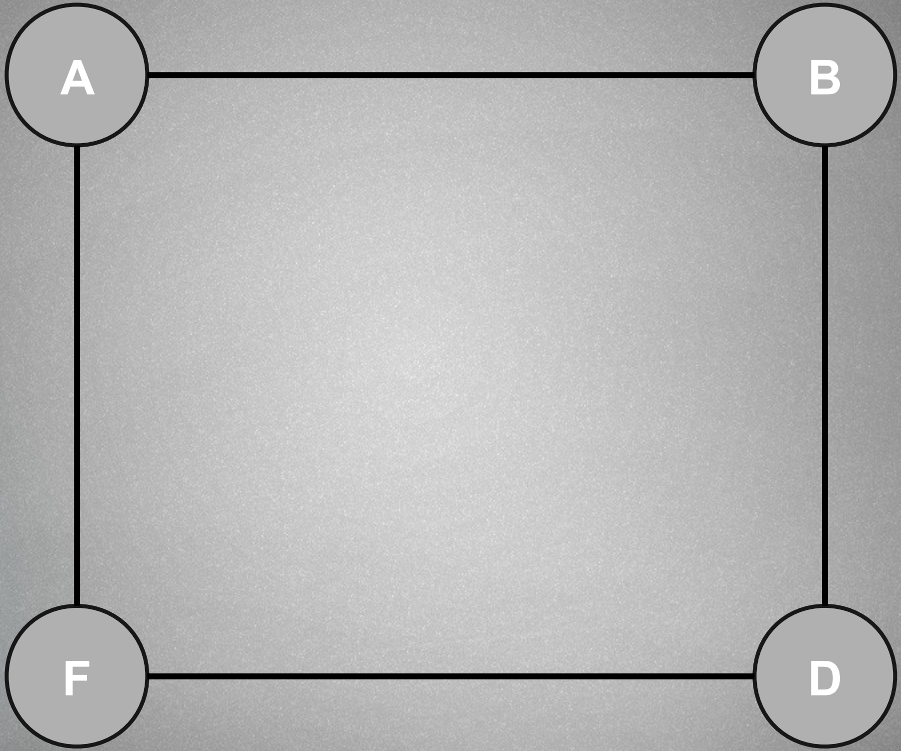
SPANNING TREES

Kruskal algorithm

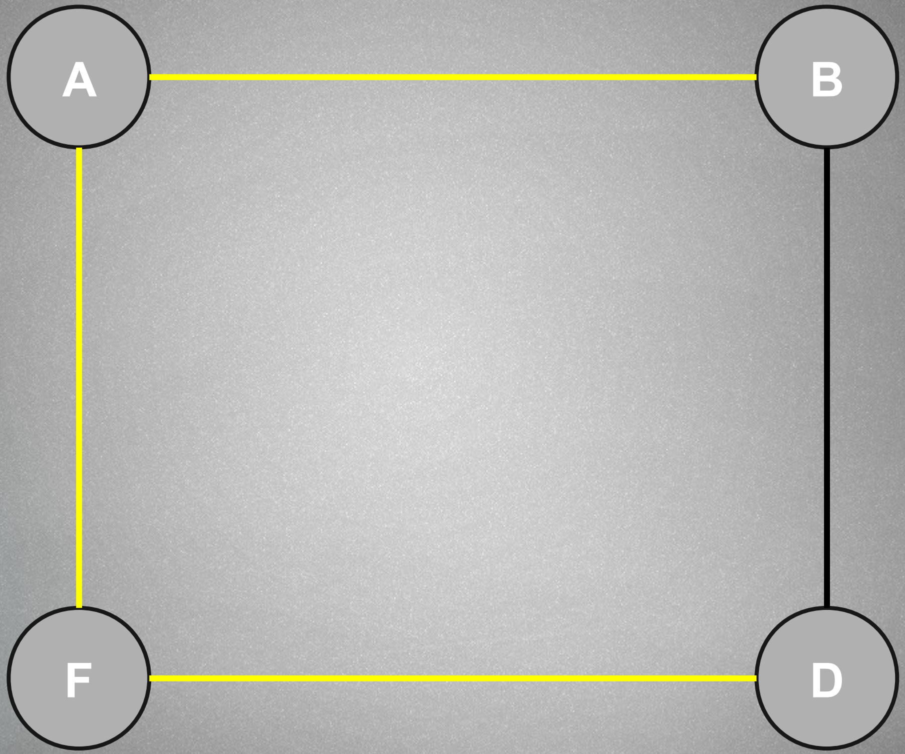
Spanning trees

- A spanning tree of an undirected **G** graph is a subgraph that includes all the vertices of **G**
- In general, a tree may have several spanning trees
- We can assign a weight to each edge
- A minimum spanning tree is then a spanning tree with weight less than or equal to the weight of every other spanning tree
- Has lots of applications: in big data analysis, clustering algorithms, finding minimum cost for a telecommunications company laying cable to a new neighborhood
- Standard algorithms: Prim’s-Jarnik, Kruskal → greedy algorithms

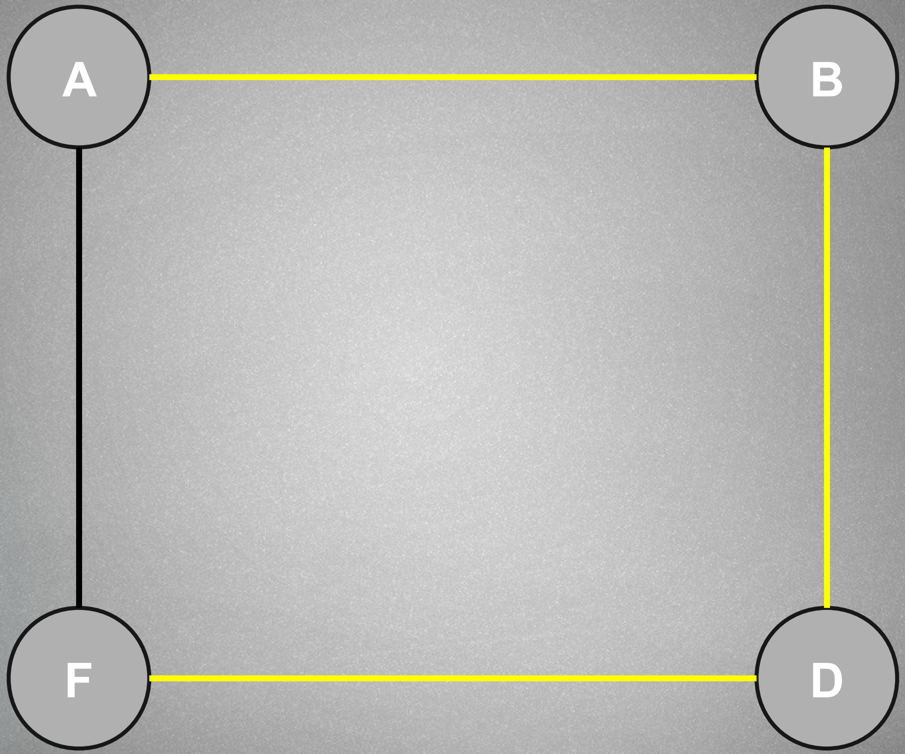
A graph may have several spanning tree !!!



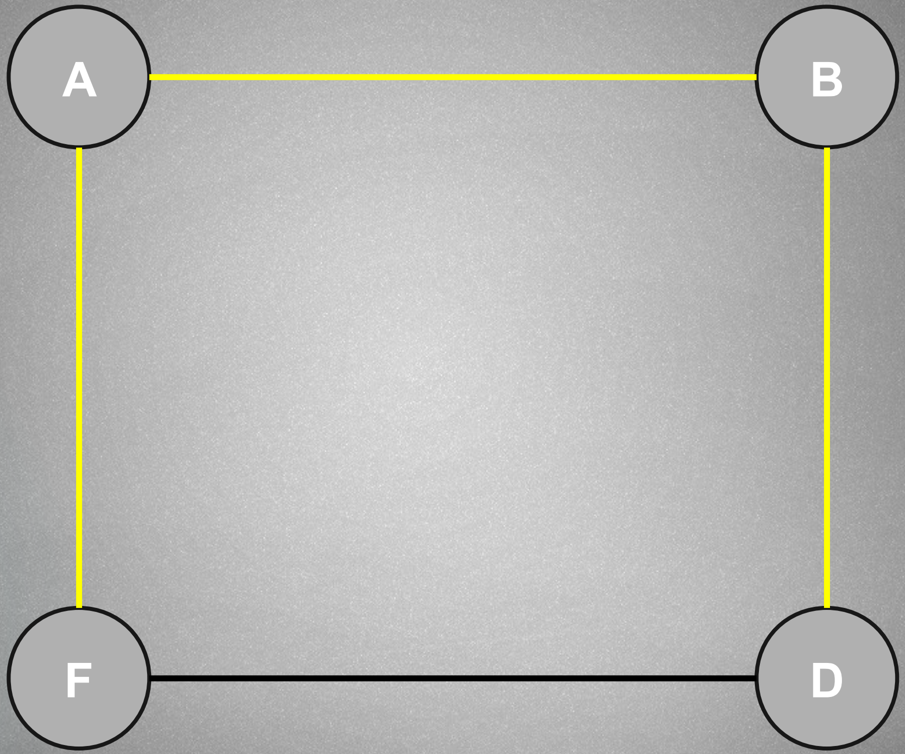
A graph may have several spanning tree !!!



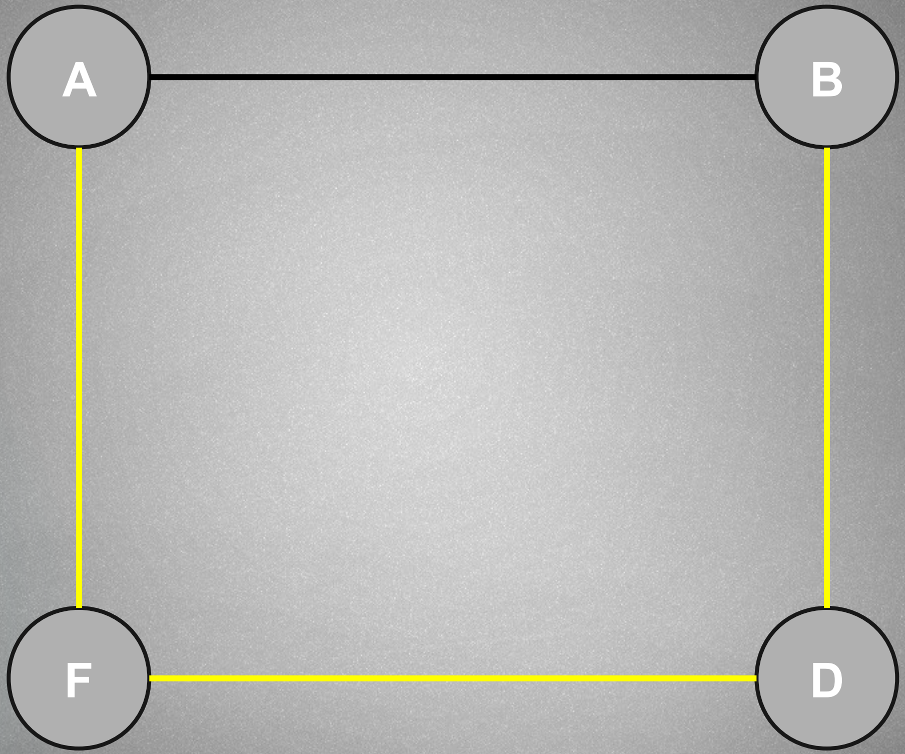
A graph may have several spanning tree !!!



A graph may have several spanning tree !!!



A graph may have several spanning tree !!!



Usually we are looking for the minimum spanning tree: the spanning tree where the sum of edge weights is the lowest possible !!!

Kruskal-algorithm

- We sort the edges according to their edge weights
- It can be done in **O(N*logN)** with mergesort or quicksort
- Union find data structure: „disjoint set"

We start adding edges to the MST and we want to make sure there will be no cycles in the spanning tree. It can be done in **O(logV)** with the help of union find data structure

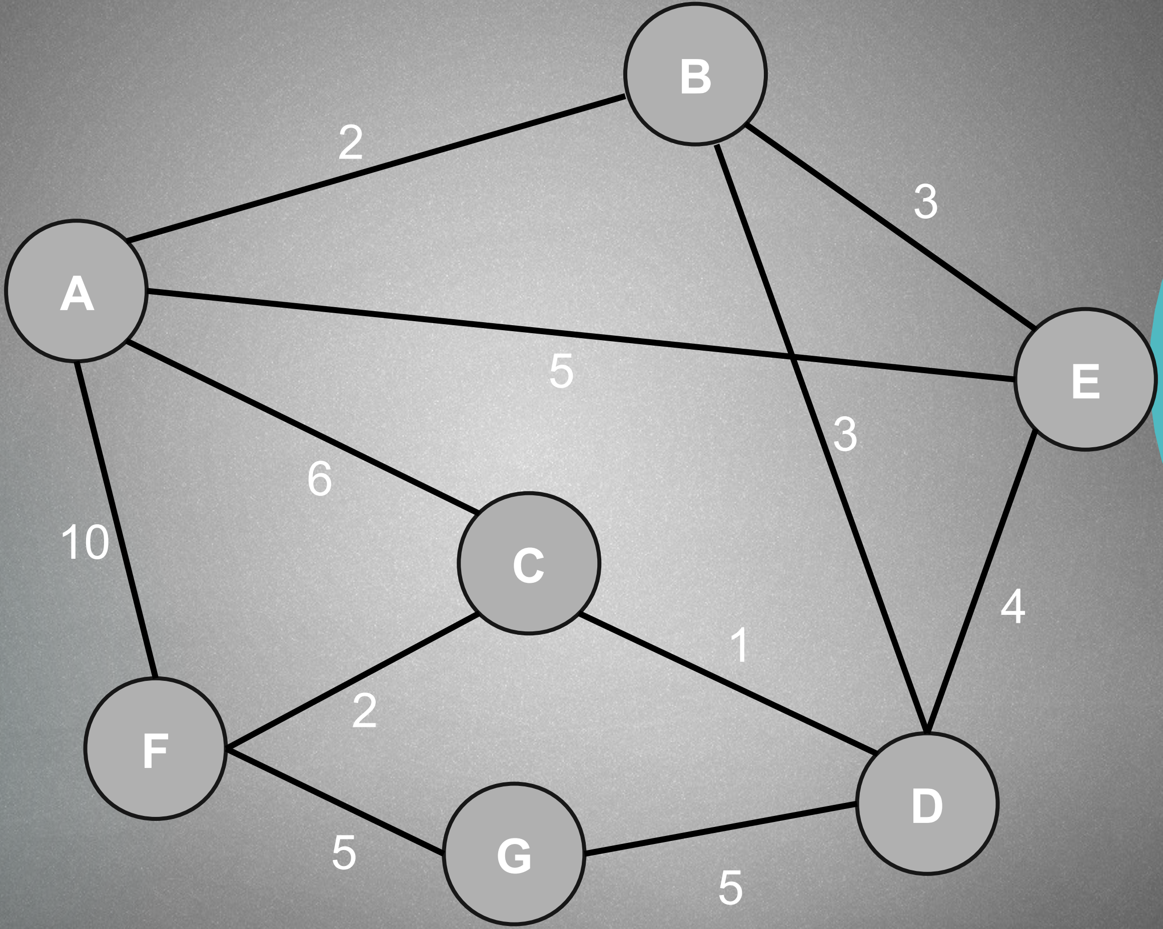
// we could use a heap instead sorting the edges in the beginning but the running time would be the same. So sometimes Kruskal's algorithm is implemented with priority queues

Kruskal-algorithm

- Worst case running time: **O(E*logE)**, so we can use it for huge graphs too
- If the edges are sorted: the algorithm will be quasi-linear
- If we multiply the weights with a constant or add a constant to the edge weights: the result will be the same

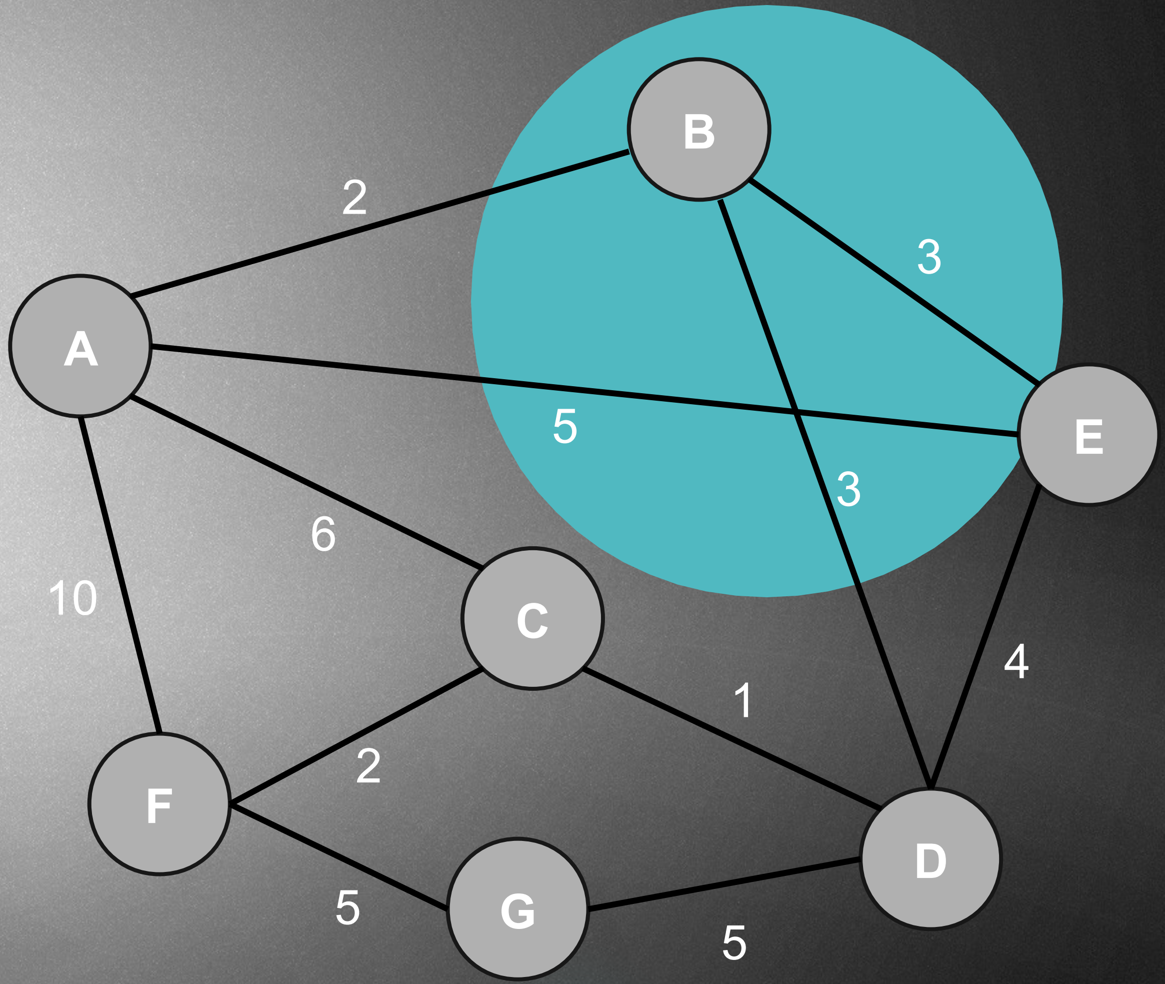
  // in physics, an invariant is a property of the system that remains unchanged under some transformation.

  In Kruskal algorithm, spanning trees are invariant under the transformation of these weights (addition, multiplication)


A
B
C
D
E
F
G
2
3
5
3
6
10
1
4
2
5
5

We have to sort the edges: 1, 2, 2, 3, 3, 4, 5, 5, 5, 6, 10

On every iteration we have to make sure whether by adding the new edge → will there be a cycle or not ...



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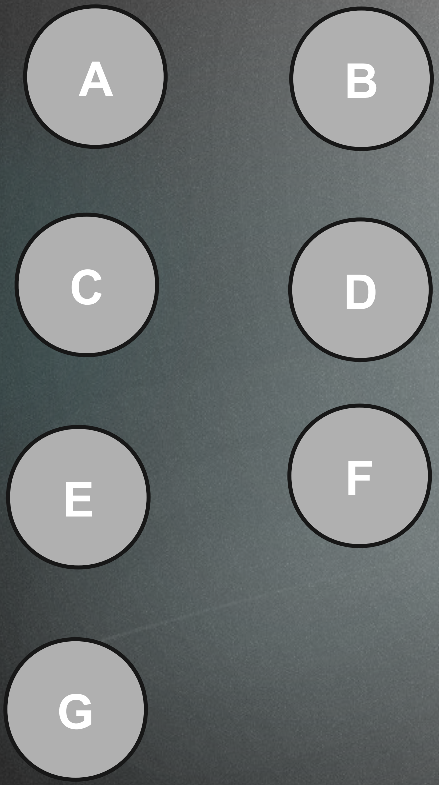
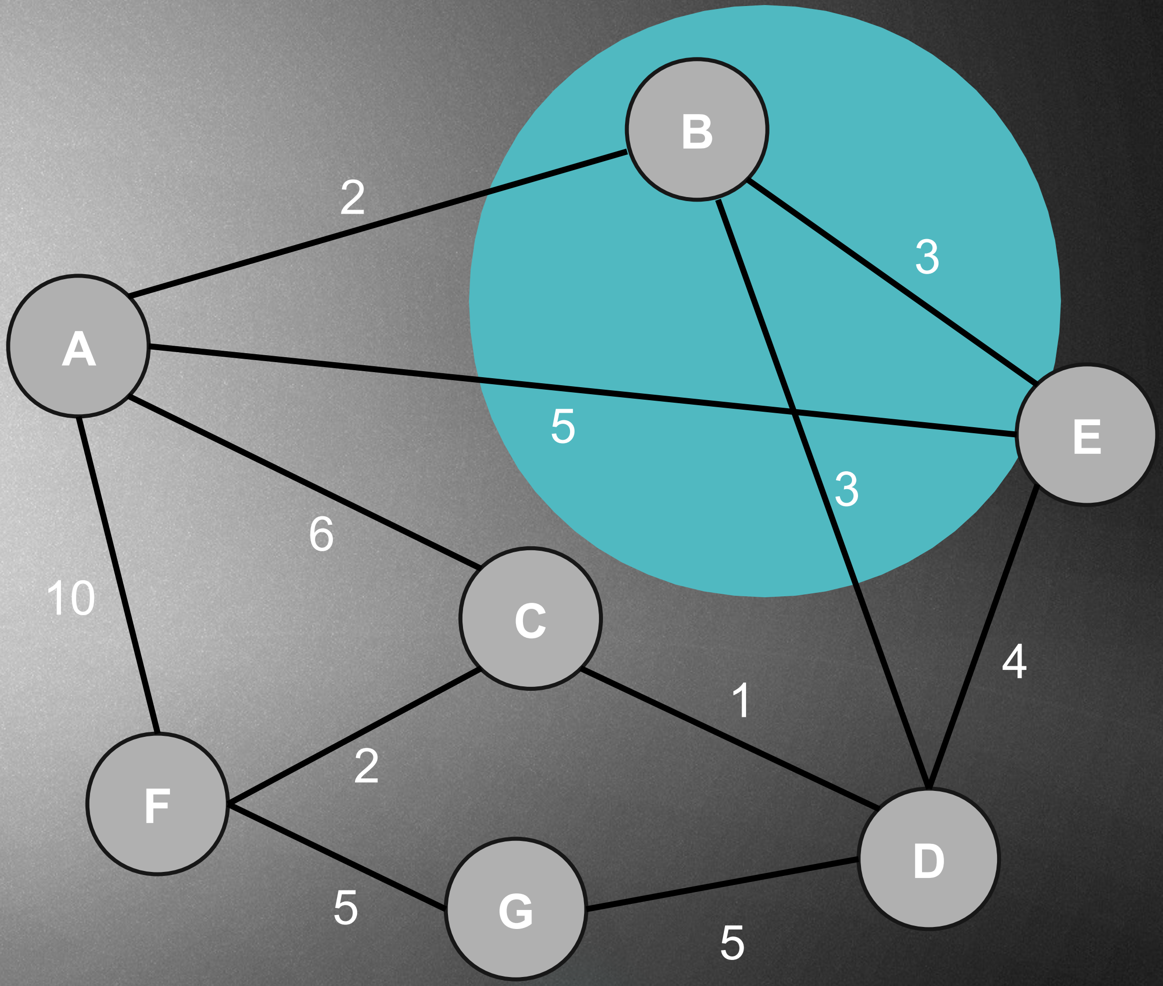
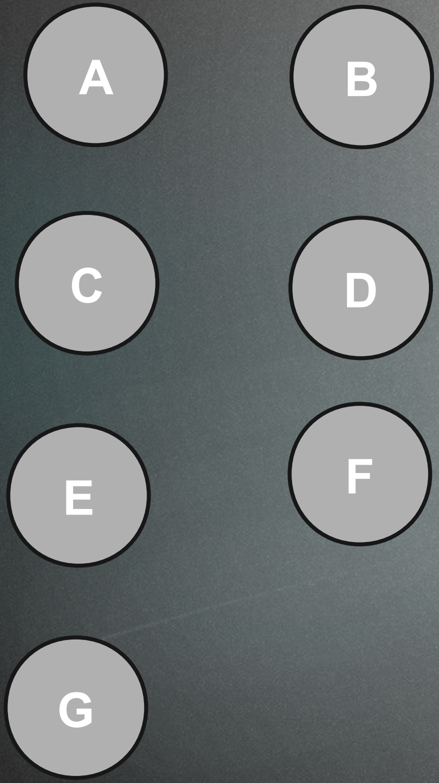
Disjoint sets

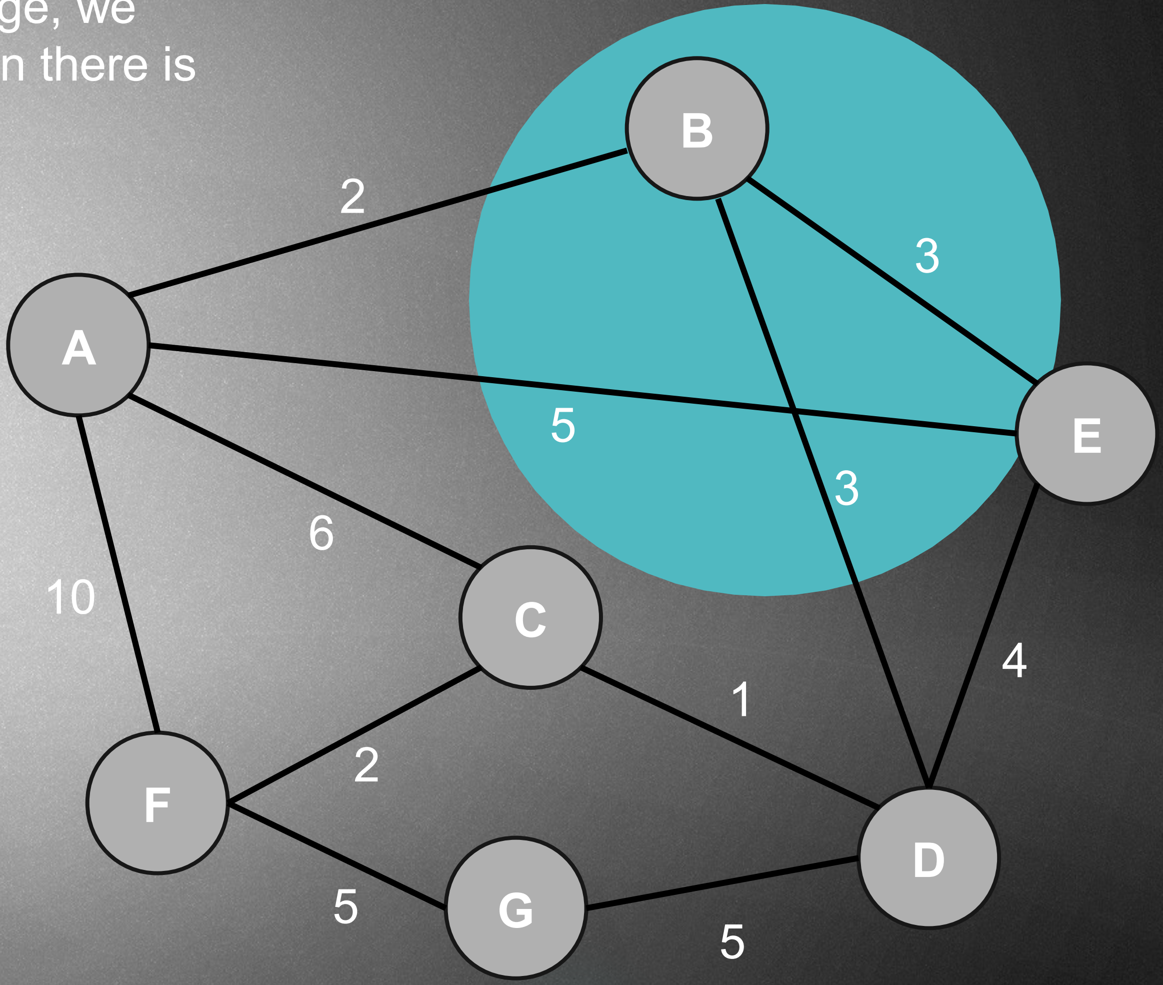




We have to sort the edges: 1, 2, 2, 3, 3, 4, 5, 5, 5, 6, 10

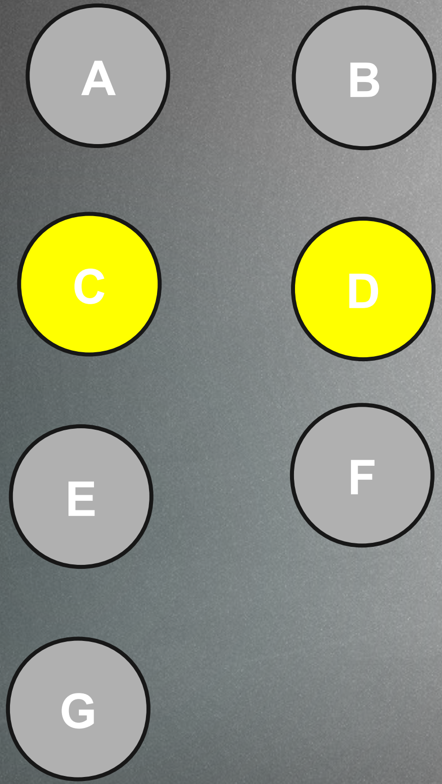
Disjoint sets: at the beginning we have as many sets as the number of vertices. When adding an edge, we merge two sets together ... the algorithm stops when there is only a single set remains

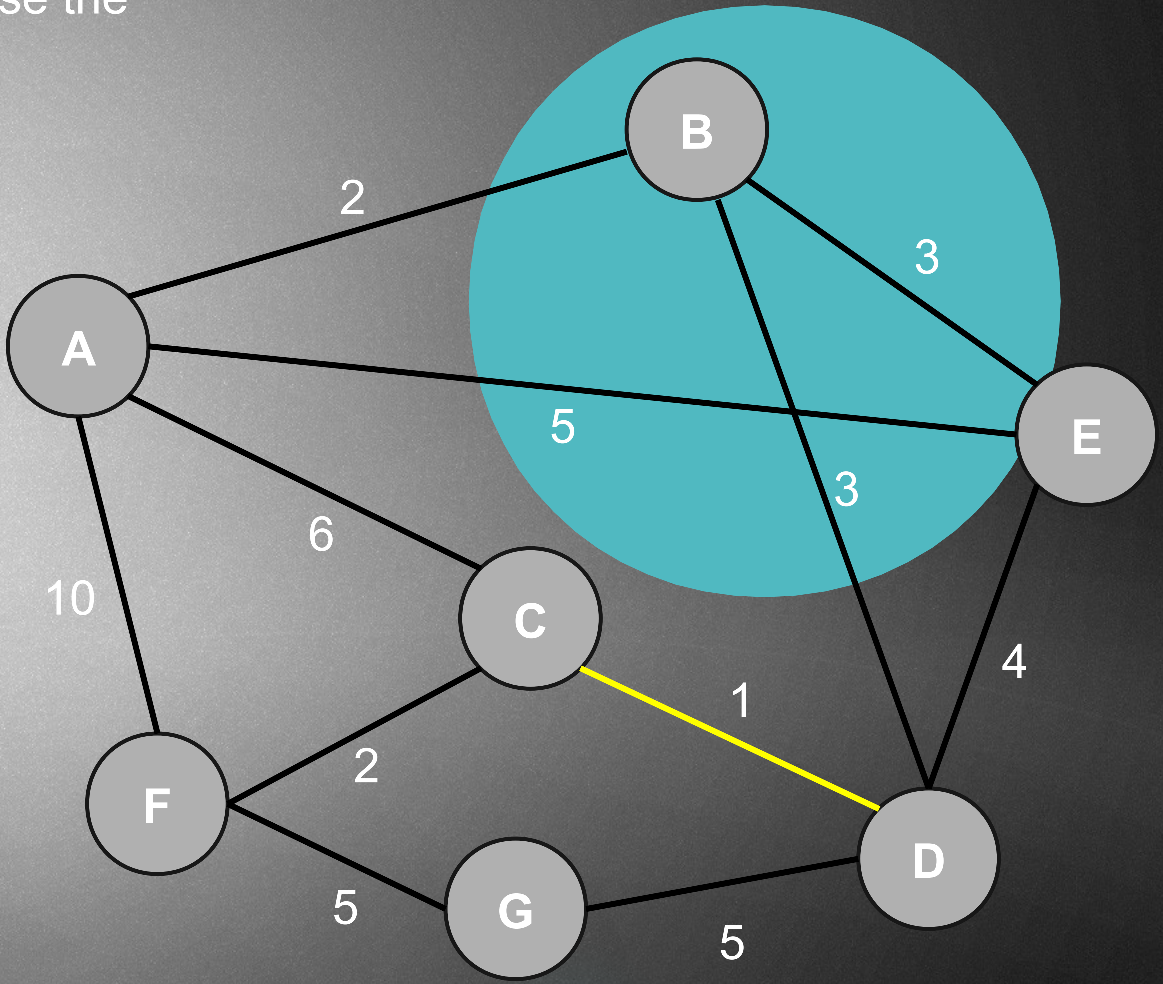




We have to sort the edges: 1, 2, 2, 3, 3, 4, 5, 5, 5, 6, 10

We can add the edge to the spanning tree because the the node are in distinct sets !!!





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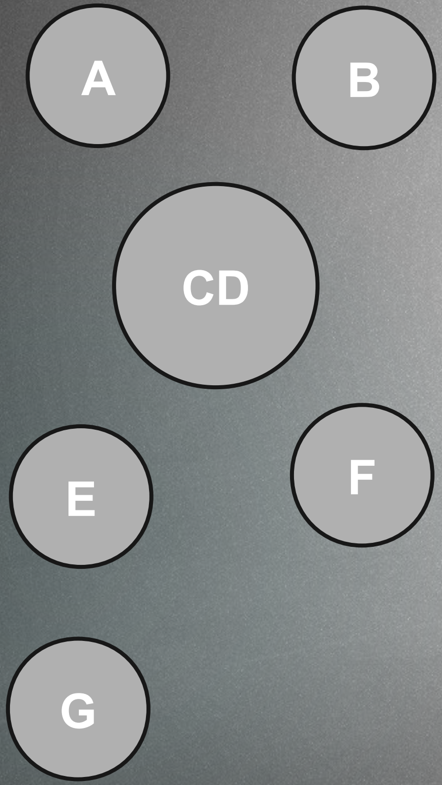

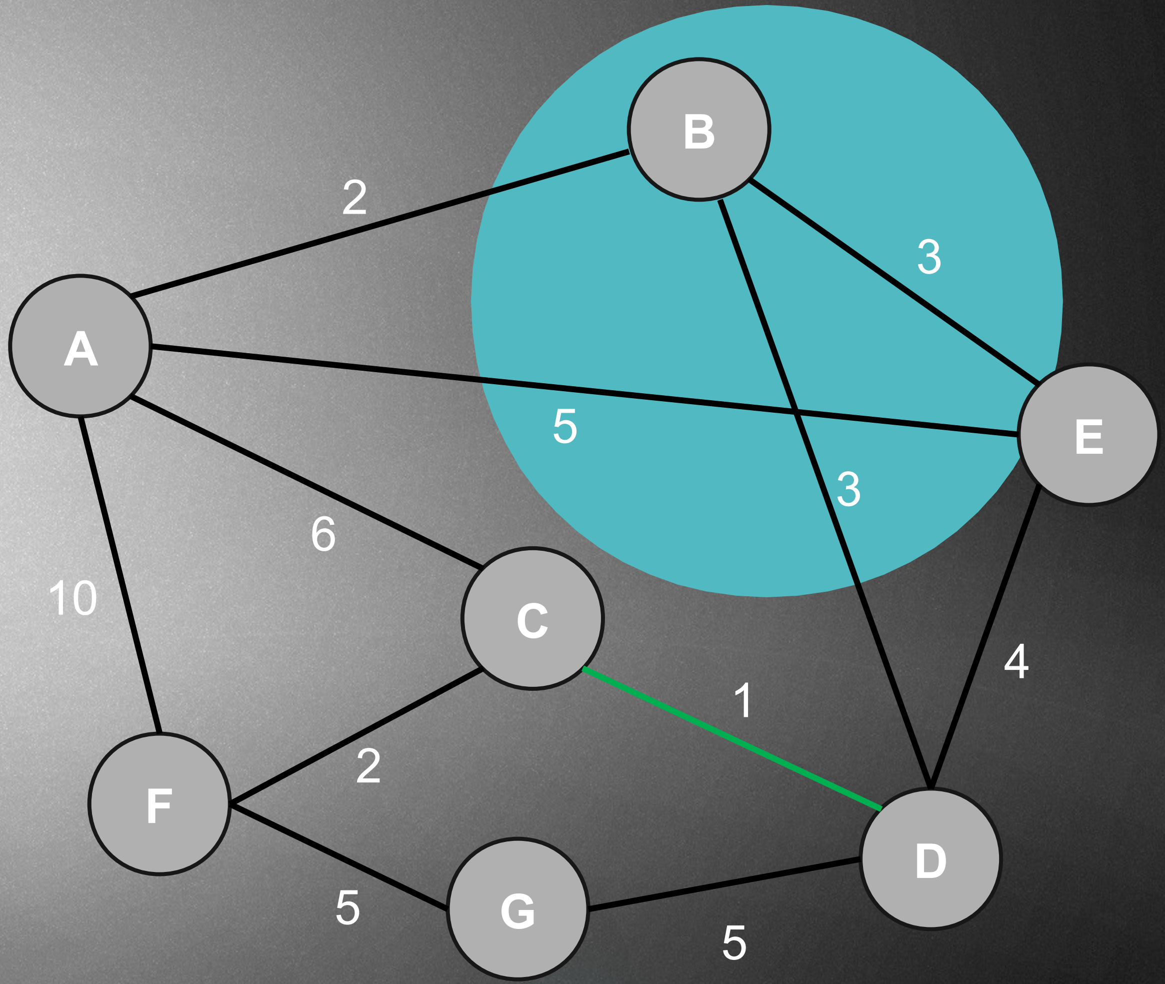

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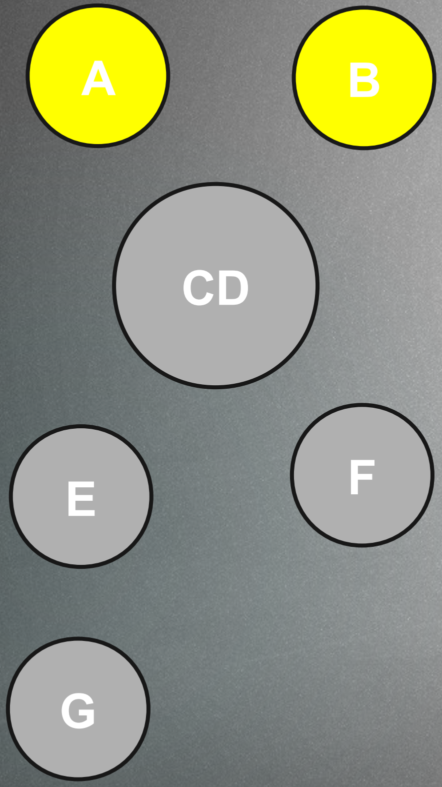

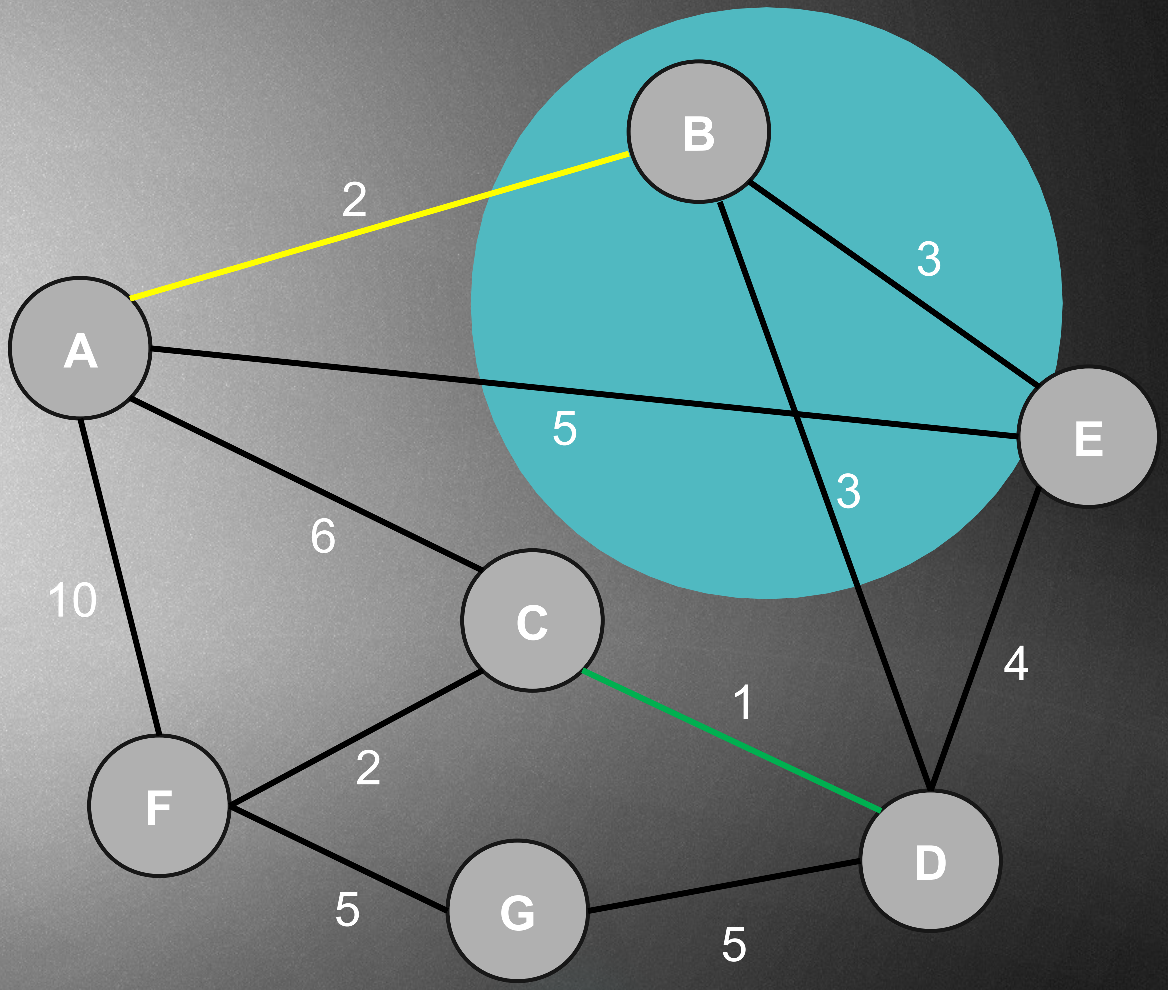

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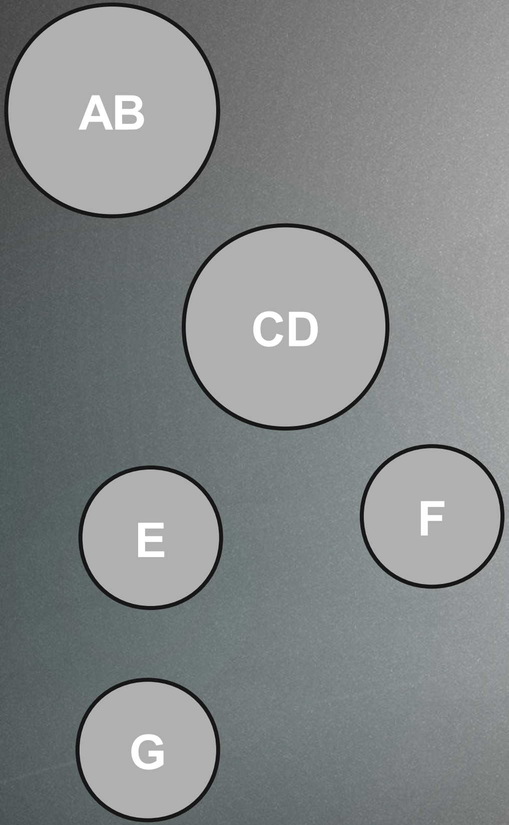

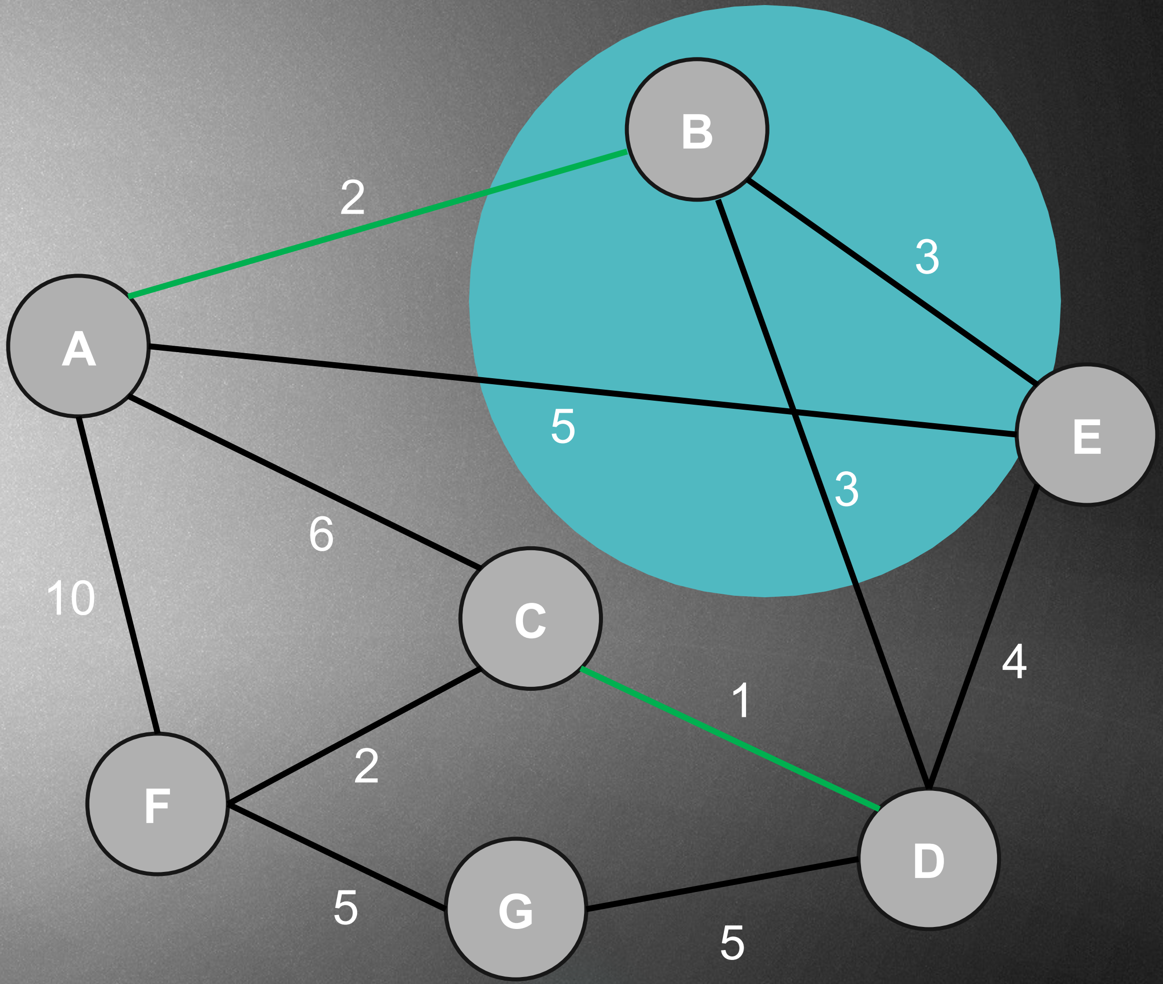

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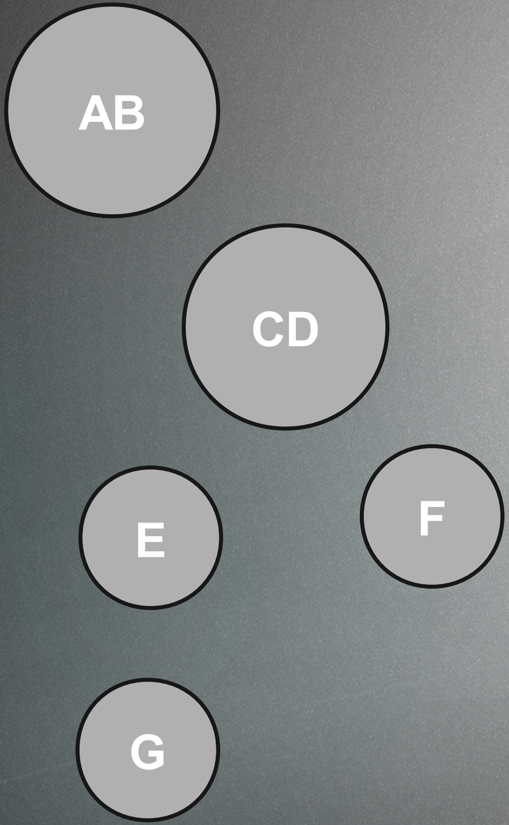

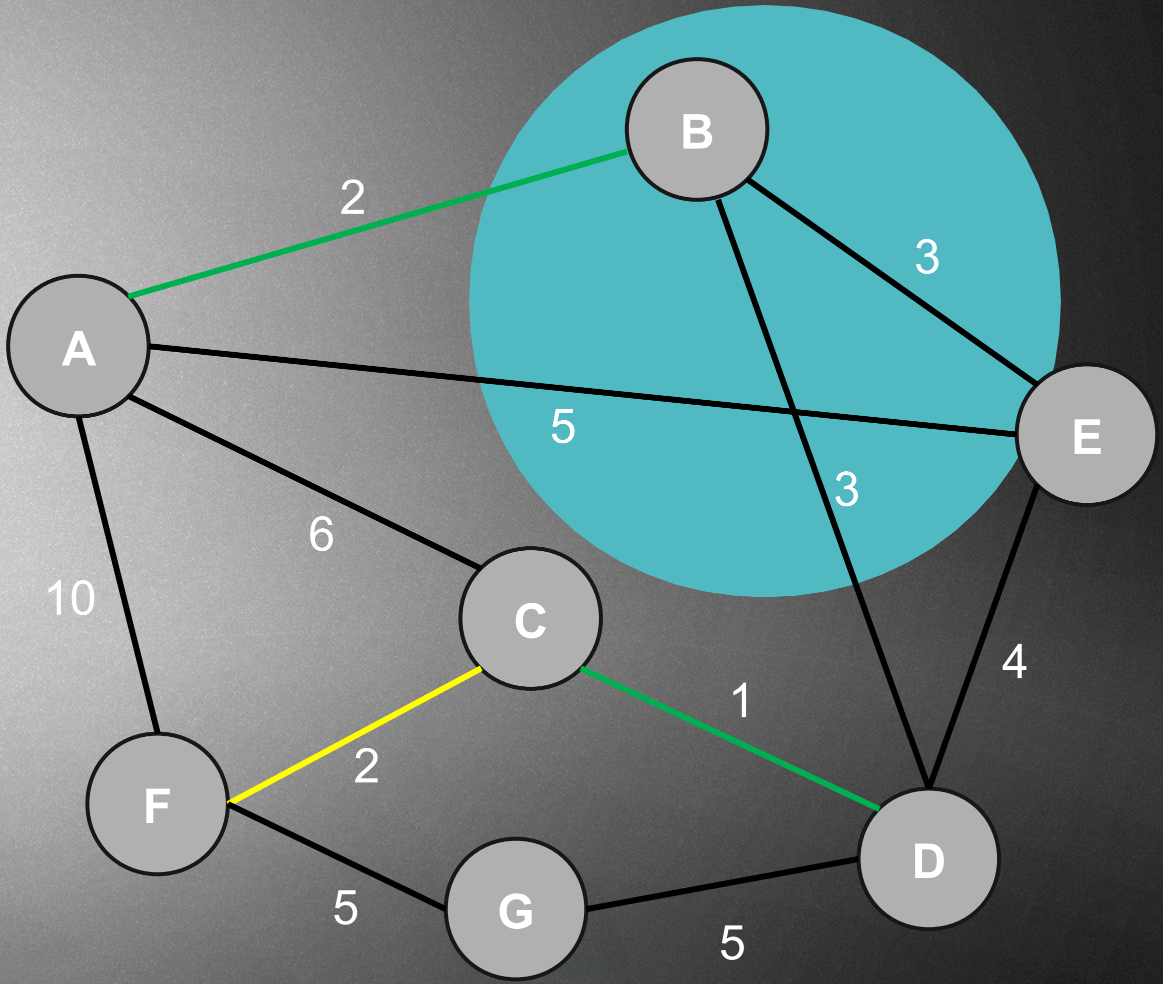

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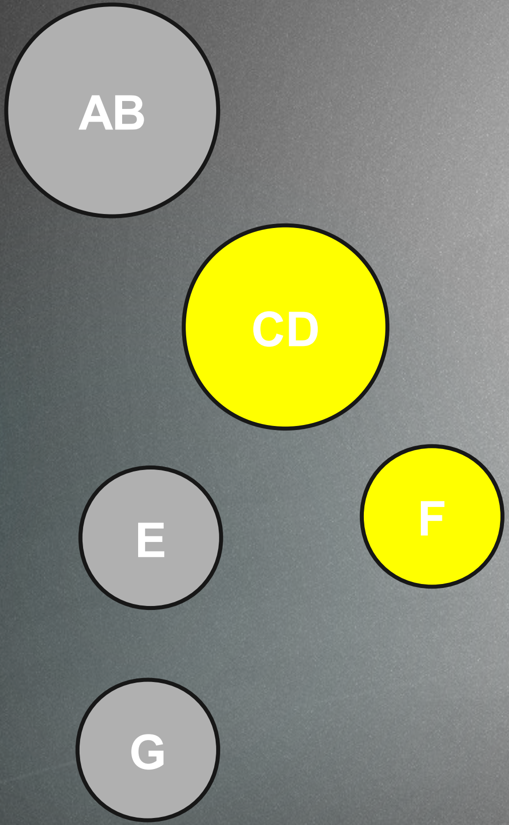

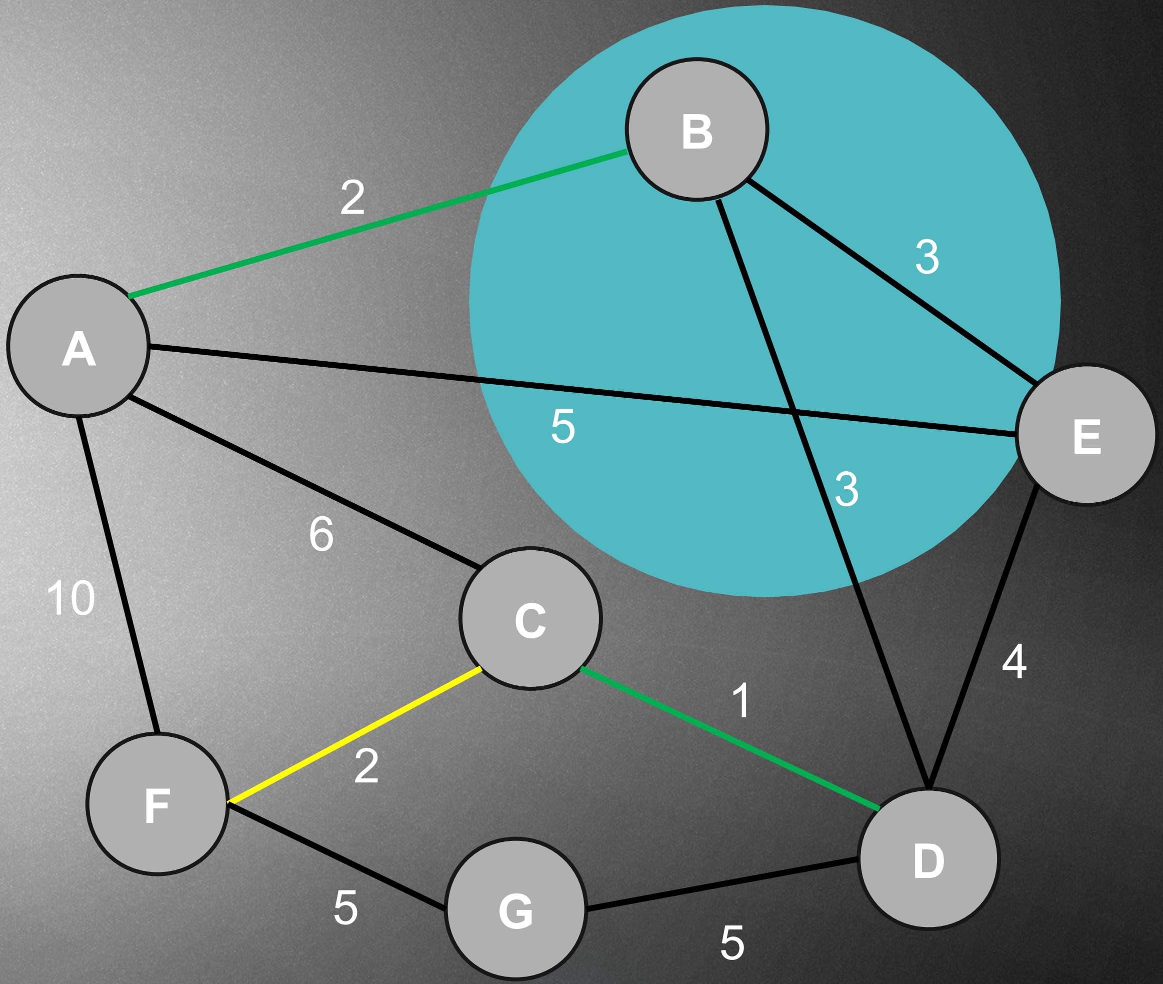

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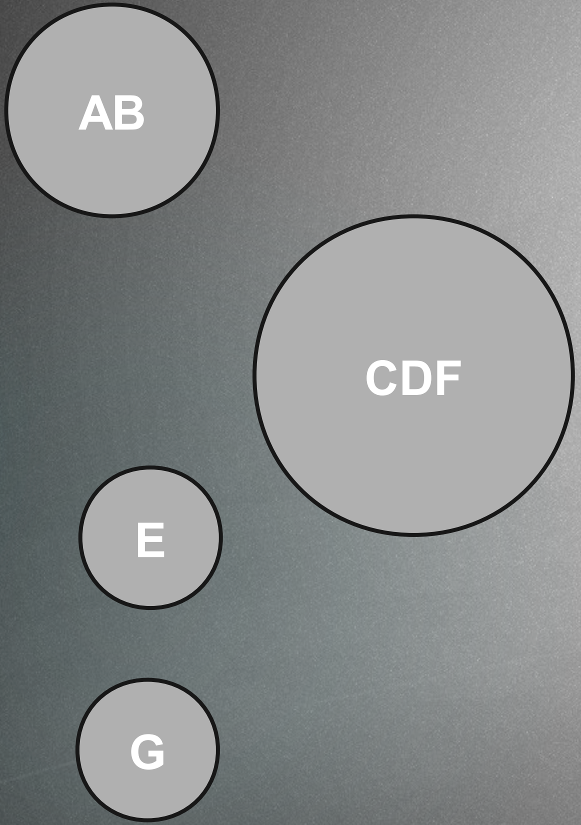

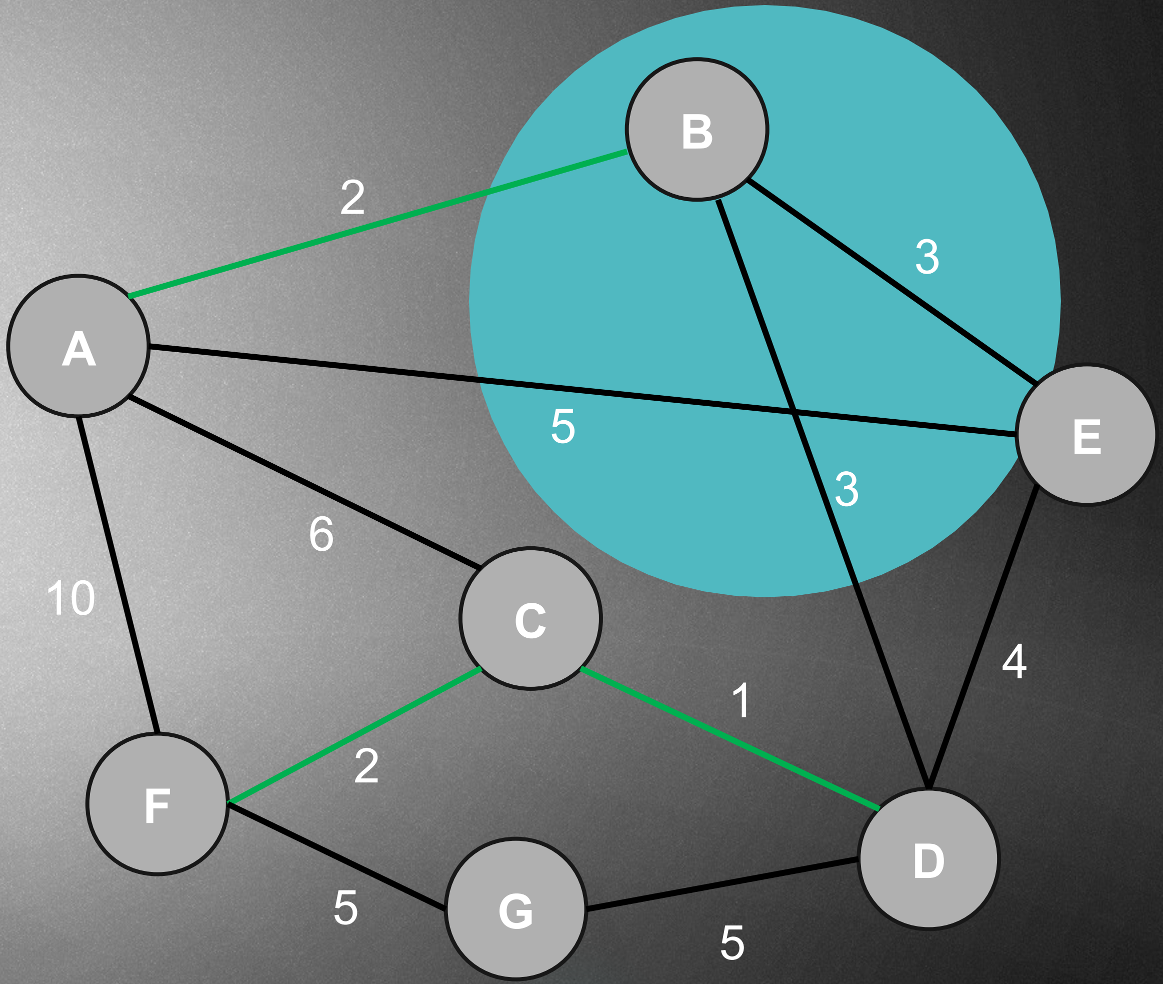

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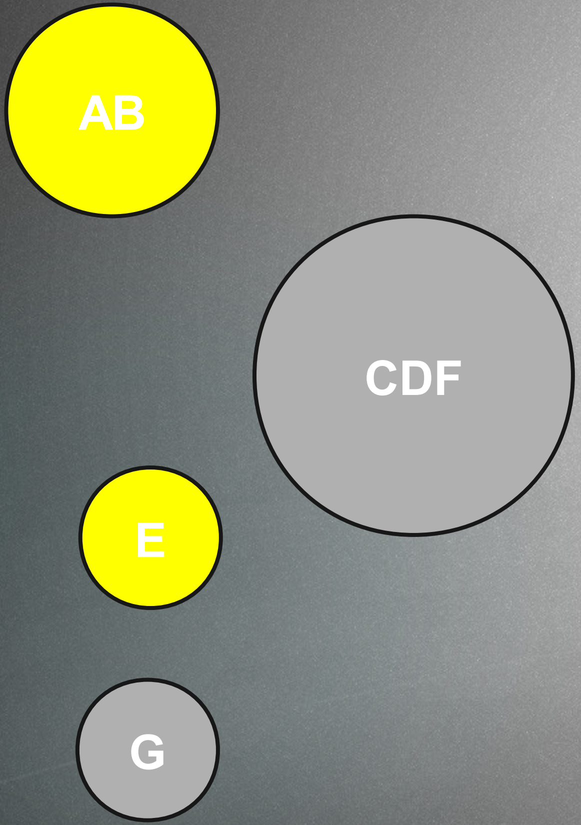

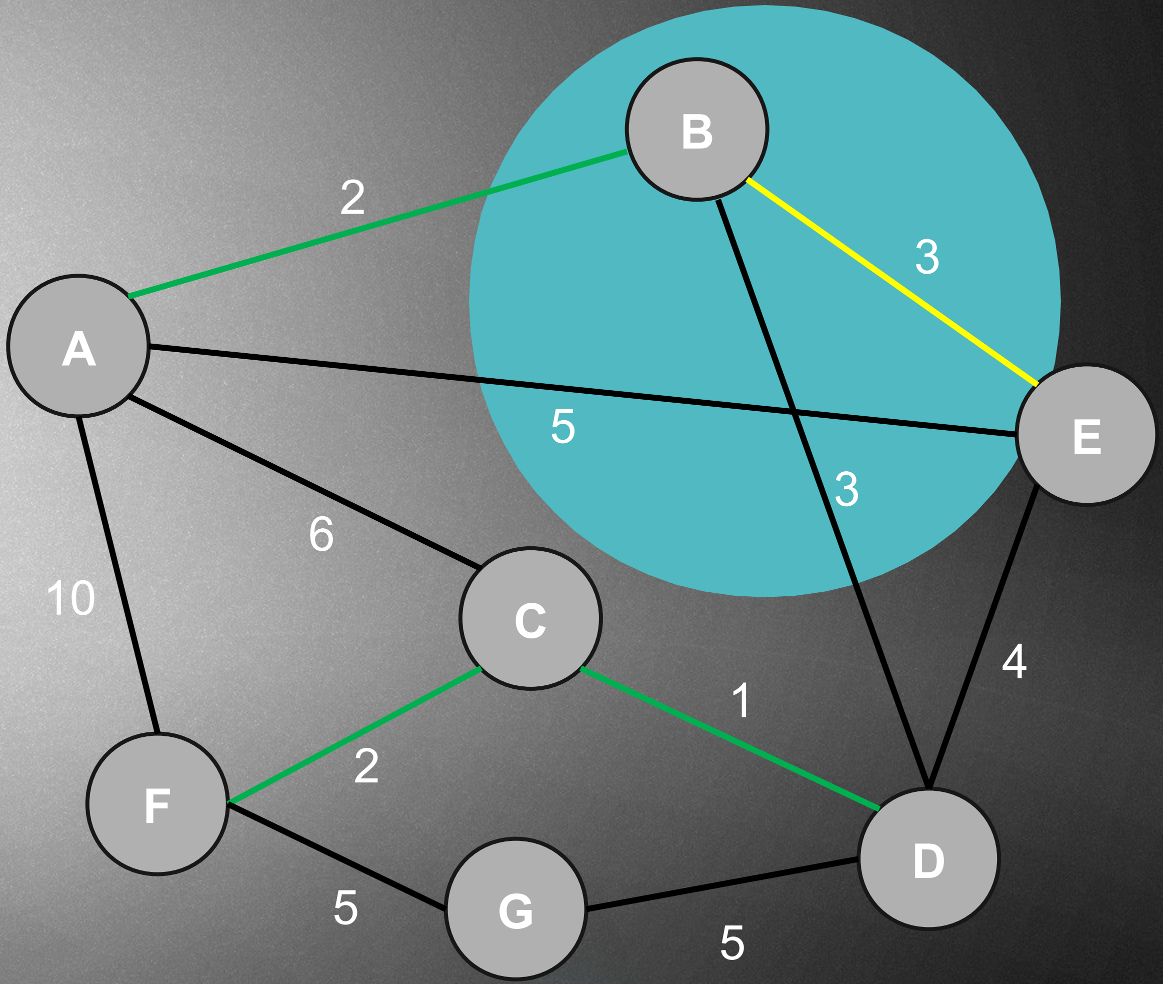

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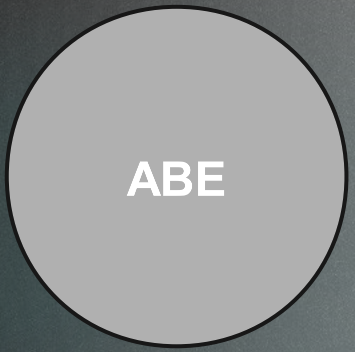

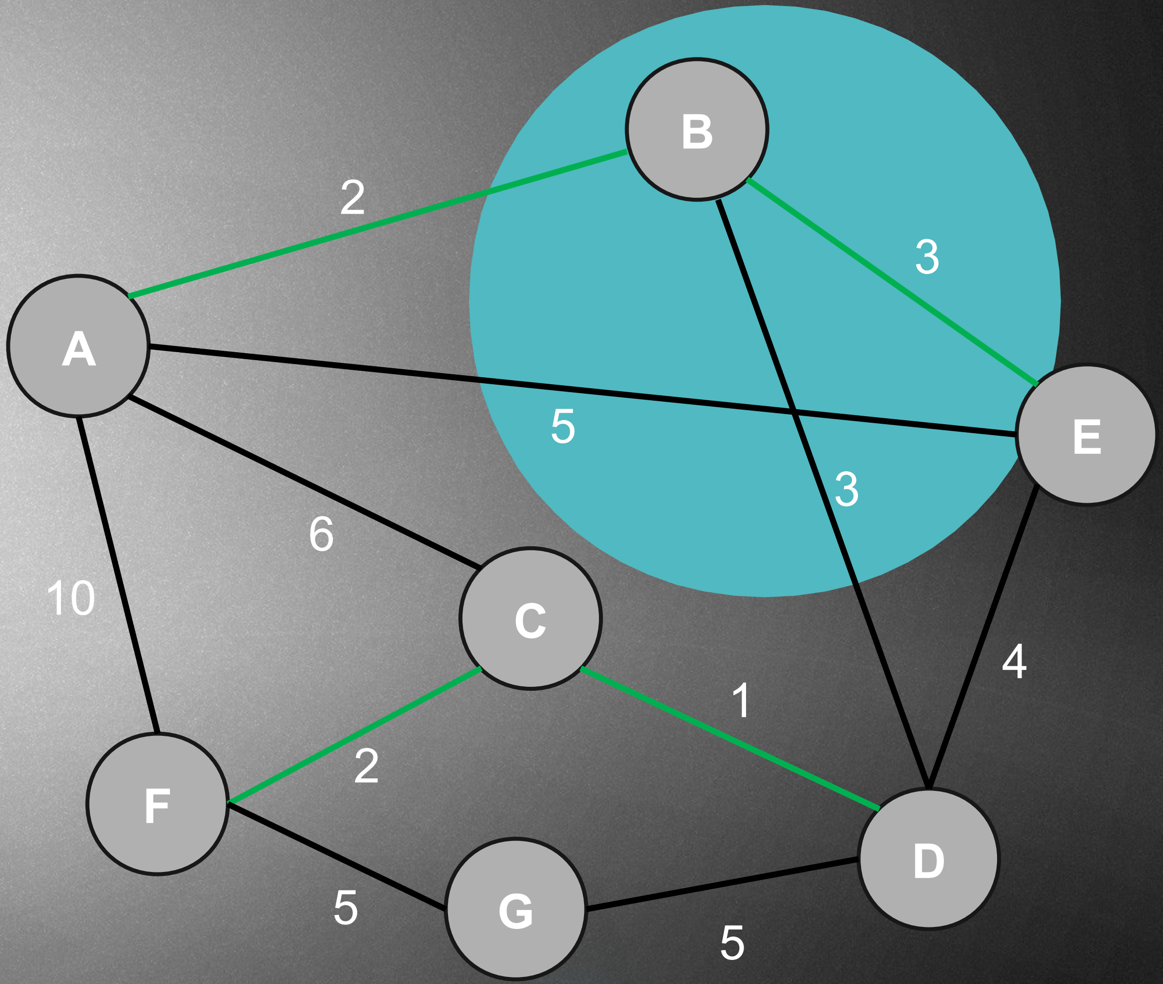

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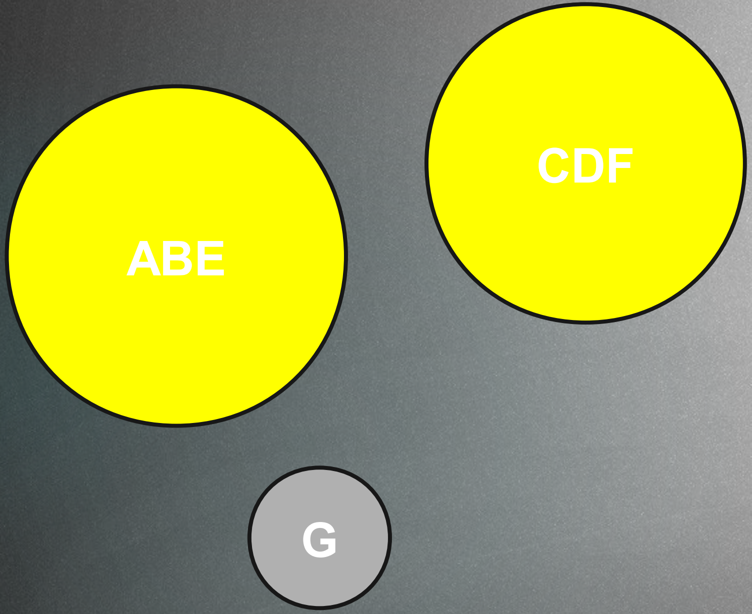

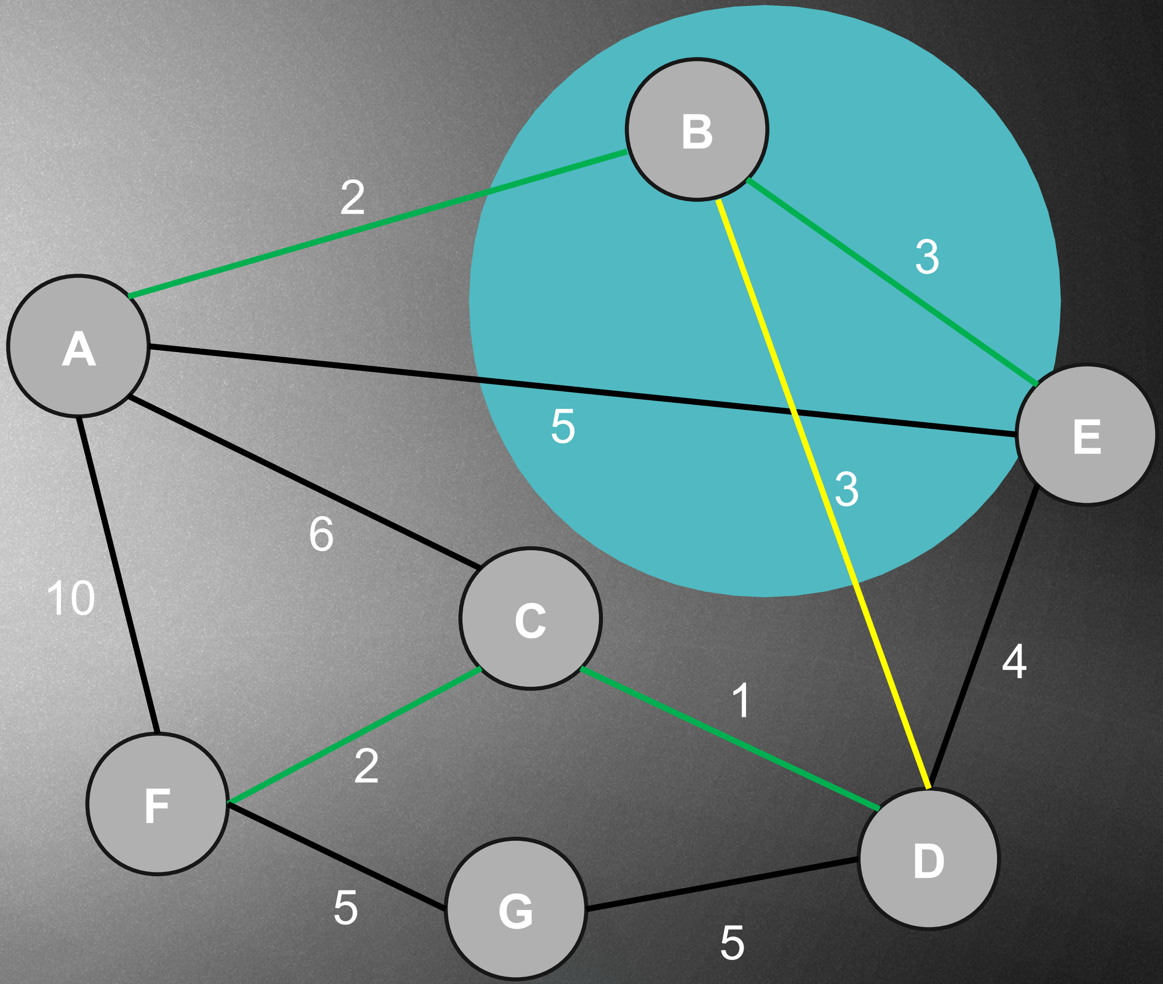

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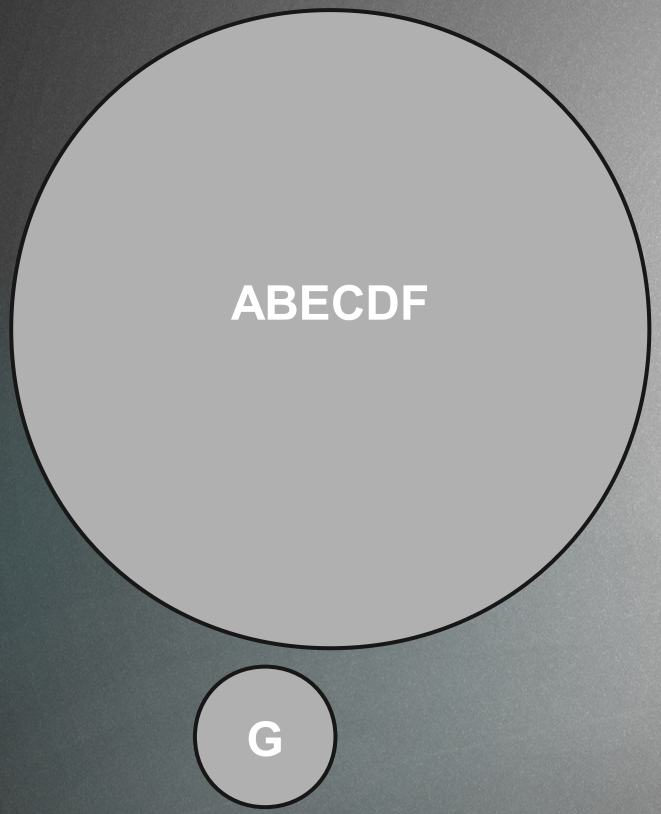

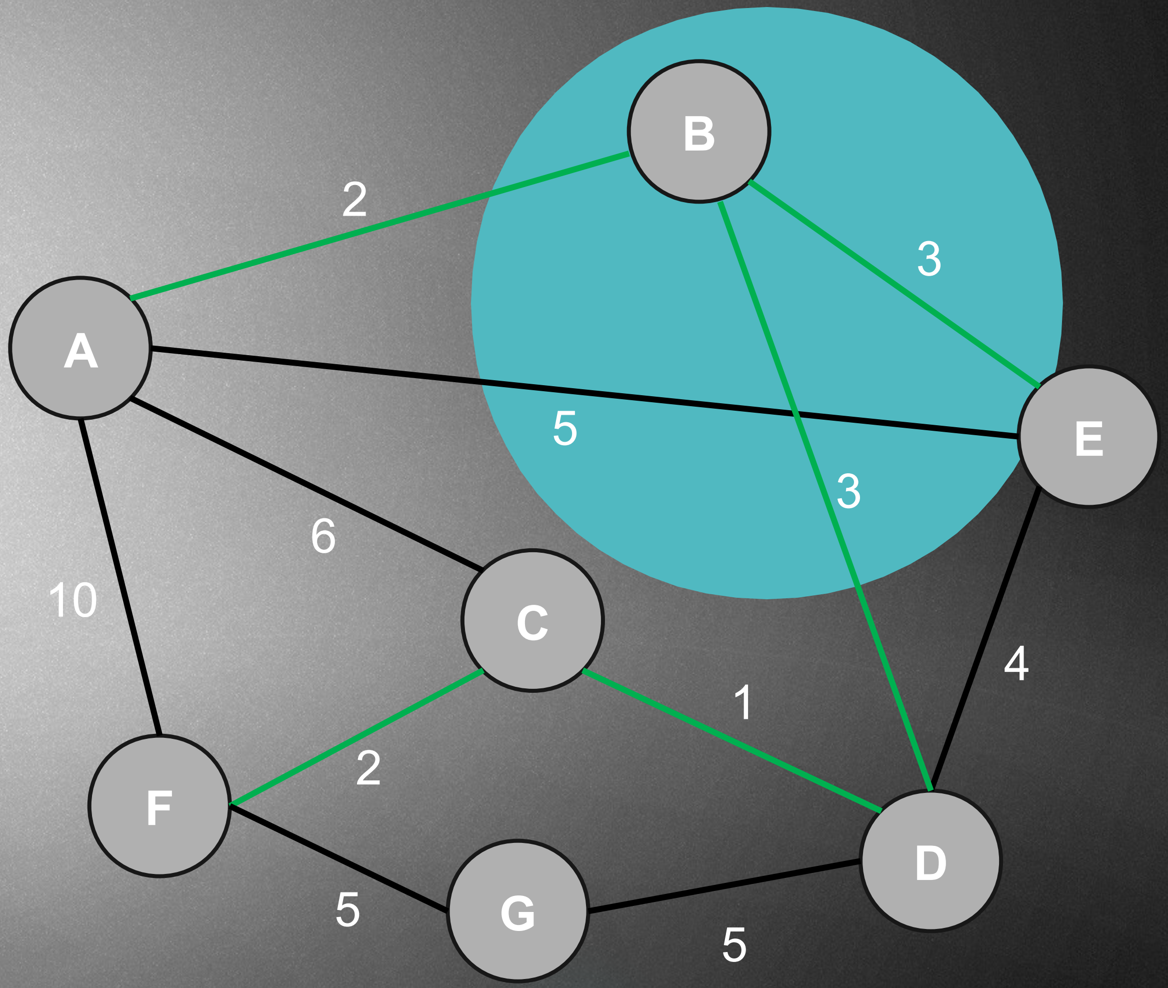

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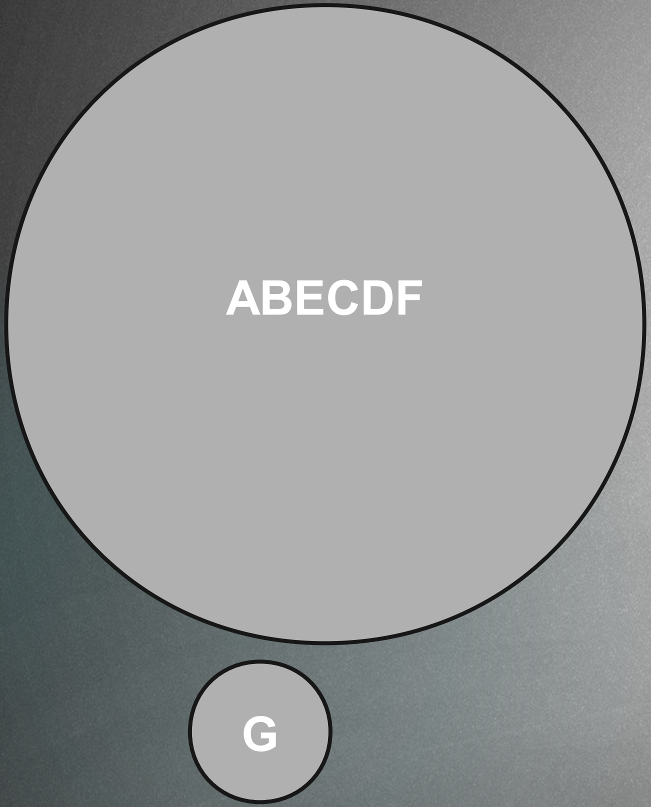

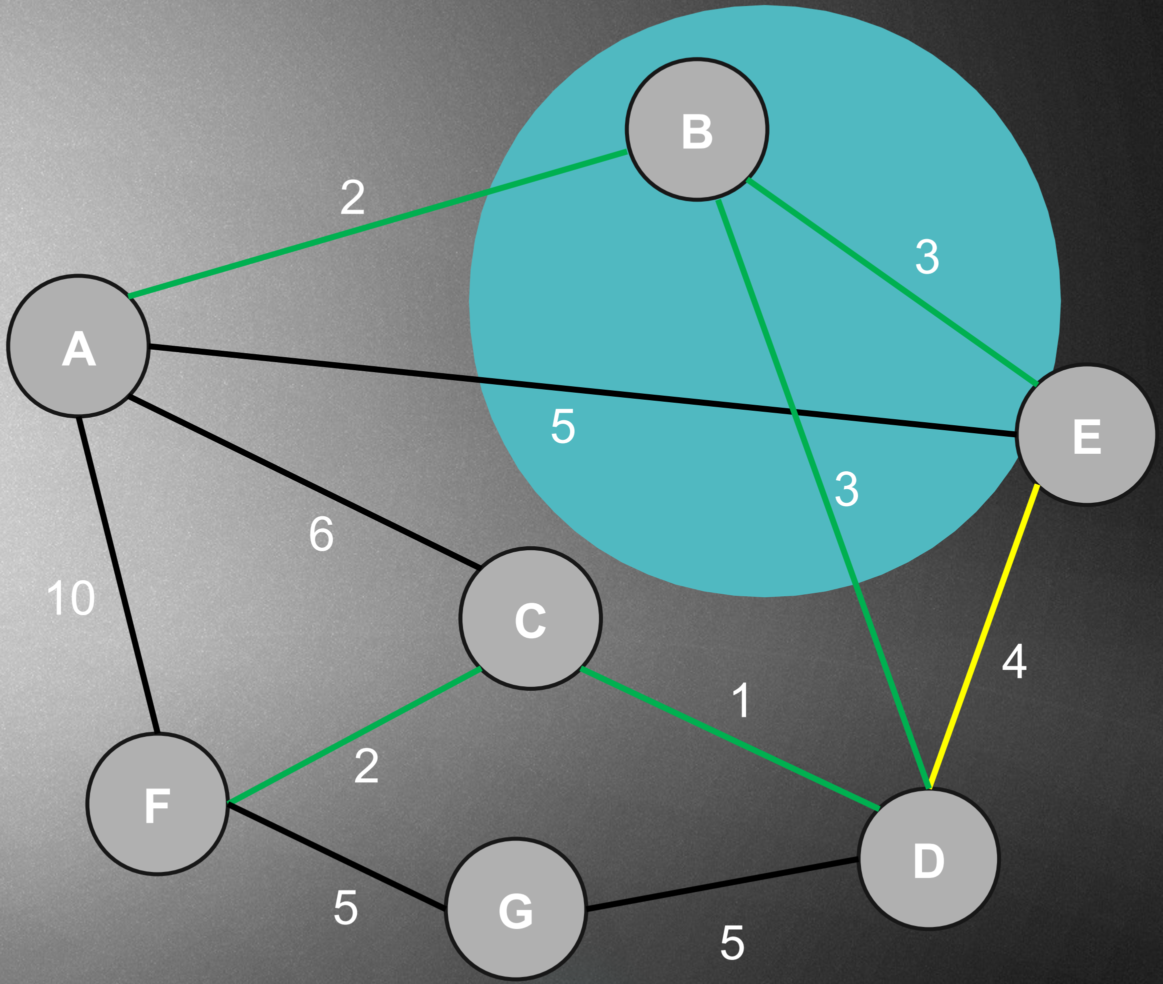

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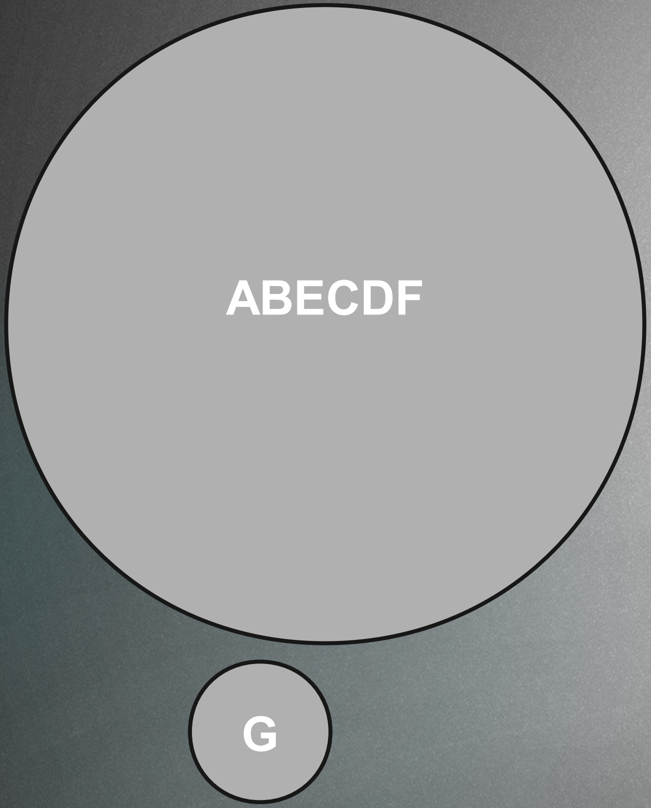

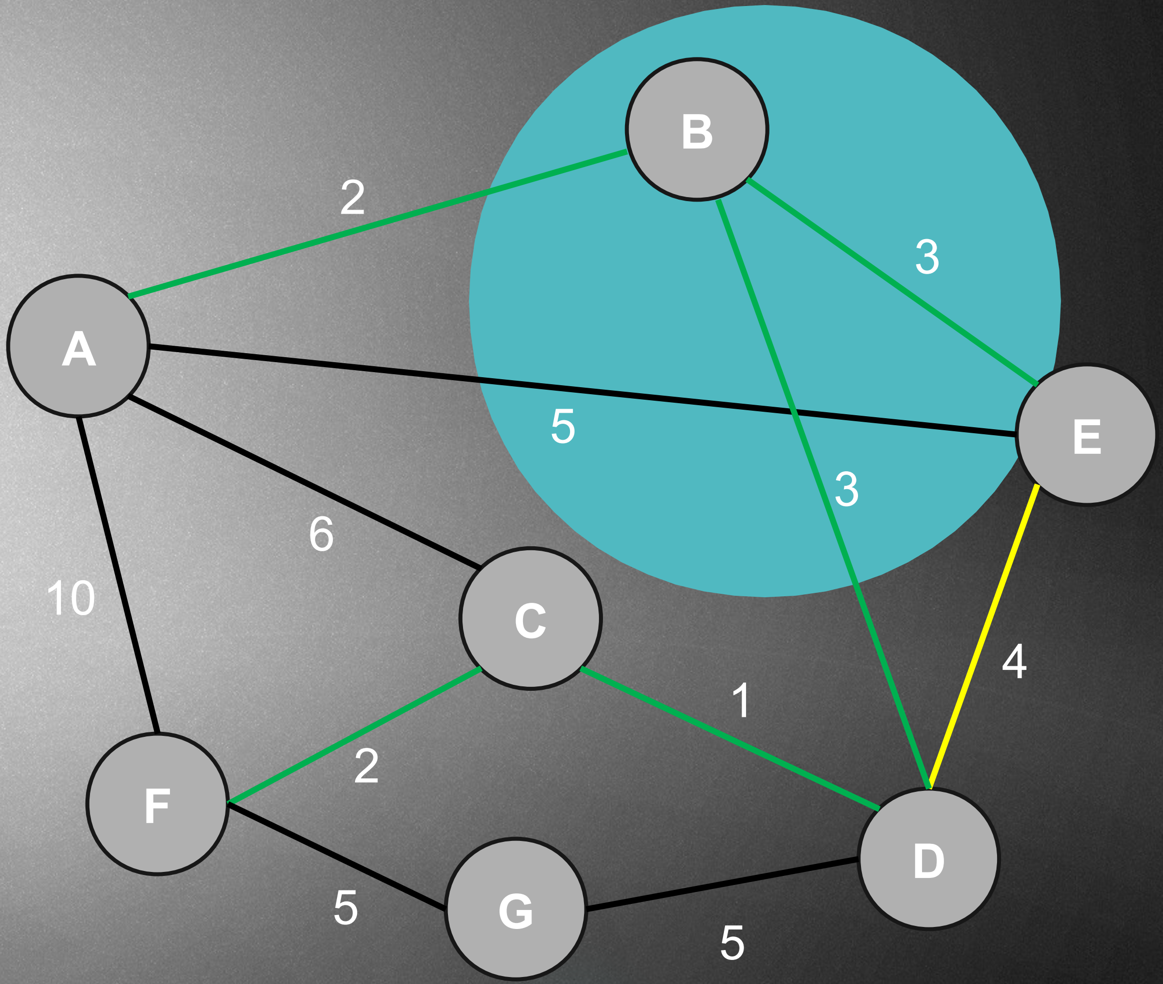

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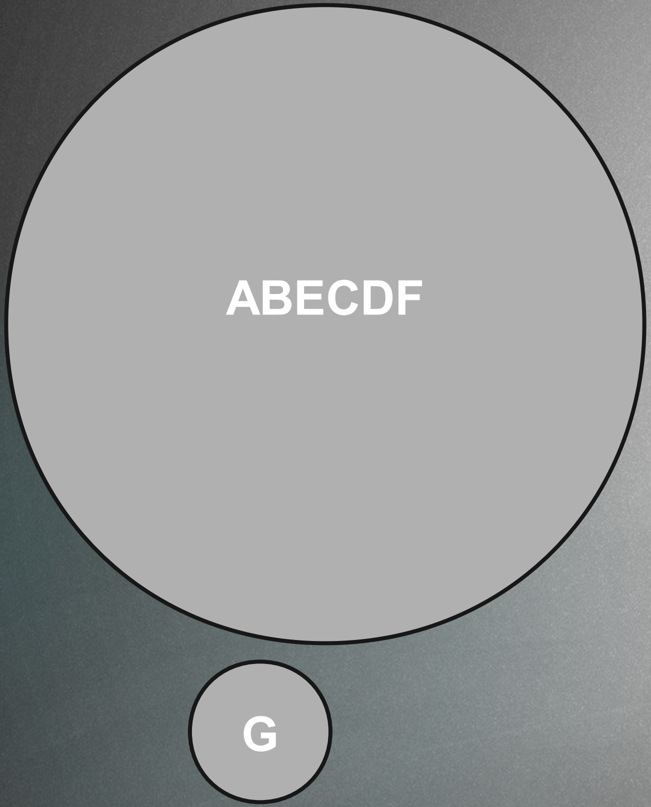

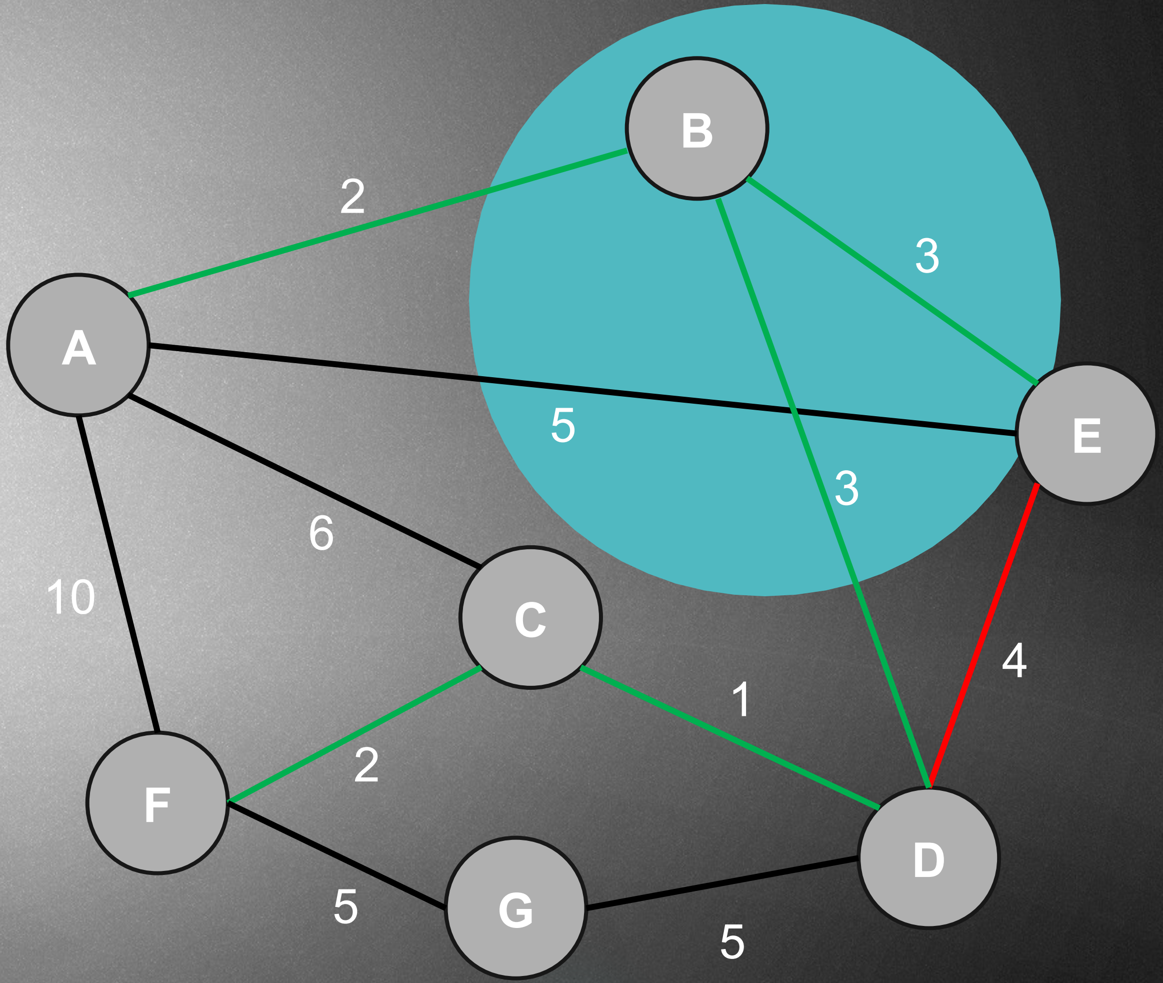

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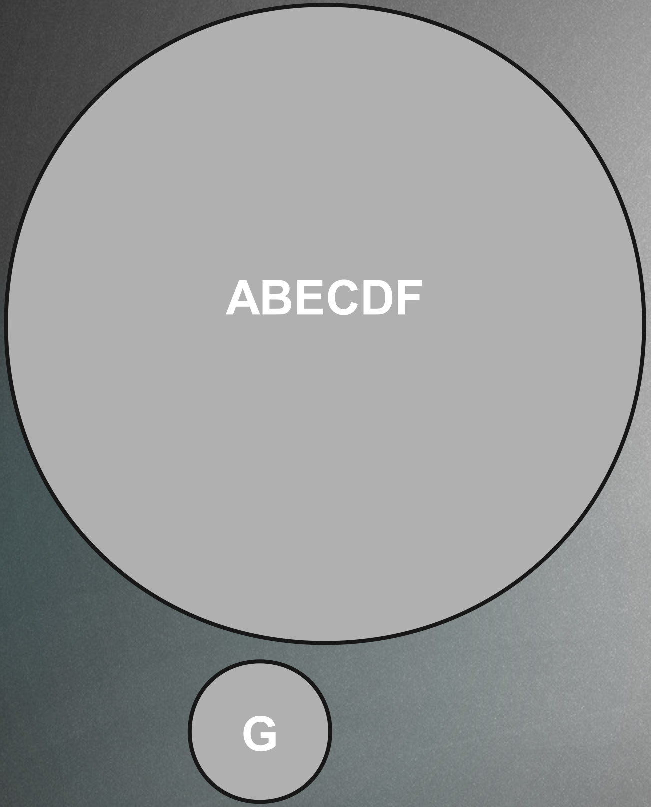

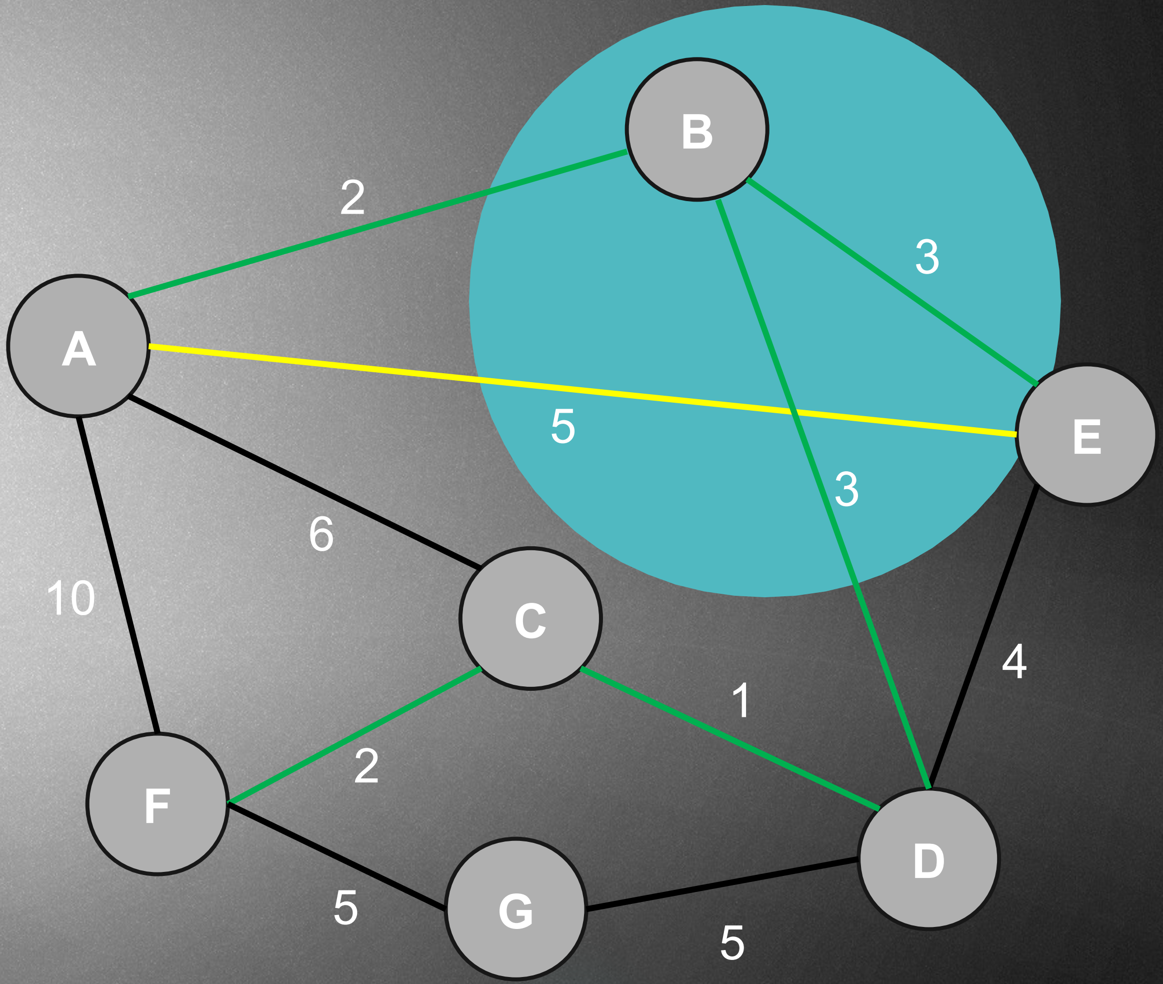

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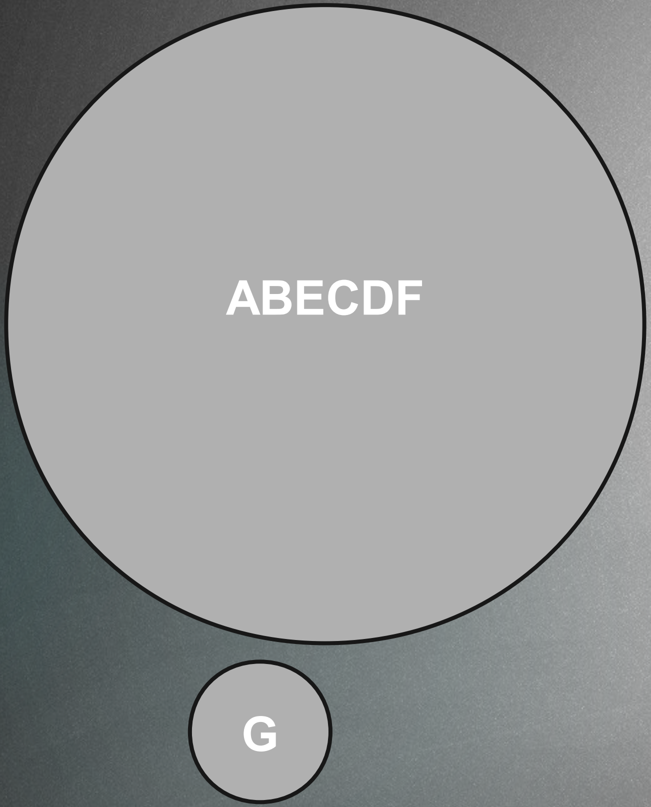

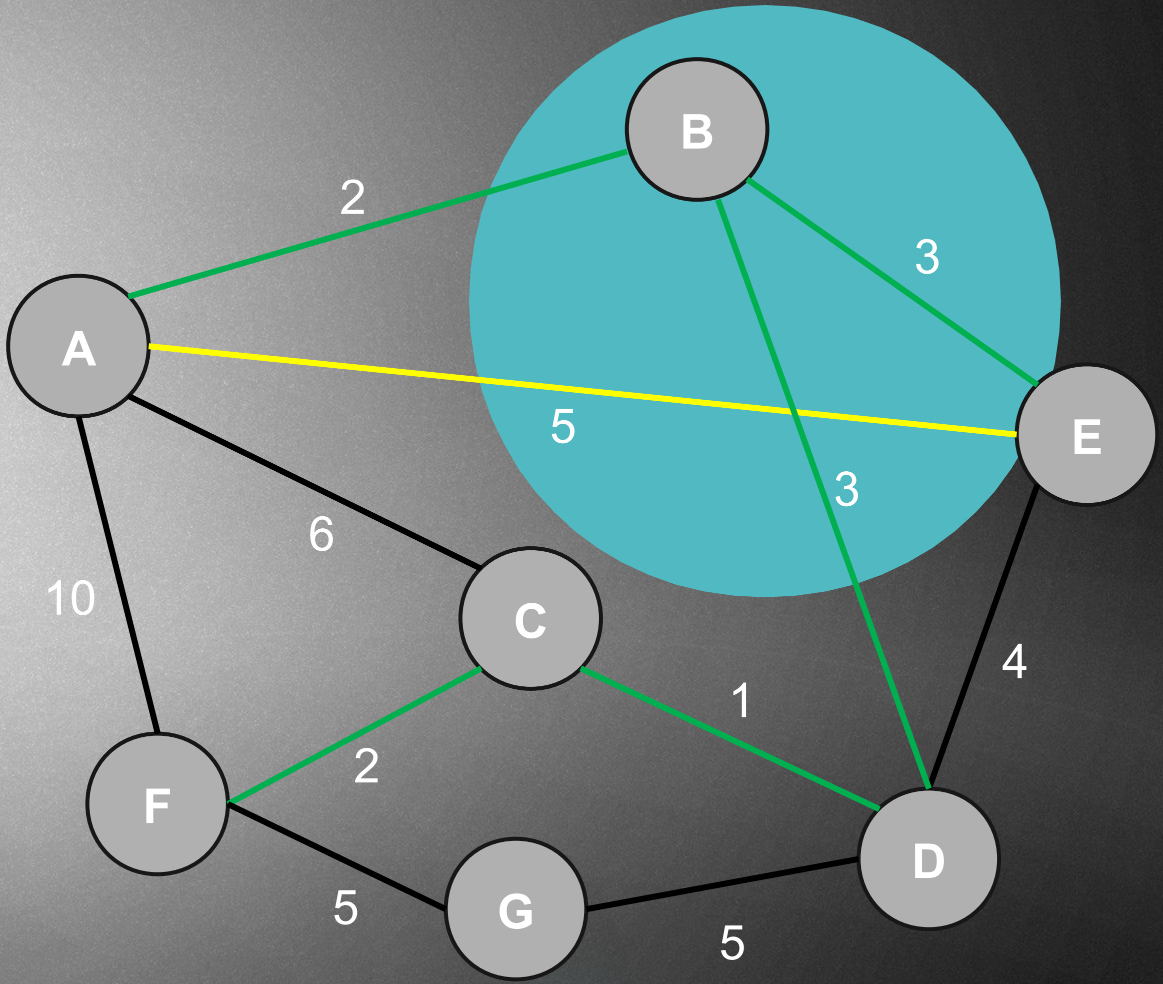

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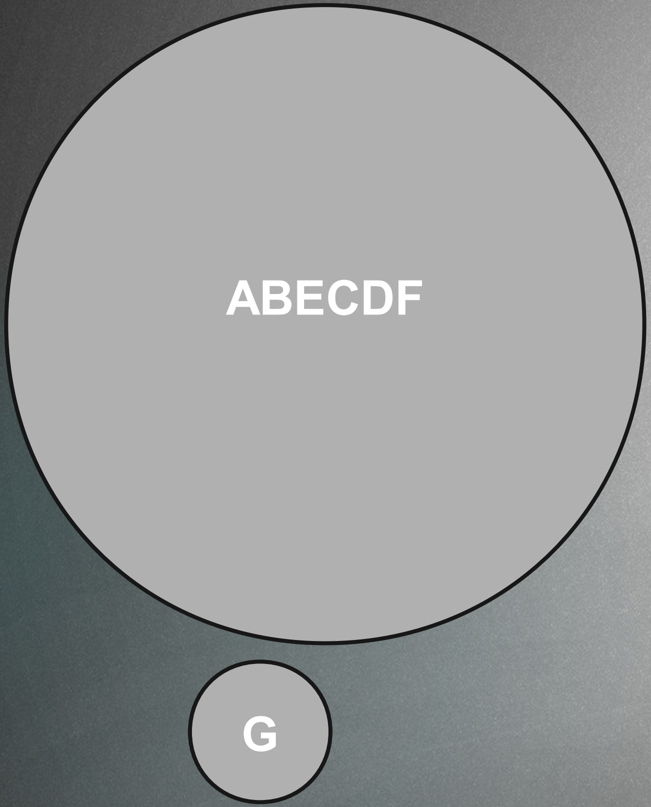

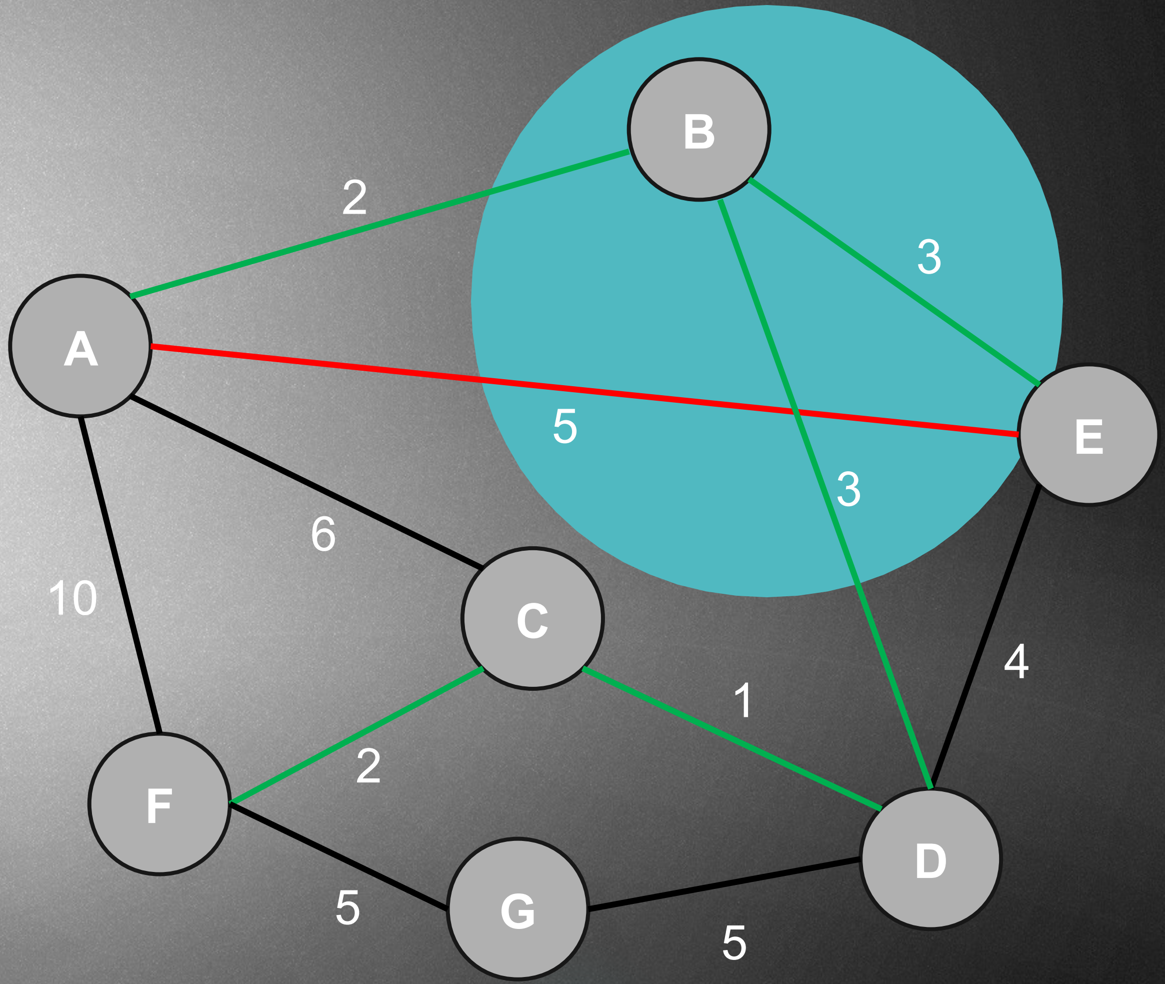

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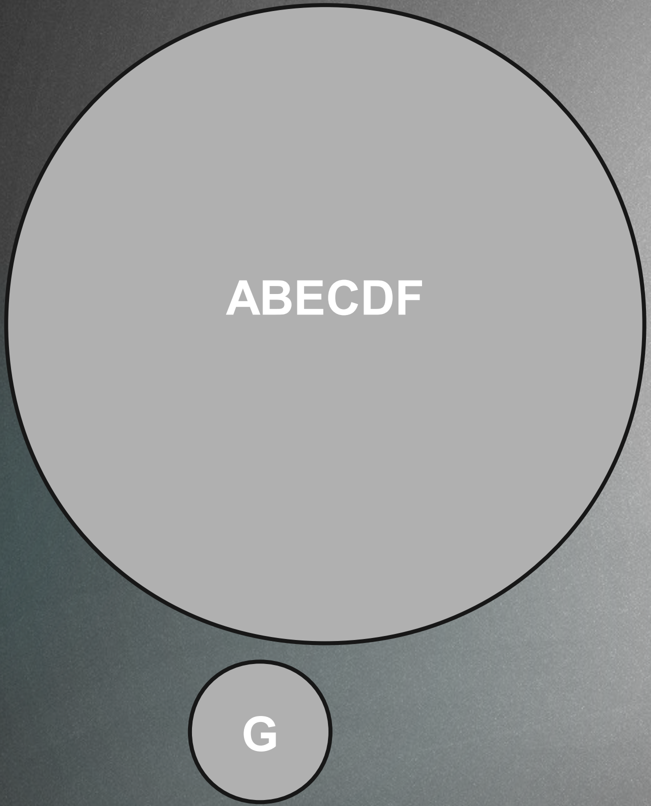

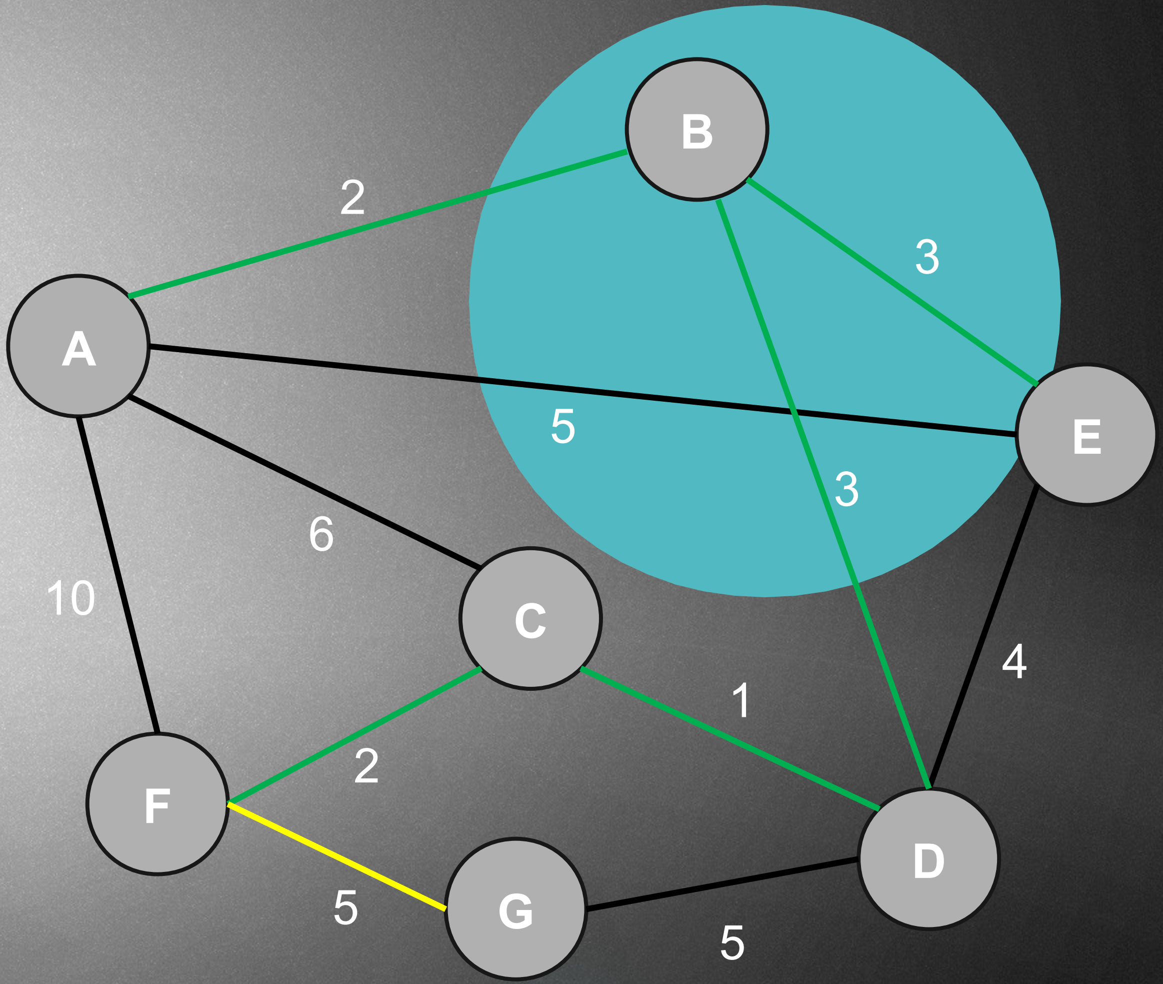

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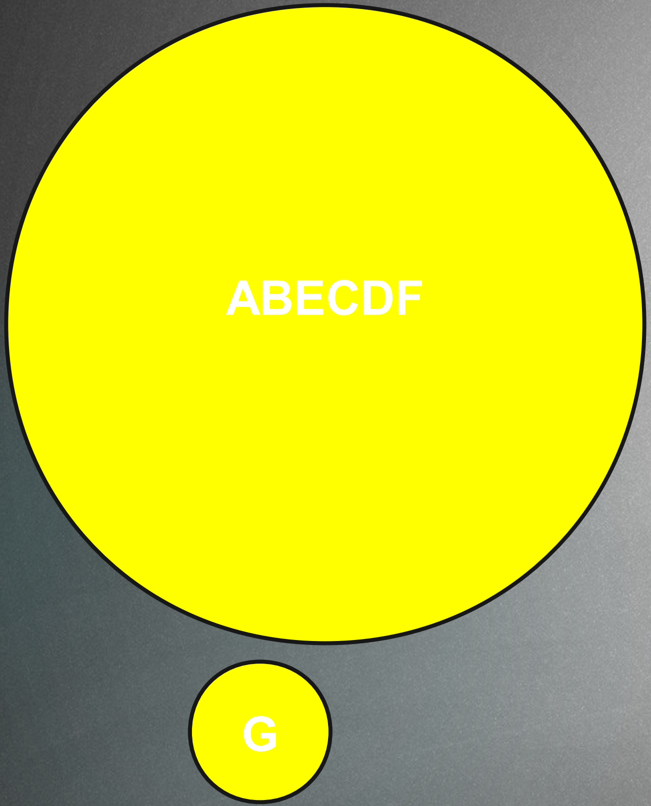

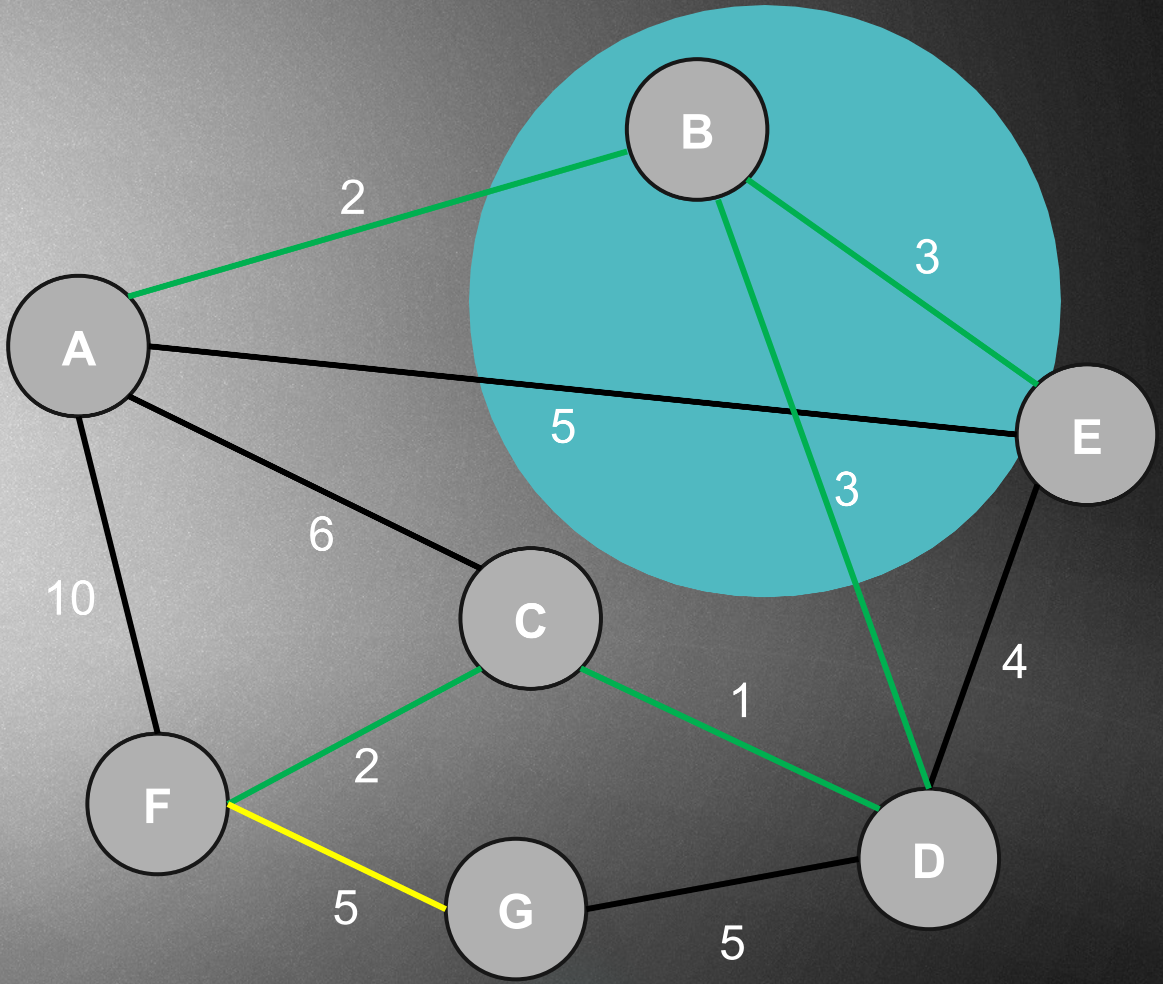

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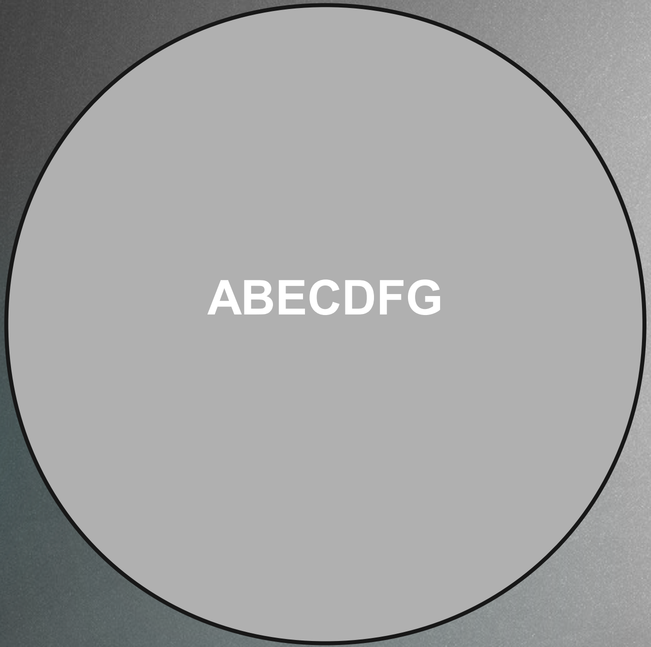

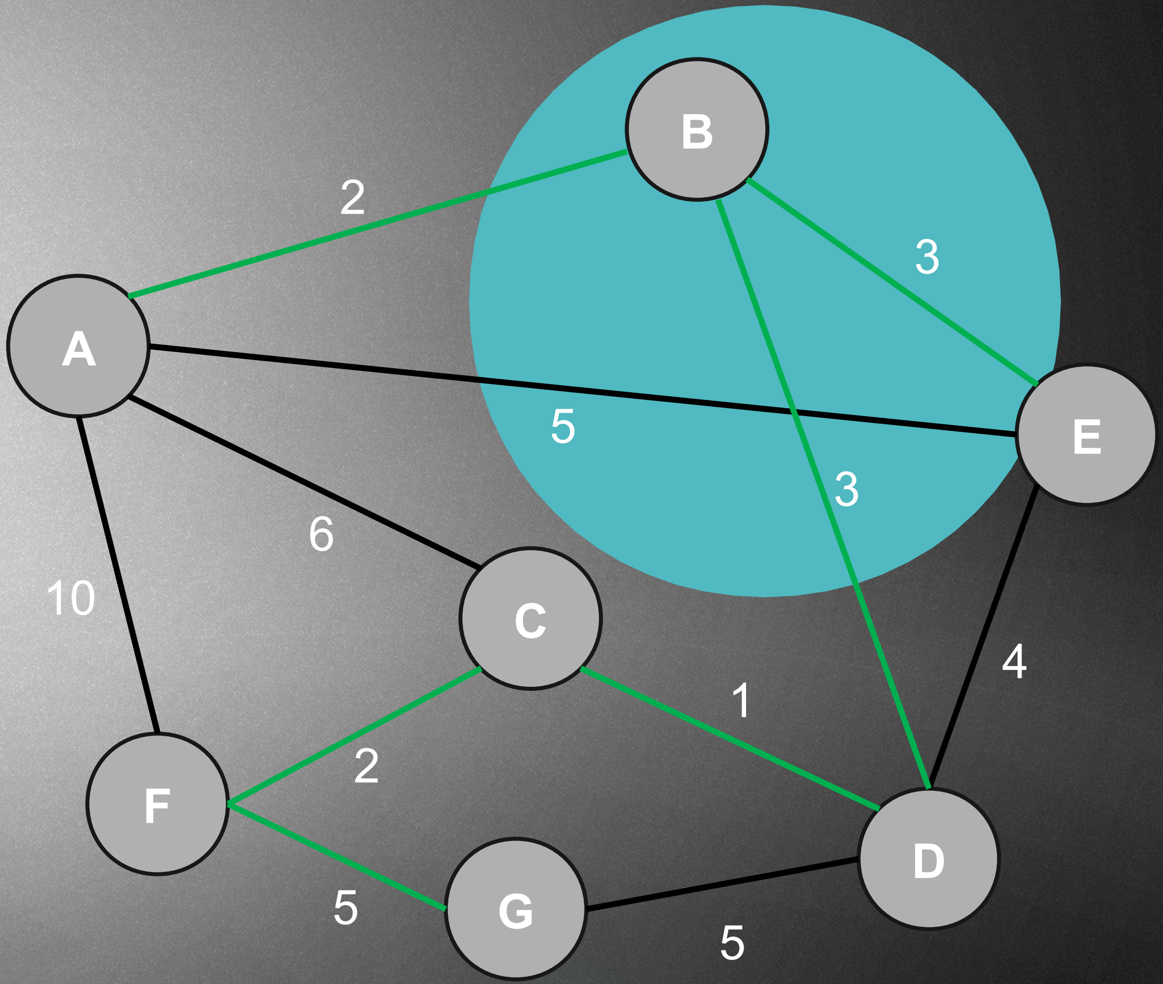

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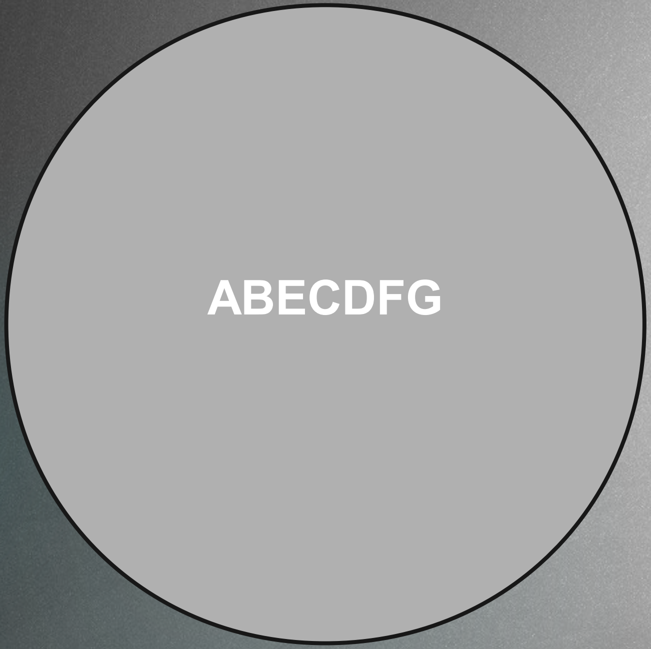
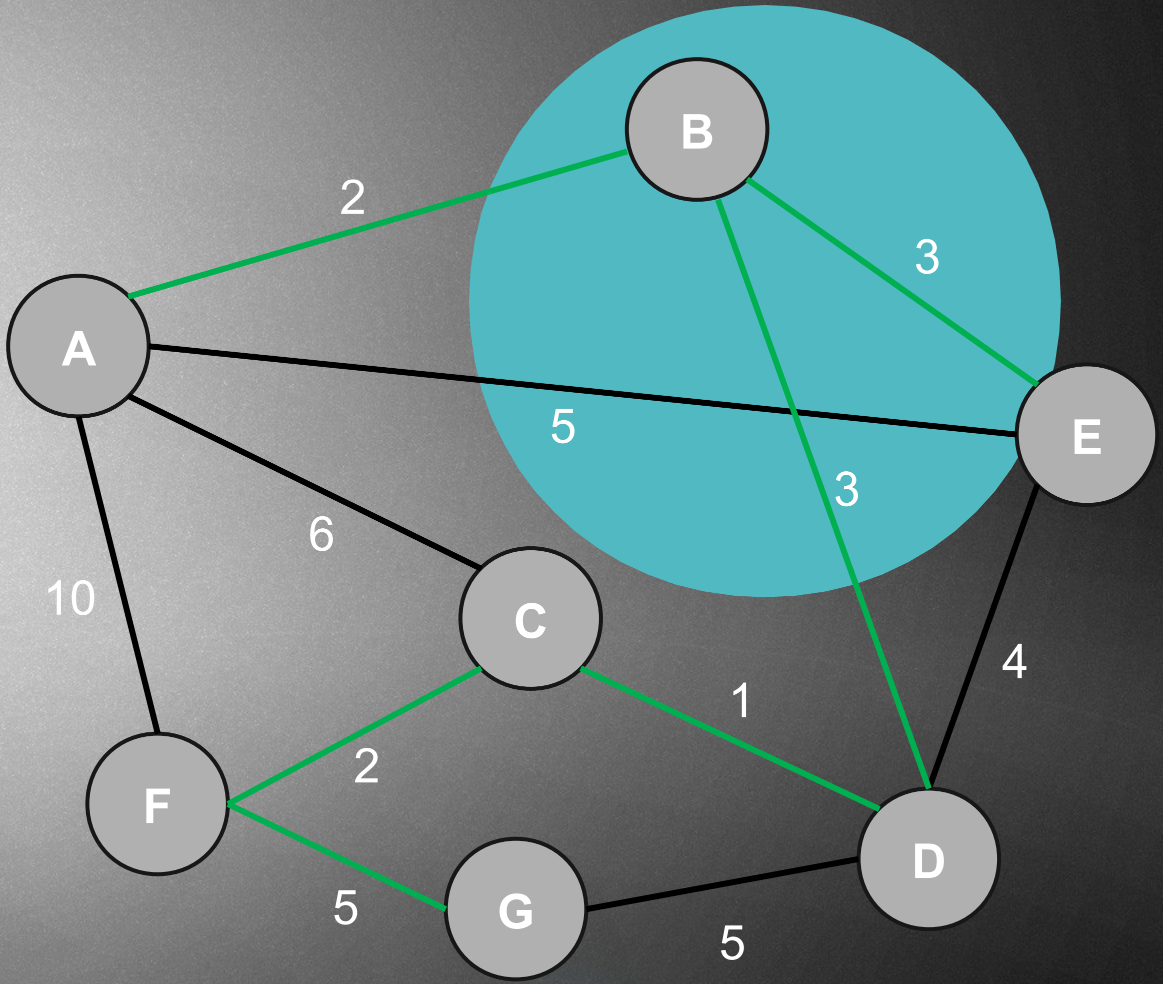




We have a single set so it is the end of the algorithm,
Minimal cost is: 2+3+3+1+2+5=16

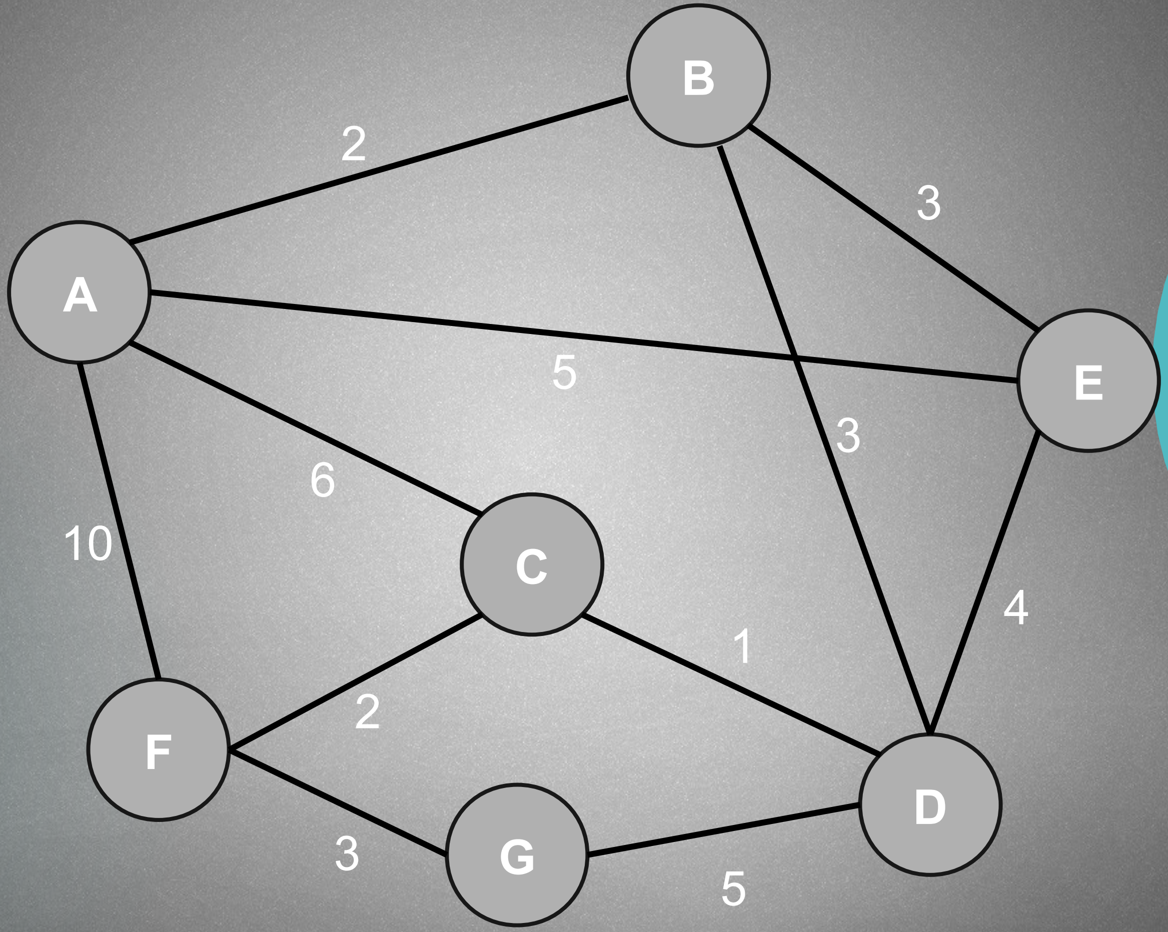
SPANNING TREES

PRIMS algorithm

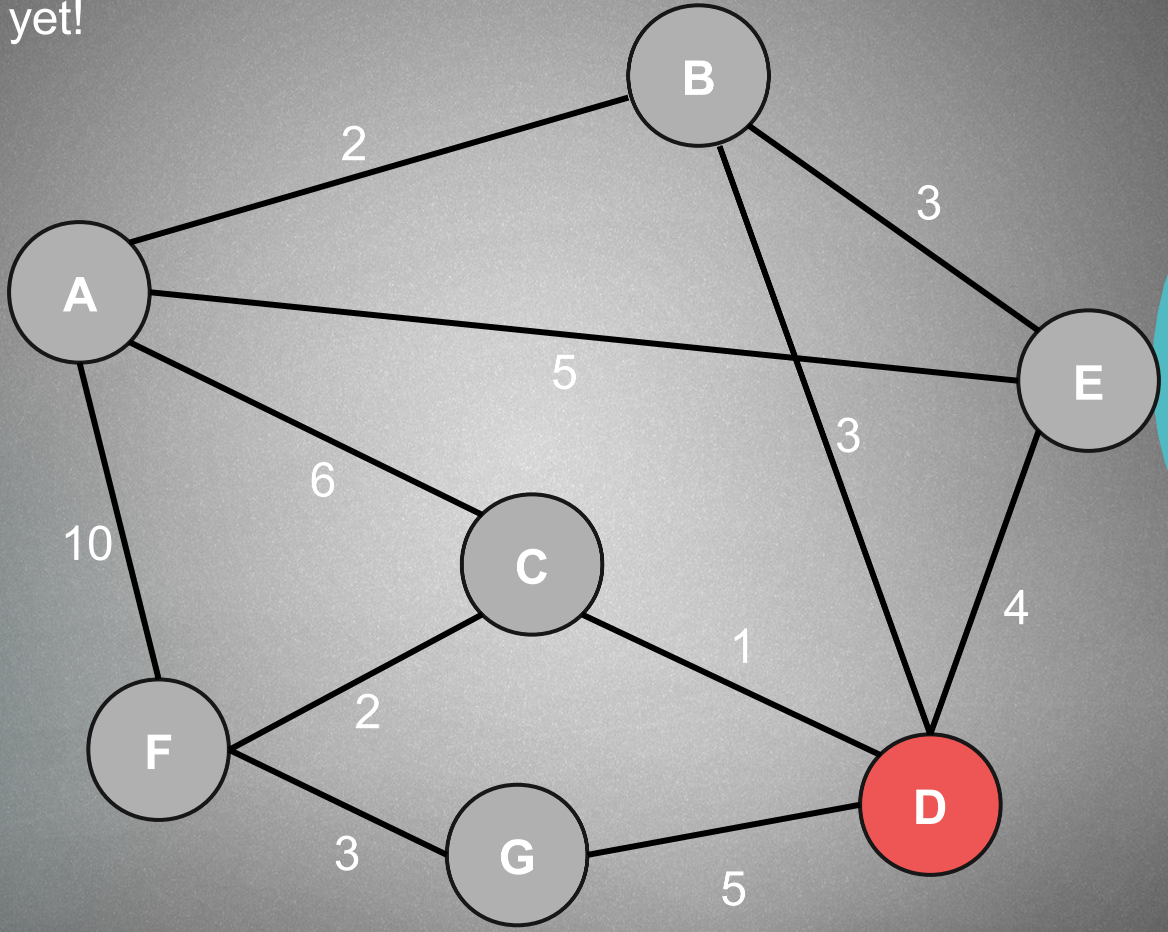
Prim-Jarnik algorithm

- In Kruskal implementation we build the spanning tree separately, adding the smallest edge to the spanning tree if there is no cycle
- In Prims algorithm we build the spanning tree from a given vertex, adding the smallest edge to the MST
- Kruskal → edge based   Prims → vertex based !!!
- There are two implementations: lazy and eager
- Lazy implementation: add the new neighbour edges to the heap without deleting its content
- Eager implementation: we keep updating the heap if the distance from a vertex to the MST has changed
- Average running time: **O(E*logE)** but we need additional memory space **O(E)**
- Worst case: **O(E*logV)**

Start at a random vertex!



We consider edges to vertexes
we have not visited yet!



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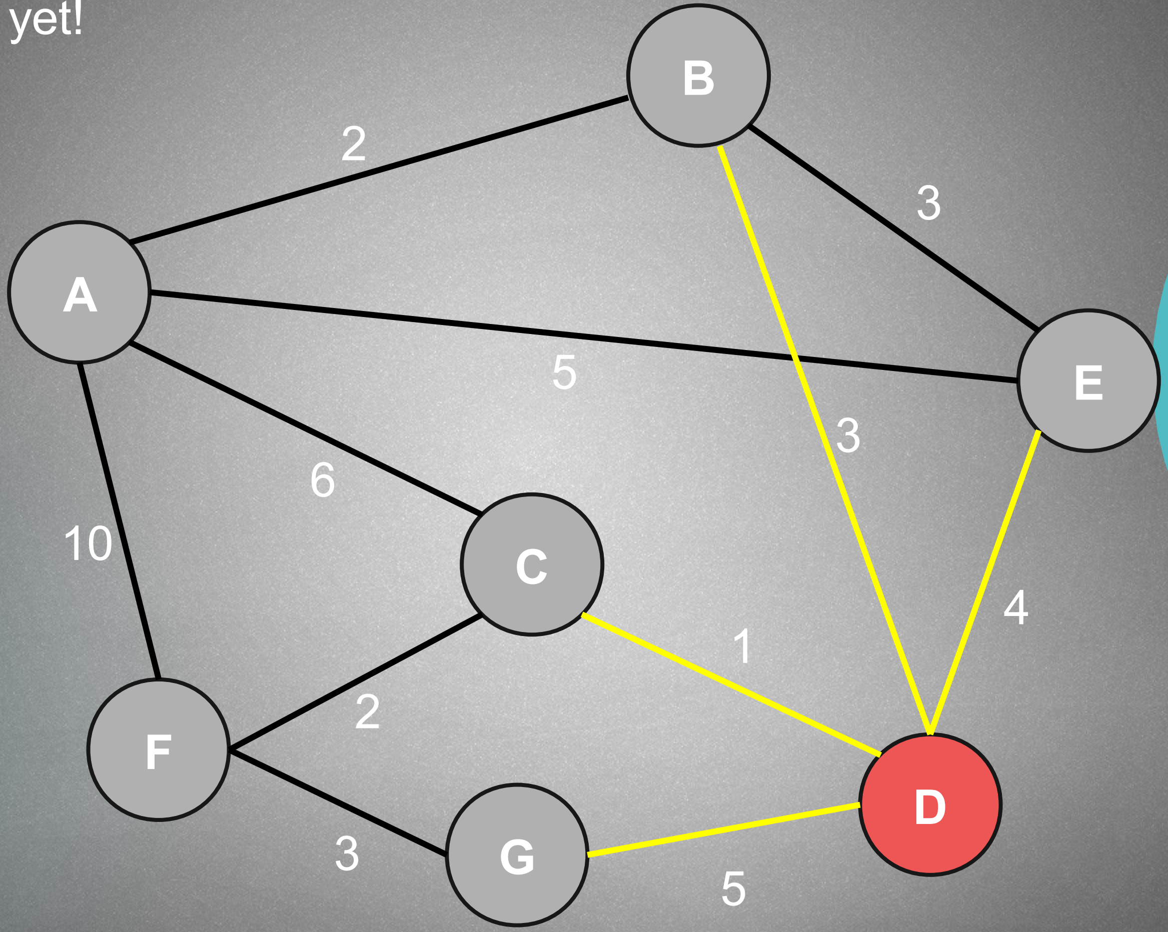

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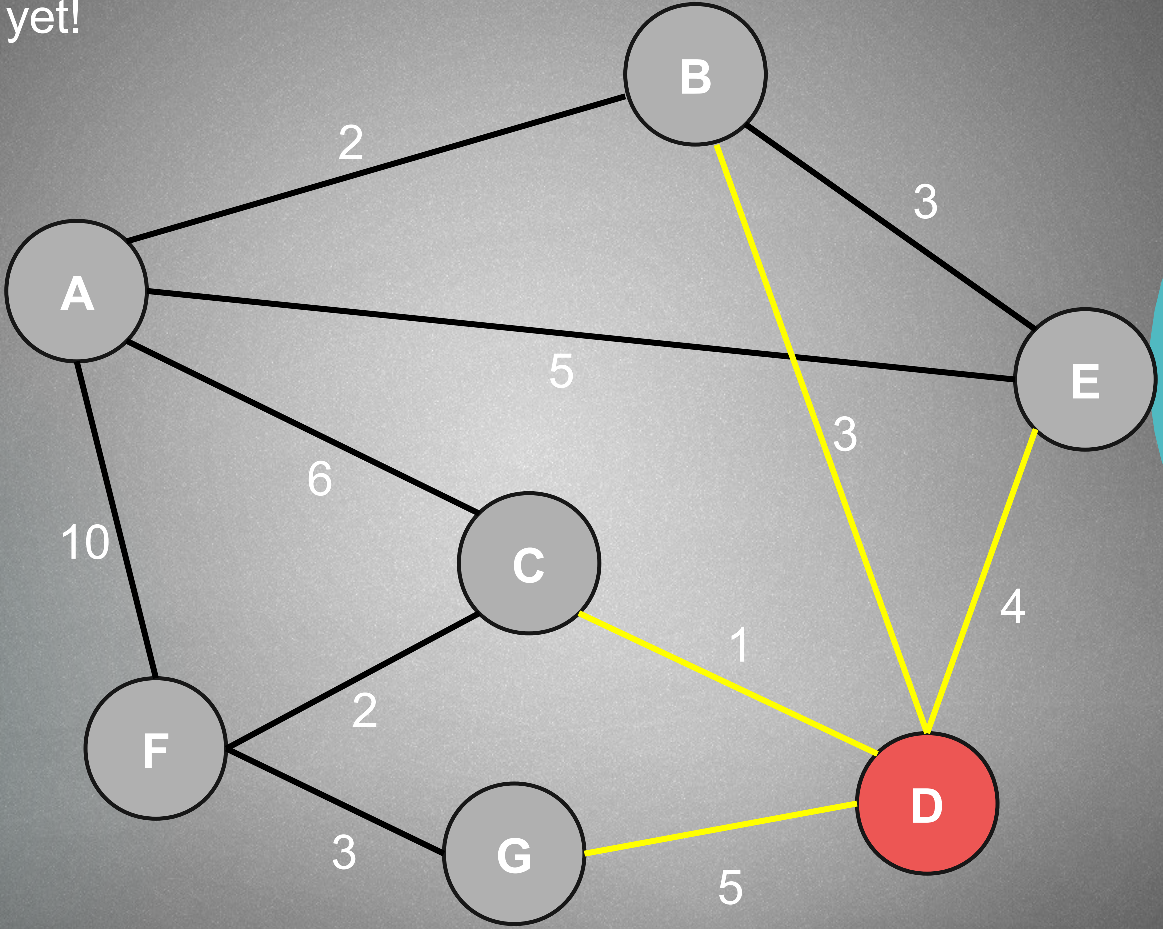
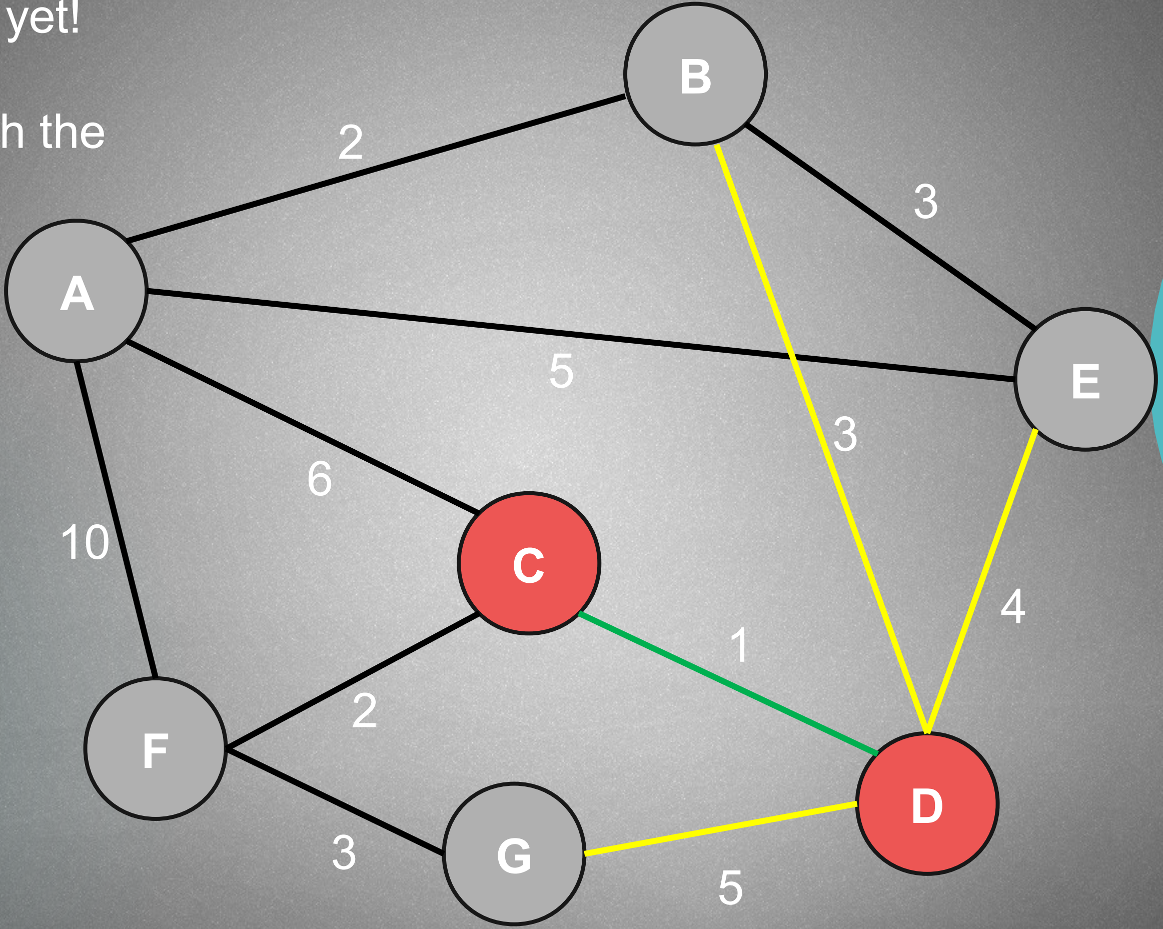


Heap content: G – 5 , C – 1 , B – 3 , E – 3

We consider edges to vertexes we have not visited yet!

We pick the one with the lowest cost



Heap content: G – 5 , **C – 1** , B – 3 , E – 3

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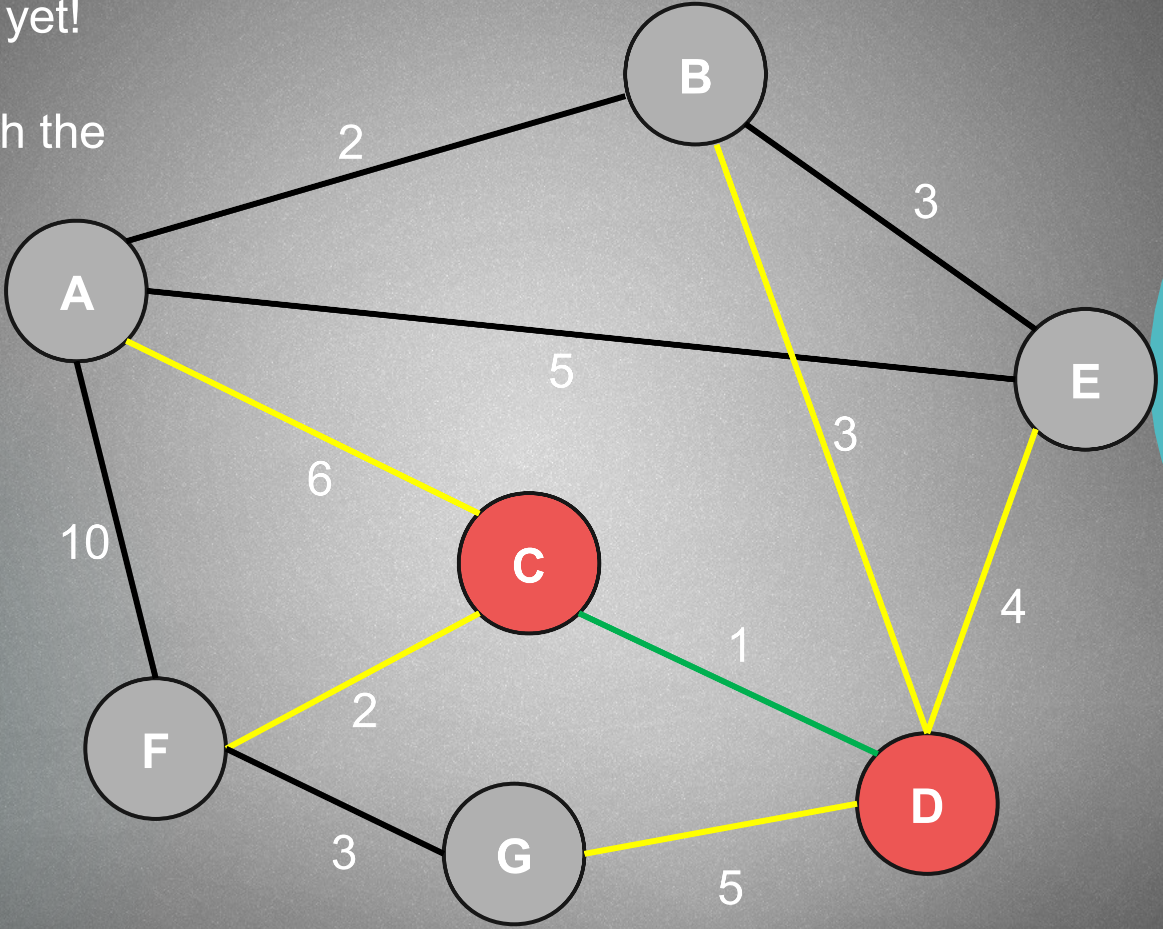
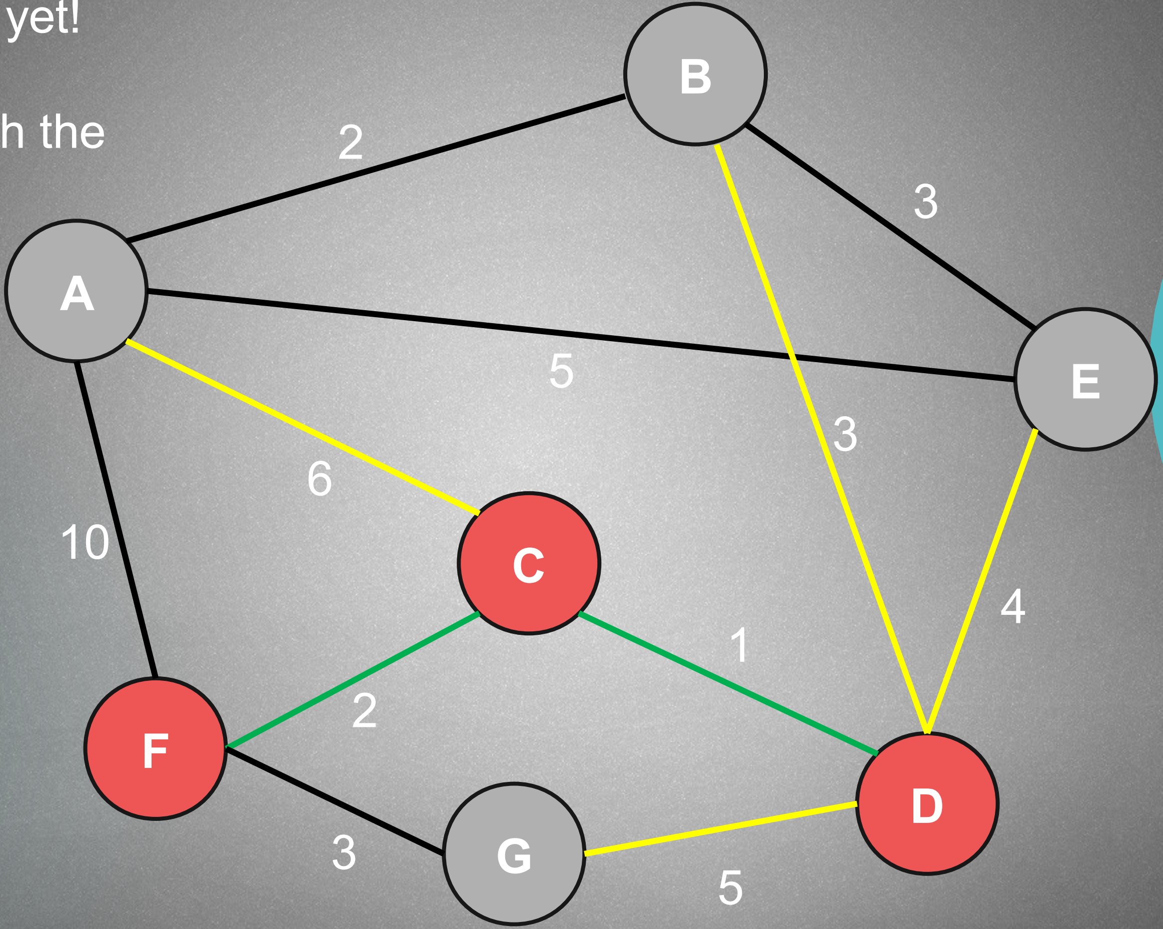


Heap content: G – 5 , B – 3 , E – 3 , A – 6 , F – 2

We consider edges to vertexes we have not visited yet!

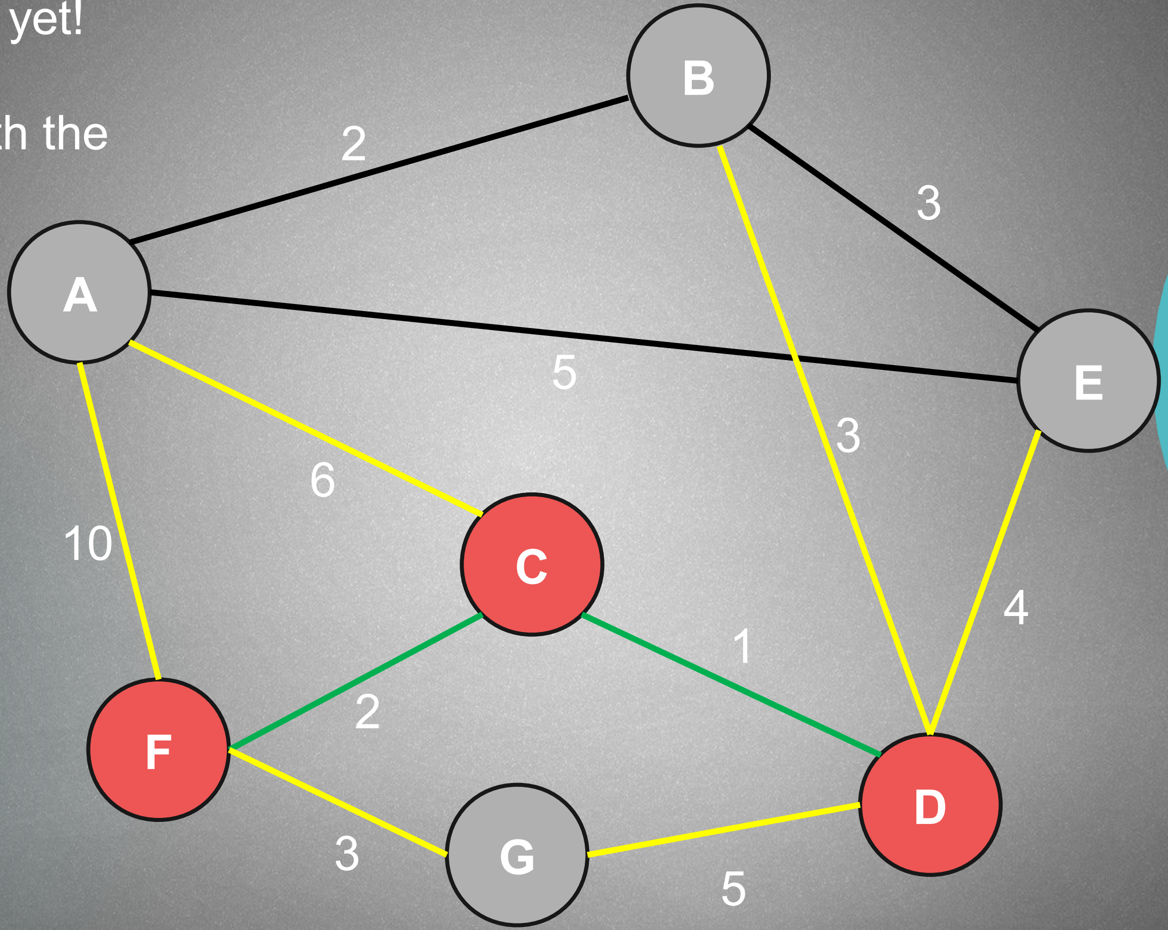
We pick the one with the lowest cost



Heap content: G – 5 , B – 3 , E – 3 , A – 6 , **F – 2**

We consider edges to vertexes we have not visited yet!

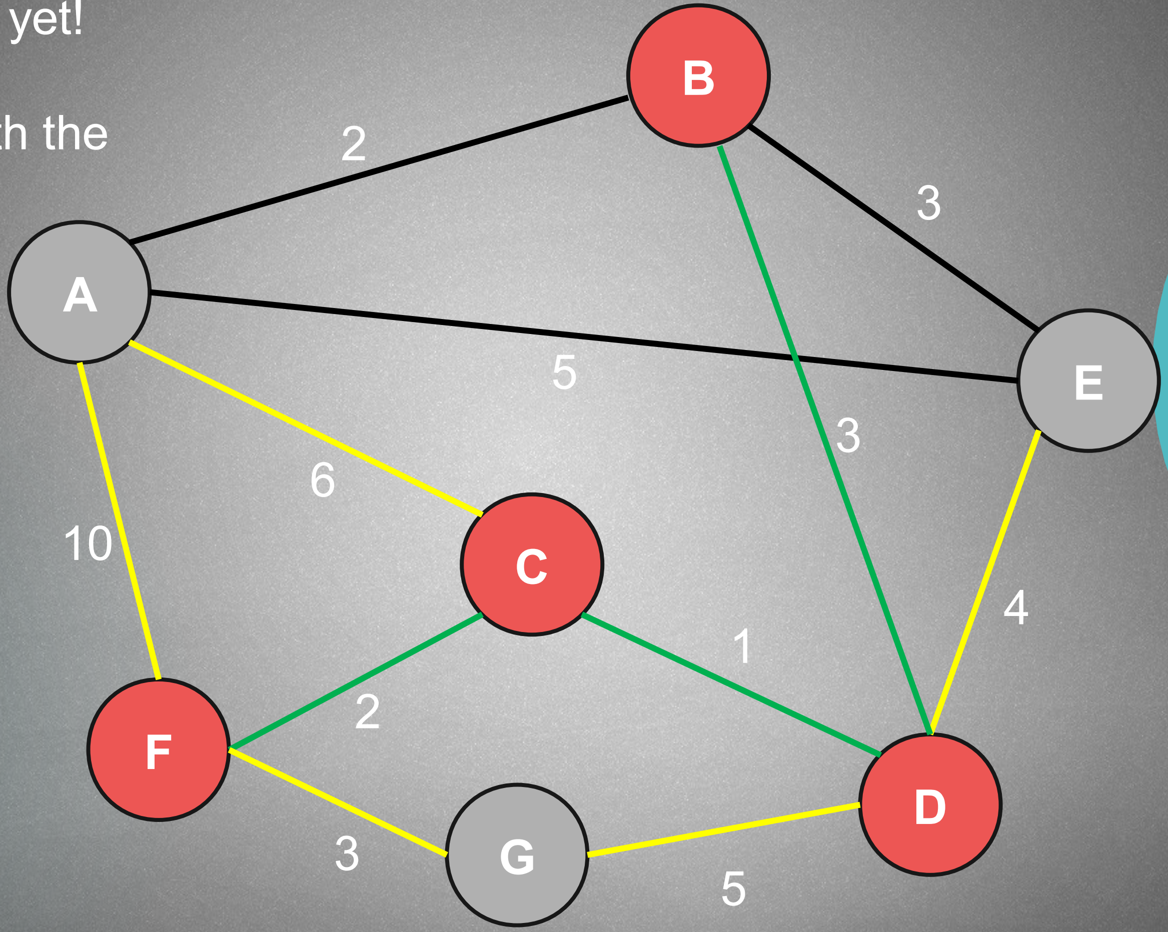
We pick the one with the lowest cost



Heap content: G – 5 , B – 3 , E – 3 , A – 6 , A – 10 , G – 3

We consider edges to vertexes we have not visited yet!

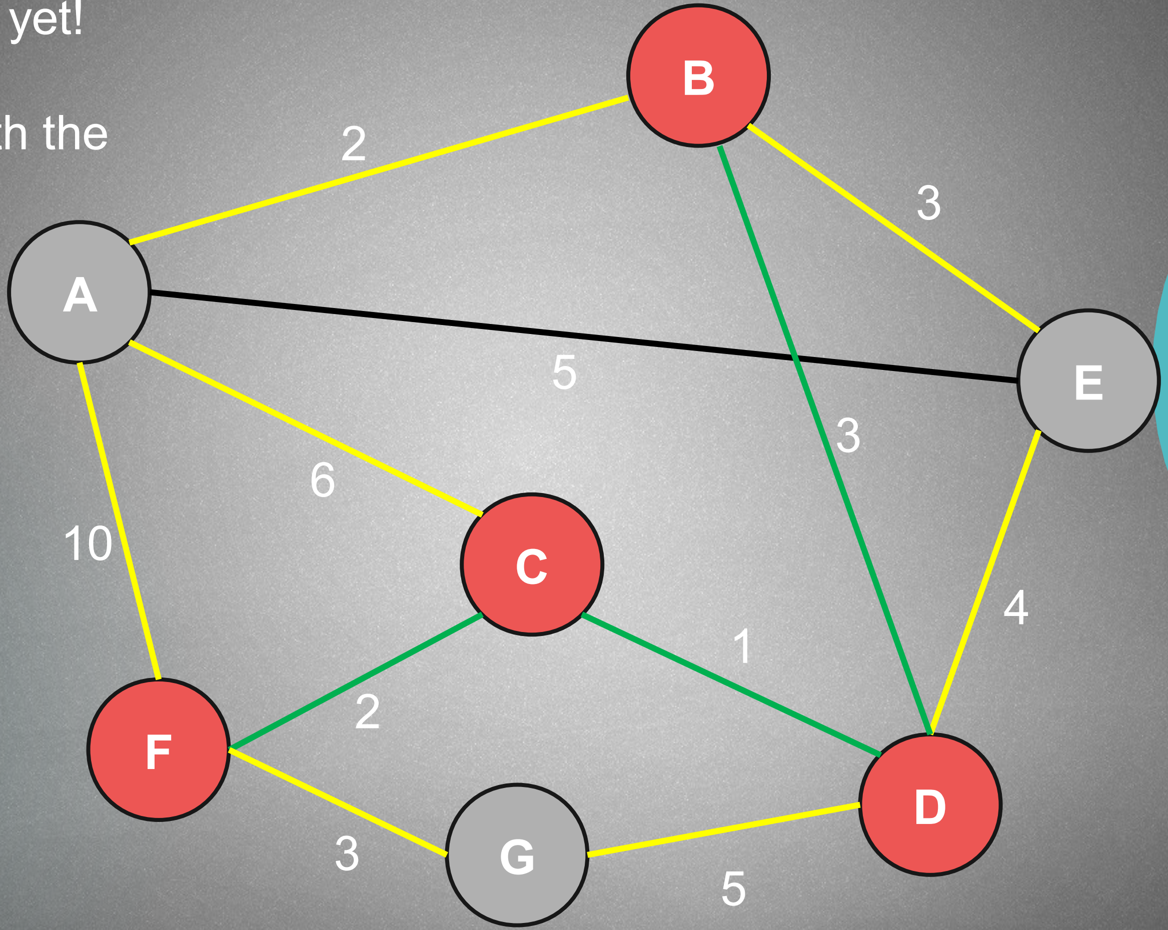
We pick the one with the lowest cost



Heap content: G – 5 , **B – 3** , E – 3 , A – 6 , A – 10 , G – 3

We consider edges to vertexes we have not visited yet!

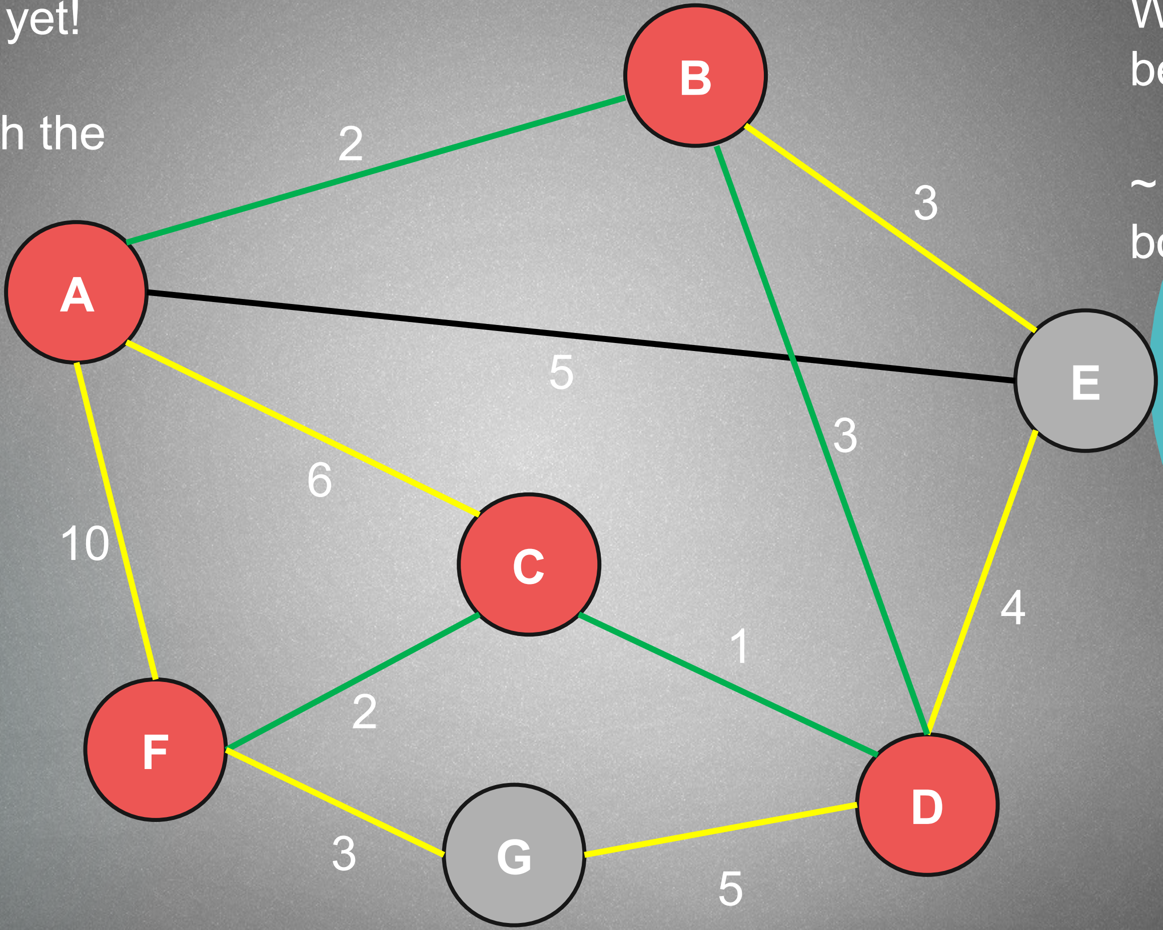
We pick the one with the lowest cost



Heap content: G – 5 , E – 3 , A – 6 , A – 10 , G – 3 , A – 2 , E – 3

We consider edges to vertexes we have not visited yet!

We pick the one with the lowest cost



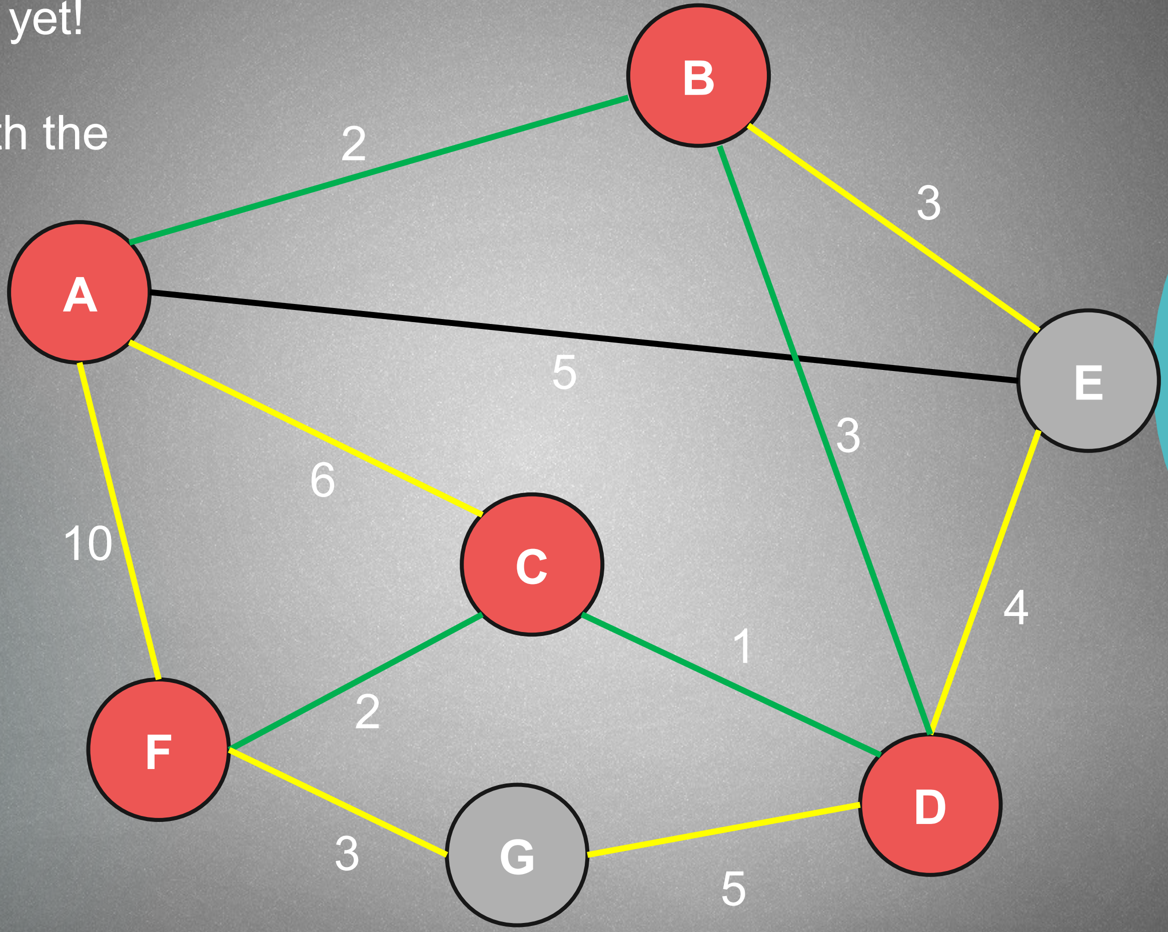
We do not consider edge between A and E

~ because we have considered both vertexes !!!

Heap content: G – 5 , E – 3 , A – 6 , A – 10 , G – 3 , A – 2 , E – 3

We consider edges to vertexes we have not visited yet!

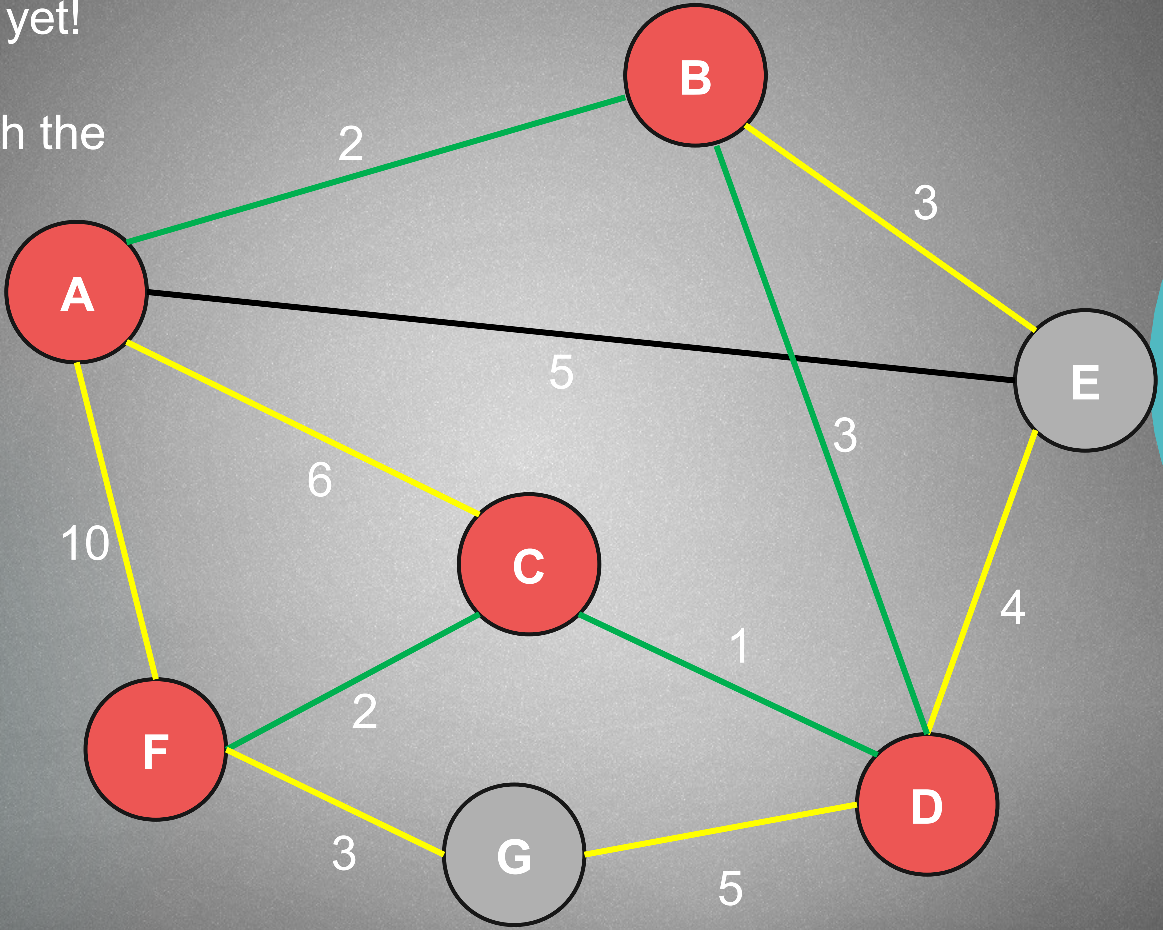
We pick the one with the lowest cost



Heap content: G – 5 , E – 3 , A – 6 , A – 10 , G – 3 , **A – 2** , E – 3

We consider edges to vertexes we have not visited yet!

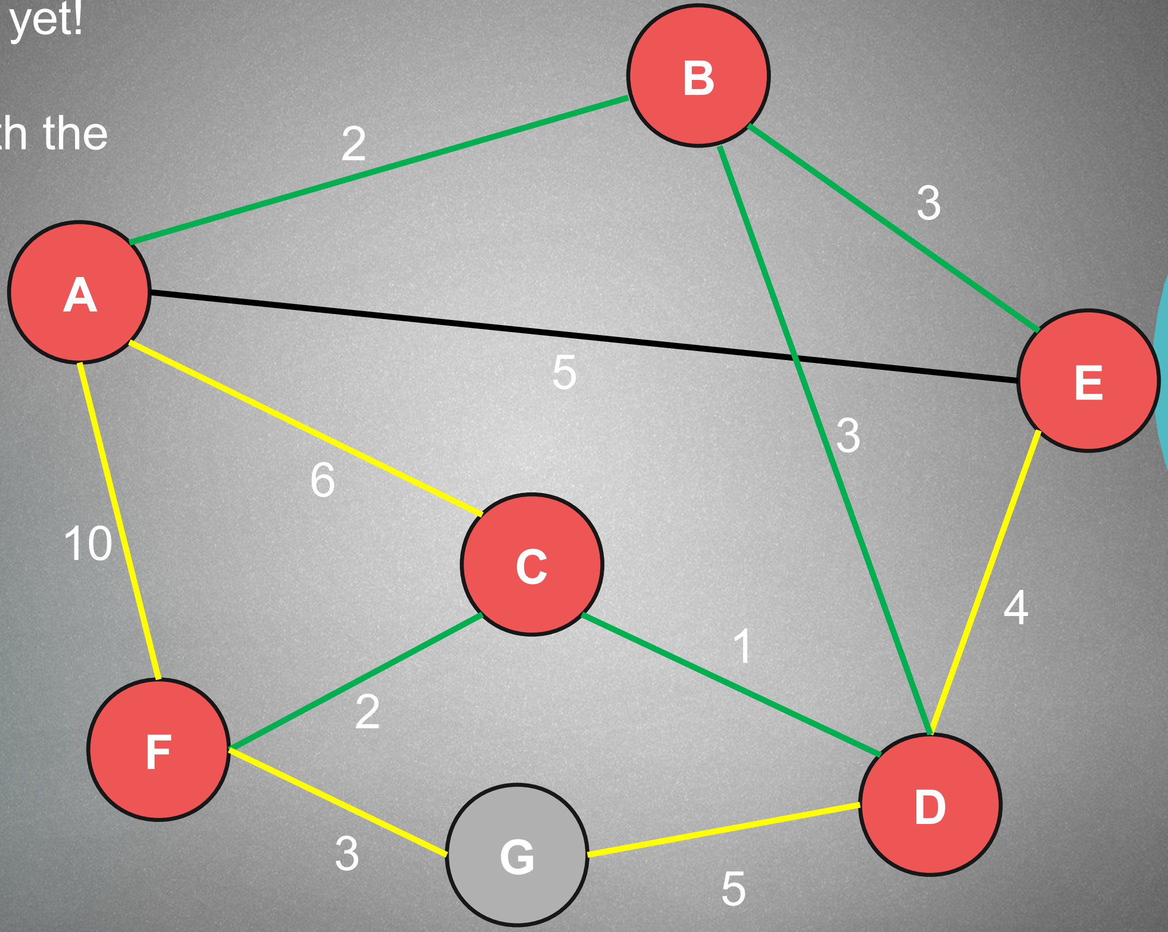
We pick the one with the lowest cost



Heap content: G – 5 , E – 3 , A – 6 , A – 10 , G – 3 , E – 3

We consider edges to vertexes we have not visited yet!

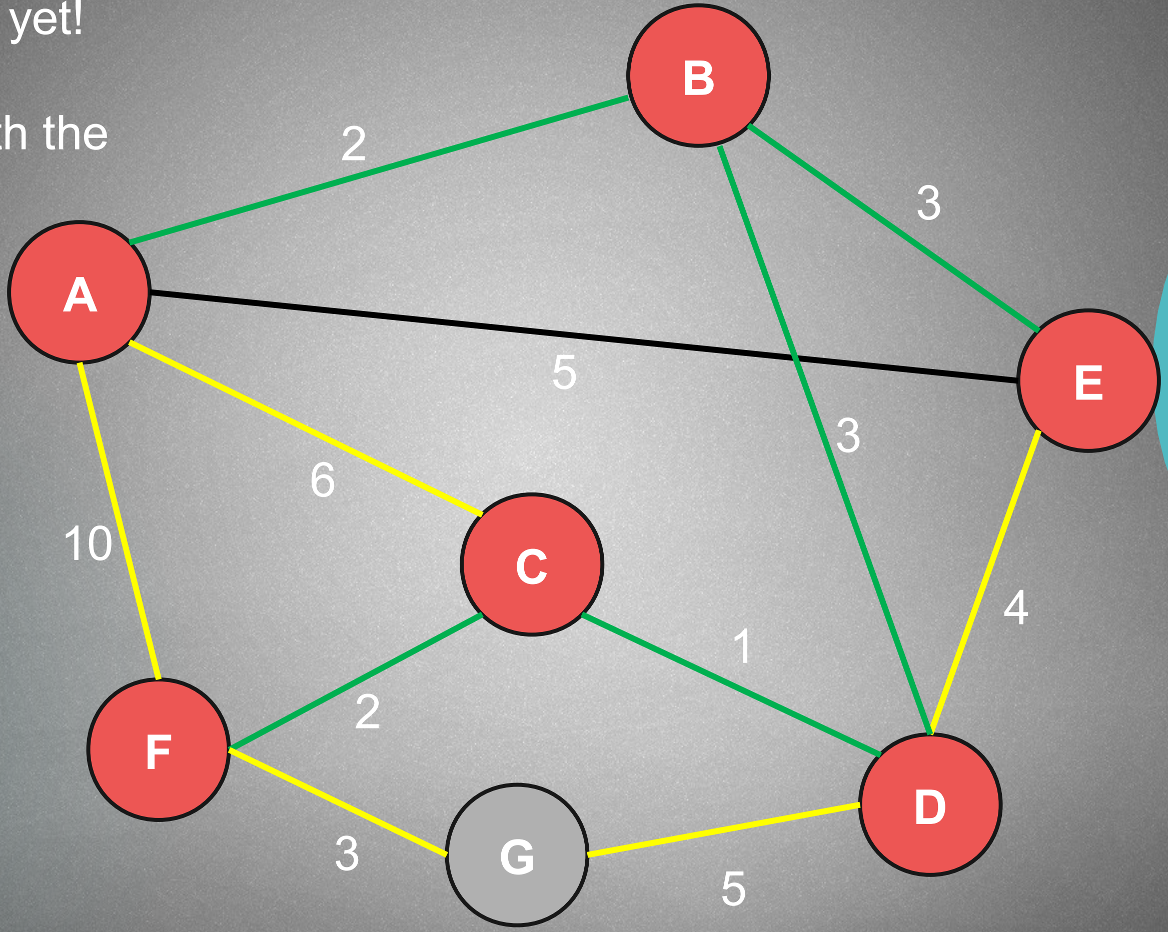
We pick the one with the lowest cost



Heap content: G – 5 , E – 3 , A – 6 , A – 10 , G – 3 , **E – 3**

We consider edges to vertexes we have not visited yet!

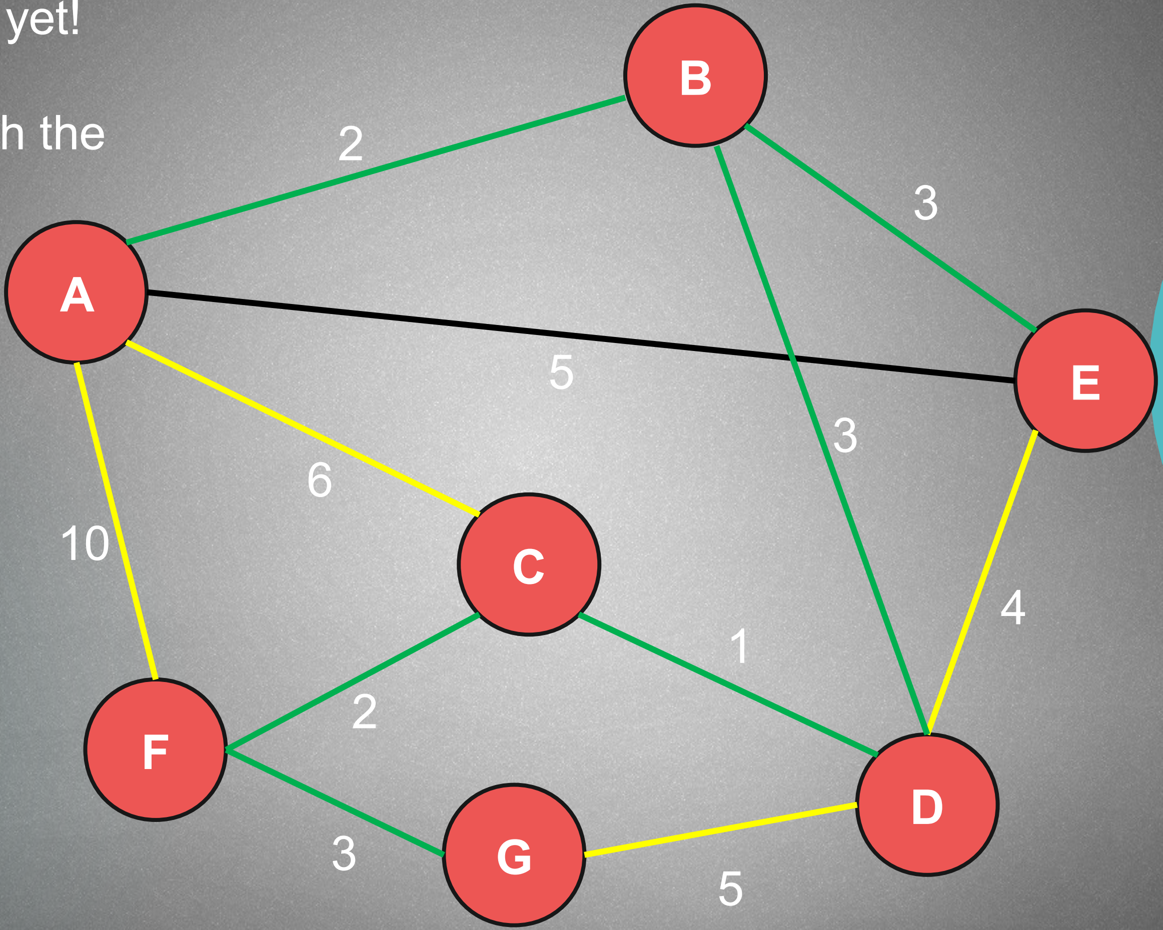
We pick the one with the lowest cost



Heap content: G – 5 , E – 3 , A – 6 , A – 10 , G – 3

We consider edges to vertexes we have not visited yet!

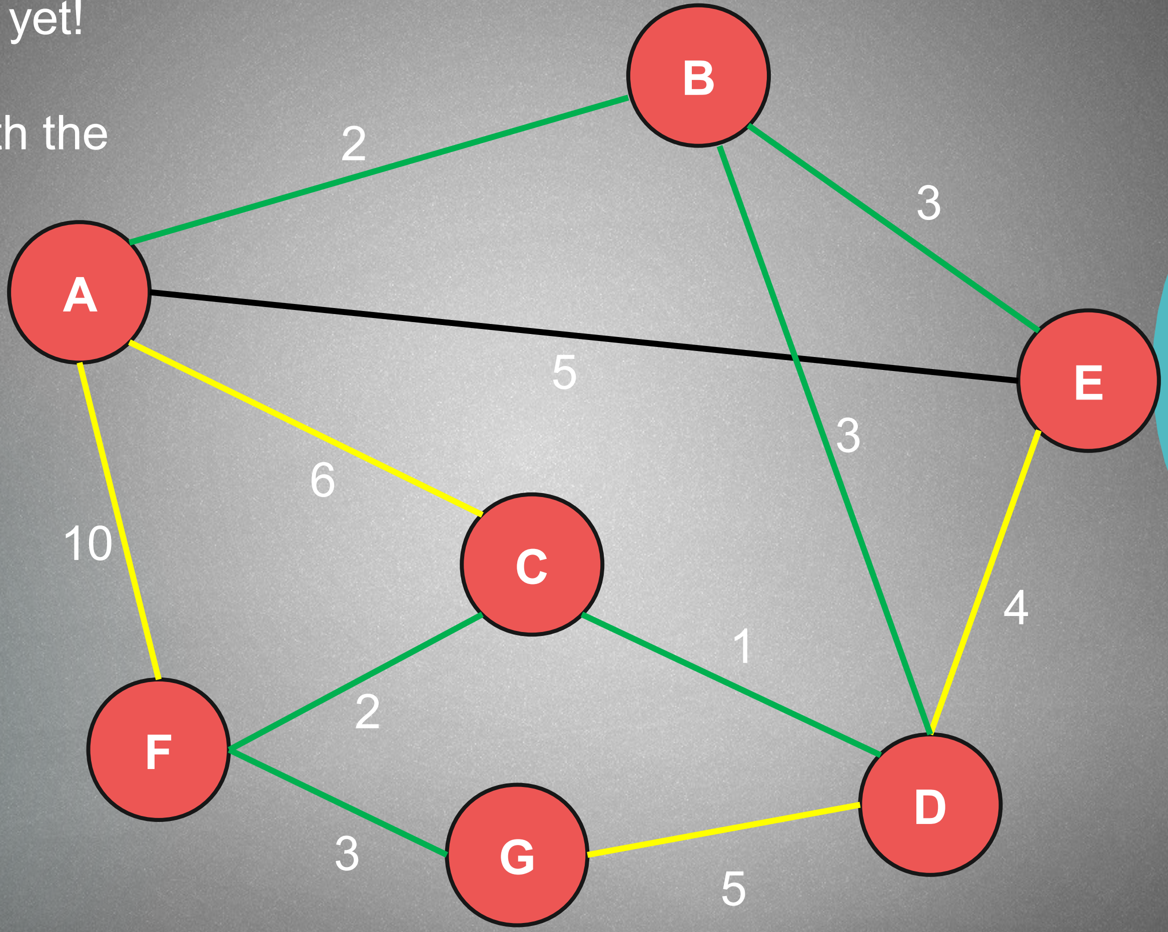
We pick the one with the lowest cost



Heap content: G – 5 , E – 3 , A – 6 , A – 10 , **G – 3**

We consider edges to vertexes we have not visited yet!

We pick the one with the lowest cost



We have visited all the nodes, so we are done! The minimum cost: 14

Prims VS Kruskal

- Prim's algorithm is significantly faster in the limit when you've got a really dense graph with many more edges than vertices
- Kruskal performs better in typical situations (sparse graphs) because it uses simpler data structures
- Kruskal can have better performance if the edges can be sorted in linear time or the edges are already sorted
- Prim's better if the number of edges to vertices is high (dense graphs)

Applications

- K means
- Telecommunications
- Recommendation systems

K-means clustering

- Important unsupervised machine learning technique
- We have a dataset: we want to find patterns !!!
- For examle: want to classify and cluster the points in the 2D plane
- We construct a minimal spanning tree –> leave the first k edge
- Result: we will have k clusters !!!