \dots m_N, u_1, u_2, \dots u_N), f_2, \dots f_K\}$ . We will iteratively solve the problem by guessing the best parameter for each step and checking  $\chi^2$ . Let  $\mathbf{m}^0$  denote our initial measured data, and  $\mathbf{m}$  represent the 'guess' of the data in each iterative step.

$$\chi^2(\mathbf{m}) = (\mathbf{m}^0 - \mathbf{m})^\dagger V^{-1}(\mathbf{m}^0 - \mathbf{m}) + 2\lambda^\dagger \mathbf{f}(\mathbf{m}, \mathbf{u}). \quad (11)$$

Here, the Lagrange multiplier  $\lambda = \{\lambda_1, \lambda_2, \dots \lambda_K\}$  is not just a number but a column vector with  $k$  elements, corresponding to each kinematic constraint in  $\mathbf{f}$ . Our task is to minimize  $\chi^2$  to obtain the best guesses in statistically fair method.



# $\chi^2$ Minimization

By (partially)differentiating with respect to all variables involved, we obtain the gradients of  $\chi^2$ . Setting all of them to zero indicates that we have reached a minimum point of  $\chi^2$ . We have 3 sets of gradient equations:

$$\nabla_{\mathbf{m}} = -2V^{-1}(\mathbf{m}^0)(\mathbf{m}^0 - \mathbf{m}) + 2\mathbf{F}_{\mathbf{m}}^{\dagger}(\mathbf{m}, \mathbf{u})\lambda = 0 \quad (12)$$

$$\nabla_{\mathbf{u}} = 2\mathbf{F}_{\mathbf{u}}^{\dagger}(\mathbf{m}, \mathbf{u})\lambda = 0 \quad (13)$$

$$\nabla_{\lambda} = \mathbf{f}(\mathbf{m}, \mathbf{u}). \quad (14)$$

Here, the subscripts denote partial derivatives. i.e.  $((\mathbf{F}_m)_{ki} \equiv \frac{\partial f_k}{\partial m_i})$ .



# Processing Iterative Steps.

We can express the following equations based on the ones provided above:

$$\mathbf{V}^{-1}(\mathbf{m}^0)(\mathbf{m}^{\nu+1} - \mathbf{m}^0) + (\mathbf{F}_m^\dagger)^\nu \lambda^{\nu+1} = 0 \quad (15)$$

$$(\mathbf{F}_u^\dagger)^\nu \lambda^{\nu+1} = 0 \quad (16)$$

$$\mathbf{f}^\nu + \mathbf{F}_m^\nu(\mathbf{m}^{\nu+1} - \mathbf{m}^\nu) + \mathbf{F}_u^\nu(\mathbf{u}^{\nu+1} - \mathbf{u}^\nu) = 0. \quad (17)$$

Equation (17) is not a direct consequence of Equation (13) but rather a linear approximation to proceed with our iteration step. Expanding the  $\nabla_\lambda$  term with a Taylor series leads to this equation. Note that, as our parameters  $\mathbf{m}$  and  $\mathbf{u}$  are updated during the step, our constraint matrix  $\mathbf{f}$  should also be updated during the iteration. Here,  $\lambda$  should be indexed as  $\nu + 1$  since it is a parameter to be guessed in the next step.





# Solving the Equation(1)

Multiplying  $\mathbf{V}$  to Equation (15) leads to:

$$\mathbf{m}^{\nu+1} - \mathbf{m}^0 = -V(\mathbf{m}^0)(\mathbf{F}_m^\dagger)^\nu \lambda^{\nu+1}. \quad (18)$$

Substituting Equation (18) into Equation (17), we get:

$$\begin{aligned} \mathbf{F}_u^\nu (\mathbf{u}^{\nu+1} - \mathbf{u}^\nu) &= -\mathbf{f}^\nu - \mathbf{F}_m^\nu (-V(\mathbf{m}^0)(\mathbf{F}_m^\dagger)^\nu \lambda^{\nu+1} + \mathbf{m}^0 - \mathbf{m}^\nu) \\ &= S\lambda^{\nu+1} - R \end{aligned} \quad (19)$$

where  $S \equiv \mathbf{F}_m^\nu V(\mathbf{m}^0)(\mathbf{F}_m^\dagger)^\nu$  and  $R \equiv \mathbf{f}^\nu + \mathbf{F}_m^\nu (\mathbf{m}^0 - \mathbf{m}^\nu)$ . Multiplying  $(\mathbf{F}_u^\dagger)^\nu S^{-1}$  and substituting Equation (16), we get:

$$(\mathbf{F}_u^\dagger)^\nu S^{-1} \mathbf{F}_u^\nu (\mathbf{u}^{\nu+1} - \mathbf{u}^\nu) = \cancel{(\mathbf{F}_u^\dagger)^\nu \lambda^{\nu+1}}^0 - (\mathbf{F}_u^\dagger)^\nu S^{-1} R. \quad (20)$$



## Solving the Equation(2)

Then we naturally obtain:

$$\mathbf{u}^{\nu+1} = \mathbf{u}^{\nu} - ((\mathbf{F}_u^{\dagger})^{\nu} S^{-1} \mathbf{F}_u^{\nu})^{-1} (\mathbf{F}_u^{\dagger})^{\nu} S^{-1} R. \quad (21)$$

and from Equation (19)

$$\lambda^{\nu+1} = S^{-1} (\mathbf{F}_u^{\nu} (\mathbf{u}^{\nu+1} - \mathbf{u}^{\nu}) + R) \quad (22)$$

. For a summary, we have obtained all equations to proceed to the next step. All other matrices in the equation can be calculated from parameters of the current step, and  $\chi^2$  can be evaluated from (11) .

$$\begin{cases} \mathbf{u}^{\nu+1} = \mathbf{u}^{\nu} - ((\mathbf{F}_u^{\dagger})^{\nu} S^{-1} \mathbf{F}_u^{\nu})^{-1} (\mathbf{F}_u^{\dagger})^{\nu} S^{-1} R & (21) \\ \lambda^{\nu+1} = S^{-1} (\mathbf{F}_u^{\nu} (\mathbf{u}^{\nu+1} - \mathbf{u}^{\nu}) + R) & (22) \\ \mathbf{m}^{\nu+1} = \mathbf{m}^0 - V(\mathbf{m}^0) (\mathbf{F}_m^{\dagger})^{\nu} \lambda^{\nu+1} & (18) \end{cases}$$

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# Evolution of Variance Matrix

Take a look at Eq.(18). We see that  $m^{\nu+1}$  is an addition(subtraction) of some parameters to the initially measured data. As we already know the error, i.e. Variance matrix, of initial data, we can estimate how error propagates through the fitting process.

$$V(m) = J_{m,m^0} V(m^0) J_{m,m^0}^\dagger \quad (23)$$

We need to calculate the Jacobian,

$$J_{m,m^0}(i,j) = \frac{\partial m_i}{\partial m_j^0} \quad (24)$$



# Evolution of Variance Matrix.

To begin with, let us express Eq (18) in terms of  $m^0$ . At the moment we will drop the superscript  $v$ . As  $\mathbf{f}(\mathbf{m}, \mathbf{u})$  is a constant on  $m^0$ ,  $\mathbf{F}_m$  also will be a constant to  $m^0$ . Then we only need to consider the derivatives of  $\lambda$ . By substituting (21) ,

$$\lambda = S^{-1}(-\mathbf{F}_u((\mathbf{F}_u^\dagger S^{-1} \mathbf{F}_u)^{-1}(\mathbf{F}_u^\dagger S^{-1} R) + R) \quad (25)$$

and we have

$$R \equiv \mathbf{f} + \mathbf{F}_m(\mathbf{m}^0 - \mathbf{m}) \rightarrow \frac{\partial R}{\partial m^0} = \mathbf{F}_m \quad (26)$$

so that

$$\frac{\partial \lambda}{\partial m^0} = S^{-1}(-\mathbf{F}_u((\mathbf{F}_u^\dagger S^{-1} \mathbf{F}_u)^{-1} \mathbf{F}_u^\dagger S^{-1} \mathbf{F}_m) + \mathbf{F}_m). \quad (27)$$



# Evolution of Variance Matrix

Now define the symmetric matrices  $G \equiv \mathbf{F}_m^\dagger S^{-1} \mathbf{F}_m$ ,  $U \equiv (\mathbf{F}_u^\dagger S^{-1} \mathbf{F}_u)^{-1}$  and  $H \equiv \mathbf{F}_m^\dagger S^{-1} \mathbf{F}_u$ . Then we have expressions for  $\frac{\partial \lambda}{\partial m^0}$  hence

$$\begin{aligned} J_{m,m^0} &= I - V(m^0) \mathbf{F}_m^\dagger \frac{\partial \lambda}{\partial m^0} = I - V \mathbf{F}_m^\dagger (-S^{-1} \mathbf{F}_u U^{-1} H^\dagger + S^{-1} \mathbf{F}_m) \\ &= I - V(G - HUH^\dagger) \end{aligned} \quad (28)$$

If we let  $C = G - HUH^\dagger$ , we obtain

$$V(m) = J_{m,m^0} V J_{m,m^0}^\dagger = V - 2VCV + VCVCV. \quad (29)$$

You might want to neglect higher order term, but please keep 2nd order term at the moment. Some materials like [1] had an error in this part.



# Pull distribution

It is better to check the pull distribution to check the quality of a fit. By defining the residual  $\epsilon = m - m^0$  and its variance  $V(\epsilon)$ , pull is defined as:

$$P(\epsilon) = \epsilon / \sqrt{V(\epsilon)} \quad (30)$$

and

$$V(\epsilon) \equiv V(m) + V(m^0) - 2\text{Cov}(m, m^0). \quad (31)$$

We have already calculated  $J_{m,m^0}$ . Then we directly get the covariance matrix.



# Pull distribution

The covariance can be estimated as:

$$\text{Cov}(m, m^0) = J_{m, m^0} V(m) = V - VCV. \quad (32)$$

If we substitute this and Eq.(29) into Eq.(31), we get

$$V(\epsilon) = VCVCV. \quad (33)$$

Note that 2nd order term affects the variance.



## Example: 3-C Fit

Assuming a decay of  $\Lambda \rightarrow p\pi^-$ , let's represent the momentum of those particles in spherical coordinates:

$$\vec{P}_\Lambda = (P_\Lambda, \theta_\Lambda, \phi_\Lambda), \dots \quad (34)$$

Then the kinematic constraints can be expressed as:

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} -P_\Lambda \sin \theta_\Lambda \cos \phi_\Lambda + P_p \sin \theta_p \cos \phi_p + P_\pi \sin \theta_\pi \cos \phi_\pi \\ -P_\Lambda \sin \theta_\Lambda \sin \phi_\Lambda + P_p \sin \theta_p \sin \phi_p + P_\pi \sin \theta_\pi \sin \phi_\pi \\ -P_\Lambda \cos \theta_\Lambda + P_p \cos \theta_p + p_\pi \cos \theta_\pi \\ -\sqrt{P_\Lambda^2 + m_\Lambda^2} + \sqrt{P_p^2 + m_p^2} + \sqrt{P_\pi^2 + m_\pi^2} \end{pmatrix}. \quad (35)$$





## Example: 3-C Fit

We have unmeasured and measured variables as:

$$\mathbf{u} = \{P_{\Lambda}\}; \quad \mathbf{m} = \{\theta_{\Lambda}, \phi_{\Lambda}, P_p, \theta_p, \phi_p, P_{\pi}, \theta_{\pi}, \phi_{\pi}\}. \quad (36)$$

Since there is 1 unmeasured variable with 4 kinematical constraints, this is a  $4-1 = 3$ -Constrained fit. Let us substitute Eq (35) and (36) into Eq (11)s and its resulting equations. We get  $\mathbf{F}_u$  and  $\mathbf{F}_m$  as

$$\mathbf{F}_u = \begin{pmatrix} \frac{\partial f_1}{\partial P_{\Lambda}} \\ \frac{\partial P_{\Lambda}}{\partial f_2} \\ \frac{\partial P_{\Lambda}}{\partial f_3} \\ \frac{\partial P_{\Lambda}}{\partial f_4} \end{pmatrix}; \quad \mathbf{F}_m = \begin{pmatrix} \frac{\partial f_1}{\partial \theta_{\Lambda}} & \dots & \frac{\partial f_1}{\partial \phi_{\pi}} \\ \frac{\partial f_2}{\partial \theta_{\Lambda}} & \dots & \frac{\partial f_2}{\partial \phi_{\pi}} \\ \frac{\partial f_3}{\partial \theta_{\Lambda}} & \dots & \frac{\partial f_3}{\partial \phi_{\pi}} \\ \frac{\partial f_4}{\partial \theta_{\Lambda}} & \dots & \frac{\partial f_4}{\partial \phi_{\pi}} \end{pmatrix} \quad (37)$$

ΩH? We have all the matrices to calculate in each step. By applying an appropriate variance matrix and employing  $\chi^2$  selection criteria, we can kinematically fit the particles.



# References

 O. Skjeggstad Frodesen A.G. **Probability and Statistics in Particle Physics**. Columbia University Press, 1980. ISBN: 8200019063.

