Question 1 (TV Scheduling, 30 points). *John's favorite TV show aired* 60 *episodes during its first calendar year on the air. Show that at least two of these episodes must have first aired within a week of each other.*

Split the year up into 1-week periods beginning on January 1st. This divides the year into 52-weeks plus one or two left over days at the end. This divides the year into 53 periods each at most a week long. Since $60 > 53$, the pigeonhole principle tells us that at least two of the episodes must have first aired in the same period. Since any two times in a single period are at most a week apart from each other, these two episodes must have aired within a week of one another.

Question 2 (Stirling Recurrence, 35 points). *Show that the following relation holds for all $n \geq k \geq 1$:*

$$S(n,k) = \sum_{m=0}^{n-k} \binom{n-1}{m} S(n-1-m, k-1).$$

[Hint: Count the number of partitions of $[n]$ into k parts where the element n ends up in a set of size $m+1$.]

Note that $S(n,k)$ is the number of partitions of the set $[n]$ into k parts. Each such partition must put the element n into a part of size $m+1$ for $n-k \geq m \geq 0$ (the upper bound holds because there must be at least $k-1$ elements not in this part in order to fill up the remaining parts). Hence by the addition rule $S(n,k)$ is a sum over m of the number of partitions in which n is in a part of size $m+1$. We can enumerate such partitions in the following way. First we note that there are $\binom{n-1}{m}$ ways to select the other m elements to go in the same part as n. Once those are chosen, there are $S(n-1-m, k-1)$ ways to partition the remaining elements of $[n]$ into $k-1$ parts. Thus, the number of partitions for given m is the product $\binom{n-1}{m} S(n-1-m, k-1)$. Thus, taking the sum over m we get

$$S(n,k) = \sum_{m=0}^{n-k} \binom{n-1}{m} S(n-1-m, k-1).$$

Question 3 (Fibonacci Formula, 35 points)**.** *Define the Fibonacci numbers by the recurrence relation*

$$F_0 = 1,\ F_1 = 1,\ F_n = F_{n-1} + F_{n-2} \text{ for all } n \geq 2.$$

Show that

$$F_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k}$$

for all integers $n \geq 0$. (Recall that $\lfloor x \rfloor$ is the largest integer less than or equal to x).

We proceed by strong induction on n.

Base Case: $n = 0$ or $n = 1$.

For $n = 0$, we have that

$$F_0 = 1 = \binom{0}{0}.$$

As desired.

For $n = 1$, we have that

$$F_1 = 1 = \binom{1}{0}.$$

As desired.

Inductive Step: Assume that

$$F_m = \sum_{k=0}^{\lfloor m/2 \rfloor} \binom{m-k}{k}$$

for all $m < n$. Note that the sum over k can be extended over all integers, because all of the terms not listed would be 0. We have that

$$F_n = F_{n-1} + F_{n-2} = \sum_k \binom{n-1-k}{k} + \sum_k \binom{n-2-k}{k}.$$

Where the second equality is by the inductive hypothesis. Decreasing the index k by 1 in the second sum above we find that

$$F_n = \sum_k \binom{n-1-k}{k} + \binom{n-1-k}{k-1} = \sum_k \binom{n-k}{k}.$$

Which is what we wanted to show. This completes the inductive step and proves our result.