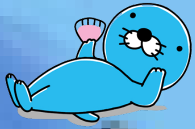
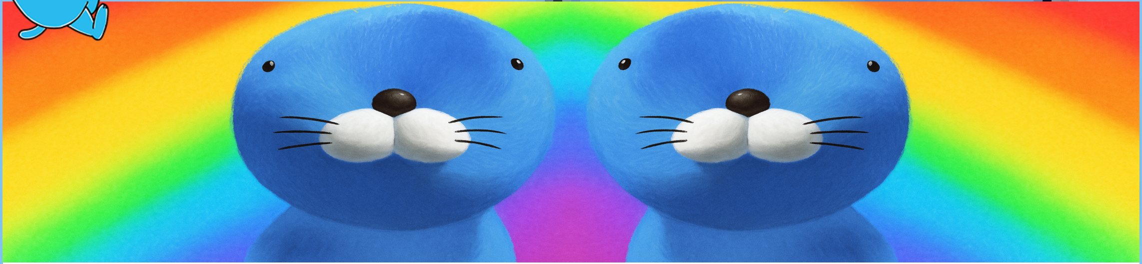





# Notes on Kinematic Fit

**The template for this note is provided here[2] [1]**
**November 14, 2025**

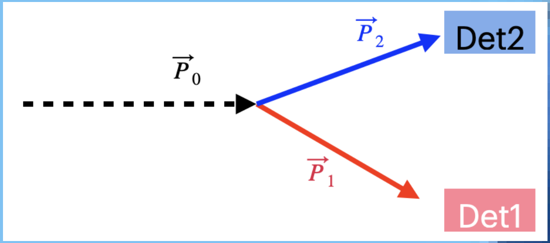
[1] Bono University

# Outline

1. Introduction
2. Example: Mass-Constraint Fit
3. Applications: $\Xi$ decay at J-PARC
4. Tricks
5. Appendix

# Introduction

# Measurement Error




Assume a beam with momentum $\vec{P}^-$ decays into $\vec{P}_1$ and $\vec{P}_2$. Measured momentum are smeared due to detector resolution, leading to unbalance in the momentum conservation.

$$\vec{P}_0 = \vec{P}_1 + \vec{P}_2; \quad \vec{P}_{1,meas} + \vec{P}_{2,meas} \neq \vec{P}_0 \tag{1}$$

We can define the $\chi^2$ to quantitatively represent our measurement error. However, we can't derive meaningful expressions from this $\chi^2$.

$$\chi^2 = \frac{(P_1 - P_{1,meas})^2}{\sigma_1^2} + \frac{(P_2 - P_{2,meas})^2}{\sigma_2^2} \tag{2}$$

# Constrained Optimization with The Lagrange Multiplier

By incorporating the *Kinematic Constraints*, specifically *momentum conservation*, we involve additional knowledge to (2). This is known as the *Lagrange Multiplier*

$$\chi^2 = \frac{(P_{1,KF} - P_{1,meas})^2}{\sigma_1^2} + \frac{(P_{2,KF} - P_{2,meas})^2}{\sigma_2^2} + 2\lambda(\mathbf{P_{1,KF}} + \mathbf{P_{2,KF}} - \mathbf{P_0}) \tag{3}$$

Now we have meaningful expressions to minimize $\chi^2$, hence get better estimations for the measurement.

$$\frac{1}{2}\frac{\partial\chi 2}{\partial P_{1,KF}} = \frac{(P_{1,KF} - P_{1,meas})}{\sigma_1^2} + \lambda = 0 \tag{4}$$

$$\frac{1}{2}\frac{\partial\chi 2}{\partial P_{2,KF}} = \frac{(P_{2,KF} - P_{2,meas})}{\sigma_2^2} + \lambda = 0 \tag{5}$$

$$\frac{1}{2}\frac{\partial\chi 2}{\partial\lambda} = (P_{1,KF} + P_{2,KF} - P_0) = 0 \tag{6}$$

# Why Better Resolution?

By solving the equations 4,5,6 and defining $\delta_i = P_{i,meas} - P_i$, we obtain the following expressions:

$$\lambda = \frac{P_{1,meas} + P_{2,meas} - P_0}{\sigma_1^2 + \sigma_2^2} = \frac{\delta_1 + \delta_2}{\sigma_1^2 + \sigma_2^2} \tag{7}$$

$$P_{1,KF} = P_{1,meas} - \sigma_1^2 \lambda \tag{8}$$

$$P_{2,KF} = P_{2,meas} - \sigma_2^2 \lambda \tag{9}$$

$$\begin{aligned} < P_{1,KF} - P_1 > &=< P_{1,KF} - P_{1,meas} + \delta_1 >=< -\sigma_1^2 \lambda + \delta_1 > \\ &=< \frac{-\sigma_1^2}{\sigma_1^2 + \sigma_2^2}(\delta_1 + \delta_2) + \delta_1 >=< \frac{\sigma_2^2 \delta_1 - \sigma_1^2 \delta_2}{\sigma_1^2 + \sigma_2^2} > \end{aligned} \tag{10}$$

$$\sigma_{1,KF}^2 =< (P_{1,KF} - P_1)^2 >= \frac{\sigma_2^4 \cancelto{\sigma_1^2}{< \delta_1^2 >} + \sigma_1^4 \cancelto{\sigma_2^2}{< \delta_2^2 >}}{(\sigma_1^2 + \sigma_2^2)^2} = \frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2} < \sigma_1^2 \tag{11}$$

# The Covariance After Kinematic Fit

$$cov(P_1, P_2)_{KF} =< \delta_{1,KF}\delta_{2,KF} >=< (\delta_1 - \sigma_1^2\lambda)(\delta_2 - \sigma_2^2\lambda) >$$

$$= \sigma_1^2\sigma_2^2 \cancelto{\frac{1}{\sigma_1^2+\sigma_2^2}}{< \lambda^2 >} - \frac{\sigma_1^2 < \delta_2^2 > + \sigma_2^2 < \delta_1^2 >}{\sigma_1^2 + \sigma_2^2} = -\frac{\sigma_1^2\sigma_2^2}{\sigma_1^2 + \sigma_2^2} \tag{12}$$

$$V = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix} \rightarrow V_{KF} = \begin{pmatrix} \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2} & -\frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2} \\ -\frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2} & \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2} \end{pmatrix} \tag{13}$$

- Improved momentum resolution
- Negative correlation between $P_1$ and $P_2$

# Generalization to Multi-Variables

Assume that we have a set of measured data $\mathbf{m^0}$, unknown parameters $\mathbf{u^0}$ and constraints $\mathbf{f}^0$.

$$\begin{aligned} \mathbf{m^0} &= \{m_1^0, m_2^0 \ldots m_N^0\}; \quad \mathbf{u^0} = \{u_1^0, u_2^0 \ldots u_J^0\} \\ \mathbf{f} &= \{f_1(m_1^0, m_2^0, \ldots m_N^0, u_1^0, u_2^0, \ldots u_N^0), f_2^0, \ldots f_K^0\} \end{aligned} \tag{14}$$

Let $\mathbf{m}^0$ denote our initial measured data, and $\mathbf{m}$ represent the 'guess' of the data in each iterative step, just alike $P_{KF}$s in the previous example. Equation (3) is generalized to:

$$\chi^2(\mathbf{m}) = (\mathbf{m}^0 - \mathbf{m})^\dagger V^{-1}(\mathbf{m}^0 - \mathbf{m}) + 2\lambda^\dagger \mathbf{f}(\mathbf{m}, \mathbf{u}). \tag{15}$$

Here, the Lagrange multiplier $\lambda = \{\lambda_1, \lambda_2, \ldots \lambda_K\}$ is not just a number but a column vector with k elements, corresponding to each kinematic constraint in $\mathbf{f}$.

# $\chi^2$ Minimization

We want to solve the equation

$$\vec{\nabla}\chi^2 = 0 \tag{16}$$

to obtain the minimized state. The differential term are listed within three groups.

$$\nabla_{\mathbf{m}} = -2V^{-1}(\mathbf{m}^0)(\mathbf{m}^0 - \mathbf{m}) + 2\mathbf{F}^{\dagger}_{\mathbf{m}}(\mathbf{m},\mathbf{u})\lambda = 0 \tag{17}$$

$$\nabla_{\mathbf{u}} = 2\mathbf{F}^{\dagger}_{\mathbf{u}}(\mathbf{m},\mathbf{u})\lambda = 0 \tag{18}$$

$$\nabla_{\lambda} = \mathbf{f}(\mathbf{m},\mathbf{u}). \tag{19}$$

Here, the subscripts denote partial derivatives. i.e. $((\mathbf{F}_m)_{ki} \equiv \frac{\partial f_k}{\partial m_i})$.

## User Should Define...


| $\mathbf{m}$ | $\mathbf{u}$ | $\mathbf{f}$ | $\mathbf{V}$ | $\mathbf{F_m}, \mathbf{F_u}$ |
|---|---|---|---|---|
| Measured Data | Unknown parameters | Constraints | Covariance Matrix | Derivatives |

# Pull distribution

A bias or resolution miss-estimation is revealed by observing the *Pull distribution* of each measurements. However, we cannot evaluate the 'true' value of measurement, hence pull for the real data is not accessible. Instead, from the the residual $\epsilon = m - m^0$ and its variance $V(\epsilon)$, we observe the pull(of the residual) as :

$$P(\epsilon) = \epsilon / \sqrt{V(\epsilon)} \tag{20}$$

and

$$V(\epsilon) \equiv V(m) + V(m^0) - 2Cov(m, m^0). \tag{21}$$

The variance of the fitted variables, $V(m)$, is evaluated as

$$V(m) = J_{m,m^0} V(m^0) J^{\dagger}_{m,m^0} \tag{22}$$

where $J_{m,m^0}$ is the Jacobian for $m$ and $m^0$. Detailed calculations are provided in the appendix.

# Example: Mass-Constraint Fit

# Example: $\Lambda \to p\pi$, Defining Variables and Constraints

Assume a decay of $\Lambda \to p\pi^-$. We define the measurements and unknowns as:

$$\mathbf{m} = \{P_p, \theta_p, \phi_p, P_\pi, \theta_\pi, \phi_\pi\}; \quad \mathbf{u} = \{P_\Lambda, \theta_\Lambda, \phi_\Lambda\} \tag{23}$$

Then we define the energy-momentum constraint equation as:

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} -P_\Lambda \sin\theta_\Lambda \cos\phi_\Lambda + P_p \sin\theta_p \cos\phi_p + P_\pi \sin\theta_\pi \cos\phi_\pi \\ -P_\Lambda \sin\theta_\Lambda \sin\phi_\Lambda + P_p \sin\theta_p \sin\phi_p + P_\pi \sin\theta_\pi \sin\phi_\pi \\ -P_\Lambda \cos\theta_\Lambda + P_p \cos\theta_p + p_\pi \cos\theta_\pi \\ -\sqrt{P_\Lambda^2 + m_\Lambda^2} + \sqrt{P_p^2 + m_p^2} + \sqrt{P_\pi^2 + m_\pi^2} \end{pmatrix}. \tag{24}$$

where the mass constraint is naturally implemented in energy term.
Since we have 3 unmeasured variable with 4 kinematical constraints, this is a 4-3 = 1-Constrained fit.

# Example: $\Lambda \to p\pi$, The Derivatives

We get $\mathbf{F_u}$ and $\mathbf{F_m}$ as

$$\mathbf{F_u} = \begin{pmatrix} \frac{\partial f_1}{\partial P_\Lambda} & \frac{\partial f_1}{\partial \theta_\Lambda} & \frac{\partial f_1}{\partial \phi_\Lambda} \\ \frac{\partial f_2}{\partial P_\Lambda} & \frac{\partial f_2}{\partial \theta_\Lambda} & \frac{\partial f_2}{\partial \phi_\Lambda} \\ \frac{\partial f_3}{\partial P_\Lambda} & \frac{\partial f_3}{\partial \theta_\Lambda} & \frac{\partial f_3}{\partial \phi_\Lambda} \\ \frac{\partial f_4}{\partial P_\Lambda} & \frac{\partial f_4}{\partial \theta_\Lambda} & \frac{\partial f_4}{\partial \phi_\Lambda} \end{pmatrix}; \quad \mathbf{F_m} = \begin{pmatrix} \frac{\partial f_1}{\partial P_p} & \cdots & \frac{\partial f_1}{\partial \phi_\pi} \\ \frac{\partial f_2}{\partial P_p} & \cdots & \frac{\partial f_2}{\partial \phi_\pi} \\ \frac{\partial f_3}{\partial P_p} & \cdots & \frac{\partial f_3}{\partial \phi_\pi} \\ \frac{\partial f_4}{\partial P_p} & \cdots & \frac{\partial f_4}{\partial \phi_\pi} \end{pmatrix} \tag{25}$$

We have all the matrices to calculate in each step. By applying an appropriate variance matrix and employing $\chi^2$ selection criteria, we can do kinematic fit for the particles.

# Example: $\Xi \to \Lambda\pi$, $\Lambda \to p\pi$

We require two mass constraints for $\Xi \to \Lambda\pi$; $\quad \Lambda \to p\pi$. In this case, careful considerations on the selection of variables. We will select

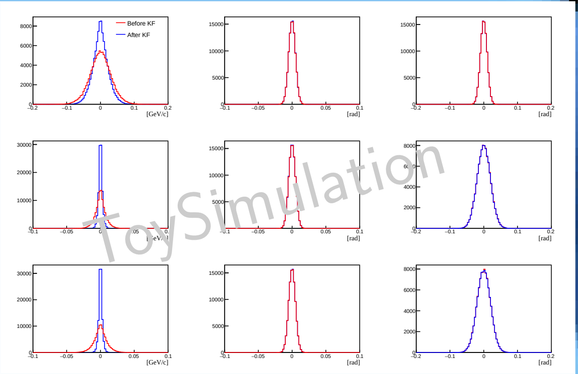
$$\mathbf{u} = \{P_\Xi, \theta_\Xi, \phi_\Xi\}; \quad \mathbf{m} = \{P_p, \theta_p, \phi_p, P_\pi, \theta_\pi, \phi_\pi\} \tag{26}$$

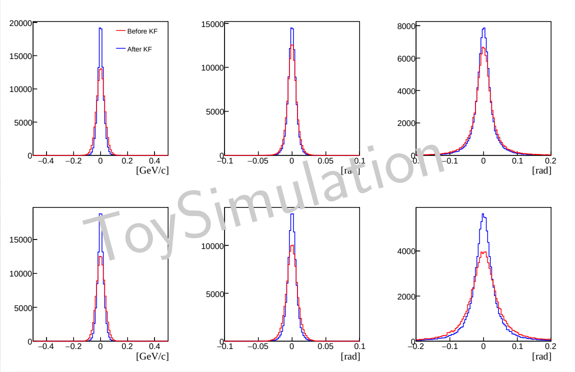
and define five constraints as:

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{pmatrix} = \begin{pmatrix} -P_{\Xi,x} + P_{p,x} + P_{\pi_\Lambda,x} + P_{\pi_\Xi,x} \\ -P_{\Xi,y} + P_{p,y} + P_{\pi_\Lambda,y} + P_{\pi_\Xi,y} \\ -P_{\Xi,z} + P_{p,z} + P_{\pi_\Lambda,z} + P_{\pi_\Xi,z} \\ -E_\Lambda + E_p + E_{\pi_\Lambda} \\ -E_\Xi + E_p + E_{\pi_\Lambda} + E_{\pi_\Xi} \end{pmatrix}. \tag{27}$$

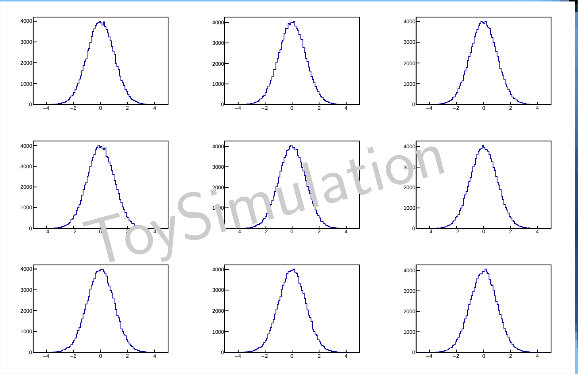
$\Lambda$ variables($\{P_\Lambda, \theta_\Lambda, \phi_\Lambda\}$) are not selected in $\mathbf{u}$ to reduce matrix dimension. That is, **we don't have explicit terms** related to $\vec{P}_\Lambda$, i.e. $-P_{\Lambda,x} + P_{p,x} + P_{\pi_\Lambda,x}$ etc., because $\vec{P}_\Lambda$ are neither unmeasured nor measured variables in our choice of parameters.

# Kinematics Restoration



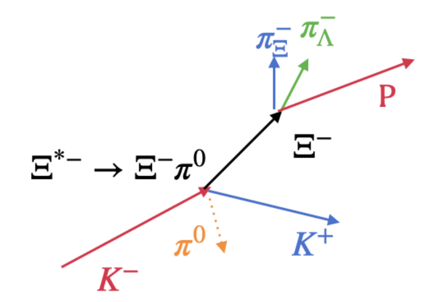

# Pull Distribution

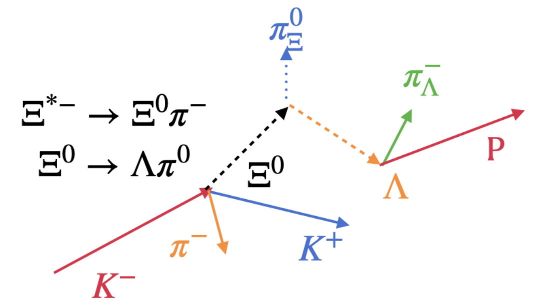



$$P(X) = \frac{X_{KF} - X_0}{\sqrt{V(X_{KF} - X_0)}} \quad (28)$$

- Pull distribution shows the normalized amount of parameter adjustment.
- Gaussian distribution with $\sigma = 1$ implies good understandings in covariance of the measurement.
- In practice, resolution can be iteratively scaled by 1./ pull width

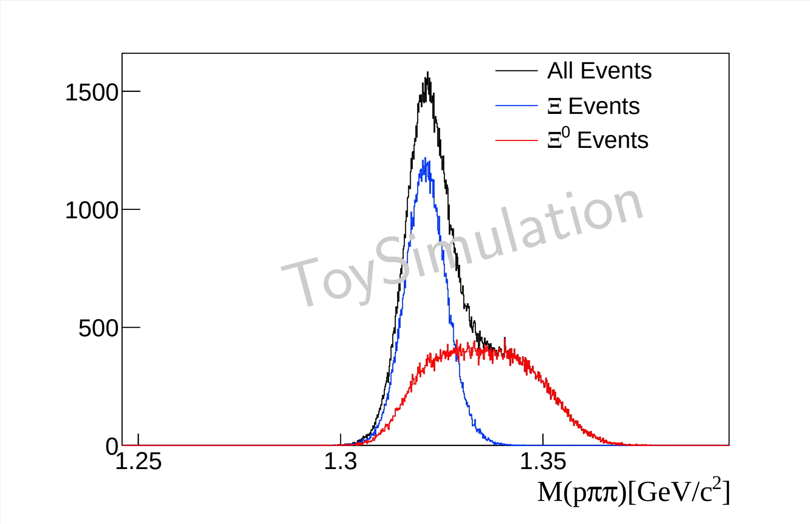
# $\Xi^*(1530) \to \Xi\pi^0$ and $\Xi^*(1530) \to \Xi^0\pi^-$ Separation



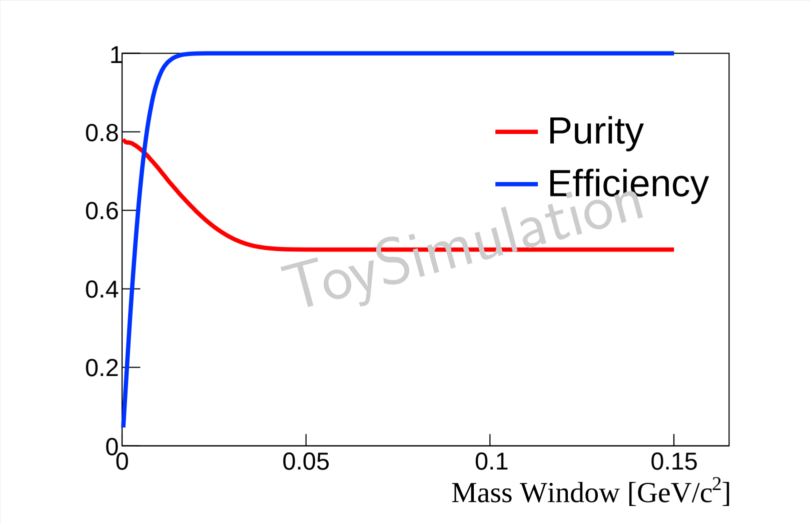




- Two decay channel of $\Xi^*$ share the same decay product.
- Separation criteria should be defined to distinguish combinatorial backgrounds.
- Kinematic Fit result can provide another selection criteria based on Kinematics.

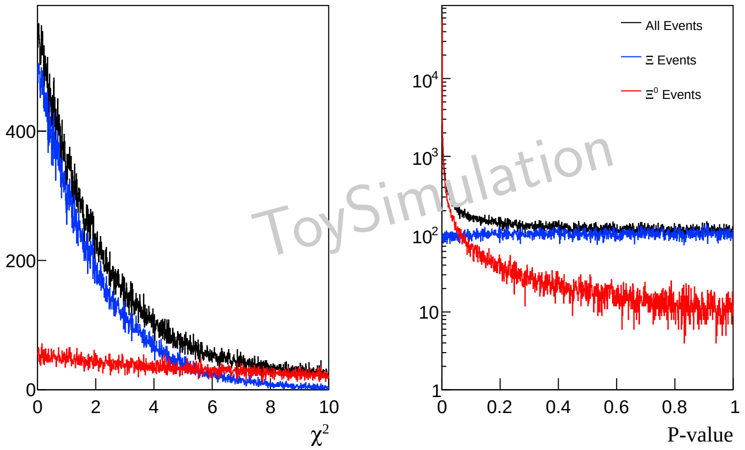
# Invariant Mass




# Mass Window Selection



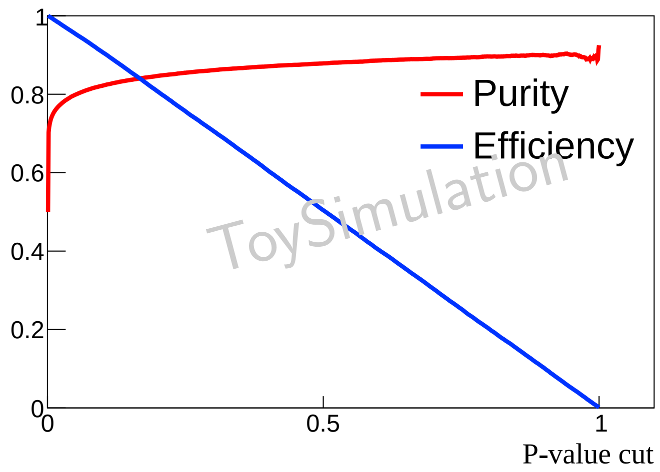

# The P-value



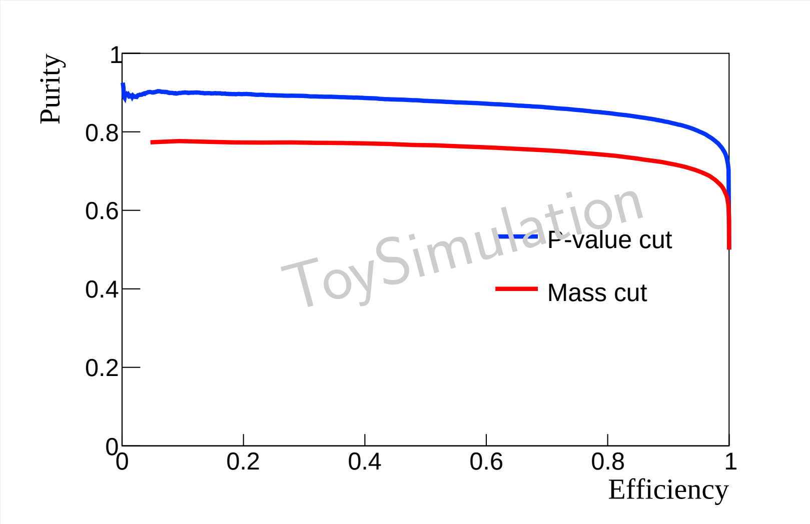
400
200
0
0
2
4
6
8
10
$\chi^2$
All Events
Ξ Events
$\Xi^0$ Events
$10^4$
$10^3$
$10^2$
10
1
0
0.2
0.4
0.6
0.8
1
P-value
ToySimulation


왜?
? ?

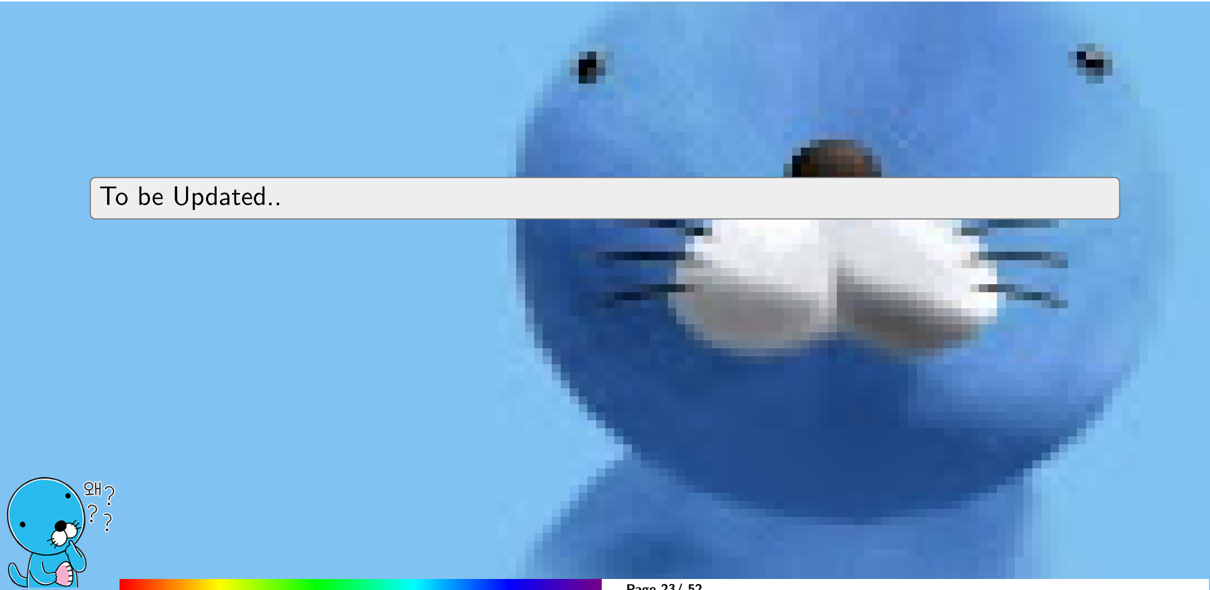
# P-value Selection




# P-value vs Invariant Mass

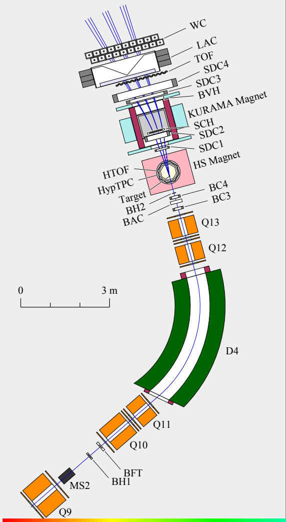



# Example: MassVertex-Constraint Fit

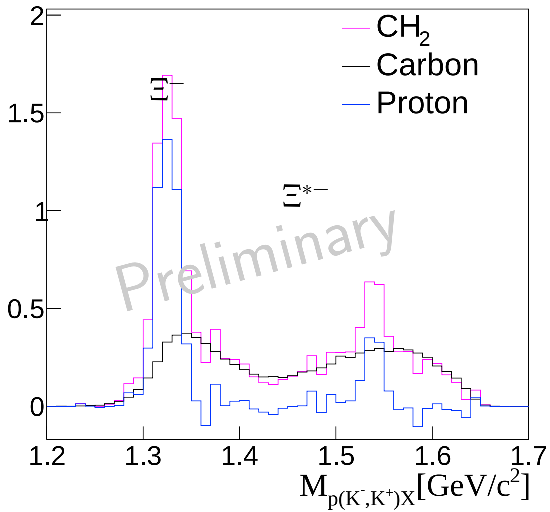


# Applications: $\Xi$ decay at J-PARC

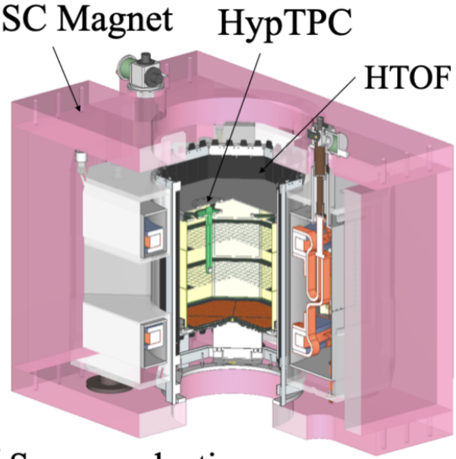
# $p(K^-, K^+)X$ at J-PARC E42



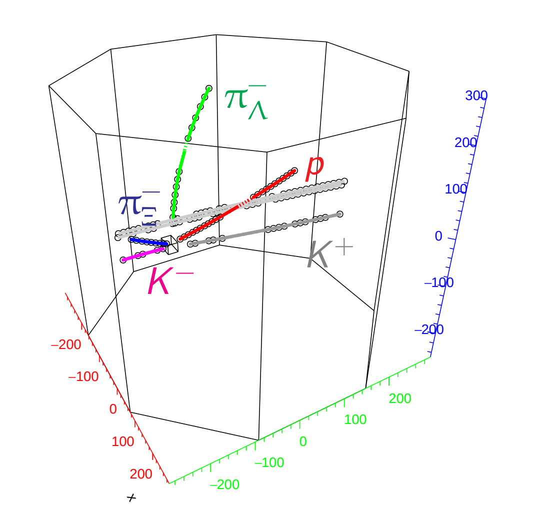




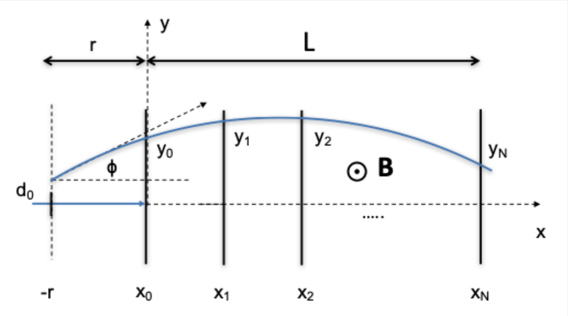
# HypTPC




Superconducting
Hyperon Spectrometer



# Gluckstern Formula




Z. Drasal, W. Riegler, Nucl. Instrum. Methds. A, 910, 127-132 (2018)

$$\frac{\sigma_{P_T}}{P_T} \simeq \frac{P_T}{0.3L^2B}\sqrt{\frac{720}{N+4}}\sigma_T \quad (29)$$

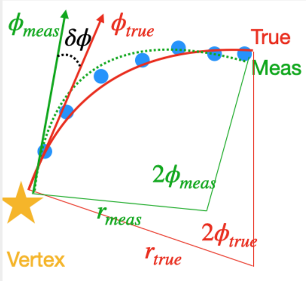
$$\frac{\sigma_{P_T,m.s}}{P_T} \simeq \frac{0.0136^1}{0,3\beta BL}\sqrt{\frac{d_{tot}}{X_0}} \quad (30)$$

Units in GeV/c

- Momentum resolution comprises geometrical term and scattering term
- In practice, empirical rescaling factor should be multiplied

[1] G.R. Lynch and O.I Dahl, Nucl. Instrum. Methods B58, 6 (1991).

# Covariance Matrix in Helix Track



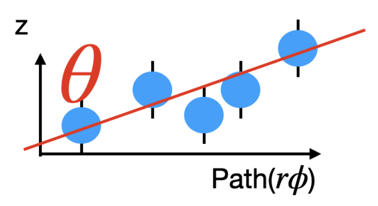

- Variance in momentum modifies the curvature of the helix $\longrightarrow$ direction at the vertex changes.
- 'Position' of the helix is defined from the TPC hits. $\longrightarrow$ Center-of-gravity should be fixed.

Denote the tangent angle at the center be $\phi_0$ and path length to the vertex $l$.

$$\phi = \phi_0 \pm \frac{l}{2r}; \quad \delta\phi = \pm\frac{l}{2r}\frac{\delta r}{r} = \pm\frac{l}{r}\frac{\delta p_T}{p_T} \tag{31}$$

$$\sigma_\phi^2 = \langle\delta\phi\delta\phi\rangle = \frac{l^2}{r^2 p_T^2}\sigma_{p_T}^2; \quad \mathrm{Cov}(\phi, p_T) = \langle\delta\phi\delta p_T\rangle = \pm\frac{l}{rp_T}\sigma_{p_T}^2 \tag{32}$$

# Covariance Matrix in Helix Track




$$h(t) : \{r\cos(\phi) - c_x, r\sin\phi - c_y, dz * r\phi - z_0\} \tag{33}$$

- The 'pitch' parameter, dz, is the slope along the circular trajectory

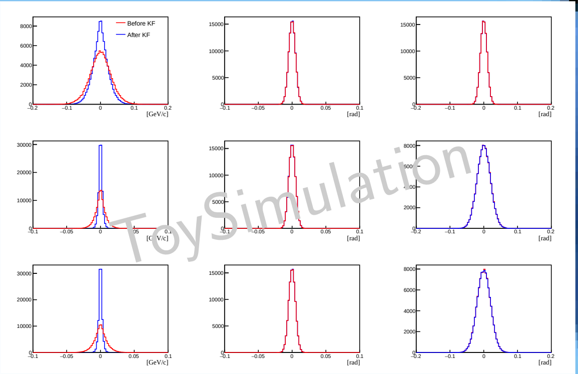
$\theta = \frac{\pi}{2} - \arctan(dz)$, we estimate the variance of $\theta$ based on the fitting error of dz. The error is estimated from the slope error of a linear fit:

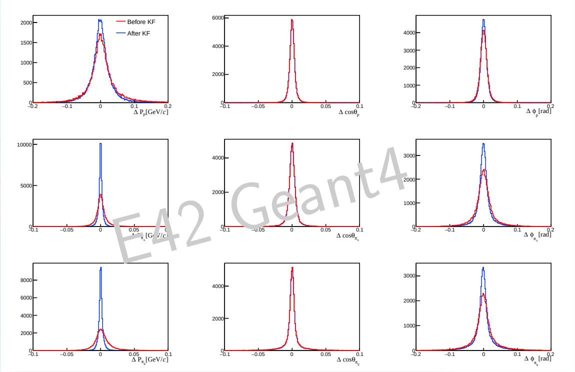
$$\sigma_{dz}^2 = \frac{\sum \delta_z^2/(n-2)}{\sum (x-\bar{x})^2} \simeq \frac{n\sigma_z^2/(n-2)}{nL^2/12}; \quad \sigma_\theta = \frac{\partial dz}{\partial \theta}\sigma_{dz} = \frac{1}{1+dz^2}\sigma_{dz}.$$

Note that, the momentum $p_z = p_T dz$ would also have some covariance with $\theta$,

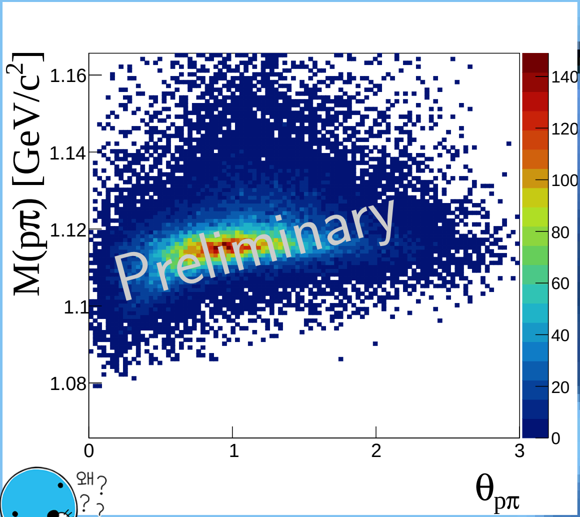
$$\langle \delta p \delta\theta \rangle = dz \cancelto{0}{\langle \delta p_T \delta\theta \rangle} + p_T \langle \delta dz \delta\theta \rangle = \frac{p_T}{1+dz^2}\sigma_{dz}^2.$$

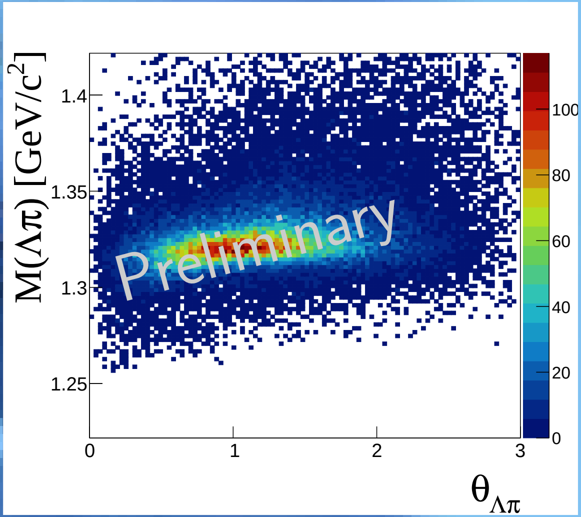
# ϕ Restoration from Diagonal Component



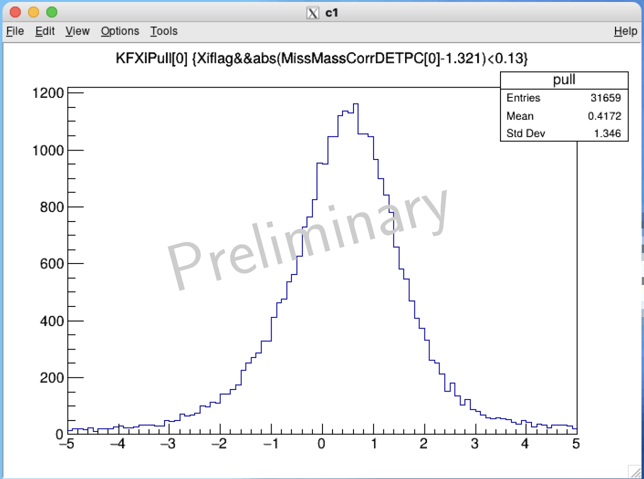




# Invariant Mass Bending from Momentum Bias

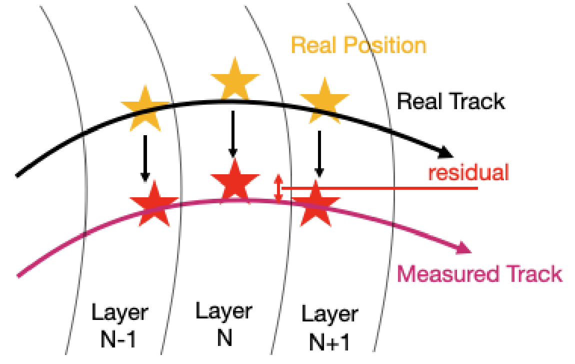


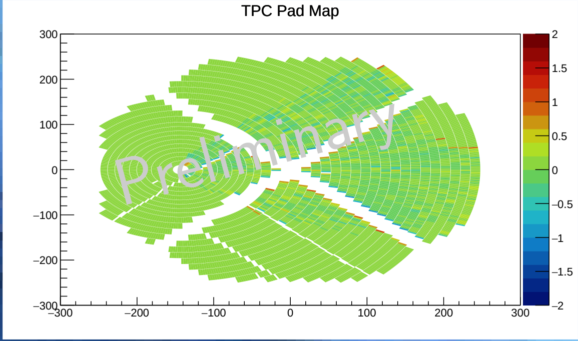




# Off-center Pull Distribution




# Position Residual?

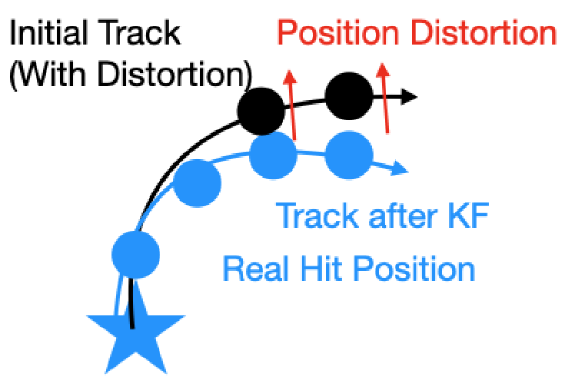


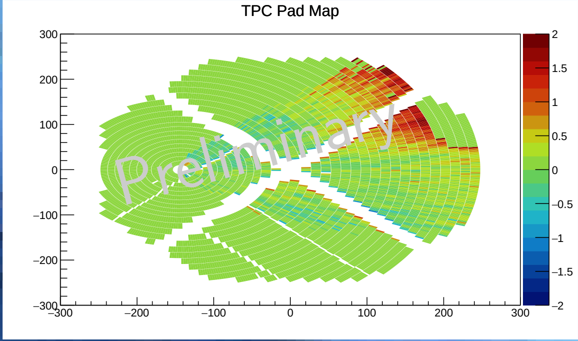





- Local, but simultaneous shift cannot be detected from position residual measurement.
- External reference for track should be provided to estimate 'true' trajectory

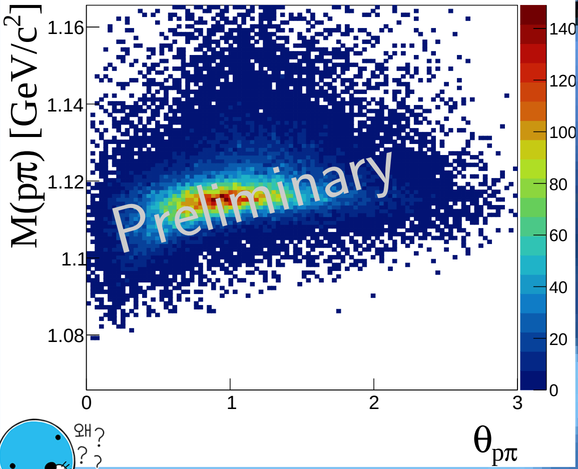
# Position Residual from KF Track

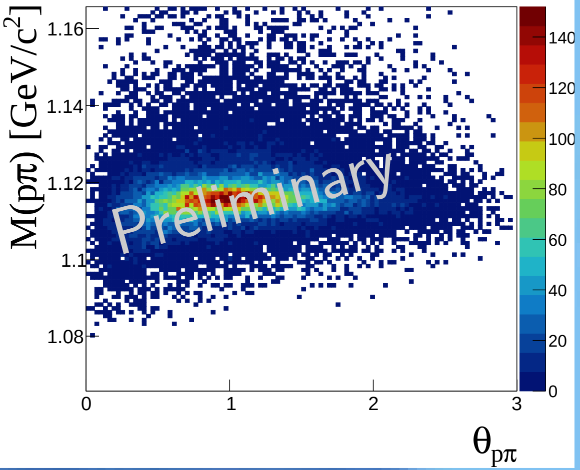



- From Kinematic fit, 'true' momentum, hence trajectory is estimated

# Λ After Position Correction

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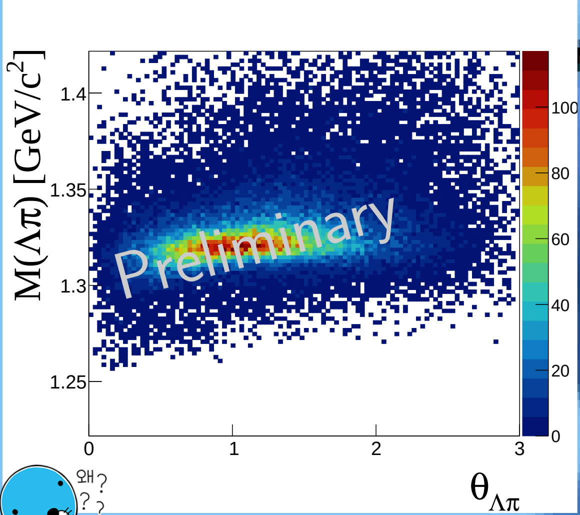
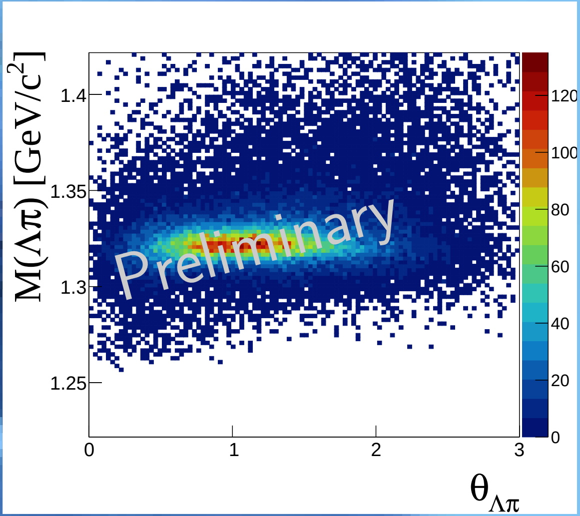
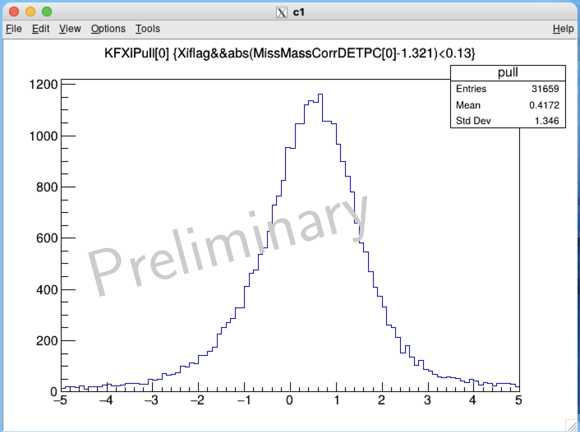
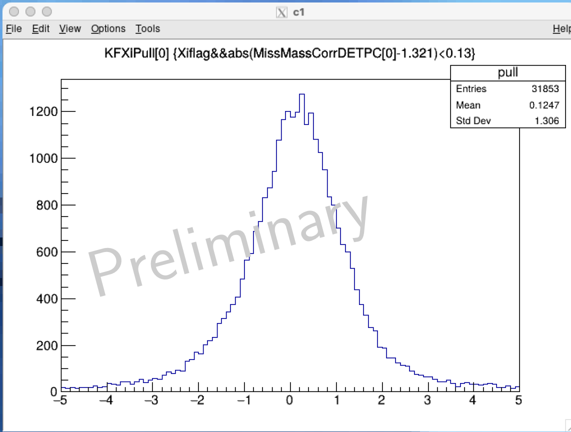



# Pull distribution After Position Correction




# Tricks

# Variance Normalization

$$V = \begin{pmatrix} 10^{12} & 0.9 \\ 0.9 & 10^{-12} \end{pmatrix} \rightarrow V^{-1} = ? \tag{34}$$

While taking an inverse of the variance, matrix elements with different order may be added together, leading to possible numerical unstability.

$$\tilde{V} = SVS^T; S \equiv \frac{1}{\sqrt{V_{ij}}}\delta_{ij} \rightarrow \tilde{V} = \begin{pmatrix} 1 & Cov(01)/\sigma_1\sigma_2 & \cdots \\ Cov(01)/\sigma_1\sigma_2 & 1 & \\ \cdots & & \cdots \end{pmatrix} \tag{35}$$

We can take out scaling factors in S. Measurement vectors could share the same problem, so they should also be scaled. We rewrite equation (15)

$$\chi^2 = dM^\dagger V^{-1} dM + \cdots = d\tilde{M}^\dagger \tilde{V}^{-1} d\tilde{M} + \cdots ; \quad d\tilde{M} = S(M - M_0) \tag{36}$$

# Off-diagonal Reduction

$$V = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \rightarrow V^{-1} = ? \tag{37}$$

Adding off-diagonal term could make matrix uninvertable. Also we require $\chi^2 = dMV^{-1}dM > 0$; $V^{-1}$ (hence V)should be *Positive Definite*. Then, we can 'damp' the offdiagonal elements.

$$while(IsPositiveDefinite(V)) \tag{38}$$

$$V_{ij} \rightarrow V_{ij} - \alpha(\delta_{ij} - 1)V_{ij} \tag{39}$$

## Property of Positive Definite Matrix

All Eigenvalues are Positive! TMatrixD well-supports eigenvalues, so we can just use it.

# Summary

- Measurement error can be reduced by correlating the measurements with physical constraints; This process is Kinematic Fit.
- By observing statistical parameters, we can estimate many physical context other than error reduction.(i.e. Bias estimation, S/N separation, etc...)
- Proper understandings of covariance matrix is required for Kinematic Fit.

# Appendix

# Minimization Steps.

The coupled differential equations (17),(19) and (18) will be solved iteratively. For each $\nu$th step,

$$V^{-1}(\mathbf{m}^0)(\mathbf{m}^{\nu+1} - \mathbf{m}^0) + (\mathbf{F}_m^\dagger)^\nu \lambda^{\nu+1} = 0 \tag{40}$$

$$(\mathbf{F}_u^\dagger)^\nu \lambda^{\nu+1} = 0 \tag{41}$$

$$\mathbf{f}^\nu + \mathbf{F}_m^\nu(\mathbf{m}^{\nu+1} - \mathbf{m}^\nu) + \mathbf{F}_u^\nu(\mathbf{u}^{\nu+1} - \mathbf{u}^\nu) = 0. \tag{42}$$

Equation (42) is not a direct consequence of Equation (18) but rather a *linear approximation*.

Note that, as we are determining the parameters $\mathbf{m}, \mathbf{u}$ and $\lambda$, they are indexed as $\nu + 1$, while constraint terms(i.e. $\mathbf{f}, \mathbf{F}_\mu$ and $\mathbf{F}_\nu$) are calculated from current step, $\nu$. This fit is basically using Newton's method.

# Coupled Equation Solving (1)

Multiplying **V** to Equation (40) leads to:

$$\mathbf{m}^{\nu+1} - \mathbf{m}^0 = -V(\mathbf{m}^0)(\mathbf{F}_m^\dagger)^\nu \lambda^{\nu+1}. \tag{43}$$

Substituting Equation (43) into Equation (42),

$$\begin{aligned}\mathbf{F}_u^\nu(\mathbf{u}^{\nu+1} - \mathbf{u}^\nu) &= -\mathbf{f}^\nu - \mathbf{F}_m^\nu(-V(\mathbf{m}^0)(\mathbf{F}_m^\dagger)^\nu \lambda^{\nu+1} + \mathbf{m}^0 - \mathbf{m}^\nu) \\ &= S\lambda^{\nu+1} - R \end{aligned} \tag{44}$$

where we define the *constraiint covariance* $S$ and *residual* $R$ as:

$$S \equiv \mathbf{F}_\mathbf{m}^\nu V(\mathbf{m}^0)(\mathbf{F}_\mathbf{m}^\dagger)^\nu; \quad R \equiv \mathbf{f}^\nu + \mathbf{F}_\mathbf{m}^\nu(\mathbf{m}^0 - \mathbf{m}^\nu) \tag{45}$$

Multiplying $(\mathbf{F}_u^\dagger)^\nu S^{-1}$ into (44), we get:


$$(\mathbf{F}_u^\dagger)^\nu S^{-1} \mathbf{F}_u^\nu(\mathbf{u}^{\nu+1} - \mathbf{u}^\nu) = \cancelto{0,\ \because (41)}{(\mathbf{F}_u^\dagger)^\nu \lambda^{\nu+1}} - (\mathbf{F}_u^\dagger)^\nu S^{-1} R. \tag{46}$$

# Coupled Equation Solving (2)

Then we naturally derive the expressions

$$\mathbf{u}^{\nu+1} = \mathbf{u}^{\nu} - ((\mathbf{F}_u^{\dagger})^{\nu} S^{-1} \mathbf{F}_u^{\nu})^{-1} (\mathbf{F}_u^{\dagger})^{\nu} S^{-1} R. \tag{47}$$

and from (44),

$$\lambda^{\nu+1} = S^{-1}(\mathbf{F}_u^{\nu}(\mathbf{u}^{\nu+1} - \mathbf{u}^{\nu}) + R). \tag{48}$$

For a summary, we have obtained all equations to proceed to the next step. All other matrices in the equation can be calculated from parameters of the current step, and $\chi^2$ can be evaluated from (15) .

$$\begin{cases} \mathbf{u}^{\nu+1} = \mathbf{u}^{\nu} - ((\mathbf{F}_u^{\dagger})^{\nu} S^{-1} \mathbf{F}_u^{\nu})^{-1} (\mathbf{F}_u^{\dagger})^{\nu} S^{-1} R & (47) \\ \lambda^{\nu+1} = S^{-1}(\mathbf{F}_u^{\nu}(\mathbf{u}^{\nu+1} - \mathbf{u}^{\nu}) + R) & (48) \\ \mathbf{m}^{\nu+1} = \mathbf{m}^{0} - V(\mathbf{m}^{0})(\mathbf{F}_m^{\dagger})^{\nu} \lambda^{\nu+1} & (43) \end{cases}$$

# Covariance Matrix Propagation

We estimate the covariance for the fitted variables, $m$,

$$V(m) = J_{m,m^0} V(m^0) J^{\dagger}_{m,m^0} \tag{49}$$

To solve (49) the Jacobian should be determined.

$$J_{m,m^0(i,j)} = \frac{\partial m_i}{\partial m^0_j} \tag{50}$$

# Jacobian

To begin with, let us express Eq (43) in terms of $m^0$. At the moment we will drop the superscript $\nu$. As $\mathbf{f}(\mathbf{m},\mathbf{u})$is a constant on $m^0$, $\mathbf{F}_{\mathbf{m}}$ also will be a constant to $m^0$. Then we only need to consider the derivatives of $\lambda$. By substituting (47) ,

$$\lambda = S^{-1}(-\mathbf{F}_u(((\mathbf{F}_u^\dagger)S^{-1}\mathbf{F}_u)^{-1}(\mathbf{F}_u^\dagger)S^{-1}R) + R) \tag{51}$$

and the residual matrix is:

$$R \equiv \mathbf{f} + \mathbf{F}_{\mathbf{m}}(\mathbf{m}^0 - \mathbf{m}) \rightarrow \frac{\partial R}{\partial m^0} = \mathbf{F}_m \tag{52}$$

Now we obtain the derivative of $\lambda$ as:

$$\frac{\partial \lambda}{\partial m^0} = S^{-1}(-\mathbf{F}_u((\mathbf{F}_u^\dagger S^{-1}\mathbf{F}_u)^{-1}\mathbf{F}_u^\dagger S^{-1}\mathbf{F}_m) + \mathbf{F}_m). \tag{53}$$

# Jacobian

Now define the symmetric matrices

$$G \equiv \mathbf{F}_{\mathbf{m}}^{\dagger} S^{-1} \mathbf{F}_{\mathbf{m}}; \quad U \equiv (\mathbf{F}_{u}^{\dagger} S^{-1} \mathbf{F}_{u})^{-1}; \quad H \equiv \mathbf{F}_{m}^{\dagger} S^{-1} \mathbf{F}_{\mathbf{u}} \tag{54}$$

We have expressions for $\frac{\partial \lambda}{\partial m^0}$. Equation (50) is determined as:

$$\begin{aligned} J_{m,m^0} &= I - V(m^0)\mathbf{F}_{m}^{\dagger}\frac{\partial \lambda}{\partial m^0} = I - V\mathbf{F}_{m}^{\dagger}(-S^{-1}\mathbf{F}_{u}U^{-1}H^{\dagger} + S^{-1}\mathbf{F}_{m}) \\ &= I - V(G - HUH^{\dagger}) \end{aligned} \tag{55}$$

If we let $C = G - HUH^{\dagger}$, we obtain

$$V(m) = J_{m,m^0} V J_{m,m^0}^{\dagger} = V - 2VCV + VCVCV. \tag{56}$$

We would keep 2nd order term at the moment. Some materials like [1] had an error in this part.

# Variance of the Unknowns

Just like how we derived Eq.(56) we can estimate the variance matrix of the unknowns.

$$V_U = J_{u,m0} V J_{u,m0}^T \tag{57}$$

$J_{u,m0}$ can be obtained from Eq.(47). Defining

$$K \equiv ((\mathbf{F}_u^\dagger)^\nu S^{-1} \mathbf{F}_u^\nu)^{-1} (\mathbf{F}_u^\dagger)^\nu S^{-1} \tag{58}$$

we write:

$$J_{u^{\nu+1},m^0} = \frac{\partial u^{\nu+1}}{\partial m^0} = \frac{\partial u^\nu}{\partial m^0} - K \frac{\partial R}{\partial m^0} \simeq -K \mathbf{F}_m. \tag{59}$$

Note that we only have initial "Guess" for the unknowns; In principle, it is not a driven value from measurements. Then, $\frac{\partial u^0}{\partial m_0} = 0$. Also, we approximate that the terms in 2nd or higher iterations are negligible: $J_{u,m^0} \simeq J_{u^0,m^0}$.

# Pull distribution

The covariance in Equation (21) is estimated as:

$$Cov(m, m^0) = J_{m,m^0} V(m) = V - VCV. \tag{60}$$

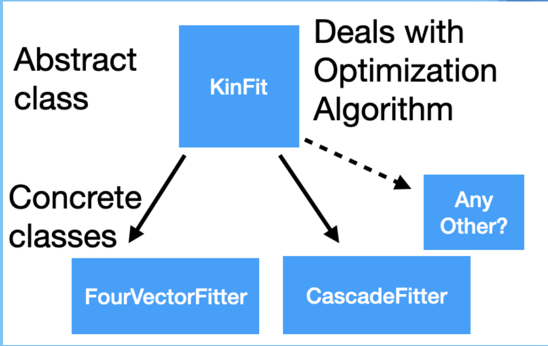
If we substitute this and Eq.(56) into Eq.(21), we get

$$V(\epsilon) = VCVCV. \tag{61}$$

Note that 2nd order term affects the covariance matrix of the correction.

# KinFit Package

$$\chi^2 = \delta M^T V^{-1} \delta M + 2\lambda f(M, U) \quad (62)$$




**KinFIt provides...**

- Minimize $\chi^2$
- Calculate pulls and p-values

**Users should...**

- Assign proper variables for M and U
- Define physical constraints
- Write the derivatives, $F_M$ and $F_U$ by hand

# References


- O. Skjeggestad Frodesen A.G. **Probability and Statistics in Particle Physics**. Columbia University Press, 1980. ISBN: 8200019063.
- B. Kang. **Bono-Beamer**. URL: https://github.com/kangbm94/Bono-Beamer.