Math 184A - Homework 1

Solutions

April 15, 2018

1. (*Chebyshev Polynomials*) We will prove that for all positive integer n, there exist a pair of polynomials $T_n(x)$ and $S_n(x)$, such that $\cos(n\theta) = T_n(\cos\theta)$ and $\sin(n\theta) = \sin\theta S_n(\cos\theta)$. We will induct on n.

Base case : For $n = 1$, let $T_1(x) = x$ and $S_1(x) = 1$. It is easy to verify that $\cos\theta = T_1(\cos\theta)$ and $\sin\theta = \sin\theta S_1(\sin\theta)$.

Induction : Assume that for any arbitrary natural number n, there exist a pair of polynomial, $T_n(x)$ and $S_n(x)$, such that $\cos(n\theta) = T_n(\cos\theta)$ and $\sin(n\theta) = \sin\theta S_n(\cos\theta)$. Using angle sum formula and the induction hypothesis, we can define the pair of polynomials $T_{n+1}(x)$ and $S_{n+1}(x)$ as done below

$$\begin{aligned}
\cos((n+1)\theta) &= \cos(\theta)\cos(n\theta) - \sin(\theta)\sin(n\theta) \\
&= \cos(\theta)T_n(\cos(\theta)) - \sin^2(\theta)S_n(\cos(\theta)) \\
&= \cos(\theta)T_n(\cos(\theta)) - (1 - \cos^2(\theta))S_n(\cos(\theta)) \\
&= T_{n+1}(\cos\theta)
\end{aligned}$$

$$\begin{aligned}
\sin((n+1)\theta) &= \cos(\theta)\sin(n\theta) + \sin(\theta)\cos(n\theta) \\
&= \cos(\theta)\sin(\theta)S_n(\cos(\theta)) + \sin(\theta)T_n(\cos(\theta)) \\
&= \sin(\theta)(\cos(\theta)S_n(\cos(\theta)) + T_n(\cos(\theta))) \\
&= \sin(\theta)S_{n+1}(\cos(\theta))
\end{aligned}$$

2. (*Twenty Questions*) Observe that $1,040,000 < 2^{20} < 1,050,000$, so now on we will solve the problem of "n-Questions" with M secret objects.

(a) We want to show that if $M > 2^n$, there does not exist a strategy to ask n yes/no question to guarantee that you can successfully guess the secret object.

Given a strategy, each secret object gives rise to a sequence of pair of question/answer. Consider the set S of possible sequence of pairs

$$(Q_1, \epsilon_1), (Q_2, \epsilon_2), ..., (Q_i, \epsilon_i), ..., (Q_n, \epsilon_n)$$

where Q_i denotes the i^th question and ϵ_i is the Y/N answer to Q_i. Note that the question Q_i depends on the answers to the previous questions.

I claim that number of such sequences $|S| \leq 2^n$. We will prove this by induction.

Proof. Base case : Observe that we are given a strategy, so the first question Q_1 is fixed. Hence the number of possible pair $(Q_1, \epsilon_1) \leq 2^1$ (since ϵ can take at most two values).

Induction : Suppose the number of distinct subsequence of first k question/answer is less that 2^k. Observe that Q_{k+1} solely depends on the subsequence $(Q_1, \epsilon_1), (Q_2, \epsilon_2), ..., (Q_k, \epsilon_k)$ and for each such question there are at most two possible values of ϵ_{k+1}. Hence the number of distinct subsequence of first $k = 1$ question/answer is less than 2^{k+1}. Hence $|S| \leq 2^n$ $\square$

Since $M > 2^n \geq |S|$ (apply pigeon-hole principle), there exist two secret objects which gives rise to exactly same sequence $(Q_1, \epsilon_1), (Q_2, \epsilon_2), ..., (Q_n, \epsilon_n)$ of question and answer. So it is impossible tell apart such two secret objects using our strategy (i.e. we cannot guarantee to find the secret).

Remark : We have also shown that you cannot have a search algorithm (where a step is a binary decision) faster than $O(\log n)$.

(b) Suppose $M \leq 2^n$, label the set of secret objects as $A_0, A_1, ..., A_{M-1}$. So we can assume that the secret objects is from the set of integers $\{0, 1, ..., M-1\}$. Observe that if you write these number in binary (i.e base 2), then these numbers have at most n digits because $M - 1 \leq 2^n - 1$.

Now let the questions Q_i be : Under the labeling as above, does the number assigned to the secret object has i^{th} digit 0 in its binary (base 2) expansion?

It is easy to see the n questions certainly describe the number assigned to the secret object, hence we can find the secret for sure.

3. (*Mini-Checkerboard Colorings*)
(a) $2^{16} = 65536$: We have to color the oriented 4×4 checker board using two colors. There are total 16 squares to be colored (white or black) and any coloring of these 16 squares gives a distinct coloring of the oriented checker board. There are 2 choices of colors for each squares, hence there are 2^{16} ways of coloring.

(b) $2^4 = 16$: There are 4 rows in the checker board and each of them has to be monochromatic (i.e. each square in a row has same color). There are 2 choices of color for each row, so the total number of coloring is 2^4.

(c) $\binom{16}{5} + \binom{16}{6} = 12376$ = number of ways of choosing 5 squares (and color black) + number of ways of choosing 6 squares (and color black).

(d) $\binom{4}{2}(2^4 - 1)^2 = 1350$: There are $\binom{4}{2}$ ways of picking 2 rows which are exactly those having black color, which is $2^4 - 1$ ways each. Observe that remaining 2 rows have to be colored white.

(e) $4!(\binom{4}{0}\binom{4}{1}\binom{4}{2}\binom{4}{3} + \binom{4}{0}\binom{4}{1}\binom{4}{2}\binom{4}{4} + \binom{4}{0}\binom{4}{1}\binom{4}{3}\binom{4}{4} + \binom{4}{0}\binom{4}{2}\binom{4}{3}\binom{4}{4} + \binom{4}{1}\binom{4}{2}\binom{4}{3}\binom{4}{4}) = 6144$:
The set of 4 numbers describing the number of black squares are

$$\{0, 1, 2, 3\}, \{0, 1, 2, 4\}, \{0, 1, 3, 4\}, \{0, 2, 3, 4\}, \{1, 2, 3, 4\}$$

For each such set, there are 4! ways of ordering it. Once you have an order say (a_1, a_2, a_3, a_4), where a_i denote the number of black in the i^{th} row, the number of ways of coloring is $\prod_{i=1}^{4} \binom{4}{a_i}$

(f) $\binom{4}{2}^4 = 1296$: There are $\binom{4}{2}$ ways of coloring each column such that it has exactly 2 white color.

(g) $4! = 24$: It is the number permutation of $[4]$, where a permutation $(a_1, ..., a_4)$ corresponds to coloring the squares $(1, a_1), (2, a_2), (3, a_3)$ and $(4, a_4)$, where (i,j) refers to the square at i^{th} row and j^{th} column.

(h) $\sum_{i=0}^{4} \binom{4}{i} \frac{4!}{(4-i)!} = 209$: There are $\binom{4}{i}$ ways of choosing the rows which will have exactly one white square (and the remaining rows have all black squares). Given the i rows, there are exactly $\frac{4!}{(4-i)!}$ ways to assign white color to exactly one square per row, with distinct column numbers.