MATH 184A

HW3 Solutions

1.a. By relabeling we can assume that $a_1 = n$. We then consider a permutation in which $a_1, \dots, a_k$ are in the same cycle. When we put this permutation into canonical cycle notation, the cycle containing $a_1, \dots, a_k$ will start with n, and will be the last cycle since n is the largest value in $[n]$. Thus when we apply the bijection from canonical cycle notation to one-line notation, we will get a permutations in which $a_2, \dots, a_k$ appear after n. Furthermore the preimage of any permutation in which $a_2, \dots, a_k$ appear after n in the one-line notation will be a permutation in which $a_1, \dots, a_k$ all appear in a cycle together. Therefore we can answer this problem by counting the number of permutations in one-line notation where $a_2, \dots, a_k$ appear after n. To count this we first order $a_1, a_2, \dots a_k$ in such a way that $a_1 = n$ appears first, then we order the remaining $n-k$ elements of $[n]$, and finally we interlace these two orderings. There are $(k-1)!$ ways to order $a_1, a_2, \dots a_k$ with n first. There are $(n-k)!$ ways to order the other $n-k$ elements. Finally there are $\binom{n}{k}$ ways to interlace these orderings because we have to choose k positions out of n for $a_1, \dots a_k$, and the remaining positions will be taken by the other $n-k$ elements. Then since these choices are independent, by the generalized product rule the total number is

$$(k-1)!(n-k)!\binom{n}{k},$$

which simplifies down to

$$\frac{n!}{k}$$

1.b. Similarly to 1.a. we can assume that $i = n$ and $j = n-1$. When we put such a permutation into canonical cycle notation, the last cycle will start with n, and the second to last cycle will start with $n-1$. Therefore when we apply the bijection to one-line notation, n will be in position $n-a+1$, and $n-1$ will be in position $n-a-b+1$. Furthermore if n and $n-1$ are in these positions, then their preimage will have n and $n-1$ in distinct cycles of length a and b respectively. The number of such permutations will be $(n-2)!$ since we know the position of n and $n-1$, and there are $(n-2)!$ ways to choose the position of $1, \dots, n-2$.

2.a. From the formula in the book we know that

$$\frac{n!}{a_1 a_2 \dots a_k b_1! b_2! \dots b_n!}$$

is the number of permutations of n with cycles of length $a_1, \dots, a_k$. On the other hand, any permutation of n with k-cycles will give a partition of n into k parts by taking the lengths of the cycles. Thus if we sum over all partitions of n into k parts we will count all of the permutations with k-cycles which is exactly $c(n,k)$.

2.b. We note that a partition of n into $n-10$ parts will have at most 10 parts of size greater than one, furthermore the sum of non-one parts is at most 20. Thus for $n \geq 20$, we can enumerate all partitions of n into $n-10$ parts by adding $n-20$ ones to partitions of 20 into 10 parts. Modifying the equation from 2.a. using this will give us

$$\sum \frac{n!}{a_1 a_2 \dots a_k (b_1 + n - 20)! b_2! \dots b_n!},$$

where the sum is over partitions of 20 into 10 parts. Rewriting this gives

$$\sum \frac{(n)(n-1)\dots(n-20+b_1+1)}{a_1a_2\dots a_kb_2!\dots b_n!}$$

The top is a polynomial in n of degree $20-b_1$, and the bottom is a constant, and the index of summation does not depend on n, so the sum is a polynomial of degree at most 20, and thus the result is shown.

3.a. Consider constructing a path $(v_0 = v, v_1, \dots, v_m)$ in G one step at a time. Since $\deg(v) \geq 3$, we have at least three choices for v_1. Then for $m-1 \geq i > 1$ there are at least two vertices x, y neighboring v_i that are different from v_{i-1}, since $\deg(v_i) \geq 3$. Furthermore $x, y \neq v_j$ for all $j < i$. To see this, assume by way of contradiction that there exists $j < i$ such that either $v_j = x$ or $v_j = y$, and pick j be maximal. Without loss of generality assume that $v_j = x$. Then $(v_j, v_{j+1}, \dots v_i, x)$ is a cycle of length $(i-j+1) \leq m$, which is a contradiction since we assume that all cycles are of length larger than m. Thus there are at least 2 options v_{i+1}. By this reasoning there at least $3 * 2^{m-1} > 2^m$ paths starting at v_0.

3.b. If G contains a cycle of length at most $\lceil \log_2(n) \rceil$, then we are done. If not, we can apply 3.a. with $m = \lceil \log_2(n) \rceil$ to conclude that for any vertex v there are at least $2^{\lceil \log_2(n) \rceil} \geq 2^{\log_2(n)} = n$ paths of length $\lceil \log_2(n) \rceil$ starting at v. Since these paths start at v, they cannot end at v, thus there are $n-1$ possible endpoints for these paths. Therefore by the Pigeon Hole Principle two of these path have the same endpoint. Call these paths $P = (v, p_1, p_2, \dots, p_m)$ and $Q = (v, q_1, \dots, q_m)$. We then consider the set S of vertices that appear in both P and Q other than v. The set S is non-empty since it contains the common endpoint $p_m = q_m$. Choose i minimal such that $p_i \in S$ and let j be such that $p_i = q_j$. Then $(v, p_1, \dots p_i = q_j, q_{j-1}, q_{j-2}, \dots q_1, v)$ is a cycle of length at most $2\lceil \log_2(n) \rceil$, by the minimality of i.