Practice Final

1: About rates Consider the iteration

$$x_{k+1} = x_k + \cos(x_k).$$

1. What are the fixed points?
2. Classify each fixed point as attractive or repulsive.
3. For the attractive fixed points, what is the rate of convergence?

Answer:

1. The fixed points are $x = (n + 0.5)\pi$ for integer n.
2. The error iteration near $n\pi$ is

$$e_{k+1} = e_k - \sin(n\pi)e_k + O(|e_k|^2)$$

For n even, $\sin(n\pi) = 1$ and the iteration converges quadratically. For n odd, $\sin(n\pi) = -1$ and the iteration diverges.

2: Rewrite for accuracy Consider $f(x) \equiv \sqrt{1+x} - \sqrt{1-x}$ when $x \ll 1$. Write a routine to compute f without catastrophic cancellation.

Answer: Multiply and divide by the conjugate:

$$f(x) = \frac{(1+x) - (1-x)}{\sqrt{1+x} + \sqrt{1-x}} = \frac{2x}{\sqrt{1+x} + \sqrt{1-x}}$$

The latter expression does not suffer cancellation.

3: Q-less QR MATLAB's sparse QR solver computes a "Q-less" QR decomposition, i.e. a sparse R is computed explicitly but not the dense Q factor. Given a sparse triangular R and a sparse A, describe:

- How would one efficiently solve a least squares problem involving A?
- Solves with Q-less QR iteration are less well-behaved than those involving standard QR, and so a typical implementation will do a step of iterative refinement. Write a MATLAB fragment to describe iterative refinement in this setting.

Answer:

```
% Solve the least squares problem
x = R\(R'\(A'*b));

% Iterative refinement
resid = b-A*x;
dx = R\(R'\(A'*resid));
x = x + dx;
```

4: Funky fill Consider an SPD matrix with the nonzero pattern

$$A = \begin{bmatrix} \times & & & \times \\ & \times & \times & \times \\ & \times & \times & \\ \times & \times & & \times \end{bmatrix}.$$

What is the nonzero structure of the Cholesky factor?

Answer: The factorization has one fill element:

$$R = \begin{bmatrix} \times & & & \times \\ & \times & \times & \times \\ & & \times & * \\ & & & \times \end{bmatrix}.$$

5: Frobenius Write a line of MATLAB to minimize $\|XA - B\|_F^2$ where $X \in \mathbb{R}^{n \times m}$ is unknown and $A \in \mathbb{R}^{m \times p}$ and $B \in \mathbb{R}^{n \times p}$ are given, with $m < p$.

Answer:

```
X = B/A;
```

6: Residual Suppose $r = Ax - b$ is the residual in a least squares problem. Given only b and the Q factor in an economy QR decomposition of A, what is $\|r\|$?

Answer: $\|r\| = \sqrt{\|b\|^2 - \|Q^T b\|^2}$

7: Constrained LS Write a routine to minimize $\|Ax - b\|^2$ subject to $\sum_{j=1}^n x_j = 1$.

Answer: There are several approaches. Maybe the simplest is a Lagrange multiplier formulation

```
e = ones(size(A,2),1);
x = [A'*A, e; e',  0]\[A'*b; 1];
```

One can also manipulate this to get something involving QR.

8: Transfer trouble Suppose $A = QHQ^T$ is upper Hessenberg. Argue that the Hessenberg form provides a way of computing the transfer function $h(s) = c^T(A - sI)^{-1}b$ in $O(n^2)$ time for arbitrary s. Give MATLAB code; you may assume backslash with a Hessenberg matrix requires $O(n^2)$ time (because it does!) – explain why.

Answer: The LU decomposition of a Hessenberg matrix only ever has to work with two rows at a time – the Schur complements remain upper Hessenberg throughout. Therefore, the cost is n updates of $O(n)$ cost each, or $O(n^2)$. Using this, and observing that

$$(A - sI)^{-1} = Q(H - sI)^{-1}Q^T,$$

we have

```
bhat = Q'*b;
chat = Q'*c;
h = bhat'*((H-s*I)\chat);
```

9: Diagonal decisions Consider the block 2×2 matrix

$$M = \begin{bmatrix} A & B \\ B & D \end{bmatrix}$$

where $A, B, D \in \mathbb{R}^{n\times n}$ are orthogonal. Describe an efficient algorithm for computing all the eigenvalues of M.

Answer: Compute the eigenvalues of the 2-by-2 matrices

$$\begin{bmatrix} a_i & b_i \\ b_i & d_i \end{bmatrix},$$

which are just roots of the polynomials

$$(a_i - \lambda)(d_i - \lambda) = b_i^2$$

10: Jacobi jumble What is the rate of convergence of Jacobi on a diagonal matrix?

Answer: M is the whole matrix, so it converges in one step.

11: Killer Krylov Suppose $\hat{A} = A + uv^T$. Given A^{-1} as a preconditioner, how many steps of a Krylov subspace method are required to solve a system with $\hat{A}$, and why?

Answer: $\hat{A}^{-1}A$ has two eigenvalues, so the iteration converges in two steps.

12: Line search Does the Armijo condition for a line search guarantee that $\|x_{k+1} - x_*\| < \|x_k - x_*\|$? Why or why not?

Answer: The Armijo condition controls the residual, not the error. The error might actually increase.

13: Simple solver Suppose $(A + \eta \operatorname{diag}(x))x = b$. Under what conditions does the fixed point iteration

$$Ax_{k+1} = b - \eta \operatorname{diag}(x_k)x_k$$

converge? Give a bound on the rate of convergence.

Answer: The error iteration is

$$Ae_{k+1} = -\eta 2 \operatorname{diag}(x_*)e_k + O(\|e_k\|^2)$$

Multiplying through by A^{-1} and taking norms gives

$$\|e_{k+1}\| \leq 2|\eta|\|A^{-1}\|\|x_*\|\|e_k\| + O(\|e_k\|^2).$$

Therefore, we have

$$\|e_{k+1}\| \leq \gamma\|e_k\| + O(\|e_k\|^2)$$

where $\gamma = 2|\eta|\|A^{-1}\|\|x_*\|$.

14: Differential deal Suppose H is positive semi-definite, and consider the trust region step

$$(H + \lambda I)p = -\nabla\phi.$$

show that at $\lambda = 0$, $dp/d\lambda = H^{-2}\nabla\phi$.

Answer: Recall that

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0} (H + \lambda I)^{-1} = -H^{-1}IH^{-1} = -H^{-2}.$$

15: Modified Gauss-Newton Consider the Gauss-Newton-like iteration

$$p_k = M^{-1} J(x_k)^T r(x_k)$$

where M is a fixed positive definite matrix that we hope approximates the matrix $J^T J$ at the minimizer of $\phi(x) = \|r(x)\|^2/2$.

1. Write a short MATLAB code to implement the iteration efficiently. You may take $O(mn^2)$ setup time, but you should only require $O(mn)$ time per step.
2. Show that p_k is a descent direction (so the iteration will converge with line search).
3. Give conditions on M and r such that the iteration is guaranteed to be locally convergent without line search. You may assume the Jacobian satisfies a Lipschitz condition $\|J(x) - J(y)\| \leq \gamma \|x - y\|$.

Answer: The iteration might look like

```
R = chol(M);
for k = 1:maxiter
  resid = J(x)'*r(x);
  x = x - R'\(R\(resid));
end
```

Note that

$$p_k^T \nabla \phi(x_k) = -\nabla \phi(x_k)^T J M^{-1} J^T \nabla \phi(x_k) < 0$$

by positive definiteness of M^{-1}, assuming x_k is not a stationary point.

The error iteration is

$$e_{k+1} = e_k - M^{-1} J(x_* + e_k)^T r(x_* + e_k)$$

which, using the Lipschitz condition, yields

$$\|e_{k+1}\| \leq (\|I - M^{-1} J_*^T J\| + \|M^{-1}\| \gamma) \|e_k\|$$

Therefore, a sufficient condition for convergence is that

$$\|I - M^{-1} J_*^T J_*\| + \|M^{-1}\| \gamma < 1$$