Math 184A Solutions 2

Spring 2018

Question 1. *The left hand side count the number of partitions of n into k parts. The right hand side count the number of partitions of $n-k$ into at most k parts. We will construct a bijection between these two sets of objects.*
For each partition of n into k parts, subtracting 1 from each part (and removing the parts of size 0) will result in a partition of $n-k$ into at most k parts. For example, $(5,3,2,2,1,1,1)$ will become $(4,2,1,1)$.
On the other hand, for each partition of $n-k$ into at most k parts, adding 1 to each part and then adding additional parts of size 1 until there are k total will produce a partition of n into k parts. This is an inverse of the first map.
So this is a bijection.

Question 2. *Consider the Ferrers shape of a partition of n into parts of distinct sizes. The rightmost column should have 1 square, and consecutive columns will differ by at most 1 square. This is because, if the rightmost column has size at least 2, then the size of top row and the next row will be the same, and thus the partition will not have distinct parts. Similarly, if there are two adjacent columns differing by at least 2, then there will exist two rows of the same size. So the conjugate of this shape is the Ferrers shape of a partition of n so that the adjacent parts have sizes differing by at most 1 and the smallest part has size 1.*
Similarly, the conjugate of the Ferrers shape of a partition of n so that the adjacent parts have sizes differing by at most 1 and the smallest part has size 1, is the Ferrers shape of a partition of n into parts of distinct sizes.
So we have establish a bijection between the set of partitions of n into parts of distinct sizes and the set of partitions of n so that the adjacent parts have sizes differing by at most 1 and the smallest part has size 1. This implies they have the same size.

Question 3. *(a)We will construct a bijection between the set of compositions of n into odd parts and the set of compositions of $n-1$ into parts of size 1 and 2.*
For any compositions of n into odd parts, we replace each odd part of size $2k+1$ with $(2,2...,2,1)$. And then delete the last part (at the very end of the composition), which is 1. This process result in a partitions of $n-1$ into parts of size 1 and 2. For example, $(5,3,3,1,3)$ will become $(2,2,1,2,1,2,1,1,2)$.
On the other hand, for every composition of $n-1$ into parts of size 1 and 2, add a part of size 1 at the end, then read the partition from left to right, whenever you read a 1, combine it with all the 2s before it consecutively to get an odd part. This is an inverse of the first map.
So this is a bijection.

(b)Denote A_n to be the number of compositions of n into odd parts. By part (a), this is also the number of compositions of $n-1$ into parts of size 1 and 2. We will prove $A_n = F_n$ by induction.
Base case: We can check that $A_1 = A_2 = 1$, Which implies the equation holds for $n = 1$ and 2.
Inductive step: Now assume $n \geq 3$, and the equation holds for $n-1$ and $n-2$. We will construct a bijection between the set of compositions of n into parts of size 1 and 2 and the set of compositions of $n-1$ and $n-2$ into parts into size 1 and 2.
For any composition of n into parts into size 1 and 2, delete the first part will result in a composition of $n-1$ or $n-2$ into parts of size 1 and 2. On the other hand, add a part of size 1 or 2 at the beginning will reverse this process, so it is a bijection.
This implies $A_n = A_{n-1} + A_{n-2} = F_{n-1} + F_{n-2} = F_n$. The equation holds for n. This finishes the induction.

Question 4. *(a)For any composition of* n *into* k *parts, deleting the first part will result in a composition of* m *into* $k-1$ *parts, for some* $0 \le m \le n-1$.
On the other hand, for any composition of $m(0 \le m \le n-1)$ *into* $k-1$ *parts, adding a part of size* $n-m$ *at the beginning will result in a composition of* n *into* k *parts, which is an inverse of the first process.*
So this is a bijection, between the set of compositions of n *into* k *parts and the compositions of numbers* $0 \le m < n$ *into* $k-1$ *parts. This implies the desired conclusion.*

(b)The number of compositions of n *into* k *parts is* $\binom{n-1}{k-1}$. *Use this fact and part (a) will give us the desired equation.*

(c)

$$P_m(n) = \sum_{i=0}^{n} i^m = \sum_{i=0}^{n}\sum_{k=0}^{m} k!S(m,k)\binom{i}{k} = \sum_{k=0}^{m}\sum_{i=0}^{n} k!S(m,k)\binom{i}{k} = \sum_{k=0}^{m} k!S(m,k)\binom{n+1}{k+1}$$