Computational methods and applications (AMS 147)

Homework 7 - Due Friday, March 15th (**no late submission accepted**)

Please submit to CANVAS a .zip file that includes the following Matlab functions:

`AB2.m`

`solve_ODE_system.m`

For the Extra Credit part, scan your notes into one PDF file `scan.pdf`, and attach it to your submission (as a separate file, not as part of the .zip file)

Exercise 1 (50 points) Consider the the following initial value problem for an n-dimensional system of ordinary differential equations (ODEs)

$$\begin{cases} \dfrac{d\boldsymbol{y}(t)}{dt} = \boldsymbol{f}(\boldsymbol{y}(t), t) \\ \boldsymbol{y}(0) = \boldsymbol{y}_0 \end{cases} \tag{1}$$

Here, $\boldsymbol{f} : \mathbb{R}^n \times [0, T] \to \mathbb{R}^n$, and $\boldsymbol{y} : [0, T] \to \mathbb{R}^n$ (T is the period of integration). Write a Matlab function that computes the numerical solution of (1) by using the two-step Adams-Bashforth (AB2) scheme

$$\boldsymbol{u}_{n+1} = \boldsymbol{u}_n + \frac{\Delta t}{2} \left[3\boldsymbol{f}(\boldsymbol{u}_n, t_n) - \boldsymbol{f}(\boldsymbol{u}_{n-1}, t_{n-1})\right] \qquad n = 1, 2, ... \tag{2}$$

To start-up AB2, i.e., to compute $\boldsymbol{u}_1$, use the Heun method

$$\boldsymbol{u}_1 = \boldsymbol{u}_0 + \frac{\Delta t}{2} \left[\boldsymbol{f}(\boldsymbol{u}_0, t_0) + \boldsymbol{f}(\boldsymbol{u}_0 + \Delta t \boldsymbol{f}(\boldsymbol{u}_0, t_0), t_1)\right], \qquad \boldsymbol{u}_0 = \boldsymbol{y}_0. \tag{3}$$

The Matlab function implementing the scheme (2)-(3) should be of the form

```
function [y,t] = AB2(fun,y0,NSTEPS,DT,IOSTEP)
```

Input:

`fun`: function handle representing $\boldsymbol{f}(\boldsymbol{y}, t)$

`y0`: column vector representing the initial condition $\boldsymbol{y}_0$

`NSTEPS`: total number of steps

`DT`: time step

`IOSTEP`: Input/output step. The numerical solution is saved in the output matrix `y` every `IOSTEP` steps.

Output:

`y`: $n \times S$ matrix collecting the time snapshots of the solution to (1). Note that the total number of snapshots S (including the initial condition) is `floor(NSTEPS/IOSTEP)+1`.

`t`: vector collecting the time instants at which the solution is saved in the output matrix `y`.

Exercise 2 (50 points) Consider the following three-dimensional nonlinear dynamical system

$$\begin{cases} \dfrac{dy_1}{dt} = -y_1 + y_2 y_3 \\ \dfrac{dy_2}{dt} = -y_2 + (y_3 - 2) y_1 \\ \dfrac{dy_3}{dt} = 1 - y_1 y_2 \end{cases} \tag{4}$$

It is known that the solution to (4) is chaotic in time and it settles on a strange attractor. By using the function `AB2.m` you coded in Exercise 1, compute the numerical solution to (4). To this end, set `NSTEPS`=1e5, `DT`= 1e-3, `IOSTEP`=50, `y0`=[1; 2; 3], and write a function

```
function [y,t]=solve_ODE_system()
```

Output:

`y`: $3 \times S$ matrix collecting S time snapshots of the solution to (4). Note that S=`floor(NSTEPS/IOSTEP)+1=2001`.

`t`: $1 \times S$ vector collecting the time instants at which the solution is saved in the output matrix `y`.

The function `solve_ODE_system` should also return the following items:

1. The graphs of $y_1(t)$, $y_2(t)$ and $y_3(t)$ versus time in `figure(1)`.
2. A three-dimensional plot of the curve $(y_1(t), y_2(t), y_3(t))$ in `figure(2)` (use the Matlab command `plot3()` - see the documentation).

Extra Credit (10 points) Consider the following initial value problem for one ODE

$$\begin{cases} \dfrac{dy}{dt} = f(y,t) \\ y(0) = y_0 \end{cases} \tag{5}$$

Prove that the leapfrog method

$$u_{n+1} = u_{n-1} + 2\Delta t f(u_n, t_n), \qquad n = 1, 2, ... \tag{6}$$

is consistent with order two, i.e., the truncation error goes to zero as $\mathcal{O}(\Delta t^2)$.
Hint: A substitution of the exact solution $y(t)$ into (7) yields:

$$y(t_{n+1}) = y(t_{n-1}) + 2\Delta t f(y(t_n), t_n) + \Delta t \tau_{n+1}(\Delta t), \qquad n = 1, 2, ... \tag{7}$$

where $\tau_{n+1}(\Delta t)$ is the local truncation error at time t_{n+1}. Use Taylor series expansions of $y(t_{n+1})$ and $y(t_{n-1})$ to compute a Taylor series expansion of $\tau_{n+1}(\Delta t)$, and show that the leading term is of order Δt^2.