Question 1 (Recurrence, 15 points). *Let a_n be given by the recurrence relation*

$$a_n = \begin{cases} 1 & \textit{if } n = 1 \\ 5 & \textit{if } n = 2 \\ a_{n-1} + 2a_{n-2} & \textit{for } n > 2. \end{cases}$$

Show that $a_n = 2^n + (-1)^n$ for all positive integers n.

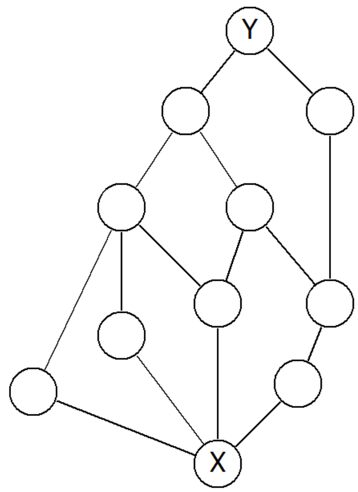
We prove that $a_n = 2^n + (-1)^n$ by strong induction on n. If $n = 1$, we have $a_n = 1 = 2^1 + (-1)^1$. If $n = 2$, we have $a_n = 5 = 2^2 + (-1)^2$.

For $n > 2$ assume that $a_m = 2^m + (-1)^m$ holds for all $1 \leq m < n$. Then

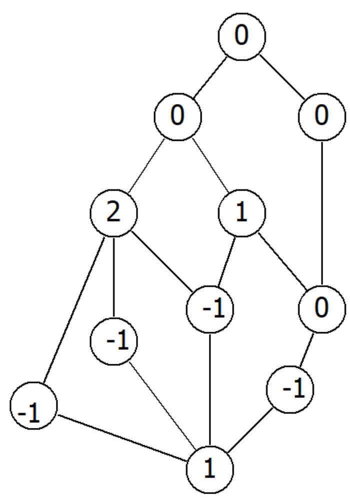
$$a_n = a_{n-1} + 2a_{n-2} = (2^{n-1} + (-1)^{n-1}) + 2(2^{n-2} + (-1)^{n-2}) = 2^{n-2}(2+2) + (-1)^{n-2}(-1+2) = 2^n + (-1)^n.$$

This completes our inductive step and proves our result.

Question 2 (Mobius Computation, 15 points)**.** *Bellow is the Hasse diagram for a partial order. Compute* $\mu(X, Y)$*.*



Computing $\mu(X, A)$ for various A using $\mu(X, X) = 1$ and $\mu(X, A) = -\sum_{X \le Z < A} \mu(X, Z)$, we get that the values of $\mu(X, A)$ are as given below:



Thus, $\mu(X, Y) = 0$.

Question 3 (Tile Coloring, 15 points). *Jean has a sequence of n tiles lined up in a row. He would like to paint them all red, blue or green with the added requirement that all of the green colored tiles should be to the left of all blue colored tiles. In how many different ways can Jean accomplish this? For full credit, you should give an explicit formula.*

We split into two cases. If there is a green tile, the rightmost green tile can be in any of n positions. The tiles to the left can then be colored green or red and the tiles to the right can be colored blue or red, for a total of 2^{n-1} options for each choice of the rightmost green tile. If there are no green tiles, any of the n tiles can be colored red or blue for a total of 2^n options. Thus, the final answer is $(n+2)2^{n-1}$.

Question 4 (Elements in the Same Cycle and not, 15 points)**.** *Let $n > k > 0$ be integers. Show that the number of permutations of $[n]$ with exactly k cycles and with 1 and 2 in the same cycle is the same as the number of permutations of $[n]$ into $k+1$ cycles and 1 and 2 in different cycles.*

We exhibit a bijection between these sets. Given a permutation with 1 and 2 in the same cycle, let this cycle be $(1a_1a_2\ldots a_n2b_1b_2\ldots b_m)$. We produce a permutation with $k+1$ cycles and 1 and 2 in different parts by replacing this with the pair of cycles $(1a_1a_2\ldots a_n)$and $(2b_1b_2\ldots b_m)$, and leaving the other cycles the same. It is easy to see that this is invertible by merging together the cycles containing 1 and 2.

Question 5 (Generating Functions, 20 points)**.** *Let $a(n)$ be the number of triples of integers n_1, n_2, n_3 with $n = n_1 + n_2 + n_3$, $n_1 \geq n_2 \geq 0$, and n_3 positive and odd. Give an explicit formula for the generating function $\sum_{n=0}^{\infty} a(n)x^n$.*

Letting $n_1 = n_2 + m$, we note that this is the same as the number of triples m, n_2, n_3 non-negative integers with $n = (n_2 + m) + n_2 + n_3 = m + 2n_2 + n_3$ and with n_3 odd. Equivalently, this is the number of ways of writing n as the sum of a non-negative integer, an even, non-negative integer and a positive odd integer. The generating function is thus given by the product of the generating functions for these three terms, which is

$$\left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^2}\right)\left(\frac{x}{1-x^2}\right) = \frac{x}{(1-x)(1-x^2)^2}.$$

Question 6 (Separated Partitions, 20 points)**.** *Show that the number of set partitions of* $[n]$ *with* $1, 2,$ *and* 3 *in distinct parts is* $B(n) - 3B(n-1) + 2B(n-2)$.

Let A be the set of set partitions of $[n]$ with 1 and 2 in the same part, B the set with 2 and 3 in the same part, and C the set with 1 and 3 in the same part. We want the number of set partitions of $[n]$ not in $A \cup B \cup C$. This is $B(n) - |A \cup B \cup C|$. By Inclusion-Exclusion, this is $B(n) - |A| - |B| - |C| + |A \cap B| + |A \cap C| + |B \cap C| - |A \cap B \cap C|$. Now A is the collection of set partitions of $[n]$ with 1 and 2 in the same part. We claim that this is in bijection with the set partitions of $\{1, 3, 4, 5, \ldots, n\}$ simply by adding 2 to the part containing 1. Thus, $|A| = B(n-1)$. Similarly, $|B| = |C| = B(n-1)$. Now it is not hard to see that $A \cap B = A \cap C = B \cap C = A \cap B \cap C$ equals the set of partitions of $[n]$ with $1, 2$ and 3 all in the same part. By logic similar to the above there are $B(n-2)$ of these. Thus, the final count is $B(n) - 3B(n-1) + 2B(n-2)$.