Math 184A HW 4 Solutions

Fall 2015

Question 1 (Permutation Parity, 40 points). .

(a) *Show that for any $n > 1$ that the number of permutations of $[n]$ with an even number of cycles is equal to the number of permutations of $[n]$ with an odd number of cycles using identities relating to $c(n, k)$.[20 points]*

(b) *Find a bijection between the permutations of $[n]$ with an even number of cycles and those with an odd number of cycles. [Hint: If 1 and 2 are in different cycles, merge the cycles together, if they are in the same cycle, split the cycle in two.][20 points]*

Solution.

(a) There are two solutions for 1(a).

The 1^{st} proof

Let O_n be the number of permutations of $[n]$ with an odd number of cycles and let E_n be the number of permutations of $[n]$ with an even number of cycles.

We know the equation that

$$\sum_{k=1}^{n} c(n,k)x^k = x(x+1)\cdots(x+n-1).$$

Let $x = -1$, we have

$$\sum_{k=1}^{n} c(n,k)(-1)^k = (-1)(-1+1)\cdots(-1+n-1).$$

LHS $= E_n - O_n$ while RHS $= 0$, which gives us $E_n = O_n$.

The 2^{nd} proof

In the textbook we can find the identity about $c(n, k)$ that

$$c(n,k) = c(n-1,k-1) + (n-1)c(n-1,k)$$

We can come up with an mathematical induction proof with a similar idea. We induct on n to prove the number of permutations of $[n]$ with an even number of cycles is equal to the number of permutations of $[n]$ with an odd number of cycles.

Initial Step. *When $n = 2$, there are only two permutations*

$$\pi_1 = (1)(2), \pi_2 = (12).$$

π_1 has 2 cycles while π_2 has 1 cycle. So $O_2 = E_2 = 1$.

Inductive Step. *Suppose $O_n = E_n$ is true for an $n > 1$, then let's check the identity for n+1. To get a permutation of $[n+1]$ with an odd number of cycles, we can either let the entry n form a cycle by it self, or insert the entry n into the cycles of a permutation of $[n]$. We have E_n permutations in the first way, while $n \times O_n$ permutations in the second way, so $O_{n+1} = E_n + n \times O_n$.*

Similarly, to get a permutation of $[n+1]$ with an even number of cycles, we can either let the entry n form a cycle by it self, or insert the entry n into the cycles of a permutation of $[n]$. We have O_n permutations in the first way, while $n \times E_n$ permutations in the second way, so $E_{n+1} = O_n + n \times E_n$.

Since $O_n = E_n$, we have $O_{n+1} = E_{n+1}$.

(b) We denote the set of permutations of $[n]$ with an even number of cycles by $\mathbb{E}_n$ while denote the set of permutations of $[n]$ with an odd number of cycles by $\mathbb{O}_n$.

Define a map $\phi : \mathbb{E}_n \to \mathbb{O}_n$ that for $\forall \pi \in \mathbb{E}_n$, if 1 and 2 are in different cycles, then $\phi(\pi)$ merge the cycles together, if they are in the same cycle, then $\phi(\pi)$ split the cycle in two. Actually the map ϕ can change the number of cycles in a permutation from even to odd, and it's not hard to show ϕ is bijective (showing it is $1-1$ and onto). Thus we get a bijection between the permutations of $[n]$ with an even number of cycles and those with an odd number of cycles.

□

Question 2 (Cycles and Powers, 20 points)**.** *Let π be a permutation of $[n]$ consisting of only one cycle of length n. Let $r > 0$ be an integer. In terms of r and n, describe the cycle structure of π^r.*

Solution. We use the notation $\overline{a}$ denote the remainder of a when divided by n.

$$a \equiv \overline{a} \pmod{n}$$

Suppose $\pi = (a_1\ a_2\ \cdots\ a_n)$, let's look at the first cycle of the permutation π^r. It will be

$$c_1 = (a_1\ a_{\overline{1+r}}\ a_{\overline{1+2r}}\ \cdots).$$

If $\gcd(n, r) = 1$, the set $\{\overline{1+kr}\}$ can go over the set $[n]$, thus the length of cycle c_1 is n and there is only one cycle $c_1 = (a_1\ a_{\overline{1+r}}\ a_{\overline{1+2r}}\ a_{\overline{1+3r}}\ \cdots)$ in π^r. (1 length-n cycle in π^r.)

If $\gcd(n, r) = d > 1$, the set $\{\overline{1+kr}\}$ can only be the element with a form $1 + kd$ in set $[n]$, thus the length of cycle c_1 is $\frac{n}{d}$, and there will be d such cycles in π^r. (d length-$\frac{n}{d}$ cycles in π^r.)

□

Question 3 (Stirling Number Lower Bound, 30 points)**.** .

(a) Show that the number of permutations of $[n]$ with k cycles so that each of $1, 2, 3, \ldots, k$ is in a different cycle is $\frac{(n-1)!}{(k-1)!}$. Use this to show that

$$c(n, k) \geq \frac{(n-1)!}{(k-1)!}.$$

[15 points]

(b) Show this formula directly using the relation

$$\sum_{k=1}^{n} c(n, k) x^k = x(x+1) \cdots (x+n-1).$$

[15 points]

Solution.

(a) We denote the set of permutations of $[n]$ with k cycles so that each of $1, 2, 3, \ldots, k$ is in a different cycle by $\overline{S_{n,k}}$. There are $(n-1)!$ permutations that fixes the first entry 1. Choose such a permutation π then we can construct a permutations of $[n]$ with k cycles so that each of $1, 2, 3, \ldots, k$ is in a different cycle by adding parentheses before every integer $1, 2, \cdots, k$.

For example we take $n = 10$ and $k = 4$, let $\pi = 1\ 6\ 5\ 3\ 9\ 7\ 4\ 8\ 2\ 10$, we can construct a permutation $\pi' = (1\ 6\ 5)\ (3\ 9\ 7)\ (4\ 8)\ (2\ 10)$.

Actually every permutation from $\overline{S_{n,k}}$ can be created by the $(n-1)!$ permutations, and since the cycle with entry 1 is fixed at the first place, the rest $k-1$ cycles can be created in any order, causing $(k-1)!$ redundancy for every permutation from $\overline{S_{n,k}}$. Thus,

$$|\overline{S_{n,k}}| = \frac{(n-1)!}{(k-1)!}.$$

The number $c(n,k) \geq |\overline{S_{n,k}}| = \frac{(n-1)!}{(k-1)!}$.

(b) By the relation

$$\sum_{k=1}^{n} c(n,k)x^k = x(x+1)\cdots(x+n-1),$$

We see $c(n,k)$ is the coefficient of x^k in the polynomial $(x)(x+1)\cdots(x+n-1)$. To calculate the coefficient, we choose k parentheses to pick x's, and pick constant from the rest $(n-k)$ parentheses, then add such terms up, i.e.

$$c(n,k) = \text{coeff of } x^k = \sum_{1\leq a_1<a_2<\cdots<a_{n-k}\leq n-1} a_1a_2\cdots a_{n-k}.$$

Since every term $a_1a_2\cdots a_{n-k}$ is positive and

$$a_1a_2\cdots a_{n-k}|_{a_1=k,a_2=k+1,\ldots,a_{n-k}=n-1} = k\times(k+1)\times(k+2)\times\cdots\times(n-1) = \frac{(n-1)!}{(k-1)!},$$

we have

$$c(n,k) \geq \frac{(n-1)!}{(k-1)!}.$$

□

Question 4 (Triplet Permutation, 10 points)**.** *How many permutations of* $[3n]$ *have only cycles of length 3?*

Solution.

We denote the set of permutations of $[3n]$ have only cycles of length 3 by $\widetilde{S_{3n}}$.

Given any permutation $\pi = \pi_1\pi_2\cdots\pi_{3n}$ of $[3n]$, we can construct a $\tilde{\pi} \in \widetilde{S_{3n}}$ by

$$\tilde{\pi} = (\pi_1\pi_2\pi_3)(\pi_4\pi_5\pi_6)\cdots(\pi_{3n-2}\pi_{3n-1}\pi_{3n}).$$

Actually every permutations of $[3n]$ having only cycles of length 3 is constructed with two kinds of redundancy. The first kind of redundancy is that the cycle can rotate, e.g. $(\pi_1\pi_2\pi_3) = (\pi_3\pi_1\pi_2)$, this causes 3^n times repetitions the second kind of redundancy is that the cycles can permute, e.g. $(\pi_1\pi_2\pi_3)(\pi_4\pi_5\pi_6) = (\pi_4\pi_5\pi_6)(\pi_1\pi_2\pi_3)$, this causes $n!$ times repetitions. In total every permutation $\tilde{\pi} \in \widetilde{S_{3n}}$ can appear $3^n \times n!$ times, so

$$\#\text{ of permutations of } [3n] \text{ having only cycles of length } 3 = \frac{3n!}{3^n \times n!}.$$

□