CS4220 Project 2

Due: March 18, 2015

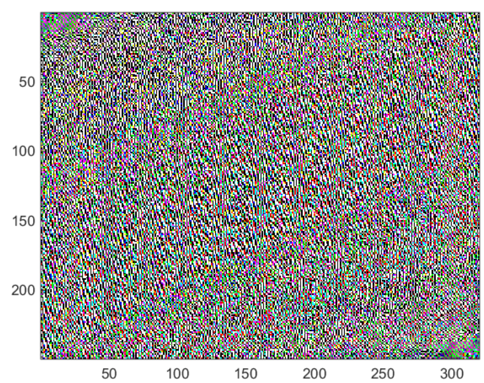
1. The Naive Approach

The simplest reconstruction approach is to simply solve

$$V^{\text{naive}} \approx H^{-1} V^{\text{blur}}.$$

1. What does the reconstructed image look like?

   Solution: The picture is full of small speckles, indicating a complete failure in restoring the blurred image:



2. For this image, give an estimated bound on $\|HV^{\text{orig}} - V^{\text{blur}}\|_F / \|V^{\text{blur}}\|_F$, and use the MATLAB function `condest` to estimate the condition number of H. Use these estimates to explain the picture.

   Solution: The ratio of $\|HV^{\text{orig}} - V^{\text{blur}}\|_F / \|V^{\text{blur}}\|_F$ provides a measure of the residual error relative to the norm of the blurred image matrix. In the naive approach, $\frac{1/2}{\|V^{blur}\|_F}$, note that the error comes from rounding. On the other hand, the estimated condition number of the image transformation matrix H is a huge number: `4.2227E6`, which indicates that we have a very ill-conditioned numerical problem — the system matrix is nearly singular. This explains the reason for the complete failure of the naive approach to the image-deburring problem. Now we can provide an explanation to the apparently good (tiny) value of the residual error of the linear equation:

$$HV = V^{\text{blur}}.$$

Because the problem is so ill-conditioned, the very small residual error itself does not guarantee us the quality of the solution, due to the fact that the system matrix H is nearly singular.

2. Tikhonov regularization

A better approach to deblurring the image is *Tikhonov regularization*:

$$V^{\text{tik}} = \text{argmin}_V \, \|HV - V^{\text{blur}}\|_F^2 + \beta^2 \|V\|_F^2. \tag{1}$$

Before we proceed to solve the Tikhonov reguarlized equation, we first replace the Frobenius norms by the ℓ-2 norms in Eq.(1), then the problem is immediately recognized as a least-squares problem modified with the regularization parameter β:

$$(H^T H + \beta^2 I)V = H^T V^{\text{blur}}. \tag{2}$$

If we stick with the Frobenius instead of the ℓ-2 norms, the matrix derivative of the Tikhonov regularization equation is more complicated, but fundamentally we seek the solution of the following equation:

$$\frac{\partial V^{\text{tik}}}{\partial V} = 0. \tag{3}$$

The derivative has to be carefully obtained in the following steps:

$$\begin{aligned} \frac{\partial V^{\text{tik}}}{\partial V} &= 2(HV - V^{\text{blur}})^T H + 2\beta^2 V^T = 0 \\ \left(V^T H^T - (V^{\text{blur}})^T\right) H &+ \beta^2 V^T = 0 \\ V^T (H^T H + \beta^2 I) &= (V^{\text{blur}})^T H \\ (H^T H + \beta^2 I) V &= H^T V^{\text{blur}} \end{aligned} \tag{4}$$

This is exactly the same as Eq.(2).

1. Show that the Tikhonov regularized problem is equivalent to

$$V^{\text{tik}} = \text{argmin}_V \left(\left\| \begin{bmatrix} H \\ \beta I \end{bmatrix} V - \begin{bmatrix} V^{\text{blur}} \\ 0 \end{bmatrix} \right\|_F^2 \right) \tag{5}$$

The Frobenius norm of an $m \times n$ matrix A is defined as the square root of the sum of the absolute squares of all its elements:

$$\|A\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2}\,, \quad \text{or} \quad \|A\|_F^2 = \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2. \tag{6}$$

The arguments inside the matrix norm $\|\cdot\|_F$ in Eq.(5) can be combined according to the block-matrix manipulation rules as follows:

$$\begin{bmatrix} H \\ \beta I \end{bmatrix} V - \begin{bmatrix} V^{\text{blur}} \\ 0 \end{bmatrix} = \begin{bmatrix} HV - V^{\text{blur}} \\ \beta V \end{bmatrix}. \tag{7}$$

Since the Frobenius norm of a matrix requires the sum of the squares of all its elements, the Frobenius norm of the matrix in Eq.(7) is obviously equal to the sum of the Frobenius norms of its *submatrices*:

$$\left\| \begin{bmatrix} HV - V^{\text{blur}} \\ \beta V \end{bmatrix} \right\|_F^2 = \left\| HV - V^{\text{blur}} \right\|_F^2 + \beta^2 \left\| V \right\|_F^2 .$$

Combining the above equation with Eq.(7), we have got what we want to show.

2. Write an analytic formula for the singular values of the reguarlized matrix

$$\begin{bmatrix} H \\ \beta I \end{bmatrix}.$$

Call the Tikhonov regularized matrix as matrix A:

$$A = \begin{bmatrix} H \\ \beta I \end{bmatrix}.$$

By definition, the singular values of A are the square roots of eigenvalues of $A^T A$, which can be easily obtained by multiplying the block matrices as follows:

$$A^T A = \begin{bmatrix} H^T \beta I \end{bmatrix} \begin{bmatrix} H \\ \beta I \end{bmatrix} = H^T H + \beta^2 I. \tag{8}$$

The eigenvalues of $H^T H$ are σ_i^2, and the associated eigenvectors are v_i, $i = 1, 2, \ldots, n$, we can express the following according to the definition of eigenvalues and vectors:

$$(H^T H + \beta^2 I) v_i = (\sigma_i^2 + \beta^2) v_i \, ,$$

so the eigenvalues of matrix $H^T H + \beta^2 I$ is simply $\sigma_i^2 + \beta^2$. Consequently, the singular values of matrix $\begin{bmatrix} H \\ \beta I \end{bmatrix}$ are

$$\sigma_A = \sqrt{\sigma_i^2 + \beta^2},$$

where σ_i is the i-th singular value of matrix H.

3. What would be the complexity of solving this ordinary least-squares system using dense QR factorization?

   For each iteration step the algorithm computes the QR factorization of a full $n \times n$ matrix, so each step has complexity of $\mathcal{O}(n^3)$. Convergence of the algorithm can be arbitrarily slow if the eigenvalues are very close to each other.

3. Landweber Iteration

The *Landweber iteration* is a fixed point iteration of the form:

$$V^{(k+1)} = V^{(k)} + \alpha H^T (V^{\text{blur}} - HV^{(k)}). \tag{9}$$

1. Show that the fixed point iteration converges to the solution to the normal equations whenever

   $$0 < \alpha < \frac{2}{\sigma_1^2},$$

   where σ_1 is the largest singular value of H. Computing σ_1^2 can be expensive, but there is a cheap upper bound:

   $$\sigma_1^2 \leq \|H\|_1 \|H\|_\infty.$$

   In this particular case, $\|H\|_1 = \|H\|_\infty = 1$ and σ_1^2 turns out to be just a smidge less than one.

   Proof. We can rewrite Eq.(9) in the following

   $$V^{(k+1)} = (I - \alpha H^T H)V^{(k)} + \alpha H^T V^{\text{blur}}, \tag{10}$$

   and we can derive the next equation easily from Eq.(10):

   $$V^{(k+1)} - V^k = (I - \alpha H^T H)(V^{(k)} - V^{(k-1)}). \tag{11}$$

   ℓ-2 norms of matrices are implied everywhere in this proof. Based on Eq.(11), we can calculate the ℓ-2 norms as follows:

   $$\|V^{(k+1)} - V^k\| = \|(I - \alpha H^T H)(V^{(k)} - V^{(k-1)})\| \tag{12}$$
   $$\leq \|(I - \alpha H^T H)\| \cdot \|(V^{(k)} - V^{(k-1)})\|. \tag{13}$$

   For brevity, we designate $(I - \alpha H^T H)$ as matrix M, and the eigenvalues of M as λ_i. The absolute value of the eigenvalues $|\lambda_i|$ of $M = (I - \alpha H^T H)$ is related to the eigenvalues (or singular values) of $H^T H$ as follows:

   $$|\lambda_i| = |1 - \alpha \sigma_i^2|, \quad i = 1, \ldots, n, \tag{14}$$

where σ_i is the i-th singular value of H (the square root of the eigenvalue of $H^T H$), and the index has been arranged such that $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq 0$. It can be shown that the ℓ-2 norm of M is the *largest* square root of the eigenvalue of $M^T M$, which is $(|\lambda_1| \cdot |\lambda_1|)^{1/2}$, and using λ_1 from Eq.(14):

$$\|(I - \alpha H^T H)\|_2 = \left((|1 - \alpha\sigma_1^2|)^2 \right)^{1/2}. \tag{15}$$

Substitute Eq.(15) into the inequality in Eq.(13), we have

$$\frac{\|V^{(k+1)} - V^k\|}{\|(V^{(k)} - V^{(k-1)})\|} \leq |1 - \alpha\sigma_1^2|.$$

The Landweber fixed point iteration converges if this ratio is less than 1:

$$\frac{\|V^{(k+1)} - V^k\|}{\|(V^{(k)} - V^{(k-1)})\|} \leq |1 - \alpha\sigma_1^2| < 1.$$

Consequently we need

$$-1 < 1 - \alpha\sigma_1^2 < 1,$$

and the bounds for α:

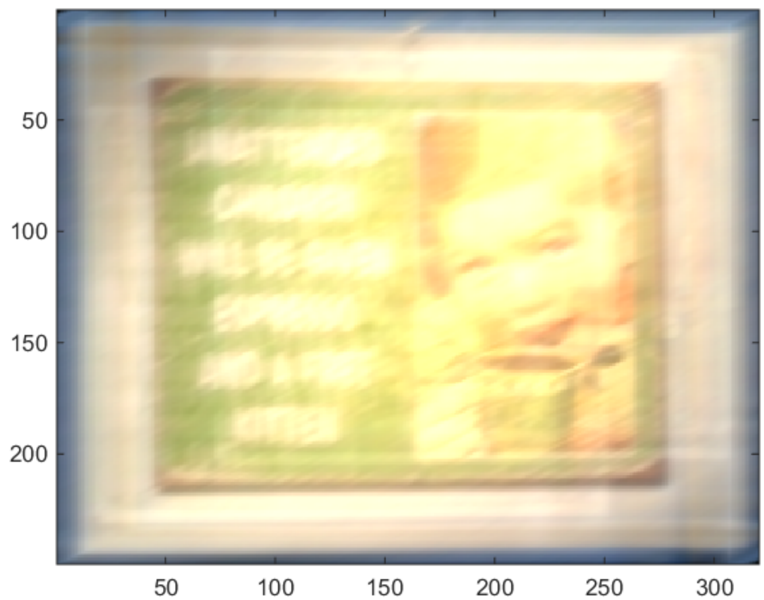
$$0 < \alpha < \frac{2}{\sigma_1^2}.$$

□

2. My script uses a default value of $\alpha = 2$. Do you get better results with other α?

   I have run a number of Landweber iterations from 2.0 to 0.5, the picture quality starts to deteriorate slightly for $\alpha < 1.0$. For $1 < \alpha < 2$, no significant changes have been observed (maybe a little bit brighter at $\alpha = 1.5$ and 1.0.

3. After around the 19 iterations, one barely starts making out the words. But it takes close to 100 iterations to get a image of reasonable quality.



4. The Tikhonov regularization method clearly has better image quality. Run time as generated by `tic toc` in MATLAB shows Tikhonov: 6.27426s and Landweber: 7.07217s (100 iterations) .

4. LSQR Iteration

The LSQR iteration is a Krylov subspace iteration that minimizes the residual for least-squares problems or ill-posed linear systems. Like the Landweber iteration, LSQR tends to have a regularizing effect when it is not run all the way to convergence.

1. Compare the run time, image quality, and residual error (measured in Frobenius norm) for LSQR as compared to Landweber with $\alpha = 2$. Comment on what you see for 30 iterations and for 100 iterations.

```
Run time:                Residual Error                 Quality
LSQR:       7.31017 s    5.6040e-04 (200 iterations)    Best
Tikhonov:   7.21675 s    8.1889e-04                     very good
Landweber:  5.96845 s    0.2083    (100 iterations)     good
```

LSQR with 200 iterations provides the best quality of the image, with Tikhonov being very close, the Landweber iteration has a lower quality. But if we tuned the β parameter in Landweber iterations, we can get a substantially improved quality (for instance at $\beta = 0.05$).

For LSQR, the differences between 30 iterations and 100 iterations are visible. Both images are clear, but the image after 100 iterations are smoother.

Restored image using LSQR for 200 iterations.