# Math 184A Homework 5

Fall 2015

This homework is due Monday November 9th in discussion section. Remember to justify your work even if the problem does not explicitly say so. Writing your solutions in LaTeXis recommend though not required. **Optional Practice Problems:** (do not turn in) Chapter 7 problems 4, 9, 10, 12, 13. If you want more practice on applying Inclusion-Exclusion, you can also try problems 16, 18, 19. These unfortunately, do not have solutions written in the book, but should be relatively easy.

**Question 1** (Points in Separate Cycles, 25 points)**.** *Show that the number of permutations of* $[n]$ *so that* $1, 2, 3, \ldots, k$ *are in separate cycles is* $\frac{n!}{k!}$. *Note that this differs from the problem last week where we also required that there by exactly* $k$ *cycles. [Hint: instead of* $1, 2, \ldots, k$ *consider* $n, n-1, \ldots, n-k+1$. *What does it mean for the canonical cycle notation for a permutation for these to be in different cycles?]*

*Proof.* **By Induction**: This is an point that many people make mistakes on: the problem statement has *two* parameters, $n$ and $k$, and if we want to do induction we need to do it on only one of those variables. We choose to do induction on $n$. This means that we treat $k$ as being fixed. This is important because it tells us what our base case is! **The base case is not** $n = 1, k = 1$. Treating $k$ as fixed, our base case is $n = k$. In this case there is only one way to put $1, \ldots, k$ into their own cycles since those are all the elements, so we see $1 = 1!/1!$ and this settles the base case.

Now suppose we have shown that the number of such permutations of $[n]$ is $n!/k!$, we want to show it for permutations of $[n+1]$. Once we start with a permutation of $[n]$ such that $1, 2, \ldots, k$ are in separate cycles, we need only decide where to place $n+1$ in order to obtain every such permutation of $[n+1]$. We assume that permutations are written so that the least number of a cycle is the first number. There are $n+1$ possible placements of $n+1$. I can either place $n+1$ in an existing cycle, in which case I place it to the right of a number in the representation of a permutation for $[n]$. This gives $n$ possibilities. You should be skeptical about this operation, since there is potential ambiguity. For example, if I am placing 5 to the right of 3 in the permutation

$$(123)(4)$$

does this mean that I get (123)(54) or do I get (1235)(4)? We resolve the ambiguity by requiring that the resulting thing have its cycles start with the least number of the cycles. This means that $n+1$ can never begin a cycle, since it is larger than all the other elements. Hence in our example the resulting thing should be (1235)(4). The other possibility is if $n+1$ is in its own cycle, and this gives one possibility. Altogether there are $n+1$ ways to place the element $n+1$.

Convince yourself that every permutation of $[n+1]$ which has $1, 2, \ldots, k$ in their own cycles can be obtained from such a permutation of $[n]$ in this way, and that if I start with different permutations of $[n]$ I get different permutations of $[n+1]$. By induction we started with $n!/k!$ such permutations of $[n]$, and our procedure gives $n+1$ new permutations for each of those, hence we have $(n+1) \cdot n!/k! = (n+1)!/k!$ permutations of $[n+1]$ as desired.

**By Hint**: Convince yourself that the number of permutations with $1, \ldots, k$ in separate cycles is the same as the number of permutations with $n, n-1, \ldots, n-k+1$ in separate cycles (a bijection between the two sets is obtained by, for example, the permutation $(1\ n)(2\ n-1)\ldots(k\ n-k+1)$). Now suppose we have a permutation with $n, n-1, \ldots n-k+1$ in separate cycles, and further assume that it is written in canonical form, so cycles begin with their largest element in ascending order. This means that $n$ will lead some cycle, and it must be the last cycle that occurs. Since $n-1$ is in a different cycle than $n$, it too will lead a cycle, and that cycle will come right before the one lead by $n$. Therefore the canonical cycle form must look like

$$(\dots)\dots(n-k+1\dots)(n-k+2\dots)\dots(n-1\dots)(n\dots),$$

where the dots are filled in with elements between 1 and $n-k$. We have a lemma which states that if we remove parentheses from permutations written in canonical cycle notation and treat the resulting thing as a permutation in one line notation, this operation is a bijection. In our case this means that our resulting permutation is just a sequence of numbers with the property that $n-k+1$ occurs before $n-k+2$ which occurs before $n-k+3$ etc. Hence we need to choose which spots are taken up by $n, n-1, \dots, n-k+1$ and once we choose these places we put them in order. This gives $\binom{n}{k}$ choices. Then we may place the remaining $n-k$ elements in any order in the remaining spots, which gives $(n-k)!$ ways. Hence we have $(n-k)! \cdot \binom{n}{k} = n!/k!$ such permutations. $\square$

**Question 2** (Useless Poker Hands, 25 points)**.** *How many four card poker hands have the "high card" valuation? That is hands of four cards so that*

- *No two cards in the hand have the same rank.*
- *Not all cards have the same suit (i.e. the hand is not a flush).*
- *Not all cards have different suits (i.e. the hand is not a flash).*
- *The ranks of the cards are not all consecutive (i.e. the hand is not a straight).*

*Make sure that you don't double count anything. It is possible for a hand to be both a straight and a flush or both a straight and a flash. Please give your answer both in terms of a formula involving powers and binomial coefficients. Note that the numerical answer is 160,740. It is recommended that you use this to check your work.*

*Solution.* Let us define the following events:

$$H_1 = \{\text{No two cards have the same rank}\}, \tag{1}$$
$$H_2 = \{\text{Not all cards have the same suit}\}, \tag{2}$$
$$H_3 = \{\text{Not all cards have different suits}\}, \tag{3}$$
$$H_4 = \{\text{The hand is not a straight}\}. \tag{4}$$

We want $|H_1 \cap H_2 \cap H_3 \cap H_4|$. Note that $H_1 \cap H_2 \cap H_3 \cap H_4 = (H_1 \cap H_2) \cap (H_1 \cap H_3) \cap (H_1 \cap H_4)$, so we define $H_i' = H_i \cap H_1$ for $i = 2, 3, 4$. We do this since it makes the computation much simpler, since we are restricting ourselves only to hands where all the ranks are different. Then we are computing $|H_2' \cap H_3' \cap H_4'|$. To do this, we take the total number of hands where the ranks are different and subtract the hands where one of $H_2'$, $H_3'$, or $H_4'$ does not occur. There are $\binom{13}{4} \cdot 4^4$ total hands where all the ranks are different. Then we want to subtract $|H_2'^c \cup H_3'^c \cup H_4'^c|$ which we compute using inclusion exclusion (again we only consider hands where the ranks are all different). We must compute the sizes of these sets individually and then all possible intersections.

Note $H_2'^c$ is the set of flushes with all cards of different ranks. This is just equal to the set of flushes, of which there are $4 \cdot \binom{13}{4}$. $H_3'^c$ is the set of flash hands with all different ranks. There are $\binom{13}{4} \cdot 4!$ such hands. $H_4'^c$ is the set of straights with different ranks, which is just the set of straights. There are 10 ways to choose the ranks and then for each card we may choose the suit freely. So there are $10 \cdot 4^4$ such hands.

We now must compute the double intersections. $H_2'^c \cap H_3'^c$ is the set of flushes which are flash - an impossibility, so this contributes nothing. $H_2'^c \cap H_4'^c$ is the set of straight flushes, of which there are $10 \cdot 4$ (choose the ranks, then the suit). $H_3'^c \cap H_4'^c$ is the set of flash straights, of which there are $10 \cdot 4!$. We have the triple intersection to consider, but this is easily seen to be impossible and so contributes nothing. Putting everything together we obtain

$$\binom{13}{4} 4^4 - \left( \binom{13}{4} \cdot 4 + \binom{13}{4} \cdot 4! + 10 \cdot 4^4 \right) + (10 \cdot 4 + 10 \cdot 4!).$$

$\square$

**Question 3** (Approximate Inclusion-Exclusion, 25 points)**.** *Let $A_1, A_2, \ldots, A_n$ be finite sets. Show for even integers $m$ that*

$$|A_1 \cup A_2 \cup \ldots \cup A_n| \geq \sum_{k=1}^{m} (-1)^{k+1} \sum_{i_1 < i_2 < \ldots < i_k} |A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}|,$$

*and that for odd integers $m$*

$$|A_1 \cup A_2 \cup \ldots \cup A_n| \leq \sum_{k=1}^{m} (-1)^{k+1} \sum_{i_1 < i_2 < \ldots < i_k} |A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}|.$$

*Solution.* The idea is to count how many times a given $x \in A_1 \cup \cdots \cup A_n$ is counted on either side of the equation. Clearly $|A_1 \cup \ldots A_n|$ counts $x$ exactly once. Then it is enough to show that for any $x$ the right side of the inequality undercounts $x$ when $m$ is even and overcounts $x$ when $m$ is odd.

Let us list all the sets out of the $A_i$ which contain $x$, say $x \in A_{j_1}, x \in A_{j_2}, \ldots x \in A_{j_\ell}$, so that $x$ belongs to $\ell$ of the sets. This means that on the right side of the inequality, $x$ is counted only by those intersections indexed by some combination of $j_1, j_2, \ldots, j_\ell$, and in each such intersection it is counted exactly once. What this means is that the terms on the right are simply counting the number of combinations of the sets which contain $x$, and hence the contribution of $x$ to the right side is:

$$\sum_{k=1}^{m} (-1)^{k+1} \binom{\ell}{k}.$$

We want to show that this sum is $\leq 1$ when $m$ is even and $\geq 1$ when $m$ is odd. Using the recurrence relation for the binomial coefficients, we see that this sum is equal to

$$\sum_{k=1}^{m} (-1)^{k+1} \left( \binom{\ell - 1}{k} + \binom{\ell - 1}{k - 1} \right),$$

which one observes is a telescoping sum, equal to:

$$1 + (-1)^{m+1} \binom{\ell - 1}{m},$$

which is $\leq 1$ for $m$ even and $\geq 1$ for $m$ odd. □

**Question 4** (No Repeated Letter Count, 25 points)**.** *How many anagrams of 'AABBCCDD' (that is strings with 2 A's, 2 B's, 2 C's and 2 D's) have no repeated letters (that is letters that appear twice in a row)?*

*Solution.* We take all anagrams and subtract those with repeats. The number of all anagrams is the multinomial coefficent $\binom{8}{2\,2\,2\,2}$, and to count repeats we use inclusion exclusion. Let $R_A, R_B$, etc. denote the events where $A$ is repeated, $B$ is repeated, and so on. We want to subtract $|R_A \cup R_B \cup R_C \cup R_D|$, so we have:

$$|R_A \cup R_B \cup R_C \cup R_D| = \sum_{k=1}^{4} (-1)^{k+1} \sum_{i_1 < \ldots i_k} |R_{i_1} \cap \cdots \cap R_{i_k}|,$$

so we need to count how many ways we can get anagrams with repeats. We observe that all the $k$ fold intersections have the same size since, for example, there should be an equal number of anagrams with repeated $A$'s and $B$'s as there are anagrams with repeated $C$'s and $D$'s. So we only need to do 4 calculations. Lets calculate $|R_A|$: We have to repeat $A$, so we can think of $AA$ as being its own letter that we have to place. We now are in a situation where we have 7 objects to order, and $B$, $C$, and $D$ can be interchanged without altering the anagram. Hence there are $7!/(2!)^3$ such anagrams. Note that there are 4 such sets we have to count - $R_A, R_B, R_C, R_D$, they all have the same size, so the contribution to the inclusion exclusion formula is $\binom{4}{1} \cdot 7!/2^3$.

We now compute the size of a double intersection, $|R_A \cap R_B|$, using similar reasoning. We treat $AA$ and $BB$ as their own letters, so we now have to order 6 objects and $C$ and $D$ each have two copies, so in any anagram we may interchange the copies without changing the resulting anagram. Then we see there are $6!/2^2$ such anagrams. There are $\binom{4}{2}$ ways to choose a double intersection $|R_i \cap R_j|$, and they all have the same size, so this contributes $\binom{4}{2} \cdot 6!/2^2$.

For the triple intersections, we have 5 objects: $AA$, $BB$, $CC$, and two copies of $D$, so we order them and divide by the ways to interchange the copies of $D$: This gives $5!/2$. There are $\binom{4}{3}$ ways to choose a triple intersection. For the last case where every letter is repeated, we are just ordering 4 objects, which gives $4!$ anagrams, and there is only one quadruple intersection $|R_A \cap R_B \cap R_C \cap R_D|$. Putting it all together we see that the number of anagrams without repeats is given by

$$\binom{8}{2\,2\,2\,2} - \binom{4}{1} \cdot 7!/2^3 + \binom{4}{2} \cdot 6!/2^2 - \binom{4}{3} \cdot 5!/2 + \binom{4}{4} \cdot 4!.$$

$\square$