Math 184A Homework 6

Spring 2018

Question 1 (Limiting Exponent, 50 points). .

(a) *Show that for any permutation q and any positive integers n and m that*

$$S_n(q)S_m(q) \leq S_{n+m}(q).$$

[25 points]

(b) *Use this to prove that for any permutation q that*

$$L(q) = \lim_{n\to\infty} \sqrt[n]{S_n(q)}$$

exists and is finite. [You may use the result mentioned in class that for each q there is a constant C_q so that $S_n(q) \leq C_q^n$] [25 points]

Solution. *s*

(a) *Left hand side is the number of pair (p_n, p_m) such that p_n is the permutation of $[n]$ avoiding q and p_m is the permutation of $[m]$ avoiding q. Right hand side is the number of permutation of $[n+m]$ that avoid q.The idea is to construct an injective map from left hand side to right hand side.*
Suppose the permutation $q = a_1a_2 \dots a_n$.There is two cases: $a_1 > a_n$ or $a_1 < a_n$.
If $a_1 < a_n$, for any pair $(p_n, p_m), p_n = b_1b_2 \dots b_n, p_m = c_1c_2 \dots c_m$, we map it to

$$(b_1+m)(b_2+m)\dots(b_n+m)c_1c_2\dots c_m$$

the first n entries are always larger then the last m entries, which implies that it avoid q.
If $a_1 > a_n$, similar idea, we maps the pair to

$$b_1b_2\dots b_n(c_1+n)(c_2+n)\dots(c_m+n)$$

(b) *Let $f_n = logS_n(q)$, then we have $f_n \geq 0$, $f_n + f_m \leq f_{n+m}$, $\frac{f_n}{n} \leq C_q$. We want to prove that $\lim_{n\to\infty} \frac{f_n}{n}$ exists. Since $\frac{f_n}{n}$ is bounded above, the $\sup_{n\geq 1} \frac{f_n}{n}$ is finite. We can assume $\sup_{n\geq 1} \frac{f_n}{n} = a$. So now it remains to show that $\liminf_{n\to\infty} \frac{f_n}{n} \geq a$.*
For $\forall\epsilon > 0$, there exist a m such that $\frac{f_m}{m} \geq a - \epsilon$. Hence for n large enough, $n = km + r, 0 \leq r \leq m-1$,

$$\frac{f_n}{n} \geq \frac{km\frac{f_m}{m} + f_r}{km + r} \geq \frac{km}{km+m}(a-\epsilon) = \frac{k}{k+1}(a-\epsilon) \to \ a - \epsilon \ as \ n \to \infty$$

So $\liminf_{n\to\infty} \frac{f_n}{n} \geq a - \epsilon$ which implies that $\liminf_{n\to\infty} \frac{f_n}{n} \geq a$

Question 2 (Avoiding 132 and 4321, 50 points). *Let $S_n(132, 4321)$ be the number of permutations of $[n]$ that avoid both 132 and 4321. Show that*

$$S_n(132, 4321) = 2\binom{n}{4} + \binom{n+1}{3} + 1.$$

Solution. *We use induction here.*
Base case: when $n = 1, 2, 3, 4$, we can check that the equation holds.
Inductive step: when $n = 5$, we consider the position of n.
If its position is at ith place, where $1 \leq i \leq n-1$, then the right hand side is nonempty. Because it avoid 132, entries at left hand side are always larger than the entries at right hand side. Furthermore, at least 1 of them avoid pattern 21, that is, at least one of them is increasing. Otherwise, there will be 4 entries with pattern 4321. So here we have 2 cases: left hand side increasing or right hand side increasing. When the left hand side is increasing, right hand side avoid pattern 132 and 321. If there is 3 entries with pattern 321, n plus these 3 entries will have pattern 4321. Similarly, if the right hand side is increasing, then left hand side should avoid 132 and 321. If there is 3 entries with pattern 321, these 3 entries plus an entry at right hand side(which is nonempty) will have pattern 4321.
If n's position is at nth place, then the left hand side should avoid 132 and 4321.
Therefore we have the following recursive formula.

$$f_n(132, 4321) = \sum_{i=1}^{n-1} (f_{i-1}(132, 321) + f_{n-i}(132, 321) - 1) + f_{n-1}(132, 4321)$$

Now we want to compute $f_n(132, 321)$. Similar idea as above, consider the position of n.
If its positon is at ith place, where $1 \leq i \leq n-1$, then both side have to be increasing.
If its position is at nth place,then the left hand side avoid 132 and 321.
This give us a recursive formula

$$f_n(132, 321) = n - 1 + f_{n-1}(132, 321)$$

which give us

$$f_n(132, 321) = \binom{n}{2} + 1$$

Put this in the previous equation, and then use the inductive assumption, we have

$$f_n(132, 4321) = \sum_{i=1}^{n-1} (\binom{i-1}{2} + \binom{n-i}{2} + 1) + 2\binom{n-1}{4} + \binom{n}{3} + 1 = 2\binom{n}{4} + \binom{n+1}{3} + 1$$

So the equation holds for n.

Question 3 (Extra credit, 1 point)**.** *Approximately how much time did you spend on this homework?*