# Math 103A: Homework 2 solutions

1. Solution to I.2, Q2
   $(a \star b) \star c = b \star c = a$
   $a \star (b \star c) = a \star a = a$
   We cannot conclude based on the above computation if $\star$ is associative. We would need to verify $(x \star y) \star z = x \star (y \star z)$ $\forall x, y, z \in S$.

2. Solution to I.2, Q4
   $\star$ is not commutative because $b \star e = c$, whereas $e \star b = b$.

3. Solution to I.2, Q17
   $\star$ satisfies condition 1 since $a - b$ is uniquely defined for any $a, b \in \mathbb{Z}^+$.
   $\star$ does not satisfy condition 2. For example: $1 - 2 = -1$, and $-1 \notin \mathbb{Z}^+$.
   Hence, $\star$ does not give a binary operation on $\mathbb{Z}^+$.

4. Solution to I.2, Q24
   (a) False. Take $(\mathbb{Z}, +)$. $1 + 1 \neq 1$.
   (b) True. Let $b \star c = d \in S$. Since $\star$ is commutative, $a \star d = d \star a$.
   (c) False. As discussed on page 24, multiplication of $n$ x $n$ matrices is an associative binary operation. Take $a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $c = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.
   (d) False. We already saw several important binary operations on sets of functions and matrices.
   (e) False. The commutative property needs to hold for **every** pair of elements in S.
   (f) True. To be noncommutative, for example, there must exist $a, b \in S$ such that $a \star b \neq b \star a$. This cannot happen if S contains only one element.
   (g) True. See page 24.
   (h) True. See page 24.
   (i) True. See page 24.
   (j) False. See page 24.

5. Solution to I.2 Q26

$$\begin{aligned}
(a \star b) \star (c \star d) &= (c \star d) \star (a \star b) && \text{Commutative} \\
&= [(c \star d) \star a] \star b && \text{Associative} \\
&= [(d \star c) \star a] \star b && \text{Commutative}
\end{aligned}$$

6. Solution to I.2 Q36
   Let $h_1, h_2 \in H$. Need to show $h_1 \star h_2 \in H$.

$$\begin{aligned}
(h_1 \star h_2) \star x &= h_1 \star (h_2 \star x) && \text{Associative} \\
&= h_1 \star (x \star h_2) && h_2 \in H \\
&= (h_1 \star x) \star h_2 && \text{Associative} \\
&= (x \star h_1) \star h_2 && h_1 \in H \\
&= x \star (h_1 \star h_2) && \text{Associative}
\end{aligned}$$

7. Solution to I.2 Q37
   Let $h_1, h_2 \in H$. Need to show $h_1 \star h_2 \in H$.

$$\begin{aligned}
(h_1 \star h_2) \star (h_1 \star h_2) &= h_1 \star (h_2 \star (h_1 \star h_2)) && \text{Associative} \\
&= h_1 \star ((h_2 \star h_1) \star h_2) && \text{Associative} \\
&= h_1 \star ((h_1 \star h_2) \star h_2) && \text{Commutative} \\
&= h_1 \star (h_1 \star (h_2 \star h_2)) && \text{Associative} \\
&= h_1 \star (h_1 \star h_2) && h_2 \in H \\
&= (h_1 \star h_1) \star h_2 && \text{Associative} \\
&= h_1 \star h_2 && h_1 \in H
\end{aligned}$$

8. Solution to I.3 Q2
   $\phi$ is an isomorphism. It is clearly a bijection of sets. Furthermore, $-(m+n) = \phi(m+n) = \phi(m) + \phi(n) = -m - n = -(m+n)$

9. Solution to I.3 Q4
   $\phi$ is not an isomorphism since it fails the homomorphism property. For example $\phi(2) = 2 + 1 = 3$, but $\phi(1) + \phi(1) = 2 + 2 = 4$.

10. Solution to I.3 Q17
    (a) Define $\star$ on $\mathbb{Z}$ as $m \star n = (m-1)(n-1) + 1$. Then $\phi(m) \star \phi(n) = (m+1) \star (n+1) = mn + 1 = \phi(mn)$ as needed. The identity element is $\phi(1) = 2$.

    (b) Define $\star$ on $\mathbb{Z}$ as $m \star n = (m+1)(n+1) - 1$. Then $\phi(m \star n) = \phi((m+1)(n+1) - 1) = (m+1)(n+1) = \phi(m)\phi(n)$ as needed.The identity element is 0 since $\phi(0) = 1$.

11. Solution to I.3 Q19
    (a) Define $\star$ on $\mathbb{Q}$ as $x \star y = \frac{(x+1)(y+1)}{3} - 1$. Then $\phi(x) \star \phi(y) = (3x-1) \star (3y-1) = \frac{(3x)(3y)}{3} - 1 = 3xy - 1 = \phi(xy)$ as needed. The identity element is $\phi(1) = 3 - 1 = 2$.

    (b) Define $\star$ on $\mathbb{Q}$ as $x \star y = \frac{(3x-1)(3y-1)+1}{3}$. Then $\phi(x \star y) = \phi\left(\frac{(3x-1)(3y-1)+1}{3}\right) = (3x-1)(3y-1) = \phi(x)\phi(y)$ as needed. The identity element is $\frac{2}{3}$ since $\phi(\frac{2}{3}) = 2 - 1 = 1$.

12. Solution to I.3 Q27

First we prove that $\psi \circ \phi$ is a bijection between $S$ and $S''$.
Onto: Let $s_2 \in S''$. Since $\psi$ is onto, there exists $s_1 \in S'$ such that $\psi(s_1) = s_2$. Furthermore, since $\phi$ is also onto, there exists $s \in S$ such that $\phi(s) = s_1$. So $\psi \circ \phi(s) = \psi(\phi(s)) = \psi(s_1) = s_2$ as needed.
One-to-one: Assume $\psi \circ \phi(s) = \psi \circ \phi(t)$, i.e. $\psi(\phi(s)) = \psi(\phi(t))$. Since $\psi$ is one-to-one, $\phi(s) = \phi(t)$, and since $\phi$ is one-to-one, $s = t$ as needed.

Now we prove that $\psi \circ \phi$ satisfies the homomorphism property.
$\psi \circ \phi(s \star t) = \psi(\phi(s \star t)) = \psi(\phi(s) \star' \phi(t)) = \psi(\phi(s)) \star'' \psi(\phi(t)) = \psi \circ \phi(s) \star'' \psi \circ \phi(s)$ as needed. Here, the second and third equalities hold because $\phi$ and $\psi$ satisfy the homomorphism property.