CS4220 PS5

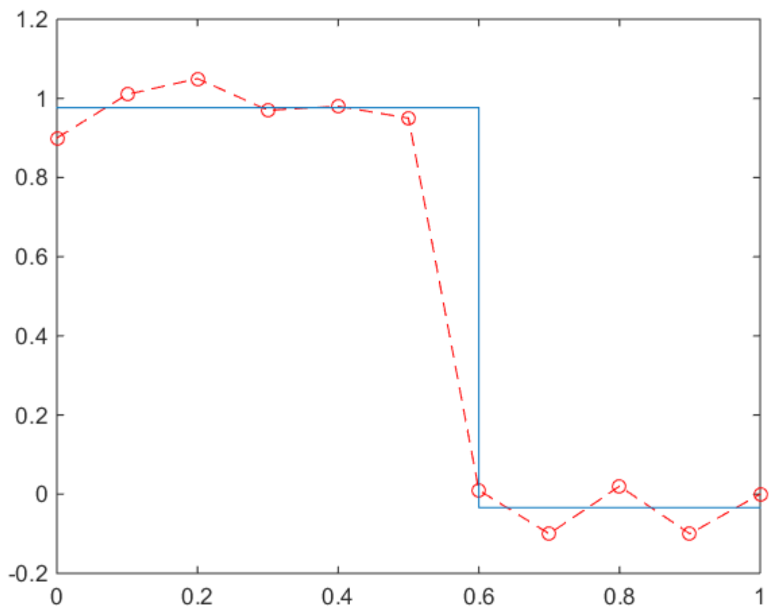
Due: March 2, 2015

1 (Ascher and Greif, 6.4, Problem 1) The given data:

t	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
b	0.9	1.01	1.05	0.97	0.98	0.95	0.01	-0.1	0.02	-0.1	0.0

Solution:

1. From the plot, we can see the break approximately occurs at 0.5 to 0.6. The red data points are the data.



2. For each segment in the piecewise constant function, $p_0(t) = v(t) = x_1$, where t and $v(t)$ (which is b here) are the independent and dependent variables, respectively, and x_1 is the constant term of $p_0(t)$ to be determined by the least-squares method. We write out the equation and obtain x in

$$Ax = \begin{pmatrix} 1 \\ . \\ . \\ . \\ 1 \end{pmatrix} x = \begin{pmatrix} b_1 \\ b_2 \\ . \\ . \\ b_n \end{pmatrix}, \quad n \text{ is the number of data points.}$$

A is an $n \times 1$ matrix with all elements being 1. We can compute either by MATLAB or by hand calculation to get the constants. Here is the MATLAB code:

```
t1 =[0.0:0.1:0.5]';
b1=[0.9 1.01 1.05 0.97 0.98 0.95]';

t2 =[0.6:0.1:1.0]';
b2=[0.01 −0.1 0.02 −0.1 0.0]';

A=ones(length(t1),1);
x1 = (A'*A)\(A'*b1)
A=ones(length(t2),1);
x2 = (A'*A)\(A'*b2)
x1 =
   0.976666666666667
>> x2
x2 =
  −0.034000000000000
```

$$v(t) = 0.9767, \quad 0 \leq t \leq 0.6,$$
$$v(t) = -0.0340, \quad 0.6 \leq t \leq 1.0.$$

The piecewise constant fit is plotted as the blue lines in the figure.

2. QR to SVD Suppose $A = QR$ is an economy QR factorization. Show that the singular values of A are those of R.

Solution: Let $A \in \mathbb{R}^{m\times n}$, $m > n$. An economy QR factorization produces $A = QR$, where $Q \in \mathbb{R}^{m\times n}$ is orthogonal, and $R \in \mathbb{R}^{n\times n}$. The singular values of A are the square of the eigenvalues of the symmetric matrix $A^T A \in \mathbb{R}^{n\times n}$. Because $Q^T Q = I$ (Q is orthogonal), we have

$$A^T A = (QR)^T QR = R^T Q^T QR = R^T R.$$

So we see $A^T A = R^T R$, therefore the singular values of A are the same as the singular values of $R^T R \in \mathbb{R}^{n\times n}$.

3: Vector projector Suppose $A \in \mathbb{R}^{m \times n}$ where $m > n$ has full column rank. Given A and a vector b, write one line of MATLAB to compute the element c in the range space of A that is nearest to b (in the Euclidean norm).

Solution: We want to solve the normal equation, which guarantees that c is the normal projection of b in the range space of A:

$$(A^T A)c = A^T b,$$

and the one-line MATLAB code is:

```
c = (A'*A)\(A'*b);
```

4: Generally speaking Often, we use least squares to construct models of the world. We assume that the "truth" is

$$Ax = b,$$

but what we measure is the first few rows of A and b (which we write as A_1 and b_1), and those measurements are corrupted by noise. Suppose we have A exactly, but only get the noisy partial right hand side $\hat{b}_1 = b_1 + e_1$, from which we form

$$\text{minimize } \|A_1 \hat{x} - \hat{b}_1\|^2.$$

Our goal in this problem is to use the error analysis ideas in Section 6.2 to figure out the inherited error in the reconstruction of $\hat{b}_2 = A_2 \hat{x}$.

1. Let $e_2 = \hat{b}_2 - b_2$. Argue *briefly* that $e_2 = A_2 A_1^\dagger e_1$.
2. Show that
$$\frac{\|e_2\|}{\|b_2\|} \leq \kappa(A_2 A_1^\dagger) \frac{\|e_1\|}{\|b_1\|}.$$

Things get somewhat more complicated if we also allow the entries of A to be contaminated by error, though the same basic ingredients come into play.

Solution:

1. The truth of the model is represented by

$$A_1 x = b_1, \quad A_2 x = b_2 \quad \ldots$$

   The noise or error is defined as

$$e_1 = \hat{b}_1 - b_1, \quad e_1 = \hat{b}_2 - b_2, \quad \ldots$$

The least-squares approximation method provides the solution to $A_1\hat{x} = \hat{b}_1$ as follows:

$$\hat{x} = (A_1^T A_1)^{-1} A_1^T x = A_1^\dagger b_1,$$

where $A_1^\dagger = (A_1^T A_1)^{-1} A_1^T$ is called the *pseudo-inverse* of matrix A_1, and obviously

$$A_1^\dagger A_1 = (A_1^T A_1)^{-1} A_1^T A_1 = I.$$

But in general $A_1 A_1^\dagger \neq I$ unless A_1 is invertible.

Now from the definition of e_2 and the least-squares solution, we get

$$\begin{aligned} e_2 = \hat{b}_2 - b_2 &= A_2\hat{x} - b_2 \\ &= A_2(A_1^\dagger b_1) - b_2 \\ &= A_2 A_1^\dagger b_1 - A_2 x \\ &= A_2(A_1^\dagger b_1 - x) \end{aligned} \tag{1}$$

Similarly to Eq.(1), we have:

$$e_1 = \hat{b}_1 - b_1 = A_1\hat{x} - b_1 = A_1(A_1^\dagger b_1 - x). \tag{2}$$

Now left multiply $A_1^\dagger$ to Eq.(2):

$$A_1^\dagger e_1 = A_1^\dagger A_1(A_1^\dagger b_1 - x) = I(A_1^\dagger b_1 - x) = (A_1^\dagger b_1 - x). \tag{3}$$

Substituting Eq.(3) into Eq.(1), we arrive at the result for the inherited error e_2:

$$e_2 = A_2 A_1^\dagger e_1.$$

2. Show that

$$\frac{\|e_2\|}{\|b_2\|} \leq \kappa(A_2 A_1^\dagger)\frac{\|e_1\|}{\|b_1\|}.$$

Proof. We consider 2-norms throughout this part of the problem. For any vector u, we can use the following inequality

$$\sigma_{\min}(A_2 A_1^\dagger)\|u\| \leq \|A_2 A_1^\dagger u\| \leq \sigma_{\max}(A_2 A_1^\dagger)\|u\|, \tag{4}$$

where $\sigma(A_2 A_1^\dagger)$ stands for the singular value of matrix $A_2 A_1^\dagger$.

Substituting $u = e_1$ into Eq.(4) and use the result $e_2 = A_2 A_1^\dagger e_1$ obtained in the first part of the problem, we have

$$\sigma_{\min}(A_2 A_1^\dagger)\|e_1\| \leq \|A_2 A_1^\dagger e_1\| = \|e_2\| \leq \sigma_{\max}(A_2 A_1^\dagger)\|e_1\|,$$

hence we establish

$$\|e_2\| \leq \sigma_{\max}(A_2 A_1^\dagger)\|e_1\|. \tag{5}$$

Now substituting $u = b_1$ into Eq.(4), and recognizing $A_2 A_1^\dagger b_1 = b_2$, we have

$$\begin{aligned}\sigma_{\min}(A_2 A_1^\dagger)\|b_1\| \leq \|A_2 A_1^\dagger b_1\| &= \|A_2(A_1^\dagger A_1 x)\| \\ &= \|A_2 x\| = \|b_2\| \leq \sigma_{\max}(A_2 A_1^\dagger)\|b_1\|,\end{aligned}$$

therefore we establish

$$\|b_2\| \geq \sigma_{\min}(A_2 A_1^\dagger)\|b_1\|. \tag{6}$$

Dividing Eq.(5) by Eq.(6), we obtain the following

$$\frac{\|e_2\|}{\|b_2\|} \leq \frac{\sigma_{\max}(A_2 A_1^\dagger)}{\sigma_{\min}(A_2 A_1^\dagger)} \frac{\|e_1\|}{\|b_1\|}. \tag{7}$$

By definition, the condition number of a non-invertible matrix, like $A_2 A_1^\dagger$, is the ratio of the maximum singular value to the minimum singular value:

$$\kappa(A_2 A_1^\dagger) \equiv \frac{\sigma_{\max}(A_2 A_1^\dagger)}{\sigma_{\min}(A_2 A_1^\dagger)}.$$

After plugging in the definition of the condition number into Eq.(7), we have completed the proof:

$$\frac{\|e_2\|}{\|b_2\|} \leq \kappa(A_2 A_1^\dagger) \frac{\|e_1\|}{\|b_1\|}.$$

□