Math 184A Homework 2

Spring 2018

This homework is due on gradescope by Friday April 20th at 11:59pm. Remember to justify your work even if the problem does not explicitly say so. Writing your solutions in LaTeXis recommend though not required.

Question 1 (Partition Recurrence, 15 points)**.** *We didn't mention in class any method to compute partition numbers, but there is a relatively simple recurrence relation that can be used for them. Prove that for all $n \geq k \geq 1$ that*

$$p_k(n) = \sum_{i=0}^{k} p_i(n-k).$$

Question 2 (Partitions with Sequential Part Sizes, 15 points)**.** *Show that the number of partitions of n into parts of distinct sizes is the same as the number of partitions of n so that the adjacent parts have sizes differing by at most 1 (so in particular $a_i \geq a_{i+1} \geq a_i - 1$) and the smallest part has size 1.*

Question 3 (Compositions and Fibonacci Numbers, 30 points)**.** .

(a) Show that the number of compositions of n into odd parts is the same as the number of compositions of $n-1$ into parts of size 1 and 2 for all $n \geq 1$. [15 points]

(b) Define the Fibonacci numbers by the recurrence relation $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 3$. Show that the number of compositions of n into odd parts is F_n for all $n \geq 0$. [15 points]

Question 4 (Summation Polynomials, 40 points)**.** .

(a) Show that the number of compositions of n into k parts is the sum of m going from 0 to $n-1$ of the number of compositions of m into $k-1$ parts. [10 points]

(b) Show that for any n and k that

$$\sum_{i=0}^{n} \binom{i}{k} = \binom{n+1}{k+1}.$$

[10 points]

(c) Recall that

$$x^m = \sum_{k=0}^{m} k! S(m,k) \binom{x}{k}.$$

We would like to come up with a formula for

$$\sum_{i=0}^{n} i^m = P_m(n).$$

In particular, we claim that for each m, we claim that $P_m(n)$ is a polynomial in n. For example,

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2},$$

so $P_1(n) = n(n+1)/2$. Using the above formula and the result in part (b), give a formula for $P_m(n)$ in terms of Stirling numbers, and binomial coefficients. [20 points]

Question 5 (Extra credit, 1 point)**.** *Approximately how much time did you spend on this homework?*