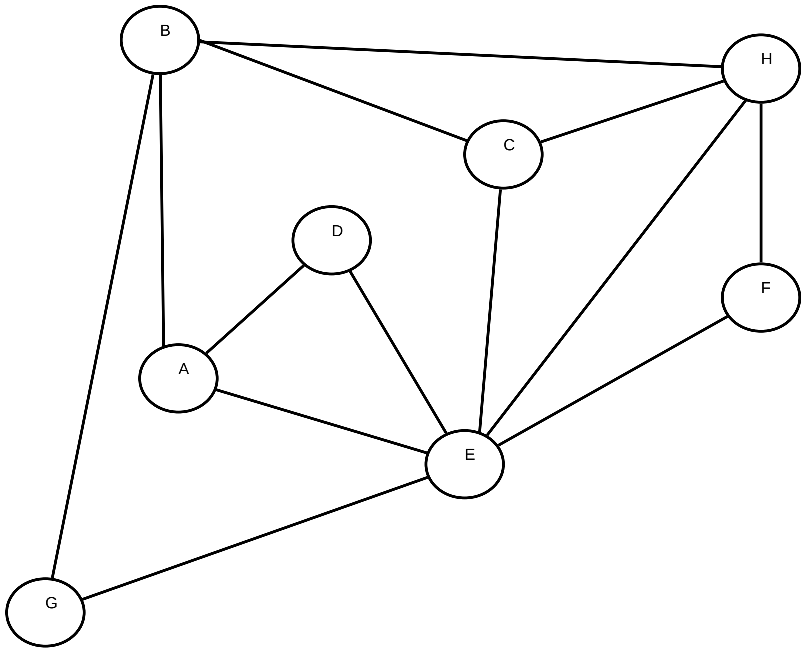
Question 1 (Eulerian Trail, 15 points). *Either exhibit an Eulerian trail (not necessarily closed) in the graph below or show that one does not exist:*



We note that $ABGEADEFHECBHC$ is such a path.

Question 2 (Restricted Permutation Counting, 15 points). *How many permutations* $\pi : [8] \to [8]$ *are there so that* $\pi(1), \pi(2), \pi(3)$ *are all at most* 5*?*

Answer: $5 \cdot 4 \cdot 3 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 7200$. There are 5 possibilities for $\pi(1)$. Given that, there are 4 remaining possibilities for $\pi(2)$. Given that there are 3 remaining possibilities for $\pi(3)$. Then $\pi(4)$ can be any of the 5 remaining possibilities, $\pi(5)$ any of 4, $\pi(6)$ any of 3, $\pi(7)$, and of 2, and $\pi(8)$ is then forced to be the one remaining possibility. The answer is then determined by the product rule.

Question 3 (Stirling Number Computation, 15 points). *Compute the unsigned Stirling number of the first kind* $c(6,3)$.

Answer: 225.

We use the recurrence relation $c(n,k) = (n-1)c(n-1,k) + c(n-1,k-1)$ to compute the following table:

$n\backslash k$	0	1	2	3	4	5	6
0	1	0	0	0	0	0	0
1	0	1	0	0	0	0	0
2	0	1	1	0	0	0	0
3	0	2	3	1	0	0	0
4	0	6	11	6	1	0	0
5	0	24	50	35	10	1	0
6	0	120	274	225	85	15	1

Alternate Solution: A permutation of [6] with exactly 3 cycles, must have cycles of lengths $1,1,4$ or $1,2,3$, or $2,2,2$. The number of permutations of each type is

$$\frac{6!}{1^2 \cdot 4^1 \cdot 2!} = 90, \frac{6!}{1^1 \cdot 2^1 \cdot 3^1} = 120, \frac{6!}{2^3 \cdot 3!} = 15.$$

Thus, the total number is $90 + 120 + 15 = 225$.

Question 4 (Near Integer Multiples, 15 points). *Let x and y be real numbers and let n be an integer. Show that there is an integer m with $1 \leq m \leq n^2$ so that both mx and my are within $1/n$ of an integer.*

Hint: Find m, m' so that the fractional parts of mx and $m'x$ are close as are the fractional parts of my and $m'y$.

Divide $[0,1] \times [0,1]$ into n^2 regions of the form $[a/n, (a+1)/n] \times [b/n, (b+1)/n]$ where $n > a, b \geq 0$ are integers. Let α_m be the pair given by the fractional part of mx and the fractional part of my. By the pigeonhole principle, there must be m, m' between 0 and n^2 so that α_m and $\alpha_{m'}$ are in the same region (there are $n^2 + 1$ such m, and only n^2 regions). If this is the case the fractional parts of mx and $m'x$ differ by at most $1/n$. Therefore, $(m - m')x$ must be within $1/n$ of an integer. Similarly, $(m - m')y$ must be within $1/n$ of an integer. Therefore $|m - m'|$ satisfies our conditions.

Question 5 (Generating Function Computation, 20 points)**.** *(a) Let a_n, $n \geq 0$ be the sequence given by $a_0 = 0$, $a_1 = 0$ and $a_{n+2} = a_{n+1} - a_n + 1$. Find the ordinary generating function for this sequence.*

Answer: $\frac{x^2}{1-2x+2x^2-x^3}$.

Let $A(x) = \sum_{n=0}^{\infty} a_n a^n$ be the generating function. We have that

$$\begin{aligned}
A(x) &= \sum_{n=0}^{\infty} a_n x^n \\
&= x^2 \sum_{n=0}^{\infty} a_{n+2} x^2 \\
&= x^2 \sum_{n=0}^{\infty} (a_{n+1} - a_n + 1) x^n \\
&= x^2 \sum_{n=0}^{\infty} a_{n+1} x^n - 2x^2 \sum_{n=0}^{\infty} a_n x^n + x^2 \sum_{n=0}^{\infty} x^n \\
&= xA(x) - x^2 A(x) + x^2/(1-x) \\
&= x^2/(1-x) + (x - 2x^2)A(x).
\end{aligned}$$

Rearranging, we get that

$$A(x)(1 - x + x^2) = x^2/(1-x),$$

and so

$$A(x) = \frac{x^2}{1 - 2x + 2x^2 - x^3}.$$

(b) Let a_n be the number of n-letter strings using the 26 letters of the English alphabet so that each letter appears an even number of times. Find the exponential generating function for this sequence. Answer: $((e^x + e^{-x})/2)^{26}$.

We note that a_n can be thought of as the number of partitions of $[n]$ into 26 labelled subsets of even size, each subset representing the locations of a given letter. We define an E-structure on a set to be that set if the set has even cardinality. We note that this is the number of ways of splitting $[n]$ into 26 parts and putting an E-structure on each part. Thus, the exponential generating function for a_n is the 26^{th} power of the exponential generating function for the number of E-structures. This is

$$\left(\sum_{n \text{ even}} \frac{x^n}{n!} \right)^{26} = \left(\frac{e^x + e^{-x}}{2} \right)^{26} = \cosh(x)^{26}.$$

Question 6 (Set Coloring, 20 points). *Let S be a finite set. Let $S_1, S_2, \ldots, S_m$ be subsets of S. Let $p(n)$ be the number of ways that the points of S can be colored with n colors so that for no S_i are all the elements of S_i colored with the same color. Show that $p(n)$ is a polynomial in n.*

Let $A(n)$ be the set of all colorings of S with n colors. Let $A_i(n)$ be the set of all such colorings so that all of the elements of S_i are the same color. We have that

$$p(n) = |A(n)| - |A_1(n) \cup A_2(n) \cup \ldots \cup A_m(n)|.$$

By inclusion exclusion, this is

$$|A(n)| + \sum_{k=1}^{m} (-1)^k \sum_{1 \leq i_1 < \ldots < i_k \leq m} |A_{i_1}(n) \cap \ldots \cap A_{i_k}(n)|.$$

Now, $|A(n)| = n^|S|$ is a polynomial in n. We note that $A_{i_1}(n) \cap \ldots \cap A_{i_k}(n)$ are the set of colorings of S so that a number of pairs of elements are all required to be the same color. We claim that the size of this set is a power of n. In particular, we note that the first element can be colored any color we want. This might then force some other elements to be that same color, and those might force further elements. Once all such elements have been painted, we pick another uncolored element and can pick one of n colors. This might force some to be the same color as it, so we repeat the coloring of them and so on. We note though that there are some ℓ elements of S that can all be colored independently of each other any of the n colors and that these colorings will force the colors of all other elements of S. Therefore, $|A_{i_1}(n) \cap \ldots \cap A_{i_k}(n)| = n^k$. Therefore, $p(n)$ is a sum of a number of powers of n, and is thus a polynomial in n.