Homework 5 Solution

Spring 2018

Question 1 (Permutation Identity, 50 points)**.** .

(a) Let $p_{d,o}(n)$ be the number of partitions of n into distinct, odd parts. Find a formula for the generating function $\sum_{n=0}^{\infty} p_{d,o}(n)x^n$. [20 points]

(b) Let $p_{even}(n)$ (resp. $p_{odd}(n)$) denote the total number of partitions of n into an even (resp. odd) number of parts. Find a formula for the generating function $\sum_{n=0}^{\infty}(-1)^n(p_{even}(n) - p_{odd}(n))x^n$. [20 points]

(c) Show that the above generating functions are the same and conclude $p_{d,o}(n) = |p_{even}(n) - p_{odd}(n)|$ for all n. [10 points]

[Note: the generating functions above may need to involve infinite products.]

Solution. *(a) We consider the infinite product*

$$(1 + x)(1 + x^3)(1 + x^5)\ldots = \prod_{k=0}^{\infty}(1 + x^{2k+1}).$$

We pick x^{2j+1} from jth parentheses on the left hand side to form the coefficient of x^n. Then we have

$$(2j_1 + 1) + (2j_2 + 1) + \ldots + (2j_m + 1) = n.$$

which give us a partition into distinct, odd parts because each j_i can be only pick by once. And it is easy to see that every partition into distinct, odd parts have a corresponding way to pick the parentheses. Therefore, the above infinite product is exactly the generating function for $\sum_{n=0}^{\infty} p_{d,o}(n)x^n$.

(b) We add one more parameter into our generating function for partition.

We considering the following infinite product

$$G(u, x) = \prod_{k=1}^{\infty}(\frac{1}{1 - ux^k}) = (1 + ux + u^2x^2 + \ldots)(1 + ux^2 + u^2x^4 + \ldots)\ldots.$$

Note that, for the term $u^k x^{kj}$ (we pick k copies of j in partition),the power of u is recording the number of copies of j we picked. Therefore, the coefficient of $u^m x^n$ will be the number of partition of n into m parts.

Especially, when $u = -1$, the positive coefficient in $G(-1, x)$ is $\sum_{n=0}^{\infty} p_{even}(n)x^n$ and the negative coefficient in $G(-1, x)$ is $\sum_{n=0}^{\infty}(-1)p_{odd}(n)x^n$. This gives us

$$G(-1, x) = \prod_{k=1}^{\infty}(\frac{1}{1 + x^k}) = \sum_{n=0}^{\infty}(p_{even}(n) - p_{odd}(n))x^n.$$

Moreover,

$$G(-1, -x) = \prod_{k=1}^{\infty}(\frac{1}{1 + (-1)^k x^k}) = \sum_{n=0}^{\infty}(-1)^n(p_{even}(n) - p_{odd}(n))x^n.$$

(c) *We want to show*

$$\prod_{k=0}^{\infty}(1+x^{2k+1}) = \prod_{k=1}^{\infty}(\frac{1}{1+(-1)^k x^k}).$$

It is equivalent to show (replacing x by $-x$)

$$\prod_{k=0}^{\infty}(1-x^{2k+1}) = \prod_{k=1}^{\infty}(\frac{1}{1+x^k}).$$

$$\begin{aligned}\prod_{k=1}^{\infty}(\frac{1}{1+x^k}) &= \prod_{k=1}^{\infty}(\frac{1-x^k}{1-x^{2k}}) \\ &= \frac{1-x}{}\frac{1-x^2}{1-x^2}\frac{1-x^3}{}\frac{1-x^4}{1-x^4}\dots \\ &= (1-x)(1-x^3)\dots \\ &= \prod_{k=0}^{\infty}(1-x^{2k+1})\end{aligned}$$

Question 2 (Generating Functions, 50 points)**.** *Find expressions for the following generating functions: [10 points each]*

(a) *Let a_n be the number of compositions of n into odd parts where each part is colored either red or blue (for example $a_1 = 2$ since you can write 1 as either a red 1 or a blue 1). Give the ordinary generating function $\sum_{n=0}^{\infty} a_n x^n$.*

(b) *Let b_n be the difference between the number of compositions of n into an even number of parts of size 1 and 2, and the number of compositions of n into an odd number of parts of size 1 and 2. Give a closed form for the ordinary generating function $\sum_{n=0}^{\infty} b_n x^n$. Use this to give an explicit formula for b_n.*

(c) *Let $c_n = nC_n$ where C_n is the Catalan number. Give an explicit formula for the ordinary generating function $\sum_{n=0}^{\infty} c_n x^n$. [Hint: you might want to consider differentiating the generating function for the Catalan numbers.]*

(d) *Let d_n be the number of set partitions of $[n]$ into sets of size exactly 2 [Note: d_n will be 0 if n is odd]. Find a formula for d_n, and use it to obtain an explicit form for the exponential generating function $\sum_{n=0}^{\infty} d_n x^n/n!$.*

(e) *Let e_n be a sequence with $e_{n+2} = e_{n+1} + ne_n$ for all $n \geq 0$. Give a differential equation satisfied by the exponential generating function $E(x) = \sum_{n=0}^{\infty} e_n x^n/n!$.*

Solution. *(a) Let b_n be the number of compositions of n into 1 odd parts where each part is colored either red or blue. Clearly,*

$$b_n = \begin{cases} 2 & n \text{ is odd} \\ 0 & n \text{ is even} \end{cases}$$

Therefore, let $b_0 = 0$, the generating function $B(x) = \sum_{n=0}^{\infty} b_n x^n = 2x + 2x^3 + 2x^5 \dots = \frac{2x}{1-x^2}$.

Now, by Theorem 8.13 in the book,

$$\sum_{n=0}^{\infty} a_n x^n = \frac{1}{1-B(x)} = \frac{1}{1-\frac{2x}{1-x^2}} = \frac{1-x^2}{1-2x-x^2}$$

with $a_0 = 1$.

(b) *We note that the coefficient of* x^n *in* $(x+x^2)^m$ *is the number of composition of* n *into* m *parts of size 1 and 2. Then if we donate* $p_{even}(n)$ *(resp.* $p_{odd}(n)$*) be the number of composition of* n *into even (resp. odd) number of parts of size 1 and 2,*

$$\sum_{n=0}^{\infty} p_{even}(n)x^n = 1 + (x+x^2)^2 + (x+x^2)^4 + \dots$$

and

$$\sum_{n=0}^{\infty} p_{odd}(n)x^n = (x+x^2) + (x+x^2)^3 + \dots.$$

Therefore,

$$\begin{aligned}\sum_{n=0}^{\infty} b_n x^n &= \sum_{n=0}^{\infty}(p_{even}(n) - p_{odd}(n))x^n \\ &= \prod_{k=0}^{\infty}(-1)^k(x+x^2)^k \\ &= \frac{1}{1+x+x^2}.\end{aligned}$$

Additionally, notice

$$\frac{1}{1+x+x^2} = \frac{1-x}{1-x^3} = (1-x)(1+x^3+x^6+\dots) = 1 - x + x^3 - x^4 + x^6 - x^7 + \dots.$$

This will give us

$$b_n = \begin{cases} 1 & n \equiv 0 \\ -1 & n \equiv 1 \\ 0 & n \equiv 2 \end{cases} \qquad \mathit{mod}\ 3$$

(c) *The generating function for Catalan number is*

$$\sum_{n=0}^{\infty} C_n x^n = \frac{2}{1+\sqrt{1-4x}}.$$

Taking derivative on both sides, we get

$$\sum_{n=1}^{\infty} nC_n x^{n-1} = \frac{d}{dx}\frac{2}{1+\sqrt{1-4x}}.$$

$$\implies \sum_{n=1}^{\infty} c_n x^n = x * \frac{d}{dx}\frac{2}{1+\sqrt{1-4x}}$$

$$\implies \sum_{n=0}^{\infty} c_n x^n = \frac{4x}{\sqrt{1-4x}(1+\sqrt{1-4x})^2}.$$

(d) *when* n *is odd,* d_n *is 0.*
When $n = 2k$ *for* $k = 0, 1, 2, \dots,$ $b_{2k} = \binom{2k}{2,2,\dots,2}/k!$. *We divide it by* $k!$ *because of the number of ways to*

pick same k' 2-element subset. Therefore, $b_{2k} = \frac{(2k)!}{2^k k!}$ *for* $k = 0, 1, 2, \ldots$. *Then,*

$$\begin{aligned}\sum_{n=0}^{\infty} d_n x^n/n! &= \sum_{k=0}^{\infty} d_{2k} x^{2k}/2k! \\ &= \sum_{k=0}^{\infty} \frac{(2k)!}{2^k k!} x^{2k}/2k! \\ &= \sum_{k=0}^{\infty} \frac{1}{2^k k!} x^{2k} \\ &= e^{\frac{x^2}{2}}\end{aligned}$$

(e) *Taking derivative of* $E(x)$*, we get*

$$E'(x) = \sum_{n=1}^{\infty} e_n x^{n-1}/(n-1)! = e_1 + \sum_{n=1}^{\infty} e_{n+1} x^n/n!.$$

Moreover,

$$xE'(x) = x \sum_{n=1}^{\infty} n e_n x^{n-1}/n! = \sum_{n=1}^{\infty} n e_n x^n/n!.$$

If we add above 2 equality, we get

$$E'(x) + xE'(x) - e_1 = \sum_{n=1}^{\infty} (e_{n+1} + n e_n) x^n/n! = \sum_{n=1}^{\infty} e_{n+2} x^n/n!.$$

However,

$$E''(x) = e_2 + \sum_{n=1}^{\infty} e_{n+2} x^n/n!$$

This gives

$$E'(x) + xE'(x) - e_1 = E''(x) - e_2.$$

As the result, $E(x)$ *satisfies the differential euqation*

$$f'' - (1+x)f' + e_1 - e_2 = 0.$$