Solutions to HW1 of Math 103A, Fall 2018

P.8 Q11

$$(a,1),(a,2),(a,c),(b,1),(b,2),(b,c),(c,1),(c,2),(c,c).$$

Q12

(a) defines a function which is neither one-to-one nor onto.

(b) defines a function which is neither one-to-one nor onto.

(c) does not define a function.

(d) defines a function that is ono-to-one and onto.

(e) defines a function that is neither one-to-one nor onto.

(f) does not define a function.

Q14

(a) Define $f_1 : [0,1] \to [0,2], f_1(x) = 2x$.

(b) Define $f_2 : [1,3] \to [5,25], f_2(x) = 10x - 5$.

(c) Define $f_3 : [a,b] \to [c,d]$,

$$f_3(x) = c + \frac{d-c}{b-a}(x-a).$$

Of course there are some other choices for the bijections. Here we only used affine functions.

Q16

(a) $\mathscr{P}(\emptyset) = \{\emptyset\}$ and $|\mathscr{P}(\emptyset)| = 1$.

(b) $\mathscr{P}(\{a\}) = \{\emptyset, \{a\}\},\ |\mathscr{P}(\{a\})| = 2$.

(c) $\mathscr{P}(\{a,b\}) = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}, |\mathscr{P}(\{a,b\})| = 4$.

(d) $\mathscr{P}(\{a,b,c\}) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}\}, |\mathscr{P}(\{a,b,c\})| = 8$.

Q17 We conjecture that

$$|\mathscr{P}(A)| = 2^s.$$

For every subset of A, an element of A can either be in this subset or not. So the cardinality is 2^s. To be more explicit, suppose $A = \{a_1, \cdots, a_s\}$ and we can construct a map f from $\mathscr{P}(A)$ to $\{0,1\}^s$ which means the s products of $\{0,1\}$ by letting the i'th component of $f(S)$ be 1 if $a_i \in S$ and be 0 otherwise. One can easily see that f is a bijection. Thus

$$|\mathscr{P}(A)| = |\{0,1\}^s| = 2^s.$$

Q19

Suppose we have an injective $f : A \to \mathscr{P}(A)$. We can show that f cannot be onto. Define

$$S = \bigcup_{x \in A, x \notin f(x)} \{x\}.$$

If there is $y \in A$ such that $f(y) = S$, then either $y \in S$ or $y \notin S$. If $y \notin S$, then by the definition of S, $y \notin S = f(y)$ and thus $y \in S$, which is a contradiction. But if $y \in S$, then y is one of the elements that lie outside their image which means $y \notin f(y) = S$, which is a contradiction. Therefore, S is not in the range of f and f cannot be onto. Thus the cardinality of $\mathscr{P}(A)$ is always strictly greater than that of A.

If we do have a set S of everything, then $\mathscr{P}(S)$ is of greater cardinality and this cannot happen. Note that the argument above is just a variation of the standard proof of Russell's paradox.

Q30 $\mathscr{R}$ is not an equivalence relation for it is not symmetric. For example, if we put $x = 3, y = 1$, then $x\mathscr{R}y$ but $y < x$ so we do not have $y\mathscr{R}x$.

Q31 $\mathscr{R}$ is an equivalence relation.
$x\mathscr{R}x$ as $|x| = |x|$.
If $x\mathscr{R}y$, then $|x| = |y|, |y| = |x|$ and thus $y\mathscr{R}x$.
If $x\mathscr{R}y, y\mathscr{R}z$, then $|x| = |y|, |y| = |z|$, then $|x| = |z|$ and $x\mathscr{R}z$.

Q32 $\mathscr{R}$ is not an equivalence relation for it is not transitive. Consider $x = 0, y = 3, z = 6$. $|x - y| \leq 3, |y - z| \leq 3$ and hence $x\mathscr{R}y, y\mathscr{R}z$. But $|x - z| = 6 > 3$ and we do not have $x\mathscr{R}z$.