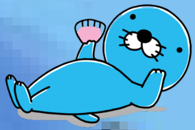

Notes on Kinematic Fit

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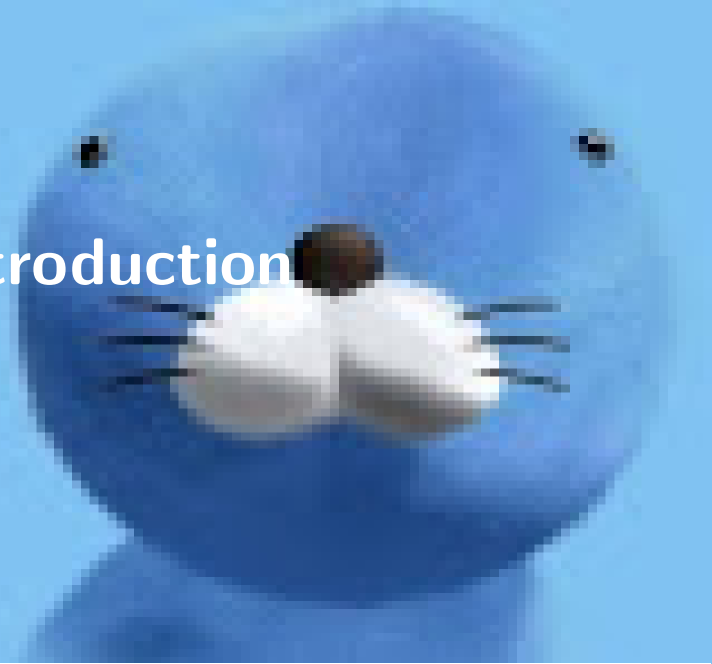
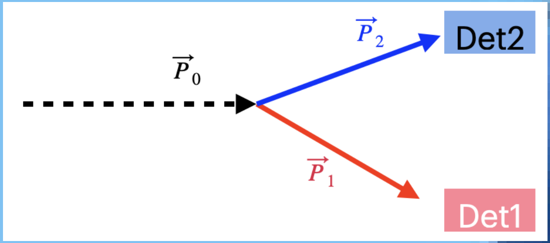
Outline

1. Introduction
2. Example: Mass-Constraint Fit
3. Applications: Ξ decay at J-PARC
4. Applications to Position Correction
5. Tricks
6. Calculations

Introduction



Measurement Error



Assume a beam with momentum $\vec{P}^-$ decays into $\vec{P}_1$ and $\vec{P}_2$. Measured momentum are smeared due to detector resolution, leading to unbalance in the momentum conservation.

$$\vec{P}_0 = \vec{P}_1 + \vec{P}_2; \quad \vec{P}_{1,meas} + \vec{P}_{2,meas} \neq \vec{P}_0 \tag{1}$$

We can define the χ^2 to quantitatively represent our measurement error. However, we can't derive meaningful expressions from this χ^2.

$$\chi^2 = \frac{(P_1 - P_{1,meas})^2}{\sigma_1^2} + \frac{(P_2 - P_{2,meas})^2}{\sigma_2^2} \tag{2}$$

Constrained Optimization with The Lagrange Multiplier

By incorporating the *Kinematic Constraints*, specifically *momentum conservation*, we involve additional knowledge to (2). This is known as the *Lagrange Multiplier*

$$\chi^2 = \frac{(P_{1,KF} - P_{1,meas})^2}{\sigma_1^2} + \frac{(P_{2,KF} - P_{2,meas})^2}{\sigma_2^2} + 2\lambda(\mathbf{P_{1,KF}} + \mathbf{P_{2,KF}} - \mathbf{P_0}) \tag{3}$$

Now we have meaningful expressions to minimize χ^2, hence get better estimations for the measurement.

$$\frac{1}{2}\frac{\partial \chi 2}{\partial P_{1,KF}} = \frac{(P_{1,KF} - P_{1,meas})}{\sigma_1^2} + \lambda = 0 \tag{4}$$

$$\frac{1}{2}\frac{\partial \chi 2}{\partial P_{2,KF}} = \frac{(P_{2,KF} - P_{2,meas})}{\sigma_2^2} + \lambda = 0 \tag{5}$$

$$\frac{1}{2}\frac{\partial \chi 2}{\partial \lambda} = (P_{1,KF} + P_{2,KF} - P_0) = 0 \tag{6}$$

Why Better Resolution?

By solving the equations 4,5,6 and defining $\delta_i = P_{i,meas} - P_i$, we obtain the following expressions:

$$\lambda = \frac{P_{1,meas} + P_{2,meas} - P_0}{\sigma_1^2 + \sigma_2^2} = \frac{\delta_1 + \delta_2}{\sigma_1^2 + \sigma_2^2} \tag{7}$$

$$P_{1,KF} = P_{1,meas} - \sigma_1^2 \lambda \tag{8}$$

$$P_{2,KF} = P_{2,meas} - \sigma_2^2 \lambda \tag{9}$$

$$\begin{aligned} < P_{1,KF} - P_1 > &=< P_{1,KF} - P_{1,meas} + \delta_1 >=< -\sigma_1^2 \lambda + \delta_1 > \\ &=< \frac{-\sigma_1^2}{\sigma_1^2 + \sigma_2^2}(\delta_1 + \delta_2) + \delta_1 >=< \frac{\sigma_2^2 \delta_1 - \sigma_1^2 \delta_2}{\sigma_1^2 + \sigma_2^2} > \end{aligned} \tag{10}$$

$$\sigma_{1,KF}^2 =< (P_{1,KF} - P_1)^2 >= \frac{\sigma_2^4 \cancelto{\sigma_1^2}{< \delta_1^2 >} + \sigma_1^4 \cancelto{\sigma_2^2}{< \delta_2^2 >}}{(\sigma_1^2 + \sigma_2^2)^2} = \frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2} < \sigma_1^2 \tag{11}$$

The Covariance After Kinematic Fit

$$
\begin{aligned}
cov(P_1, P_2)_{KF} &= < \delta_{1,KF}\delta_{2,KF} > = < (\delta_1 - \sigma_1^2\lambda)(\delta_2 - \sigma_2^2\lambda) > \\
&= \sigma_1^2\sigma_2^2 \cancelto{\frac{1}{\sigma_1^2+\sigma_2^2}}{<\lambda^2>} - \frac{\sigma_1^2 < \delta_2^2 > + \sigma_2^2 < \delta_1^2 >}{\sigma_1^2+\sigma_2^2} = -\frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2}
\end{aligned}
\tag{12}
$$

$$
V = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix} \rightarrow V_{KF} = \begin{pmatrix} \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2} & -\frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2} \\ -\frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2} & \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2} \end{pmatrix} \tag{13}
$$

- Improved momentum resolution
- Negative correlation between P_1 and P_2

Generalization to Multi-Variables

Assume that we have a set of measured data $\mathbf{m^0}$, unknown parameters $\mathbf{u^0}$ and constraints $\mathbf{f}^0$.

$$\begin{aligned}\mathbf{m^0} &= \{m_1^0, m_2^0 \ldots m_N^0\}; \quad \mathbf{u^0} = \{u_1^0, u_2^0 \ldots u_J^0\} \\ \mathbf{f} &= \{f_1(m_1^0, m_2^0, \ldots m_N^0, u_1^0, u_2^0, \ldots u_N^0), f_2^0, \ldots f_K^0\}\end{aligned} \tag{14}$$

Let $\mathbf{m}^0$ denote our initial measured data, and $\mathbf{m}$ represent the 'guess' of the data in each iterative step, just alike P_{KF}s in the previous example. Equation (3) is generalized to:

$$\chi^2(\mathbf{m}) = (\mathbf{m}^0 - \mathbf{m})^\dagger V^{-1}(\mathbf{m}^0 - \mathbf{m}) + 2\lambda^\dagger \mathbf{f}(\mathbf{m}, \mathbf{u}). \tag{15}$$

Here, the Lagrange multiplier $\lambda = \{\lambda_1, \lambda_2, \ldots \lambda_K\}$ is not just a number but a column vector with k elements, corresponding to each kinematic constraint in $\mathbf{f}$.

χ^2 Minimization

We want to solve the equation

$$\vec{\nabla}\chi^2 = 0 \tag{16}$$

to obtain the minimized state. The differential term are listed within three groups.

$$\nabla_{\mathbf{m}} = -2V^{-1}(\mathbf{m}^0)(\mathbf{m}^0 - \mathbf{m}) + 2\mathbf{F}^\dagger_{\mathbf{m}}(\mathbf{m},\mathbf{u})\lambda = 0 \tag{17}$$

$$\nabla_{\mathbf{u}} = 2\mathbf{F}^\dagger_{\mathbf{u}}(\mathbf{m},\mathbf{u})\lambda = 0 \tag{18}$$

$$\nabla_{\lambda} = \mathbf{f}(\mathbf{m},\mathbf{u}). \tag{19}$$

Here, the subscripts denote partial derivatives. i.e. $((\mathbf{F}_m)_{ki} \equiv \frac{\partial f_k}{\partial m_i})$.

User Should Define...

$\mathbf{m}$	$\mathbf{u}$	$\mathbf{f}$	$\mathbf{V}$
Measured Data	Unknown parameters	Constraints	Covariance Matrix

Pull distribution

A bias or resolution miss-estimation is revealed by observing the *Pull distribution* of each measurements. By defining the residual $\epsilon = m - m^0$ and its variance $V(\epsilon)$, the pull is defined as:

$$P(\epsilon) = \epsilon/\sqrt{V(\epsilon)} \tag{20}$$

and

$$V(\epsilon) \equiv V(m) + V(m^0) - 2Cov(m, m^0). \tag{21}$$

The variance of the fitted variables, $V(m)$, is evaluated as

$$V(m) = J_{m,m^0} V(m^0) J^{\dagger}_{m,m^0} \tag{22}$$

where J_{m,m^0} is the Jacobian for m and m^0. Detailed calculations are provided in the appendix.

Example: Mass-Constraint Fit

Example: $\Lambda \to p\pi$, Defining Variables and Constraints

Assume a decay of $\Lambda \to p\pi^-$. We define the measurements and unknowns as:

$$\mathbf{m} = \{P_p, \theta_p, \phi_p, P_\pi, \theta_\pi, \phi_\pi\}; \quad \mathbf{u} = \{P_\Lambda, \theta_\Lambda, \phi_\Lambda\} \tag{23}$$

Then we define the energy-momentum constraint equation as:

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} -P_\Lambda \sin\theta_\Lambda \cos\phi_\Lambda + P_p \sin\theta_p \cos\phi_p + P_\pi \sin\theta_\pi \cos\phi_\pi \\ -P_\Lambda \sin\theta_\Lambda \sin\phi_\Lambda + P_p \sin\theta_p \sin\phi_p + P_\pi \sin\theta_\pi \sin\phi_\pi \\ -P_\Lambda \cos\theta_\Lambda + P_p \cos\theta_p + p_\pi \cos\theta_\pi \\ -\sqrt{P_\Lambda^2 + m_\Lambda^2} + \sqrt{P_p^2 + m_p^2} + \sqrt{P_\pi^2 + m_\pi^2} \end{pmatrix}. \tag{24}$$

where the mass constraint is naturally implemented in energy term.
Since we have 3 unmeasured variable with 4 kinematical constraints, this is a 4-3 = 1-Constrained fit.

Example: $\Lambda \to p\pi$, The Derivatives

We get $\mathbf{F_u}$ and $\mathbf{F_m}$ as

$$\mathbf{F_u} = \begin{pmatrix} \frac{\partial f_1}{\partial P_\Lambda} & \frac{\partial f_1}{\partial \theta_\Lambda} & \frac{\partial f_1}{\partial \phi_\Lambda} \\ \frac{\partial f_2}{\partial P_\Lambda} & \frac{\partial f_2}{\partial \theta_\Lambda} & \frac{\partial f_2}{\partial \phi_\Lambda} \\ \frac{\partial f_3}{\partial P_\Lambda} & \frac{\partial f_3}{\partial \theta_\Lambda} & \frac{\partial f_3}{\partial \phi_\Lambda} \\ \frac{\partial f_4}{\partial P_\Lambda} & \frac{\partial f_4}{\partial \theta_\Lambda} & \frac{\partial f_4}{\partial \phi_\Lambda} \end{pmatrix}; \quad \mathbf{F_m} = \begin{pmatrix} \frac{\partial f_1}{\partial P_p} & \cdots & \frac{\partial f_1}{\partial \phi_\pi} \\ \frac{\partial f_2}{\partial P_p} & \cdots & \frac{\partial f_2}{\partial \phi_\pi} \\ \frac{\partial f_3}{\partial P_p} & \cdots & \frac{\partial f_3}{\partial \phi_\pi} \\ \frac{\partial f_4}{\partial P_p} & \cdots & \frac{\partial f_4}{\partial \phi_\pi} \end{pmatrix} \tag{25}$$

We have all the matrices to calculate in each step. By applying an appropriate variance matrix and employing χ^2 selection criteria, we can do kinematic fit for the particles.

Example: $\Xi \to \Lambda\pi$, $\Lambda \to p\pi$

We require two mass constraints for $\Xi \to \Lambda\pi;\quad \Lambda \to p\pi$. In this case, careful considerations on the selection of variables. We will select

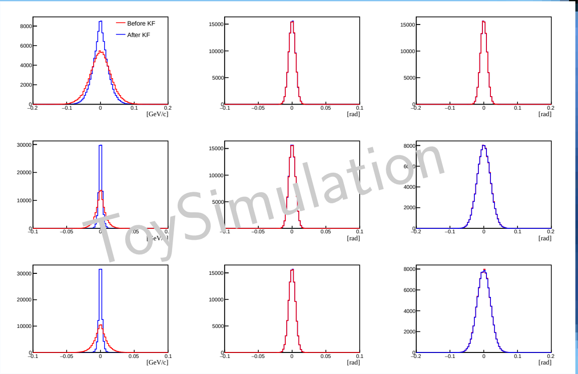
$$\mathbf{u} = \{P_\Xi, \theta_\Xi, \phi_\Xi\}; \quad \mathbf{m} = \{P_p, \theta_p, \phi_p, P_\pi, \theta_\pi, \phi_\pi\} \tag{26}$$

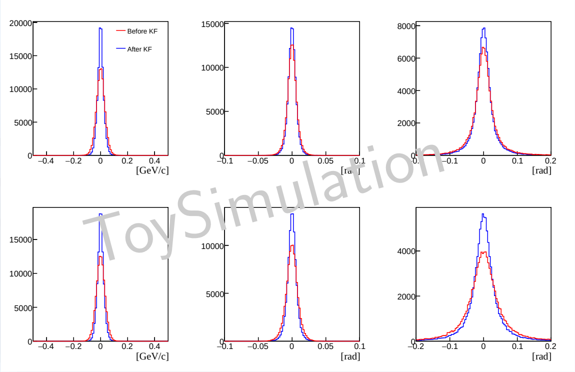
and define five constraints as:

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{pmatrix} = \begin{pmatrix} -P_{\Xi,x} + P_{p,x} + P_{\pi_\Lambda,x} + P_{\pi_\Xi,x} \\ -P_{\Xi,y} + P_{p,y} + P_{\pi_\Lambda,y} + P_{\pi_\Xi,y} \\ -P_{\Xi,z} + P_{p,z} + P_{\pi_\Lambda,z} + P_{\pi_\Xi,z} \\ -E_\Lambda + E_p + E_{\pi_\Lambda} \\ -E_\Lambda + E_p + E_{\pi_\Lambda} + E_{\pi_\Xi} \end{pmatrix}. \tag{27}$$

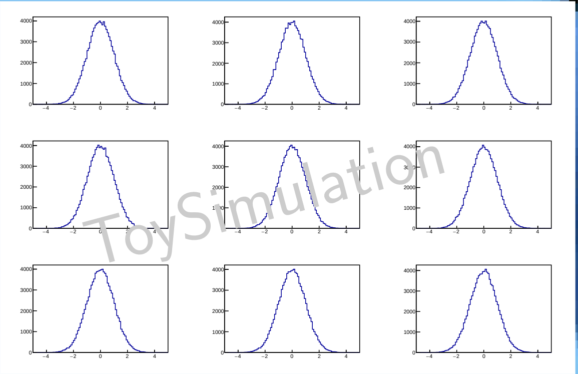
Λ variables are not selected in $\mathbf{u}$ to avoid negative DoF.($N_f - N_u = 5 - 6 = -1$)
Note that **we don't have explicit terms** related to $\vec{P}_\Lambda$, i.e. $-P_{\Lambda,x} + P_{p,x} + P_{\pi_\Lambda,x}$ etc., because $\vec{P}_\Lambda$ are neither unmeasured nor measured variables in our choice of parameters.

Kinematics Restoration





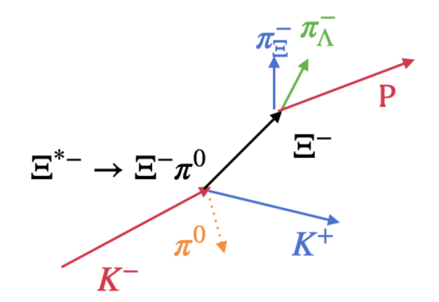
Pull Distribution

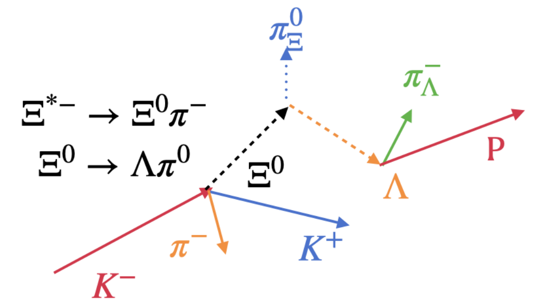


$$P(X) = \frac{X_{KF} - X_0}{\sqrt{V(X_{KF} - X_0)}} \tag{28}$$

- Pull distribution shows the normalized amount of parameter adjustment.
- In practice, resolution can be iteratively scaled by 1./ pull width

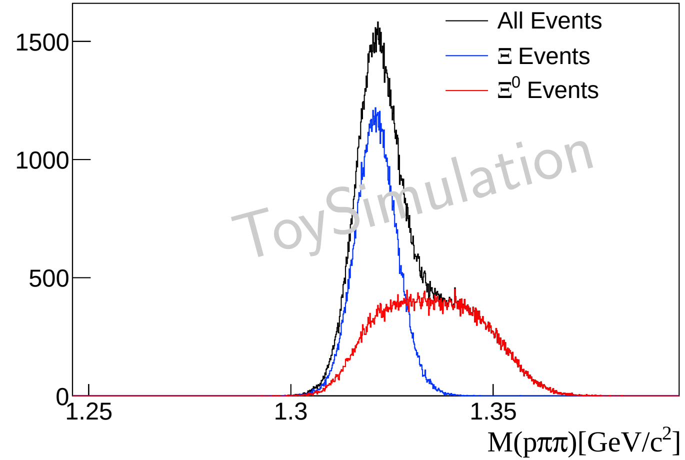
$\Xi^*(1530) \to \Xi\pi^0$ and $\Xi^*(1530) \to \Xi^0\pi^-$ Separation



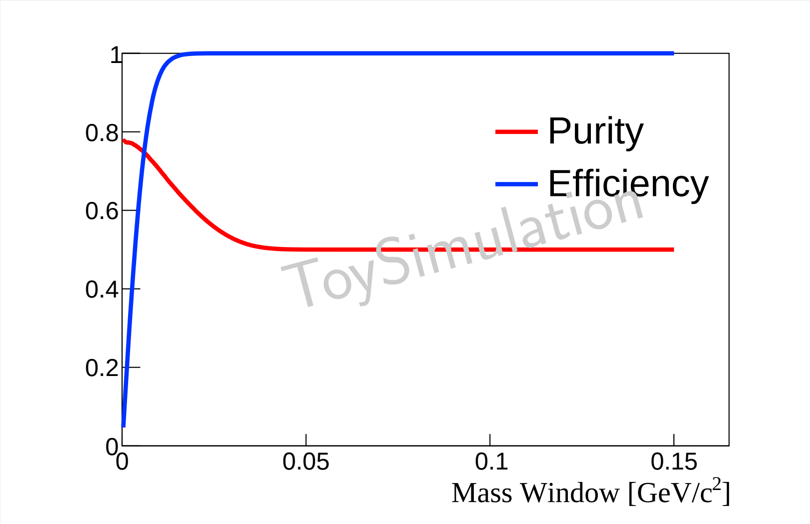


- Two decay channel of Ξ^* share the same decay product.
- Separation criteria should be defined to distinguish combinatorial backgrounds.
- Kinematic Fit result can provide another selection criteria based on Kinematics.

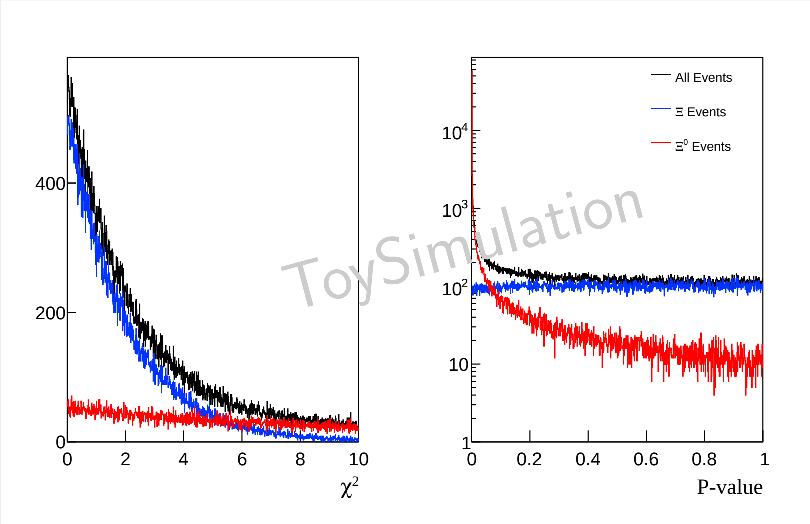
Invariant Mass



Mass Window Selection



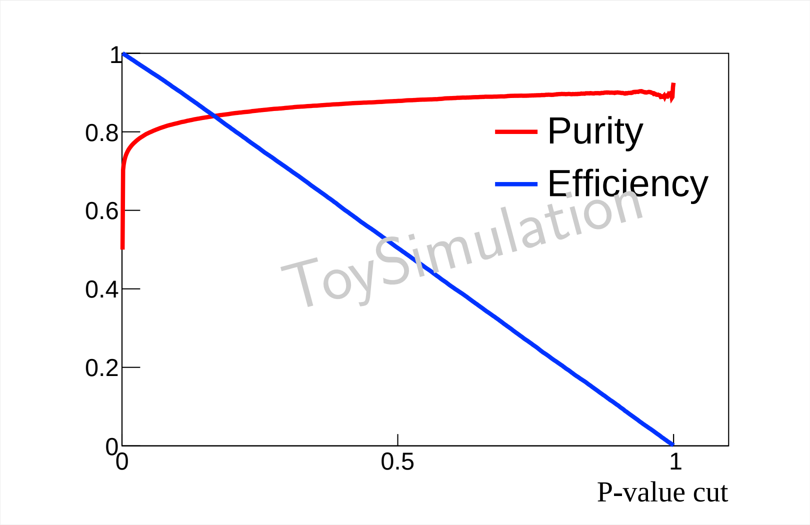
The P-value



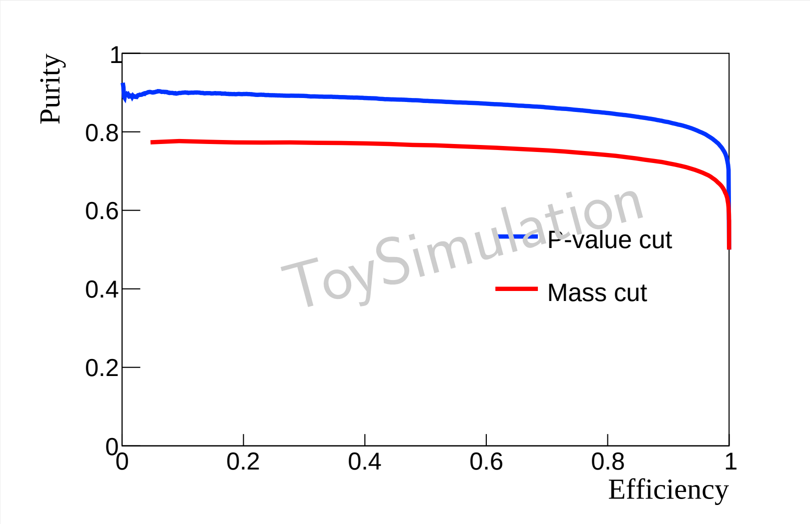
400
200
0
0
2
4
6
8
10
χ^2
ToySimulation
10^4
10^3
10^2
10
1
0
0.2
0.4
0.6
0.8
1
P-value
All Events
Ξ Events
Ξ^0 Events

왜?
? ?

P-value Selection

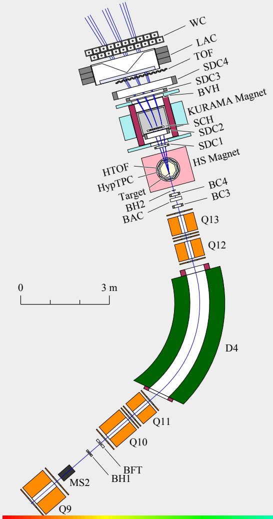


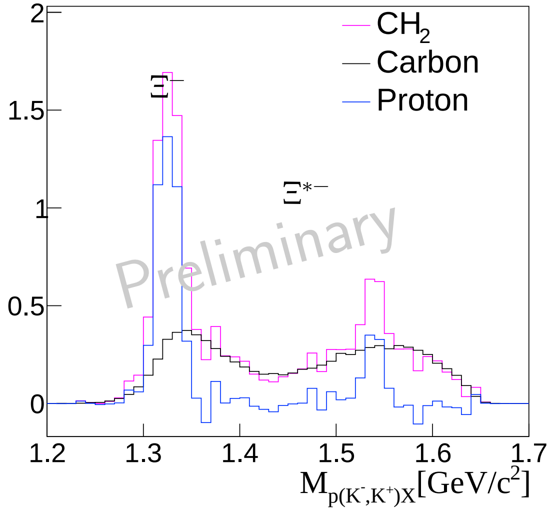
P-value vs Invariant Mass



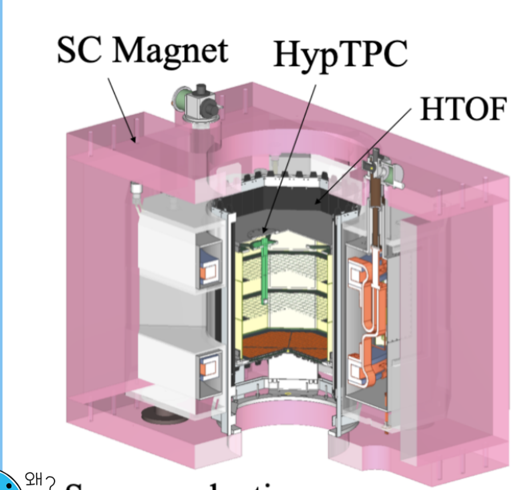
Applications: Ξ decay at J-PARC

$p(K^-, K^+)X$ at J-PARC E42

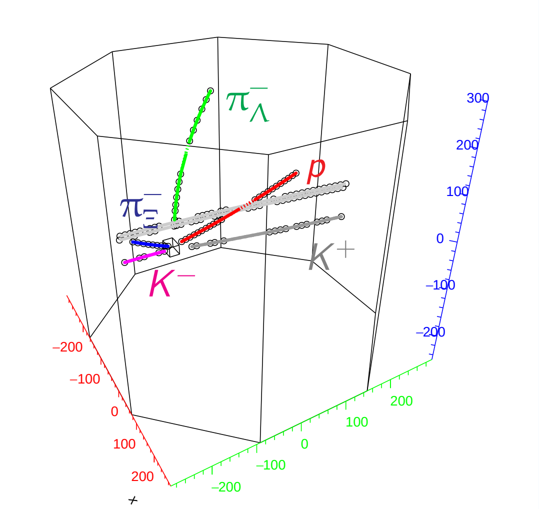




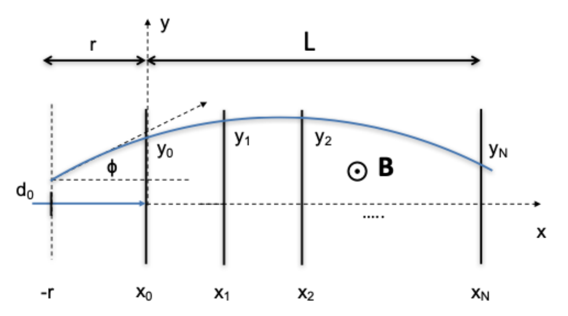
HypTPC



Superconducting
Hyperon Spectrometer



Gluckstern Formula



Z. Drasal, W. Riegler, Nucl. Instrum. Methds. A, 910, 127-132 (2018)

$$\frac{\sigma_{P_T}}{P_T} \simeq \frac{P_T}{0.3L^2B}\sqrt{\frac{720}{N+4}}\sigma_T \quad (29)$$

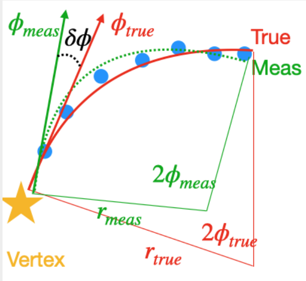
$$\frac{\sigma_{P_T,m.s}}{P_T} \simeq \frac{0.0136^1}{0,3\beta BL}\sqrt{\frac{d_{tot}}{X_0}} \quad (30)$$

Units in GeV/c

- Momentum resolution comprises geometrical term and scattering term
- In practice, empirical rescaling factor should be multiplied

[1] G.R. Lynch and O.I Dahl, Nucl. Instrum. Methods B58, 6 (1991).

Covariance Matrix in Helix Track



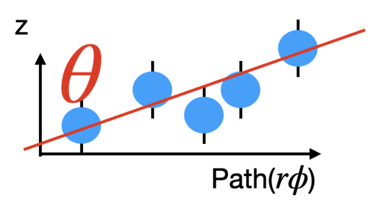
- Variance in momentum modifies the curvature of the helix $\rightarrow$ direction at the vertex changes.
- 'Position' of the helix is defined from the TPC hits. $\rightarrow$ Center-of-gravity should be fixed.

Denote the tangent angle at the center be ϕ_0 and path length to the vertex l.

$$\phi = \phi_0 \pm \frac{l}{2r}; \quad \delta\phi = \pm\frac{l}{2r}\frac{\delta r}{r} = \pm\frac{l}{r}\frac{\delta p_T}{p_T} \tag{31}$$

$$\sigma_\phi^2 = \langle\delta\phi\delta\phi\rangle = \frac{l^2}{r^2 p_T^2}\sigma_{p_T}^2; \quad \mathrm{Cov}(\phi, p_T) = \langle\delta\phi\delta p_T\rangle = \pm\frac{l}{r p_T}\sigma_{p_T}^2 \tag{32}$$

Covariance Matrix in Helix Track



$$h(t) : \{r\cos(\phi) - c_x, r\sin\phi - c_y, dz * r\phi - z_0\} \tag{33}$$

- The 'pitch' parameter, dz, is the slope along the circular trajectory

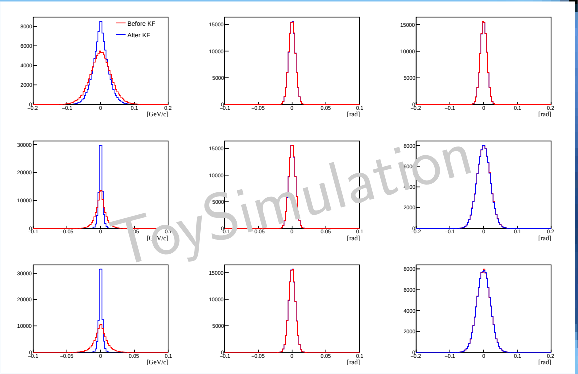
$\theta = \frac{\pi}{2} - \arctan(dz)$, we estimate the variance of θ based on the fitting error of dz. The error is estimated from the slope error of a linear fit:

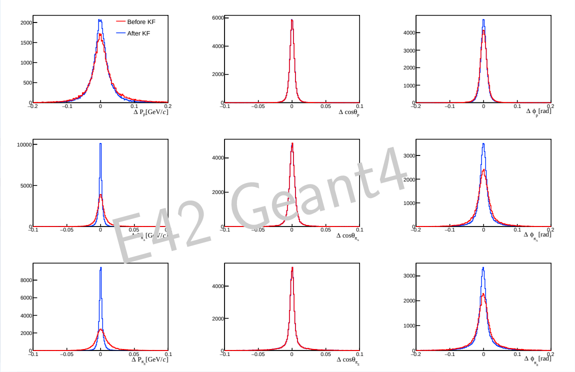
$$\sigma_{dz}^2 = \frac{\sum \delta_z^2/(n-2)}{\sum (x-\bar{x})^2} \simeq \frac{n\sigma_z^2/(n-2)}{nL^2/12}; \quad \sigma_\theta = \frac{\partial dz}{\partial\theta}\sigma_{dz} = \frac{1}{1+dz^2}\sigma_{dz}.$$

Note that, the momentum $p_z = p_T dz$ would also have some covariance with θ,

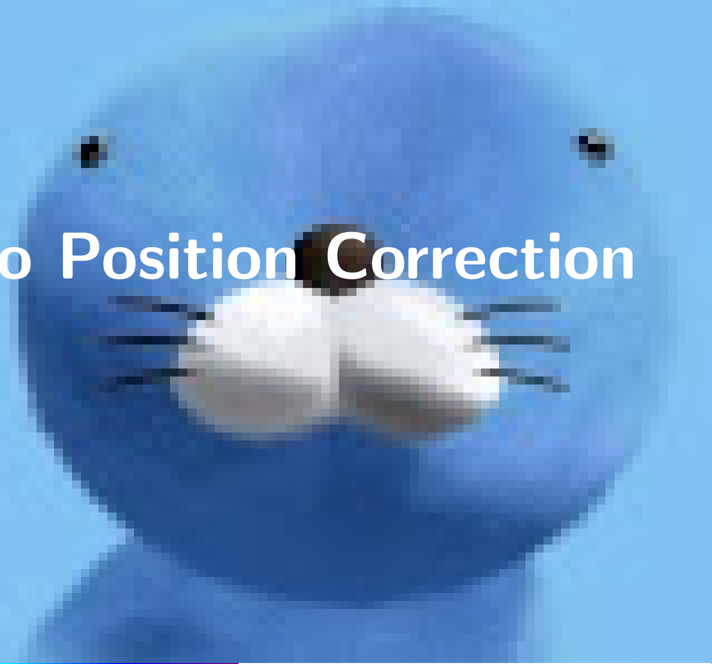
$$\langle\delta p \delta\theta\rangle = dz\cancelto{0}{\langle\delta p_T \delta\theta\rangle} + p_T\langle\delta dz \delta\theta\rangle = \frac{p_T}{1+dz^2}\sigma_{dz}^2.$$

ϕ Restoration from Diagonal Component

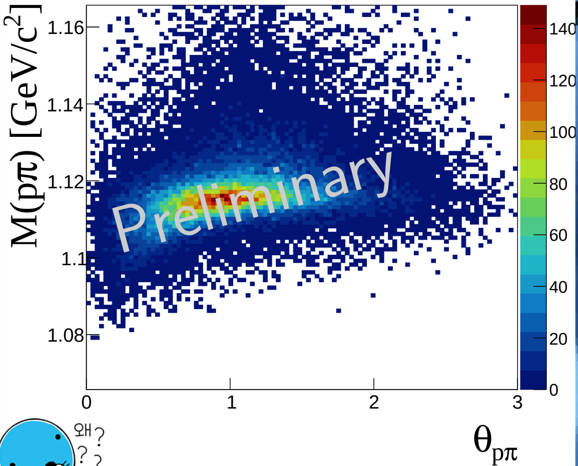


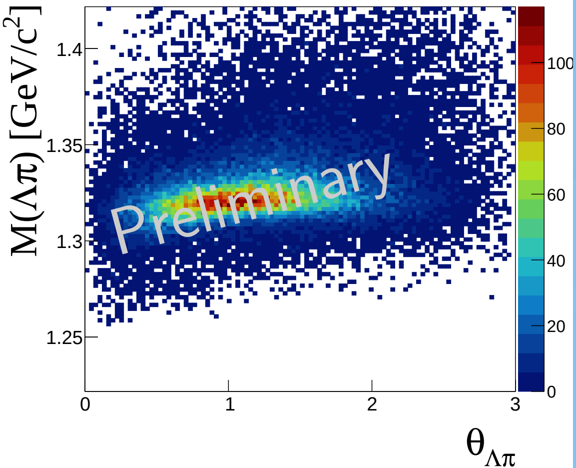


Applications to Position Correction

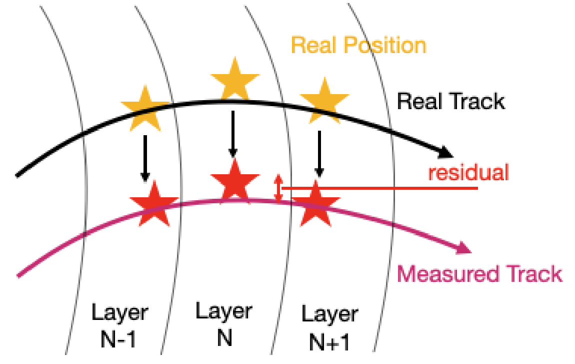


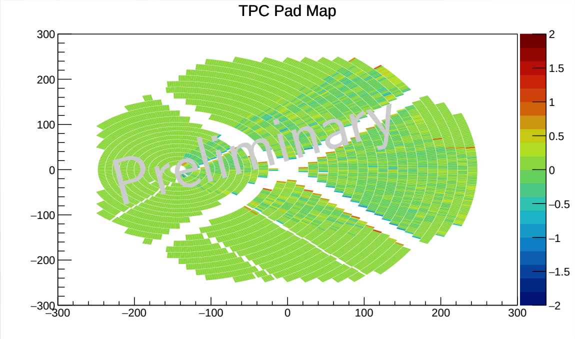
Momentum Bias from Position Shift





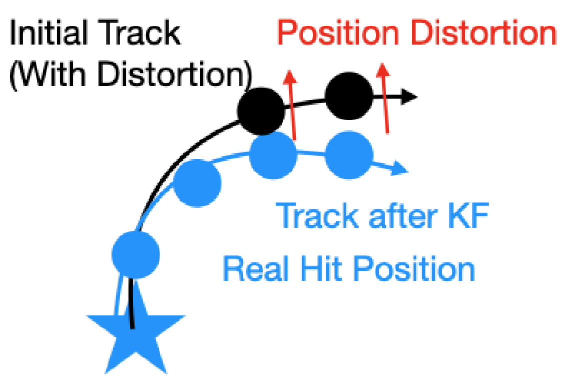
Position Residual?

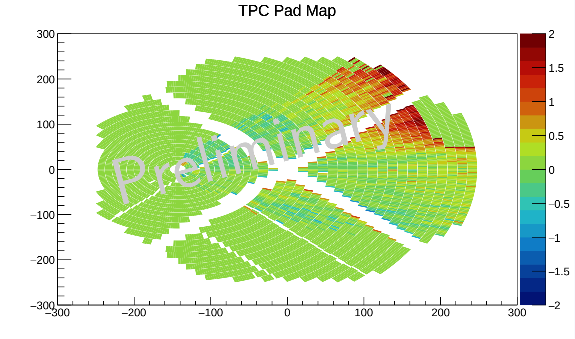




- Local, but simultaneous shift cannot be detected from position residual measurement.
- External reference for track should be provided to estimate 'true' trajectory

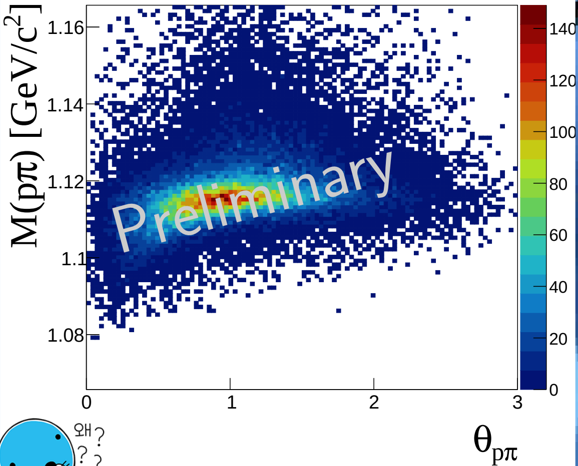
Position Residual from KF Track

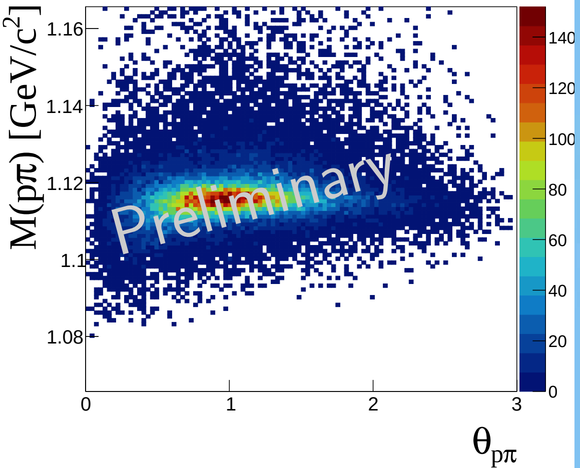




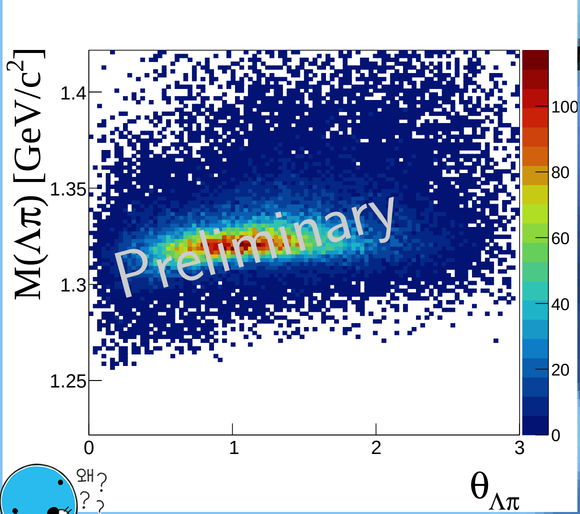
- From Kinematic fit, 'true' momentum, hence trajectory is estimated

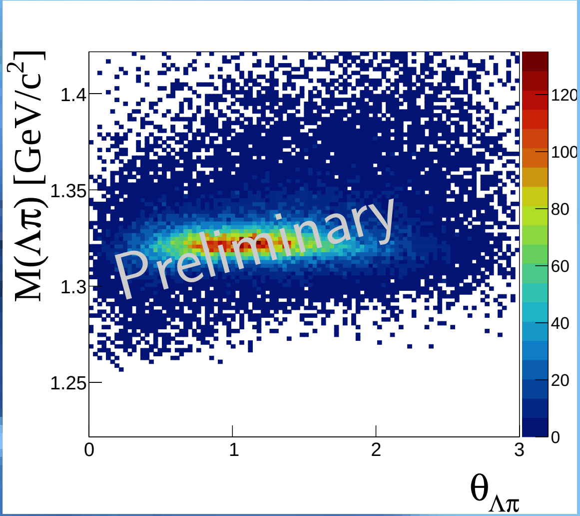
Λ After Position Correction





Ξ After Position Correction





Tricks

Variance Normalization

$$V = \begin{pmatrix} 10^{12} & 0.9 \\ 0.9 & 10^{-12} \end{pmatrix} \rightarrow V^{-1} = ? \tag{34}$$

While taking an inverse of the variance, matrix elements with different order may be added together, leading to possible numerical unstability.

$$\tilde{V} = SVS^T; S \equiv \frac{1}{\sqrt{V_{ij}}}\delta_{ij} \rightarrow \tilde{V} = \begin{pmatrix} 1 & Cov(01)/\sigma_1\sigma_2 & \cdots \\ Cov(01)/\sigma_1\sigma_2 & 1 & \\ \cdots & & \cdots \end{pmatrix} \tag{35}$$

We can take out scaling factors in S. Measurement vectors could share the same problem, so they should also be scaled. We rewrite equation (15)

$$\chi^2 = dM^\dagger V^{-1} dM + \cdots = d\tilde{M}^\dagger \tilde{V}^{-1} d\tilde{M} + \cdots; \quad d\tilde{M} = S(M - M_0) \tag{36}$$

Off-diagonal Reduction

$$V = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \to V^{-1} = ? \tag{37}$$

Adding off-diagonal term could make matrix uninvertable. Also we require $\chi^2 = dMV^{-1}dM > 0$; V^{-1} (hence V)should be *Positive Definite*. Then, we can 'damp' the offdiagonal elements.

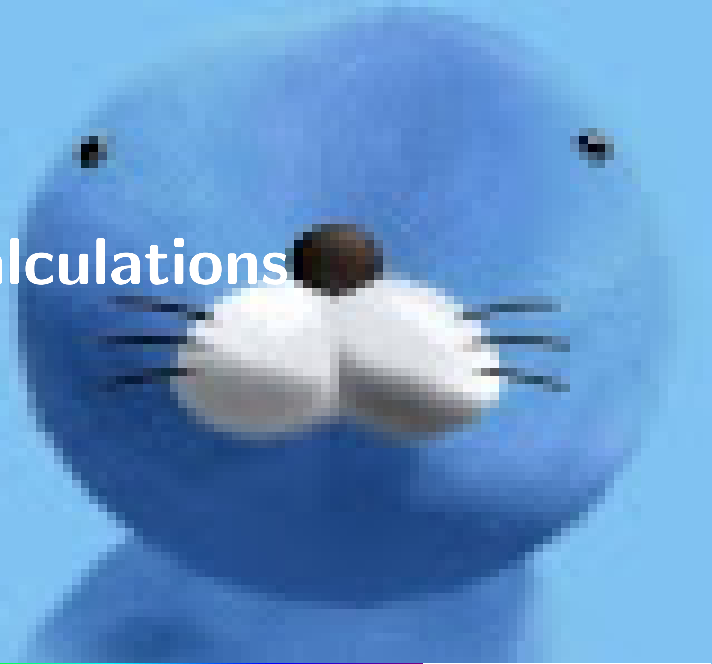
$$while(IsPositiveDefinite(V)) \tag{38}$$

$$V_{ij} \to V_{ij} - \alpha(\delta_{ij} - 1)V_{ij} \tag{39}$$

Property of Positive Definite Matrix

All Eigenvalues are Positive! TMatrixD well-supports eigenvalues, so we can just use it.

Calculations



Processing Iterative Steps.

We can express the following equations based on the ones provided above:

$$V^{-1}(\mathbf{m}^0)(\mathbf{m}^{\nu+1} - \mathbf{m}^0) + (\mathbf{F}_m^\dagger)^\nu \lambda^{\nu+1} = 0 \tag{40}$$

$$(\mathbf{F}_u^\dagger)^\nu \lambda^{\nu+1} = 0 \tag{41}$$

$$\mathbf{f}^\nu + \mathbf{F}_m^\nu(\mathbf{m}^{\nu+1} - \mathbf{m}^\nu) + \mathbf{F}_u^\nu(\mathbf{u}^{\nu+1} - \mathbf{u}^\nu) = 0. \tag{42}$$

Equation (42) is not a direct consequence of Equation (18) but rather a *linear approximation* to proceed with our iteration step. Expanding the ∇_λ term with a Taylor series leads to this equation. Note that, as our parameters $\mathbf{m}$ and $\mathbf{u}$ are updated during the step, our constraint matrix $\mathbf{f}$ should also be updated during the iteration. Here, λ should be indexed as $\nu + 1$ since it is a parameter to be guessed in the next step.

Solving the Equation(1)

Multiplying **V** to Equation (40) leads to:

$$\mathbf{m}^{\nu+1} - \mathbf{m}^0 = -V(\mathbf{m}^0)(\mathbf{F}_m^\dagger)^\nu \lambda^{\nu+1}. \tag{43}$$

Substituting Equation (43) into Equation (42), we get:

$$\begin{aligned} \mathbf{F}_u^\nu(\mathbf{u}^{\nu+1} - \mathbf{u}^\nu) &= -\mathbf{f}^\nu - \mathbf{F}_m^\nu(-V(\mathbf{m}^0)(\mathbf{F}_m^\dagger)^\nu \lambda^{\nu+1} + \mathbf{m}^0 - \mathbf{m}^\nu) \\ &= S\lambda^{\nu+1} - R \end{aligned} \tag{44}$$

where$S \equiv \mathbf{F}_\mathbf{m}^\nu V(\mathbf{m}^0)(\mathbf{F}_\mathbf{m}^\dagger)^\nu$ and $R \equiv \mathbf{f}^\nu + \mathbf{F}_\mathbf{m}^\nu(\mathbf{m}^0 - \mathbf{m}^\nu)$. Multiplying $(\mathbf{F}_u^\dagger)^\nu S^{-1}$ and substituting Equation (41), we get:

$$(\mathbf{F}_u^\dagger)^\nu S^{-1}\mathbf{F}_u^\nu(\mathbf{u}^{\nu+1} - \mathbf{u}^\nu) = \cancelto{0}{(\mathbf{F}_u^\dagger)^\nu \lambda^{\nu+1}} - (\mathbf{F}_u^\dagger)^\nu S^{-1} R. \tag{45}$$

Solving the Equation(2)

Then we naturally obtain:

$$\mathbf{u}^{\nu+1} = \mathbf{u}^{\nu} - ((\mathbf{F}_u^{\dagger})^{\nu} S^{-1} \mathbf{F}_u^{\nu})^{-1} (\mathbf{F}_u^{\dagger})^{\nu} S^{-1} R. \tag{46}$$

and from Equation (44)

$$\lambda^{\nu+1} = S^{-1}(\mathbf{F}_u^{\nu}(\mathbf{u}^{\nu+1} - \mathbf{u}^{\nu}) + R) \tag{47}$$

. For a summary, we have obtained all equations to proceed to the next step. All other matrices in the equation can be calculated from parameters of the current step, and χ^2 can be evaluated from (15) .

$$\begin{cases} \mathbf{u}^{\nu+1} = \mathbf{u}^{\nu} - ((\mathbf{F}_u^{\dagger})^{\nu} S^{-1} \mathbf{F}_u^{\nu})^{-1} (\mathbf{F}_u^{\dagger})^{\nu} S^{-1} R & (46) \\ \lambda^{\nu+1} = S^{-1}(\mathbf{F}_u^{\nu}(\mathbf{u}^{\nu+1} - \mathbf{u}^{\nu}) + R) & (47) \\ \mathbf{m}^{\nu+1} = \mathbf{m}^{0} - V(\mathbf{m}^{0})(\mathbf{F}_m^{\dagger})^{\nu} \lambda^{\nu+1} & (43) \end{cases}$$

Evolution of the Variance Matrix

Take a look at Eq.(43). We see that $m^{\nu+1}$ is an addition(subtraction) of some parameters to the initially measured data. As we already know the error, i.e. Variance matrix, of initial data, we can estimate how error propagates through the fitting process.

$$V(m) = J_{m,m^0} V(m^0) J^{\dagger}_{m,m^0} \tag{48}$$

We need to calculate the Jacobian,

$$J_{m,m^0(i,j)} = \frac{\partial m_i}{\partial m_j^0} \tag{49}$$

Evolution of the Variance Matrix.

To begin with, let us express Eq (43) in terms of m^0. At the moment we will drop the superscript ν. As $\mathbf{f}(\mathbf{m},\mathbf{u})$is a constant on m^0, $\mathbf{F}_\mathbf{m}$ also will be a constant to m^0. Then we only need to consider the derivatives of λ. By substituting (46) ,

$$\lambda = S^{-1}(-\mathbf{F}_u(((\mathbf{F}_u^\dagger)S^{-1}\mathbf{F}_u)^{-1}(\mathbf{F}_u^\dagger)S^{-1}R) + R) \tag{50}$$

and we have

$$R \equiv \mathbf{f} + \mathbf{F}_\mathbf{m}(\mathbf{m}^0 - \mathbf{m}) \rightarrow \frac{\partial R}{\partial m^0} = \mathbf{F}_m \tag{51}$$

so that

$$\frac{\partial \lambda}{\partial m^0} = S^{-1}(-\mathbf{F}_u((\mathbf{F}_u^\dagger S^{-1}\mathbf{F}_u)^{-1}\mathbf{F}_u^\dagger S^{-1}\mathbf{F}_m) + \mathbf{F}_m). \tag{52}$$

Evolution of the Variance Matrix

Now define the symmetric matrices $G \equiv \mathbf{F}^\dagger_{\mathbf{m}} S^{-1} \mathbf{F}_{\mathbf{m}}$, $U \equiv (\mathbf{F}^\dagger_u S^{-1} \mathbf{F}_u)^{-1}$ and $H \equiv \mathbf{F}^\dagger_m S^{-1} \mathbf{F}_{\mathbf{u}}$. Then we have expressions for $\frac{\partial \lambda}{\partial m^0}$ hence

$$
\begin{aligned}
J_{m,m^0} &= I - V(m^0)\mathbf{F}^\dagger_m \frac{\partial \lambda}{\partial m^0} = I - V\mathbf{F}^\dagger_m(-S^{-1}\mathbf{F}_u U^{-1} H^\dagger + S^{-1}\mathbf{F}_m) \\
&= I - V(G - HUH^\dagger)
\end{aligned} \tag{53}
$$

If we let $C = G - HUH^\dagger$, we obtain

$$
V(m) = J_{m,m^0} V J^\dagger_{m,m^0} = V - 2VCV + VCVCV. \tag{54}
$$

You might want to neglect higher order term, but please keep 2nd order term at the moment. Some materials like [1] had an error in this part.

Variance of the Unknowns

Just like how we derived Eq.(54) we can estimate the variance matrix of the unknowns.

$$V_U = J_{u,m0} V J_{u,m0}^T \tag{55}$$

$J_{u,m0}$ can be obtained from Eq.(46). Denoting $((\mathbf{F}_u^\dagger)^\nu S^{-1} \mathbf{F}_u^\nu)^{-1} (\mathbf{F}_u^\dagger)^\nu S^{-1}$ as K,

$$J_{u^{\nu+1},m^0} = \frac{\partial u^{\nu+1}}{\partial m^0} = \frac{\partial u^\nu}{\partial m^0} - K \frac{\partial R}{\partial m^0} \simeq -K\mathbf{F}_m. \tag{56}$$

Note that we only have initial "Guess" for the unknowns; In principle, it is not a driven value from measurements. Then, $\frac{\partial u^0}{\partial m_0} = 0$. Also, we approximate that the terms in 2nd or higher iterations are negligible: $J_{u,m^0} \simeq J_{u^0,m^0}$.

Pull distribution

A bias or resolution miss-estimation is revealed by observing the *Pull distribution* of each measurements. By defining the residual $\epsilon = m - m^0$ and its variance $V(\epsilon)$, the pull is defined as:

$$P(\epsilon) = \epsilon/\sqrt{V(\epsilon)} \tag{57}$$

and

$$V(\epsilon) \equiv V(m) + V(m^0) - 2Cov(m, m^0). \tag{58}$$

The variance of the fitted variables, $V(m)$, is evaluated as

$$V(m) = J_{m,m^0} V(m^0) J^{\dagger}_{m,m^0} \tag{59}$$

where J_{m,m^0} is the Jacobian for m and m^0. Detailed calculations are provided in the appendix.

Pull distribution

The covariance can be estimated as:

$$Cov(m, m^0) = J_{m,m^0} V(m) = V - VCV. \tag{60}$$

If we substitute this and Eq.(54) into Eq.(58), we get

$$V(\epsilon) = VCVCV. \tag{61}$$

Note that 2nd order term affects the variance.

Applications

Example: Chained Mass-Constraint Fit

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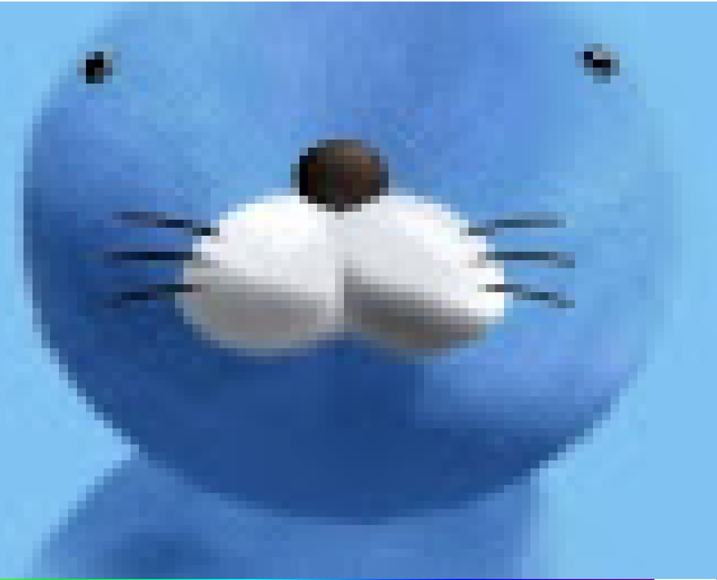
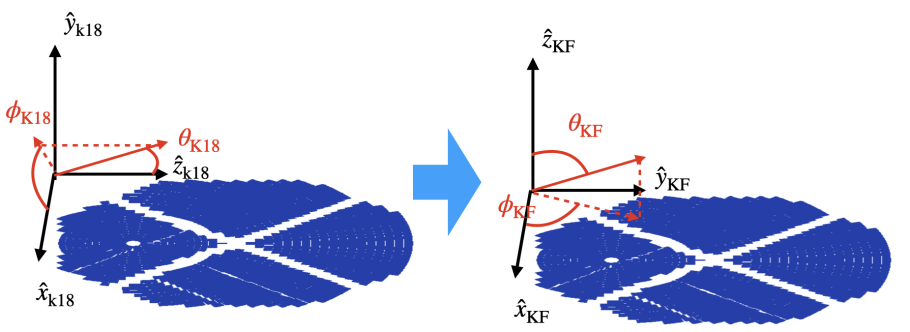
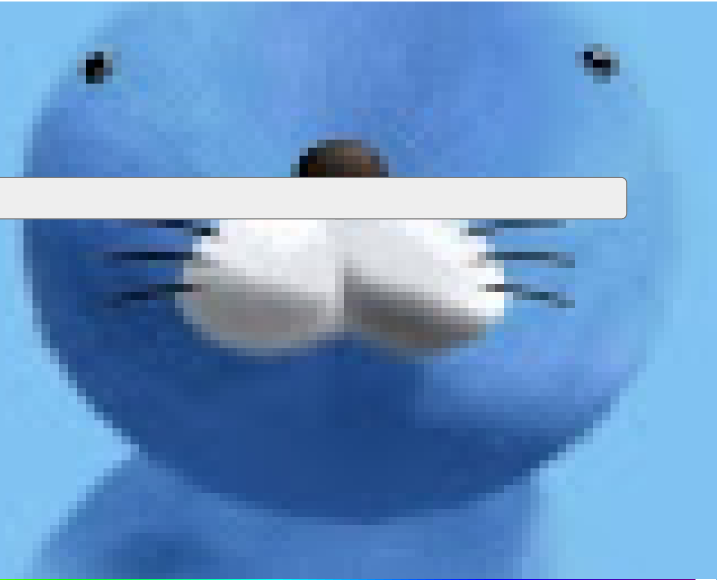


Mass-Constraint Kinematic Fit with HypTPC.



For Kinematic FIt in HypTPC analysis, we want our coordinate system to be aligned with $\vec{B}$, so that our covariance matrix representation fits the representations in KF coordinate. We correlated ϕ angle with p as an feature of helix fit, where ϕ is the angle lying on the circle of the helix.

Example: MassVertex-Constraint Fit



To be Updated..

References

O. Skjeggestad Frodesen A.G. **Probability and Statistics in Particle Physics**. Columbia University Press, 1980. ISBN: 8200019063.