**Question 1** (Fibonacci Identity, 15 points)**.** *Recall that the Fibonacci numbers are defined by* $F_1 = F_2 = 1$ *and* $F_{n+2} = F_{n+1} + F_n$ *for all* $n \geq 0$*. Prove that*

$$\sum_{k=1}^{n} F_{2k-1} = F_{2n}$$

*for all positive integers* $n$*.*

We proceed by induction on $n$. When $n = 1$, we note that the equation says that $F_1 = F_2$, which is true. We now proceed to the inductive step. We assume that

$$\sum_{k=1}^{n} F_{2k-1} = F_{2n}.$$

Adding $F_{2n+1}$ to each side, we have that

$$F_{2n+1} + \sum_{k=1}^{n} F_{2k-1} = F_{2n} + F_{2n+1} = F_{2n+2}.$$

Rewriting this, we have that

$$\sum_{k=1}^{n+1} F_{2k-1} = F_{2(n+1)}.$$

This completes the inductive step, and proves our result.

**Question 2** (Equal Length Cycles, 15 points)**.** *How many permutations of* [10] *have all cycles the same length? Any closed form solution is acceptable, you do not need to compute the actual integer.*

We note that the cycles must be of length 1,2, 5 or 10 (since the length must divide 10). Recall that the number of permutations of $[n]$ with $a_i$ cycles of length $i$ for each $i$ is

$$\frac{n!}{1^{a_1}2^{a_2}\cdots n^{a_n}a_1!a_2!\cdots a_n!}.$$

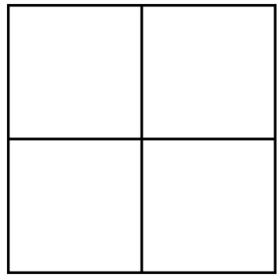
We need to consider the cases where

- We have 10 cycles of length 1.
- We have 5 cycles of length 2.
- We have 2 cycles of length 5.
- We have 1 cycle of length 10.

Using the above formula, the number of permutations with each of these structures are $\frac{10!}{10!}$, $\frac{10!}{2^5 5!}$, $\frac{10!}{5^2 2!}$, and $\frac{10!}{10}$, respectively. The final answer is just the sum of these, which is

$$\frac{10!}{10!}+\frac{10!}{2^5 5!}+\frac{10!}{5^2 2!}+\frac{10!}{10}=1+9!/16+9!/5+9!=458137.$$

**Question 3** (Square Coloring, 15 points)**.** *I have a pallet of $n$ colors. In how many ways can I color the squares in the diagram below with these colors so that no two adjacent regions are given the same color? Note: the diagonally opposite regions are not adjacent.*



We break into cases based on the total number of distinct colors used.

If four distinct colors are used, there are $n$ ways to choose the color of the first square, then $n-1$ ways to choose the color of the second, then $n-2$ for the third and $n-3$ for the last, for a total of $n(n-1)(n-2)(n-3)$ colorings.

If three distinct colors are used, some pair of diagonally opposite squares are the same color. There are 2 ways to choose which pair. There are then $n$ ways to choose the colors of these squares, and $n-1$ and $n-2$ ways to choose the colors of the other two squares.

If there are two distinct colors, each set of diagonally opposite squares must be the same color. There are $n$ ways to color the first and $n-1$ for the second.

Adding these cases up, the final answer is $n(n-1)(n-2)(n-3) + 2n(n-1)(n-2) + n(n-1)$.

Note: This problem can also be solved with inclusion-exclusion with a bit more effort.

**Question 4** (Generating Function Construction, 15 points)**.** *Let* $a_n$ *be the number of compositions of* $n$ *into an even number of parts each of which is at most* 5*. Give an explicit formula for the generating function*

$$\sum_{n=0}^{\infty} a_n x^n.$$

Note that the generating function for the number of compositions of $n$ into one part of size at most 5 is $y := x+x^2+x^3+x^4+x^5$. By the standard interpretation of the product of ordinary generating functions, we find that the generating function for the number of compositions into $k$ such parts of $(x+x^2+x^3+x^4+x^5)^k = y^k$. We need to sum this over all even values of $k$, giving

$$\sum_{n=0}^{\infty} y^{2k} = \frac{1}{1-y^2} = \frac{1}{1-(x+x^2+x^3+x^4+x^5)^2}.$$

**Question 5** (Identical Parts, 20 points)**.** *Suppose that you are given more than $n/4$ compositions of $n$ into three parts. Show that at least two parts of these compositions are the same (these could be two parts from the same composition or parts from different compositions). Proving that this can be done with a greater number of compositions (i.e. replacing $n/4$ in the problem statement with something larger) will be worth partial credit.*

Note that each composition has at least 2 parts of size at most $n/2$ (for otherwise two parts would have size at least $n/2$, and then the sum of the sizes of the parts would be more than $n$). Therefore among any $m$ such compositions, we will have at least $2m$ parts of size at most $n/2$. Since there are only $n/2$ many values that these parts can take, if $m > n/4$, we have $2m > n/2$, and by the pigeonhole principle, two of the parts must have the same size.

Note that you can actually do better than this. If we have $m$ such compositions with all parts distinct, in total they have $3m$ distinct parts. The sum of these parts must be at least $1+2+\ldots+3m = 3m(3m+1)/2$. On the other hand, this sum will be exactly $n$ for each composition, and so the total sum will be $nm$. Therefore, if no two parts have the same size, it must be the case that $nm \geq 3m(3m+1)/2$, or that $m \leq (2n+3)/9$. Therefore if we have more than $(2n+3)/9$ such compositions, some pair of parts must be the same.

**Question 6** (Ramsey Lower Bound, 20 points). *(a) Prove that for any integers $n, m, k$ that*

$$R(n, m+k-1) \geq R(n, m) + R(n, k) - 1.$$

*(b) Use this to show that $R(3, 3n+1) \geq 8n+1$ for all $n \geq 1$. You may use the result from part (a), even if you did not complete that part.*

(a) It suffices to show that for $N = R(n, m) + R(n, k) - 2$ that there is a coloring of the edges of $K_N$ so that there is no red $K_n$ and no blue $K_{m+k-1}$. On the other hand, letting $N_1 = R(n, m) - 1$ and $N_2 = R(n, k) - 1$, we know that there is a coloring of $K_{N_1}$ with no red $K_n$ nor blue $K_m$ and a coloring of $K_{N_2}$ with no red $K_n$ or blue $K_k$.

Noting that $N = N_1 + N_2$, we color $K_N$ by coloring the edges between the first $N_1$ vertices in the way we colored $K_{N_1}$ described above, coloring the edges between the last $N_2$ vertices in the way we colored the $K_{N_2}$ described above, and coloring all remaining edges blue.

If this graph had a red $K_n$, it cannot contain vertices in both the first and second halves (because the edge between would be blue), but this implies that one half had a red $K_n$, which contradicts our assumptions.

If this graph had a blue $K_{m+k-1}$, it would have to either have $m$ vertices in the first half or $k$ vertices in the second half. The former would mean our $K_{N_1}$ had a blue $K_m$, and the latter that our $K_{N_2}$ had a blue $K_k$, either of which contradicts our assumptions.

This completes the proof.

(b) To show that $R(3, 3n+1) \geq 8n+1$, we proceed by induction on $n$. For $n = 1$, we have $R(3, 3n+1) = R(3, 4) = 9 \geq 8 \cdot 1 + 1$. Assuming that $R(3, 3n+1) \geq 8n+1$, we have by the above that

$$R(3, 3(n+1)+1) = R(3, (3n+1)+4-1) \geq R(3, 3n+1) + R(3, 4) - 1 \geq (8n+1) + 9 - 1 = 8(n+1) + 1.$$

This finishes the inductive step and proves our result.