Math 184A Homework 7

Spring 2018

This homework is due on gradescope by Friday June 8th at 11:59pm. Remember to justify your work even if the problem does not explicitly say so. Writing your solutions in LaTeXis recommend though not required.

Question 1 (Avoidance Bounds, 20 points)**.** *From the book we know that* $S_n(1432) \leq 9^n$*. Find a constant* C *so that* $S_n(321456987) \leq C^n$ *for all* n*.*

Solution. *First let's prove,*

Claim 1. $S_n(123) \leq 9^n$

Proof. *Since any permutation that avoids 321 must also avoid 1432, we have* $S_n(321) \leq S_n(1432) \leq 9^n$*, and by reflection we have* $S_n(123) = S_n(321)$*. Hence the claim holds.*

Claim 2. $S_n((31245) \oplus 1) = S_n(321456) \leq 36^n$

Proof. *Since 3214 is the reverse of 4123, which is the complement of 1432, we have*

$$S_n((321) \oplus 1) = S_n(3214) = S_n(4123) = S_n(1432) \leq 9^n$$

By claim 1,

$$S_n(1 \oplus (12)) = S_n(123) \leq 9^n$$

By theorem 14.17,

$$S_n(321456) = S_n((321) \oplus 1 \oplus (12)) \leq (\sqrt{9} + \sqrt{9})^{2n} = 36^n$$

By theorem 14.17 again,

$$S_n(321456987) = S_n((32145) \oplus 1 \oplus (321)) \leq (\sqrt{36} + \sqrt{9})^{2n} = (6+3)^{2n} = 81^n$$

Hence $S_n(321456987) \leq C^n$ *holds for all* n *if we take* $C = 81$.

Question 2 (Hill Avoidance, 40 points)**.** *Let a* k*-hill in a permutation be a subsequence of* $2k-1$ *of the entries the first* k *of which are in increasing order and the last* k *of which are in decreasing order. Note that a* k*-hill is not a single pattern. For example, a* 2*-hill is either an instance of the pattern* 132 *or an instance of the pattern* 231*.*

(a) *Show that the number of permutations of* $[n]$ *with no* 2*-hill is* 2^{n-1}*. [15 points]*

   Solution. *The number of permutations with no 2-hill* $= S_n(132, 231) = \sum_{m=1}^{n}$(*the number of* n*-permutations that avoid* $(132, 231)$ *where* n *is located at the* m*-th position*)*. Suppose* n *is located at the* m*-th position.*

   (1) *If* $m = 1$, n *cannot engage in any forbidden patterns with entries that appear after* n*, hence for the whole permutation to avoid* $(132, 231)$*, it suffices to have the last* $n-1$ *entries to avoid* $(132, 231)$*. Hence the number of* n*-permutations that avoid* $(132, 231)$ *where* n *is located at the* m*-th position* $= S_{n-1}(132, 231)$*.*

(2) *If $1 < m < n$, then in order to avoid 132 any entries located at the right of n must be greater than those at the left of n, otherwise there exists (a, b) such that $a < b < n$ where a(resp. b) is at the right(resp. left) of n, then (a, b, n) will form a 132 pattern. Similarly, to avoid 231 any entries located at the right of n must be greater than those at the left of n. No permutation can satisfy both of these two requirements, hence the number of n-permutations that avoid $(132, 231)$ where n is located at the m-th position $= 0$.*

(3) *If $m = n$, n cannot engage in any forbidden patterns with entries that appear before n, hence for the whole permutation to avoid $(132, 231)$, it suffices to have the first $n-1$ entries to avoid $(132, 231)$. Hence the number of n-permutations that avoid $(132, 231)$ where n is located at the m-th position $= S_{n-1}(132, 231)$.*

Therefore we have, $S_n(132, 231) = 2 * S_{n-1}(132, 231)$, solving the recurrence relation with the initial condition $S_1(132, 231) = 1 = 2^0$, we obtain $S_n(132, 231) = 2^{n-1}$ for all n.

(b) *Show that the number of permutations of $[n]$ with no k-hill is at most $(4(k-1)^2)^n$. [Hint: try to find a decreasing sequence among elements that are the largest of a k-term increasing subsequence.] [25 points]*

Solution. *Let us say that an entry x is of left order i if x is the top of an increasing subsequence of length i, but there is no increasing subsequence of length $i+1$ whose top is x.*
Let us say that an entry x is of right order i if x is the top of an decreasing subsequence of length i, but there is no decreasing subsequence of length $i+1$ whose top is x.
The for all i, elements of left order i must form a decreasing subsequence and elements of right order i must form a increasing subsequence.
For any permutation that avoids a k-hill, the minimum of the left order and the right order of x is at most $k-1$ for all $x \in [n]$ [other wise there will be a k-hill].
We define A_m^l to be the set of all elements contained in $[n]$ whose left order is m and whose right order is greater than or equal to m for $1 \leq m \leq n$.
And we define A_m^r to be the set of all elements contained in $[n]$ whose right order is m and whose right order is greater than m for $1 \leq m \leq n$.
For any k-hill avoiding permutation, $\{A_m^l\}_{m=1}^{k-1} \cup \{A_m^r\}_{m=1}^{k-1}$ forms a partition of $[n]$. Therefore, any k-hill avoiding permutation can be decomposed into $k-1$ classes of increasing subsequence and $k-1$ classes of decreasing subsequence.There are $(2(k-1))^n$ ways to partition the elements into $2(k-1)$ classes and there are less than $(2(k-1))^n$ ways to assign each position to one of the subsequences, completing the proof.

Question 3 (Marriage Lemma, 40 points)**.** *The Marriage Lemma states that if you are given two sets S and T of size n and a set E of pairs of one element of each set, then there is a matching between S and T (namely a set of n pairs from E using each element of S and each element of T exactly once) unless there is some subset c so that the total number of elements of T that pair with some element of S' is less than $|S'|$.*

Prove the Marriage Lemma using Dilworth's Theorem.

Solution. *For all $s \in S$, let A_s be the set of elements $t \in T$ such that (s, t) is contained in E.*

(1) *If for all subsets $S' \subset S$, we have $|\bigcup_{s \in S'} A_s| \geq |S'|$*

Consider the poset P whose elements are those of $S \cup T$, where $t \leq s$ if and only if $t \in T, s \in S$ such that $t \in A_s$, and no other comparisons hold.

Claim. *Any Chain of P can consist of at most one element from S(resp. T)*

Proof. *Any two elements of S(resp. T) are not comparable, therefore they cannot be in the same chain.*

Claim. *The size of the largest antichain is n.*

Proof. *Assume for sake of contradiction that there exists an antichain of size $n+1$, suppose this antichain contain i elements from S and j elements from T, we call these elements $\{s_1, s_2 \ldots s_i\}$ and $\{t_1, t_2 \ldots t_j\}$.*

Notice that $i, j \geq 1$ *and* $i + j = n + 1 \Rightarrow j = n + 1 - i > n - i$.
For $1 \leq m \leq j$, t_m *is not comparable with* s_k *for* $1 \leq k \leq i$ *by the definition of an antichain* $\Rightarrow t_m \notin \bigcup_{k=1}^{i} A_{s_k} \Rightarrow t_m \in T \setminus \bigcup_{k=1}^{i} A_{s_k} \Rightarrow \{t_1, t_2 \dots t_j\} \subseteq T \setminus \bigcup_{k=1}^{i} A_{s_k} \Rightarrow j = |\{t_1, t_2 \dots t_j\}| \leq |T \setminus \bigcup_{k=1}^{i} A_{s_k}| = |T| - |\bigcup_{k=1}^{i} A_{s_k}| \leq n - |\{s_1, s_2 \dots s_i\}| \leq n - i$ *by our previous assumption.*
But $j > n - i$ *and* $j \leq n - i$ *cannot both hold, contradiction! Hence the size of the largest antichain is at most* n.
We show that there exists an antichain of size n.
Consider the set S, *notice that any two elements of* S *are not comparable, hence* S *is an antichain, and* $|S| = n$.
By Claim 1, any chain in P *can consist of at most* 2 *elements. By Dilworth's Theorem, the number of chains in the minimum chain cover of* P *is* n. *To cover all* $2n$ *elements in* P, *every chain in the minimum chain cover must consist of exactly* 2 *elements, namely, one from* S *and one from* T. *Since the chains are disjoint, we have therefore proved that there exist a matching between* S *and* T.

(2) If there exists subset $S' \subset S$ *such that* $|\bigcup_{s \in S'} A_s| < |S'|$
Suppose $|S'| = k \leq n$, *we can number the elements of* $S' = \{s_1, s_2 \dots s_k\}$.
Assume for sake of contradiction that there exist a matching between S *and* T, *then for* $1 \leq i \leq k$, *there exist* $t_i \in T$ *such that* $(s_i, t_i) \in E$. *By our definition of* A_s *we must have*

$$\{t_1, t_2 \dots t_k\} \subset \bigcup_{i=1}^{k} A_{s_i} = \bigcup_{s \in S'} A_s$$

But,

$$k = |\{t_1, t_2 \dots t_k\}| \leq |\bigcup_{i=1}^{k} A_{s_i}| = |\bigcup_{s \in S'} A_s| < |S'| = k$$

Contradiction ! Therefore there does not exist a matching between S *and* T.

Question 4 (Extra credit, 1 point)**.** *Free point!*