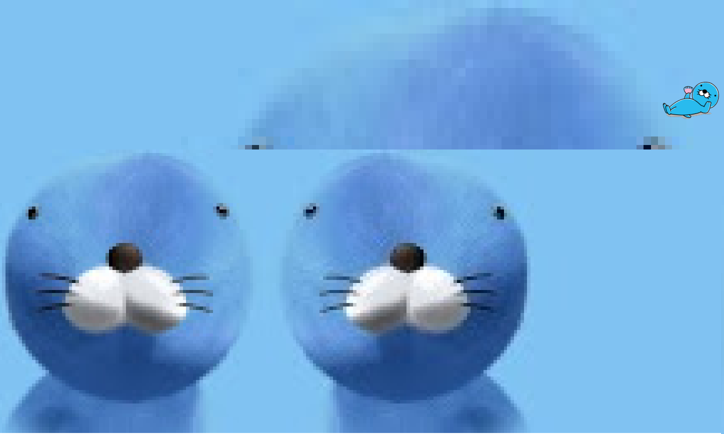

Notes on Kinematic Fit

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June 6, 2024

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Introduction

Assume that we have measured a momentum of two particles, which decay from a mother particle with an exact momentum P_0. In the real world, every measurement inherently carries some errors. Consequently, the measured momentum may not satisfy momentum conservation:

$$P_0 = P_1 + P_2; \quad P_{1,meas} + P_{2,meas} \neq P_0 \tag{1}$$

However, since these measurements did not incorporate our prior knowledge from physics, we can make a more informed estimate of the measured parameters. If the momentum resolution of particles 1 and 2 is well-known, then we can express χ^2 as

$$\chi^2 = \frac{(P_1 - P_{1,meas})^2}{\sigma_1^2} + \frac{(P_2 - P_{2,meas})^2}{\sigma_2^2} \tag{2}$$

Introduction

By incorporating the *Kinematic Constraints*, specifically *momentum conservation*, into our example, we introduce additional terms known as *Lagrange Multiplier* to Equation (3):

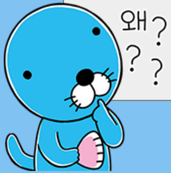
$$\chi^2 = \frac{(P_1 - P_{1,meas})^2}{\sigma_1^2} + \frac{(P_2 - P_{2,meas})^2}{\sigma_2^2} + 2\lambda(\mathbf{P_{1,meas}} + \mathbf{P_{2,meas}} - \mathbf{P_0}) \tag{3}$$

We then proceed to evaluate the conditions for local minima, i.e. setting the partial derivatives equal to zero:

$$\frac{1}{2}\frac{\partial\chi 2}{\partial P_1} = \frac{(P_1 - P_{1,meas})}{\sigma_1^2} + \lambda = 0 \tag{4}$$

$$\frac{1}{2}\frac{\partial\chi 2}{\partial P_2} = \frac{(P_2 - P_{2,meas})}{\sigma_2^2} + \lambda = 0 \tag{5}$$

$$\frac{1}{2}\frac{\partial\chi 2}{\partial \lambda} = (P_1 + P_2 - P_0) = 0 \tag{6}$$



Introduction

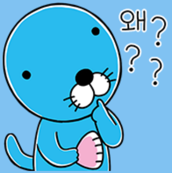
By solving the equations 4,5,6, we obtain the following expressions:

$$P_1 = \frac{\sigma_2^2 P_{1,meas} - \sigma_1^2 P_{2,meas} + \sigma_1^2 P_0}{\sigma_1^2 + \sigma_2^2} \tag{7}$$

$$P_2 = \frac{\sigma_1^2 P_{2,meas} - \sigma_2^2 P_{1,meas} + \sigma_2^2 P_0}{\sigma_1^2 + \sigma_2^2} \tag{8}$$

$$\lambda = \frac{P_{1,meas} + P_{2,meas} - P_0}{\sigma_1^2 + \sigma_2^2}. \tag{9}$$

Now, we have obtained the 'corrected' measurements with minimized χ^2, which incorporates momentum conservation. Let us delve into the interpretation of these equations.



Introduction

In a straightforward interpretation, λ can be viewed as a kind of 'normalized variance' of the kinematic constraint. It quantifies the error in momentum conservation$(P_{1,meas} + P_{2,meas} - P_0)$ relative to the overall resolution$(\sigma_1^2 + \sigma_2^2)$. Equation (4) implies that

$$P_1 = P_{1,meas} - \sigma_1^2 \lambda, \tag{10}$$

suggesting that the corrected momentum(P_1) is essentially the measured momentum$(P_{1,meas})$ augmented by a term proportional to the detector resolution and the normalized error of the kinematic constraint. Thus, we can assert that we have applied a statistically fair correction to the momentum, taking into account both the detector resolution and kinematic constraints.

Fitting in General

Assume that you have a set of measurements, $\mathbf{m} = \{m_1, m_2 \dots m_N\}$, and some unmeasured data, $\mathbf{u} = \{u_1, u_2 \dots u_J\}$ to be estimated. Kinematic constraints can be represented by sets of equations $\mathbf{f} = \{f_1(m_1, m_2, \dots m_N, u_1, u_2, \dots u_N), f_2, \dots f_K\}$. We will iteratively solve the problem by guessing the best parameter for each step and checking χ^2. Let $\mathbf{m}^0$ denote our initial measured data, and $\mathbf{m}$ represent the 'guess' of the data in each iterative step.

$$\chi^2(\mathbf{m}) = (\mathbf{m}^0 - \mathbf{m})^\dagger V^{-1}(\mathbf{m}^0 - \mathbf{m}) + 2\lambda^\dagger \mathbf{f}(\mathbf{m},\mathbf{u}). \tag{11}$$

Here, the Lagrange multiplier $\lambda = \{\lambda_1, \lambda_2, \dots \lambda_K\}$ is not just a number but a column vector with k elements, corresponding to each kinematic constraint in $\mathbf{f}$. Our task is to minimize χ^2 to obtain the best guesses in statistically fair method.

χ^2 Minimization

By (partially)differentiating with respect to all variables involved, we obtain the gradients of χ^2. Setting all of them to zero indicates that we have reached a minimum point of χ^2. We have 3 sets of gradient equations:

$$\nabla_{\mathbf{m}} = -2V^{-1}(\mathbf{m}^0)(\mathbf{m}^0 - \mathbf{m}) + 2\mathbf{F}^{\dagger}_{\mathbf{m}}(\mathbf{m}, \mathbf{u})\lambda = 0 \tag{12}$$

$$\nabla_{\mathbf{u}} = 2\mathbf{F}^{\dagger}_{\mathbf{u}}(\mathbf{m}, \mathbf{u})\lambda = 0 \tag{13}$$

$$\nabla_{\lambda} = \mathbf{f}(\mathbf{m}, \mathbf{u}). \tag{14}$$

Here, the subscripts denote partial derivatives. i.e. $((\mathbf{F}_m)_{ki} \equiv \frac{\partial f_k}{\partial m_i})$.

Processing Iterative Steps.

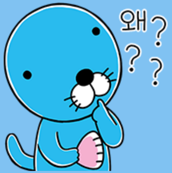
We can express the following equations based on the ones provided above:

$$V^{-1}(\mathbf{m}^0)(\mathbf{m}^{\nu+1} - \mathbf{m}^0) + (\mathbf{F}_m^\dagger)^\nu \lambda^{\nu+1} = 0 \tag{15}$$

$$(\mathbf{F}_u^\dagger)^\nu \lambda^{\nu+1} = 0 \tag{16}$$

$$\mathbf{f}^\nu + \mathbf{F}_m^\nu(\mathbf{m}^{\nu+1} - \mathbf{m}^\nu) + \mathbf{F}_u^\nu(\mathbf{u}^{\nu+1} - \mathbf{u}^\nu) = 0. \tag{17}$$

Equation (17) is not a direct consequence of Equation (13) but rather a linear approximation to proceed with our iteration step. Expanding the ∇_λ term with a Taylor series leads to this equation. Note that, as our parameters $\mathbf{m}$ and $\mathbf{u}$ are updated during the step, our constraint matrix $\mathbf{f}$ should also be updated during the iteration. Here, λ should be indexed as $\nu + 1$ since it is a parameter to be guessed in the next step.



Solving the Equation(1)

Multiplying **V** to Equation (15) leads to:

$$\mathbf{m}^{\nu+1} - \mathbf{m}^0 = -V(\mathbf{m}^0)(\mathbf{F}_m^\dagger)^\nu \lambda^{\nu+1}. \tag{18}$$

Substituting Equation (18) into Equation (17), we get:

$$\begin{aligned}\mathbf{F}_u^\nu(\mathbf{u}^{\nu+1} - \mathbf{u}^\nu) &= -\mathbf{f}^\nu - \mathbf{F}_m^\nu(-V(\mathbf{m}^0)(\mathbf{F}_m^\dagger)^\nu \lambda^{\nu+1} + \mathbf{m}^0 - \mathbf{m}^\nu) \\ &= S\lambda^{\nu+1} - R \end{aligned} \tag{19}$$

where$S \equiv \mathbf{F}_\mathbf{m}^\nu V(\mathbf{m}^0)(\mathbf{F}_\mathbf{m}^\dagger)^\nu$ and $R \equiv \mathbf{f}^\nu + \mathbf{F}_\mathbf{m}^\nu(\mathbf{m}^0 - \mathbf{m}^\nu)$. Multiplying $(\mathbf{F}_u^\dagger)^\nu S^{-1}$ and substituting Equation (16), we get:

$$(\mathbf{F}_u^\dagger)^\nu S^{-1} \mathbf{F}_u^\nu(\mathbf{u}^{\nu+1} - \mathbf{u}^\nu) = \cancelto{0}{(\mathbf{F}_u^\dagger)^\nu \lambda^{\nu+1}} - (\mathbf{F}_u^\dagger)^\nu S^{-1} R. \tag{20}$$

Solving the Equation(2)

Then we naturally obtain:

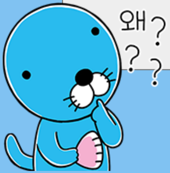
$$\mathbf{u}^{\nu+1} = \mathbf{u}^{\nu} - ((\mathbf{F}_u^{\dagger})^{\nu} S^{-1} \mathbf{F}_u^{\nu})^{-1} (\mathbf{F}_u^{\dagger})^{\nu} S^{-1} R. \qquad (21)$$

and from Equation (19)

$$\lambda^{\nu+1} = S^{-1}(\mathbf{F}_u^{\nu}(\mathbf{u}^{\nu+1} - \mathbf{u}^{\nu}) + R) \qquad (22)$$

. For a summary, we have obtained all equations to proceed to the next step. All other matrices in the equation can be calculated from parameters of the current step, and χ^2 can be evaluated from (11) .

$$\begin{cases} \mathbf{u}^{\nu+1} = \mathbf{u}^{\nu} - ((\mathbf{F}_u^{\dagger})^{\nu} S^{-1} \mathbf{F}_u^{\nu})^{-1} (\mathbf{F}_u^{\dagger})^{\nu} S^{-1} R & (21) \\ \lambda^{\nu+1} = S^{-1}(\mathbf{F}_u^{\nu}(\mathbf{u}^{\nu+1} - \mathbf{u}^{\nu}) + R) & (22) \\ \mathbf{m}^{\nu+1} = \mathbf{m}^{0} - V(\mathbf{m}^{0})(\mathbf{F}_m^{\dagger})^{\nu} \lambda^{\nu+1} & (18) \end{cases}$$



Evolution of Variance Matrix

Take a look at Eq.(18). We see that $m^{\nu+1}$ is an addition(subtraction) of some parameters to the initially measured data. As we already know the error, i.e. Variance matrix, of initial data, we can estimate how error propagates through the fitting process.

$$V(m) = J_{m,m^0} V(m^0) J^{\dagger}_{m,m^0} \tag{23}$$

We need to calculate the Jacobian,

$$J_{m,m^0(i,j)} = \frac{\partial m_i}{\partial m_j^0} \tag{24}$$

Evolution of Variance Matrix.

To begin with, let us express Eq (18) in terms of m^0. At the moment we will drop the superscript ν. As $\mathbf{f}(\mathbf{m},\mathbf{u})$is a constant on m^0, $\mathbf{F_m}$ also will be a constant to m^0. Then we only need to consider the derivatives of λ. By substituting (21) ,

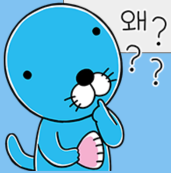
$$\lambda = S^{-1}(-\mathbf{F}_u(((\mathbf{F}_u^\dagger)S^{-1}\mathbf{F}_u)^{-1}(\mathbf{F}_u^\dagger)S^{-1}R)+R) \tag{25}$$

and we have

$$R \equiv \mathbf{f} + \mathbf{F_m}(\mathbf{m}^0 - \mathbf{m}) \rightarrow \frac{\partial R}{\partial m^0} = \mathbf{F}_m \tag{26}$$

so that

$$\frac{\partial \lambda}{\partial m^0} = S^{-1}(-\mathbf{F}_u((\mathbf{F}_u^\dagger S^{-1}\mathbf{F}_u)^{-1}\mathbf{F}_u^\dagger S^{-1}\mathbf{F}_m) + \mathbf{F}_m). \tag{27}$$



Evolution of Variance Matrix

Now define the symmetric matrices $G \equiv \mathbf{F}_{\mathbf{m}}^{\dagger} S^{-1} \mathbf{F}_{\mathbf{m}}$, $U \equiv (\mathbf{F}_u^{\dagger} S^{-1} \mathbf{F}_u)^{-1}$ and $H \equiv \mathbf{F}_m^{\dagger} S^{-1} \mathbf{F}_{\mathbf{u}}$. Then we have expressions for $\frac{\partial \lambda}{\partial m^0}$ hence

$$
\begin{aligned}
J_{m,m^0} &= I - V(m^0)\mathbf{F}_m^{\dagger} \frac{\partial \lambda}{\partial m^0} = I - V\mathbf{F}_m^{\dagger}(-S^{-1}\mathbf{F}_u U^{-1} H^{\dagger} + S^{-1}\mathbf{F}_m) \\
&= I - V(G - HUH^{\dagger}) \qquad (28)
\end{aligned}
$$

If we let $C = G - HUH^{\dagger}$, we obtain

$$
V(m) = J_{m,m^0} V J_{m,m^0}^{\dagger} = V - 2VCV + VCVCV. \qquad (29)
$$

You might want to neglect higher order term, but please keep 2nd order term at the moment. Some materials like [1] had an error in this part.

Variance of the Unknowns

Just like how we derived Eq.(29) we can estimate the variance matrix of the unknowns.

$$V_U = J_{u,m0} V J_{u,m0}^T \tag{30}$$

$J_{u,m0}$ can be obtained from Eq.(21). Denoting $((\mathbf{F}_u^\dagger)^\nu S^{-1} \mathbf{F}_u^\nu)^{-1} (\mathbf{F}_u^\dagger)^\nu S^{-1}$ as K,

$$J_{u^{\nu+1},m^0} = \frac{\partial u^{\nu+1}}{\partial m^0} = \frac{\partial u^\nu}{\partial m^0} - K \frac{\partial R}{\partial m^0} \simeq -K \mathbf{F}_m. \tag{31}$$

We have initial "Guess" for the unknowns; In principle, it is not a driven value from measurements. Then, $\frac{\partial u^0}{\partial m_0} = 0$. Also, we approximate that the terms in 2nd or higher iterations are negligible: $J_{u,m^0} \simeq J_{u^0,m^0}$.

Pull distribution

It is better to check the pull distribution to check the quality of a fit. By defining the residual $\epsilon = m - m^0$ and its variance $V(\epsilon)$, pull is defined as:,

$$P(\epsilon) = \epsilon/\sqrt{V(\epsilon)} \tag{32}$$

and

$$V(\epsilon) \equiv V(m) + V(m^0) - 2Cov(m, m^0). \tag{33}$$

We have already calculated J_{m,m^0}. Then we directly get the covariance matrix.

Pull distribution

The covariance can be estimated as:

$$Cov(m, m^0) = J_{m,m^0} V(m) = V - VCV. \tag{34}$$

If we substitute this and Eq.(29) into Eq.(33), we get

$$V(\epsilon) = VCVCV. \tag{35}$$

Note that 2nd order term affects the variance.

Example: 3-C Fit

Assuming a decay of $\Lambda \to p\pi^-$, let's represent the momentum of those particles in spherical coordinates:

$$\vec{P}_\Lambda = (P_\Lambda, \theta_\Lambda, \phi_\lambda), \cdots \tag{36}$$

Then the kinematic constraints can be expressed as:

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} -P_\Lambda \sin\theta_\Lambda \cos\phi_\Lambda + P_p \sin\theta_p \cos\phi_p + P_\pi \sin\theta_\pi \cos\phi_\pi \\ -P_\Lambda \sin\theta_\Lambda \sin\phi_\Lambda + P_p \sin\theta_p \sin\phi_p + P_\pi \sin\theta_\pi \sin\phi_\pi \\ -P_\Lambda \cos\theta_\Lambda + P_p \cos\theta_p + p_\pi \cos\theta_\pi \\ -\sqrt{P_\Lambda^2 + m_\Lambda^2} + \sqrt{P_p^2 + m_p^2} + \sqrt{P_\pi^2 + m_\pi^2} \end{pmatrix}. \tag{37}$$

Example: 3-C Fit

We have unmeasured and measured variables as:

$$\mathbf{u} = \{P_\Lambda\}; \quad \mathbf{m} = \{\theta_\Lambda, \phi_\Lambda, P_p, \theta_p, \phi_p, P_\pi, \theta_\pi, \phi_\pi\}. \tag{38}$$

Since there is 1 unmeasured variable with 4 kinematical constraints, this is a 4-1 = 3-Constrained fit. Let us substitute Eq (37) and (38) into Eq (11)s and its resulting equations. We get $\mathbf{F_u}$ and $\mathbf{F_m}$ as

$$\mathbf{F_u} = \begin{pmatrix} \frac{\partial f_1}{\partial P_\Lambda} \\ \frac{\partial f_2}{\partial P_\Lambda} \\ \frac{\partial f_3}{\partial P_\Lambda} \\ \frac{\partial f_4}{\partial P_\Lambda} \end{pmatrix}; \quad \mathbf{F_m} = \begin{pmatrix} \frac{\partial f_1}{\partial \theta_\Lambda} & \cdots & \frac{\partial f_1}{\partial \phi_\pi} \\ \frac{\partial f_2}{\partial \theta_\Lambda} & \cdots & \frac{\partial f_2}{\partial \phi_\pi} \\ \frac{\partial f_3}{\partial \theta_\Lambda} & \cdots & \frac{\partial f_3}{\partial \phi_\pi} \\ \frac{\partial f_4}{\partial \theta_\Lambda} & \cdots & \frac{\partial f_4}{\partial \phi_\pi} \end{pmatrix} \tag{39}$$

We have all the matrices to calculate in each step. By applying an appropriate variance matrix and employing χ^2 selection criteria, we can kinematically fit the particles.

References

O. Skjeggestad Frodesen A.G. **Probability and Statistics in Particle Physics**. Columbia University Press, 1980. ISBN: 8200019063.