

# Notes on Kinematic Fit

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# Outline

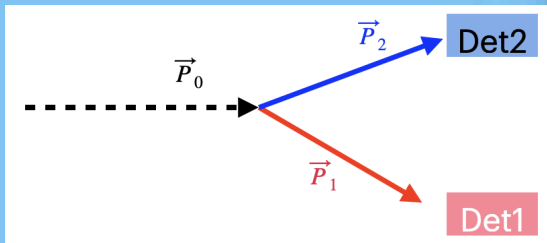
1. Introduction
2. Example: Mass-Constraint Fit
3. Applications:  $\Xi$  decay at J-PARC
4. Tricks
5. Appendix



# Introduction



# Measurement Error



Assume a beam with momentum  $\vec{P}^-$  decays into  $\vec{P}_1$  and  $\vec{P}_2$ . Measured momentum are smeared due to detector resolution, leading to unbalance in the momentum conservation.

$$\vec{P}_0 = \vec{P}_1 + \vec{P}_2; \quad \vec{P}_{1,meas} + \vec{P}_{2,meas} \neq \vec{P}_0 \quad (1)$$

We can define the  $\chi^2$  to quantitatively represent our measurement error. However, we can't derive meaningful expressions from this  $\chi^2$ .

$$\chi^2 = \frac{(P_1 - P_{1,meas})^2}{\sigma_1^2} + \frac{(P_2 - P_{2,meas})^2}{\sigma_2^2} \quad (2)$$



# Constrained Optimization with The Lagrange Multiplier

By incorporating the *Kinematic Constraints*, specifically *momentum conservation*, we involve additional knowledge to (2). This is known as the *Lagrange Multiplier*

$$\chi^2 = \frac{(P_{1,KF} - P_{1,meas})^2}{\sigma_1^2} + \frac{(P_{2,KF} - P_{2,meas})^2}{\sigma_2^2} + 2\lambda(\mathbf{P}_{1,KF} + \mathbf{P}_{2,KF} - \mathbf{P}_0) \quad (3)$$

Now we have meaningful expressions to minimize  $\chi^2$ , hence get better estimations for the measurement.

$$\frac{1}{2} \frac{\partial \chi^2}{\partial P_{1,KF}} = \frac{(P_{1,KF} - P_{1,meas})}{\sigma_1^2} + \lambda = 0 \quad (4)$$

$$\frac{1}{2} \frac{\partial \chi^2}{\partial P_{2,KF}} = \frac{(P_{2,KF} - P_{2,meas})}{\sigma_2^2} + \lambda = 0 \quad (5)$$

$$\frac{1}{2} \frac{\partial \chi^2}{\partial \lambda} = (P_{1,KF} + P_{2,KF} - P_0) = 0 \quad (6)$$



# Why Better Resolution?

By solving the equations 4,5,6 and defining  $\delta_i = P_{i,meas} - P_i$ , we obtain the following expressions:

$$\lambda = \frac{P_{1,meas} + P_{2,meas} - P_0}{\sigma_1^2 + \sigma_2^2} = \frac{\delta_1 + \delta_2}{\sigma_1^2 + \sigma_2^2} \quad (7)$$

$$P_{1,KF} = P_{1,meas} - \sigma_1^2 \lambda \quad (8)$$

$$P_{2,KF} = P_{2,meas} - \sigma_2^2 \lambda \quad (9)$$

$$\begin{aligned} \langle P_{1,KF} - P_1 \rangle &= \langle P_{1,KF} - P_{1,meas} + \delta_1 \rangle = \langle -\sigma_1^2 \lambda + \delta_1 \rangle \\ &= \langle \frac{-\sigma_1^2}{\sigma_1^2 + \sigma_2^2} (\delta_1 + \delta_2) + \delta_1 \rangle = \langle \frac{\sigma_2^2 \delta_1 - \sigma_1^2 \delta_2}{\sigma_1^2 + \sigma_2^2} \rangle \end{aligned} \quad (10)$$

$$\sigma_{1,KF}^2 = \langle (P_{1,KF} - P_1)^2 \rangle = \frac{\sigma_2^4 \delta_1^2 + \sigma_1^4 \delta_2^2 + 2\sigma_1^2 \sigma_2^2 \delta_1 \delta_2}{(\sigma_1^2 + \sigma_2^2)^2} = \frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2} < \sigma_1^2 \quad (11)$$



# The Covariance After Kinematic Fit

$$\begin{aligned}
 \text{cov}(P_1, P_2)_{KF} &= \langle \delta_{1,KF} \delta_{2,KF} \rangle = \langle (\delta_1 - \sigma_1^2 \lambda)(\delta_2 - \sigma_2^2 \lambda) \rangle \\
 &= \sigma_1^2 \sigma_2^2 \langle \lambda^2 \rangle - \frac{1}{\sigma_1^2 + \sigma_2^2} \sigma_1^2 \langle \delta_2^2 \rangle + \sigma_2^2 \langle \delta_1^2 \rangle = -\frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2} \quad (12)
 \end{aligned}$$

$$V = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix} \rightarrow V_{KF} = \begin{pmatrix} \frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2} & -\frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2} \\ -\frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2} & \frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2} \end{pmatrix} \quad (13)$$

- Improved momentum resolution
- Negative correlation between  $P_1$  and  $P_2$



# Generalization to Multi-Variables

Assume that we have a set of measured data  $\mathbf{m}^0$ , unknown parameters  $\mathbf{u}^0$  and constraints  $\mathbf{f}^0$ .

$$\begin{aligned}\mathbf{m}^0 &= \{m_1^0, m_2^0 \dots m_N^0\}; \quad \mathbf{u}^0 = \{u_1^0, u_2^0 \dots u_J^0\} \\ \mathbf{f} &= \{f_1(m_1^0, m_2^0, \dots m_N^0, u_1^0, u_2^0, \dots u_N^0), f_2^0, \dots f_K^0\}\end{aligned}\quad (14)$$

Let  $\mathbf{m}^0$  denote our initial measured data, and  $\mathbf{m}$  represent the 'guess' of the data in each iterative step, just alike  $P_{KFS}$  in the previous example. Equation (3) is generalized to:

$$\chi^2(\mathbf{m}) = (\mathbf{m}^0 - \mathbf{m})^\dagger V^{-1}(\mathbf{m}^0 - \mathbf{m}) + 2\lambda^\dagger \mathbf{f}(\mathbf{m}, \mathbf{u}). \quad (15)$$

Here, the Lagrange multiplier  $\lambda = \{\lambda_1, \lambda_2, \dots \lambda_K\}$  is not just a number but a column vector with  $k$  elements, corresponding to each kinematic constraint in  $\mathbf{f}$ .





# $\chi^2$ Minimization

We want to solve the equation

$$\vec{\nabla} \chi^2 = 0 \quad (16)$$

to obtain the minimized state. The differential term are listed within three groups.

$$\nabla_{\mathbf{m}} = -2V^{-1}(\mathbf{m}^0)(\mathbf{m}^0 - \mathbf{m}) + 2\mathbf{F}_{\mathbf{m}}^{\dagger}(\mathbf{m}, \mathbf{u})\lambda = 0 \quad (17)$$

$$\nabla_{\mathbf{u}} = 2\mathbf{F}_{\mathbf{u}}^{\dagger}(\mathbf{m}, \mathbf{u})\lambda = 0 \quad (18)$$

$$\nabla_{\lambda} = \mathbf{f}(\mathbf{m}, \mathbf{u}). \quad (19)$$

Here, the subscripts denote partial derivatives. i.e.  $((\mathbf{F}_{\mathbf{m}})_{ki} \equiv \frac{\partial f_k}{\partial m_i})$ .

## User Should Define...

$\mathbf{m}$	$\mathbf{u}$	$\mathbf{f}$	$\mathbf{V}$	$\mathbf{F}_{\mathbf{m}}, \mathbf{F}_{\mathbf{u}}$
Measured Data	Unknown parameters	Constraints	Covariance Matrix	Derivatives



# Pull distribution

A bias or resolution miss-estimation is revealed by observing the *Pull distribution* of each measurements. However, we cannot evaluate the 'true' value of measurement, hence pull for the real data is not accessible. Instead, from the the residual  $\epsilon = m - m^0$  and its variance  $V(\epsilon)$ , we observe the pull(of the residual) as :

$$P(\epsilon) = \epsilon / \sqrt{V(\epsilon)} \quad (20)$$

and

$$V(\epsilon) \equiv V(m) + V(m^0) - 2Cov(m, m^0). \quad (21)$$

The variance of the fitted variables,  $V(m)$ , is evaluated as

$$V(m) = J_{m,m^0} V(m^0) J_{m,m^0}^\dagger \quad (22)$$

where  $J_{m,m^0}$  is the Jacobian for  $m$  and  $m^0$ . Detailed calculations are provided in the appendix.



# Example: Mass-Constraint Fit



## Example: $\Lambda \rightarrow p\pi$ , Defining Variables and Constraints

Assume a decay of  $\Lambda \rightarrow p\pi^-$ . We define the measurements and unknowns as:

$$\mathbf{m} = \{P_p, \theta_p, \phi_p, P_\pi, \theta_\pi, \phi_\pi\}; \quad \mathbf{u} = \{P_\Lambda, \theta_\Lambda, \phi_\Lambda\} \quad (23)$$

Then we define the energy-momentum constraint equation as:

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} -P_\Lambda \sin \theta_\Lambda \cos \phi_\Lambda + P_p \sin \theta_p \cos \phi_p + P_\pi \sin \theta_\pi \cos \phi_\pi \\ -P_\Lambda \sin \theta_\Lambda \sin \phi_\Lambda + P_p \sin \theta_p \sin \phi_p + P_\pi \sin \theta_\pi \sin \phi_\pi \\ -P_\Lambda \cos \theta_\Lambda + P_p \cos \theta_p + P_\pi \cos \theta_\pi \\ -\sqrt{P_\Lambda^2 + m_\Lambda^2} + \sqrt{P_p^2 + m_p^2} + \sqrt{P_\pi^2 + m_\pi^2} \end{pmatrix}. \quad (24)$$

where the mass constraint is naturally implemented in energy term.

Since we have 3 unmeasured variable with 4 kinematical constraints, this is a  $4-3 = 1$ -Constrained fit.



## Example: $\Lambda \rightarrow p\pi$ , The Derivatives

We get  $\mathbf{F}_u$  and  $\mathbf{F}_m$  as

$$\mathbf{F}_u = \begin{pmatrix} \frac{\partial f_1}{\partial P_\Lambda} & \frac{\partial f_1}{\partial \theta_\Lambda} & \frac{\partial f_1}{\partial \phi_\Lambda} \\ \frac{\partial f_2}{\partial P_\Lambda} & \frac{\partial f_2}{\partial \theta_\Lambda} & \frac{\partial f_2}{\partial \phi_\Lambda} \\ \frac{\partial f_3}{\partial P_\Lambda} & \frac{\partial f_3}{\partial \theta_\Lambda} & \frac{\partial f_3}{\partial \phi_\Lambda} \\ \frac{\partial f_4}{\partial P_\Lambda} & \frac{\partial f_4}{\partial \theta_\Lambda} & \frac{\partial f_4}{\partial \phi_\Lambda} \end{pmatrix}; \quad \mathbf{F}_m = \begin{pmatrix} \frac{\partial f_1}{\partial P_p} & \cdots & \frac{\partial f_1}{\partial \phi_\pi} \\ \frac{\partial f_2}{\partial P_p} & \cdots & \frac{\partial f_2}{\partial \phi_\pi} \\ \frac{\partial f_3}{\partial P_p} & \cdots & \frac{\partial f_3}{\partial \phi_\pi} \\ \frac{\partial f_4}{\partial P_p} & \cdots & \frac{\partial f_4}{\partial \phi_\pi} \end{pmatrix} \quad (25)$$

We have all the matrices to calculate in each step. By applying an appropriate variance matrix and employing  $\chi^2$  selection criteria, we can do kinematic fit for the particles.



**Example:**  $\Xi \rightarrow \Lambda\pi, \Lambda \rightarrow p\pi$

We require two mass constraints for  $\Xi \rightarrow \Lambda\pi; \Lambda \rightarrow p\pi$ . In this case, careful considerations on the selection of variables. We will select

$$\mathbf{u} = \{P_\Xi, \theta_\Xi, \phi_\Xi\}; \quad \mathbf{m} = \{P_p, \theta_p, \phi_p, P_\pi, \theta_\pi, \phi_\pi\} \quad (26)$$

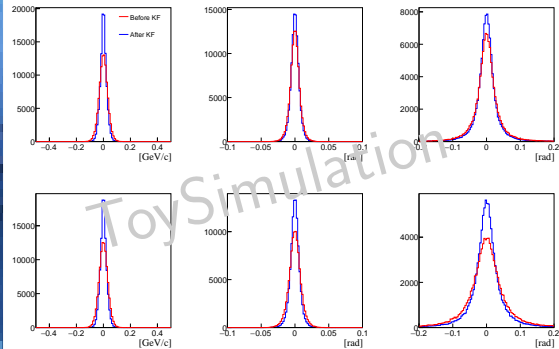
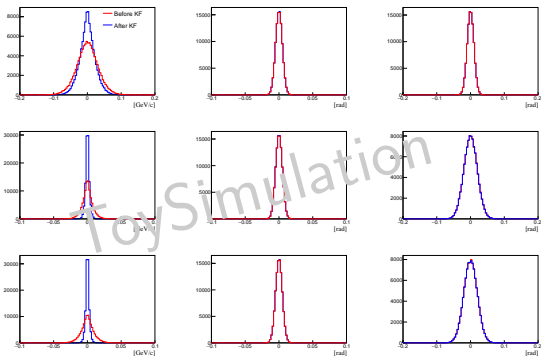
and define five constraints as:

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{pmatrix} = \begin{pmatrix} -P_{\Xi,x} + P_{p,x} + P_{\pi_\Lambda,x} + P_{\pi_\Xi,x} \\ -P_{\Xi,y} + P_{p,y} + P_{\pi_\Lambda,y} + P_{\pi_\Xi,y} \\ -P_{\Xi,z} + P_{p,z} + P_{\pi_\Lambda,z} + P_{\pi_\Xi,z} \\ -E_\Lambda + E_p + E_{\pi_\Lambda} \\ -E_\Xi + E_p + E_{\pi_\Lambda} + E_{\pi_\Xi} \end{pmatrix}. \quad (27)$$

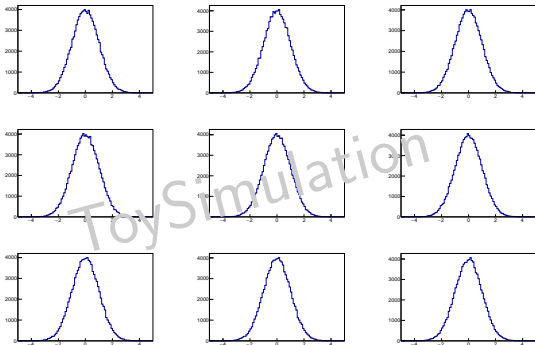
$\Lambda$  variables( $\{P_\Lambda, \theta_\Lambda, \phi_\Lambda\}$ ) are not selected in  $\mathbf{u}$  to reduce matrix dimension. That is, **we don't have explicit terms** related to  $\vec{P}_\Lambda$ , i.e.  $-P_{\Lambda,x} + P_{p,x} + P_{\pi_\Lambda,x}$  etc., because  $\vec{P}_\Lambda$  are neither unmeasured nor measured variables in our choice of parameters.



# Kinematics Restoration



# Pull Distribution



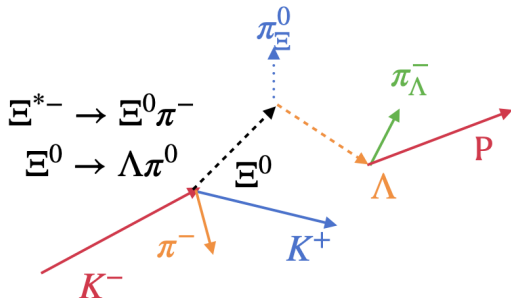
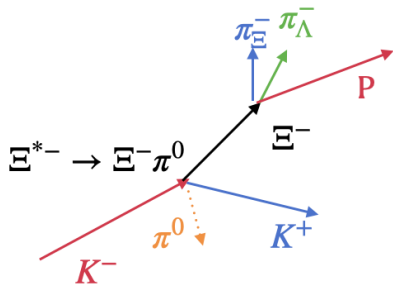
$$P(X) = \frac{X_{KF} - X_0}{\sqrt{V(X_{KF} - X_0)}} \quad (28)$$

- Pull distribution shows the normalized amount of parameter adjustment.
- Gaussian distribution with  $\sigma = 1$  implies good understandings in covariance of the measurement.
- In practice, resolution can be iteratively scaled by  $1./$  pull width





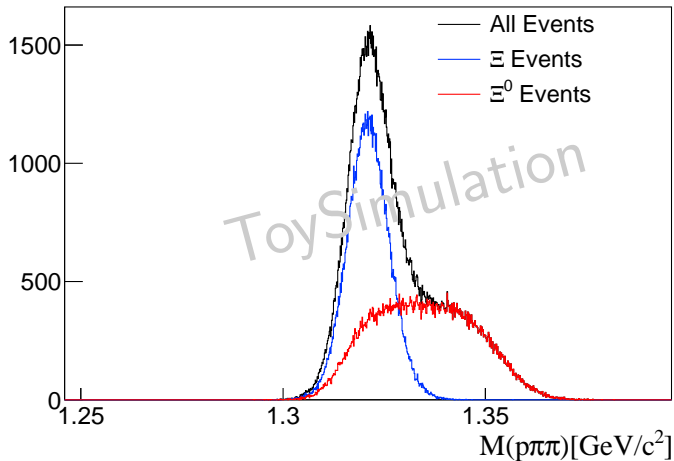
# $\Xi^*(1530) \rightarrow \Xi\pi^0$ and $\Xi^*(1530) \rightarrow \Xi^0\pi^-$ Separation



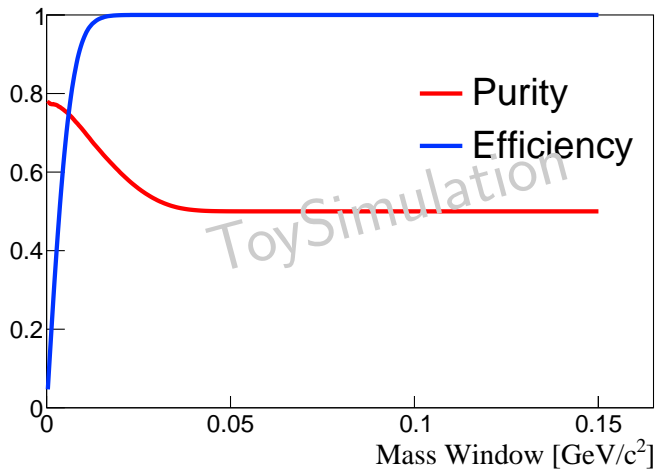
- Two decay channel of  $\Xi^*$  share the same decay product.
- Separation criteria should be defined to distinguish combinatorial backgrounds.
- Kinematic Fit result can provide another selection criteria based on Kinematics.



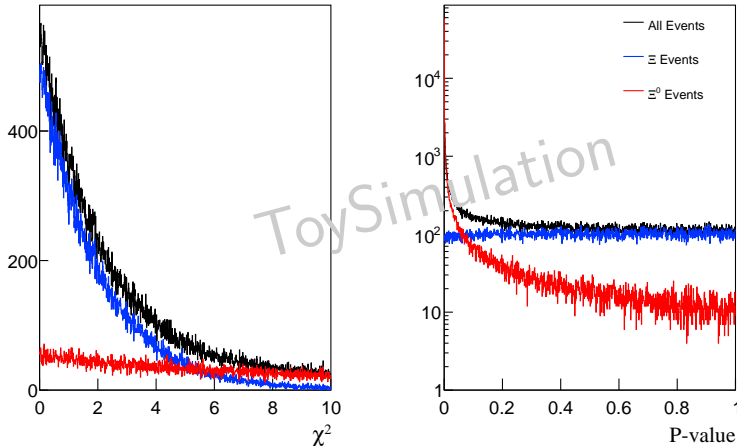
# Invariant Mass



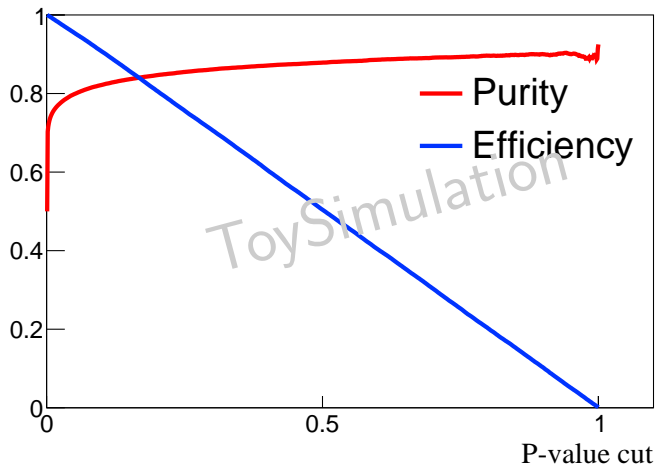
# Mass Window Selection



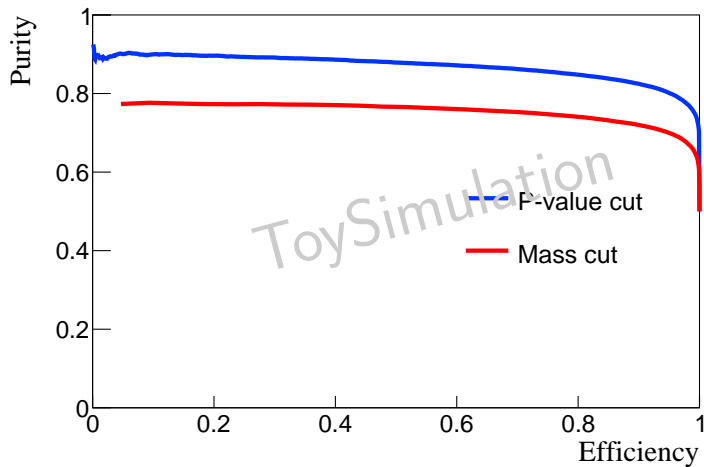
# The P-value



# P-value Selection



# P-value vs Invariant Mass



# Example: MassVertex-Constraint Fit

To be Updated..

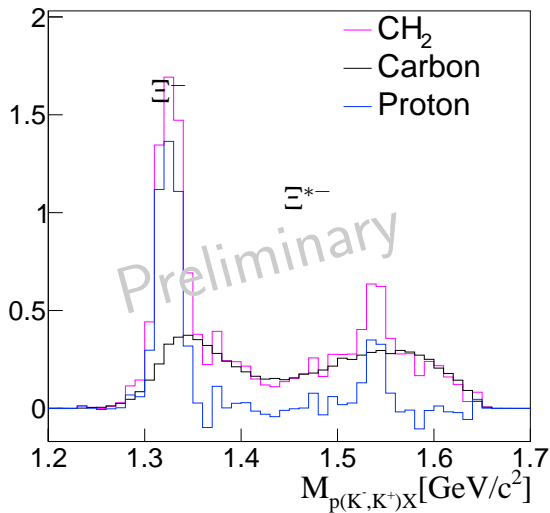
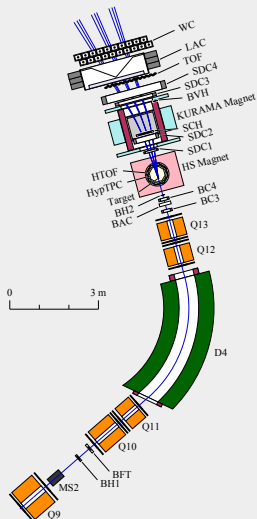


Applications:  $\Xi$  decay at J-PARC

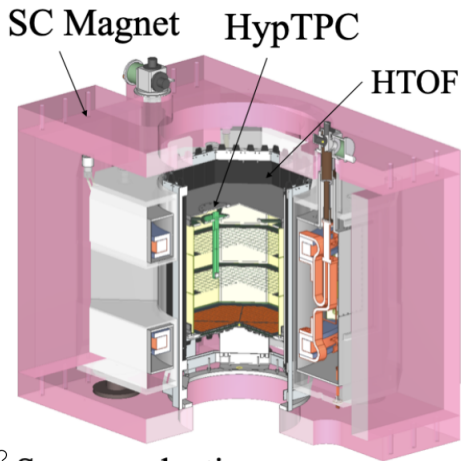




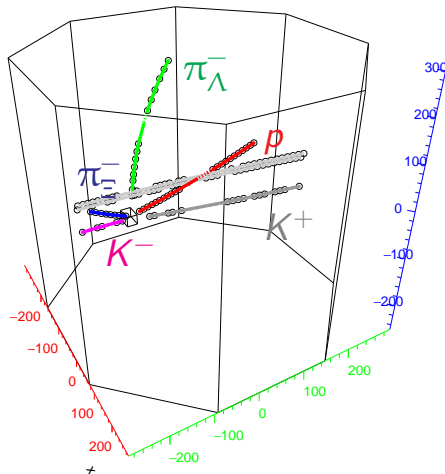
# $p(K^-, K^+)X$ at J-PARC E42



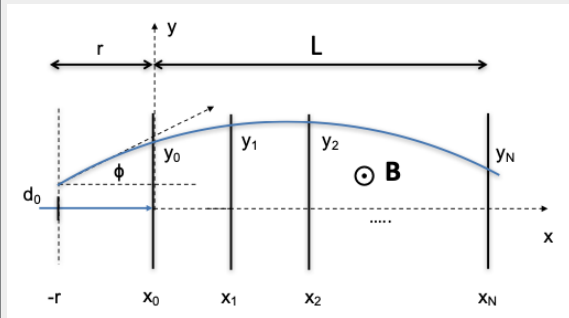
# HypTPC



Superconducting  
Hyperon Spectrometer



# Gluckstern Formula



Z. Drasal, W. Riegler, Nucl. Instrum. Methds. A, 910, 127-132 (2018)

$$\frac{\sigma_{P_T}}{P_T} \simeq \frac{P_T}{0.3L^2B} \sqrt{\frac{720}{N+4}} \sigma_T \quad (29)$$

$$\frac{\sigma_{P_T, m.s}}{P_T} \simeq \frac{0.0136^1}{0.3\beta BL} \sqrt{\frac{d_{tot}}{X_0}} \quad (30)$$

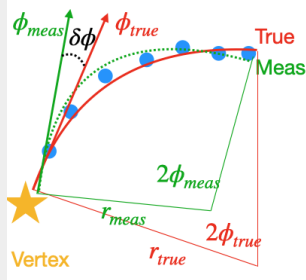
Units in GeV/c

- Momentum resolution comprises geometrical term and scattering term
- In practice, empirical rescaling factor should be multiplied

<sup>1</sup>G.R. Lynch and O.I Dahl, Nucl. Instrum. Methods B58, 6 (1991).



# Covariance Matrix in Helix Track



- Variance in momentum modifies the curvature of the helix → direction at the vertex changes.
- 'Position' of the helix is defined from the TPC hits. → Center-of-gravity should be fixed.

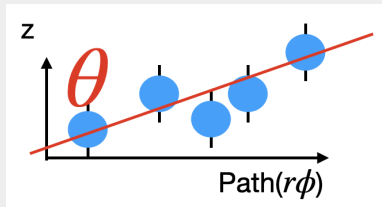
Denote the tangent angle at the center be  $\phi_0$  and path length to the vertex  $l$ .

$$\phi = \phi_0 \pm \frac{l}{2r}; \quad \delta\phi = \pm \frac{l}{2r} \frac{\delta r}{r} = \pm \frac{l}{r} \frac{\delta p_T}{p_T} \quad (31)$$

$$\sigma_\phi^2 = \langle \delta\phi \delta\phi \rangle = \frac{l^2}{r^2 p_T^2} \sigma_{p_T}^2; \quad \text{Cov}(\phi, p_T) = \langle \delta\phi \delta p_T \rangle = \pm \frac{l}{r p_T} \sigma_{p_T}^2 \quad (32)$$



# Covariance Matrix in Helix Track



$$h(t) : \{r \cos(\phi) - c_x, r \sin \phi - c_y, dz * r\phi - z_0\} \quad (33)$$

- The 'pitch' parameter,  $dz$ , is the slope along the circular trajectory

$\theta = \frac{\pi}{2} - \arctan(dz)$ , we estimate the variance of  $\theta$  based on the fitting error of  $dz$ . The error is estimated from the slope error of a linear fit:

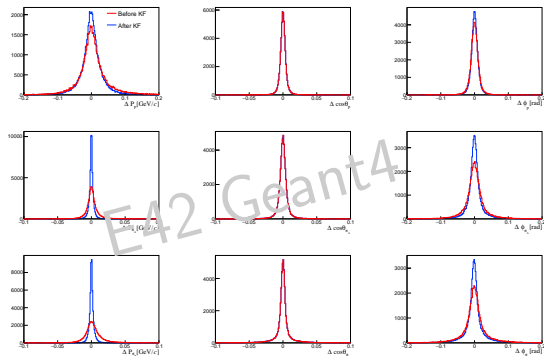
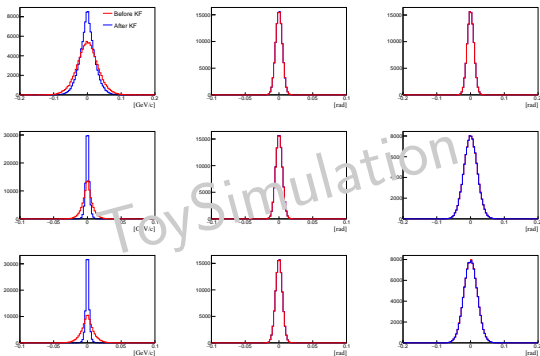
$$\sigma_{dz}^2 = \frac{\sum \delta_z^2 / (n-2)}{\sum (x - \bar{x})^2} \simeq \frac{n\sigma_z^2 / (n-2)}{nL^2/12}; \quad \sigma_\theta = \frac{\partial dz}{\partial \theta} \sigma_{dz} = \frac{1}{1 + dz^2} \sigma_{dz}.$$

Note that, the momentum  $p_z = p_T dz$  would also have some covariance with  $\theta$ ,

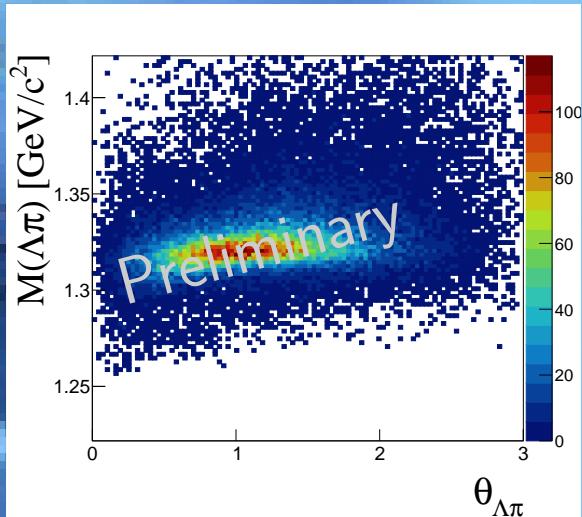
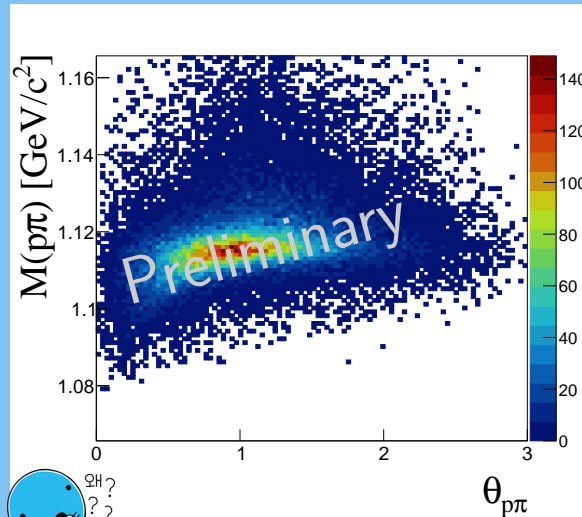
$$\langle \delta p \delta \theta \rangle = dz \langle \delta p_T \delta \theta \rangle + p_T \langle \delta dz \delta \theta \rangle = \frac{p_T}{1 + dz^2} \sigma_{dz}^2.$$



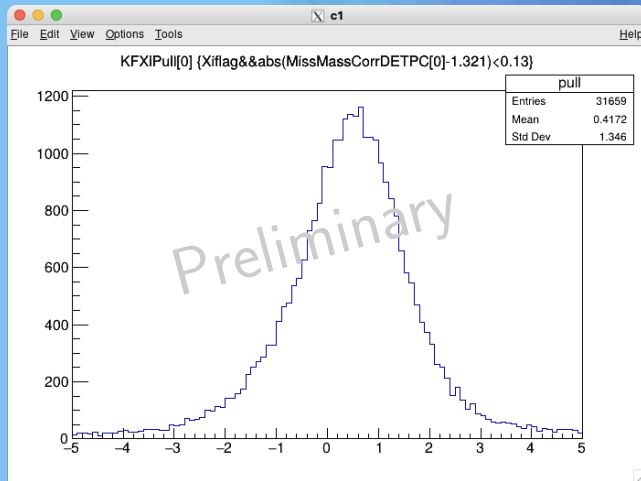
## $\phi$ Restoration from Diagonal Component



# Invariant Mass Bending from Momentum Bias

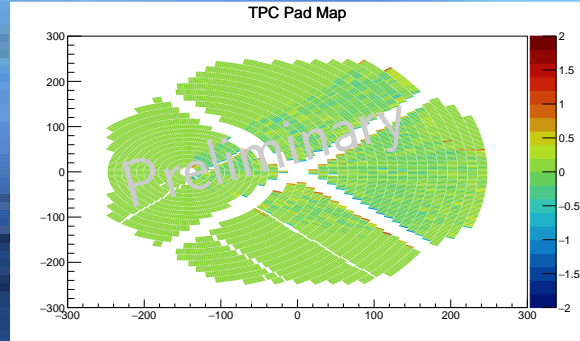
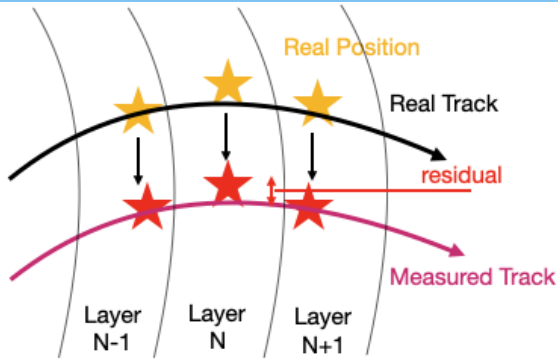


# Off-center Pull Distribution



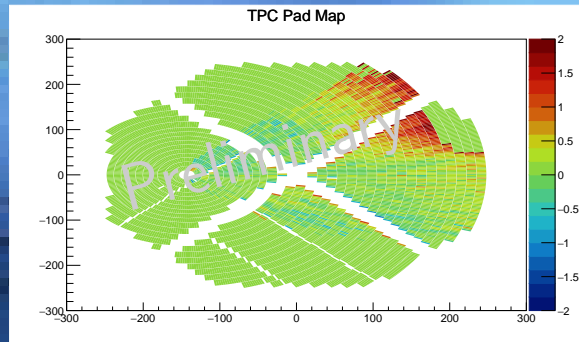
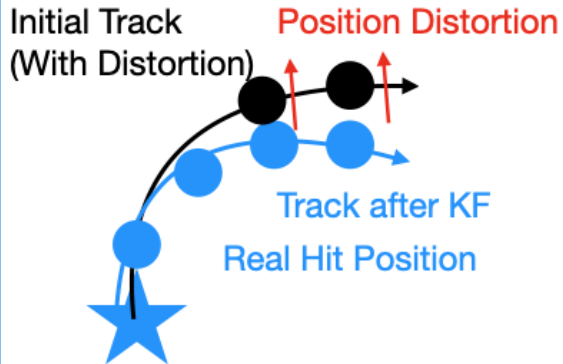


# Position Residual?



- Local, but simultaneous shift cannot be detected from position residual measurement.
- External reference for track should be provided to estimate 'true' trajectory

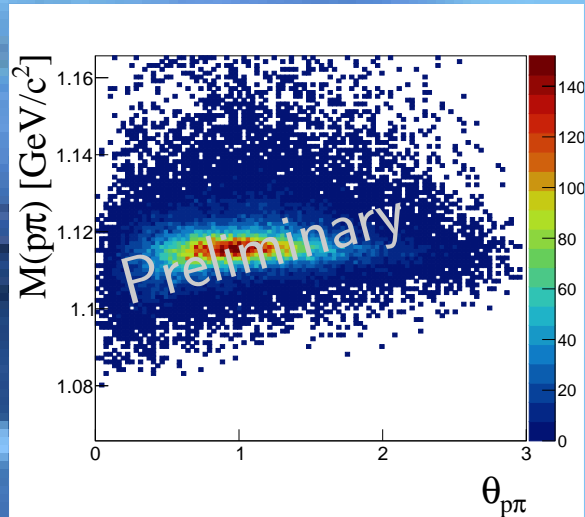
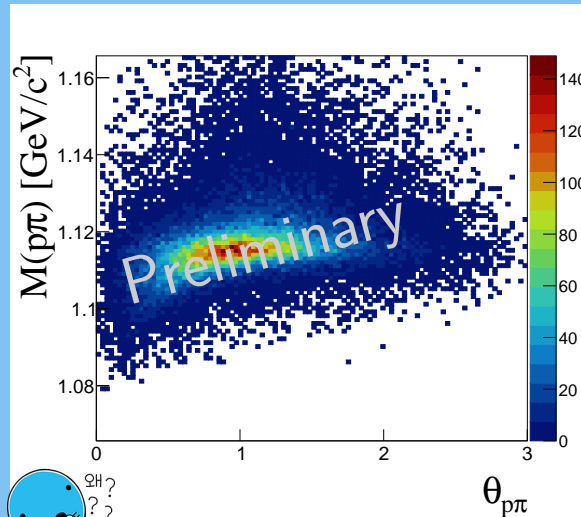
# Position Residual from KF Track



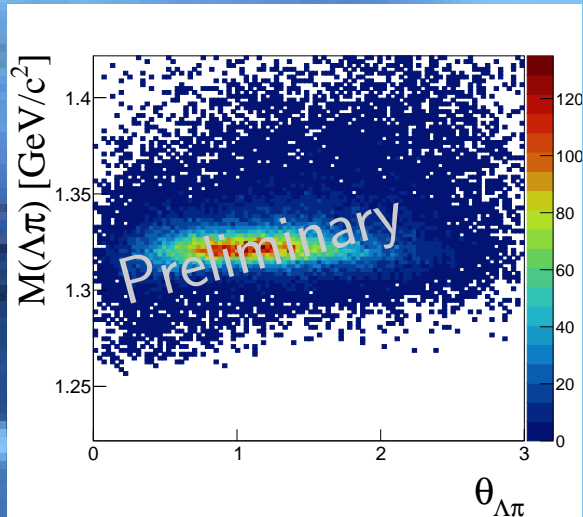
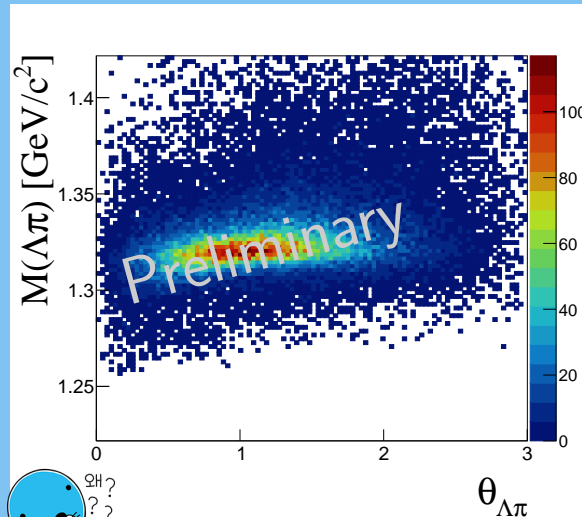
- From Kinematic fit, 'true' momentum, hence trajectory is estimated



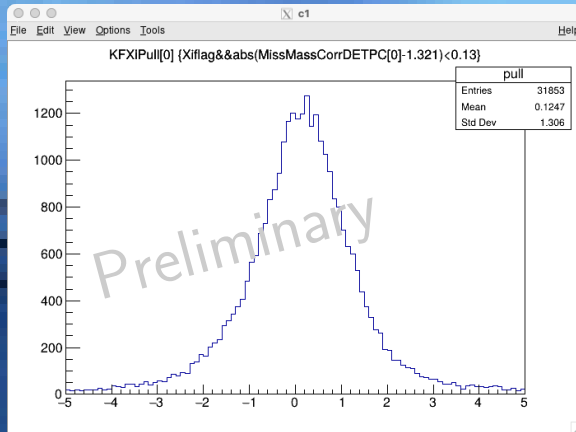
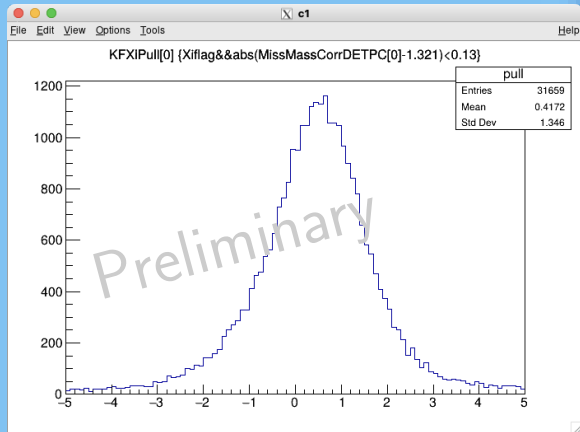
# $\Lambda$ After Position Correction



## ≡ After Position Correction



# Pull distribution After Position Correction





# Tricks



# Variance Normalization

$$V = \begin{pmatrix} 10^{12} & 0.9 \\ 0.9 & 10^{-12} \end{pmatrix} \rightarrow V^{-1} = ? \quad (34)$$

While taking an inverse of the variance, matrix elements with different order may be added together, leading to possible numerical instability.

$$\tilde{V} = SVS^T; S \equiv \frac{1}{\sqrt{V_{ij}}} \delta_{ij} \rightarrow \tilde{V} = \begin{pmatrix} 1 & \text{Cov}(01)/\sigma_1\sigma_2 & \cdots \\ \text{Cov}(01)/\sigma_1\sigma_2 & 1 & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix} \quad (35)$$

We can take out scaling factors in S. Measurement vectors could share the same problem, so they should also be scaled. We rewrite equation (15)

$$\chi^2 = dM^\dagger V^{-1} dM + \cdots = d\tilde{M}^\dagger \tilde{V}^{-1} d\tilde{M} + \cdots; \quad d\tilde{M} = S(M - M_0) \quad (36)$$



# Off-diagonal Reduction

$$V = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \rightarrow V^{-1} = ? \quad (37)$$

Adding off-diagonal term could make matrix uninvertible. Also we require  $\chi^2 = dMV^{-1}dM > 0$ ;  $V^{-1}$  (hence  $V$ ) should be *Positive Definite*. Then, we can 'damp' the offdiagonal elements.

$$\text{while}(\text{IsPositiveDefinite}(V)) \quad (38)$$

$$V_{ij} \rightarrow V_{ij} - \alpha(\delta_{ij} - 1)V_{ij} \quad (39)$$

## Property of Positive Definite Matrix

All Eigenvalues are Positive! TMatrixD well-supports eigenvalues, so we can just use it.





# Summary

- Measurement error can be reduced by correlating the measurements with physical constraints; This process is Kinematic Fit.
- By observing statistical parameters, we can estimate many physical context other than error reduction.(i.e. Bias estimation, S/N separation, etc...)
- Proper understandings of covariance matrix is required for Kinematic Fit.



# Appendix



# Minimization Steps.

The coupled differential equations (17),(19) and (18) will be solved iteratively. For each  $\nu$ th step,

$$V^{-1}(\mathbf{m}^0)(\mathbf{m}^{\nu+1} - \mathbf{m}^0) + (\mathbf{F}_m^\dagger)^\nu \lambda^{\nu+1} = 0 \quad (40)$$

$$(\mathbf{F}_u^\dagger)^\nu \lambda^{\nu+1} = 0 \quad (41)$$

$$\mathbf{f}^\nu + \mathbf{F}_m^\nu(\mathbf{m}^{\nu+1} - \mathbf{m}^\nu) + \mathbf{F}_u^\nu(\mathbf{u}^{\nu+1} - \mathbf{u}^\nu) = 0. \quad (42)$$

Equation (42) is not a direct consequence of Equation (18) but rather a *linear approximation*.

Note that, as we are determining the parameters  $\mathbf{m}$ ,  $\mathbf{u}$  and  $\lambda$ , they are indexed as  $\nu + 1$ , while constraint terms(i.e.  $\mathbf{f}$ ,  $\mathbf{F}_\mu$  and  $\mathbf{F}_\nu$ ) are calculated from current step,  $\nu$ . This fit is basically using Newton's method.



# Coupled Equation Solving (1)

Multiplying  $\mathbf{V}$  to Equation (40) leads to:

$$\mathbf{m}^{\nu+1} - \mathbf{m}^0 = -V(\mathbf{m}^0)(\mathbf{F}_m^\dagger)^\nu \lambda^{\nu+1}. \quad (43)$$

Substituting Equation (43) into Equation (42),

$$\begin{aligned} \mathbf{F}_u^\nu (\mathbf{u}^{\nu+1} - \mathbf{u}^\nu) &= -\mathbf{f}^\nu - \mathbf{F}_m^\nu (-V(\mathbf{m}^0)(\mathbf{F}_m^\dagger)^\nu \lambda^{\nu+1} + \mathbf{m}^0 - \mathbf{m}^\nu) \\ &= S\lambda^{\nu+1} - R \end{aligned} \quad (44)$$

where we define the *constraint covariance*  $S$  and *residual*  $R$  as:

$$S \equiv \mathbf{F}_m^\nu V(\mathbf{m}^0)(\mathbf{F}_m^\dagger)^\nu; \quad R \equiv \mathbf{f}^\nu + \mathbf{F}_m^\nu (\mathbf{m}^0 - \mathbf{m}^\nu) \quad (45)$$

Multiplying  $(\mathbf{F}_u^\dagger)^\nu S^{-1}$  into (44), we get:

$$(\mathbf{F}_u^\dagger)^\nu S^{-1} \mathbf{F}_u^\nu (\mathbf{u}^{\nu+1} - \mathbf{u}^\nu) = \cancel{(\mathbf{F}_u^\dagger)^\nu \lambda^{\nu+1}}^{0, \because (41)} - (\mathbf{F}_u^\dagger)^\nu S^{-1} R. \quad (46)$$



## Coupled Equation Solving (2)

Then we naturally derive the expressions

$$\mathbf{u}^{\nu+1} = \mathbf{u}^{\nu} - ((\mathbf{F}_u^{\dagger})^{\nu} S^{-1} \mathbf{F}_u^{\nu})^{-1} (\mathbf{F}_u^{\dagger})^{\nu} S^{-1} R. \quad (47)$$

and from (44),

$$\lambda^{\nu+1} = S^{-1} (\mathbf{F}_u^{\nu} (\mathbf{u}^{\nu+1} - \mathbf{u}^{\nu}) + R). \quad (48)$$

For a summary, we have obtained all equations to proceed to the next step. All other matrices in the equation can be calculated from parameters of the current step, and  $\chi^2$  can be evaluated from (15) .

$$\begin{cases} \mathbf{u}^{\nu+1} = \mathbf{u}^{\nu} - ((\mathbf{F}_u^{\dagger})^{\nu} S^{-1} \mathbf{F}_u^{\nu})^{-1} (\mathbf{F}_u^{\dagger})^{\nu} S^{-1} R & (47) \\ \lambda^{\nu+1} = S^{-1} (\mathbf{F}_u^{\nu} (\mathbf{u}^{\nu+1} - \mathbf{u}^{\nu}) + R) & (48) \\ \mathbf{m}^{\nu+1} = \mathbf{m}^0 - V(\mathbf{m}^0) (\mathbf{F}_m^{\dagger})^{\nu} \lambda^{\nu+1} & (43) \end{cases}$$



# Covariance Matrix Propagation

We estimate the covariance for the fitted variables,  $m$ ,

$$V(m) = J_{m,m^0} V(m^0) J_{m,m^0}^\dagger \quad (49)$$

To solve (49) the Jacobian should be determined.

$$J_{m,m^0}(i,j) = \frac{\partial m_i}{\partial m_j^0} \quad (50)$$



# Jacobian

To begin with, let us express Eq (43) in terms of  $m^0$ . At the moment we will drop the superscript  $v$ . As  $\mathbf{f}(\mathbf{m}, \mathbf{u})$  is a constant on  $m^0$ ,  $\mathbf{F}_m$  also will be a constant to  $m^0$ . Then we only need to consider the derivatives of  $\lambda$ . By substituting (47) ,

$$\lambda = S^{-1}(-\mathbf{F}_u(((\mathbf{F}_u^\dagger)S^{-1}\mathbf{F}_u)^{-1}(\mathbf{F}_u^\dagger)S^{-1}R) + R) \quad (51)$$

and the residual matrix is:

$$R \equiv \mathbf{f} + \mathbf{F}_m(\mathbf{m}^0 - \mathbf{m}) \rightarrow \frac{\partial R}{\partial m^0} = \mathbf{F}_m \quad (52)$$

Now we obtain the derivative of  $\lambda$  as:

$$\frac{\partial \lambda}{\partial m^0} = S^{-1}(-\mathbf{F}_u((\mathbf{F}_u^\dagger S^{-1} \mathbf{F}_u)^{-1} \mathbf{F}_u^\dagger S^{-1} \mathbf{F}_m) + \mathbf{F}_m). \quad (53)$$



# Jacobian

Now define the symmetric matrices

$$G \equiv \mathbf{F}_m^\dagger S^{-1} \mathbf{F}_m; \quad U \equiv (\mathbf{F}_u^\dagger S^{-1} \mathbf{F}_u)^{-1}; \quad H \equiv \mathbf{F}_m^\dagger S^{-1} \mathbf{F}_u \quad (54)$$

We have expressions for  $\frac{\partial \lambda}{\partial m^0}$ . Equation (50) is determined as:

$$\begin{aligned} J_{m,m^0} &= I - V(m^0) \mathbf{F}_m^\dagger \frac{\partial \lambda}{\partial m^0} = I - V \mathbf{F}_m^\dagger (-S^{-1} \mathbf{F}_u U^{-1} H^\dagger + S^{-1} \mathbf{F}_m) \\ &= I - V(G - HUH^\dagger) \end{aligned} \quad (55)$$

If we let  $C = G - HUH^\dagger$ , we obtain

$$V(m) = J_{m,m^0} V J_{m,m^0}^\dagger = V - 2VCV + VCVCV. \quad (56)$$

We would keep 2nd order term at the moment. Some materials like [1] had an error in this part.





# Variance of the Unknowns

Just like how we derived Eq.(56) we can estimate the variance matrix of the unknowns.

$$V_U = J_{u,m0} V J_{u,m0}^T \quad (57)$$

$J_{u,m0}$  can be obtained from Eq.(47). Defining

$$K \equiv ((\mathbf{F}_u^\dagger)^\nu S^{-1} \mathbf{F}_u^\nu)^{-1} (\mathbf{F}_u^\dagger)^\nu S^{-1} \quad (58)$$

we write:

$$J_{u^{\nu+1},m^0} = \frac{\partial u^{\nu+1}}{\partial m^0} = \frac{\partial u^\nu}{\partial m^0} - K \frac{\partial R}{\partial m^0} \simeq -K \mathbf{F}_m. \quad (59)$$

Note that we only have initial "Guess" for the unknowns; In principle, it is not a driven value from measurements. Then,  $\frac{\partial u^0}{\partial m^0} = 0$ . Also, we approximate that the terms in 2nd or higher iterations are negligible:  $J_{u,m^0} \simeq J_{u^0,m^0}$ .



# Pull distribution

The covariance in Equation (21) is estimated as:

$$\text{Cov}(m, m^0) = J_{m, m^0} V(m) = V - VCV. \quad (60)$$

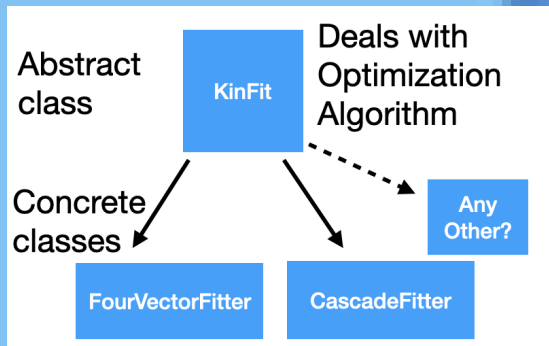
If we substitute this and Eq.(56) into Eq.(21), we get

$$V(\epsilon) = VCVCV. \quad (61)$$

Note that 2nd order term affects the covariance matrix of the correction.



# KinFit Package



$$\chi^2 = \delta M^T V^{-1} \delta M + 2\lambda f(M, U) \quad (62)$$

## KinFit provides...

- Minimize  $\chi^2$
- Calculate pulls and p-values

## Users should...

- Assign proper variables for M and U
- Define physical constraints
- Write the derivatives,  $F_M$  and  $F_U$  by hand



# References



O. Skjeggstad Frodesen A.G. **Probability and Statistics in Particle Physics**. Columbia University Press, 1980. ISBN: 8200019063.

