Section 1.1–1.3 Classification of Differential Equations.

What is a differential equation?

A differential equation is any equation that contains derivatives.

~~E.g. $y' = f(t)$~~

E.g. 1: $y' = \sin t$.

(general sol): $\Rightarrow$ $y(t) = -\cos t + C$

where C is an arbitrary constant.

E.g. 2: $y' = f(t)$

General sol: $y(t) = F(t) + C$

where $F(t)$ is the antiderivative of $f(t)$, and C is arbitrary constant.

If given an initial value $\begin{cases} y' = f(t) \\ y(t_0) = y_0 \end{cases}$

it is called an <u>initial value problem</u>, and the constant C needs to satisfy the equation

$$y(t_0) = y_0 = F(t_0) + C$$

E.g. 3. $\begin{cases} y' = \sin t \\ y(0) = 1. \end{cases}$

Sol. $y(t) = -\cos t + C$.

Since $y(0) = 1$, $-1 + C = 1 \Rightarrow C = 2$.

- Classification: involve derivatives of functions which depend on a single independent variable

1) $\boxed{\text{Ordinary differential equations (ODE)}}$
   Partial differential equation (PDE).

2) Order of differential equations.
   is the order of the highest derivative it contains.

   E.g. $y' + y^3 = \sin(y)$ (first-order)
   $y''' + y = 0$ (third-order)

3) Linear vs. nonlinear differential equations.
   An n-th order differential equation is linear if it can be written in the following form:

$$a_n(t)\, y^{(n)} + a_{n-1}(t)\, y^{(n-1)} + \dots + a_0(t)\, y = g(t).$$

Otherwise, the equation is nonlinear.

E.g. $y'' - 5y = \sin(y)$ (nonlinear)
$y' + e^t y = 0$ (linear)
$y \cdot y'' + y = 0$ (nonlinear).
$x^2 y' + (\sin x) y = \sqrt{x}$, $y' \sin x = x$ (linear) $y' + y^2 = 0$ (nonlinear).

E.g. Consider $y'' + y = 0$.
It is easy to show that $y = \sin x + \cos x$ is a solution.

Why? $y' = \cos x - \sin x$
$y'' = -\sin x - \cos x = -y$.
$\Rightarrow y'' + y = 0$.

Two questions arise:
1. Given an ODE, does a solution exist?
2. If a solution exists, is it unique?

Note that $y = \sin x + \cos x$ is not the only solution to $y'' + y = 0$.

And the practical question,

assuming a solution exists, how do we find it? (sometimes finding a solution is the easiest way to prove existence of a solution)

E.g. $\frac{dy}{dt} = 2y$.

We want to rewrite this so all y terms appear on one side of an equation and all t terms appear on the other side.

$$\frac{1}{2y}\frac{dy}{dt} = 1.$$

$$\frac{dy}{2y} = dt.$$

Now integrate $\int \frac{dy}{2y} = \int dt.$

$$\frac{1}{2}\ln|y| = \cancel{\tfrac{t^2}{2}}\, t + C$$

Now solve for y.

$$\ln|y| = 2t + 2c$$

$$|y| = e^{2t + 2c}$$

$$y = Ce^{2t}.$$

Where C is an arbitrary constant.

Check!

Ex. $\frac{dy}{dx} = 3y - 2.$

Section 2.2 Separable Equations

A first order ODE is separable if it can be written in the following form:

$$(*) \qquad M(x) + N(y)\frac{dy}{dx} = 0$$

E.g. 1: Are these ODE separable?

a) $x^2 \frac{dy}{dx} + \sin y = 0$ (Yes!)

b) $x^2 y \frac{dy}{dx} - (y-x)^2 = 0$ (No!).

(*) can be written (in differential form) as

~~$M(x)\,dx =$~~

$$N(y)\,dy = -M(x)\,dx.$$

If each side can be integrated and then if y can be isolated, we have a solution.

E.g. 2. Solve $\frac{dy}{dx} = xy^2$ such that $y(0) = -1$.

1) Separate: $\frac{1}{y^2}\frac{dy}{dx} = x.$

$$\frac{dy}{y^2} = x\,dx$$

2) Integrate $\int \frac{dy}{y^2} = \int x\,dx.$

$$-\frac{1}{y} = \frac{x^2}{2} + C$$

$$y = -\frac{1}{\frac{x^2}{2} + C}$$

Since $y(0) = -1$, $-\frac{1}{C} = -1 \Rightarrow C = 1.$

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$$y(x) = -\frac{1}{\frac{x^2}{2}+1}.$$

~~Eg.~~ Note that it may be difficult to solve for y explicitly after integrating.

E.g. Solve $\frac{dy}{dx} = \frac{y-1}{x+3}$. such that $y(-1) = 0$.

1) Seperate $\frac{dy}{y-1} = \frac{dx}{x+3}$.

2) Integrate $\int \frac{dy}{y-1} = \int \frac{dx}{x+3}$.

$$\ln|y-1| = \ln|x+3| + C.$$

since $y(-1) = 0$

$$\cancel{\ln|-1+3|}$$

$$\ln|0-1| = \ln|-1+3| + C.$$

$$\ln 1 = \ln 2 + C.$$

$$C = -\ln 2.$$

Hence,

$$\ln|y-1| = \ln|x+3| - \ln 2.$$

$$|y-1| = \frac{|x+3|}{2}.$$

So $y-1 = \frac{x+3}{2}$ or $y-1 = -\frac{x+3}{2}$.

$y = \frac{x+5}{2}$ $\boxed{y = -\frac{x+1}{2}}$

$\Rightarrow$ $y = -\frac{x+1}{2}$ because it satisfies the initial value condition $y(-1) = 0$.

E.g. Solve $y' = 2x\cos^2 y$, $y(0) = \frac{\pi}{4}$.

Sol.
$$\frac{dy}{\cancel{2x}\cos^2 y} = 2x\,dx.$$

$$\int \sec^2 y \, dy = \int 2x\,dx$$

$$\tan y = x^2 + C.$$

$$\tan\left(\tfrac{\pi}{4}\right) = 0 + C. \Rightarrow C = 1.$$

$$\Rightarrow \tan y = x^2 + 1.$$
$$y = \tan^{-1}(x^2 + 1)$$

Ex. Solve $y' = \dfrac{3x^2}{3y^2 - 4}$ and $y(1) = 0$.

Section 2.1: Linear Equations: Method of Integrating Factors

E.g. Solve for

$$\ln(t) \cdot \frac{dy}{dt} + \frac{1}{t} \cdot y = 4t, \qquad t > 0.$$

Notice that

$$\ln(t) \cdot \frac{dy}{dt} + \frac{1}{t} \cdot y = \frac{d}{dt}\left(\ln(t) \cdot y \right)$$

so $$\frac{d}{dt}\left(\ln(t) \cdot y \right) = 4t.$$

$$\Rightarrow \quad \ln(t)\, y = \underbrace{2t^2 + C}_{\int 4t\, dt.}$$

$$\Rightarrow \quad y = \frac{2t^2 + C}{\ln t}.$$

is the solution.

The above example is a special case because most first order ODEs cannot be solved directly using the same method.

However, one might be able to find a function $u(t)$ such that by multiplying $u(t)$ on both sides of the ODE one can solve it using the same method.

this $u(t)$ is called the <u>integrating factor</u>.

<u>Example 2</u>: Solve for $\frac{dy}{dt} - \frac{1}{t} y = t^2$.

we take $u(t) = \frac{1}{t}$ and multiply $u(t)$ on both sides:

$$\frac{1}{t}\frac{dy}{dt} - \frac{1}{t^2} y = t.$$

$$\frac{d}{dt}\left(\frac{1}{t}\cdot y\right) = t.$$

this function is an antiderivation of function $f(t) = t$

$$\frac{1}{t} y = \frac{t^2}{2} + C$$

$$y = \frac{t^3}{3} + Ct$$

where C is an arbitrary constant.

<u>Q</u>: How can we find $u(t)$?

Let's consider the same example $\frac{dy}{dt} - \frac{1}{t} y = t^2$.

⑨

$u(t)$ is an integrating factor if after we multiply ~~the equation both sides by of~~ both sides of the equation by $u(t)$,

$$u(t)\,\frac{dy}{dt} - \frac{1}{t}\,u(t)\,y = t^2 u(t).$$

LHS becomes $\frac{d}{dt}(u(t)\cdot y)$.

$\Rightarrow$ the following needs to be true

$$\frac{du}{dt} = -\frac{1}{t}\,u(t).$$

To solve for $u(t)$, there are two ways:

1) Take a guess, try many functions for $u(t)$, and find out which one works.
2) Use the method that solves separable equations:

$$\frac{du}{dt} = -\frac{1}{t}\,u(t).$$

$$\Rightarrow \quad \frac{du}{u(t)} = -\frac{1}{t}\,dt.$$

$$\int \frac{du}{u} = -\int \frac{1}{t}\,dt$$

$$\ln|u| = -\ln t + C.$$

We can choose $C = 0$.

$$\Rightarrow \quad \ln|u| = -\ln t.$$

$$|u| = \frac{1}{t}.$$

$$u = \frac{1}{t} \quad \text{or} \quad -\frac{1}{t}.$$

In general, given a first order linear ODE in standard form

$$\frac{dy}{dt} + p(t)\, y = g(t),$$

we multiplying both sides by the integrating factor $u(t)$

$$u(t)\frac{dy}{dt} + p(t)u(t)\, y = g(t)u(t).$$

If we have $u'(t) = p(t)u(t)$, then we can write the above as

$$u(t)\frac{dy}{dt} + u'(t)\, y = g(t)u(t).$$

$$(u(t)\cdot y)' = g(t)u(t).$$

so $$u(t)\cdot y = \int g(t)u(t)\, dt + C$$

hence $$y = \frac{1}{u(t)}\left(\int g(t)u(t)\, dt + C\right)$$

How to find $u(t)$?

$$u'(t) = u(t)\, p(t).$$

$$\frac{u'(t)}{u(t)} = p(t).$$

$$\Rightarrow \quad \ln|u(t)| = \int p(t)\, dt + C$$

$$u(t) = e^{\int p(t)\, dt + C}$$

$\Rightarrow$ Choose $C = 0$,

$$u(t) = e^{\int p(t)\, dt}$$

$$\boxed{\text{So, the general solution to } \frac{dy}{dt} + p(t)\, y = g(t) \text{ is } y(t) = \frac{\int g(t)u(t)\, dt + C}{u(t)} \text{ where } u(t) = e^{\int p(t)\, dt}}$$

Example: Solve $y' + 2xy = e^{-x^2}$ and $y(0) = 2$.

1) Find $u(x) = e^{\int 2x\,dx} = e^{x^2}$.

2) $e^{x^2} y' + 2x e^{x^2} y = 1$.

$$(e^{x^2} y)' = 1.$$

$$e^{x^2} y = x + C$$

$$y = x e^{-x^2} + C e^{-x^2}.$$

[Check: $y' = e^{-x^2} - 2x^2 e^{-x^2} + C(-2x) e^{-x^2}$.

$y' + 2xy = e^{-x^2} - 2x^2 e^{-x^2} - 2xCe^{-x^2}$

$+ 2x^2 e^{-x^2} + 2xCe^{-x^2} = e^{-x^2}$ ✓.]

Since $y(0) = 2$,

$$0 + Ce^0 = 2.$$

$$C = 2.$$

$\Rightarrow \quad y(x) = x e^{-x^2} + 2 e^{-x^2}.$

Section 2.4. Differences between Linear and nonlinear equations

Questions of Existence and uniqueness.
1) Does a solution exist?
2) If so, is it unique?
3) For what values of the independent variable does the solution exist?

Theorem: (linear) $\begin{cases} y' + p(x)\,y = g(x), & \alpha < x < \beta. \\ y(x_0) = y_0 & \alpha < x_0 < \beta. \end{cases}$

has a unique solution $y = y(x)$, $\alpha < x < \beta$, provided that $p(x)$ and $g(x)$ are continuous on (α, β).

Recall. General sol. $y(x) = \dfrac{1}{u(x)}\left[\int u(x)g(x)\,dx + C\right]$

where $u(x) = e^{\int p(x)\,dx}$.

E.g. Find the solution of $\begin{cases} x^2 y' - xy = \frac{1}{x} \\ y(1) = -1 \end{cases}$.

and determine for what values of x the solution is ~~valid~~. unique.

Sol. 1) Put the equation into stand form:

$$y' - \frac{1}{x} y = \frac{1}{x^3}.$$

$$\Rightarrow \quad p(x) = -\frac{1}{x} \quad \text{and} \quad g(x) = \frac{1}{x^3}.$$

2) Find $u(x) = \exp\left(\int \left(-\frac{1}{x}\right) dx\right) = \exp\left[-\ln|x|\right] = \frac{1}{|x|}.$

3) $y = \frac{1}{u(x)} \int u(x) g(x)\, dx = |x| \cancel{\frac{1}{|x|}} \int \frac{1}{|x|\, x^3}\, dx$

For $x > 0$, $y = x \cancel{\frac{1}{x}} \int \frac{1}{x^4}\, dx = x\left[\left(-\frac{1}{3x^3}\right) + C\right].$

$$\Rightarrow y = -\frac{1}{3x^2} + Cx.$$

When $x = 1$, $y(1) = -1$.

$$\Rightarrow -\frac{1}{3} + C = -1$$

$$C = -\frac{2}{3}.$$

$$y(x) = -\frac{1}{3x^2} - \frac{2}{3}x$$

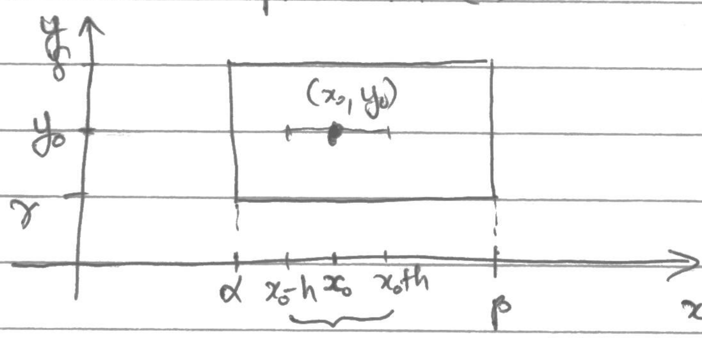
and the solution is ~~valid~~ unique for $x > 0$

since $p(x)$ and $g(x)$ are continuous on $(0, \infty)$.

Theorem. (Nonlinear equation).

$$\begin{cases} \dfrac{dy}{dx} = f(x,y) \\ \\ y(x=x_0) = y_0 \end{cases} \qquad (*)$$

Let the function f and $\dfrac{\partial f}{\partial y}$ be continuous in some rectangle $\alpha < x < \beta$, $\gamma < y < \delta$ containing the point (x_0, y_0). Then, in some interval $x_0 - h < x < x_0 + h$ contained in $\alpha < x < \beta$, there is a unique solution $y = \phi(x)$ of the initial value problem $(*)$.



↳ interval in which the solution $y = \phi(x)$ is guaranteed to exist uniquely.

E.g. Given $\dfrac{dy}{dx} = \dfrac{1+x^2}{3y - y^2}$, does it satisfy the condition of the theorem?

Sol. $f(x,y) = \dfrac{1+x^2}{3y-y^2}$ is continuous if $3y - y^2 \neq 0$.

$$y(3-y) \neq 0$$

$$y \neq 0 \text{ or } y \neq 3.$$

$$\frac{\partial f}{\partial y} = -\frac{1+x^2}{(3y-y^2)}(3-2y) \Rightarrow \text{continuous if } y \neq 0,\ y \neq 0.$$

E.g. Solve $y' = y^{1/3}$, $y(0) = 0$, for $x \geq 0$.

Sol. Separable

$$\Rightarrow \quad y^{-1/3}\, dy = dx.$$

$$\int y^{-1/3}\, dy = \int dx$$

$$\frac{3}{2} y^{2/3} = x + C$$

$$y^{2/3} = \frac{2}{3}(x+C)$$

$$y = \pm \left[\frac{2}{3}(x+C)\right]^{3/2}.$$

Using $y(0) = 0$,

$$0 = \pm\left[\frac{2}{3}C\right]^{3/2} \quad \Rightarrow \quad C = 0.$$

So

$$y = \begin{cases} \phi_1(x) = \left(\frac{2}{3}x\right)^{3/2} & x \geq 0 \\ \phi_2(x) = -\left(\frac{2}{3}x\right)^{3/2} & x \geq 0 \end{cases}$$

In fact, $y = \phi_3(x) = 0$ for $x \geq 0$ is also a solution.

~~For an arbitrary positive t_0 the functions~~

$$\cancel{y = \chi(x) = \begin{cases} 0, & 0 < t < t_0 \\ \pm & \end{cases}}$$

What is going on here?

$f(x,y) = y^{1/3}$ is continuous for all y.

$$\frac{\partial}{\partial y} f(x,y) = \frac{d}{dy} y^{1/3} = \frac{1}{3} y^{-2/3}$$

which is not defined when $y = 0$.

Thus, the theorem fails when γ and δ have opposite sign or if one is zero.

However, if $y_0 \neq 0$, then $y' = y^{1/3}$, $y(x_0) = y_0$ does have a unique solution which passes through (x_0, y_0).

E.g. $y' = 2xy^2$, $y(0) = y_0$.

Sol. $y^{-2}\,dy = 2x\,dx$

$$\int y^{-2}\,dy = \int 2x\,dx$$

$$-y^{-1} = x^2 + C$$

$$y = -\frac{1}{x^2+C}$$

Initial condition implies:

$$y_0 = y(0) = -\frac{1}{0+C}$$

$$\Rightarrow C = -\frac{1}{y_0} \quad \text{if } y_0 \neq 0.$$

So $y = -\dfrac{1}{x^2 - \frac{1}{y_0}} = \dfrac{y_0}{1 - x^2 y_0}$

and we need $1 - x^2 y_0 \neq 0 \Rightarrow x^2 \neq \dfrac{1}{y_0}$

$\Rightarrow x \neq \pm\sqrt{\dfrac{1}{y_0}}$ if $y_0 > 0$

If $y_0 = 0$, then $y = 0$ (why?)

Thus, if $y_0 \neq 0 \Rightarrow y = \dfrac{y_0}{1 - x^2 y_0}$

with $-\sqrt{\dfrac{1}{y_0}} < x < \sqrt{\dfrac{1}{y_0}}$ if $y_0 > 0$

and $-\infty < x < \infty$ if $y_0 < 0$.