# Math 184A Homework 3

Fall 2016

This homework is due on gradescope by Friday November 11th at 11:59pm. Remember to justify your work even if the problem does not explicitly say so. Writing your solutions in LaTeXis recommend though not required.

**Question 1** (Combinatorial Identity, 20 points). *Come up with a combinatorial proof of the following identity for $n \geq 2m > 0$:*

$$\sum_{k=m}^{n-m} \binom{n}{k} c(k,m)c(n-k,m) = \binom{2m}{m} c(n,2m).$$

**Question 2** (Generating Functions, 50 points). .

(a) *Consider the sequence defined by the recurrence, $a_0 = 0, a_1 = 3$ and*

$$a_{n+2} = a_{n+1} + 2a_n - 6$$

*for $n \geq 0$. Find a formula for the generating function $A(x) = \sum_{n=0}^{\infty} a_n x^n$. [10 points]*

(b) *Using this generating function find a formula for $a_n$ (you will want to find a partial fractions decomposition). [10 points]*

(c) *Consider the sequence defined by the recurrence, $b_0 = 0$ and*

$$b_n = n + \frac{2}{n} \sum_{i=0}^{n-1} b_i.$$

*Find a differential equation satisfied by the generating function $B(x) = \sum_{n=0}^{\infty} b_n x^n$ (you do not have to solve it). You may need to use the identity that*

$$\sum_{n=0}^{\infty} (n+1)^2 x^n = \frac{1+x}{(1-x)^3}.$$

*Note: For those of you interested in computer science, $b_n$ is related to the runtime of the quicksort algorithm. [15 points]*

(d) *It turns out that the generating function above is given by*

$$B(x) = \frac{2\log\left(\frac{1}{1-x}\right) - x}{(1-x)^2}.$$

*Use this to give a formula for $b_n$. You may need to use the harmonic numbers $H_k = \sum_{n=1}^{k} \frac{1}{n} \approx \log(k)$ to express your answer. Recall that $\log(1/(1-x)) = \sum_{n=1}^{\infty} x^n/n$. [15 points]*

**Question 3** (Partition Generating Functions, 30 points)**.** *(a) Let $a_n$ be the number of integer partitions of $n$ into distinct parts. Show that this sequence has the generating function*

$$\sum a_n x^n = (1+x)(1+x^2)(1+x^3)\cdots = \prod_{n=1}^{\infty}(1+x^n).$$

*[10 points]*

*(b) Let $b_n$ be the number of integer partitions of $n$ into odd parts. Show that this sequence has the generating function*

$$\sum b_n x^n = \frac{1}{(1-x)(1-x^3)(1-x^5)\cdots} = \prod_{n=1}^{\infty}\frac{1}{1-x^{2n-1}}.$$

*[10 points]*

*(c) Show directly that the above generating functions are equal. [10 points]*

**Question 4** (Extra credit, 1 point)**.** *Approximately how much time did you spend working on this homework?*