Question 1 (Permutation Notation, 30 points). *Consider the string $S = 425613$.*

(a) If S is interpreted as a permutation of $[6]$ *in the standard notation, how would you write this permutation in canonical cycle form? [15 points]*

When considered as a permutation π we have $\pi(1) = 4, \pi(2) = 2, \pi(3) = 5, \pi(4) = 6, \pi(5) = 1, \pi(6) = 3$. The cycles are therefore (14635) and (2). For canonical cycle notation we must sort these cycles as (63514) and (2) and write them in increasing order as (2)(63514).

(b) If S is interpreted as a permutation of $[6]$ *in canonical cycle form, how would you write this permutation in the standard notation? [15 points]*

When written in canonical cycle form, the last cycle must be everything after the 6, or (613). The next cycle will be (5), then (42). We now need to evaluate what the permutation does to various inputs. We find that $\pi(1) = 3, \pi(2) = 4, \pi(3) = 6, \pi(4) = 2, \pi(5) = 5, \pi(6) = 1$. So when written in standard notation, the permutation is 346251.

Question 2 (Binomial Identity, 35 points)**.** *Prove the following identity for all integers $n \geq k \geq 0$.*

$$2^k \binom{n}{k} = \sum_{i=0}^{k} \binom{n}{n-k, i, k-i}.$$

Recall here that $\binom{n}{n-k,i,k-i}$ is the multinomial coefficient.

We note that

$$\binom{n}{n-k, i, k-i} = \frac{n!}{(n-k)!i!(k-i)!} = \left(\frac{n!}{(n-k)!k!}\right)\left(\frac{k!}{i!(k-i)!}\right) = \binom{n}{k}\binom{k}{i}.$$

Therefore the right hand side of the above is

$$\sum_{i=0}^{k} \binom{n}{k}\binom{k}{i} = \binom{n}{k}\sum_{i=0}^{k}\binom{k}{i}1^i 1^{k-i} = \binom{n}{k}(1+1)^k = 2^k\binom{n}{k}.$$

Alternate Solution: We claim that both sides count the number of ways given n objects to paint k of them either blue or red. On the one hand there are $\binom{n}{k}$ ways to select the k objects to paint, and once this is done, 2^k ways to paint each of the k selected objects one of the two colors. Therefore, there are $2^k\binom{n}{k}$ ways to do this.

On the other hand, if i is the number of objects painted blue, and $k-i$ the number painted red, there are $\binom{n}{n-k,i,k-i}$ ways to choose i objects to paint blue and $k-i$ to paint red. Summing over i gives the total.

Question 3 (Unique Inclusion-Exclusion, 35 points)**.** *Let $A_1, A_2, \ldots, A_n$ be finite sets. Let S be the set of elements x that are in exactly one of the A_i. So for example if $A_1 = \{1,2,3\}$ and $A_2 = \{1,3,5\}$, then $S = \{2,5\}$, since $1,3$ are in more than one of the A's. Show that*

$$|S| = \sum_{k=1}^{n} (-1)^{k+1} k \sum_{1 \le i_1 < i_2 < \ldots < i_k \le n} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}|.$$

Let S_i be the set of elements in A_i but in none of the other A_j's. We note that $S = \bigcup_{i=1}^n S_i$. Furthermore, since the S_i are disjoint, we have that $|S| = \sum_{i=1}^n |S_i|$.

Now $S_i = A_i - (A_i \cap A_1) \cup (A_i \cap A_2) \cup \cdots \cup (A_i \cap A_n)$. Therefore we have that

$$|S_i| = |A_i| - |(A_i \cap A_1) \cup (A_i \cap A_2) \cup \cdots \cup (A_i \cap A_n)|.$$

By inclusion-exclusion this is

$$\begin{aligned}
|A_i| - \sum_{k=1}^{n-1} (-1)^{k+1} \sum_{\substack{1 \le i_1 < \cdots < i_k \\ i_j \ne i}} |(A_i \cap A_{i_1}) \cap \cdots \cap (A_i \cap A_{i_k})| &= |A_i| - \sum_{k=1}^{n-1} (-1)^{k+1} \sum_{\substack{1 \le i_1 < \cdots < i_k \\ i_j \ne i}} |A_{i_1} \cap \cdots \cap A_{i_k} \cap A_i| \\
&= \sum_{k=0}^{n-1} (-1)^k \sum_{\substack{1 \le i_1 < \cdots < i_k \\ i_j \ne i}} |A_{i_1} \cap \cdots \cap A_{i_k} \cap A_i| \\
&= \sum_{k=1}^{n} (-1)^k \sum_{\substack{1 \le i_1 < \cdots < i_k \\ i \text{ equals some } i_j}} |A_{i_1} \cap \cdots \cap A_{i_k}|.
\end{aligned}$$

Adding the $|S_i$ together we find that

$$\begin{aligned}
|S| &= \sum_{i=1}^{n} |S_i| \\
&= \sum_{i=1}^{n} \sum_{k=1}^{n} (-1)^k \sum_{\substack{1 \le i_1 < \cdots < i_k \\ i \text{ equals some } i_j}} |A_{i_1} \cap \cdots \cap A_{i_k}| \\
&= \sum_{k=1}^{n} (-1)^k \sum_{1 \le i_1 < \cdots < i_k} \sum_{i \in \{i_1, i_2, \ldots, i_k\}} |A_{i_1} \cap \cdots \cap A_{i_k}| \\
&= \sum_{k=1}^{n} (-1)^k \sum_{1 \le i_1 < \cdots < i_k} k |A_{i_1} \cap \cdots \cap A_{i_k}| \\
&= \sum_{k=1}^{n} (-1)^k k \sum_{1 \le i_1 < \cdots < i_k} |A_{i_1} \cap \cdots \cap A_{i_k}|.
\end{aligned}$$