Math 184A Homework 3

Fall 2016

This homework is due on gradescope by Friday October 28th at 11:59pm. Remember to justify your work even if the problem does not explicitly say so. Writing your solutions in LaTeXis recommend though not required.

Question 1 (Cycle Structure Questions, 30 points)**.** *(a) Given k distinct elements $a_1, \ldots, a_k \in [n]$, how many permutations of $[n]$ have $a_1, \ldots, a_k$ all in the same cycle? Hint: consider $a_1 = n$. [15 points]*

(b) Given $i, j \in [n]$ and $1 \leq a, b \leq n$ with $a + b \leq n$, how many permutations of $[n]$ have i and j in distinct cycles of length a and b respectively? Hint: consider $i = n, j = n - 1$ and write the permutation in canonical cycle notation. [15 points]

Question 2 (Stirling Number Formula, 30 points)**.** *(a) Show that $c(n, k)$ is the sum over partitions $a_1 + a_2 + \ldots + a_k = n$ of n into k parts of*

$$\frac{n!}{a_1 a_2 \cdots a_k b_1! b_2! \cdots b_n!}$$

where b_i is the number of parts of size exactly i in the partition. Hint: Let $a_1, \ldots, a_k$ be the sizes of the parts. [15 points]

(b) Show that there is a polynomial $q(n)$ so that for all sufficiently large n that $c(n, n - 10) = q(n)$. Hint: Note that almost all parts of the partition must be 1. [15 points]

Question 3 (Short Cycles, 40 points)**.** *Let G be a graph with n vertices, each of which has degree at least 3.*

(a) Assuming that there are no cycles of length less than or equal to m, show that for each $v \in G$ that there are at least 2^m paths of length m starting at v. [20 points]

(b) Prove that G contains a cycle of length at most $2\lceil \log_2(n) \rceil$. Hint: Find two paths starting at v of length at most $\lceil \log_2(n) \rceil$ that have the same endpoint. [20 points]

Question 4 (Extra credit, 1 point)**.** *Approximately how much time did you spend working on this homework?*