

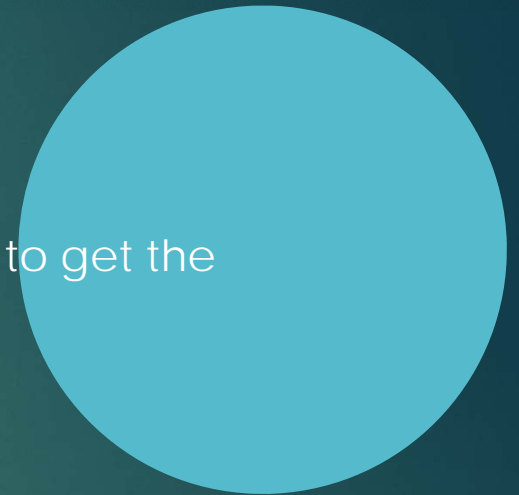
COIN CHANGE PROBLEM

DYNAMIC PROGRAMMING



Coin change problem

- ▶ Given a set of coins $v[]$ for example $\{1,2,3\}$
- ▶ Given an M amount \rightarrow the total
- ▶ How many ways the coins $v[]$ can be combined in order to get the total M ?
- ▶ The order of coins does not matter !!!
- ▶ This is the coin change problem



Coin change problem

Coins $v[] \rightarrow \{1, 2, 3\}$

Total amount $M \rightarrow 4$

Solution to the coin change problem:

$\{1,1,1,1\} \{1,1,2\} \{1,3\} \{2,2\}$

The order of coins does not matter !!!

For exmple $\{1,3\} = \{3,1\}$



Recursion

- ▶ The naive approach is to use a simple recursive method / function
- ▶ For every single coin we have two options: include it in our solution or exclude it
- ▶ Problems: time complexity + overlapping subproblems
- ▶ Exponential time complexity: $O(2^N)$ where **N** is the number of coins
- ▶ For every coin we have **2** options whether to take it or not

Dynamic programming

We have to create a solution matrix:

`dpTable[numOfCoins+1][totalAmount+1]`
 rows columns

We have to define the base cases:

- if totalAmount is 0 → there is 1 way to make the change
Because we do not include any coin !!!
- if numOfCoins is 0 → there is 0 way to change the amount
In this case there is no solution !!!

Complexity: $O(v \cdot M)$

Dynamic programming

We have to create a solution matrix:

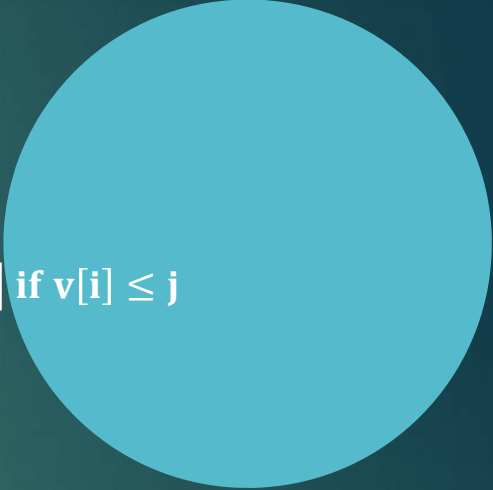
`dpTable[numOfCoins+1][totalAmount+1]`
 rows columns

For every coin: make a decision whether to include it or not

Check if the coin value is less than or equal to the amount needed

If yes → then we will find ways by including that coin and excluding that coin

- 1.) include the coin: reduce the amount by coin value and use the subproblem solution // **totalAmount - v[i]**
- 2.) exclude the coin: solution for the same amount without considering that coin


$$\text{dpTable}[i][j] = \begin{cases} 0 & \text{if } i = 0 \\ 1 & \text{if } j = 0 \\ \text{dpTable}[i - 1][j] + \text{dpTable}[i][j - v[i - 1]] & \text{if } v[i] \leq j \\ \text{dpTable}[i - 1][j] & \text{if } v[i] > j \end{cases}$$

If the coin value is smaller than the amount: it means we can consider that coin !!!

Base cases

$$\text{dpTable}[i][j] = \begin{cases} 0 & \text{if } i = 0 \\ 1 & \text{if } j = 0 \\ \text{dpTable}[i - 1][j] + \text{dpTable}[i][j - v[i - 1]] & \text{if } v[i] \leq j \\ \text{dpTable}[i - 1][j] & \text{if } v[i] > j \end{cases}$$

How many ways the first i coins can be combined in order to get the total j ?

If the coin value is greater than the amount: it means we can not consider that coin !!!

$M = 4$
 $v[] = \{1,2,3\}$

$\text{numOfColumns} = M+1$
 $\text{numOfRows} = v.\text{length}+1$


	0	1	2	3	4
0					
1					
2					
3					

coins

totals

$M = 4$
 $v[] = \{1, 2, 3\}$

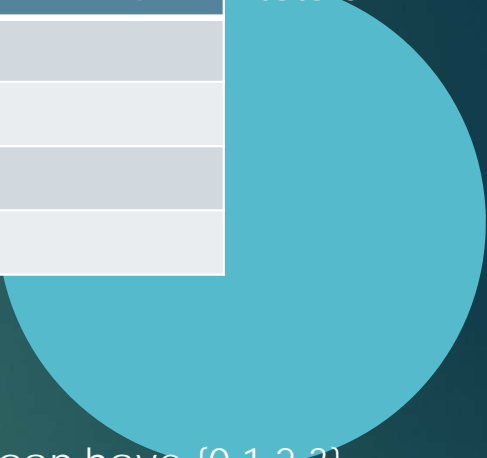
$\text{numOfColumns} = M + 1$
 $\text{numOfRows} = v.\text{length} + 1$

	0	1	2	3	4	totals
0						
1						
2						
3						
coins						

Subproblems: we consider the totals $\{0, 1, 2, 3, 4\}$ step by step when we can have $\{0, 1, 2, 3\}$ coins at the same time !!! We solve the subproblems and combine them for the final solution

$M = 4$
 $v[] = \{1,2,3\}$

$\text{numOfColumns} = M+1$
 $\text{numOfRows} = v.\text{length}+1$

		M=0	M=1	M=2	M=3	M=4	totals
Take no coins	0						
Take 1	1						
Take 1 or 2	2						
Take 1 or 2 or 3	3						
coins							

Subproblems: we consider the totals $\{0,1,2,3,4\}$ step by step when we can have $\{0,1,2,3\}$ coins at the same time !!! We solve the subproblems and combine them for the final solution

$M = 4$
 $v[] = \{1,2,3\}$

$\text{numOfColumns} = M+1$
 $\text{numOfRows} = v.\text{length}+1$

	0	1	2	3	4
0	0	0	0	0	0
1					
2					
3					

$M = 4$
 $v[] = \{1,2,3\}$

$\text{numOfColumns} = M+1$
 $\text{numOfRows} = v.\text{length}+1$

	0	1	2	3	4
0	1	0	0	0	0
1	1				
2	1				
3	1				

$M = 4$
 $v[] = \{1, 2, 3\}$

$\text{numOfColumns} = M + 1$
 $\text{numOfRows} = v.\text{length} + 1$

	0	1	2	3	4
0	1	0	0	0	0
1	1	1			
2	1				
3	1				

$M = 4$
 $v[] = \{1,2,3\}$

$\text{numOfColumns} = M+1$
 $\text{numOfRows} = v.\text{length}+1$

	0	1	2	3	4
0	1	0	0	0	0
1	1	1	1		
2	1				
3	1				

$M = 4$
 $v[] = \{1,2,3\}$

$\text{numOfColumns} = M+1$
 $\text{numOfRows} = v.\text{length}+1$

	0	1	2	3	4
0	1	0	0	0	0
1	1	1	1	1	
2	1				
3	1				

$M = 4$
 $v[] = \{1,2,3\}$

$\text{numOfColumns} = M+1$
 $\text{numOfRows} = v.\text{length}+1$

	0	1	2	3	4
0	1	0	0	0	0
1	1	1	1	1	1
2	1				
3	1				

$M = 4$
 $v[] = \{1, 2, 3\}$

$\text{numOfColumns} = M + 1$
 $\text{numOfRows} = v.\text{length} + 1$

	0	1	2	3	4
0	1	0	0	0	0
1	1	1	1	1	1
2	1				
3	1				

$$\text{dpTable}[i][j] = \begin{cases} \text{dpTable}[i-1][j] + \text{dpTable}[i][j - v[i-1]] & \text{if } v[i] \leq j \\ \text{dpTable}[i-1][j] & \text{if } v[i] > j \end{cases}$$

What does it mean simply?

If the given $v[i] > j \rightarrow$ copy the content of the box above the current

Else: $\text{dpTable}[i][j] = \text{value of box above the current}$
 $+ (\text{value in same row} - v[i])$

$M = 4$
 $v[] = \{1, 2, 3\}$

$\text{numOfColumns} = M + 1$
 $\text{numOfRows} = v.\text{length} + 1$

	0	1	2	3	4
0	1	0	0	0	0
1	1	1	1	1	1
2	1	1			
3	1				

$$\text{dpTable}[i][j] = \begin{cases} \text{dpTable}[i-1][j] + \text{dpTable}[i][j - v[i-1]] & \text{if } v[i] \leq j \\ \text{dpTable}[i-1][j] & \text{if } v[i] > j \end{cases}$$

$M = 4$
 $v[] = \{1, 2, 3\}$

$\text{numOfColumns} = M + 1$
 $\text{numOfRows} = v.\text{length} + 1$

	0	1	2	3	4
0	1	0	0	0	0
1	1	1	1	1	1
2	1	1	2		
3	1				

$$\text{dpTable}[i][j] = \begin{cases} \text{dpTable}[i-1][j] + \text{dpTable}[i][j - v[i-1]] & \text{if } v[i] \leq j \\ \text{dpTable}[i-1][j] & \text{if } v[i] > j \end{cases}$$

$$\text{dpTable}[2][2] = \text{dpTable}[1][2] + \text{dpTable}[2][0]$$

$M = 4$
 $v[] = \{1, 2, 3\}$

$\text{numOfColumns} = M + 1$
 $\text{numOfRows} = v.\text{length} + 1$

	0	1	2	3	4
0	1	0	0	0	0
1	1	1	1	1	1
2	1	1	2	2	
3	1				

$$\text{dpTable}[i][j] = \begin{cases} \text{dpTable}[i-1][j] + \text{dpTable}[i][j - v[i-1]] & \text{if } v[i] \leq j \\ \text{dpTable}[i-1][j] & \text{if } v[i] > j \end{cases}$$

$$\text{dpTable}[2][3] = \text{dpTable}[1][3] + \text{dpTable}[2][1]$$

$M = 4$
 $v[] = \{1, 2, 3\}$

$\text{numOfColumns} = M + 1$
 $\text{numOfRows} = v.\text{length} + 1$

	0	1	2	3	4
0	1	0	0	0	0
1	1	1	1	1	1
2	1	1	2	2	3
3	1				

$$\text{dpTable}[i][j] = \begin{cases} \text{dpTable}[i-1][j] + \text{dpTable}[i][j - v[i-1]] & \text{if } v[i] \leq j \\ \text{dpTable}[i-1][j] & \text{if } v[i] > j \end{cases}$$

$$\text{dpTable}[2][4] = \text{dpTable}[1][4] + \text{dpTable}[2][2]$$

$M = 4$
 $v[] = \{1, 2, 3\}$

$\text{numOfColumns} = M + 1$
 $\text{numOfRows} = v.\text{length} + 1$

	0	1	2	3	4
0	1	0	0	0	0
1	1	1	1	1	1
2	1	1	2	2	3
3	1	1			

$$\text{dpTable}[i][j] = \begin{cases} \text{dpTable}[i-1][j] + \text{dpTable}[i][j - v[i-1]] & \text{if } v[i] \leq j \\ \text{dpTable}[i-1][j] & \text{if } v[i] > j \end{cases}$$

$$\text{dpTable}[3][1] = \text{dpTable}[2][1]$$

$M = 4$
 $v[] = \{1, 2, 3\}$

$\text{numOfColumns} = M + 1$
 $\text{numOfRows} = v.\text{length} + 1$

	0	1	2	3	4
0	1	0	0	0	0
1	1	1	1	1	1
2	1	1	2	2	3
3	1	1	2		

$$\text{dpTable}[i][j] = \begin{cases} \text{dpTable}[i-1][j] + \text{dpTable}[i][j - v[i-1]] & \text{if } v[i] \leq j \\ \text{dpTable}[i-1][j] & \text{if } v[i] > j \end{cases}$$

$$\text{dpTable}[3][2] = \text{dpTable}[2][2]$$

$M = 4$
 $v[] = \{1, 2, 3\}$

$\text{numOfColumns} = M + 1$
 $\text{numOfRows} = v.\text{length} + 1$

	0	1	2	3	4
0	1	0	0	0	0
1	1	1	1	1	1
2	1	1	2	2	3
3	1	1	2	3	

$$\text{dpTable}[i][j] = \begin{cases} \text{dpTable}[i-1][j] + \text{dpTable}[i][j - v[i-1]] & \text{if } v[i] \leq j \\ \text{dpTable}[i-1][j] & \text{if } v[i] > j \end{cases}$$

$$\text{dpTable}[3][3] = \text{dpTable}[2][3] + \text{dpTable}[3][0]$$

$M = 4$
 $v[] = \{1, 2, 3\}$

$\text{numOfColumns} = M + 1$
 $\text{numOfRows} = v.\text{length} + 1$

	0	1	2	3	4
0	1	0	0	0	0
1	1	1	1	1	1
2	1	1	2	2	3
3	1	1	2	3	4

$$\text{dpTable}[i][j] = \begin{cases} \text{dpTable}[i - 1][j] + \text{dpTable}[i][j - v[i - 1]] & \text{if } v[i] \leq j \\ \text{dpTable}[i - 1][j] & \text{if } v[i] > j \end{cases}$$

$$\text{dpTable}[3][4] = \text{dpTable}[2][4] + \text{dpTable}[3][1]$$

$M = 4$
 $v[] = \{1, 2, 3\}$

$\text{numOfColumns} = M + 1$
 $\text{numOfRows} = v.\text{length} + 1$

	0	1	2	3	4
0	1	0	0	0	0
1	1	1	1	1	1
2	1	1	2	2	3
3	1	1	2	3	4

$$\text{dpTable}[i][j] = \begin{cases} \text{dpTable}[i - 1][j] + \text{dpTable}[i][j - v[i - 1]] & \text{if } v[i] \leq j \\ \text{dpTable}[i - 1][j] & \text{if } v[i] > j \end{cases}$$

SO WE HAVE SOLVED OUR PROBLEM WITHOUT RECALCULATING THE
SAME PROBLEMS OVER AND OVER AGAIN !!!