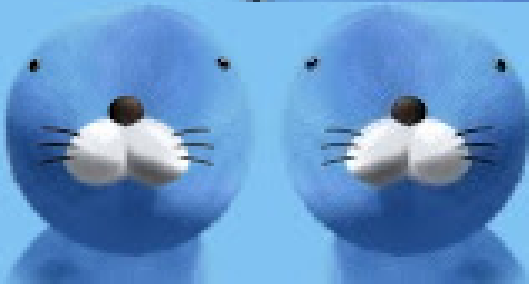


난 다 좋아



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Notes on Kinematic Fit

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Introduction

Assume that we have measured a momentum of two particles, which decay from a mother particle with an exact momentum P_0 . In the real world, every measurement inherently carries some errors. Consequently, the measured momentum may not satisfy momentum conservation:

$$P_0 = P_1 + P_2; \quad P_{1,meas} + P_{2,meas} \neq P_0 \quad (1)$$

However, since these measurements did not incorporate our prior knowledge from physics, we can make a more informed estimate of the measured parameters. If the momentum resolution of particles 1 and 2 is well-known, then we can express χ^2 as

$$\chi^2 = \frac{(P_1 - P_{1,meas})^2}{\sigma_1^2} + \frac{(P_2 - P_{2,meas})^2}{\sigma_2^2} \quad (2)$$



Introduction

By incorporating the *Kinematic Constraints*, specifically *momentum conservation*, into our example, we introduce additional terms known as *Lagrange Multiplier* to Equation (3):

$$\chi^2 = \frac{(P_1 - P_{1,meas})^2}{\sigma_1^2} + \frac{(P_2 - P_{2,meas})^2}{\sigma_2^2} + 2\lambda(P_{1,meas} + P_{2,meas} - P_0) \quad (3)$$

We then proceed to evaluate the conditions for local minima, i.e. setting the partial derivatives equal to zero:

$$\frac{1}{2} \frac{\partial \chi^2}{\partial P_1} = \frac{(P_1 - P_{1,meas})}{\sigma_1^2} + \lambda = 0 \quad (4)$$

$$\frac{1}{2} \frac{\partial \chi^2}{\partial P_2} = \frac{(P_2 - P_{2,meas})}{\sigma_2^2} + \lambda = 0 \quad (5)$$

$$\frac{1}{2} \frac{\partial \chi^2}{\partial \lambda} = (P_1 + P_2 - P_0) = 0 \quad (6)$$



Introduction

By solving the equations 4,5,6, we obtain the following expressions:

$$P_1 = \frac{\sigma_2^2 P_{1,meas} - \sigma_1^2 P_{2,meas} + \sigma_1^2 P_0}{\sigma_1^2 + \sigma_2^2} \quad (7)$$

$$P_2 = \frac{\sigma_1^2 P_{2,meas} - \sigma_2^2 P_{1,meas} + \sigma_2^2 P_0}{\sigma_1^2 + \sigma_2^2} \quad (8)$$

$$\lambda = \frac{P_{1,meas} + P_{2,meas} - P_0}{\sigma_1^2 + \sigma_2^2}. \quad (9)$$

Now, we have obtained the 'corrected' measurements with minimized χ^2 , which incorporates momentum conservation. Let us delve into the interpretation of these equations.

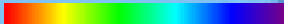


Introduction

In a straightforward interpretation, λ can be viewed as a kind of 'normalized variance' of the kinematic constraint. It quantifies the error in momentum conservation($P_{1,meas} + P_{2,meas} - P_0$) relative to the overall resolution($\sigma_1^2 + \sigma_2^2$). Equation (4) implies that

$$P_1 = P_{1,meas} + \sigma_1^2 \lambda, \quad (10)$$

suggesting that the corrected momentum(P_1) is essentially the measured momentum($P_{1,meas}$) augmented by a term proportional to the detector resolution and the normalized error of the kinematic constraint. Thus, we can assert that we have applied a statistically fair correction to the momentum, taking into account both the detector resolution and kinematic constraints.



Fitting in General

Assume that you have a set of measurements, $\mathbf{m} = \{m_1, m_2 \dots m_N\}$, and some unmeasured data, $\mathbf{u} = \{u_1, u_2 \dots u_J\}$ to be estimated. Kinematic constraints can be represented by sets of equations $\mathbf{f} = \{f_1(m_1, m_2, \dots m_N, u_1, u_2, \dots u_N), f_2, \dots f_K\}$. We will iteratively solve the problem by guessing the best parameter for each step and checking χ^2 . Let \mathbf{m}^0 denote our initial measured data, and \mathbf{m} represent the 'guess' of the data in each iterative step.

$$\chi^2(\mathbf{m}) = (\mathbf{m}^0 - \mathbf{m})^T V^{-1} (\mathbf{m}^0 - \mathbf{m}) + 2\lambda^T \mathbf{f}(\mathbf{m}, \mathbf{u}). \quad (11)$$

Here, the Lagrange multiplier $\lambda = \{\lambda_1, \lambda_2, \dots \lambda_K\}$ is not just a number but a column vector with k elements, corresponding to each kinematic constraint in \mathbf{f} . Our task is to minimize χ^2 to obtain the best guesses in statistically fair method.



χ^2 Minimization

By (partially)differentiating with respect to all variables involved, we obtain the gradients of χ^2 . Setting all of them to zero indicates that we have reached a minimum point of χ^2 . We have 3 sets of gradient equations:

$$\nabla_{\mathbf{m}} = -2V^{-1}(\mathbf{m}^0)(\mathbf{m}^0 - \mathbf{m}) + 2\mathbf{F}_{\mathbf{m}}^T(\mathbf{m}, \mathbf{u})\lambda = 0 \quad (12)$$

$$\nabla_{\mathbf{u}} = 2\mathbf{F}_{\mathbf{u}}^T(\mathbf{m}, \mathbf{u})\lambda = 0 \quad (13)$$

$$\nabla_{\lambda} = \mathbf{f}(\mathbf{m}, \mathbf{u}). \quad (14)$$

Here, the subscripts denote partial derivatives. i.e. $((\mathbf{F}_m)_{ki} \equiv \frac{\partial f_k}{\partial m_i})$.



Processing Iterative Steps.

We can express the following equations based on the ones provided above:

$$\mathbf{V}^{-1}(\mathbf{m}^0)(\mathbf{m}^{\nu+1} - \mathbf{m}^0) + (\mathbf{F}_m^T)^\nu \lambda^{\nu+1} = 0 \quad (15)$$

$$(\mathbf{F}_u^T)^\nu \lambda^{\nu+1} = 0 \quad (16)$$

$$\mathbf{f}^\nu + \mathbf{F}_m^\nu(\mathbf{m}^{\nu+1} - \mathbf{m}^\nu) + \mathbf{F}_u^\nu(\mathbf{u}^{\nu+1} - \mathbf{u}^\nu) = 0. \quad (17)$$

Equation (17) is not a direct consequence of Equation (13) but rather a linear approximation to proceed with our iteration step. Expanding the ∇_λ term with a Taylor series leads to this equation. Note that, as our parameters \mathbf{m} and \mathbf{u} are updated during the step, our constraint matrix \mathbf{f} should also be updated during the iteration. Here, λ should be indexed as $\nu + 1$ since it is a parameter to be guessed in the next step.



Solving the Equation(1)

Multiplying \mathbf{V} to Equation (15) leads to:

$$\mathbf{m}^{\nu+1} - \mathbf{m}^0 = -V(\mathbf{m}^0)(\mathbf{F}_m^T)^\nu \lambda^{\nu+1}. \quad (18)$$

Substituting Equation (18) into Equation (17), we get:

$$\begin{aligned} \mathbf{F}_u^\nu (\mathbf{u}^{\nu+1} - \mathbf{u}^\nu) &= -\mathbf{f}^\nu - \mathbf{F}_m^\nu (-V(\mathbf{m}^0)(\mathbf{F}_m^T)^\nu \lambda^{\nu+1} + \mathbf{m}^0 - \mathbf{m}^\nu) \\ &= S\lambda^{\nu+1} - R \end{aligned} \quad (19)$$

where $S \equiv \mathbf{F}_m^\nu V(\mathbf{m}^0)(\mathbf{F}_m^T)^\nu$ and $R \equiv \mathbf{f}^\nu + \mathbf{F}_m^\nu (\mathbf{m}^0 - \mathbf{m}^\nu)$. Multiplying $(\mathbf{F}_u^T)^\nu S^{-1}$ and substituting Equation (16), we get:

$$(\mathbf{F}_u^T)^\nu S^{-1} \mathbf{F}_u^\nu (\mathbf{u}^{\nu+1} - \mathbf{u}^\nu) = \cancel{(\mathbf{F}_u^T)^\nu \lambda^{\nu+1}}^0 - (\mathbf{F}_u^T)^\nu S^{-1} R. \quad (20)$$

왜?
??



Solving the Equation(2)

Then we naturally obtain:

$$\mathbf{u}^{\nu+1} = \mathbf{u}^{\nu} - ((\mathbf{F}_u^T)^{\nu} S^{-1} \mathbf{F}_u^{\nu})^{-1} (\mathbf{F}_u^T)^{\nu} S^{-1} R. \quad (21)$$

and from Equation (19)

$$\lambda^{\nu+1} = S^{-1} (\mathbf{F}_u^{\nu} (\mathbf{u}^{\nu+1} - \mathbf{u}^{\nu}) + R) \quad (22)$$

. For a summary, we have obtained all equations to proceed to the next step. All other matrices in the equation can be calculated from parameters of the current step, and χ^2 can be evaluated from (11) .

$$\begin{cases} \mathbf{u}^{\nu+1} = \mathbf{u}^{\nu} - ((\mathbf{F}_u^T)^{\nu} S^{-1} \mathbf{F}_u^{\nu})^{-1} (\mathbf{F}_u^T)^{\nu} S^{-1} R & (21) \\ \lambda^{\nu+1} = S^{-1} (\mathbf{F}_u^{\nu} (\mathbf{u}^{\nu+1} - \mathbf{u}^{\nu}) + R) & (22) \\ \mathbf{m}^{\nu+1} = \mathbf{m}^0 - V(\mathbf{m}^0) (\mathbf{F}_m^T)^{\nu} \lambda^{\nu+1} & (18) \end{cases}$$

왜?
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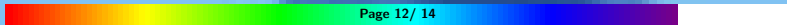
Evolution of Variance Matrix

- to be filled...



Pull distribution

- to be filled...



Example: 3-C Fit

Assuming a decay of $\Lambda \rightarrow p\pi^-$, let's represent the momentum of those particles in spherical coordinates:

$$\vec{P}_\Lambda = (P_\Lambda, \theta_\Lambda, \phi_\Lambda), \dots \quad (23)$$

Then the kinematic constraints can be expressed as:

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} -P_\Lambda \sin \theta_\Lambda \cos \phi_\Lambda + P_p \sin \theta_p \cos \phi_p + P_\pi \sin \theta_\pi \cos \phi_\pi \\ -P_\Lambda \sin \theta_\Lambda \sin \phi_\Lambda + P_p \sin \theta_p \sin \phi_p + P_\pi \sin \theta_\pi \sin \phi_\pi \\ -P_\Lambda \cos \theta_\Lambda + P_p \cos \theta_p + p_\pi \cos \theta_\pi \\ -\sqrt{P_\Lambda^2 + m_\Lambda^2} + \sqrt{P_p^2 + m_p^2} + \sqrt{P_\pi^2 + m_\pi^2} \end{pmatrix}. \quad (24)$$



Example: 3-C Fit

We have unmeasured and measured variables as:

$$\mathbf{u} = \{P_{\Lambda}\}; \quad \mathbf{m} = \{\theta_{\Lambda}, \phi_{\Lambda}, P_p, \theta_p, \phi_p, P_{\pi}, \theta_{\pi}, \phi_{\pi}\}. \quad (25)$$

Since there is 1 unmeasured variable with 4 kinematical constraints, this is a $4-1 = 3$ -Constrained fit. Let us substitute Eq (24) and (25) into Eq (11)s and its resulting equations. We get $\mathbf{F}_{\mathbf{u}}$ and $\mathbf{F}_{\mathbf{m}}$ as

$$\mathbf{F}_{\mathbf{u}} = \begin{pmatrix} \frac{\partial f_1}{\partial P_{\Lambda}} \\ \frac{\partial f_2}{\partial P_{\Lambda}} \\ \frac{\partial f_3}{\partial P_{\Lambda}} \\ \frac{\partial f_4}{\partial P_{\Lambda}} \end{pmatrix}; \quad \mathbf{F}_{\mathbf{m}} = \begin{pmatrix} \frac{\partial f_1}{\partial \theta_{\Lambda}} & \cdots & \frac{\partial f_1}{\partial \phi_{\pi}} \\ \frac{\partial f_2}{\partial \theta_{\Lambda}} & \cdots & \frac{\partial f_2}{\partial \phi_{\pi}} \\ \frac{\partial f_3}{\partial \theta_{\Lambda}} & \cdots & \frac{\partial f_3}{\partial \phi_{\pi}} \\ \frac{\partial f_4}{\partial \theta_{\Lambda}} & \cdots & \frac{\partial f_4}{\partial \phi_{\pi}} \end{pmatrix} \quad (26)$$

ΩH? We have all the matrices to calculate in each step. By applying an
? appropriate variance matrix and employing χ^2 selection criteria,
? we can kinematically fit the particles.

