Math 184A Homework 3

Fall 2015

This homework is due Monday October 26th in discussion section. Remember to justify your work even if the problem does not explicitly say so. Writing your solutions in $\LaTeX$is recommend though not required.

Question 1 (Different Types of Compositions, 40 points)**.** .

(a) *How many compositions of n are there into k parts each of size at least 2? [20 points]*

(b) *How many compositions of n are there into k odd parts? [20 points]*

Solution. (a) Firstly we fill each part with one. There are k parts so that takes up k of the n balls that we have. What we are left with is a composition of $n-k$ into k parts. This is given by

$$\binom{n-k-1}{k-1}.$$

(b) Firstly observe that such a composition is possible only if either n and k are both even, or else if n and k are both odd. Supposing we are in one of those cases, let us remove one ball from each part. What we are now left with is a *weak* composition of $n-k$ into k parts in which each part has an even number (possibly zero) of balls. Note that it is a weak composition since some part may have initially had only one ball, so when we remove it that part will be empty.
Now we have a weak composition of $n-k$ into k parts with each part having an even number of balls. Divide the number of balls in each part by 2: we are left with a weak composition of $(n-k)/2$ into k parts. Conversely, given a weak composition of $(n-k)/2$ into k parts, by doubling the number of balls in each part and then adding one ball to each part we end up with a composition of n into k odd parts. It is evident that these processes are inverse to each other. Hence we obtain a bijection between compositions of n into k odd parts and weak compositions of $(n-k)/2$ into k parts. Therefore there are

$$\binom{(n-k)/2+k-1}{k-1}$$

such compositions if n and k have the same parity (either both odd or both even) and zero otherwise.

□

Question 2 (Number of Parts in a Set Partition, 20 points)**.** *Let P_n be the sum over all set partitions of the set $[n]$ of the number of parts in the partition. For example, since the set partitions of $[3]$ are $\{1,2,3\},\{1\}\{2,3\},\{1,2\}\{3\},\{1,3\}\{2\},\{1\}\{2\}\{3\}$, we have that $P_3 = 1+2+2+2+3 = 10$. Show that*

$$P_n = B(n+1) - B(n).$$

[Hint: Either write both sides in terms of Stirling numbers of the second kind, or find a way to relate set partitions of $[n+1]$ to set partitions of $[n]$ with a specified part.]

Solution. Let us first prove this by writing both sides in terms of Stirling numbers of the second kind. Given a partition of $[n]$ into k parts, the contribution to P_n is exactly k. There are $S(n,k)$ such partitions by definition, and hence we obtain the formula:

$$P_n = \sum_{k=1}^{n} kS(n,k).$$

On the other hand, we have the formula for the Bell numbers $B(n) = \sum_{k=1}^{n} S(n,k)$, whence

$$B(n+1) - B(n) = \left(\sum_{k=1}^{n+1} S(n+1,k)\right) - \left(\sum_{k=1}^{n} S(n,k)\right).$$

We then use the recurrence relation $S(n+1,k) = S(n,k-1) + kS(n,k)$ to obtain:

$$\begin{aligned}
B(n+1) - B(n) &= \left(\sum_{k=1}^{n+1} S(n+1,k)\right) - \left(\sum_{k=1}^{n} S(n,k)\right) \\
&= \left(\sum_{k=1}^{n+1} S(n,k) + kS(n,k)\right) - \left(\sum_{k=1}^{n} S(n,k)\right) \\
&= \sum_{k=1}^{n} kS(n,k) \\
&= P_n.
\end{aligned}$$

We now give a second method of proof. What we wish to show is $P_n + B(n) = B(n+1)$. We proceed in a manner similar to the proof of the recurrence relation for the Stirling numbers $S(n,k)$: Given a partition of $[n]$, we want to obtain a partition of $[n+1]$, and we focus on where we are going to place $n+1$. There are two choices: Either $n+1$ goes into a subset by itself or else we place it into one of the existing subsets. For each partition of $[n]$, there is exactly one way to obtain a partition of $[n+1]$ by placing $n+1$ into a singleton subset, hence we obtain $B(n)$ such partitions of $[n+1]$ in which the element $n+1$ is by itself. Otherwise we place $n+1$ into one of the existing subsets in the partition of $[n]$. The number of choices for this is equal to the number of parts of the partition of $[n]$. Summing over all possible partitions of $[n]$ we obtain P_n distinct partitions of $[n+1]$. Every possible partition of $[n+1]$ is obtained in this manner, so

$$B(n+1) = P_n + B(n).$$

□

Question 3 (Partitions into Distinct Parts, 20 points)**.** *Show that the number of integer partitions of n into k distinct parts is equal to $p_k(n - k(k-1)/2)$. [Hint: find numbers that you can add to the parts of an arbitrary partition into k parts to ensure that they are distinct.]*

Solution. Consider the Ferrer's diagram of an integer partition of n into k distinct parts. We claim that there is always a "staircase" in such a partition, meaning the following: As the partition is into distinct parts, we must have

$$\#\text{ of boxes in row } k-1 > \#\text{ of boxes in row } k,$$

and hence by removing one box from row $k-1$ we still have a Ferrer's diagram, meaning the number of boxes in rows is still non-increasing as we go down.

Having removed one box from row $k-1$, we may now remove *two* boxes from row $k-2$ and still have a Ferrer's diagram, by similar reasoning. In general, we may remove j boxes from row $k-j$, and this is how we obtain the "staircase" inside our Ferrer's diagram (draw a picture). Thus by removing the staircase, we obtain a Ferrer's diagram with k rows. The number of boxes we have removed is equal to $1 + 2 + \cdots + (k-1) = k(k-1)/2$. This procedure gives a map between partitions of n into k distinct parts and $n - k(k-1)/2$ into k parts, and by inserting a staircase into the latter partition we see that this map has an inverse, and so is a bijection. Therefore the number of partitions of n into k distinct parts is equal to $p_k(n - k(k-1)/2)$.

□

Question 4 (Average Number of Fixed Points, 20 points)**.** *What is the average number of fixed points that a permutation of $[n]$ has? [Hint: count the number of pairs (π, x) of a permutation $\pi : [n] \to [n]$ and an $x \in [n]$ so that $\pi(x) = x$.]*

Solution. We claim that the average number of fixed points in a permutation of $[n]$ is equal to 1. As with any instance of taking an average, we must sum all the events we wish to measure, in this case fixed points, and divide by the total number of possible events, here the number of permutations. The number of permutations of $[n]$ is equal to $n!$. To compute the total number of fixed points we do the following: For each $x \in [n]$, we count how many permutations of $[n]$ fix x, and call this number $\pi(x, n)$. By summing over all $x \in [n]$, we obtain the total number of fixed points over all permutations of $[n]$. Thus our answer is:

$$\frac{\text{total number of fixed points}}{\text{total number of permutations}} = \frac{\sum_{x=1}^{n} \pi(x, n)}{n!}.$$

So it remains to compute $\pi(x, n)$. Note for any permutation that fixes x, we may think of it as first fixing x, and then permuting the remaining $n - 1$ elements of $[n]$. Hence there are $(n - 1)!$ permutations of $[n]$ which fix x. That is,

$$\pi(x, n) = (n - 1)!$$

for all $x \in [n]$. Then we obtain

$$\begin{aligned} \frac{\sum_{x=1}^{n} \pi(x, n)}{n!} &= \frac{\sum_{x=1}^{n} (n-1)!}{n!} \\ &= \frac{n \cdot (n-1)!}{n!} \\ &= \frac{n!}{n!} = 1. \end{aligned}$$

□

Question 5 (Extra credit, 1 point)**.** *Approximately how much time did you spend working on this homework?*