Problem Set 7

All parts are due on April 26, 2020 at 6PM. Please write your solutions in the LaTeX and Python templates provided. Aim for concise solutions; convoluted and obtuse descriptions might receive low marks, even when they are correct. Solutions should be submitted on the course website, and any code should be submitted for automated checking on `alg.mit.edu`.

Please solve each of the following problems using **dynamic programming**. For each problem, be sure to define a set of subproblems, relate the subproblems recursively, argue the relation is acyclic, provide base cases, construct a solution to the original problem from the subproblems, and analyze running time. Correct but inefficient dynamic programs will be awarded significant partial credit.

Problem 7-1. [15 points] **Effective Campaigning**

Representative Zena Torr is facing off against Senator Kong Grossman in a heated presidential primary: a sequence of n head-to-head state contests, one per day for n days. Each state contest $i \in \{1, \ldots, n\}$ has a known positive integer **delegate count** d_i, and a **projected delegate count** $z_i < d_i$ that Rep. Torr would win if she took no further action. There are $D = \sum_i d_i$ total delegates and Rep. Torr needs at least $\lfloor D/2 \rfloor + 1$ delegates to win. Unfortunately, Rep. Torr is projected to lose the race, since $\sum_i z_i < \lfloor D/2 \rfloor + 1$, so she needs to take action. Rep. Torr has a limited but effective election team which can **campaign** in at most one state per day. If the team campaigns on day i, they will win all d_i delegates in state i, but they will **not be able to campaign at all** for two days after day i, as it will take time to relocate. Describe an $O(n)$-time algorithm to determine whether it is possible for Rep. Torr to win the primary contest by campaigning effectively.

Problem 7-2. [15 points] **Caged Cats**

Ting Kiger is an eccentric personality who owns n pet tigers and n^2 cages.

- Each tiger i has known positive integer **age** a_i and **size** s_i (no two have the same age or size).
- Each cage j has known positive integer **capacity** c_j and **distance** d_j from Ting's bedroom (no two have the same capacity or distance).

Ting needs to assign each tiger its own cage.

- Ting **favors older tigers** and wants them to sleep closer to his bedroom, i.e., any two tigers x and y with ages $a_x < a_y$ must be assigned to cages X and Y respectively such that $d_Y < d_X$.
- A tiger i assigned to cage c_j will experience positive **discomfort** $s_i - c_j$ if $s_i > c_j$, but will not experience any discomfort if $s_i \leq c_j$.

Describe an $O(n^3)$-time algorithm to assign tigers to cages that favors older tigers and minimizes the total discomfort experienced by the tigers.

Problem 7-3. [15 points] **Odd Paths**

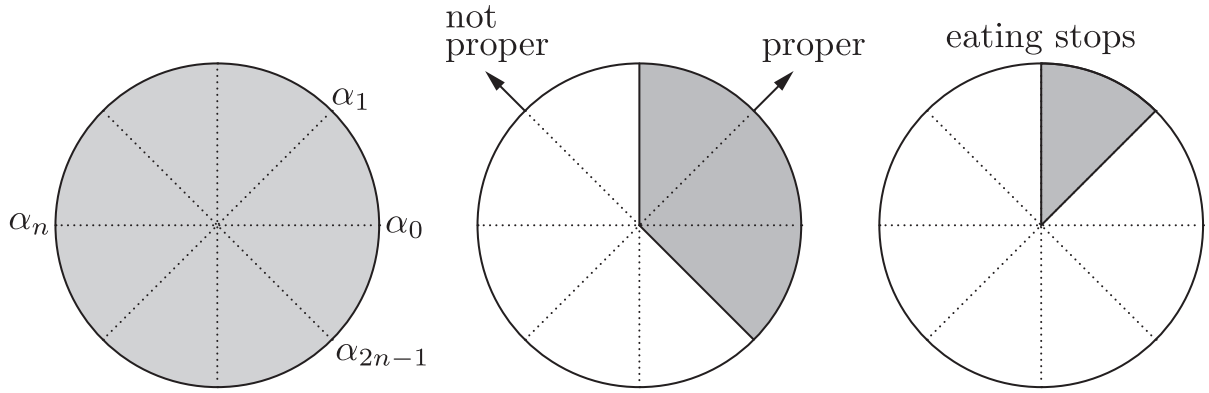
Given a weighted directed acyclic graph $G = (V, E, w)$ with integer weights and two vertices $s, t \in V$, describe a linear-time algorithm to determine the number of paths from s to t having **odd weight**. When solving this problem, you may assume that a single machine word is large enough to hold any integer computed during your algorithm.

Problem 7-4. [15 points] **Pizza Partitioning**

Liza Pover and her little brother Lie Pover want to share a round pizza pie that has been cut into $2n$ equal sector slices along rays from the center at angles $\alpha_i = i\pi/n$ for $i \in \{0, 1, \ldots, 2n\}$, where $\alpha_0 = \alpha_{2n}$. Each slice i between angles α_i and α_{i+1} has a known integer tastiness t_i (which might be negative). To be "fair" to her little brother, Liza decides to eat slices in the following way:

- They will each take turns choosing slices of pizza to eat: Liza starts as **the chooser**.
- If there is only one slice remaining, the chooser eats that slice, and eating stops.
- Otherwise the chooser does the following:
  - Angle α_i is **proper** if there is at least one uneaten slice on either side of the line passing through the center of the pizza at angle α_i.
  - The chooser picks any number $i \in \{1, \ldots, 2n\}$ where α_i is proper, and eats all uneaten slices counter-clockwise around the pizza from angle α_i to angle $\alpha_i + \pi$.
  - Once the chooser has eaten, the other sibling becomes the chooser, and eating continues.

Liza wants to maximize the total tastiness of slices she will eat. Describe an $O(n^3)$-time algorithm to find the maximum total tastiness Liza can guarantee herself via this selection process.



Problem 7-5. [40 points] **Shorting Stocks**

Bordan Jelfort is a short seller at a financial trading firm. He has collected **stock price information** from s different companies $C = (c_0, \ldots, c_{s-1})$ for n consecutive days. Stock price information for a company c_i is a chronological sequence $P_i = (p_0, \ldots, p_{nk-1})$ of nk **prices**, where each price is a positive integer and prices $\{p_{kj}, \ldots, p_{kj+k-1}\}$ all occur on day j for $j \in \{0, \ldots, n-1\}$. The **shorting value** of a company is the length of the longest chronological subsequence of strictly decreasing prices for that company that **doesn't skip days**: if the sequence contains two prices on different days i and j with $i < j$, then the sequence must also contain at least one price from every day in $\{i, \ldots, j\}$.

(a) [15 points] Describe an $O(snk^2)$-time algorithm to determine which company c_i has the highest shorting value, and return a longest subsequence S of decreasing subsequences of prices from P_i that doesn't skip days.

(b) [25 points] Write a Python function `short_company(C, P, n, k)` that implements your algorithm from part (a) using the template code provided. You can download the code template and some test cases from the website. Submit your code online at `alg.mit.edu`.

```
def short_company(C, P, n, k):
    '''
    Input:  C | Tuple of s = |C| strings representing names of companies
            P | Tuple of s lists each of size nk representing prices
            n | Number of days of price information
            k | Number of prices in one day
    Output: c | Name of a company with highest shorting value
            S | List containing a longest subsequence of
              | decreasing prices from c that doesn't skip days
    '''
    c = C[0]
    S = []
    ##################
    # YOUR CODE HERE #
    ##################
    return (c, S)
```