

Math 184A Homework 5

Spring 2018

This homework is due on gradescope by Friday May 18th at 11:59pm. Remember to justify your work even if the problem does not explicitly say so. Writing your solutions in L^AT_EX is recommended though not required.

Question 1 (Permutation Identity, 50 points). .

- (a) Let $p_{d,o}(n)$ be the number of partitions of n into distinct, odd parts. Find a formula for the generating function $\sum_{n=0}^{\infty} p_{d,o}(n)x^n$. [20 points]
- (b) Let $p_{\text{even}}(n)$ (resp. $p_{\text{odd}}(n)$) denote the total number of partitions of n into an even (resp. odd) number of parts. Find a formula for the generating function $\sum_{n=0}^{\infty} (-1)^n (p_{\text{even}}(n) - p_{\text{odd}}(n))x^n$. [20 points]
- (c) Show that the above generating functions are the same and conclude $p_{d,o}(n) = |p_{\text{even}}(n) - p_{\text{odd}}(n)|$ for all n . [10 points]

[Note: the generating functions above may need to involve infinite products.]

Question 2 (Generating Functions, 50 points). Find expressions for the following generating functions: [10 points each]

- (a) Let a_n be the number of compositions of n into odd parts where each part is colored either red or blue (for example $a_1 = 2$ since you can write 1 as either a red 1 or a blue 1). Give the ordinary generating function $\sum_{n=0}^{\infty} a_n x^n$.
- (b) Let b_n be the difference between the number of compositions of n into an even number of parts of size 1 and 2, and the number of compositions of n into an odd number of parts of size 1 and 2. Give a closed form for the ordinary generating function $\sum_{n=0}^{\infty} b_n x^n$. Use this to give an explicit formula for b_n .
- (c) Let $c_n = nC_n$ where C_n is the Catalan number. Give an explicit formula for the ordinary generating function $\sum_{n=0}^{\infty} c_n x^n$. [Hint: you might want to consider differentiating the generating function for the Catalan numbers.]
- (d) Let d_n be the number of set partitions of $[n]$ into sets of size exactly 2 [Note: d_n will be 0 if n is odd]. Find a formula for d_n , and use it to obtain an explicit form for the exponential generating function $\sum_{n=0}^{\infty} d_n x^n / n!$.
- (e) Let e_n be a sequence with $e_{n+2} = e_{n+1} + ne_n$ for all $n \geq 0$. Give a differential equation satisfied by the exponential generating function $E(x) = \sum_{n=0}^{\infty} e_n x^n / n!$.

Question 3 (Extra credit, 1 point). Approximately how much time did you spend on this homework?