Math 184A Homework 2 Solution

Question 1 (Colored Balls in Bins, 30 points). *Consider the problem of placing unlabelled red and blue colored balls into labelled bins. As the balls are unlabelled you cannot distinguish one from the other except by color. You can distinguish two arrangements based on the number of red and blue balls in each bin.*

(a) *Show that the number of ways to put n red and m blue balls into k bins is*

$$\binom{n+k-1}{k-1}\binom{m+k-1}{k-1}.$$

[10 points]

(b) *Suppose that you have a total of N balls that can be colored either color. Show that the number of distinct ways to color these balls and put them into k bins is*

$$\binom{N+2k-1}{2k-1}.$$

Hint: Think of breaking each bin into two sub-bins each storing the balls of only one color. [10 points]

(c) *Show that*

$$\binom{N+2k-1}{2k-1} = \sum_{n=0}^{N}\binom{n+k-1}{k-1}\binom{N-n+k-1}{k-1}.$$

[10 points]

Solution.

(a) The number of ways to put n red balls into k bins is

$$\binom{n+k-1}{k-1},$$

while the number of ways to put m blue balls into k bins is

$$\binom{m+k-1}{k-1}.$$

Since we do not consider the order of the balls in the bins, we can first put the red balls and then do the blue balls, and these events are independent. We can get the final answer by multiplying the two binomial coefficients together. The number of ways to put n red and m blue balls into k bins is

$$\binom{n+k-1}{k-1}\binom{m+k-1}{k-1}.$$

(b) By the hint, we break bin i into two sub-bins, i^r and i^b, letting i^r stores red balls in bin i and i^b stores blue balls in bin i. So if a ball is colored red and put into bin i, we say the ball is placed in bin i^r; if a ball is colored blue and put into bin i, we say the ball is placed in bin i^b. This is another explanation of choose color and box for each of the N balls. The number of ways to choose color and box for each of the N balls is equal to the number of ways to place the N balls into the $2k$ bins, thus the number is

$$\binom{N+2k-1}{2k-1}.$$

(c) We want to count the number of ways to choose color and box for each of the N balls twice to prove the two numbers are equal.

In part (b), we have showed that this number is equal to

$$\binom{N+2k-1}{2k-1}.$$

Here we count in another way. Suppose that there are n red balls in the N balls, then there are $N-n$ blue balls. By part (a), we know that the number of ways to put n red and $N-n$ blue balls into k bins is

$$\binom{n+k-1}{k-1}\binom{N-n+k-1}{k-1}.$$

Since the number of red balls n can be any integer in $[N]=\{1,\ldots,N\}$, the total number of ways to choose colors and bins for the N balls is

$$\sum_{n=0}^{N}\binom{n+k-1}{k-1}\binom{N-n+k-1}{k-1}.$$

Since the two method above is counting the same number, we have

$$\binom{N+2k-1}{2k-1}=\sum_{n=0}^{N}\binom{n+k-1}{k-1}\binom{N-n+k-1}{k-1}.$$

□

Question 2 (Bell Numbers and Restricted Growth Sequences, 20 points)**.** *Call a sequence $a_1, a_2, \ldots, a_n$ of positive integers a* restricted growth sequence *if $a_1 = 1$ and for each $i > 0$*

$$a_i \leq \max_{j<i}(a_j) + 1.$$

In particular the i^{th} term of the sequence is at most one more than the largest previous term. Show that $B(n)$ counts the number of restricted growth sequences of length n. Hint: Show that these sequences are in bijection to set partitions. In particular, think of a_i as the label for the part containing the element i.

Solution. In this question, we want to show that $B(n)$ counts the number of restricted growth sequences of length n. Since $B(n)$ counts the number of set partitions of set $[n]=\{1,\ldots,n\}$, we need to prove that the number of restricted growth sequences of length n is equal to the number of set partitions of set $[n]$.

By the hint, we need to construct a bijection from restricted growth sequences of length n to set partitions of set $[n]$. We think of a_i as the label for the part containing the element i to map the sequence into a set partition. Following is an example of this process.

Given a restricted growth sequence, $a_1, a_2, a_3, a_4, a_5, a_6, a_7 = 1, 2, 2, 1, 3, 2, 3$, of length 7, we take a_i as the label for the part containing the element i. Then, each element of set [7] has a label.

label	1	2	2	1	3	2	3
[7]	1	2	3	4	5	6	7

Then, the element in [7] labeled i will be put into the i^{th} part, and we can get a set partition: $\{1,4\},\{2,3,6\},\{5,7\}$, and the parts are organized in a minimum element increasing order of the parts. By the restricted growth requirement, we see than when we put the elements from 1 to n into different parts, we start with putting 1 into part 1. When $a_i = \max_{j<i}(a_j)+1$, we are opening a new part next to the biggest part, thus this map is well defined from the set of restricted growth sequences of length n to the set of set partitions of $[n]$ and there will not be any empty parts.

To show that this is a bijection, we need to prove that this is an injection and a surjection.

Injection: We want to show that two different sequences have different images. For two different sequences $a_1,\ldots,a_n$ and $b_1,\ldots,b_n$, they are different by at least one digit. Their images are the same if and only if each element i in $[n]$ are in the same part(since the parts are organized in a minimum element increasing order). Let the i^{th} location be the first in which $a_i \neq b_i$. If $a_i = a_j$ for some $j < i$ then in that partition, i and j are in the same part. Therefore, the two set partitions cannot be the same unless $b_i = b_j = a_j = a_i$. If on the other hand a_i is not equal to a_j for any $j < i$, the only possibility is that $a_i = \max_{j<i} a_j + 1$. This also means that in this partition, i is not in the same part as any $j < i$. If the two partitions are the same, then this must also hold for the other partition, and thus b_i cannot equal b_j for any $j < i$. For this to occur, we must have $b_i = \max_{j<i} b_j + 1 = \max_{j<i} a_j + 1 = a + i$. This proves injectivity.

Surjection: We need to show that each set partition of $[n]$ has a preimage in the set of restricted growth sequences. For any given set partition, we first organize the parts in a minimum element increasing order and label the parts with $1, 2, 3, \ldots$, then we can get the sequence by taking a_i as the label for the part containing the element i. For example, for a set partition $\{6\}, \{1, 3, 4\}, \{2, 5\}$, we first organize the parts in a minimum element increasing order: $\{1, 3, 4\}, \{2, 5\}, \{6\}$, then we label a_1, a_3, a_4 with 1, label a_1, a_5 with 2 and label a_6 with 3, and reconstruct the sequence $a_1, a_2, a_3, a_4, a_5, a_6 = 1, 2, 1, 1, 2, 3$.

We still need to show that the sequence is a restricted growth sequence. It is clear that this sequence begin with 1, since the number 1 must be placed in the 1^{st} part. For the second requirement, we consider a_i, the label of i. If i is in the first part of a set partition, then $a_i = 1 = a_1 \leq \max_{j<i}(a_j) + 1$. Otherwise, since the parts are organized in a minimum element increasing order, there is some $j < i$ in the previous part. The label $a_i = a_j + 1 \leq \max_{j<i}(a_j) + 1$. So in this way, we can always reconstruct a preimage of any set partition, thus this is a surjection.

Since this is a surjection and an injection, this map is a bijection. Thus we know that the number of restricted growth sequences of length n is equal to the number of set partitions of set $[n]$, which is the Bell number, $B(n)$. □

Question 3 (Stirling Number Formula, 20 points)**.** *Find a closed form formula for $S(n, n-2)$ for all $n \geq 2$. Hint: think about which parts do not consist of single elements.*

Solution. By definition, $S(n, n-2)$ is the number of set partitions of set $[n]$ into $n-2$ parts(no empty part allowed). There can be at least one part and at most 2 parts containing more than one element. Thus we have two cases.

1 There is only one part containing more than one element. This part should contain 3 elements and every other element is by itself. The number of such set partitions is equal to the number of ways to choose 3 elements from set $[n]$, which is $\binom{n}{3}$.

2 There are two parts containing more than one element. These two part should contain 2 elements and every other element is by itself. We first choose 2 elements from the n elements to form the first pair, then choose another 2 elements from the remaining $n-2$ elements, and since the two pairs don't have an order, we divide it by 2 to kill the redundancy. We have $\frac{\binom{n}{2}\binom{n-2}{2}}{2}$ such set partitions.

In summary, $S(n, n-2) = \binom{n}{3} + \frac{\binom{n}{2}\binom{n-2}{2}}{2}$. □

Question 4 (Compositions and Partitions, 30 points)**.** *Recall that the number of ways to place n unlabelled balls into k non-empty bins is given by $\binom{n-1}{k-1}$. The number of ways of putting n unlabelled balls into k non-empty unlabelled bins is $p_k(n)$. Show that*

$$\binom{n-1}{k-1} \geq p_k(n) \geq \binom{n-1}{k-1}/k!$$

Hint: Each partition gives rise to several possible compositions. Try to prove bounds on how many compositions per partition there are.

Solution. Let $a_k(n)$ be the number of compositions of n into k non-zero parts. We learned that $a_k(n) = \binom{n-1}{k-1}$. What we need to prove is that

$$a_k(n) \geq p_k(n) \geq a_k(n)/k!,$$

or

$$a_k(n) \geq p_k(n) \quad \text{and} \quad k!p_k(n) \geq a_k(n).$$

Since each partition itself is a composition, we know that the number of partitions of n into k parts is no more than the number of compositions of n into k parts, and the first inequality is clear.

For the second inequality, $k!p_k(n) \geq a_k(n)$, we apply the hint. Each composition has an underlying partition, which is just put the parts in a size decreasing order. For example, the composition 2321 has an underlying partition 3221. Thus each partition gives rise to several possible compositions. For a partition with k parts, it can give rise to at most $k!$ compositions, so we have at most $k!p_k(n)$ compositions of n into k non-zero parts, proves the second inequality that $k!p_k(n) \geq a_k(n)$. $\square$

Note: Using this we can show that

$$p(n) \geq p_k(n) \geq \binom{n-1}{k-1}/k! \geq (n-k)^{k-1}/k!(k-1)! \geq \left(\frac{n-k}{k^2}\right)^{k-1}/k.$$

Taking $k = \sqrt{n}/2$, it is not hard to show that $p(n) \geq 2^{\sqrt{n}/2}$, which is a pretty decent estimate as $p(n)$ actually grows like $C^{\sqrt{n}}$ for some constant C.