Homework due Thursday, November 15, at noon.

(1) (a) Let $\sigma_1 = (123)(4567)$ and $\sigma_2 = (1256)(347)$ be two permutations in S_9. Find a permutation τ so that $\sigma_1 = \tau\sigma_2\tau^{-1}$.

(b) (Bonus problem) Let $\sigma, \tau \in S_n$. Further, assume that $\sigma = (i_1, i_2, \ldots, i_\ell)$ is a cycle of length ℓ. Prove that

$$\tau\sigma\tau^{-1} = (\tau(i_1), \tau(i_2), \ldots, \tau(i_\ell)).$$

In particular, $\tau\sigma\tau^{-1}$ is also a cycle of length ℓ.

(2) List all the elements of the group A_4.

(3) Define $\mathrm{sgn} : S_n \to \mathbb{Z}_2$ by

$$\mathrm{sgn}(\sigma) = \begin{cases} 0 & \text{if } \sigma \text{ is even} \\ 1 & \text{if } \sigma \text{ is odd} \end{cases}$$

Prove that sgn is a homomorphism.

(4) (a) Let $\sigma = (1234)(567)$ be an element in S_8. Find the order of σ. (Recall: the order of an element g in a finite group G is the smallest positive integer m so that $g^m = e$.)

(b) Generalize your observation from part (a) as follows. Suppose $\sigma = \tau\mu$ is an element in S_n where τ and μ are *disjoint* cycles of length t and m respectively. Assume that g.c.d.$(t, m) = 1$. Find the order of σ. (Hint: Use the fact that τ and μ are disjoint to show that if $\sigma^k = \iota$, then k is divisible both by t and m.)

(5) (a) Show that S_5 is generated by the following list of transpositions.

$$\{(1, 2), (2, 3), (3, 4), (4, 5)\}.$$

(b) (Bonus problem) Show that S_n is generated by the following list of transpositions.

$$\{(1, 2), (2, 3), (3, 4), \ldots, (i, i + 1), \ldots, (n - 1, n)\}.$$

(6) Let G be a group and let $H \leq G$ be a subgroup.

(a) Define $N_G(H) = \{g \in G | gHg^{-1} = H\}$. Prove that $N_G(H)$ is a subgroup of G.

(b) Define $Z_G(H) = \{g \in G | gh = hg, \forall h \in H\}$. Prove that $Z_G(H)$ is a subgroup of G.

(c) Let $g_1, g_2 \in G$. Prove that $g_1Hg_1^{-1} = g_2Hg_2^{-1}$ if and only if $g_1N_G(H) = g_2N_G(H)$.

(7) Exercise 10 page 101: 3, 6, 15, 28, 29, 36, 39