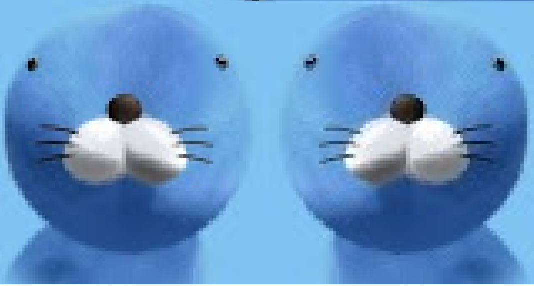


Notes on Baryon Polarization

January 8, 2024

Spin State Representation of Baryon Decay

Consider a decay process where spin-$\frac{1}{2}$ baryon(Ξ^-) decays into another spin-$\frac{1}{2}$ baryon (Λ) and a spin-0 meson(π^-). In quantum mechanics, this decay $\Xi^- \to \Lambda\pi^-$ can be modeled as a transition from the initial state $|\Xi^-\rangle$ to the final state $|\Lambda, \pi^-\rangle$. Since π^- is a spin-0 meson, we focues on the orbital angular momentum carried by π^-. In the S-wave transition π^- carries no angular momentum($l = 0$), while in the P-wave transition it carries an angular momentum of $l = 1$. The initial state of Ξ^- can be defined as $\Xi^- = |\frac{1}{2}, \frac{1}{2}\rangle$. The decay process is then represented as:

$$\begin{aligned} |\frac{1}{2}, \frac{1}{2}\rangle_\Xi = A|\frac{1}{2}, \frac{1}{2}\rangle_\Lambda |0,0\rangle_\pi Y_0^0 + B[C^{\frac{1}{2},\frac{1}{2}}_{\frac{1}{2},\frac{1}{2},1,0}|\frac{1}{2}, \frac{1}{2}\rangle_\Lambda |1,0\rangle_\pi Y_1^0 \\ + C^{\frac{1}{2},\frac{1}{2}}_{\frac{1}{2},-\frac{1}{2},1,1}|\frac{1}{2}, -\frac{1}{2}\rangle_\Lambda |1,1\rangle \pi Y_1^1] \end{aligned} \tag{1}$$

Spin State Representation of Baryon Decay

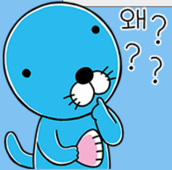
Here A and B represent the transition amplitudes of S-wave and P-wave, respectively. The terms $C^{J,M}_{j1,m1,j2,m2}$ are Clebsch-Gordan coefficients, and $Y^m_l(\theta,\phi)$s denote spherical harmonics. Note that A and B are complex numbers. Focusing on the spin state of Λ, we rewrite (1) as:

$$|\Lambda\rangle = (a + b\cos\theta)|\frac{1}{2},\frac{1}{2}\rangle_\Lambda + be^{i\phi}\sin\theta|\frac{1}{2},-\frac{1}{2}\rangle_\Lambda. \tag{2}$$

The decay spectrum of Λ can be expressed as:

$$\langle\Lambda||\Lambda\rangle = (a + b\cos\theta)^*(a + b\cos\theta)\Lambda_\uparrow + (be^{i\phi})^* be^{i\phi}\Lambda_\downarrow \tag{3}$$

Here, $\Lambda_\uparrow$ and $\Lambda_\downarrow$ represent the spin-up and spin-down states of Λ, respectively.



Baryon Decay Parameters

If we not distinguish $\Lambda_\uparrow$ and $\Lambda_\downarrow$, (3) turns into:

$$\Lambda(\theta, \phi) = |a|^2 + |b|^2 + 2\mathrm{Re}[a^* b] \cos\theta = 1 + \alpha \cos\theta. \tag{4}$$

The wavefunction should be normalized at the final calculation, but at the moment we will require $|a|^2 + |b|^2 = 1$ and introduce decay parameters:

$$\alpha = \mathrm{Re}[a^* b], \beta = \mathrm{Im}[a^* b], \gamma = |a|^2 - |b|^2. \tag{5}$$

Remind that our coordinate system here is the Center-of-Mass frame of Ξ^-, with spin pointing the z axis as we already set the initial state as $|\Xi\rangle = |\frac{1}{2}, \frac{1}{2}\rangle$. θ and ϕ corresponds to the spherical variables of Λ momentum at CoM of Ξ^-.

Spin State of Daughter Baryon

Now time has come to calculate the spin state of Λ in **CoM of** Ξ^-. The spinnor representation of Λ, as given in (2), is:

$$|\Lambda\rangle = \begin{pmatrix} a + b\cos\theta \\ be^{i\phi}\sin\theta \end{pmatrix} \tag{6}$$

and the direction of spin, or, the *polarization* of Λ is

$$\vec{P} = \langle\Lambda|\hat{S}|\Lambda\rangle. \tag{7}$$

Here we defined the spin-operator $\hat{S} = \{\sigma_x, \sigma_y, \sigma_z\}$, where σ_is are the *Pauli matrices*. Our task is to calculate each component of $\vec{P}$, *at Center of Mass of* Λ.

Calculating P_x

We start with calculating P_x. $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and

$$
\begin{aligned}
P_x &= \langle \Lambda | \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} | \Lambda \rangle \\
&= (a + b\cos\theta)^* b e^{i\phi} \sin\theta + (b e^{i\phi} \sin\theta)^* (a + b\cos\theta) \\
&= (a^* b e^{i\phi} + a b^* e^{-i\phi}) \sin\theta + |b|^2 \sin\theta\cos\theta (e^{i\phi} + e^{-i\phi}) \\
&= (2\mathrm{Re}[a^* b] \cos\phi - 2\mathrm{Im}[a^* b] \sin\phi) \sin\theta + 2|b|^2 \sin\theta\cos\theta\cos\phi \\
&= [\alpha\cos\phi - \beta\sin\phi + (1-\gamma)\cos\theta\cos\phi] \sin\theta. \qquad (8)
\end{aligned}
$$

Here we used $2|b|^2 = |a|^2 + |b|^2 - (|a|^2 - |b|^2) = 1 - \gamma$.

Calculating P_y

$$
\begin{aligned}
P_y &= \langle\Lambda| \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} |\Lambda\rangle \\
&= -i(a + b\cos\theta)^* b e^{i\phi} \sin\theta + i(b e^{i\phi}\sin\theta)^*(a + b\cos\theta) \\
&= -i[(a^* b e^{i\phi} - a b^* e^{-i\phi})\sin\theta + |b|^2 \sin\theta\cos\theta(e^{i\phi} - e^{-i\phi})] \\
&= -i[(2i\mathrm{Im}[a^* b]\cos\phi + 2i\mathrm{Re}[a^* b]\sin\phi)\sin\theta + 2i|b|^2\sin\theta\cos\theta\sin\phi] \\
&= [\beta\cos\phi + \alpha\sin\phi + (1-\gamma)\cos\theta\sin\phi]\sin\theta. \qquad (9)
\end{aligned}
$$

Calculating P_z

$$
\begin{aligned}
P_z &= \langle\Lambda| \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} |\Lambda\rangle \\
&= (a + b\cos\theta)^*(a + b\cos\theta) - (b^* e^{-i\phi}) b e^{i\phi} \\
&= |a|^2 + (a^* b + a b^*)\cos\theta + |b|^2\cos^2\theta - |b|^2\sin^2\theta \\
&= \frac{1}{2}(1+\gamma) + 2\mathrm{Re}[a^* b]\cos\theta + \frac{\cos^2\theta - \sin^2\theta}{2}(1-\gamma) \\
&= \frac{1}{2}(1+\gamma) + \alpha\cos\theta + \frac{\cos^2\theta - \sin^2\theta}{2}(1-\gamma). \qquad (10)
\end{aligned}
$$

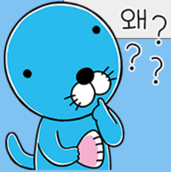
CoM Frame of Λ

Untill now we calculated Λ polarization vector in CoM of Ξ. In order to obtain meaningful results, we need to represent it in Λ CoM. We would define the orientation of Λ CoM frame as:

$$\begin{cases} \hat{z}_\Lambda = \hat{\Lambda} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \\ \hat{x}_\Lambda = \frac{\hat{P}_\Xi \times \hat{\Lambda}}{|\hat{P}_\Xi \times \hat{\Lambda}|} = (-\sin\phi, \cos\phi, 0) \\ \hat{y}_\Lambda = \hat{z}_\Lambda \times \hat{x}_\Lambda = (-\cos\theta\cos\phi, -\cos\theta\sin\phi, \sin\theta) \end{cases} \tag{11}$$

and obtain the representation of $\vec{P}$ as:

$$\begin{cases} P_{x,\Lambda} = \vec{P} \cdot \hat{x}_\Lambda \\ P_{y,\Lambda} = \vec{P} \cdot \hat{y}_\Lambda \\ P_{z,\Lambda} = \vec{P} \cdot \hat{z}_\Lambda \end{cases} \tag{12}$$

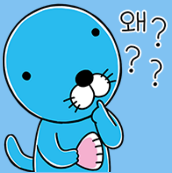


$P_{x,\Lambda}$

We should calculate each component of $\vec{P}_\Lambda$. Remind that, we stated by setting polarization of Ξ,$\vec{P}_\Xi = (0,0,1)$. However, Λ CoM is defined relative to $\vec{P}_\Xi$ and $\hat{\Lambda}$. Then, without loss of generality, we represent the polarization of Λ in terms of Ξ polarization and Λ momentum direction.

$$\begin{aligned} P_{x,\Lambda} &= -\sin\phi[\alpha\cos\phi - \beta\sin\phi + (1-\gamma)\cos\theta\cos\phi]\sin\theta \\ &\quad + \cos\phi[\beta\cos\phi + \alpha\sin\phi + (1-\gamma)\cos\theta\sin\phi]\sin\theta \\ &= \beta\sin\theta = \beta(\vec{P}_\Xi \times \hat{\Lambda})\cdot\hat{x} \end{aligned} \tag{13}$$

Here we used the fact that, From (11), $\vec{P}_\Xi \times \hat{\Lambda} = |\vec{P}_\Xi \times \hat{\Lambda}|\hat{x} = \sin\theta\hat{x}$.



$P_{y,\Lambda}$

$$
\begin{aligned}
P_{y,\Lambda} = & -\cos\theta\cos\phi[\alpha\cos\phi - \beta\sin\phi + (1-\gamma)\cos\theta\cos\phi]\sin\theta. \\
& -\cos\theta\sin\phi[\beta\cos\phi + \alpha\sin\phi + (1-\gamma)\cos\theta\sin\phi]\sin\theta. \\
& +\sin\theta[\frac{1}{2}(1+\gamma) + \alpha\cos\theta + \frac{\cos^2\theta - \sin^2\theta}{2}(1-\gamma)] \\
= & \sin\theta[\frac{1}{2}(1+\gamma) + (1-\gamma)(\frac{\cos^2\theta - \sin^2\theta}{2} - \cos^2\theta)] \\
= & \sin\theta\gamma = \gamma\hat{\Lambda}\times(\vec{P}_{\Xi}\times\hat{\Lambda})\cdot\hat{y}
\end{aligned}
\tag{14}
$$

$P_{z,\Lambda}$

$$
\begin{aligned}
P_{z,\Lambda} =& \sin\theta\cos\phi[\alpha\cos\phi - \beta\sin\phi + (1-\gamma)\cos\theta\cos\phi]\sin\theta \\
&+ \sin\theta\sin\phi[\beta\cos\phi + \alpha\sin\phi + (1-\gamma)\cos\theta\sin\phi]\sin\theta \\
&+ \cos\theta[\frac{1}{2}(1+\gamma) + \alpha\cos\theta + \frac{\cos^2\theta - \sin^2\theta}{2}(1-\gamma)] \\
=& \alpha + \cos\theta[\frac{1+\gamma}{2} + (1-\gamma)(\frac{\cos^2\theta - \sin^2\theta}{2} + \sin^2\theta)] \\
=& \alpha + \cos\theta
\end{aligned} \tag{15}
$$

Summary

To summarize (13) - (15), we have

$$\vec{P}_\Lambda \propto \beta \vec{P_\Xi} \times \hat{\Lambda} + \gamma \hat{\Lambda} \times (\vec{P_\Xi} \times \hat{\Lambda}) + (\alpha + \cos\theta)\hat{\Lambda}$$

However, we should note that $|\vec{P}|_\Lambda = 1$. We need a normalization. Referring to (4) normalization parameter seems to be obvious, but we will show the derivation here. Remind that $\alpha^2 + \beta^2 + \gamma^2 = 1$.

$$\begin{aligned} N &= \sqrt{(\alpha + \cos\theta)^2 + (\beta \sin\theta)^2 + (\gamma \sin\theta)^2} \\ &= \sqrt{\alpha^2 + 2\alpha\cos\theta + \cos^2\theta + \cancelto{1-\alpha^2}{(\beta^2 + \gamma^2)}(1 - \cos^2\theta)} \\ &= \sqrt{1 + 2\alpha\cos\theta + \alpha^2\cos^2\theta} = 1 + \alpha\cos\theta \end{aligned} \tag{16}$$

Summary

Now we obtained normalization parameter. Then we have

$$\vec{P}_\Lambda = \frac{(\alpha + \cos\theta)\hat{\Lambda} + \beta(\vec{P}_\Xi \times \hat{\Lambda}) + \gamma\hat{\Lambda} \times (\vec{P}_\Xi \times \hat{\Lambda})}{1 + \alpha\cos\theta} \quad (17)$$