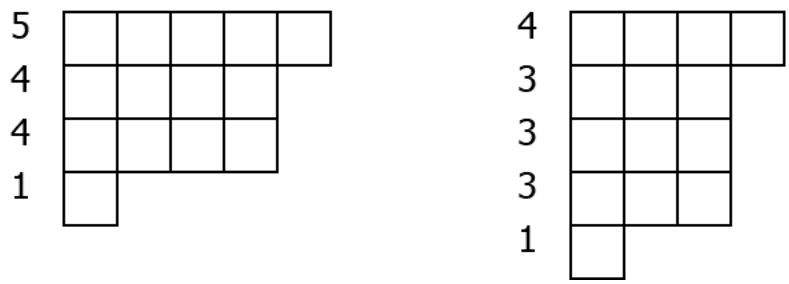
Question 1 (Conjugate Partitions, 30 points). *What is the conjugate of the partition* $5+4+4+1$ *of* 14*?*

The answer is $4+3+3+3+1$. The Ferrer's diagram for the partition and its conjugate are shown below.



Question 2 (Semi-Sorted Permutations, 35 points)**.** *How many permutations* $\pi : \{1, 2, \ldots, 2n\} \to \{1, 2, \ldots, 2n\}$ *have* $\pi(1) < \pi(2), \pi(3) < \pi(4), \ldots, \pi(2n-1) < \pi(2n)$*? Justify your answer.*

The answer is $\frac{(2n)!}{2^n}$.

To see this, note that there are $\binom{2n}{2}$ possible pairs of elements that could be $\{\pi(1), \pi(2)\}$. Once we have picked the pair, $\pi(1)$ must be the smaller element and $\pi(2)$ the larger one. Once those have been selected, there are similarly $\binom{2n-2}{2}$ ways to pick $\pi(3) < \pi(4)$ from the remaining $2n-2$ values. There are then $\binom{2n-4}{2}$ ways to pick $\pi(5) < \pi(6)$, $\binom{2n-6}{2}$ ways to pick $\pi(7) < \pi(8)$ and so on, up to $\binom{2}{2}$ ways to pick $\pi(2n-1) < \pi(2n)$. Thus, the total number of such permutations is

$$\binom{2n}{2}\binom{2n-2}{2}\cdots\binom{2}{2} = \frac{(2n!)(2n-2)!\cdots 2!}{2(2n-2)!2(2n-4)!\cdots 2(0)!} = \frac{(2n)!}{2^n},$$

where the last step is by cancelling terms.

Question 3 (Pile Splitting Game, 35 points). *Locke and Sabetha are playing a game. The game starts with a single pile of N stones and the players take turns (starting with Locke) splitting every pile with more than one stone into two smaller piles. A player loses the game if at the start of their turn all piles have size 1.*

So for example, if $N = 5$, Locke might divide it into piles of size 2 and 3. Sabetha on her turn could split the pile of size 2 into two of size 1 and the pile of size 3 into one of size 2 and one of size 1. Locke would then split the pile of size 2 into two of size 1 (leaving the singleton piles alone), and Sabetha would lose on her next turn.

Show that Sabetha has a winning strategy in this game if and only if N is one less than a power of 2. [Hint: Show by induction that if the largest pile has size n that the player on move has a winning strategy unless $n + 1$ is a power of 2, in which case their opponent does.]

We prove by strong induction on n that if the game is in any state where the largest number of stones in any pile is equal to n, then the player on move has a forced win if and only if $n + 1$ is not a power of 2.

For the base case, we note that if the largest pile has size $n = 1$, then all piles are of size 1, and the player on move will lose instantly. This corresponds to $n + 1$ being a power of 2.

Next assume that this holds for any collection of piles all smaller than size n. Suppose that 2^m is the smallest power of 2 more than n. If n is not $2^m - 1$, then $n \leq 2^m - 2$. Therefore, the largest pile can be split into a pile of size $2^{m-1} - 1$ and a smaller pile, and all other piles can be split into piles of size at most $2^{m-1} - 1$. Therefore, by the inductive hypothesis, after making this move, the next player is in a losing position, and thus the player originally on move has a winning strategy.

On the other hand, if $n = 2^m - 1$, after splitting the largest pile in two, you will end up with at least one size of size at least 2^{m-1}. Thus, the largest remaining pile will have size strictly between $2^{m-1} - 1$ and $2^m - 1$. Thus, again by the inductive hypothesis, after making any first move, the other player will have a winning strategy. Thus, in this case, the initial player will not have a forced win.

This completes the inductive step.

Therefore, since the initial setup has largest pile of size N, Locke has a winning strategy unless $N + 1$ is a power of 2.