
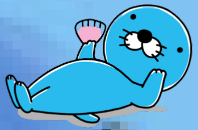

# Notes on Kinematic Fit

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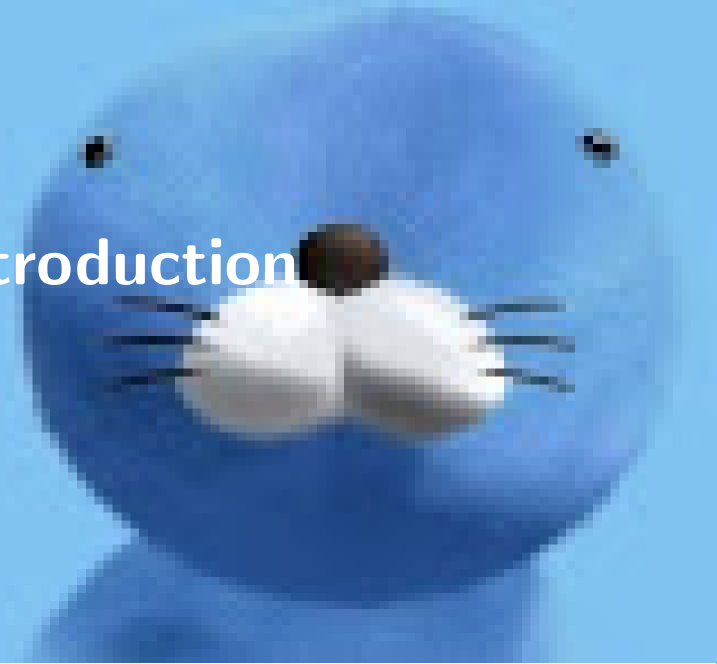
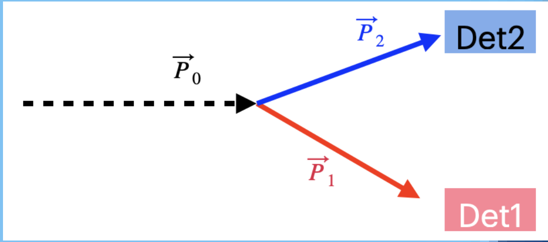
# Outline

1. Introduction
2. Example: Mass-Constraint Fit
3. Applications: $\Xi$ decay at J-PARC
4. Tricks
5. Appendix



# Introduction

# Measurement Error




Assume a beam with momentum $\vec{P}^-$ decays into $\vec{P}_1$ and $\vec{P}_2$. Measured momentum are smeared due to detector resolution, leading to unbalance in the momentum conservation.

$$\vec{P}_0 = \vec{P}_1 + \vec{P}_2; \quad \vec{P}_{1,meas} + \vec{P}_{2,meas} \neq \vec{P}_0 \tag{1}$$

We can define the $\chi^2$ to quantitatively represent our measurement error. However, we can't derive meaningful expressions from this $\chi^2$.

$$\chi^2 = \frac{(P_1 - P_{1,meas})^2}{\sigma_1^2} + \frac{(P_2 - P_{2,meas})^2}{\sigma_2^2} \tag{2}$$

# Constrained Optimization with The Lagrange Multiplier

By incorporating the *Kinematic Constraints*, specifically *momentum conservation*, we involve additional knowledge to (2). This is known as the *Lagrange Multiplier*

$$\chi^2 = \frac{(P_{1,KF} - P_{1,meas})^2}{\sigma_1^2} + \frac{(P_{2,KF} - P_{2,meas})^2}{\sigma_2^2} + 2\lambda(\mathbf{P_{1,KF}} + \mathbf{P_{2,KF}} - \mathbf{P_0}) \tag{3}$$

Now we have meaningful expressions to minimize $\chi^2$, hence get better estimations for the measurement.

$$\frac{1}{2}\frac{\partial\chi 2}{\partial P_{1,KF}} = \frac{(P_{1,KF} - P_{1,meas})}{\sigma_1^2} + \lambda = 0 \tag{4}$$

$$\frac{1}{2}\frac{\partial\chi 2}{\partial P_{2,KF}} = \frac{(P_{2,KF} - P_{2,meas})}{\sigma_2^2} + \lambda = 0 \tag{5}$$

$$\frac{1}{2}\frac{\partial\chi 2}{\partial \lambda} = (P_{1,KF} + P_{2,KF} - P_0) = 0 \tag{6}$$

# Why Better Resolution?

By solving the equations 4,5,6 and defining $\delta_i = P_{i,meas} - P_i$, we obtain the following expressions:

$$\lambda = \frac{P_{1,meas} + P_{2,meas} - P_0}{\sigma_1^2 + \sigma_2^2} = \frac{\delta_1 + \delta_2}{\sigma_1^2 + \sigma_2^2} \tag{7}$$

$$P_{1,KF} = P_{1,meas} - \sigma_1^2 \lambda \tag{8}$$

$$P_{2,KF} = P_{2,meas} - \sigma_2^2 \lambda \tag{9}$$

$$\begin{aligned} < P_{1,KF} - P_1 > &=< P_{1,KF} - P_{1,meas} + \delta_1 >=< -\sigma_1^2 \lambda + \delta_1 > \\ &=< \frac{-\sigma_1^2}{\sigma_1^2 + \sigma_2^2}(\delta_1 + \delta_2) + \delta_1 >=< \frac{\sigma_2^2 \delta_1 - \sigma_1^2 \delta_2}{\sigma_1^2 + \sigma_2^2} > \end{aligned} \tag{10}$$

$$\sigma_{1,KF}^2 =< (P_{1,KF} - P_1)^2 >= \frac{\sigma_2^4 \cancelto{\sigma_1^2}{< \delta_1^2 >} + \sigma_1^4 \cancelto{\sigma_2^2}{< \delta_2^2 >}}{(\sigma_1^2 + \sigma_2^2)^2} = \frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2} < \sigma_1^2 \tag{11}$$

# The Covariance After Kinematic Fit

$$cov(P_1, P_2)_{KF} =< \delta_{1,KF}\delta_{2,KF} >=< (\delta_1 - \sigma_1^2\lambda)(\delta_2 - \sigma_2^2\lambda) >$$

$$= \sigma_1^2\sigma_2^2 \cancelto{\frac{1}{\sigma_1^2+\sigma_2^2}}{<\lambda^2>} - \frac{\sigma_1^2 < \delta_2^2 > + \sigma_2^2 < \delta_1^2 >}{\sigma_1^2 + \sigma_2^2} = -\frac{\sigma_1^2\sigma_2^2}{\sigma_1^2 + \sigma_2^2} \tag{12}$$

$$V = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix} \rightarrow V_{KF} = \begin{pmatrix} \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2} & -\frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2} \\ -\frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2} & \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2} \end{pmatrix} \tag{13}$$

- Improved momentum resolution
- Negative correlation between $P_1$ and $P_2$

# Generalization to Multi-Variables

Assume that we have a set of measured data $\mathbf{m^0}$, unknown parameters $\mathbf{u^0}$ and constraints $\mathbf{f}^0$.

$$
\begin{aligned}
\mathbf{m^0} &= \{m_1^0, m_2^0 \ldots m_N^0\}; \quad \mathbf{u^0} = \{u_1^0, u_2^0 \ldots u_J^0\} \\
\mathbf{f} &= \{f_1(m_1^0, m_2^0, \ldots m_N^0, u_1^0, u_2^0, \ldots u_N^0), f_2^0, \ldots f_K^0\}
\end{aligned} \tag{14}
$$

Let $\mathbf{m}^0$ denote our initial measured data, and $\mathbf{m}$ represent the 'guess' of the data in each iterative step, just alike $P_{KF}$s in the previous example. Equation (3) is generalized to:

$$
\chi^2(\mathbf{m}) = (\mathbf{m}^0 - \mathbf{m})^\dagger V^{-1}(\mathbf{m}^0 - \mathbf{m}) + 2\lambda^\dagger \mathbf{f}(\mathbf{m}, \mathbf{u}). \tag{15}
$$

Here, the Lagrange multiplier $\lambda = \{\lambda_1, \lambda_2, \ldots \lambda_K\}$ is not just a number but a column vector with k elements, corresponding to each kinematic constraint in $\mathbf{f}$.

# $\chi^2$ Minimization

We want to solve the equation

$$\vec{\nabla}\chi^2 = 0 \tag{16}$$

to obtain the minimized state. The differential term are listed within three groups.

$$\nabla_{\mathbf{m}} = -2V^{-1}(\mathbf{m}^0)(\mathbf{m}^0 - \mathbf{m}) + 2\mathbf{F}^{\dagger}_{\mathbf{m}}(\mathbf{m}, \mathbf{u})\lambda = 0 \tag{17}$$

$$\nabla_{\mathbf{u}} = 2\mathbf{F}^{\dagger}_{\mathbf{u}}(\mathbf{m}, \mathbf{u})\lambda = 0 \tag{18}$$

$$\nabla_{\lambda} = \mathbf{f}(\mathbf{m}, \mathbf{u}). \tag{19}$$

Here, the subscripts denote partial derivatives. i.e. $((\mathbf{F}_m)_{ki} \equiv \frac{\partial f_k}{\partial m_i})$.

## User Should Define...

| $\mathbf{m}$ | $\mathbf{u}$ | $\mathbf{f}$ | $\mathbf{V}$ | $\mathbf{F_m}, \mathbf{F_u}$ |
|---|---|---|---|---|
| Measured Data | Unknown parameters | Constraints | Covariance Matrix | Derivatives |

# Pull distribution

A bias or resolution miss-estimation is revealed by observing the *Pull distribution* of each measurements. However, we cannot evaluate the 'true' value of measurement, hence pull for the real data is not accessible. Instead, from the the residual $\epsilon = m - m^0$ and its variance $V(\epsilon)$, we observe the pull(of the residual) as :

$$P(\epsilon) = \epsilon / \sqrt{V(\epsilon)} \tag{20}$$

and

$$V(\epsilon) \equiv V(m) + V(m^0) - 2Cov(m, m^0). \tag{21}$$

The variance of the fitted variables, $V(m)$, is evaluated as

$$V(m) = J_{m,m^0} V(m^0) J^{\dagger}_{m,m^0} \tag{22}$$

where $J_{m,m^0}$ is the Jacobian for $m$ and $m^0$. Detailed calculations are provided in the appendix.

# Example: Mass-Constraint Fit

# Example: $\Lambda \to p\pi$, Defining Variables and Constraints

Assume a decay of $\Lambda \to p\pi^-$. We define the measurements and unknowns as:

$$\mathbf{m} = \{P_p, \theta_p, \phi_p, P_\pi, \theta_\pi, \phi_\pi\}; \quad \mathbf{u} = \{P_\Lambda, \theta_\Lambda, \phi_\Lambda\} \tag{23}$$

Then we define the energy-momentum constraint equation as:

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} -P_\Lambda \sin\theta_\Lambda \cos\phi_\Lambda + P_p \sin\theta_p \cos\phi_p + P_\pi \sin\theta_\pi \cos\phi_\pi \\ -P_\Lambda \sin\theta_\Lambda \sin\phi_\Lambda + P_p \sin\theta_p \sin\phi_p + P_\pi \sin\theta_\pi \sin\phi_\pi \\ -P_\Lambda \cos\theta_\Lambda + P_p \cos\theta_p + p_\pi \cos\theta_\pi \\ -\sqrt{P_\Lambda^2 + m_\Lambda^2} + \sqrt{P_p^2 + m_p^2} + \sqrt{P_\pi^2 + m_\pi^2} \end{pmatrix}. \tag{24}$$

where the mass constraint is naturally implemented in energy term.
Since we have 3 unmeasured variable with 4 kinematical constraints, this is a 4-3 = 1-Constrained fit.

# Example: $\Lambda \to p\pi$, The Derivatives

We get $\mathbf{F_u}$ and $\mathbf{F_m}$ as

$$\mathbf{F_u} = \begin{pmatrix} \frac{\partial f_1}{\partial P_\Lambda} & \frac{\partial f_1}{\partial \theta_\Lambda} & \frac{\partial f_1}{\partial \phi_\Lambda} \\ \frac{\partial f_2}{\partial P_\Lambda} & \frac{\partial f_2}{\partial \theta_\Lambda} & \frac{\partial f_2}{\partial \phi_\Lambda} \\ \frac{\partial f_3}{\partial P_\Lambda} & \frac{\partial f_3}{\partial \theta_\Lambda} & \frac{\partial f_3}{\partial \phi_\Lambda} \\ \frac{\partial f_4}{\partial P_\Lambda} & \frac{\partial f_4}{\partial \theta_\Lambda} & \frac{\partial f_4}{\partial \phi_\Lambda} \end{pmatrix}; \quad \mathbf{F_m} = \begin{pmatrix} \frac{\partial f_1}{\partial P_p} & \cdots & \frac{\partial f_1}{\partial \phi_\pi} \\ \frac{\partial f_2}{\partial P_p} & \cdots & \frac{\partial f_2}{\partial \phi_\pi} \\ \frac{\partial f_3}{\partial P_p} & \cdots & \frac{\partial f_3}{\partial \phi_\pi} \\ \frac{\partial f_4}{\partial P_p} & \cdots & \frac{\partial f_4}{\partial \phi_\pi} \end{pmatrix} \tag{25}$$

We have all the matrices to calculate in each step. By applying an appropriate variance matrix and employing $\chi^2$ selection criteria, we can do kinematic fit for the particles.

# Example: $\Xi \to \Lambda\pi$, $\Lambda \to p\pi$

We require two mass constraints for $\Xi \to \Lambda\pi$; $\Lambda \to p\pi$. In this case, careful considerations on the selection of variables. We will select

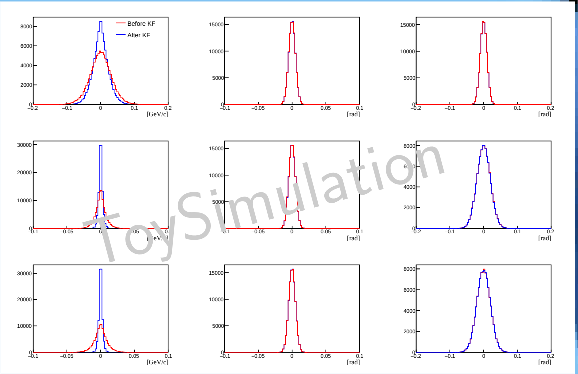
$$\mathbf{u} = \{P_\Xi, \theta_\Xi, \phi_\Xi\}; \quad \mathbf{m} = \{P_p, \theta_p, \phi_p, P_\pi, \theta_\pi, \phi_\pi\} \tag{26}$$

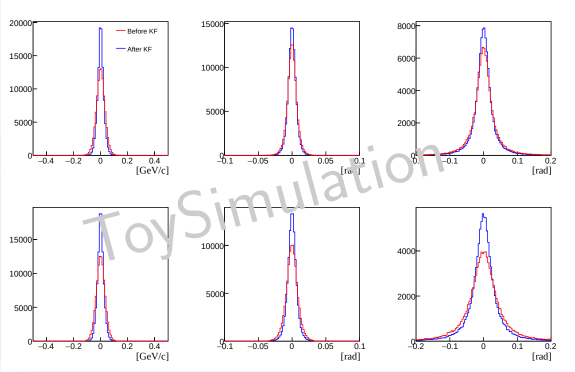
and define five constraints as:

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{pmatrix} = \begin{pmatrix} -P_{\Xi,x} + P_{p,x} + P_{\pi_\Lambda,x} + P_{\pi_\Xi,x} \\ -P_{\Xi,y} + P_{p,y} + P_{\pi_\Lambda,y} + P_{\pi_\Xi,y} \\ -P_{\Xi,z} + P_{p,z} + P_{\pi_\Lambda,z} + P_{\pi_\Xi,z} \\ -E_\Lambda + E_p + E_{\pi_\Lambda} \\ -E_\Lambda + E_p + E_{\pi_\Lambda} + E_{\pi_\Xi} \end{pmatrix}. \tag{27}$$

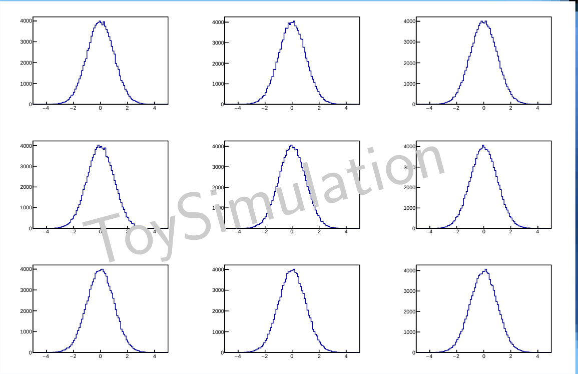
$\Lambda$ variables are not selected in $\mathbf{u}$ to avoid negative DoF.($N_f - N_u = 5 - 6 = -1$)
Note that **we don't have explicit terms** related to $\vec{P}_\Lambda$, i.e. $-P_{\Lambda,x} + P_{p,x} + P_{\pi_\Lambda,x}$ etc., because $\vec{P}_\Lambda$ are neither unmeasured nor measured variables in our choice of parameters.

# Kinematics Restoration



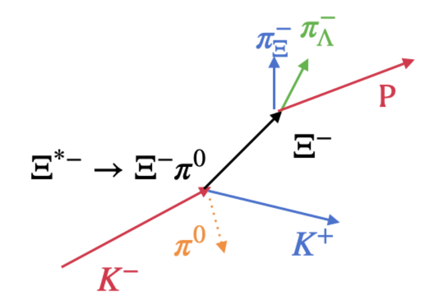

# Pull Distribution

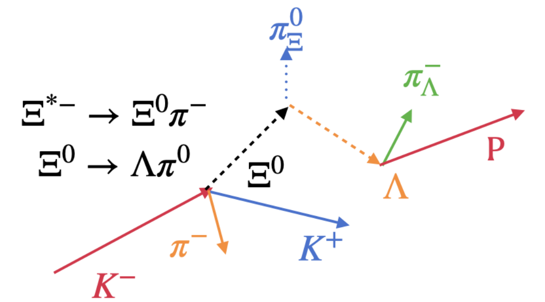



$$P(X) = \frac{X_{KF} - X_0}{\sqrt{V(X_{KF} - X_0)}} \quad (28)$$

- Pull distribution shows the normalized amount of parameter adjustment.
- In practice, resolution can be iteratively scaled by 1./ pull width

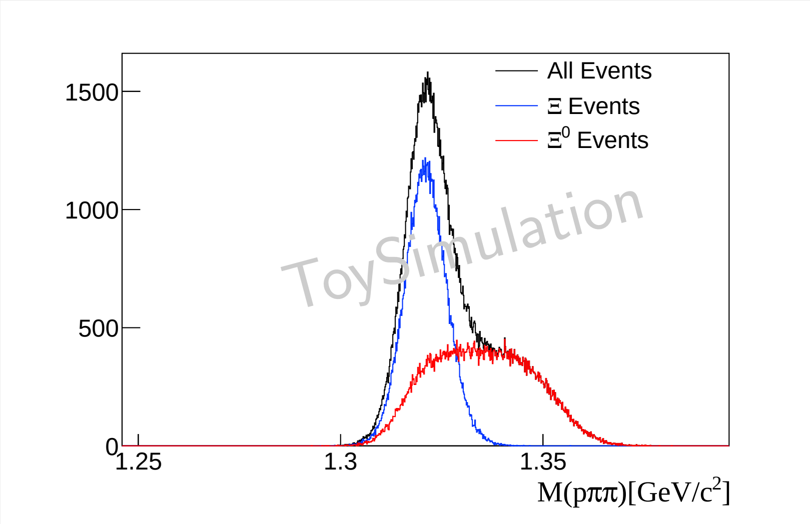
# $\Xi^*(1530) \to \Xi\pi^0$ and $\Xi^*(1530) \to \Xi^0\pi^-$ Separation



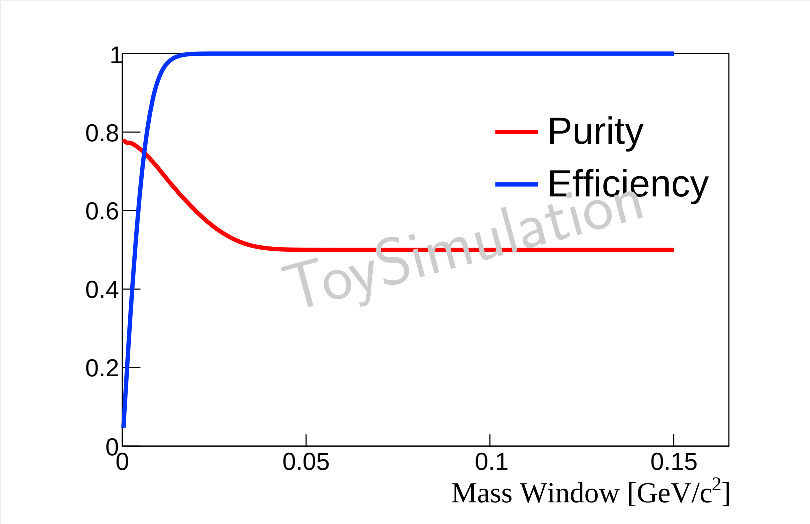




- Two decay channel of $\Xi^*$ share the same decay product.
- Separation criteria should be defined to distinguish combinatorial backgrounds.
- Kinematic Fit result can provide another selection criteria based on Kinematics.

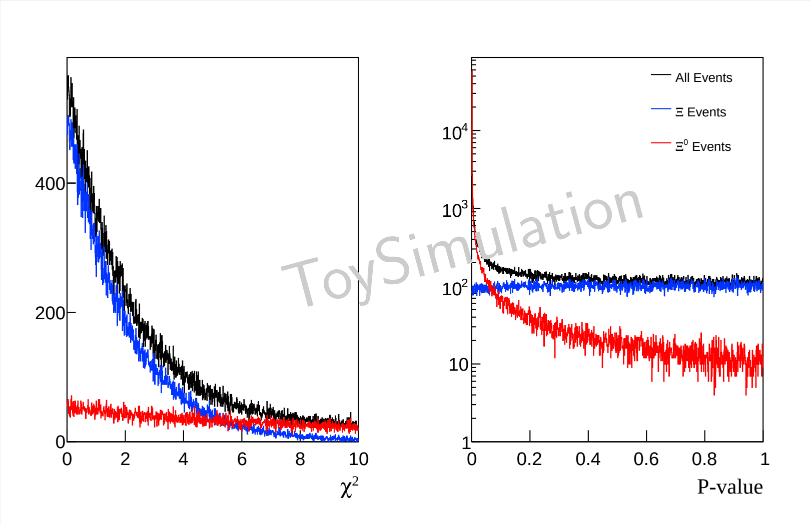
# Invariant Mass




# Mass Window Selection



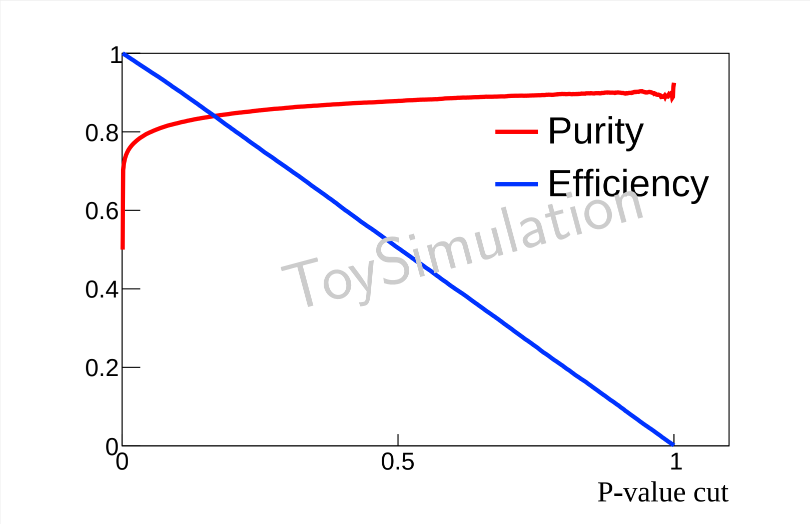

# The P-value



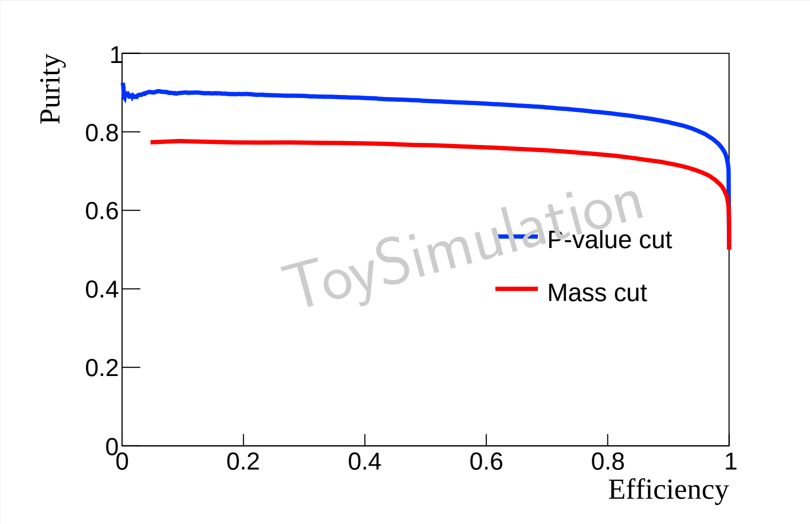
400
200
0
0
2
4
6
8
10
$\chi^2$
ToySimulation
All Events
Ξ Events
$\Xi^0$ Events
$10^4$
$10^3$
$10^2$
10
1
0
0.2
0.4
0.6
0.8
1
P-value


왜?
? ?

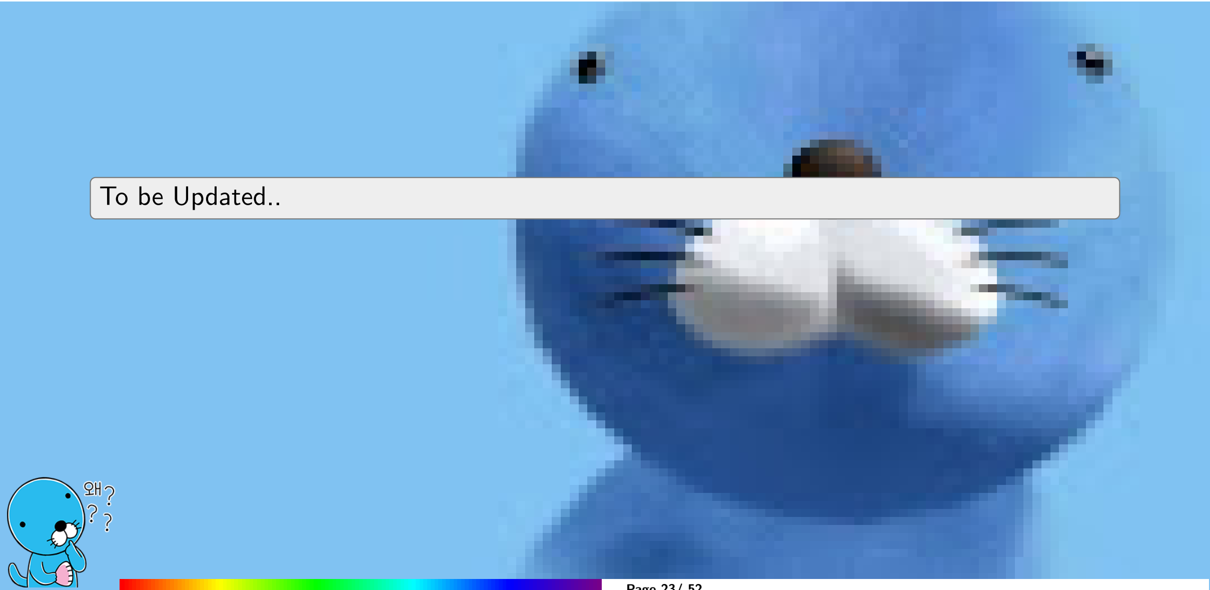
# P-value Selection




# P-value vs Invariant Mass



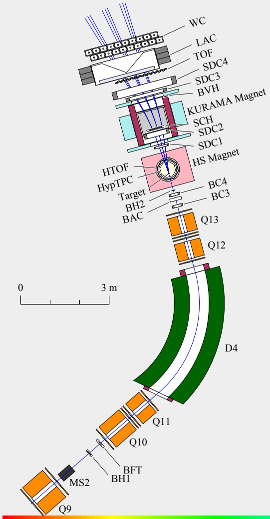

# Example: MassVertex-Constraint Fit



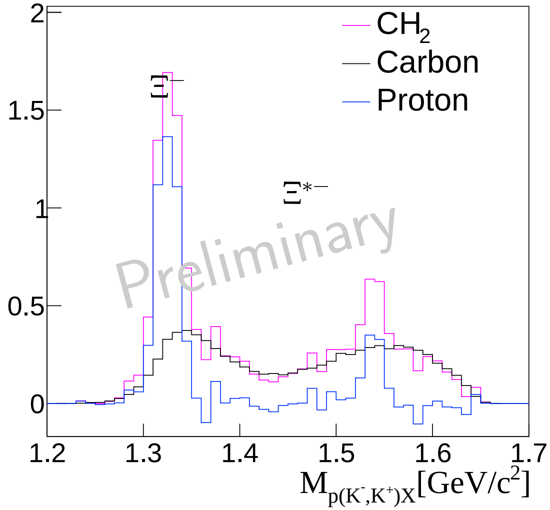

To be Updated..

# Applications: $\Xi$ decay at J-PARC

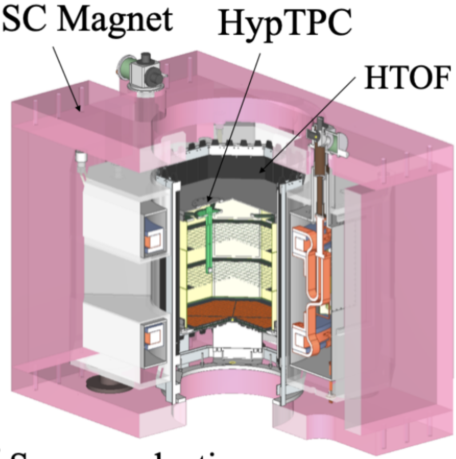
# $p(K^-, K^+)X$ at J-PARC E42



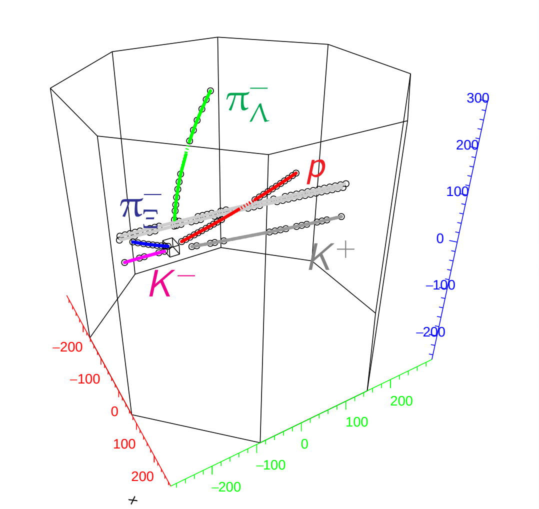




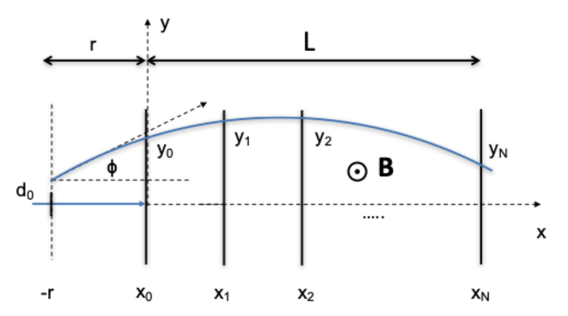
# HypTPC




Superconducting
Hyperon Spectrometer

# Gluckstern Formula




Z. Drasal, W. Riegler, Nucl. Instrum. Methds. A, 910, 127-132 (2018)

$$\frac{\sigma_{P_T}}{P_T} \simeq \frac{P_T}{0.3L^2B}\sqrt{\frac{720}{N+4}}\sigma_T \quad (29)$$

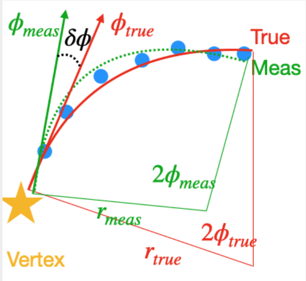
$$\frac{\sigma_{P_T,m.s}}{P_T} \simeq \frac{0.0136^1}{0,3\beta BL}\sqrt{\frac{d_{tot}}{X_0}} \quad (30)$$

Units in GeV/c

- Momentum resolution comprises geometrical term and scattering term
- In practice, empirical rescaling factor should be multiplied

[1] G.R. Lynch and O.I Dahl, Nucl. Instrum. Methods B58, 6 (1991).

# Covariance Matrix in Helix Track



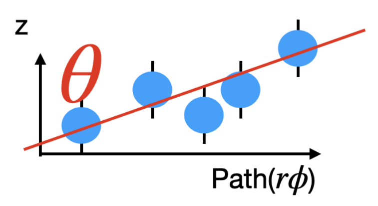

- Variance in momentum modifies the curvature of the helix $\longrightarrow$ direction at the vertex changes.
- 'Position' of the helix is defined from the TPC hits. $\longrightarrow$ Center-of-gravity should be fixed.

Denote the tangent angle at the center be $\phi_0$ and path length to the vertex $l$.

$$\phi = \phi_0 \pm \frac{l}{2r}; \quad \delta\phi = \pm\frac{l}{2r}\frac{\delta r}{r} = \pm\frac{l}{r}\frac{\delta p_T}{p_T} \tag{31}$$

$$\sigma_\phi^2 = \langle\delta\phi\delta\phi\rangle = \frac{l^2}{r^2 p_T^2}\sigma_{p_T}^2; \quad \mathrm{Cov}(\phi, p_T) = \langle\delta\phi\delta p_T\rangle = \pm\frac{l}{r p_T}\sigma_{p_T}^2 \tag{32}$$

# Covariance Matrix in Helix Track




$$h(t):\{r\cos(\phi)-c_x, r\sin\phi-c_y, dz*r\phi-z_0\} \tag{33}$$

- The 'pitch' parameter, dz, is the slope along the circular trajectory

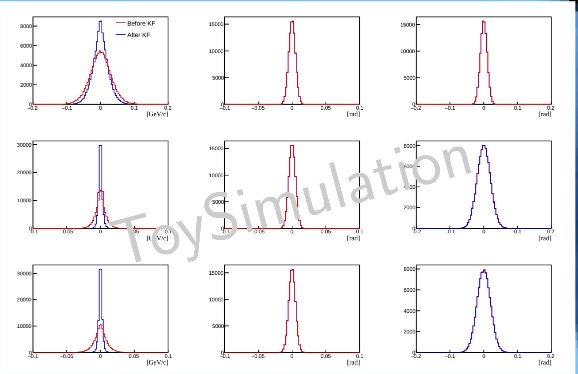
$\theta=\frac{\pi}{2}-\arctan(dz)$, we estimate the variance of $\theta$ based on the fitting error of dz. The error is estimated from the slope error of a linear fit:

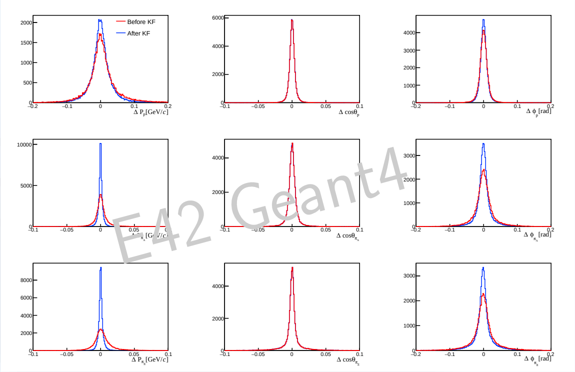
$$\sigma_{dz}^2=\frac{\sum\delta_z^2/(n-2)}{\sum(x-\bar{x})^2}\simeq\frac{n\sigma_z^2/(n-2)}{nL^2/12};\quad \sigma_\theta=\frac{\partial dz}{\partial\theta}\sigma_{dz}=\frac{1}{1+dz^2}\sigma_{dz}.$$

Note that, the momentum $p_z=p_T dz$ would also have some covariance with $\theta$,

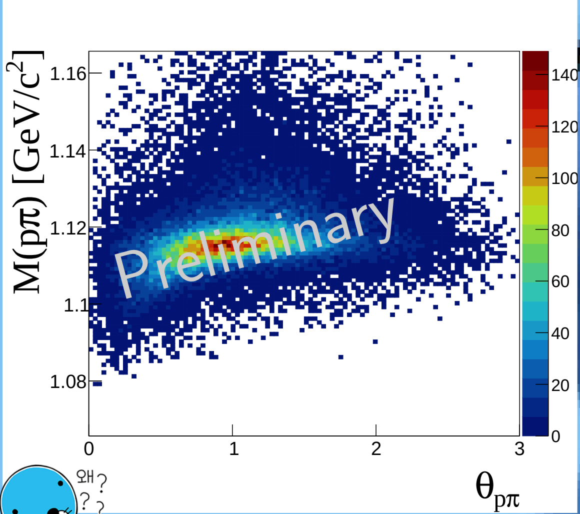
$$\langle\delta p\delta\theta\rangle=dz\cancelto{0}{\langle\delta p_T\delta\theta\rangle}+p_T\langle\delta dz\delta\theta\rangle=\frac{p_T}{1+dz^2}\sigma_{dz}^2.$$

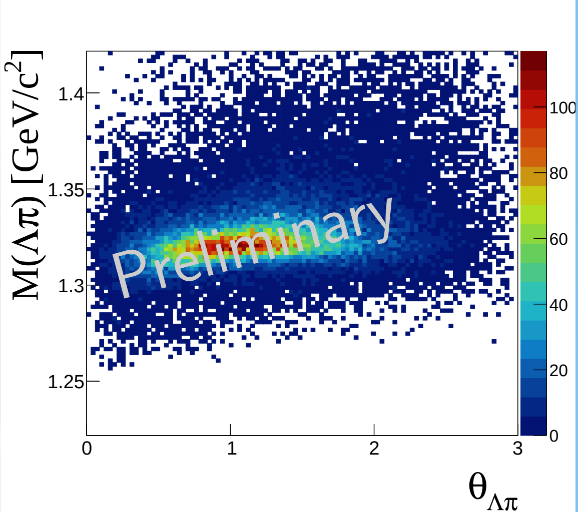
# ϕ Restoration from Diagonal Component



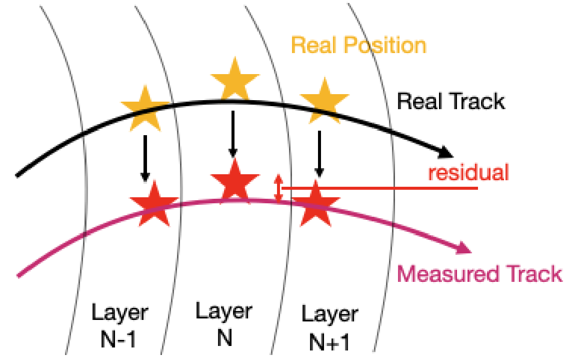



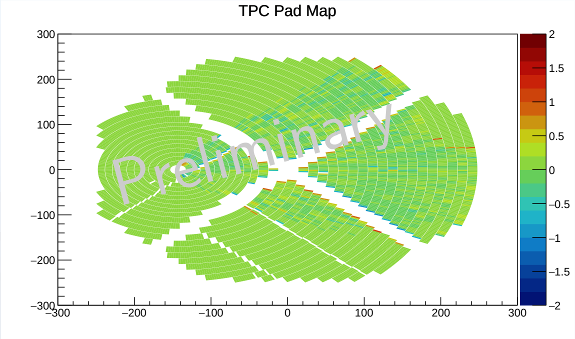

# Momentum Bias from Position Shift




# Position Residual?

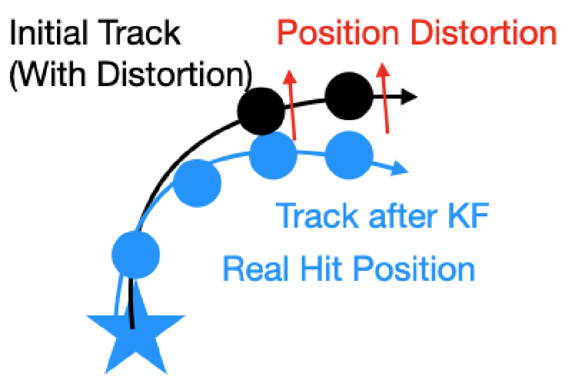


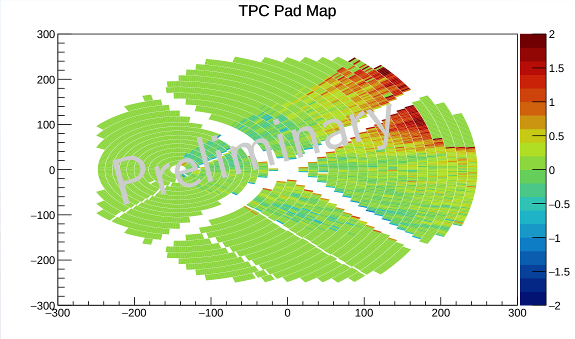





- Local, but simultaneous shift cannot be detected from position residual measurement.
- External reference for track should be provided to estimate 'true' trajectory

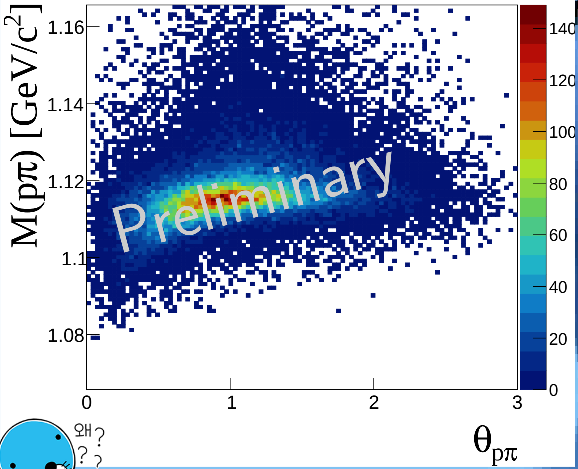
# Position Residual from KF Track

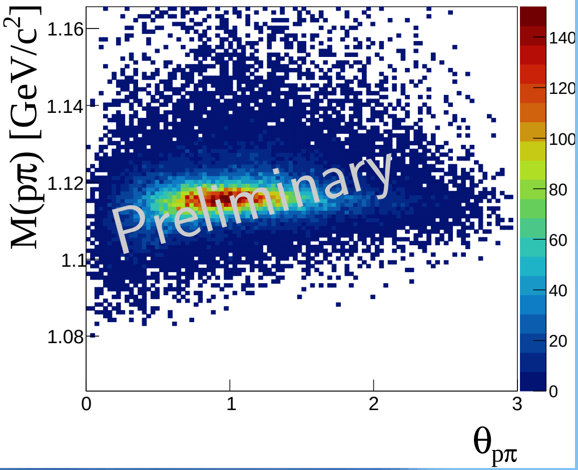


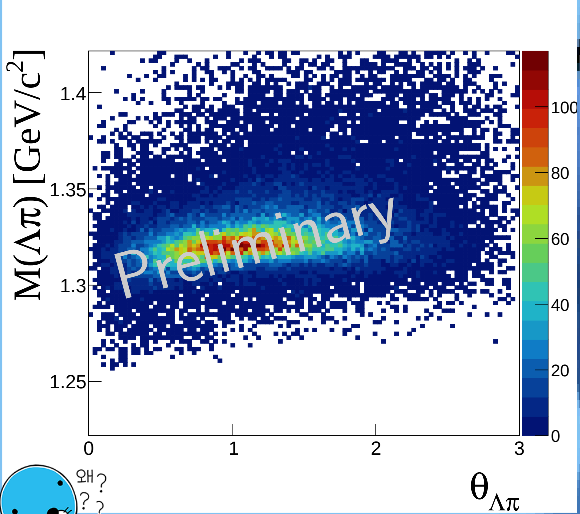

- From Kinematic fit, 'true' momentum, hence trajectory is estimated

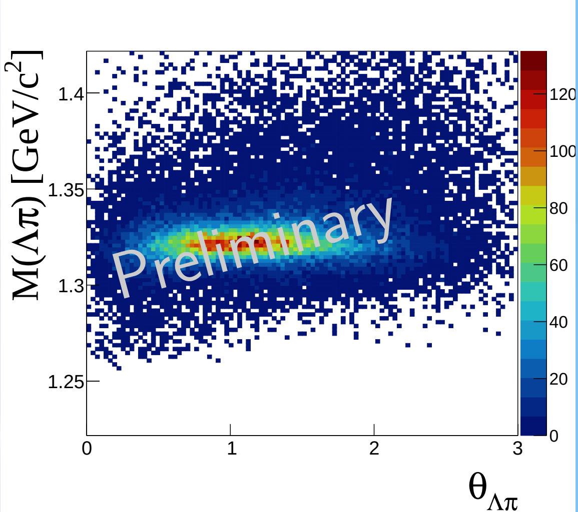
# Λ After Position Correction




# Ξ After Position Correction




# Tricks

# Variance Normalization

$$V = \begin{pmatrix} 10^{12} & 0.9 \\ 0.9 & 10^{-12} \end{pmatrix} \to V^{-1} = ? \tag{34}$$

While taking an inverse of the variance, matrix elements with different order may be added together, leading to possible numerical unstability.

$$\tilde{V} = SVS^T; S \equiv \frac{1}{\sqrt{V_{ij}}}\delta_{ij} \to \tilde{V} = \begin{pmatrix} 1 & Cov(01)/\sigma_1\sigma_2 & \cdots \\ Cov(01)/\sigma_1\sigma_2 & 1 & \\ \cdots & & \cdots \end{pmatrix} \tag{35}$$

We can take out scaling factors in S. Measurement vectors could share the same problem, so they should also be scaled. We rewrite equation (15)

$$\chi^2 = dM^\dagger V^{-1} dM + \cdots = d\tilde{M}^\dagger \tilde{V}^{-1} d\tilde{M} + \cdots ; \quad d\tilde{M} = S(M - M_0) \tag{36}$$

# Off-diagonal Reduction

$$V = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \rightarrow V^{-1} = ? \tag{37}$$

Adding off-diagonal term could make matrix uninvertable. Also we require $\chi^2 = dMV^{-1}dM > 0$; $V^{-1}$ (hence V)should be *Positive Definite*. Then, we can 'damp' the offdiagonal elements.

$$while(IsPositiveDefinite(V)) \tag{38}$$

$$V_{ij} \rightarrow V_{ij} - \alpha(\delta_{ij} - 1)V_{ij} \tag{39}$$

**Property of Positive Definite Matrix**

All Eigenvalues are Positive! TMatrixD well-supports eigenvalues, so we can just use it.

# Appendix

# Iterative Minimization Steps.

The coupled differential equations (17),(19) and (18) will be solved iteratively. For each $\nu$th step we would rewrite it as:

$$V^{-1}(\mathbf{m}^0)(\mathbf{m}^{\nu+1} - \mathbf{m}^0) + (\mathbf{F}_m^\dagger)^\nu \lambda^{\nu+1} = 0 \quad (40)$$

$$(\mathbf{F}_u^\dagger)^\nu \lambda^{\nu+1} = 0 \quad (41)$$

$$\mathbf{f}^\nu + \mathbf{F}_m^\nu(\mathbf{m}^{\nu+1} - \mathbf{m}^\nu) + \mathbf{F}_u^\nu(\mathbf{u}^{\nu+1} - \mathbf{u}^\nu) = 0. \quad (42)$$

Equation (42) is not a direct consequence of Equation (18) but rather a *linear approximation* to proceed with our iteration step.
Note that, as we are determining the parameters $\mathbf{m}, \mathbf{u}$ and $\lambda$, they are indexed as $\nu + 1$, while constraint terms(i.e. $\mathbf{f}, \mathbf{F}_\mu$ and $\mathbf{F}_\nu$) are calculated from current step, $\nu$. This fit is basically using Newton's method.

# Solving the Equation(1)

Multiplying **V** to Equation (40) leads to:

$$\mathbf{m}^{\nu+1} - \mathbf{m}^0 = -V(\mathbf{m}^0)(\mathbf{F}_m^\dagger)^\nu \lambda^{\nu+1}. \tag{43}$$

Substituting Equation (43) into Equation (42),

$$\begin{aligned} \mathbf{F}_u^\nu(\mathbf{u}^{\nu+1} - \mathbf{u}^\nu) &= -\mathbf{f}^\nu - \mathbf{F}_m^\nu(-V(\mathbf{m}^0)(\mathbf{F}_m^\dagger)^\nu \lambda^{\nu+1} + \mathbf{m}^0 - \mathbf{m}^\nu) \\ &= S\lambda^{\nu+1} - R \end{aligned} \tag{44}$$

where we define the *constraiint covariance* $S$ and *residual* $R$ as:

$$S \equiv \mathbf{F}_\mathbf{m}^\nu V(\mathbf{m}^0)(\mathbf{F}_\mathbf{m}^\dagger)^\nu; \quad R \equiv \mathbf{f}^\nu + \mathbf{F}_\mathbf{m}^\nu(\mathbf{m}^0 - \mathbf{m}^\nu) \tag{45}$$

Multiplying $(\mathbf{F}_u^\dagger)^\nu S^{-1}$ and substituting Equation (41), into (44), we get:


$$(\mathbf{F}_u^\dagger)^\nu S^{-1}\mathbf{F}_u^\nu(\mathbf{u}^{\nu+1} - \mathbf{u}^\nu) = \cancelto{0}{(\mathbf{F}_u^\dagger)^\nu \lambda^{\nu+1}} - (\mathbf{F}_u^\dagger)^\nu S^{-1} R. \tag{46}$$

# Solving the Equation(2)

Then we naturally obtain the expressions

$$\mathbf{u}^{\nu+1} = \mathbf{u}^{\nu} - ((\mathbf{F}_u^{\dagger})^{\nu} S^{-1} \mathbf{F}_u^{\nu})^{-1} (\mathbf{F}_u^{\dagger})^{\nu} S^{-1} R. \tag{47}$$

and

$$\lambda^{\nu+1} = S^{-1}(\mathbf{F}_u^{\nu}(\mathbf{u}^{\nu+1} - \mathbf{u}^{\nu}) + R). \tag{48}$$

For a summary, we have obtained all equations to proceed to the next step. All other matrices in the equation can be calculated from parameters of the current step, and $\chi^2$ can be evaluated from (15) .

$$\begin{cases} \mathbf{u}^{\nu+1} = \mathbf{u}^{\nu} - ((\mathbf{F}_u^{\dagger})^{\nu} S^{-1} \mathbf{F}_u^{\nu})^{-1} (\mathbf{F}_u^{\dagger})^{\nu} S^{-1} R & (47) \\ \lambda^{\nu+1} = S^{-1}(\mathbf{F}_u^{\nu}(\mathbf{u}^{\nu+1} - \mathbf{u}^{\nu}) + R) & (48) \\ \mathbf{m}^{\nu+1} = \mathbf{m}^{0} - V(\mathbf{m}^{0})(\mathbf{F}_m^{\dagger})^{\nu} \lambda^{\nu+1} & (43) \end{cases}$$

# Covariance Matrix Propagation

If we observe Eq.(43), we notice that $m^{\nu+1}$ is an addition(subtraction) of some parameters to the initial measurement. As we already know the error, i.e. Variance matrix, of initial data, and dealt with full statistical treatment, we can estimate how error propagates through the fitting process.

$$V(m) = J_{m,m^0} V(m^0) J^{\dagger}_{m,m^0} \tag{49}$$

To solve (49) the Jacobian should be .

$$J_{m,m^0(i,j)} = \frac{\partial m_i}{\partial m^0_j} \tag{50}$$

# Jacobian

To begin with, let us express Eq (43) in terms of $m^0$. At the moment we will drop the superscript $\nu$. As $\mathbf{f}(\mathbf{m},\mathbf{u})$is a constant on $m^0$, $\mathbf{F_m}$ also will be a constant to $m^0$. Then we only need to consider the derivatives of $\lambda$. By substituting (47) ,

$$\lambda = S^{-1}(-\mathbf{F}_u(((\mathbf{F}_u^\dagger)S^{-1}\mathbf{F}_u)^{-1}(\mathbf{F}_u^\dagger)S^{-1}R) + R) \tag{51}$$

and the residual matrix is:

$$R \equiv \mathbf{f} + \mathbf{F_m}(\mathbf{m^0} - \mathbf{m}) \rightarrow \frac{\partial R}{\partial m^0} = \mathbf{F}_m \tag{52}$$

Now we obtain the derivative of $\lambda$ as:

$$\frac{\partial \lambda}{\partial m^0} = S^{-1}(-\mathbf{F}_u((\mathbf{F}_u^\dagger S^{-1}\mathbf{F}_u)^{-1}\mathbf{F}_u^\dagger S^{-1}\mathbf{F}_m) + \mathbf{F}_m). \tag{53}$$

# Jacobian

Now define the symmetric matrices

$$G \equiv \mathbf{F}_{\mathbf{m}}^{\dagger} S^{-1} \mathbf{F}_{\mathbf{m}}; \quad U \equiv (\mathbf{F}_u^{\dagger} S^{-1} \mathbf{F}_u)^{-1}; \quad H \equiv \mathbf{F}_m^{\dagger} S^{-1} \mathbf{F}_{\mathbf{u}} \tag{54}$$

We have expressions for $\frac{\partial \lambda}{\partial m^0}$. Equation (50) is determined as:

$$\begin{aligned} J_{m,m^0} &= I - V(m^0)\mathbf{F}_m^{\dagger} \frac{\partial \lambda}{\partial m^0} = I - V\mathbf{F}_m^{\dagger}(-S^{-1}\mathbf{F}_u U^{-1} H^{\dagger} + S^{-1}\mathbf{F}_m) \\ &= I - V(G - HUH^{\dagger}) \end{aligned} \tag{55}$$

If we let $C = G - HUH^{\dagger}$, we obtain

$$V(m) = J_{m,m^0} V J_{m,m^0}^{\dagger} = V - 2VCV + VCVCV. \tag{56}$$

We would keep 2nd order term at the moment. Some materials like [1] had an error in this part.

# Variance of the Unknowns

Just like how we derived Eq.(56) we can estimate the variance matrix of the unknowns.

$$V_U = J_{u,m0} V J_{u,m0}^T \tag{57}$$

$J_{u,m0}$ can be obtained from Eq.(47). Defining

$$K \equiv ((\mathbf{F}_u^\dagger)^\nu S^{-1} \mathbf{F}_u^\nu)^{-1} (\mathbf{F}_u^\dagger)^\nu S^{-1} \tag{58}$$

we write:

$$J_{u^{\nu+1},m^0} = \frac{\partial u^{\nu+1}}{\partial m^0} = \frac{\partial u^\nu}{\partial m^0} - K \frac{\partial R}{\partial m^0} \simeq -K\mathbf{F}_m. \tag{59}$$

Note that we only have initial "Guess" for the unknowns; In principle, it is not a driven value from measurements. Then, $\frac{\partial u^0}{\partial m_0} = 0$. Also, we approximate that the terms in 2nd or higher iterations are negligible: $J_{u,m^0} \simeq J_{u^0,m^0}$.

# Pull distribution

The covariance in Equation (21) is estimated as:

$$Cov(m, m^0) = J_{m,m^0} V(m) = V - VCV. \tag{60}$$

If we substitute this and Eq.(56) into Eq.(21), we get

$$V(\epsilon) = VCVCV. \tag{61}$$

Note that 2nd order term affects the covariance matrix of the correction.

# Applications

# Example: Chained Mass-Constraint Fit

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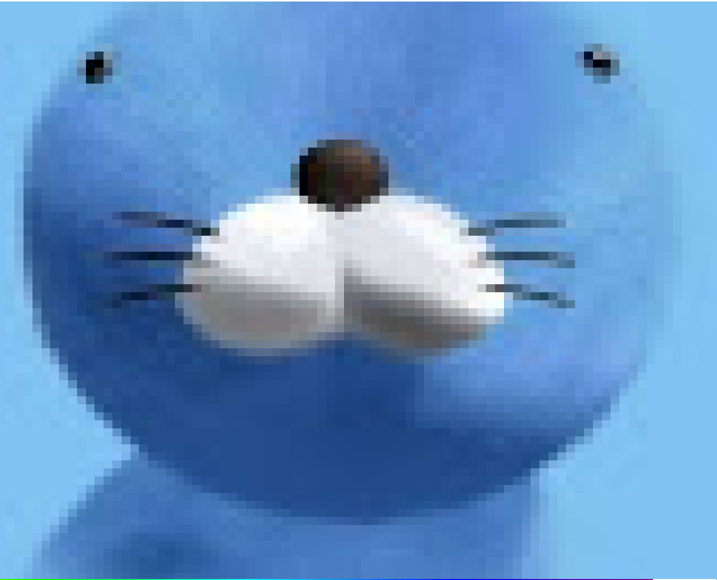
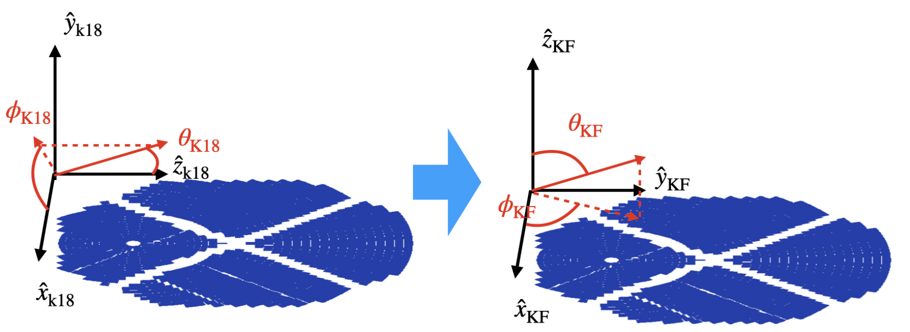


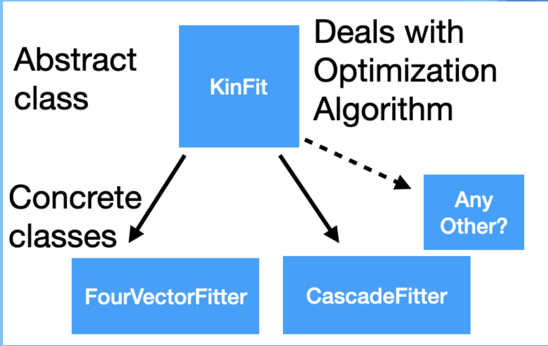
# Mass-Constraint Kinematic Fit with HypTPC.




For Kinematic FIt in HypTPC analysis, we want our coordinate system to be aligned with $\vec{B}$, so that our covariance matrix representation fits the representations in KF coordinate. We correlated $\phi$ angle with $p$ as an feature of helix fit, where $\phi$ is the angle lying on the circle of the helix.

# KinFit package

$$\chi^2 = \delta M^T V^{-1} \delta M + 2\lambda f(M, U) \tag{62}$$




**KinFIt provides...**

- Minimize $\chi^2$
- Calculate pulls and p-values

**Users should...**

- Assign proper variables for M and U
- Define physical constraints
- Write the derivatives, $F_M$ and $F_U$ by hand

# References

O. Skjeggestad Frodesen A.G. **Probability and Statistics in Particle Physics**. Columbia University Press, 1980. ISBN: 8200019063.