

Solution for Homework 6

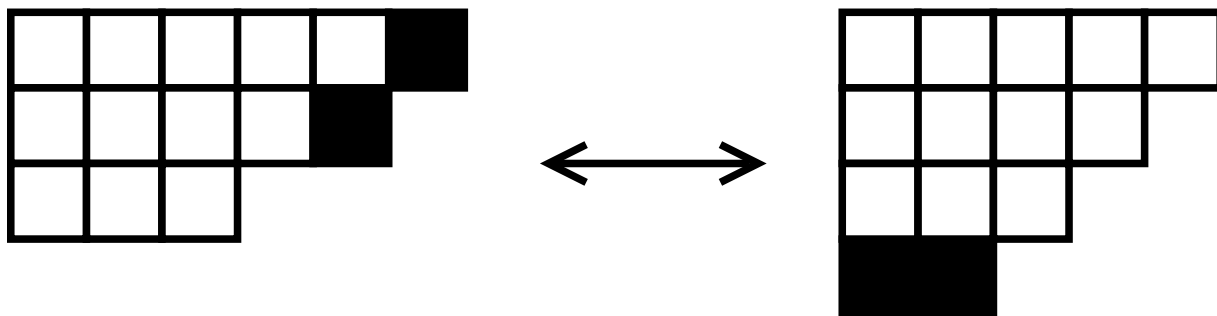
Fall 2015

This homework is due Monday November 23rd in discussion section. Remember to justify your work even if the problem does not explicitly say so. Writing your solutions in L^AT_EX is recommended though not required. If you cannot solve one part of a problem, you may still use the results from it in later parts of the same problem.

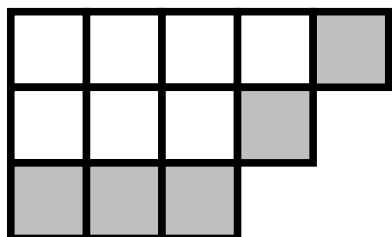
Optional Practice Problems: (do not turn in) Chapter 8 problems 1, 2, 7, 8, 9.

Question 1 (Pentagonal Number Theorem, 80 points). .

- (a) Let $F(x)$ be the function $(1-x)(1-x^2)(1-x^3)\cdots$. If $F(x) = \sum_{n=0}^{\infty} f_n x^n$ show that f_n equals the sum over integer partitions λ of n into parts of distinct sizes of $(-1)^{\text{Number of parts of } \lambda}$. [20 points]
- (b) It is actually possible to show pair up terms in the above sum in a way to show that most of them cancel out. In particular, consider the Ferrer's diagram. Suppose that the a largest parts all differ in size by 1 and that the smallest part has size b . If $a < b$, we can remove 1 from each of the a largest parts, creating a new part of size a . If $b \geq a$, we remove the smallest part and add 1 to each of the b largest parts (see figure below). Show that this method allows us to pair up most of partitions of n into distinct parts into



pairs where one element of each pair has an even number of parts and one has an odd number of parts. In particular, show that the only partitions that are not paired in this way are those where a equals the number of parts and is equal to either b or $b-1$, as shown below. [20 points]



- (c) Show that the pairings in part (b) allow us to cancel almost all the terms in the formula for f_n in part (a). In particular, show that

$$F(x) = \sum_{n=-\infty}^{\infty} (-1)^n x^{n(3n-1)/2} = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \dots$$

Note the surprising amount of cancellation showing up in this formula. [20 points]

(d) Use the above and the fact that the generating function for the partition numbers is given by

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{n=1}^{\infty} \frac{1}{1-x^n}$$

to prove the following recurrence for the partition numbers:

$$p(n) = \sum_{m \in \mathbb{Z}, n \neq 0} (-1)^{m+1} p(n - m(3m-1)/2) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + \dots$$

Note that this formula can be used to compute partition numbers fairly efficiently. In particular, if you have already computed $p(1), p(2), \dots, p(n)$, you can then compute $p(n+1)$ as a sum of approximately \sqrt{n} of these other terms. [20 points]

Solution.

(a) First we introduce some new notations about partition.

$\lambda \vdash_d n$ means λ is a partition of n with distinct part.

$l(\lambda)$ denotes the number of parts of λ .

$P_d(n)$ is the set of partitions of n into all distinct parts.

$p_d(n)$ is the number of partitions of n into all distinct parts.

Then, our object is to show

$$f_n = \sum_{\lambda \vdash_d n} (-1)^{l(\lambda)}.$$

In the textbook (Page 161) we see the generating function for $p_d(n)$ that

$$G(x) = \sum_{n \geq 0} p_d(n)x^n = \prod_{i \geq 1} (1 + x^i),$$

Which is similar to our $F(x)$

$$F(x) = \sum_{n \geq 0} f(n)x^n = \prod_{i \geq 1} (1 - x^i),$$

f_n is the coefficient of x^n . Like the statement in the text book (Page 159-161), each time a product on the RHS (a term in the expansion of $\prod_{i \geq 1} (1 - x^i)$) is a monomial of power n , we obtain a partition of n into all distinct parts. For example, $(-x^2)(-x^3)(-x^5) = (-1)^3 x^{10}$ is a monomial of power 10 related to the partition $\pi = (5, 3, 2)$. since π has 3 parts, this term's coefficient is $(-1)^3$, meaning π related term contribute $(-1)^{l(\pi)}$ to the coefficient of $x^{|\pi|}$. By summing up all the coefficients of terms of power n in RHS, we get

$$f_n = \sum_{\lambda \vdash_d n} (-1)^{l(\lambda)}.$$

(b) To solve this question, let's go over the map carefully.

ϕ is a map : $P_d(n) \rightarrow P_d(n)$. Given any $\pi \in P_d(n)$, look at the correspondent Ferrer's diagram. Suppose that the a largest parts all differ in size by 1 and that the smallest part has size b .

1 If $a < b$, we can remove 1 from each of the a largest parts, creating a new part of size a .

2 If $b \geq a$, we remove the smallest part and add 1 to each of the b largest parts.

We get a new Ferrer's diagram, if it is still a partition with distinct part, we say this method works, and call the correspondent partition $\phi(\pi)$; else, we say $\phi(\pi) = \phi$.

First we claim that,

Claim. If $\phi(\lambda) = \mu \neq \lambda$, then $\phi(\mu) = \lambda$, and the number of parts of λ and μ are different by 1.

Proof. If $\phi(\lambda) = \mu \neq \lambda$, we see the method works, so the number of parts of μ is the number of parts of $\lambda \pm 1$.

In λ , if $a_\lambda < b_\lambda$, we do the first method. Then in $\mu = \phi(\lambda)$, $b_\mu = a_\lambda$ and $a_\mu \geq a_\lambda$, so $a_\mu \geq b_\mu$.

When we apply the map ϕ again for μ , since $a_\mu \geq b_\mu$, we will use the second method, and then go back to λ , i.e. $\phi(\mu) = \lambda$.

If $a_\lambda \geq b_\lambda$, the proof is similar. □

Thus, if the method works for λ , we can pair up λ and $\phi(\lambda)$.

Then we say that if $a < \#$ of parts, then $\phi(\lambda) \neq \lambda$. You can verify that this method always works when $a < \#$ of parts.

Similarly, if $a = \#$ of parts and $a \neq b$ or $b - 1$, this method still works.

When $a = \#$ of parts and $a = b$ or $b - 1$, this method doesn't work.

If $a = \#$ of parts and $a = b$, then the diagram looks like the example given. We need to use method 2, meaning we need to move the part with size b upward. However, after this movement the number of parts becomes $b - 1 < b$, we won't get a partition, so this method doesn't work, and this partition is fixed under the map ϕ .

If $a = \#$ of parts and $a = b - 1$, we need to use method 1, meaning we need to move the staircase downward. However, after this movement the size of a^{th} part becomes $b - 1 = a$, and the new part (i.e. the $a + 1^{\text{th}}$ part) also has size a . We won't get a partition with distinct part, so this method doesn't work, and this partition is also fixed under the map ϕ .

So this method pairs up most of partitions of n into distinct parts into pairs where one element of each pair has an even number of parts and one has an odd number of parts. In particular, the only partitions that are not paired in this way are those where a equals the number of parts and is equal to either b or $b - 1$.

- (c) If λ and μ are paired up by the method, then their related terms in $F(x) = \prod_{i \geq 1} (1 - x^i)$ is of the same power, but their coefficients are inverse of each other, then they will be canceled. Only the partition that is fixed under the map ϕ won't be canceled. So

$$f(n) = \text{sum over fixed points } \lambda \text{ of } \phi \text{ in } P_d(n) \text{ of } (-1)^{l(\lambda)}.$$

Actually there is only two kinds of fixed point among all partitions of all integers as discussed in the last question.

The first kind of fixed point is $a = \#$ of parts > 0 and $a = b$, those are partitions of size $a(3a - 1)/2$, contribute a term $(-1)^a x^{a(3a-1)/2}$ to $F(x)$;

The second kind of fixed point is $a = \#$ of parts > 0 and $a = b - 1$, those are partitions of size $a(3a + 1)/2$, contribute a term $(-1)^a x^{a(3a+1)/2}$ to $F(x)$.

$n = 0$ is a special case, contribute 1 to $F(x)$.

Now we can get the formula for $F(x)$:

$$\begin{aligned} F(x) &= 1 + \sum_{a=1}^{\infty} (-1)^a (x^{a(3a-1)/2} + x^{a(3a+1)/2}) \\ &= 1 + \sum_{a=1}^{\infty} (-1)^a x^{a(3a-1)/2} + \sum_{a=-\infty}^{-1} (-1)^a x^{a(3a-1)/2} \\ &= \sum_{n=-\infty}^{\infty} (-1)^n x^{n(3n-1)/2} \\ &= 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \dots \end{aligned}$$

(d) Observing the generating function

$$P(x) = \sum_{n=0}^{\infty} p(n)x^n = \prod_{n=1}^{\infty} \frac{1}{1-x^n},$$

we can see that

$$P(x)F(x) = \prod_{n=1}^{\infty} \frac{1}{1-x^n} \times \prod_{i \geq 1} (1-x^i) = 1,$$

i.e.

$$\sum_{n=0}^{\infty} p(n)x^n \times \sum_{n=-\infty}^{\infty} (-1)^n x^{n(3n-1)/2} = 1.$$

Comparing the coefficient of $x^n (n > 0)$ both sides, we get

$$\sum_{m \in \mathbb{Z}} (-1)^m p(n - m(3m-1)/2) = 0,$$

which means

$$p(n) = \sum_{m \in \mathbb{Z}, n \neq 0} (-1)^{m+1} p(n - m(3m-1)/2) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + \dots$$

□

Question 2 (Generating Functions for Compositions, 20 points). *Let $a_{n,k}$ be the number of compositions of n into k parts (recall that this is the number of ways to write $n = x_1 + x_2 + \dots + x_k$ where the x_i are any positive integers). Show that for any k we have the generating function identity*

$$\sum_{n=0}^{\infty} a_{n,k} x^n = \left(\frac{x}{1-x} \right)^k.$$

Sum the above over k to find the generating function for a_n , the number of compositions of n into any number of parts. Provide a formula for a_n .

Solution. We need to show

$$\sum_{n=0}^{\infty} a_{n,k} x^n = \left(\frac{x}{1-x} \right)^k = (x + x^2 + x^3 + x^4 + \dots)^k.$$

Like the solution of **Example 8.9** in the text book, we solve this question by analyze the coefficient of x^n of both sides. The RHS is a sum of k -term product. If the member from the i^{th} parenthesis is x^{j_i} and their product is x^n , then we have $j_1 + j_2 + \dots + j_k = n$, thus we get a composition of n into k parts. Conversely, every composition of n into k parts can be associated to a product on the RHS, meaning that the coefficient of x^n , $a_{n,k}$, is the number of compositions of n into k parts.

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} a_{n,k} x^n = \sum_{k=1}^{\infty} \left(\frac{x}{1-x} \right)^k = \frac{\left(\frac{x}{1-x} \right)}{1 - \left(\frac{x}{1-x} \right)} = \frac{x}{1-2x} = \sum_{n=1}^{\infty} 2^{n-1} x^n,$$

So

$$a_0 = 0, a_n = 2^{n-1} \text{ for } n > 0.$$

□