Math 184A Homework 4

Spring 2018

This homework is due on gradescope by Friday May 11th at 11:59pm. Remember to justify your work even if the problem does not explicitly say so. Writing your solutions in LaTeXis recommend though not required.

Question 1 (Permutation Parity, 20 points)**.** *Let $n > 1$ be an integer and let S be a set of pairs of numbers (i, j) with $i, j \in [n]$. Say that a permutation π of $[n]$ avoids S if $\pi(i) \neq j$ for all $(i, j) \in S$. So, for example, a derangement is a permutation that avoids $\{(1, 1), (2, 2), (3, 3), \ldots, (n, n)\}$. Suppose that for any $n - 1$ elements of S that either some two share a first coordinate or some two share a second coordinate. Prove that the number of permutations that avoid S is even. [Hint: Count the number using Inclusion-Exclusion.]*

Question 2 (Size of Central Binomial Coefficients, 20 points)**.** *Show that for any $n \geq 1$*

$$4^n \geq \binom{2n}{n} \geq 4^n/(2n + 1).$$

[Hint: for the lower bound show that $\binom{2n}{n} \geq \binom{2n}{k}$ for any k.] [Note: For those who know some number theory, it is not hard to see that $\binom{2n}{n}$ is divisible by the product of all primes $n \leq p \leq 2n$. This allows one to prove rough upper bounds on the number of primes.]

Question 3 (Sums of Binomial Coefficients, 30 points)**.** .

(a) *Give a formula for $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots + \binom{n}{2\lfloor n/2 \rfloor}$ as a function of n. [Hint: use the binomial theorem. You'll need a way to make the odd terms go away.][10 points]*

(b) *Give a formula for $\binom{n}{0} + \binom{n}{3} + \binom{n}{6} + \ldots + \binom{n}{3\lfloor n/3 \rfloor}$ as a function of n. [Hint: same idea, but you might need to use complex numbers.][20 points]*

Question 4 (Linear Homogeneous Recurrence Relations, 30 points)**.** *Suppose that a sequence A_n satisfies a linear homogenous recurrence relation with constant coefficients. Namely, suppose that there are constants $C_1, C_2, \ldots, C_k$ so that*

$$A_n = C_1 A_{n-1} + C_2 A_{n-2} + \cdots + C_k A_{n-k}$$

for all $n \geq k$.

(a) *Show that the generating function $F(x) = \sum_{n=0}^{\infty} A_n x^n$ is given by a rational function in x (namely a ratio of polynomials in x). [15 points]*

(b) *Given that partial fraction decompositions, allow you to write any rational function as a polynomial plus a linear combination of terms of the form $1/(1 - b_i x)^{a_i}$, show that there's a formula expressing A_n as some linear combination of terms of the form $n^{k_i} b_i^n$ for all sufficiently large n. [15 points]*

Question 5 (Extra credit, 1 point)**.** *Approximately how much time did you spend on this homework?*