Math 184A Homework 6

Fall 2016

This homework is due on gradescope by Friday December 2nd at 11:59pm. Remember to justify your work even if the problem does not explicitly say so. Writing your solutions in LaTeXis recommend though not required.

Question 1 (Basic Generating Function Computations, 60 points)**.** *In the following when asked to give the first k coefficients of an ordinary generating function, list all elements of the power series from x^0 to x^{k-1}. The coefficients should all be written out as explicit constants. So for example, when asked to give the first 5 coefficients of $\frac{1}{1-x^2}$, an answer should be something like $1 + 0x + 1x^2 + 0x^3 + 1x^4 + \ldots$. When asked for the first k coefficients of an exponential generating function, write these terms as multiples of $x^n/n!$. So for example, the same generating function would be written as $1 + 0x^1/1! + 2x^2/2! + 0x^3/3! + 24x^4/4! + \ldots$.*

(a) What are the first seven coefficients of the ordinary generating function $\frac{2x}{1-3x^2}$? [5 points]

(b) What are the first five coefficients of the ordinary generating function $(1-3x)^{1/3}$? [5 points]

(c) What are the first five coefficients of the ordinary generation function $\frac{1}{2-x}$? [5 points]

(d) What are the first five coefficients of the ordinary generation function $\frac{\log(1+x^2)}{1-x}$ [5 points]

(e) What are the first seven coefficients of the ordinary generation function $\sqrt{1+x^2+x^3}$? [5 points]

(f) What are the first ten coefficients of the ordinary generation function $A(x)$ satisfying $A(x) = 1 + xA(x^2)$? [5 points]

(g) What are the first ten coefficients of the ordinary generation function $\left(\sum_{n=0}^{\infty} x^{n^2}\right)^3$? What is the combinatorial interpretation of the m^{th} coefficient of this generating functions? [10 points]

(h) What are the first five coefficients of the exponential generation function $\cosh(x) = (e^x + e^{-x})/2$? [5 points]

(i) What are the first five coefficients of the exponential generation function $\frac{e^x}{1-x}$? [5 points]

(j) What are the first seven coefficients of the exponential generation function $e^{x^2/2}$? What is the combinatorial interpretation of the m^{th} coefficient of this generating function? [10 points]

Question 2 (Proving Identities Using Generating Functions, 40 points)**.** *(a) Use the generating function identity*

$$\sum_{n,k} S(n,k)(x^n/n!)y^k = e^{y(e^x-1)}$$

to provide an alternative proof of the identity

$$S(n,k) = S(n-1,k-1) + kS(n-1,k).$$

[20 points]

(b) Use the generating function identity

$$\sum_n D_n x^n/n! = \frac{e^{-x}}{1-x}$$

to provide an alternative proof of the identity

$$n! = \sum_{k=0}^{n} \binom{n}{k} D_{n-k}.$$

[20 points]

Question 3 (Extra credit, 1 point)**.** *Free point!*