Homework 3 Solution

Spring 2018

Question 1 (Summation Polynomials Redux, 20 points)**.** *Recall that in the last homework we showed that*

$$\sum_{i=1}^{n} i^m = P_m(n)$$

where

$$P_m(x) = \sum_{k=0}^{m} k! S(m,k) \binom{x+1}{k+1}.$$

Suppose that we want to find the coefficients of the polynomial

$$P_m(x-1) = c_{m+1,m}x^{m+1} + c_{m,m}x^m + \ldots + c_{0,m}.$$

Show that there is a formula for the coefficients $c_{i,j}$ given as a summation involving Stirling numbers of the first and second kind.

Solution. *Recall the identity*

$$\sum_{k=0}^{n} s(n,k)x^k = (x)_n$$

Now we have

$$\begin{aligned} P_j(x-1) &= \sum_{k=0}^{j} k! S(j,k) \binom{x}{k+1} \\ &= \sum_{k=0}^{j} k! S(j,k) \frac{(x)_{k+1}}{(k+1)!} \\ &= \sum_{k=0}^{j} \frac{S(j,k)(x)_{k+1}}{k+1} \\ &= \sum_{k=0}^{j} \frac{S(j,k)}{k+1} \sum_{m=0}^{k+1} s(k+1,m)x^m \end{aligned}$$

Note that $c_{i,j}$ is the coefficient of x^i term.

$$c_{i,j} = \sum_{k=0}^{j} \frac{S(j,k)}{k+1} s(k+1,i)$$

Question 2 (Permutations Without 2-Cycles, 20 points)**.** *Give a formula for the number of permutations of a set of $2n$ elements that have no cycles of length 2. Your formula may include a single summation.*

Solution. *We count number of permutations that have a 2-cycle instead.*

Let A_{ij} be the set of all permutations in S_{2n} with cycle (ij). Now we just need to find the size of the union of the A_{ij}, which we can do using the inclusion-exclusion principle.

Note that the intersection of k distinct $A_{ij}s$ is just the set of permutations with those k distinct 2-cycles (or empty if those 2-cycles have overlapping elements). Let donate c_k be the sum of the sizes of all possible intersections of k distinct A_{ij}. Then we have

$$c_k = \frac{\binom{2n}{2}\binom{2n-2}{2}...\binom{2n-2k+2}{2} * (2n-2k)!}{k!} = \frac{(2n)!(2n-2)!...(2n-2k+2)!(2n-2k)!}{(2n-2)! * 2 * (2n-4)! * 2...(2n-2k)! * 2 * k!} = \frac{(2n)!}{2^k * k!}.$$

This is because we have $\binom{2n}{n}$ choices for the first 2-cycle. After picking the first 2-cycle, we have $\binom{2n-2}{n}$ ways to pick our second 2-cycle from remaining $2n-2$ elements. Keep doing it until we find k 2-cycles. The rest of $2n-2k$ elements can be in any permutation, which gives the factor $(2n-2k)!$. However, given any fixed k 2-cycles, the order to pick those k 2-cycle does not matter. So we divide $k!$ to avoid overcounting. Therefore, by Inclusion-Exclusion Principle, the number of permutations with at least one 2-cycle is

$$\sum_{k=1}^{n}(-1)^{k+1}c_k = \sum_{k=1}^{n}(-1)^{k+1}\frac{(2n)!}{2^k * k!}.$$

As the result, the number of permutation without 2-cycle, will be

$$(2n)! - \sum_{k=1}^{n}(-1)^{k+1}\frac{(2n)!}{2^k * k!} = \sum_{k=0}^{n}(-1)^{k}\frac{(2n)!}{2^k * k!}.$$

Question 3 (Stirling Number Identity, 20 points)**.** *Prove that*

$$c(n,k) = \sum_{m=1}^{n}(n-1)_{m-1}c(n-m,k-1).$$

Solution. *We count the number of permutations of n with k cycles where n is in cycle of length m.*

We can let n be the first elements in our m-cycle. Then the second elements in our m-cycle will be one of the $n-1$ other options. Similarly, the third elements will have $n-2$ options (every thing but n and second elements we have picked) and so on. The number of ways to choose our m-cycle will be $(n-1)_{m-1}$. And since our permutation has k cycles, the remaining elements will be in a permutation with $k-1$ cycles. This implies we have $c(n-m,k-1)$ choices for rest of elements.

Therefore, number of permutations where n is in a m-cycle will be $(n-1)_{m-1}c(n-m,k-1)$. By summing up all possible m, we get our desired equality.

Question 4 (Average Number of Cycles, 40 points)**.** .

(a) *For an ordering of the numbers from 1 to n, $a_1, a_2, \ldots, a_n$, let a record be a value i so that $a_i > a_j$ for all $j < i$. Show that the number of such orderings with exactly k records equals the number permutations of n with exactly k cycles. [20 points]*

(b) *Show that on average over permutations of $[n]$ of the number of cycles in the permutation is the harmonic number*

$$H_n = \sum_{i=1}^{n}\frac{1}{i}.$$

[20 points]

Solution. *(a) Let's donate by $q(n,k)$ the number of permutations $a_1, a_2, \ldots, a_n$ with exactly k records. We want to show $q(n,k)$ satisfies the same recursive formula in question 3.*

Let $q(0,0) = 1$. First, it is easy to check $q(1,0) = c(1,0) = 0$ and $q(1,1) = c(1,1) = 1$. If $a_{n-m+1} = n$, then $a_{n-m+2}, a_{n-m+3}, \ldots, a_n$ won't be records since they all less then a_{n-m+1}. Because we can't pick n anymore, there will be total of $(n-1)_{m-1}$ ways to pick $a_{n-m+2}, a_{n-m+3}, \ldots, a_n$.

Moreover, a_{n-m+1} will be counted once. To get total k records, there should be $k-1$ records in $a_1, a_2, \ldots, a_{n-m}$. There will be $q(n-m, k-1)$ ways to order $a_1, a_2, \ldots, a_{n-m}$ to get this number of records.

Therefore, $(n-1)_{m-1}q(n-m, k-1)$ is number of orderings with k records when $a_{n-m+1} = n$. By summing up all m, we get

$$q(n,k) = \sum_{m=1}^{n} (n-1)_{m-1} q(n-m, k-1),$$

which is exactly the same recursive formula in question 3. It is then not hard to prove that $q(n,k) = c(n,k)$ for all n, k by induction.

Alternative Solution. *By Lemma 6.15 in the book, by applying g, we can write our permutations $a_1 a_2 \ldots a_n$ in canonical cycle notation. Note that in canonical cycle notation, the first element of each cycle is the largest one in the cycle and those first elements have been written in an increasing order. That simply implies those first elements are all elements which have been recorded before we write them in canonical cycle notation. And the number of cycles will be the number of first elements which is same as number of records.*

For an example, let $s_7 = 4356172$. $4, 5, 6, 7$ are the 4 numbers we record. However, noticing

$$g(4356172) = (43)(5)(61)(72)$$

which $4, 5, 6, 7$ coincide to be the first elements in each cycle.

(b) We prove this formula by induction on n. For $n = 1$, it is trivial. Assume it is true for and average permutation of $[n-1]$. For permutations of $[n]$, if n is in 1-cycle,then remaining $n-1$ elements have H_{n-1} cycles on average. This implies the total average will increase by 1. If n is not in 1-cycle, then insert n into the permutation of $1, 2, \ldots, n-1$. Notice that n can be inserted after any of those elements in terms of canonical form and, by inserting n, it won't change the number of cycles. It is not difficult to see that any permutation where n is not fixed has a unique way to be formed by inserting n into permutation of $1, 2, \ldots, n-1$. So their average stays the same. Therefore, letting G_n be the average number of cycles of a permutation of size n, we get

$$n! * G_n = (n-1)!(G_{n-1} + 1) + (n-1)(n-1)!(G_{n-1}).$$

By simplifying above, we get

$$G_n = G_{n-1} + 1/n,$$

and it is easy to see by induction that this implies $G_n = H_n$.

Question 5 (Extra credit, 1 point). *Approximately how much time did you spend on this homework?*