Math 184A Homework 1

Fall 2015

This homework is due Monday October 5th in discussion section. Remember to justify your work even if the problem does not explicitly say so. Writing your solutions in LaTeXis recommend though not required.

Question 1 (Summation Polynomials, 40 points). *Use induction to prove the following:*

(a) Show for all positive integers n that

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

[10 points]

(b) Show that if p and q are polynomials so that

(i) $q(0) = 0$

(ii) $p(n) = q(n) - q(n-1)$ for all n

that

$$\sum_{i=1}^{n} p(i) = q(n)$$

for all positive integers n. [10 points]

(c) Show that for any polynomial p, there exists a polynomial q so that

$$\sum_{i=1}^{n} p(i) = q(n)$$

for all positive integers n. [Hint: use induction on the degree of p and note that $n^d - (n-1)^d = dn^{d-1}$ + lower order terms.] [20 points]

Solution. (a) We prove this by *induction on n*.

Base Case: The base case is $n = 1$, where we see the sum of squares up to 1 is just 1, while the right side of our equation is

$$\frac{1(2)(3)}{6} = 1,$$

as desired. This proves the base case.

Inductive Step:Now we suppose we have proved our formula for all positive integers up to $n-1$, and we want to show it is true for n. We observe that

$$\sum_{i=1}^{n} i^2 = \left(\sum_{i=1}^{n-1} i^2\right) + n^2,$$

and we now use the inductive hypothesis on the first term of the right hand side, which gives:

$$\sum_{i=1}^{n} i^2 = \frac{(n-1)(n)(2(n-1)+1)}{6} + n^2 \tag{1}$$

$$= \frac{(n-1)(n)(2n-1)+6n^2}{6} \tag{2}$$

$$= \frac{n(n+1)(2n+1)}{6}, \tag{3}$$

where we have left out some arithmetic to simplify our fraction into the desired form.

(b) **Base Case**: The base case is $n = 1$, where our desired formula is

$$p(1) = q(1),$$

which is true since by assumption $p(1) = q(1) - q(0)$ and $q(0) = 0$. So we have proven the base case.
Inductive Step: We now suppose the formula is true up to $n - 1$ and we show it is true for n. In a manner similar to the previous part, we break up our sum and finally use our knowledge of how $p(n)$ is defined:

$$\sum_{i=1}^{n} p(i) = p(n) + \sum_{i=1}^{n-1} p(i) \tag{4}$$

$$= p(n) + q(n-1) \tag{5}$$

$$= (q(n) - q(n-1)) + q(n-1) = q(n), \tag{6}$$

as desired. Note the inductive hypothesis allowed us to replace the sum $\sum_{i=1}^{n-1} p(i)$ with $q(n-1)$.

(c) Firstly, we note that the previous part of the problem means that we can reduce this problem to demonstrating for any polynomial p the existence of a polynomial q satisfying $q(0) = 0$ and $p(n) = q(n) - q(n-1)$ for all integers n. Again, we use induction, only now on the degree of the polynomial p, lets say the degree is d.
Base Case: The base case here is degree $d = 0$ (recall a degree zero polynomial is just some number). When $d = 0$, so that $p(x) = c$, we see that $q(x) = cx$ clearly works. This takes care of the base case.

Aside: Always take care to be sure what the base case is. It will depend on what the induction is on.

Inductive Step: Now suppose we have proven it for polynomials of degrees up to $d - 1$, and we want to prove it up to d. Let us say $p(x) = a \cdot x^d + p'(x)$, where a is some constant and $p'(x)$ is a polynomial of degree $< d$. Let us set $q'(x) = ax^{d+1}/(d+1)$. Then $q'(0) = 0$, and

$$p(x) - (q'(x) - q'(x-1)) = (a \cdot x^d + p'(x)) - (a \cdot x^d + \text{lower order terms}),$$

and the x^d terms cancel leaving us with some polynomial of degree $< d$ as the right hand side. Hence $p(x) - (q'(x) - q'(x-1))$ is of degree $< d$. By the inductive hypothesis, we know there exists a polynomial $q''(x)$ such that $q''(0) = 0$ and $p(n) - (q'(n) - q'(n-1)) = q''(n) - q''(n-1)$. If we now set $q(x) = q'(x) + q''(x)$, we see that $q(0) = q'(0) + q''(0) = 0$, and

$$p(n) = (q''(n) - q''(n-1)) + (q'(n) - q'(n-1)) = q(n) - q(n-1),$$

as needed. Hence, by the previous part it follows that

$$\sum_{i=1}^{n} p(i) = q(n).$$

$\square$

Question 2 (Polygonal Triangulations, 20 points). *Let $n = 2m$ for $m \geq 2$. Show by induction on m that a convex n-gon can be triangulated (that is divided into triangles whose vertices are vertices of the original polygon) in at least 2^{m-1} different ways. [Hint: First cut your polygon into a 4-gon and an $(n-2)$-gon.]*

Proof. **Base Case**:The base case is $m = 2$ which is a convex quadrilateral. This case is trivial, since either diagonal gives distinct triangulations, and these are the only ones. Hence we have $2^{m-1} = 2$ distinct triangulations.

Inductive Step: For the general case we suppose its true up to $m - 1$ and we want to show its true for m. Given a $2m$-gon, select 4 vertices who are neighbors and draw a line from the two furthest vertices, so that our $2m$-gon is divided into a convex quadrilateral and a $2(m-1)$-gon. (here we need the assumption that the n gon is convex - draw some pictures to convince yourself it won't necessarily work if it is concave somewhere). There are two ways to triangulate the quadrilateral and by the inductive hypothesis at least 2^{m-2} ways to triangulate the other part, hence we can produce at least $2 \cdot 2^{m-2} = 2^{m-1}$ triangulations of the entire $2m$-gon. $\square$

Question 3 (Proving Pigeonhole by Induction, 20 points). *Prove the pigeon hole principle by induction on the number of holes.*

Proof. **Base Case**:The base case is where we have 1 hole. If we have > 1 pigeon, then clearly the single hole has more than one pigeon in it. This takes care of the base case.

Inductive Step: Now suppose the principle holds for $n - 1$ holes, and we are given n holes and $m > n$ pigeons. Drop pigeons into holes any way you like. Lets pick a particular hole. There are two cases: Either the hole we chose has multiple pigeons in it, in which case we are done, or else that hole has no pigeons or one pigeon in it. That leaves at least $m - 1$ pigeons to be distributed amongst the remaining $n - 1$ holes. Since $m - 1 > n - 1$, this means that one of the other holes has more than one pigeon in it, by the induction hypothesis. $\square$

Question 4 (Repeating Decimals, 20 points). *Show that any rational number (that is a number of the form $\frac{n}{m}$ for integers n, m) has a decimal expansion that repeats after some point. [Hint: Consider computing the decimal expansion using the standard long division algorithm. Show that eventually, the number you are trying to divide by m on the next step repeats.]*

Proof. If we consider the steps involved in the division algorithm for the expansion of a/b, we obtain a positive remainder r at each step, with the constraint that $r < b$. Hence there are only finitely many possible remainders at each step of the algorithm. If a remainder is ever zero, the algorithm terminates, hence it has a repeating decimal expansion in the sense that it ends with an infinite string of zeroes. Otherwise, since there are only finitely many possible remainders, by the pigeon hole principle we eventually repeat a remainder. In particular we must obtain at least one repeated remainder term after doing a number of divisions equal to the divisor plus one, but possibly earlier than that.

Thinking for a moment about how the algorithm works, this means that once we hit our first repeated remainder the algorithm will cycle through a finite number of steps ad infinitum, yielding an eventually repeating decimal expansion. To see this, suppose the n'th remainder, which we will denote by r_n, is repeated, meaning for some k, $r_n = r_{n+k}$. If we write down the sequence of remainders we obtain at each step at the algorithm, part of the sequence looks like

$$\ldots, r_n, r_{n+1}, \ldots, r_{n+k-1}, r_{n+k} = r_n.$$

How do we obtain each remainder? To get the $n + 1$'st digit of the expansion, we take the remainder r_n, we take down the $n + 1$st digit in the expansion of a and bring it down to the left of r_n, and we divide the resulting thing by b. If n is large enough that the $n + 1$st digit of a is zero (always possible since a is an integer, here is where the argument fails for irrational numbers), then the thing we bring down to r_n is just zero, and we divide $10 \cdot r_n$ by b, and this yields the remainder r_{n+1}.

But the step will be the exact same once we hit r_{n+k}! We will also carry down a zero from the expansion of a, and we will divide $10 \cdot r_{n+k} = 10 \cdot r_n$ by b, which means the digit we put up in the expansion will be the same! This also means that $r_{n+k+1} = r_{n+1}$. Repeat the same argument for r_{n+1} and we see that we get the same thing yet again, and so on. So as long as the repeated digit is further out than the decimal expansion of the numerator a, which will always happen by the pigeonhole princple, we get a repeating sequence in the division algorithm. $\square$