Proj 3: MEMS Madness

1 Introduction

Unlike previous projects, most of the work in Project 3 is in getting the code right. In part, this is because much of the interesting analysis — the scaling of the pull-in, convergence of the discretized problem, analysis of model error, or direct computation of the pull-in point — requires some physics and some numerical PDE background that is probably beyond the scope of the course. Of course, I can still cover some of this in my own write-up!

2 Scaling

The finite element code actually solves the differential equations

$$\begin{aligned} EIu_{xxxx} &= -\frac{\epsilon_0 b V^2}{2(g_0+u)^2}, \quad x \in (0,L) \\ u(0) &= u_x(0) = 0 \\ u_{xx}(L) &= u_{xxx}(L) = 0 \end{aligned}$$

where

- E is Young's modulus,
- $I = bh^3/12$ is the moment of inertia for a beam with a rectangular cross-section with width b and depth h,
- ϵ_0 is the permittivity of free space, and
- g_0 is the initial gap.

The left side of the first equation and the boundary conditions are associated with Euler-Bernoulli beam theory for a cantilever. The right hand side of the first equation comes from a parallel plate approximation of the electrostatic interaction (i.e. the assumption that the electrostatic potential varies linearly across the gap at each point). All quantities have been scaled corresponding to a characteristic length scale of one micron (10^{-6} meters). Without this scaling step, MATLAB immediately complains of ill-conditioning.

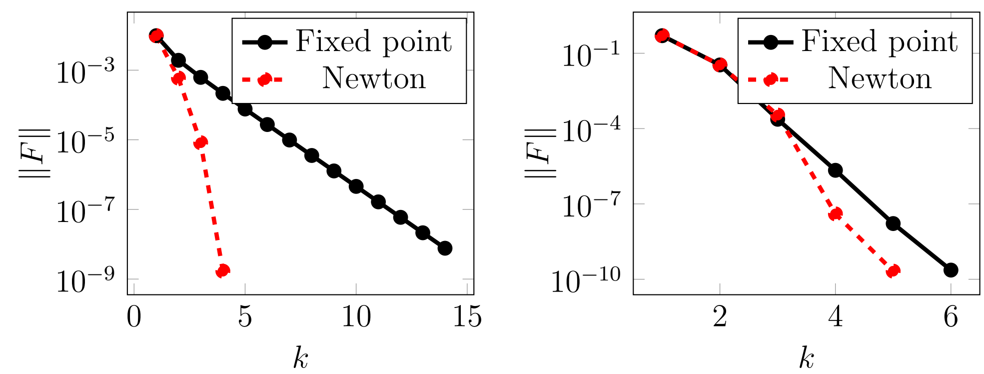

Figure 1: Left: Residual error vs iteration for fixed point and Newton iterations under voltage control at $V = 6.3$, starting from a zero initial guess for the displacement. The linear vs quadratic convergence is clear on a semi-logarithmic plot. Right: Residual for iterations under displacement control at $d = -0.5$, starting from a zero initial guess for displacement and voltage.

It's illuminating to rewrite the first equation as

$$\tilde{u}_{ssss} = -\frac{V^2/M}{2(1+\tilde{u})^2}, \quad s \in (0,1), \quad \text{where} \quad M = \frac{Eg_0h^3}{12\epsilon_0 L^4}$$

and where $s = x/L$ and $\tilde{u}(s) = u(sL)/g_0$. The pull-in voltage therefore behaves like $V = V_*\sqrt{M}$, where V_* is independent of the geometry and material parameters. The deformation can similarly be reconstructed by rescaling a single reference deformation $\tilde{u}$. This sort of scaling analysis is a useful prelude to any numerical computation – and had I given it away in advance, you might have never bothered with the sweeps in part 3!

3 Iteration convergence

Both the fixed point iteration and the Newton iterations converge as expected (see Figure 1). Under voltage control, we follow the strategy outlined in the prompt; under displacement control, we follow both Newton and the fixed point iteration

$$\begin{bmatrix} Ku - f_e(u,V) \\ u_{\text{tip}} - d \end{bmatrix} + \begin{bmatrix} K & \frac{\partial f_e}{\partial V^2}(u,V) \\ e_{\text{tip}}^T & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta V^2 \end{bmatrix} = 0.$$

Let $w_{\mathrm{tip}} = K^{-1}e_{\mathrm{tip}}$; block Gaussian elimination gives us the update

$$\begin{aligned}
f &= Ku - f_e(u, V) \\
g &= \frac{\partial f_e}{\partial V^2} \\
\Delta V^2 &= -\frac{u_{\mathrm{tip}} - d - w_{\mathrm{tip}}^T f}{w_{\mathrm{tip}}^T g} \\
\Delta u &= -K^{-1}\left(f + \Delta V^2\, g\right)
\end{aligned}$$

Though linearly convergent, this fixed-point iteration is remarkably fast, and requires only a single Cholesky factorization of the matrix K rather than the repeated LU factorizations of Newton iteration.

4 Tracing the bifurcation diagram

We show the diagram of displacement vs voltage in Figure 2. The parts of the diagram computed by the three recommended methods (fixed point and Newton with voltage control and Newton with displacement control) are shown with different colors and glyphs.

Under voltage control, both Newton's method and the fixed point iteration become sensitive close to the pull-in voltage. Though I would have been happy with a fixed-step-size strategy, in order to give these methods their best showing, I use an adaptive step control strategy that reduces the step size every time we need too many iterations, and retries each failed step with a smaller step size until the minimum step size is reached. Under this adaptive strategy, the fixed point iteration works up to a surprisingly large fraction of the pull-in voltage, while Newton iteration gets as close as we allow based on the smallest allowed step size (which we set to 10^{-4}).

Under displacement control, we require no adaptivity – everything just converges.

5 Computing pull-in

I provide two methods for computing pull-in. The first method is the one that I sketched in the prompt: use bisection on the tip displacement, where attempted Cholesky factorization of the effective stiffness matrix $K - \partial f_e/\partial u$

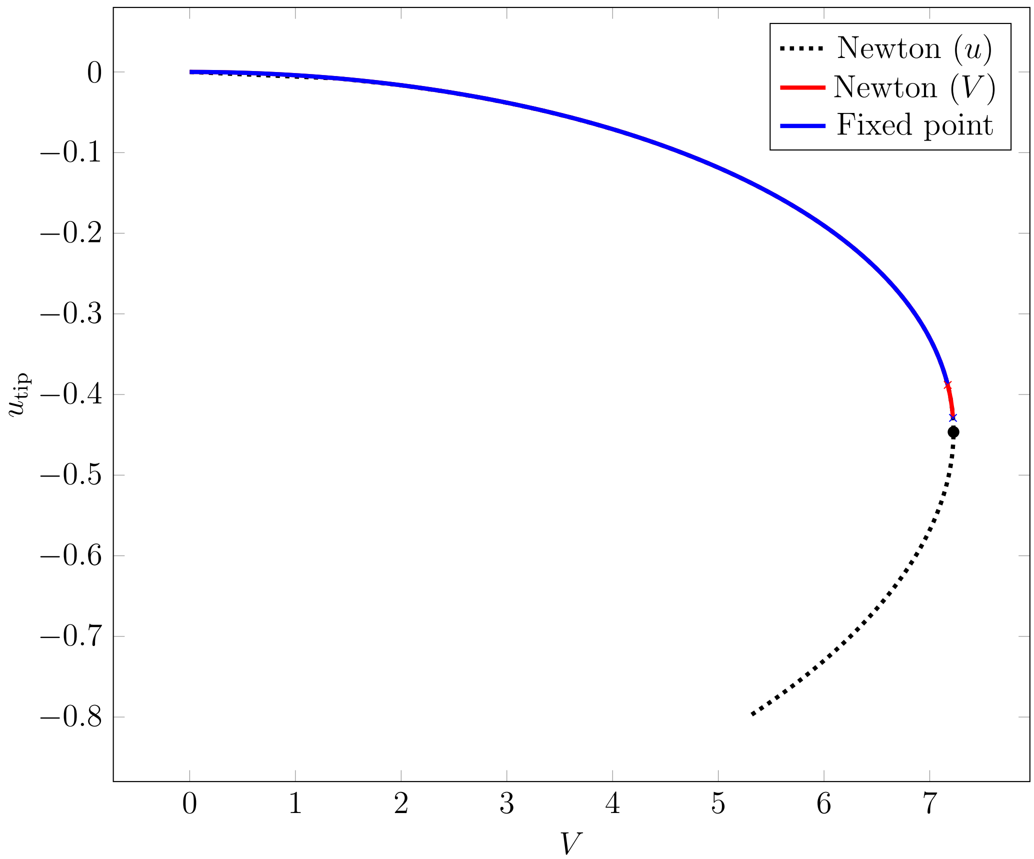

Figure 2: Tip deflection (u_{tip}) versus voltage computed by three different methods. Under voltage control, fixed point iteration behaves well up to a voltage of 7.17 (displacement -0.388), while Newton iteration behaves well up to a voltage of 7.22 (displacement -0.429). The pull-in state, marked in black, is at a voltage of about 7.2248 and tip displacement of -0.4465. Under displacement control, there were no difficulties tracing the entire curve, including the unstable states past a tip displacement of -0.4465 or so.

is used to test for stability. Because the derivative of V with respect to u is zero at the pull-in point, this method turns an $O(\epsilon)$ error in the computed displacement into an $O(\epsilon^2)$ error in the computed pull-in voltage.

6 Sweeping pull-in

The scaling analysis done at the start of this write-up actually means it is not necessary to sweep the pull-in versus geometric parameters like beam length and thickness. We can account for those effects analytically. On the other hand, doing a sweep is a good way to test that we haven't done something foolish (and is a good practice for thinking about more complicated models in which the relevant behavior depends on more than one reduced scaling parameter).

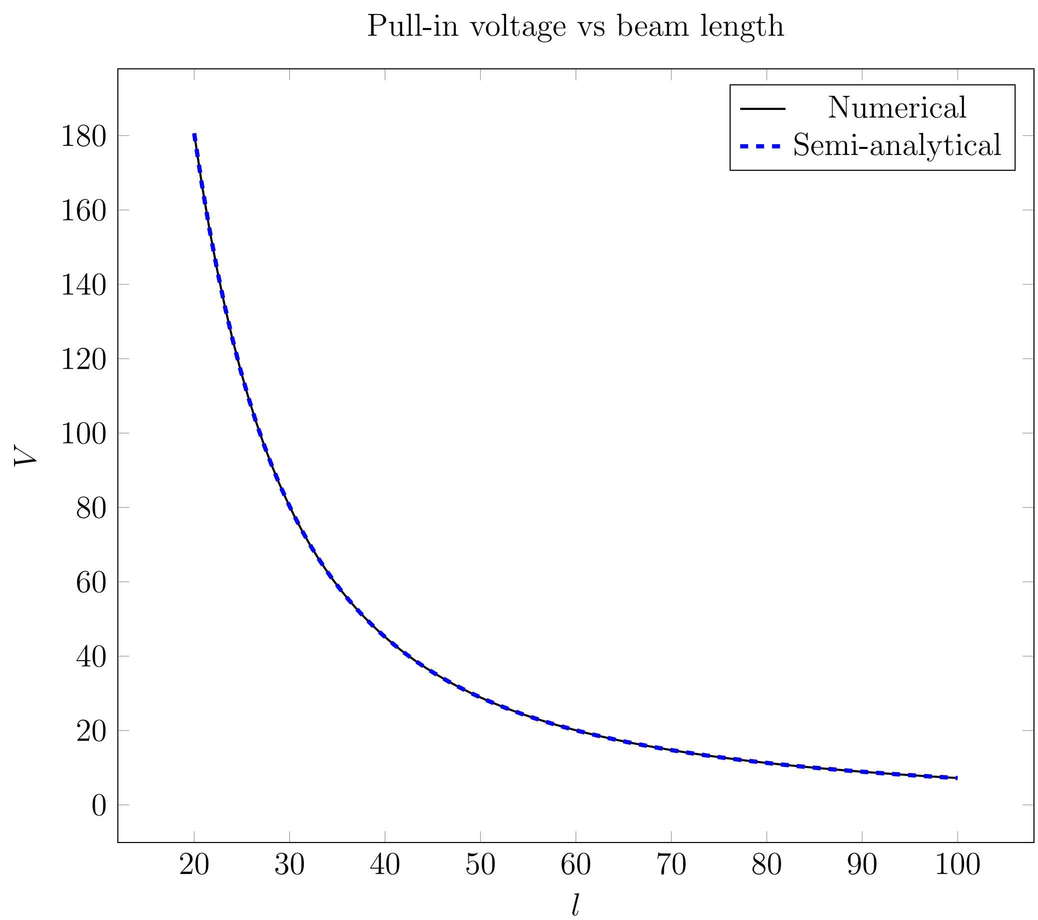

Figure 3: Pull-in voltage vs beam length. Scaling analysis says that the pull-in voltage should be proportional to L^{-2}, and this is reflected in the sweep.