Question 1 (Generating Function Computation, 30 points)**.** *What are the first* 5 *coefficients (the* x^0 *coefficient through that* x^4 *coefficient) of the generating function*

$$\frac{\sqrt{1+2x^2}}{(1+x)^2}?$$

We have that

$$\sqrt{1+2x^2} = (1+2x^2)^{1/2} = 1 + (1/2)(2x^2) + (1/2)(-1/2)/2!(2x^2)^2 + \ldots = 1 + x^2 - x^4/2 + \ldots.$$

We also have that

$$\frac{1}{(1+x)^2} = \frac{1}{(1-(-x))^2} = 1 + 2(-x) + 3(-x)^2 + 4(-x)^3 + 5(-x)^4 + \ldots = 1 - 2x + 3x^2 - 4x^3 + 5x^4 + \ldots$$

The product is therefore

$$(1 + x^2 - x^4/2 + \ldots)(1 - 2x + 3x^2 - 4x^3 + 5x^4 + \ldots) = 1 - 2x + 4x^2 - 6x^3 + (15/2)x^4 + \ldots$$

Question 2 (No Hamiltonian Paths, 35 points). *Show that if G is a graph with at least three vertices of degree 1, that G does not contain any Hamiltonian paths.*

Assume for sake of contradiction that G has vertices v_1, v_2, v_3 each of degree 1, and a Hamiltonian path P. Since P is Hamiltonian, it must pass through each v_i. Since P has only two endpoints, one of the v_i must not be an endpoint of P. Therefore P must have some u before v_i and w after. However, since v_i has degree-1, it has only one neighbor, and therefore we must have that $u = w$, which contradicts P being a Hamiltonian path.

Question 3 (Derangement Equation, 35 points)**.** *Show that*

$$n! = \sum_{k=0}^{n} \binom{n}{k} D_{n-k}$$

where D_m is the number of derangements on $[m]$.

We claim that both sides count the total number of permutations of $[n]$. The fact that this is $n!$ is standard. To get the right hand side, note that this is a sum over k of the number of permutations of n with exactly k fixed points. To count the number of such permutations, we note that there are $\binom{n}{k}$ ways to choose the set of k elements fixed by the permutation, and that having picked those elements, there are D_{n-k} ways to permute the remaining $n-k$ elements without leaving any additional fixed points. Thus,

$$n! = \sum_{k=0}^{n} \binom{n}{k} D_{n-k}.$$

Alternative Proof: Note that

$$\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k} D_{n-k} &= \sum_{k=0}^{n} \left(\frac{n!}{k!(n-k)!}\right)(n-k)!\left(\sum_{m=0}^{n-k} \frac{(-1)^m}{m!}\right) \\
&= n! \sum_{k=0}^{n} \sum_{m=0}^{n-k} \frac{(-1)^m}{m!k!}.
\end{aligned}$$

Letting $\ell = m + k$, this is

$$\begin{aligned}
n! \sum_{\ell=0}^{n} \sum_{m=0}^{\ell} \frac{(-1)^m}{m!(\ell-m)!} &= n! \sum_{\ell=0}^{n} \sum_{m=0}^{\ell} (-1)^m \binom{\ell}{m} / \ell! \\
&= n! \sum_{\ell=0}^{n} \frac{1}{\ell!} \sum_{m=0}^{\ell} (-1)^m \binom{\ell}{m} \\
&= n! \left(1 + \sum_{\ell=1}^{n} \frac{(1-1)^\ell}{\ell!}\right) \\
&= n!
\end{aligned}$$

Where the second to last line above is by the binomial theorem.