# Solutions to HW8 of Math 103A, Fall 2018

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**(1)** Let $G$ be a group and let $K \leq H \leq G$. Assume $(G : K)$ is finite. Prove that $(G : H)$ and $(H : K)$ are finite.

*Proof.* This is practice midterm II (6). $\square$

**(2)** Let $G$ be a group and let $H, K$ be two subgroups of $G$. Assume $(G : K)$ is finite. Prove that $(H : H \cap K)$ is finite.

*Proof.* Suppose $\{g_1K, \cdots, g_rK\}$ is the set of left cosets of $K$. Then

$$G = \bigcup_{i=1}^{r} g_iK, \quad g_iK \cap g_jK = \emptyset \text{ if } i \neq j.$$

Let

$$I = \{i | 1 \leq i \leq r,\ H \cap g_iK \neq \emptyset\}.$$

Clearly $I$ is finite. For every $i \in I$, there is $h_i \in H \cap g_iK$. So $h_i = g_ik_i$ for some $k_i \in K$.
For every $h \in H, h \in g_iK$ for some $i$. Then $i \in I$ and there is $k \in K$ such that $h = g_ik$. It follows that

$$h_i^{-1}h = k_i^{-1}g_i^{-1}g_ik = k_i^{-1}k \in K.$$

Since $h_i^{-1}h \in H$ as well, we have $h_i^{-1}h \in K \cap H$ and in particular, $h \in h_i(K \cap H)$. Thus

$$H \subseteq \bigcup_{i \in I} h_i(H \cap K).$$

Since $H \cap K \leq H$ and $h_i \in H$, clearly $\cup_{i \in I} h_i(H \cap K) \subseteq H$. So

$$H = \bigcup_{i \in I} h_i(H \cap K),$$

and $(H : H \cap K) \leq |I| \leq (G : H) < \infty$. $\square$

**(3)** Let $G$ be a group and let $H, K$ be two subgroups of $G$. Assume $(G : H), (G : K) < \infty$. Prove that $(G : H \cap K) < \infty$.

*Proof.* By (2) above, we know $(H : H \cap K) < \infty$. Now consider the chain of subgroups

$$H \cap K \leq H \leq G.$$

Then $(G : H \cap K) < \infty$ because $(G : H), (H : H \cap K) < \infty$. $\square$

**(4)** **(a)** Recall from HW7 (6-a), $N_G(H) \leq G$. Clearly $H \leq N_G(H) \leq G$. By (1), $(G : N_G(H))$ is finite. So there are $g_1, \cdots, g_n \in G$ so that $\{g_i N_G(H) | 1 \leq i \leq n\}$ is the set of left cosets of $N_G(H)$ in $G$. Then $g_i N_G(H) \neq g_j N_G(H)$ if $i \neq j$. By HW7 (6-c), this implies $g_i H g_i^{-1} \neq g_j H g_j^{-1}$ whenever $i \neq j$.

**(b)** For every $g \in G$, it must belong to some left coset $g_i N_G(H)$. Then $g N_G(H) = g_i N_G(H)$. By HW7 (6-c), $gHg^{-1} = g_i H g_i^{-1}$.

**(5)** (P.110 # 1)

| element | (0,0) | (0,1) | (0,2) | (0,3) | (1,0) | (1,1) | (1,2) | (1,3) |
|---|---|---|---|---|---|---|---|---|
| order | 1 | 4 | 2 | 4 | 2 | 4 | 2 | 4 |

Since $\mathbb{Z}_n$ is abelian for every integer $n$, the product $\mathbb{Z}_2 \times \mathbb{Z}_4$ is abelian.

**(6)** (P.110 #7) The orders of $3, 6, 12, 16$ in $\mathbb{Z}_4, \mathbb{Z}_{12}, \mathbb{Z}_{20}, \mathbb{Z}_{24}$ are $4, 2, 5, 3$, respectively. So the order of $(3, 6, 12, 16) = \text{l.c.m.}(4, 2, 5, 3) = 60$.

**(7)** (P.110 #13) The prime decomposition of 60 is $60 = 2^2 \times 3 \times 5$. So

$$\mathbb{Z}_{60} \cong \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \cong \mathbb{Z}_{12} \times \mathbb{Z}_5 \cong \mathbb{Z}_4 \times Z_{15} \cong \mathbb{Z}_{20} \times \mathbb{Z}_3.$$

**(8)** (P.111 #26) $24 = 2^3 \times 3, 25 = 5^2$. By the fundamental theorem of finitely generated abelian groups, the abelian groups of order 24 are

$$\mathbb{Z}_{24} \cong \mathbb{Z}_8 \times \mathbb{Z}_3, \quad \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_3, \quad \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3.$$

The abelian groups of order 25 are

$$\mathbb{Z}_{25}, \quad \mathbb{Z}_5 \times \mathbb{Z}_5.$$

Since $\gcd(24, 25) = 1$, every abelian group of order $(24)(25)$ is isomorphic to a product of a group of order 24 and a group of order 25, there are 6 abelian groups of order $(24)(25)$.

**(9)** (P.110 #29)

**(a)** It suffices to consider a combinatorial problem: what is the number $p_n$ of integer solutions to

$$\alpha_1 + \cdots + \alpha_n = n$$

with $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n \geq 0$. Because for every such solutions $(\alpha_1, \cdots, \alpha_n)$ ($\alpha_i$ might be 0 ), we get an abelian group

$$\mathbb{Z}_{p^{\alpha_1}} \times \cdots \mathbb{Z}_{p^{\alpha_n}},$$

and vise versa. So the table is

| n | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|
| number of abelian groups of order $p^n$ | 2 | 3 | 5 | 7 | 11 | 15 | 22 |

In general, there is no simple formula for $p_n$. But if you learned about combinatorics, it is not hard to find the generating function of $\{p_n\}$, which is

$$\sum_{n=0}^{\infty} p_n x^n = (1 + x + x^2 + x^3 + \cdots)(1 + x^2 + x^4 + \cdots)(1 + x^3 + x^9 + \cdots) \cdots .$$

**(10)** (P.113 #47) Let $G$ be an abelian group. Let $H$ be the subset of $G$ consisting of the identity $e$ and all elements of order 2. Show $H \leq G$.

*Proof.* By definition $e \in H$. If $x \in H, x \neq e$, then by definition, $x^2 = e$. So $x^{-1} = x \in H$. For every $x, y \in H$, as $G$ is abelina,

$$(xy)(xy) = x^2 y^2 = e.$$

So $H$ is closed. Thus $H$ is a subgroup.

□

**(11)** (P.113 #50) Since $G$ is defined to be $G = H \times K$, every element of $G$ is of form $(h, k)$ for some $h \in H, k \in K$. Let $e_H, e_K$ be identities of $H, K$, respectively. Then

$$(h, k) = (h, e_K)(e_H, k),$$

where $(h, e_K) \in H \times \{e_K\} \cong H, (e_H, k) \in \{e_H\} \times K \cong K$.
For any $h \in H, k \in K$,

$$(h, e_K)(e_H, k) = (h, k) = (e_H, k)(h, e_K).$$

And clearly

$$H \times \{e_K\} \cap \{e_H\} \times K = \{(e_H, e_K) = e_G\}.$$

**(12)** (P.113 #52) Suppose $G$ is a finite abelian group. By the fundamental theorem of finitely generated abelian groups, $G \cong \mathbb{Z}_{p_1^{r_1}} \times \cdots \times \mathbb{Z}_{p_m^{r_m}}$ for some primes $p_1, \cdots, p_m$, not necessarily distinct, and $r_i$ are positive integers.

($\Rightarrow$) Suppose $G$ is not cyclic. By Corallary 11.6, since $G$ is not cyclic, there are $p_i^{r_i}$ and $p_j^{r_j}$ with $i \neq j$ that are not coprime. But $p_k$'s are prime. We must have $p_i = p_j = p$. So $\mathbb{Z}_{p^{r_i}} \times \mathbb{Z}_{p^{r_j}}$ can be viewed a subgroup of $G$. Let $H_1 = \langle p^{r_i - 1} \rangle, H_2 = \langle p^{r_j - 1} \rangle$ be cyclic subgroups of $\mathbb{Z}_{p^{r_i}}, \mathbb{Z}_{p^{r_j}}$, respectively. Clearly $H_1 \cong H_2 \cong \mathbb{Z}_p$ and $H_1 \times H_2 \cong \mathbb{Z}_p \times \mathbb{Z}_p$ can be viewed as a subgroup of $G$.

($\Leftarrow$) If $G$ is cyclic, then any subgroup of $G$ should be cyclic as well. But $G$ contains a subgroup that is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$, which is not cyclic by Corallary 11.6.

**(13)** (P.113 #53) Let $G$ be an abelian group of order $p^k$, where $p$ is prime and $k$ is a positive integer. For any $g \in G, \langle g \rangle$ is a subgroup of $G$. By Lagrange's theorem, the order of $g$, which equals $|\langle g \rangle|$, divides $p^k$. Thus the order of $g$ is also a power of $p$.

Yes, the hypothesis of commutativity can be dropped, since in the proof above we never used the commutativity.