Homework 4 Solutions

Question 1 (Stirling Number Computation). Compute the values of $c(n,k)$ for all $1 \le k \le n \le 6$.

Answer: $c(n,n) = 1$, since the only way to split $[n]$ into n cycles is to have each cycle a singleton, which gives the identity permutation. From there, use the formula

$$c(n+1,k) = n \cdot c(n,k) + c(n,k-1).$$

So we can fill out the following table left to right, then top to bottom since $c(n,k)$ is equal to the value immediately to the left multiplied by $(n-1)$ plus the value immediately northwest. It's not too hard to figure out that $c(n,1) = (n-1)!$ and $c(n,n-1) = \binom{n}{2}$, but it's not necessary to know this.

Value of k	$c(1,k)$	$c(2,k)$	$c(3,k)$	$c(4,k)$	$c(5,k)$	$c(6,k)$
1	1	1	2	6	24	120
2		1	3	11	50	274
3			1	6	35	225
4				1	10	85
5					1	15
6						1

Question 2 (Multiple of 3 Cycles). Show that the number of permutations of $[3m]$ for which all cycles have lengths which are multiples of 3 is

$$(3m-1)(3m-2)^2(3m-4)(3m-5)^2 \cdots 2 \cdot 1^2.$$

Answer: Let $q(m) = (3m-1)(3m-2)^2(3m-4)(3m-5)^2 \cdots 2 \cdot 1^2$. Let $c_k(m)$ be the number of permutations of $[km]$ for which all cycles have lengths that are multiples of k. The book already has a proof that $c_2(m) = \text{EVEN}(2m) = (2m-1)^2(2m-3)^2 \cdots 5^2 \cdot 3^2 \cdot 1^2$ and we can use a very similar proof to show $c_3(m) = q(m)$.

Let p be a permutation with cycles that have lengths that are multiples of 3. Note that since a permutation is a bijection, (and hence injection) $p(i) = p(j)$ only if $i = j$, i.e. if $p(i) = k$, then for $j \neq i$, $p(j) \neq k$. What this means is that if we are choosing values for the permutation step-by-step, when choosing the $i+1$th value, since we've chosen i other values those i other values are not allowed. Because p has no 1-cycles, $p(1) \neq 1$, but $p(1)$ can be anything else, so we have $3m-1$ choices for $p(1)$. Because p has no 2-cycles, $p(p(1)) = p^2(1) \neq 1$, and since $p(1) \neq 1, p(p(1)) \neq p(1)$, so we have $3m-2$ choices for $p^2(1)$. We have $p^3(1) \neq p(1), p^2(1)$, but $p^3(1) = 1$ is possible, which gives us $3m-2$ choices for $p^3(1)$. If $p^3(1) = 1$, we have a completed 3-cycle and may continue our counting with the remaining $3m-3 = 3(m-1)$ elements. Continuing this process with the $3(m-1)$ other elements, we take $i \not\in \{1, p(1), p^2(1)\}$ and get $(3m-4)$ choices for $p(i)$ and $(3m-5)$ choices for $p^2(i), p^3(i)$. If $p^3(1) \neq 1$, then because we have no 4-cycles or 5-cycles we have $3m-4$ choices for $p^4(1)$ and $3m-5$ choices $p^5(1)$ and $p^6(1)$. Then, at $p^6(1)$ we again have the choice to end the cycle with $p^6(1) = 1$ or to have a cycle of length at least 9, continuing this process. It shouldn't be hard to see that the two cases $p^3(1) = 1$ and $p^3(1) \neq 1$ have the same number of choices $q(3(m-1))$ to complete the permutation for the remaining $3(m-1)$ elements, so we have overall $c_3(m) = (3m-1)(3m-2)^2 q(3(m-1))$. Continuing, we can find $q(3(m-1)) = (3m-4)(3m-5)^2 q(3(m-2))$ and so on, giving us finally that

$$c_3(m) = (3m-1)(3m-2)^2(3m-4)(3m-5)^2 \cdots 2 \cdot 1^2.$$

As a side note, it's not too hard to generalize this argument a bit to show that

$$c_k(m) = (km-1)(km-2)\cdots(km-(k-2))(km-(k-1))^2 \cdots (k-1)(k-2)\cdots 2\cdot 1^2$$

Question 3 (Two Set Inclusion-Exclusion). For sets A, B, C show that the number objects that are elements of at least two of these sets is

$$|A\cap B| + |A\cap C| + |B\cap C| - 2|A\cap B\cap C|.$$

Answer: For any two sets X and Y, let $XY = X\cap Y$, so $AB = A\cap B$ and $ABC = A\cap B\cap C$. Let $N = |A\cap B| + |A\cap C| + |B\cap C| - 2|A\cap B\cap C|$. $U = AB\cup AC\cup BC$ is precisely the set we're looking for, since every element in U is an element of at least two of A, B, and C and all elements of at least two of the sets A, B, and C must be in at least some pairing of two of the three sets, i.e. one of A and B, ($AB = A\cap B$) A and C, (AC) or B and C (BC.)

Then using the Inclusion exclusion principle,

$$\begin{aligned}
|U| = |AB\cup AC\cup BC| &= |AB| + |AC| + |BC| - (|ABAC| + |ABBC| + |ACBC|) + |ABACBC| \\
&= |AB| + |AC| + |BC| - (|ABC| + |ABC| + |ABC|) + |ABC| \\
&= |AB| + |AC| + |BC| - 2|ABC| = N
\end{aligned}$$

So we've verified that the number of objects that are elements of at least two of A, B, C is $N = |A\cap B| + |A\cap C| + |B\cap C| - 2|A\cap B\cap C|$, as requested.

Question 4 (Word Counting). How many 5-letter words have exactly two vowels (the vowels are 'a','e','i','o', and 'u' again), OR have all their letters distinct, OR have their letters appearing in alphabetical order? For example, you should count "abuzz" (two vowels and in increasing order), "alike" (all letters distinct), "aaaaa" (letters in alphabetical order), but not things like "issue".

Answer: Let A be the 5-letter words with exactly two vowels, B be the 5-letter words with all letters distinct, and C be the 5-letter words with their letters appearing in alphabetical order. Once again, let $n_{XY} = |X\cap Y|$ and $n_{XYZ} = |X\cap Y\cap Z|$. Using the Inclusion-Exclusion Principle,

$$|A\cup B\cup C| = n_A + n_B + n_C - (n_{AB} + n_{AC} + n_{BC}) + n_{ABC}.$$

From homework 1, we know $n_B = (26)_5 = 5!\binom{26}{5}$ and $n_{BC} = \binom{26}{5}$. It's not hard to see that $n_A = \binom{5}{2}5^2\cdot 21^3$, (choose the locations of the vowels, then pick among the 5 vowels twice and the 21 consonants 3 times) $n_{AB} = \binom{5}{2}(5)_2(21)_3$, (same as before, but pick distinct letters) and $n_{ABC} = \binom{5}{2}\binom{21}{3}$ (choose 2 distinct vowels and 3 distinct consonants, and there's one way to order them). The only difficult parts are n_C and n_{AC}.

For n_C, notice that if we consider each letter to be a bin labelled by that letter, any weak composition of the bins can be converted into a multiset of letters by simply including the label of each bin a number of times equal to the number of balls in the bin, i.e. $(2,0,\ldots,1,0,1,1)$ corresponds to 2 balls in the 'a' bin, 1 letter in each of the 'w', 'y', and 'z' bins, and 0 in every other bin, which converts to a, a, w, y, z. Any multiset of letters has exactly one way to sort it alphabetically, so each such multiset converts into a word sorted into alphabetical order. It should be clear that any 2 distinct weak compositions give convert to different multisets which convert to different words and that any alphabetically sorted word has a weak composition that converts to it. Hence, there's a bijection between weak compositions of 5 into 26 bins and the 5-letter words sorted in alphabetical order. So then n_C equals the number of such weak compositions, $\binom{26+5-1}{26-1} = \binom{30}{25} = \binom{30}{5}$.

Now n_{AC} is reasonably easy to calculate, since we can use the same idea by placing 2 balls into 5 bins labelled by the vowels and 3 balls into 21 bins labelled by the consonants. So then

$$n_{AC} = \binom{5+2-1}{5-1}\binom{21+3-1}{21-1} = \binom{6}{4}\binom{23}{20} = \binom{6}{2}\binom{23}{3}$$

So overall we have the number of 5-letter words that are in A, B, or C is

$$\begin{aligned}|A \cup B \cup C| &= n_A + n_B + n_C - (n_{AB} + n_{AC} + n_{BC}) + n_{ABC} \\ &= \binom{5}{2}5^2 \cdot 21^3 + (26)_5 + \binom{30}{5} - \left[\binom{5}{2}(5)_2(21)_3 + \binom{6}{2}\binom{23}{3} + \binom{26}{5}\right] + \binom{5}{2}\binom{21}{3}\end{aligned}$$