Computational methods and applications (AMS 147)

Homework 2 - Due Thursday, Jan 31st

Please submit to CANVAS a .zip file that includes the following functions:

`chord_method.m`, `test_zero.m`

For Exercise 3 and the Extra Credit part scan your notes into one PDF file `scan.pdf`, and attach it to your submission.

Exercise 1 Write a function `chord_method.m` implementing the chord method to find the zeros of a nonlinear equation. The function should be of the following form

```
function [z0,iter,res,his] = chord_method(fun,a,b,tol,Nmax)
```

Inputs:
`fun`: function handle representing $f(x)$
`a`, `b`: interval $[a, b]$ in which we believe there is a zero
`tol`: maximum tolerance allowed for the increment $|x^{(k+1)} - x^{(k)}|$
`Nmax`: maximum number of iterations allowed

Outputs:
`z0`: approximation of the zero
`iter`: number of iterations to achieve tolerance `tol` on the increment $|x^{(k+1)} - x^{(k)}|$
`res`: residual at `z0` (i.e., $|f(\texttt{z0})|$)
`his`: vector collecting all elements of the sequence $\{x^{(k)}\}_{k=0,1,..}$ (convergence history)

The function should return the numerical approximation of the zero when the increment at iteration $k+1$ is such that $|x^{(k+1)} - x^{(k)}| <$ `tol` or when the iteration number reaches the maximum value `Nmax`.

Exercise 2 Use the function you coded in Exercise 1 to compute an approximation of the smallest zero of the fifth-order Chebyshev polynomial

$$f(x) = 16x^5 - 20x^3 + 5x, \qquad x \in [-1, 1]. \tag{1}$$

To this end, set `tol`$=10^{-15}$, `Nmax`$=20000$, `a`$=-0.99$, `b`$=-0.9$, and write a function `test_zero.m` of the following form

```
function [zc,ec,x,f] = test_zero()
```

Outputs:
`zc`: numerical approximation of the zero obtained with the chord method.
`ec`: error vector with components $|x^{(k)} - z_0|$ $(k = 0, 1, ...)$, where $z_0 = \cos(9\pi/10)$ is the exact zero of (1) in

the interval $[-1, -0.9]$, while $x^{(k)}$ is the sequence converging to z_0 generated by the chord method.
`x`: row vector of 1000 evenly spaced nodes in $[-1, 1]$ (including the endopoints).
`f`: row vector representing the function (1) evaluated at `x`.

The function `test_zero()` should also produce the following three figures

1. The graph of the function (1) in `figure(1)`.
2. The plot of the convergence history, i.e., the errors $e_k = |x^{(k)} - z_0|$ versus k. This plot should be in `figure(2)`, and in a semi-log scale (use the command `semilogy`).
3. The plot of $e_{k+1} = |x^{(k+1)} - z_0|$ (y-axis) versus $e_k = |x^{(k)} - z_0|$ (x-axis) in a log-log scale (use the command `loglog`). This plot should be in `figure(3)`. Remember, for sufficiently large k, the slope of the curve in such log-log plot represents the convergence order of the sequence.

Exercise 3 Consider the following sequence

$$x^{(k+1)} = \phi\left(x^{(k)}\right), \quad \text{where} \quad \phi(x) = x + \sin(x) \tag{2}$$

is a one-dimensional map ($\phi : \mathbb{R} \to \mathbb{R}$).

1. Determine the fixed points of ϕ.
2. Show that the sequence (2) converges to π for any initial condition $x^{(0)}$ in the interval $]\pi/2, 3\pi/2[$ (excluding endpoints).
3. Determine the convergence order of the sequence (2) in a neighborhood of the fixed point π.

Extra Credit Let $f \in C^{(\infty)}([a, b])$ be a real valued function, $\alpha \in [a, b]$ such that $f(\alpha) = 0$ and $f'(\alpha) \neq 0$ (simple zero). By using the theory of fixed point iterations prove that the sequence

$$x^{(k+1)} = x^{(k)} - \frac{f(x^{(k)})}{f'(x^{(k)})} - \frac{f\left(x^{(k)} - \dfrac{f(x^{(k)})}{f'(x^{(k)})}\right)}{f'(x^{(k)})} \tag{3}$$

converges to α with order 3 for any $x^{(0)}$ in a neighborhood of α.