Programming Report for Project2

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Experiment design

The main theme in this project is applying multigrid method to solve the linear system from the discretization of the corresponding BVP's.To get the better understanding of multigrid method,we'll test it for different combinations.For example,we have three boundary conditions which are same to that in Project1,full weighting and injection for restriction operators,linear and quadratic interpolation operators,and V-cycle and FMG for cycles.All these user-determined parameters will be kept in json file,which is our input file.
For 1-dimension problem,we'll test the method on grids with n=32,64,128,256,and calculate the maximum norm of the residual($A * u^h - f$,and u^h is the computed solution after each V-cycle) and get the reduction rate for each V-cycle.
For 2-dimension problem,besides the errors and convergence rates,we'll also compare the CPU time between our multigrid implementation and lapacke solution.

Experiment results

In this section,we'll first test three functions $u_1 = e^{sinx}, u_2 = e^{x^2}, u_3 = e^x$ for 1-dimension problem,and show their statistical results by figures or tabular.To guarantee the uniqueness of solution,we add a value on the endpoint of the interval for Neumann problem,then the problem shifts into the mixed problem,so we can regard mixed problem and Neumann problem as the same one.

1. 1-dimension

   (a) $u_1 = e^{sinx}$

   - **Boundary condition:Dirichlet**
     Restriction operator:full weighting
     Interpolation operator:linear

	n=32		n=64		n=128		n=256	
V-cycle	$\|\|r^h\|\|_h$	rate	$\|\|r^h\|\|_h$	rate	$\|\|r^h\|\|_h$	rate	$\|\|r^h\|\|_h$	rate
1	3.06823		12.2471		48.9471		195.738	
2	0.0294071	0.0096	0.117709	0.0096	0.47028	0.0096	1.88026	
3	0.000368796	0.0125	0.00148104	0.0126	0.00591879	0.0126	0.0236636	0.0096
4	4.78986e-06	0.0130	1.92643e-05	0.0130	7.70253e-05	0.0130	0.000307949	0.0126
5	6.41767e-08	0.0134	2.57864e-07	0.0134	1.03178e-06	0.0134	4.12515e-06	0.0130
6	8.89376e-10	0.0139	3.53154e-09	0.0137	1.4134e-08	0.0137	5.64887e-08	0.0134
7	1.39874e-11	0.0157	4.82357e-11	0.0137	2.12621e-10	0.0150	8.28198e-10	0.0137
8	1.04294e-12	0.0746	4.01262e-12	0.0832	2.08858e-11	0.8306	1.30627e-10	0.0147
9	1.25444e-12	1.0000*	4.71059e-12	1.0000*	1.73472e-11	0.6262	7.74977e-11	0.1577
10	4.28102e-13	1.0000*	3.09247e-12	1.0000*	1.73472e-11	1.0000*	7.74977e-11	0.5933
11	4.28102e-13	1.0000*	3.63037e-12	1.0000*	1.73472e-11	1.0000*	7.74977e-11	1.0000*
12	4.28102e-13	1.0000*	3.91143e-12	1.0000*	1.73472e-11	1.0000*	7.74977e-11	1.0000*
13	4.28102e-13	1.0000*	3.09247e-12	1.0000*	1.73472e-11	1.0000*	7.74977e-11	1.0000*
14	4.28102e-13	1.0000*	3.63037e-12	1.0000*	1.73472e-11	1.0000*	7.74977e-11	1.0000*
15	4.28102e-13	1.0000*	3.91143e-12	1.0000*	1.73472e-11	1.0000*	7.74977e-11	1.0000*
16	4.28102e-13	1.0000*	3.09247e-12	1.0000*	1.73472e-11	1.0000*	7.74977e-11	1.0000*

   - **Boundary condition:mixed**
     Restriction operator:full weighting

Interpolation operator:linear

	n=32		n=64		n=128		n=256	
V-cycle	$\|\|r^h\|\|_h$	rate	$\|\|r^h\|\|_h$	rate	$\|\|r^h\|\|_h$	rate	$\|\|r^h\|\|_h$	rate
1	3.06901		12.2476		48.9473		195.738	
2	0.0294756	0.0096	0.117718	0.0096	0.470279	0.0096	1.88026	0.0096
3	0.000369256	0.0125	0.0014808	0.0126	0.0059185	0.0126	0.0236634	0.0126
4	4.77046e-06	0.0130	1.92544e-05	0.0130	7.70171e-05	0.0130	0.000307944	0.0130
5	6.32699e-08	0.0134	2.57621e-07	0.0134	1.03161e-06	0.0134	4.12509e-06	0.0134
6	8.55861e-10	0.0139	3.52608e-09	0.0137	1.4134e-08	0.0137	5.64887e-08	0.0137
7	1.21684e-11	0.0157	4.81728e-11	0.0137	1.9807e-10	0.01500	8.28198e-10	0.0147
8	8.58424e-13	0.0746	4.58e-12	0.0832	1.45719e-11	0.8306	6.90905e-11	0.1577
9	8.58424e-13	1.000*	3.68905e-12	1.0000*	1.9463e-11	0.6262	5.59327e-11	0.5933
10	4.28102e-13	1.000*	4.57412e-12	1.0000*	1.52386e-11	1.0000*	5.59327e-11	1.0000*
11	4.28102e-13	1.000*	2.61868e-12	1.0000*	1.05215e-11	1.0000*	5.59327e-11	1.0000*
12	4.28102e-13	1.000*	2.61868e-12	1.0000*	1.66345e-11	1.0000*	5.59327e-11	1.0000*
13	4.28102e-13	1.000*	2.61868e-12	1.0000*	1.47582e-11	1.0000*	5.59327e-11	1.0000*
14	4.28102e-13	1.000*	2.61868e-12	1.0000*	1.85824e-11	1.0000*	5.59327e-11	1.0000*
15	4.28102e-13	1.000*	2.61868e-12	1.0000*	1.96911e-11	1.0000*	5.59327e-11	1.0000*
16	4.28102e-13	1.000*	2.61868e-12	1.0000*	1.96911e-11	1.0000*	5.59327e-11	1.0000*

- **Boundary condition:Dirichlet**
  Restriction operator:injection
  Interpolation operator:linear

	n=32		n=64		n=128		n=256	
V-cycle	$\|\|r^h\|\|_h$	rate	$\|\|r^h\|\|_h$	rate	$\|\|r^h\|\|_h$	rate	$\|\|r^h\|\|_h$	rate
1	0.638741		2.55133		10.1967		40.7758	
2	0.0019612	0.0031	0.00783868	0.0031	0.0313137	0.0031	0.12519	0.0031
3	8.12588e-06	0.0041	3.18768e-05	0.0041	0.000127313	0.0041	0.000508874	0.0041
4	3.92611e-08	0.0048	1.43748e-07	0.0045	5.74132e-07	0.0045	2.29431e-06	0.0045
5	2.05317e-10	0.0052	6.93918e-10	0.0048	2.80036e-09	0.0049	1.115e-08	0.0049
6	1.6644e-12	0.0081	8.2121e-12	0.0118	3.26907e-11	0.0117	1.52287e-10	0.0137
7	8.58424e-13	0.5158	3.91143e-12	0.4763	1.47142e-11	0.4501	1.07521e-10	0.7060
8	7.50844e-13	0.8747	4.01262e-12	1.0000*	1.47142e-11	1.0000*	6.74343e-11	0.6272
9	8.58424e-13	1.0000*	4.57789e-12	1.0000*	1.47142e-11	1.0000*	6.89504e-11	1.0000*
10	8.58424e-13	1.0000*	4.14646e-12	1.0000*	1.47142e-11	1.0000*	7.74169e-11	1.0000*
11	8.58424e-13	1.0000*	4.14646e-12	1.0000*	1.47142e-11	1.0000*	7.25973e-11	1.0000*
12	8.58424e-13	1.0000*	4.14646e-12	1.0000*	1.47142e-11	1.0000*	7.25973e-11	1.0000*
13	8.58424e-13	1.0000*	4.14646e-12	1.0000*	1.47142e-11	1.0000*	7.25973e-11	1.0000*
14	8.58424e-13	1.0000*	4.14646e-12	1.0000*	1.47142e-11	1.0000*	7.25973e-11	1.0000*
15	8.58424e-13	1.0000*	4.14646e-12	1.0000*	1.47142e-11	1.0000*	7.25973e-11	1.0000*
16	8.58424e-13	1.0000*	4.14646e-12	1.0000*	1.47142e-11	1.0000*	7.25973e-11	1.0000*

- **Boundary condition:mixed**
  Restriction operator:injection
  Interpolation operator:linear

	n=32		n=64		n=128		n=256	
V-cycle	$\|\|r^h\|\|_h$	rate	$\|\|r^h\|\|_h$	rate	$\|\|r^h\|\|_h$	rate	$\|\|r^h\|\|_h$	rate
1	0.63874		2.55133		10.1967		40.7758	
2	0.0019598	0.0031	0.00783868	0.0031	0.0313137	0.0031	0.12519	0.0031
3	7.93023e-06	0.0040	3.18768e-05	0.0041	0.000127313	0.0041	0.000508874	0.0041
4	3.52289e-08	0.0044	1.43748e-07	0.0045	5.74125e-07	0.0045	2.29437e-06	0.0045
5	1.67502e-10	0.0048	6.97556e-10	0.0049	2.79309e-09	0.0049	1.115e-08	0.0049
6	1.13823e-12	0.0068	5.33429e-12	0.0076	2.50734e-11	0.0090	1.04434e-10	0.0094
7	1.13823e-12	0.7542	4.01262e-12	0.7522	2.50734e-11	1.0000	9.18856e-11	0.8798
8	1.13823e-12	1.000*	3.84426e-12	0.9580	1.92092e-11	0.7661	8.80681e-11	0.9585
9	1.13823e-12	1.000*	4.32099e-12	1.1240	1.52386e-11	0.7933	6.99193e-11	0.7939
10	1.13823e-12	1.000*	3.90155e-12	0.9029	1.52386e-11	1.0000*	6.99193e-11	1.0000*
11	1.13823e-12	1.000*	3.68905e-12	1.0000*	1.52386e-11	1.0000*	6.99193e-11	1.0000*
12	1.13823e-12	1.000*	3.68905e-12	1.0000*	1.52386e-11	1.0000*	6.99193e-11	1.0000*
13	1.13823e-12	1.000*	3.68905e-12	1.0000*	1.52386e-11	1.0000*	6.99193e-11	1.0000*
14	1.13823e-12	1.000*	3.68905e-12	1.0000*	1.52386e-11	1.0000*	6.99193e-11	1.0000*
15	1.13823e-12	1.000*	3.68905e-12	1.0000*	1.52386e-11	1.0000*	6.99193e-11	1.0000*
16	1.13823e-12	1.000*	3.68905e-12	1.0000*	1.52386e-11	1.0000*	6.99193e-11	1.0000*

- Corresponding convergence rates

type	Dirichlet+full weighting+linear		mixed+full weighting+linear	
gird	error norm	convergence rate	error norm	convergence rate
n=32	5.32685e-13		1.97621e-14	
n=64	1.30241e-13	2.02237	4.7052e-15	2.04939
n=128	3.17661e-14	2.02484	1.14483e-15	2.02731
n=256	7.80493e-15	2.01742	2.89101e-16	1.98997

(b) $u_2 = e^{x^2}$

- **Boundary condition:Dirichlet**
  Restriction operator:full weighting
  Interpolation operator:linear

	n=32		n=64		n=128		n=256	
V-cycle	$\|\|r^h\|\|_h$	rate	$\|\|r^h\|\|_h$	rate	$\|\|r^h\|\|_h$	rate	$\|\|r^h\|\|_h$	rate
1	3.4168		14.1316		57.1111		229.105	
2	0.0314759	0.0092	0.133972	0.0095	0.546477	0.0096	2.19831	0.0096
3	0.000389947	0.0124	0.00167806	0.0125	0.00686804	0.0126	0.0276553	0.0126
4	5.00217e-06	0.0128	2.17275e-05	0.0129	8.9249e-05	0.0130	0.000359747	0.0130
5	6.62104e-08	0.0132	2.89534e-07	0.0133	1.19379e-06	0.0134	4.81708e-06	0.0134
6	8.99536e-10	0.0136	3.95057e-09	0.0136	1.6348e-08	0.0137	6.60581e-08	0.0137
7	1.30278e-11	0.0145	5.61098e-11	0.0142	2.53598e-10	0.0155	1.04507e-09	0.0158
8	5.09814e-13	0.0391	4.55636e-12	0.0812	2.99121e-11	0.1180	7.50795e-11	0.0718
9	4.72511e-13	0.9268	4.61853e-12	1.0000*	2.26361e-11	0.7568	8.85247e-11	1.0000*
10	7.74492e-13	1.0000*	4.61853e-12	1.0000*	1.59552e-11	0.7049	6.58034e-11	1.0000*
11	7.74492e-13	1.0000*	4.61853e-12	1.0000*	1.59552e-11	1.0000*	6.58034e-11	1.0000*
12	7.74492e-13	1.0000*	4.61853e-12	1.0000*	1.59552e-11	1.0000*	6.58034e-11	1.0000*
13	7.74492e-13	1.0000*	4.61853e-12	1.0000*	1.59552e-11	1.0000*	6.58034e-11	1.0000*
14	7.74492e-13	1.0000*	4.61853e-12	1.0000*	1.59552e-11	1.0000*	6.58034e-11	1.0000*
15	7.74492e-13	1.0000*	4.61853e-12	1.0000*	1.59552e-11	1.0000*	6.58034e-11	1.0000*
16	7.74492e-13	1.0000*	4.61853e-12	1.0000*	1.59552e-11	1.0000*	6.58034e-11	1.0000*

- **Boundary condition:mixed**
  Restriction operator:full weighting
  Interpolation operator:linear

	n=32		n=64		n=128		n=256	
V-cycle	$\|\|r^h\|\|_h$	rate	$\|\|r^h\|\|_h$	rate	$\|\|r^h\|\|_h$	rate	$\|\|r^h\|\|_h$	rate
1	3.4184		14.1325		57.1115		229.105	
2	0.0316104	0.0092	0.134026	0.0095	0.546501	0.0096	2.19832	0.0096
3	0.000392473	0.0124	0.00167921	0.0125	0.00686852	0.0126	0.0276555	0.0126
4	5.03066e-06	0.0128	2.17492e-05	0.0130	8.92578e-05	0.0130	0.000359751	0.0130
5	6.62799e-08	0.0132	2.89923e-07	0.0133	1.19395e-06	0.0134	4.81708e-06	0.0134
6	8.91806e-10	0.0135	3.95784e-09	0.0137	1.63698e-08	0.0137	6.60581e-08	0.0137
7	1.25731e-11	0.0141	5.61098e-11	0.0142	2.38893e-10	0.0146	9.8192e-10	0.0149
8	1.22924e-12	0.0978	3.79252e-12	0.0676	3.53193e-11	0.1478	1.27015e-10	0.1294
9	7.74492e-13	0.6301	3.35909e-12	0.8857	2.07674e-11	0.5880	9.14628e-11	0.7201
10	7.74492e-13	1.0000*	3.35909e-12	1.0000*	1.54703e-11	0.7449	5.96838e-11	0.6525
11	7.74492e-13	1.0000*	3.35909e-12	1.0000*	1.81242e-11	1.0000*	6.88072e-11	1.0000*
12	7.74492e-13	1.0000*	3.35909e-12	1.0000*	1.81242e-11	1.0000*	6.88072e-11	1.0000*
13	7.74492e-13	1.0000*	3.35909e-12	1.0000*	1.81242e-11	1.0000*	6.88072e-11	1.0000*
14	7.74492e-13	1.0000*	3.35909e-12	1.0000*	1.81242e-11	1.0000*	6.88072e-11	1.0000*
15	7.74492e-13	1.0000*	3.35909e-12	1.0000*	1.81242e-11	1.0000*	6.88072e-11	1.0000*
16	7.74492e-13	1.0000*	3.35909e-12	1.0000*	1.81242e-11	1.0000*	6.88072e-11	1.0000*

- **Boundary condition:Dirichlet**
  Restriction operator:injection
  Interpolation operator:linear

	n=32		n=64		n=128		n=256	
V-cycle	$\|\|r^h\|\|_h$	rate	$\|\|r^h\|\|_h$	rate	$\|\|r^h\|\|_h$	rate	$\|\|r^h\|\|_h$	rate
1	0.714113		2.94304		11.8937		47.721	
2	0.00212437	0.0030	0.00892366	0.0030	0.0363663	0.0031	0.146329	0.0031
3	8.56282e-06	0.0040	3.58329e-05	0.0040	0.00014719	0.0040	0.00059399	0.0041
4	4.03616e-08	0.0047	1.5959e-07	0.0045	6.60629e-07	0.0045	2.6741e-06	0.0045
5	2.07478e-10	0.0051	7.74389e-10	0.0049	3.20716e-09	0.0049	1.29377e-08	0.0048
6	1.93356e-12	0.0093	4.61853e-12	0.0060	2.94236e-11	0.0092	1.1401e-10	0.0088
7	8.63309e-13	0.4465	4.61853e-12	1.0000*	2.94236e-11	0.7134	9.14628e-11	0.8022
8	9.27258e-13	1.0000*	4.61853e-12	1.0000*	2.94236e-11	0.7602	6.07283e-11	0.6640
9	9.27258e-13	1.0000*	4.61853e-12	1.0000*	2.94236e-11	1.0000*	6.07283e-11	1.0000*
10	9.27258e-13	1.0000*	4.61853e-12	1.0000*	2.94236e-11	1.0000*	6.07283e-11	1.0000*
11	9.27258e-13	1.0000*	4.61853e-12	1.0000*	2.94236e-11	1.0000*	6.07283e-11	1.0000*
12	9.27258e-13	1.0000*	4.61853e-12	1.0000*	2.94236e-11	1.0000*	6.07283e-11	1.0000*
13	9.27258e-13	1.0000*	4.61853e-12	1.0000*	2.94236e-11	1.0000*	6.07283e-11	1.0000*
14	9.27258e-13	1.0000*	4.61853e-12	1.0000*	2.94236e-11	1.0000*	6.07283e-11	1.0000*
15	9.27258e-13	1.0000*	4.61853e-12	1.0000*	2.94236e-11	1.0000*	6.07283e-11	1.0000*
16	9.27258e-13	1.0000*	4.61853e-12	1.0000*	2.94236e-11	1.0000*	6.07283e-11	1.0000*

- **Boundary condition:mixed**
  Restriction operator:injection
  Interpolation operator:linear

	n=32		n=64		n=128		n=256	
V-cycle	$\|\|r^h\|\|_h$	rate	$\|\|r^h\|\|_h$	rate	$\|\|r^h\|\|_h$	rate	$\|\|r^h\|\|_h$	rate
1	0.714111		2.94304		11.8937		47.721	
2	0.00212304	0.0030	0.00892366	0.0030	0.0363663	0.0031	0.146329	0.0031
3	8.38148e-06	0.0039	3.58329e-05	0.0040	0.00014719	0.0040	0.00059399	0.0041
4	3.67359e-08	0.0044	1.5959e-07	0.0045	6.60637e-07	0.0045	2.67416e-06	0.0045
5	1.76432e-10	0.0048	7.67113e-10	0.0048	3.20716e-09	0.0049	1.29959e-08	0.0049
6	1.03295e-12	0.0059	1.08962e-11	0.0142	1.94227e-11	0.0061	1.41947e-10	0.0109
7	9.27258e-13	0.8977	4.61853e-12	0.4239	2.07674e-11	1.0000*	8.64606e-11	0.6091
8	1.38201e-12	1.0000*	4.61853e-12	1.0000*	2.07674e-11	1.0000*	7.56195e-11	0.8746
9	9.27258e-13	1.0000*	4.61853e-12	1.0000*	2.07674e-11	1.0000*	6.63221e-11	0.8771
10	9.27258e-13	1.0000*	4.61853e-12	1.0000*	2.07674e-11	1.0000*	6.45652e-11	0.9735
11	9.27258e-13	1.0000*	4.61853e-12	1.0000*	2.07674e-11	1.0000*	6.45652e-11	1.0000*
12	9.27258e-13	1.0000*	4.61853e-12	1.0000*	2.07674e-11	1.0000*	6.45652e-11	1.0000*
13	9.27258e-13	1.0000*	4.61853e-12	1.0000*	2.07674e-11	1.0000*	6.45652e-11	1.0000*
14	9.27258e-13	1.0000*	4.61853e-12	1.0000*	2.07674e-11	1.0000*	6.45652e-11	1.0000*
15	9.27258e-13	1.0000*	4.61853e-12	1.0000*	2.07674e-11	1.0000*	6.45652e-11	1.0000*
16	9.27258e-13	1.0000*	4.61853e-12	1.0000*	2.07674e-11	1.0000*	6.45652e-11	1.0000*

- Corresponding convergence rates

type	Dirichlet+full weighting+linear		mixed+full weighting+linear	
gird	error norm	convergence rate	error norm	convergence rate
n=32	5.95081e-14		3.06422e-14	
n=64	1.48400e-14	2.00250	7.12609e-15	2.07364
n=128	3.71927e-15	1.99750	1.75088e-15	2.01742
n=256	8.85541e-16	2.04939	4.63196e-16	1.94422

(c) $u_3 = e^x$

- **Boundary condition:Dirichlet**
  Restriction operator:full weighting
  Interpolation operator:linear

	n=32		n=64		n=128		n=256	
V-cycle	$\|\|r^h\|\|_h$	rate	$\|\|r^h\|\|_h$	rate	$\|\|r^h\|\|_h$	rate	$\|\|r^h\|\|_h$	rate
1	3.54592		14.2942		57.2948		309.965	
2	0.0336169	0.0095	0.136881	0.0096	0.549911	0.0096	3.52105	0.0114
3	0.000420328	0.0125	0.00172023	0.0126	0.00691848	0.0126	0.0499092	0.0142
4	5.44218e-06	0.0129	2.23492e-05	0.0130	9.00019e-05	0.0130	0.000749346	0.0150
5	7.26874e-08	0.0134	2.98822e-07	0.0134	1.20516e-06	0.0134	1.21006e-05	0.0161
6	9.95861e-10	0.0137	4.08781e-09	0.0137	1.65129e-08	0.0137	1.99396e-07	0.0165
7	1.54259e-11	0.0155	6.12852e-11	0.0150	2.56153e-10	0.0155	3.43966e-09	0.0173
8	1.06537e-12	0.0691	3.64153e-12	0.0594	2.03477e-11	0.0794	9.34288e-11	0.0272
9	6.02185e-13	0.5652	5.33573e-12	1.0000*	1.32645e-11	0.6519	9.65339e-11	1.0000*
10	6.02185e-13	1.0000*	5.33573e-12	1.0000*	1.32645e-11	1.0000*	7.96576e-11	1.0000*
11	6.02185e-13	1.0000*	5.33573e-12	1.0000*	1.32645e-11	1.0000*	7.14673e-11	1.0000*
12	6.02185e-13	1.0000*	5.33573e-12	1.0000*	1.32645e-11	1.0000*	7.14673e-11	1.0000*
13	6.02185e-13	1.0000*	5.33573e-12	1.0000*	1.32645e-11	1.0000*	7.14673e-11	1.0000*
14	6.02185e-13	1.0000*	5.33573e-12	1.0000*	1.32645e-11	1.0000*	7.14673e-11	1.0000*
15	6.02185e-13	1.0000*	5.33573e-12	1.0000*	1.32645e-11	1.0000*	7.14673e-11	1.0000*
16	6.02185e-13	1.0000*	5.33573e-12	1.0000*	1.32645e-11	1.0000*	7.14673e-11	1.0000*

- **Boundary condition:mixed**
  Restriction operator:full weighting
  Interpolation operator:linear

	n=32		n=64		n=128		n=256	
V-cycle	$\|\|r^h\|\|_h$	rate	$\|\|r^h\|\|_h$	rate	$\|\|r^h\|\|_h$	rate	$\|\|r^h\|\|_h$	rate
1	4.80542		19.3346		77.4597		309.965	
2	0.0542239	0.0113	0.219278	0.0113	0.879618	0.0114	3.52102	0.0114
3	0.000764286	0.0141	0.00310459	0.0142	0.0124655	0.0142	0.0499083	0.0142
4	1.13728e-05	0.0149	4.6503e-05	0.0150	0.000187078	0.0150	0.000749317	0.0150
5	1.81603e-07	0.0160	7.49815e-07	0.0161	3.0202e-06	0.0161	1.20998e-05	0.0161
6	2.96187e-09	0.0163	1.23351e-08	0.0165	4.97422e-08	0.0165	1.99396e-07	0.0165
7	4.85336e-11	0.0164	2.08876e-10	0.0169	8.3317e-10	0.0167	3.37675e-09	0.0169
8	1.58762e-12	0.0327	5.83e-12	0.0279	3.05986e-11	0.0367	1.21822e-10	0.0361
9	7.54508e-13	0.4271	4.32898e-12	0.7425	2.21352e-11	0.7234	8.7462e-11	0.7179
10	7.53619e-13	1.0000*	4.0481e-12	0.9351	1.60321e-11	0.7243	8.4488e-11	0.9660
11	7.53619e-13	1.0000*	3.51674e-12	0.8687	1.60321e-11	1.0000*	7.14673e-11	0.8459
12	7.53619e-13	1.0000*	3.51674e-12	1.0000*	1.60321e-11	1.0000*	7.14673e-11	1.0000*
13	7.53619e-13	1.0000*	3.51674e-12	1.0000*	1.60321e-11	1.0000*	7.14673e-11	1.0000*
14	7.53619e-13	1.0000*	3.51674e-12	1.0000*	1.60321e-11	1.0000*	7.14673e-11	1.0000*
15	7.53619e-13	1.0000*	3.51674e-12	1.0000*	1.60321e-11	1.0000*	7.14673e-11	1.0000*
16	7.53619e-13	1.0000*	3.51674e-12	1.0000*	1.60321e-11	1.0000*	7.14673e-11	1.0000*

- **Boundary condition:Dirichlet**
  Restriction operator:injection
  Interpolation operator:linear

	n=32		n=64		n=128		n=256	
V-cycle	$\|\|r^h\|\|_h$	rate	$\|\|r^h\|\|_h$	rate	$\|\|r^h\|\|_h$	rate	$\|\|r^h\|\|_h$	rate
1	0.738424		2.97705		11.9345		47.7658	
2	0.00224545	0.0030	0.00911194	0.0031	0.0366076	0.0031	0.146605	0.0031
3	9.20786e-06	0.0041	3.69155e-05	0.0041	0.000148653	0.0041	0.00059571	0.0041
4	4.38594e-08	0.0048	1.65843e-07	0.0045	6.69474e-07	0.0045	2.6848e-06	0.0045
5	2.25995e-10	0.0052	8.03892e-10	0.0048	3.28273e-09	0.0049	1.31003e-08	0.0049
6	1.90781e-12	0.0084	4.82592e-12	0.0060	2.41975e-11	0.0074	9.89338e-11	0.0076
7	8.73968e-13	0.4581	4.82592e-12	1.0000*	1.69313e-11	0.6997	9.89338e-11	1.0000*
8	8.73968e-13	1.0000*	4.82592e-12	1.0000*	1.47025e-11	0.8684	9.89338e-11	1.0000*
9	1.15508e-12	1.0000*	4.82592e-12	1.0000*	1.47025e-11	1.0000*	9.89338e-11	1.0000*
10	8.73968e-13	1.0000*	4.82592e-12	1.0000*	1.47025e-11	1.0000*	9.89338e-11	1.0000*
11	8.73968e-13	1.0000*	4.82592e-12	1.0000*	1.47025e-11	1.0000*	9.89338e-11	1.0000*
12	8.73968e-13	1.0000*	4.82592e-12	1.0000*	1.47025e-11	1.0000*	9.89338e-11	1.0000*
13	8.73968e-13	1.0000*	4.82592e-12	1.0000*	1.47025e-11	1.0000*	9.89338e-11	1.0000*
14	8.73968e-13	1.0000*	4.82592e-12	1.0000*	1.47025e-11	1.0000*	9.89338e-11	1.0000*
15	8.73968e-13	1.0000*	4.82592e-12	1.0000*	1.47025e-11	1.0000*	9.89338e-11	1.0000*
16	8.73968e-13	1.0000*	4.82592e-12	1.0000*	1.47025e-11	1.0000*	9.89338e-11	1.0000*

- **Boundary condition:mixed**
  Restriction operator:injection
  Interpolation operator:linear

	n=32		n=64		n=128		n=256	
V-cycle	$\|\|r^h\|\|_h$	rate	$\|\|r^h\|\|_h$	rate	$\|\|r^h\|\|_h$	rate	$\|\|r^h\|\|_h$	rate
1	0.738423		2.97705		11.9345		47.7658	
2	0.00224405	0.0031	0.00911194	0.0031	0.0366076	0.0031	0.146605	0.0031
3	9.01217e-06	0.0041	3.69155e-05	0.0041	0.000148653	0.0041	0.00059571	0.0041
4	3.98258e-08	0.0045	1.65841e-07	0.0045	6.69481e-07	0.0045	2.6848e-06	0.0045
5	1.90129e-10	0.0048	8.03892e-10	0.0048	3.26818e-09	0.0049	1.30421e-08	0.0049
6	1.15108e-12	0.0066	5.33573e-12	0.0066	2.87965e-11	0.0088	1.01965e-10	0.0078
7	9.98313e-13	0.9815	4.82592e-12	0.9045	1.93752e-11	0.6728	8.33276e-11	0.8172
8	8.20677e-13	0.0822	4.82592e-12	1.0000*	1.93752e-11	1.0000*	8.33276e-11	1.0000*
9	8.20677e-13	1.0000*	4.82592e-12	1.0000*	1.93752e-11	1.0000*	8.33276e-11	1.0000*
10	8.20677e-13	1.0000*	4.82592e-12	1.0000*	1.93752e-11	1.0000*	8.33276e-11	1.0000*
11	8.20677e-13	1.0000*	4.82592e-12	1.0000*	1.93752e-11	1.0000*	8.33276e-11	1.0000*
12	8.20677e-13	1.0000*	4.82592e-12	1.0000*	1.93752e-11	1.0000*	8.33276e-11	1.0000*
13	8.20677e-13	1.0000*	4.82592e-12	1.0000*	1.93752e-11	1.0000*	8.33276e-11	1.0000*
14	8.20677e-13	1.0000*	4.82592e-12	1.0000*	1.93752e-11	1.0000*	8.33276e-11	1.0000*
15	8.20677e-13	1.0000*	4.82592e-12	1.0000*	1.93752e-11	1.0000*	8.33276e-11	1.0000*
16	8.20677e-13	1.0000*	4.82592e-12	1.0000*	1.93752e-11	1.0000*	8.33276e-11	1.0000*

- Corresponding convergence rates

type	Dirichlet+full weighting+linear		mixed+full weighting+linear	
gird	error norm	convergence rate	error norm	convergence rate
n=32	7.70495e-14		8.43769e-15	
n=64	1.90246e-14	2.01246	1.87504e-15	2.12132
n=128	4.81635e-15	1.98746	4.57327e-16	2.02485
n=256	1.2008e-15	2.07364	1.14246e-16	2.00075

(d) To simplify the problem,we only list the result of FMG(2,1) scheme for $u(x) = e^{sinx}$ under Dirichlet boundary condition and different combinations of restriction and interpolation operators.

type	full weighting+linear		injection+linear	
grid	$\|\|e\|\|_h$	rate	$\|\|e\|\|_h$	rate
n=32	8.37953e-05		0.00515253	
n=64	2.91607e-05	0.348	0.00200433	0.389
n=128	8.07753e-06	0.277	5.57205e-04	0.278
n=256	2.69789e-06	0.234	1.86106e-04	0.334

(e) **Some analysis for the results of 1-dimension**

Here we use weighted Jacobi relaxation in a V(10,10) scheme on fine grids with n=32,64,128 and 256.The above tables have showed residual norm after each V-cycle and its corresponding reduction rate.Besides,we also list a table for each function to show its error norm after 16th V-cycle and convergence rate on four grids in the mode of full weighting and linear interpolation.

- The residual decreases rapidly for the first few V-cycles,with the value in the corresponding rate column reaching a nearly constant value,until the last few cycles.Besides,by observation,we can conclude that this constant value is dependent on the choice of relaxation time,restriction operator and interpolation operator.The reduction rate is quite small since we have applied VC(10,10) scheme.For the last few cycles,the residuals level off as the scheme reaches its level of discretization error.

- For each of the four grid sizes,the error norms decrease by a factor of approximately four as the grid size doubles(that is the convergence rate is approximately 2),which is consistent with the second-order discretization we have used.

2. 2-dimension
Although I've programmed for 2-dimension,there is a logistic error–the error norm does not converge.After countless tests,I think the main problem may lie in the construction of restriction and interpolation operators,but I can't figure it out(Maybe next time I should start the programming assignment early,and ask TA for advice).And here are some ideas for my construction of restriction and interpolation operators and they are under the reference to the textbook *A Multigrid Tutorial*.

   (a) restriction operator

   - injection
   It is defined by $I_h^{2h}v^h = v^{2h}$,where

   $$v^{2h}_{i,j} = v^h_{2i,2j} \qquad i,j = 1,2,\dots,\tfrac{n}{2}-1.$$

   - full weighting
   For $I_h^{2h}v^h = v^{2h}$,we have

   $$v^{2h}_{i,j} = \tfrac{1}{16}[v^h_{2i-1,2j-1} + v^h_{2i-1,2j+1} + v^h_{2i+1,2j+1} + v^h_{2i+1,2j-1} + 2(v^h_{2i,2j-1} + v^h_{2i,2j+1} + v^h_{2i+1,2j} + v^h_{2i-1,2j}) + 4v^h_{2i,2j}] \qquad i,j = 1,2,\dots,\tfrac{n}{2}-1.$$

   (b) interpolation operator
   Letting $I_{2h}^h v^{2h} = v^h$,and v^h can be expressed by

   $$\begin{aligned} v^h_{2i,2j} &= v^{2h}_{i,j},\\ v^h_{2i+1,2j} &= \tfrac{1}{2}(v^{2h}_{i,j} + v^{2h}_{i+1,j}),\\ v^h_{2i,2j+1} &= \tfrac{1}{2}(v^{2h}_{i,j} + v^{2h}_{i,j+1}),\\ v^h_{2i+1,2j+1} &= \tfrac{1}{4}(v^{2h}_{i,j} + v^{2h}_{i+1,j} + v^{2h}_{i,j+1} + v^{2h}_{i+1,j+1}), \qquad i,j = 0,1,\dots,\tfrac{n}{2}-1. \end{aligned}$$

For the completeness of the report,we'll next roughly analyze a table for the result of 2-dimension which is derived from *A Multigrid Tutorial*.

	$n = 16$				$n = 32$			
V-cycle	$\|\mathbf{r}^h\|_h$	ratio	$\|\mathbf{e}\|_h$	ratio	$\|\mathbf{r}^h\|_h$	ratio	$\|\mathbf{e}\|_h$	ratio
0	6.75e+02		5.45e−01		2.60e+03		5.61e−01	
1	4.01e+00	0.01	1.05e−02	0.02	1.97e+01	0.01	1.38e−02	0.02
2	1.11e−01	0.03	4.10e−04	0.04	5.32e−01	0.03	6.32e−04	0.05
3	3.96e−03	0.04	1.05e−04	0.26	2.06e−02	0.04	4.41e−05	0.07
4	1.63e−04	0.04	1.03e−04	0.98*	9.79e−04	0.05	2.59e−05	0.59
5	7.45e−06	0.05	1.03e−04	1.00*	5.20e−05	0.05	2.58e−05	1.00*
6	3.75e−07	0.05	1.03e−04	1.00*	2.96e−06	0.06	2.58e−05	1.00*
7	2.08e−08	0.06	1.03e−04	1.00*	1.77e−07	0.06	2.58e−05	1.00*
8	1.24e−09	0.06	1.03e−04	1.00*	1.10e−08	0.06	2.58e−05	1.00*
9	7.74e−11	0.06	1.03e−04	1.00*	7.16e−10	0.06	2.58e−05	1.00*
10	4.99e−12	0.06	1.03e−04	1.00*	4.79e−11	0.07	2.58e−05	1.00*
11	3.27e−13	0.07	1.03e−04	1.00*	3.29e−12	0.07	2.58e−05	1.00*
12	2.18e−14	0.07	1.03e−04	1.00*	2.31e−13	0.07	2.58e−05	1.00*
13	2.33e−15	0.11	1.03e−04	1.00*	1.80e−14	0.08	2.58e−05	1.00*
14	1.04e−15	0.45	1.03e−04	1.00*	6.47e−15	0.36	2.58e−05	1.00*
15	6.61e−16	0.63	1.03e−04	1.00*	5.11e−15	0.79	2.58e−05	1.00*

	$n = 64$				$n = 128$			
V-cycle	$\|\mathbf{r}^h\|_h$	ratio	$\|\mathbf{e}\|_h$	ratio	$\|\mathbf{r}^h\|_h$	ratio	$\|\mathbf{e}\|_h$	ratio
0	1.06e+04		5.72e−01		4.16e+04		5.74e−01	
1	7.56e+01	0.01	1.39e−02	0.02	2.97e+02	0.01	1.39e−02	0.02
2	2.07e+00	0.03	6.87e−04	0.05	8.25e+00	0.03	6.92e−04	0.05
3	8.30e−02	0.04	4.21e−05	0.06	3.37e−01	0.04	4.22e−05	0.06
4	4.10e−03	0.05	7.05e−06	0.17	1.65e−02	0.05	3.28e−06	0.08
5	2.29e−04	0.06	6.45e−06	0.91*	8.99e−04	0.05	1.63e−06	0.50
6	1.39e−05	0.06	6.44e−06	1.00*	5.29e−05	0.06	1.61e−06	0.99*
7	8.92e−07	0.06	6.44e−06	1.00*	3.29e−06	0.06	1.61e−06	1.00*
8	5.97e−08	0.07	6.44e−06	1.00*	2.14e−07	0.06	1.61e−06	1.00*
9	4.10e−09	0.07	6.44e−06	1.00*	1.43e−08	0.07	1.61e−06	1.00*
10	2.87e−10	0.07	6.44e−06	1.00*	9.82e−10	0.07	1.61e−06	1.00*
11	2.04e−11	0.07	6.44e−06	1.00*	6.84e−11	0.07	1.61e−06	1.00*
12	1.46e−12	0.07	6.44e−06	1.00*	4.83e−12	0.07	1.61e−06	1.00*
13	1.08e−13	0.07	6.44e−06	1.00*	3.64e−13	0.08	1.61e−06	1.00*
14	2.60e−14	0.24	6.44e−06	1.00*	1.03e−13	0.28	1.61e−06	1.00*
15	2.30e−14	0.88	6.44e−06	1.00*	9.19e−14	0.89	1.61e−06	1.00*

The author had used red-black Gauss-Seidel relaxation in a V-cycle scheme on fine grids with n=16,32,64 and 128 points in each dimension.And full weighting and linear interpolation are applied.

As we can see,the above results are quite similar to those in our 1-dimension discussion. By our observation,the reduction rate for residual increases gradually,but it reach a nearly constant value for the middle V-cycles,for this scheme,the value is approximately 0.07.And our previous arguments have revealed it.Besides,the sharp increase in the reduction rate for the residual during the last few V-cycles implies that the algebraic approximation is already accurate to near machine precision.After 5-7 V-cycles,the error norm levels off as the scheme reaches the level of discretization error.The last row for error of each grid size shows the final error norms decrease by a factor of four,which is consistent with the second-order discretization the author had applied.

Although we are stuck in the restriction and interpolation operator,the program is still correct from semantic perspective.Thus,we can still compare the CPU time between multigrid method and the traditional LU-decomposition in lapacke.By our observation,the time cost for multigrid method is approximately four times as much as the lapacke solution and this factor might have something to do with my programming.