Given on exam:

Prob. inequalities
NP-hard problems
Standard rubrics — DP, Graph reduction, NP-hardness...

"Prove" → we want a proof

"~~Prove~~" → we do not want a proof

Linear arrangement problem

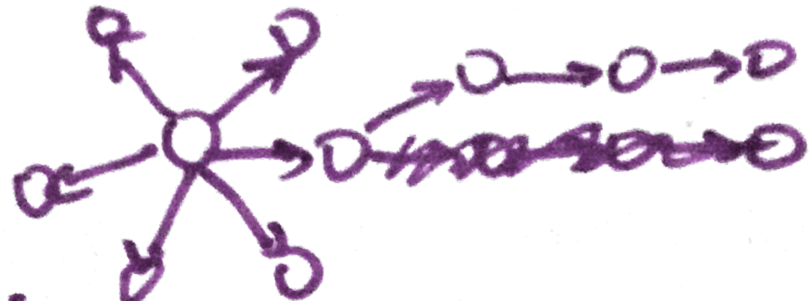
Input: Directed graph $G=(V,E)$
Output: Indexing of $V=\{v_1, v_2, \dots v_n\}$
s.t. # edges $v_i \to v_j$ with $i<j$
is maximized.

If G is a dag? Topological sort!
In general NP-hard

Question: Design a fast 2-approx algorithm.

$\boxed{\text{We know } OPT \le E.}$

~~① Sort by outdegree?~~



② Pick arbitrary ordering.

If $\ge E/2$ forward edges, done.

Else reverse everything!

SP 2015 Final #2

m soldiers

n tasks

↳ k soldiers qualified for each task

Select a set S of soldiers maximizing # tasks with ONE qualified ~~select~~ soldier in S.

ⓐ Choose each soldier with prob p.

$$E[\#\text{tasks}] = \sum_{i=1}^{n} Pr(\text{task } i \text{ is completed}]$$

$$= \boxed{n \cdot p \cdot (1-p)^{k-1} \cdot k}$$

ⓑ Best value of $p = ?$

$$\frac{d}{dp}\, p(1-p)^{k-1} = (1-p)^{k-1} - p(k-1)(1-p)^{k-2} = 0$$

$$(1-p) = p(k-1)$$

$$1 = pk - p + p$$

$$\boxed{p = 1/k}$$

$$\hookrightarrow E[\#\text{tasks}] = n \cdot \underbrace{\left(1-\frac{1}{k}\right)^{k-1}} \approx n/e$$

$$\cancel{p = (1-p)^{k-1}}$$

(c) $O(1)$-approx algo

$$E[\text{approx}] \approx 1/e \ \checkmark$$

FF: $O(E \cdot |f^*|)$ time

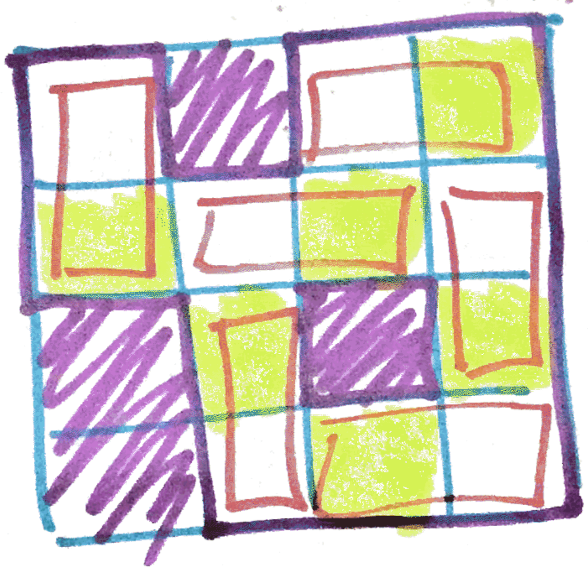
Orlin: $O(VE)$ time

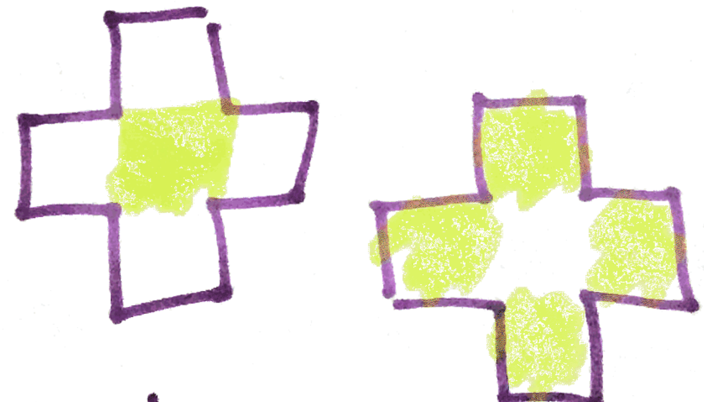
- edges have capacities and/or
  have lower bounds
- vertices have capacities and/or
  have lower bounds
  on incoming flow
  (or outgoing flow)
- multiple sources, multiple sinks,
  or no   or no

  feasible?   max. value
- vertices can have non-zero balances
- flow decomposition   - integer flow

- edge-disjoint paths
- vertex-disjoint paths
- max. bipartite matching
- disjoint path covers of dags
- ______ path covers of dags
- assignment/tuple selection

nxn checkerboard with some squares removed
cover every square exactly once with
dominoes: 2x1 or 1x2 rectangles



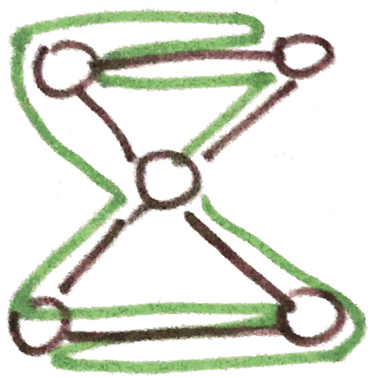


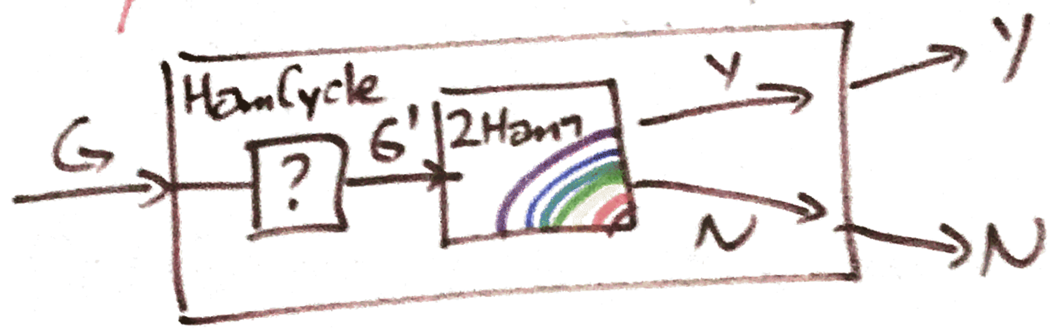
Bipartite matching $(L \cup R, E)$
L = white squares   R = black squares
E = adj squares - share boundary side
domino = edge

Cover with dominos $\Leftrightarrow$ perfect matching
time: $O(VE)$
$= \boxed{O(n^4)}$

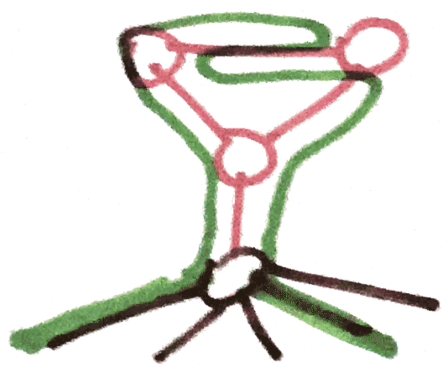
Double Hamiltonian circuit is NP-hard

Reduce from Ham cycle





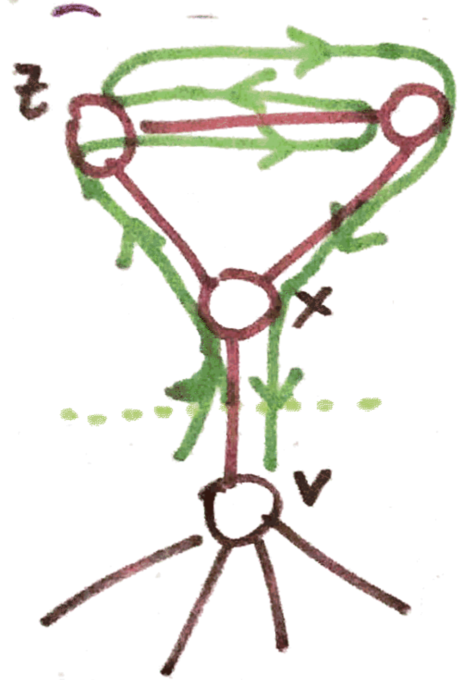
Given G(V,E) construct G'=(V',E')



Attach a lollipop to every vertex

G has Ham cycle → G' has double Ham. ✓

G' has double Ham.



case analysis:
within each gadget

2 Ham $v \to x \to z \to y \to \cancel{z \to y} \to x \to v$

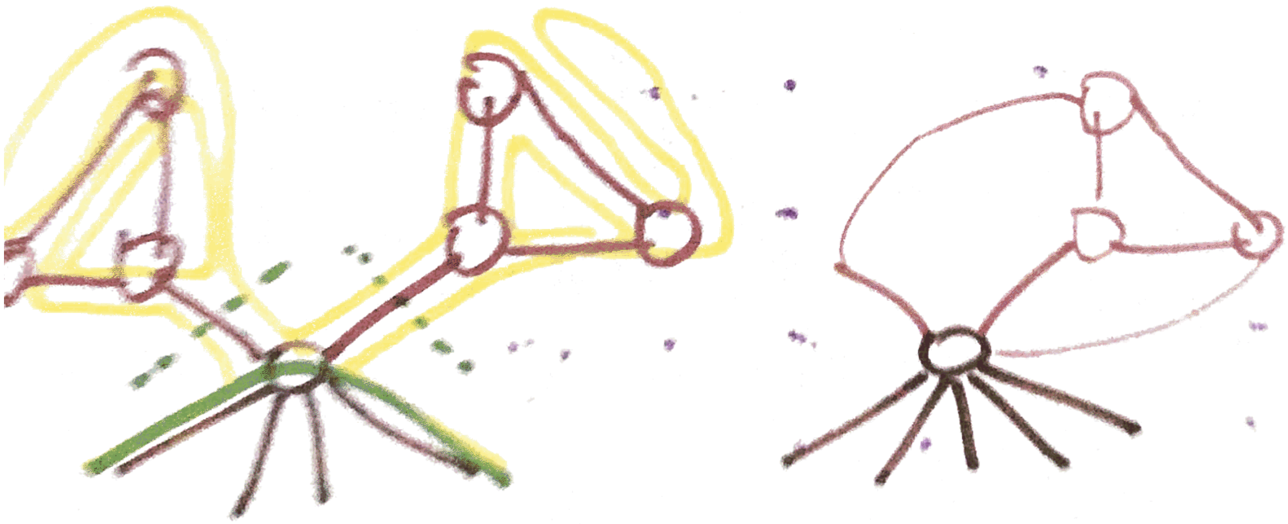
wlog

Delete gadgets, left Ham cycle in G.

poly time ✓

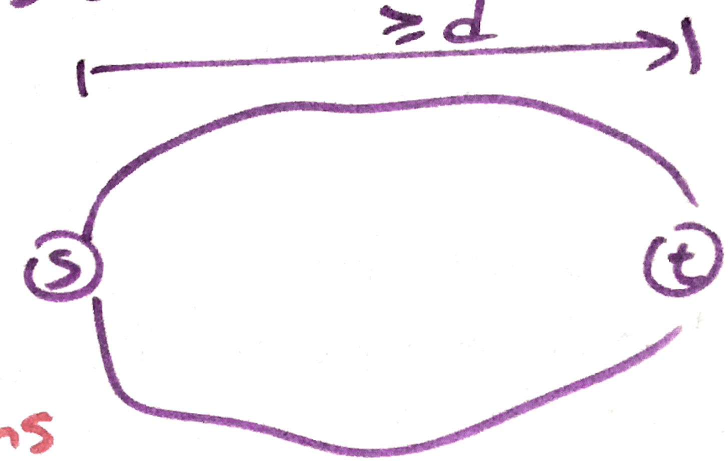
Sp 2016 Final #1

A triple Hamiltonian cycle = closed walk
that visits every vertex
exactly 3 times.



Max Flow notes problem 6.

$G=(V,E)$ Flow network
every edge has capacity 1
shortest path from s to t is $\geq d$.

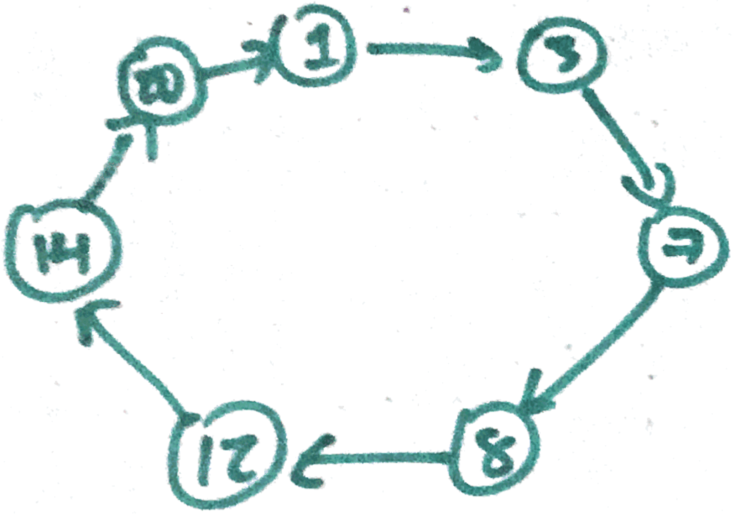


ⓐ max flow $\leq E/d$

Spose f^* is max flow
Decompose into $|f^*|$ edge-disjoint paths
each path has length $\geq d$
Total #edges covered by flow $\geq d\cdot|f^*|$
$\leq E$

$|f^*| \leq \frac{E}{d}$ □

Nuts+Bolts notes Problem 7



	1	2	3	4	5	6	7
X	3	14	7	1	12	20	8
next	3	6	7	1	2	4	5

Given → X randomly permuted

X[next[i]] is successor of X[i] in sorted order

Given x, is x in X?

9? Goal: $O(\sqrt{n})$ time

Algo:

Choose k elements of X at random.

Find largest sample smaller than x. — $O(k)$ time

Scan forward to $\geq x$ — $O(n/k)$ exp. time.

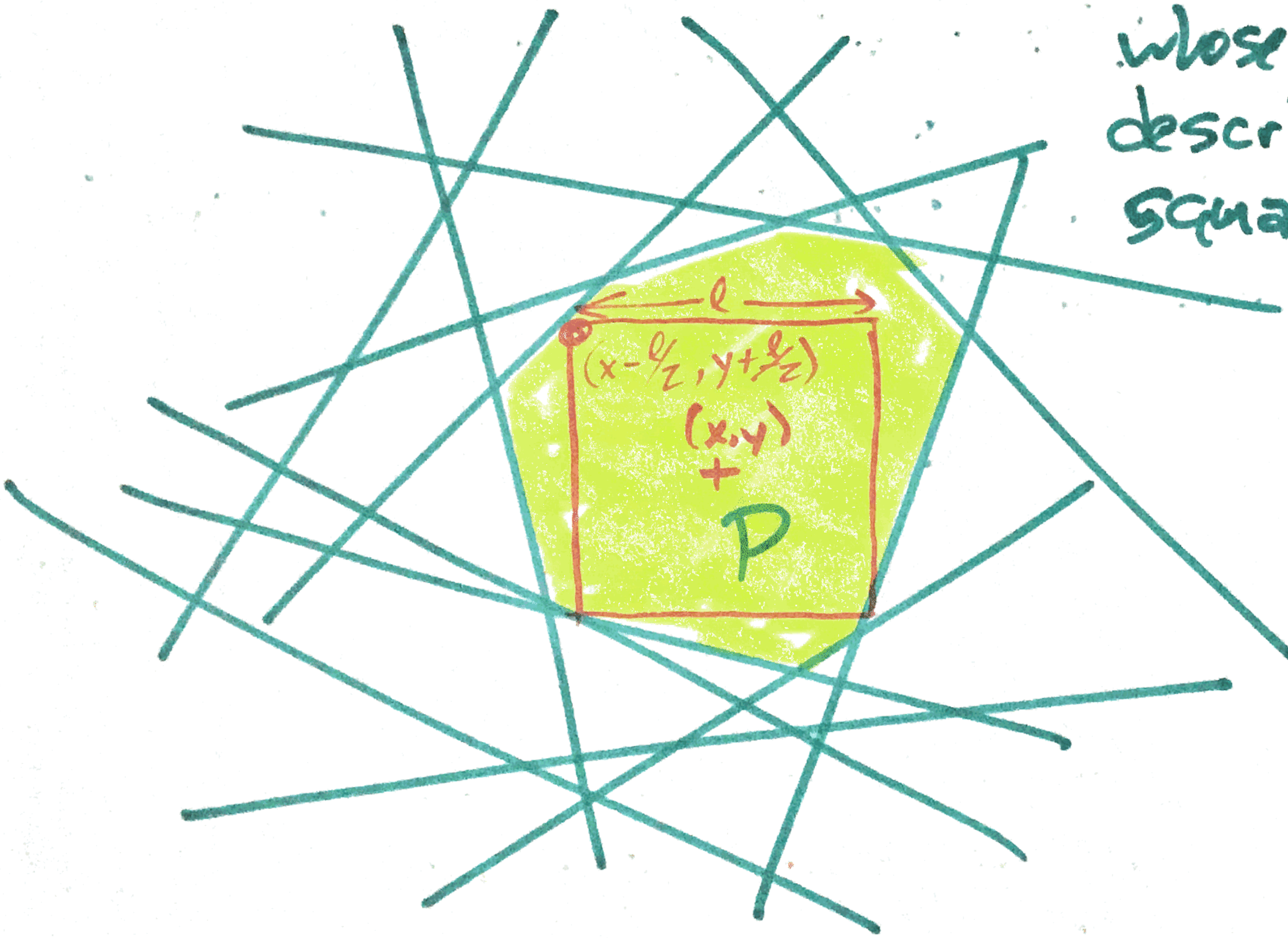
$2n/(k+1)$

Spring 2015 Final #6

Given linear inequalities

$$a_i x + b_i y \le c_i$$

Describe LP whose solution describes largest square in feasible polygon P



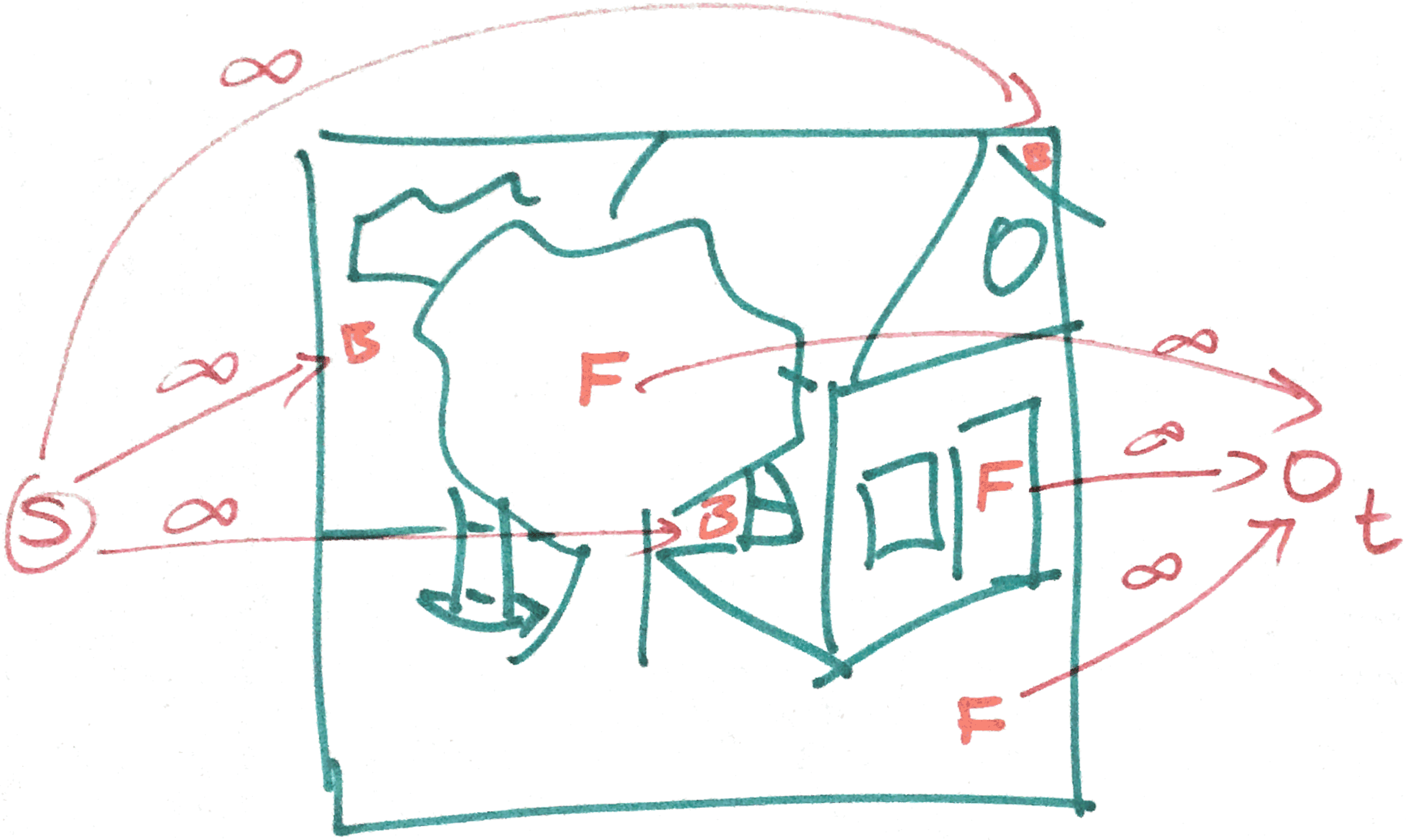
Describe $\square$ by center (x,y)
edge length ℓ

$$\max \ell$$

$$\text{s.t. } a_i(x + \ell/2) + b_i(y + \ell/2) \le c_i \quad \text{for all } i$$

4 constraints for each i

$+$	$+$
$+$	$-$
$-$	$-$



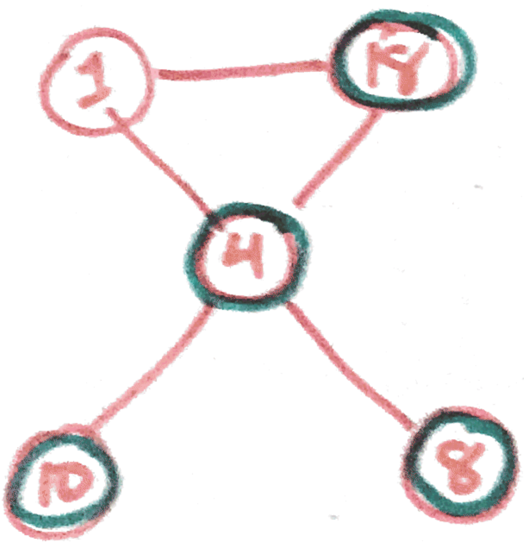
$O(n \log^5 n)$

Sp 2016 Final #5

real number between 0 and 1

Assign random priority to every vertex

$$S = \{ v \mid \text{priority}(v) \geq \min_{uv} \text{priority}(u) \}$$

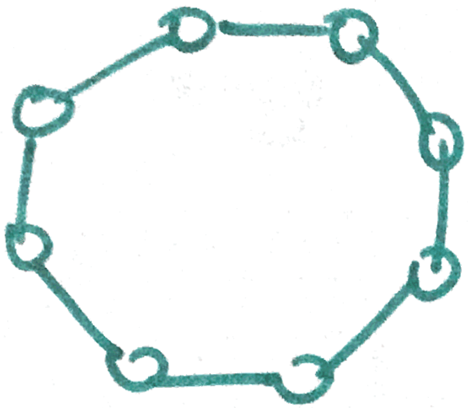


(a) $\Pr[S \text{ is a vertex cover}] = ?$

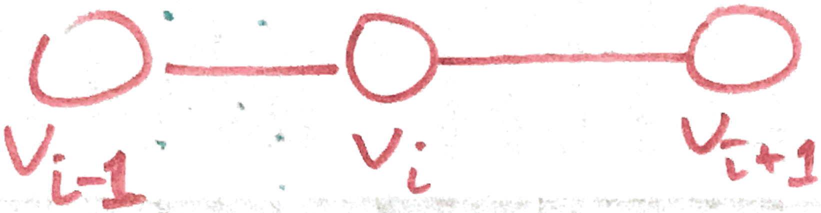
$\boxed{1}$

$\text{pri}(u) \leq \text{pri}(v)$

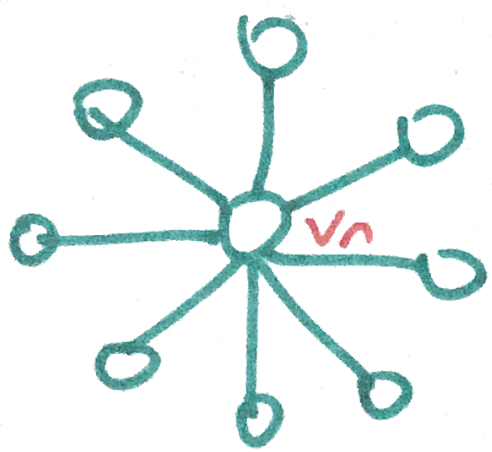
(b) G = cycle of length n



$$E[|S|] = \sum_{i=1}^{n} \Pr[v_i \in S] = \sum_{i=1}^{n} \frac{2}{3} = \boxed{\frac{2n}{3}}$$



(c) G star with $n-1$ leaves



$$E[|S|] = \sum_{i=1}^{n} \Pr[v_i \in S]$$

$$= (n-1)\Pr[\text{leaf} \in S] + \Pr[\text{center} \in S]$$

$$= \frac{n-1}{2} + \frac{n-1}{n} = \boxed{\frac{n}{2} + \frac{1}{2} - \frac{1}{n}}$$

(d) star. Choose $S_1, S_2, \ldots S_N$ independently

How large N so that some S_i is <u>min</u> vertex cover? <u>whp</u>

$$\Pr[S = \min VC] = \Pr[\text{priority}(v_n) \geq \text{priority}(v_i) \text{ for all } i]$$

$$= 1/n$$

$$\Pr[\text{min VC is one of } S_1 \ldots S_N] =$$

$$= 1 - \Pr[\text{none of } S_1 \ldots S_N \text{ is min VC}]$$

$$= 1 - \left(1 - \frac{1}{n}\right)^N \geq 1 - \frac{1}{n^\alpha}$$

if $N \geq \alpha n \ln n$

$= \Omega(n \log n)$

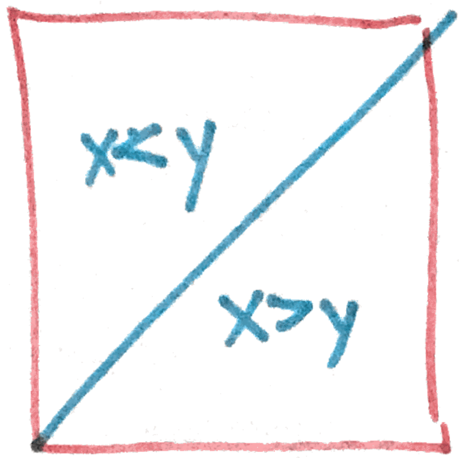
$$\boxed{N \geqq \boxed{\alpha\, n \log n}}$$

$$N/n \geqq \alpha \log n$$

$$e^{N/n} \geqq n^{\alpha}$$

$$e^{-N/n} \approx \left(1-\frac{1}{n}\right)^{N} \leqq \frac{1}{n^{\alpha}}$$

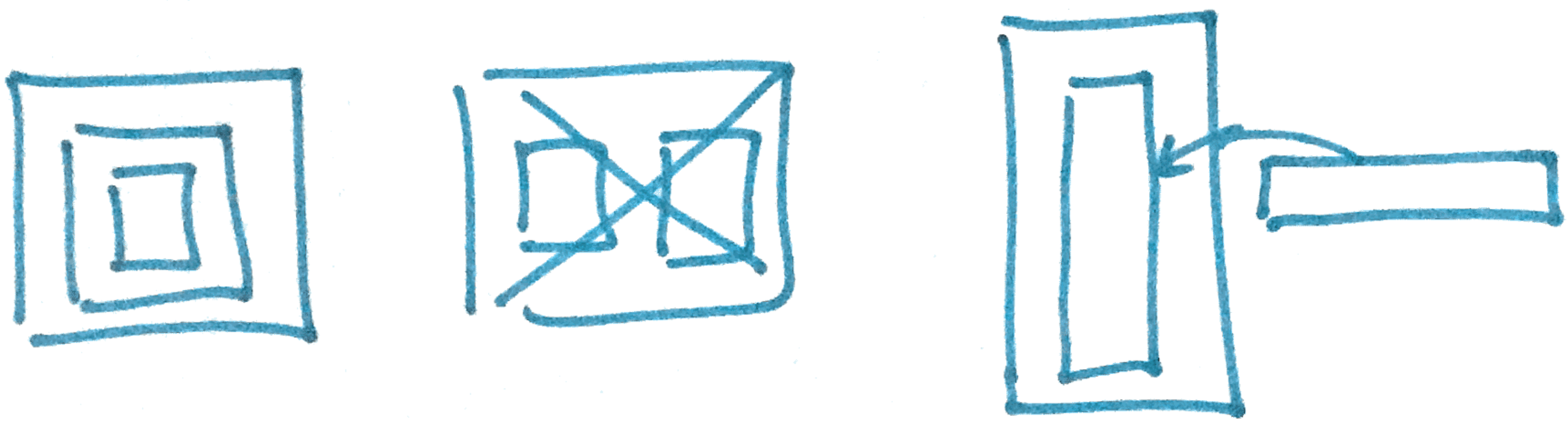
$$1-\left(1-\frac{1}{n}\right)^{N} \geq 1-\frac{1}{n^{\alpha}}$$


x<y
x>y

Spring 2016 Final #3

n boxes, h, w, d in cm

$10 \le h, w, d \le 20$



Nest boxes so that # visible boxes is as small as possibe

Solution 1

Disjoint path cover

Define dag $G=(V,E)$

V = boxes

$E = \{u \to v \mid v \text{ fits inside } u \text{ after some rotation}\}$ dag ✓

min dim v < min dim u

path = seq of nested boxes
outermost is only visible

min # visible boxes $\rightleftarrows$ min # vertex disjoint paths cover G

time $= O(VE) = \underline{O(n^3)}$ time

Solution 2: Matching

$G = (L \cup R, E)$

L = boxes

R = boxes

$E = \{u_L v_R \mid u \text{ fits inside } v\}$

max matching M in G

Intuition $uv \in M$ if v is smallest box containing u

$$\boxed{\text{visible box} = \text{unmatched vertex in } L}$$

$O(VE) = O(n^3)$ time