PRACTICE EXAM

(1) List all the elements in the group D_4 as permutations in S_4.

Solution. Recall that D_4 is the group of isometries of a square whose vertices are labeled $\{1, 2, 3, 4\}$. Therefore, we have four elements which arise as rotations by angles $\pi/2, \pi, 3\pi/2, 2\pi$; these give

$$A = \{\iota, (1, 2, 3, 4), (1, 3)(2, 4), (1, 4, 3, 2)\}.$$

We also have reflections about the x and y axes, and about the two diagonals which give $B = \{(1, 2)(3, 4), (1, 4)(23), (2, 4), (1, 3)\}$. Now $D_4 = A \cup B$.

(2) Let $\sigma = (1, 2, 3, 4, 5, 6) \in S_7$.

(a) Compute σ^{103}.
(b) Write σ^{14} as a product of disjoint cycles.

Solution.

(a) Recall that by a Lemma from class we have order of a cycle of length ℓ equals ℓ. Therefore, the order of σ is 6. Since $103 = 6 \times 17 + 1$, we have

$$\sigma^{103} = \sigma^{102}\sigma = \sigma.$$

(b) As was observed in part (a) we have $\sigma^6 = \iota$. Therefore, we have

$$\sigma^{14} = \sigma^2 = (1, 3, 5)(2, 4, 6).$$

(3) Let $H = \{\sigma \in S_5 \mid \sigma(3) = 3\}$. Show that H is a subgroup of S_5; moreover, show that H is isomorphic to S_4.

Solution. First we show that H is a subgroup. Indeed $\iota \in H$ since $\iota(3) = 3$. If $\sigma \in H$, then $\sigma(3) = 3$; hence, $\sigma^{-1}(3) = 3$. That is: $\sigma^{-1} \in H$. If $\sigma, \tau \in H$, then $\sigma\tau(3) = \sigma(\tau(3)) = \sigma(3) = 3$. Hence $\sigma\tau \in H$. We get that H is a subgroup.

Let $A = \{1, 2, 4, 5\}$. Since $|A| = 4$, a theorem from class implies that S_A is isomorphic to S_4.

We now show that $H \simeq S_A$. Define $f : H \to S_A$ by $f(\sigma) = \sigma|_A$ (the restriction of σ to A). Note that since $\sigma(3) = 3$ and σ is a bijection, we have $\sigma(A) = A$. That is $\sigma|_A \in S_A$. So f is well-defined. Moreover, given $\sigma, \tau \in H$ and $a \in A$ we have $(\sigma\tau)|_A(a) = (\sigma \circ \tau)|_A(a) = \sigma(\tau(a))$. Hence, $(\sigma\tau)|_A = \sigma|_A \circ \tau|_A$. That is $f(\sigma\tau) = f(\sigma)f(\tau)$.

We now check that f is a bijection. Suppose $f(\sigma) = f(\sigma')$. Then $\sigma(a) = \sigma'(a)$ for all $a \in A$. Further, $\sigma(3) = \sigma'(3) = 3$ since $\sigma, \sigma' \in H$. So $\sigma = \sigma'$; hence, f is one-to-one. To see f is also onto, let $\tau \in S_A$. Define $\sigma \in S_5$ by $\sigma(a) = \tau(a)$ if $a \neq 3$ and $\sigma(3) = 3$. Then $\sigma \in H$ and $f(\sigma) = \sigma|_A = \tau$. So f is onto.

These show that $H \simeq S_A \simeq S_4$ as we wanted to show.

(4) Let G be a group and let $H \leq G$ be a subgroup of G.

(a) Suppose $(G : H) = 2$. Prove that $gH = Hg$ for every $g \in G$.

(b) Prove or disprove by an example: suppose $(G : H) = 3$, then every right coset is a left coset.

Solution.
(a) See HW 7, page 106, #39.
(b) An example was discussed in class: let $G = S_3$ and $H = \{\iota, (1,2)\}$. Then $(G : H) = 3$ and we showed in class that left cosets of H in G are not right cosets.

(5) Let G be a group. Show that G is isomorphic to a subgroup of S_G.

Solution. See Theorem 8.16 in the book.

(6) Let $K \leq H \leq G$ and assume that $(G : K)$ is finite. Prove that $(G : H)$ and $(H : K)$ are finite.

Solution. Let $(G : K) = n$ and let $\{g_1K = K, g_2K, \dots, g_nK\}$ be the set of all left cosets of K in G. Since $K \subset H$, we have $g_iK \subset g_iH$ for all $1 \leq i \leq n$. Therefore,

$$G = \bigcup_{i=1}^{n} g_iK = \bigcup_{i=1}^{n} g_iH$$

This implies that every $g \in G$ belongs to *at least* one g_iH. Recall that gH and $g'H$ are either equal or disjoint for all $g, g' \in G$. Let $\{j_1, \dots, j_s\} \subseteq \{1, \dots, n\}$ be so that for every $1 \leq i \leq n$ there exists *exactly* one j_ℓ so that $g_iH = g_{j_\ell}H$. Then $\{g_{j_1}H, \dots, g_{j_s}H\}$ is the set of all left cosets of H in G. Therefore, $(G : H) = s \leq n$, in particular, $(G : H)$ is finite.

To see $(H : K)$ is also finite, we note again that $G = \bigcup_{i=1}^{n} g_iK$. Since $H \subseteq G$, we have $H \subseteq \bigcup_{i=1}^{n} g_iK$. Let $A = \{i \mid 1 \leq i \leq n, g_iK \cap H \neq \emptyset\}$. Then

$$H \subseteq \bigcup_{i \in A} g_iK.$$

Moreover, for every $i \in A$, there exists some $h_i \in H$ and some $k_i \in K$ so that $h_i = g_ik_i$. Hence $g_i = h_ik_i^{-1} \in H$. That is: $g_iH = h_iH$ and we get that

$$\bigcup_{i \in A} g_iK = \bigcup_{i \in A} h_iK \subseteq H.$$

Hence, $H = \bigcup_{i \in A} h_iK$ and $(H : K) = |A| \leq n$ is finite.