Math 184A Homework 4

Spring 2018

This homework is due on gradescope by Friday May 11th at 11:59pm. Remember to justify your work even if the problem does not explicitly say so. Writing your solutions in LaTeXis recommend though not required.

Question 1 (Permutation Parity, 20 points)**.** *Let $n > 1$ be an integer and let S be a set of pairs of numbers (i, j) with $i, j \in [n]$. Say that a permutation π of $[n]$ avoids S if $\pi(i) \neq j$ for all $(i, j) \in S$. So, for example, a derangement is a permutation that avoids $\{(1, 1), (2, 2), (3, 3), \ldots, (n, n)\}$. Suppose that for any $n - 1$ elements of S that either some two share a first coordinate or some two share a second coordinate. Prove that the number of permutations that avoid S is even. [Hint: Count the number using Inclusion-Exclusion.]*

Solution. *Let $m = |S|$. We have $S = \{(a_1, b_1), (a_2, b_2), \ldots, (a_m, b_m)\}$, where a_i, b_is are in $[n]$. Let A_i be the set of all permutations that avoids (a_k, b_k), equivalently, ${A_i}^{\text{c}}$ is the set of all permutations π that agrees on (a_i, b_i), that is, $\pi(a_i) = b_i$. For distinct i and j values, (a_i, b_i) and (a_j, b_j) cannot have both of their first and second entries equal to one another.(Namly $a_i \neq a_j$ or $b_i \neq b_j$)*

Claim 1. *If $a_i = a_j$ or $b_i = b_j$, then ${A_i}^{\text{c}} \cap {A_j}^{\text{c}} = \emptyset$, therefore $|{A_i}^{\text{c}} \cap {A_j}^{\text{c}}| = 0$.*

Proof. *If $a_i = a_j$, suppose there exists a permutation $\pi \in {A_i}^{\text{c}} \cap {A_j}^{\text{c}}$, then by definition, $\pi(a_i) = b_i$ and $\pi(a_j) = b_j$. However, we have $b_i = \pi(a_i) = \pi(a_j) = b_j \Rightarrow (a_i, b_i) = (a_i, b_j)$. Contradiction !*
If $b_i = b_j$, suppose $\exists \pi \in {A_i}^{\text{c}} \cap {A_j}^{\text{c}}$, then by definition, $\pi(a_i) = b_i = b_j = \pi(a_j)$. Since π is a permutation, π is bijective, hence $a_i = a_j \Rightarrow (a_i, b_i) = (a_i, b_j)$. Contradiction.

Claim 2. *The intersection of any k $[1 \leq k < n - 1]$ distinct ${A_i}^{\text{c}}$ sets is either empty or contains $(n - k)!$ elements.*

Proof. *Suppose ${A_i}^{\text{c}}$ and ${A_j}^{\text{c}}$ are 2 distinct sets among those k sets and we have $a_i = a_j$ or $b_i = b_j$. In either case, we have ${A_i}^{\text{c}} \cap {A_j}^{\text{c}} = \emptyset$. Since the k-sets intersection is a subset of ${A_i}^{\text{c}} \cap {A_j}^{\text{c}}$, the k-sets intersection must be empty. Suppose for all the pairs of sets ${A_i}^{\text{c}}$ and ${A_j}^{\text{c}}$, we have $a_i \neq a_j$ and $b_i \neq b_j$, then for each permutation π contained in the k-sets intersection, we have $\pi(a_i) = b_i$ for all i such that ${A_i}^{\text{c}}$ is one of those k sets, which means we require k entries of the permutation to be of fixed values, while having no restrictions whatsoever on the other entries, therefore we have $(n - k)!$ total number of permutations that satisfy the requirement.*

Claim 3. *The intersection of any k $[n - 1 \leq k \leq m]$ distinct ${A_i}^{\text{c}}$ sets is empty.*

Proof. *Since $k \geq n - 1$, by problem assumption, there exist i, j in $[m]$ s.t. either $a_i = a_j$ or $b_i = b_j$. In either case, we have we have ${A_i}^{\text{c}} \cap {A_j}^{\text{c}} = \emptyset$. Since the k-sets intersection is a subset of ${A_i}^{\text{c}} \cap {A_j}^{\text{c}}$, the k-sets intersection must be empty.*

Now by Principle of Inclusion and Exclusion, $|\bigcup_{i \in [m]} {A_i}^{\text{c}}| = \sum_{1 \leq i \leq m} |{A_i}^{\text{c}}|$ - $\sum_{1 \leq i < j \leq m} |{A_i}^{\text{c}} \cap {A_j}^{\text{c}}| + \ldots + (-1)^{n-1} |\bigcap_{i \in [m]} {A_i}^{\text{c}}|$, which is even by Claim 2 and 3.

Thus (the number of permutations that avoid S) $= |\bigcap_{i \in [m]} A_i| = |$(all n-permutations) $- \bigcup_{i \in [m]} {A_i}^{\text{c}}| = n! - |\bigcup_{i \in [m]} {A_i}^{\text{c}}|$, since $n!$ is even $\forall n \geq 2$, we have that (the number of permutations that avoid S) is even.

Question 2 (Size of Central Binomial Coefficients, 20 points)**.** *Show that for any $n \geq 1$*

$$4^n \geq \binom{2n}{n} \geq 4^n/(2n + 1).$$

[Hint: for the lower bound show that $\binom{2n}{n} \geq \binom{2n}{k}$ for any k.] [Note: For those who know some number theory, it is not hard to see that $\binom{2n}{n}$ is divisible by the product of all primes $n \leq p \leq 2n$. This allows one to prove rough upper bounds on the number of primes.]

Solution. *The number of binary string of length $2n$ is $2^{2n} = (2^2)^n = 4^n$, while the number of binary string of length $2n$ which contains k ones is $\binom{2n}{k}$. Therefore we have*

$$\sum_{k=0}^{2n} \binom{2n}{k} = 4^n$$

which gives us the right inequality,

$$\binom{2n}{n} \leq \sum_{k=0}^{2n} \binom{2n}{k} = 4^n$$

Notice that $\frac{4^n}{2n+1}$ is the average of the $2n+1$ values $\binom{2n}{0}, \binom{2n}{1}, \binom{2n}{2}, \ldots, \binom{2n}{2n}$. To prove that

$$\binom{2n}{n} \geq 4^n/(2n+1)$$

It suffices to prove that

$$\binom{2n}{n} \geq \binom{2n}{n-1} \geq \ldots \geq \binom{2n}{1} \geq \binom{2n}{0}$$

Proof. *Let $0 \leq k \leq n-1$, we have*

$$2(n-1) \geq 2k \Rightarrow 2n = 2(n-1) + 2 \geq 2(k+1) \Rightarrow 2n-k-1 \geq k+1 \Rightarrow \frac{2n-k}{k+1} \geq \frac{2n-k-1}{k+1} \geq 1$$

now

$$\binom{2n}{k} = \frac{(2n)!}{k!(2n-k)!} \leq \frac{2n-k-1}{k+1}\frac{(2n)!}{k!(2n-k)!} \leq \frac{(2n)!}{(k+1)!(2n-k-1)!} = \binom{2n}{k+1}$$

.

By symmetry, we have

$$\binom{2n}{n} \geq \binom{2n}{n+1} \geq \ldots \geq \binom{2n}{2n-1} \geq \binom{2n}{2n}$$

Hence, $\binom{2n}{n}$ is the greatest among all $2n+1$ values, hence it must be greater than the average.

Question 3 (Sums of Binomial Coefficients, 30 points). .

(a) *Give a formula for $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots + \binom{n}{2\lfloor n/2 \rfloor}$ as a function of n. [Hint: use the binomial theorem. You'll need a way to make the odd terms go away.][10 points]*

Solution. $2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k}(1)^k(1)^{n-k} = \sum_{k=0}^{n} \binom{n}{k}$
$0 = 0^n = (1+(-1))^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k}(1)^k(1)^{n-k} = \sum_{k=0}^{n} (-1)^k \binom{n}{k}$
hence we have $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{k} + \ldots + \binom{n}{n} = 2^n$
$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \ldots + (-1)^k \binom{n}{k} + \ldots + (-1)^n \binom{n}{n} = 0$
adding the two equations we obtain, $2 * (\binom{n}{0} + \binom{n}{2} + \ldots + \binom{n}{2k} + \ldots + \binom{n}{2\lfloor \frac{n}{2} \rfloor}) = 2^n \Rightarrow \binom{n}{0} + \binom{n}{2} + \ldots + \binom{n}{2k} + \ldots + \binom{n}{2\lfloor \frac{n}{2} \rfloor} = 2^{n-1}$

(b) *Give a formula for $\binom{n}{0} + \binom{n}{3} + \binom{n}{6} + \ldots + \binom{n}{3\lfloor n/3 \rfloor}$ as a function of n. [Hint: same idea, but you might need to use complex numbers.][20 points]*

Solution. *Let $w = e^{\frac{2\pi i}{3}}$, the 3rd root root of unity, which has the following properties:*

(a) $w^3 = (e^{\frac{2\pi i}{3}})^3 = e^{2\pi i} = 1$ *(By Euler's identity)*

(b) $1 + w + w^2 = \frac{w^3-1}{w-1} = \frac{0}{w-1} = 0$

(c) $w^i = w^j$ *iff* $i \equiv j \pmod 3$

Claim. $1^n + w^n + (w^2)^n = 0$ *if* $n \not\equiv 0 \pmod 3$ *and* $1^n + w^n + (w^2)^n = 3$ *if* $n \equiv 0 \pmod 3$

Proof. *if* $n \equiv 1 \pmod 3$, $w^n = w$ *(by property 1), and* $w^{2n} = w^2$ *since* $n \equiv 1 \pmod 3 \Rightarrow 2n \equiv 2 \pmod 3 \Rightarrow 1^n + w^n + (w^2)^n = 0$

if $n \equiv 2 \pmod 3$, $w^n = w^2$ *since* $n \equiv 2$ *(by property 1), and* $w^{2n} = w$ *since* $n \equiv 2 \pmod 3 \Rightarrow 2n \equiv 1 \pmod 3 \Rightarrow 1^n + w^n + (w^2)^n = 0$

if $n \equiv 0 \pmod 3$, *we have* $w^n = w^3 = 1$ *and* $(w^2)^n = (w^n)^2 = 1^2 = 1 \Rightarrow 1^n + w^n + (w^2)^n = 3$

now $(1+1)^n + (w+1)^n + (w^2+1)^n = \sum_{k=0}^{n} \binom{n}{k}(1)^k(1)^{n-k} + \sum_{k=0}^{n} \binom{n}{k}(w)^k(1)^{n-k} + \sum_{k=0}^{n} \binom{n}{k}(w)^k(1)^{n-k}$
$= \sum_{k=0}^{n}(\binom{n}{k}(1)^k(1)^{n-k} + \binom{n}{k}(w)^k(1)^{n-k} + \binom{n}{k}(w)^k(1)^{n-k}) = \sum_{k=0}^{n} \binom{n}{k}(1^k + w^k + (w^2)^k) = 3\sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} \binom{n}{3k}$
hence $\sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} \binom{n}{3k}$ *as a function of* n *is* $\frac{1}{3}((1+1)^n + ((e^{\frac{2\pi i}{3}}) + 1)^n + ((e^{\frac{2\pi i}{3}})^2 + 1)^n)$

Question 4 (Linear Homogeneous Recurrence Relations, 30 points). *Suppose that a sequence A_n satisfies a linear homogenous recurrence relation with constant coefficients. Namely, suppose that there are constants $C_1, C_2, \ldots, C_k$ so that*

$$A_n = C_1 A_{n-1} + C_2 A_{n-2} + \cdots + C_k A_{n-k}$$

for all $n \geq k$.

(a) *Show that the generating function $F(x) = \sum_{n=0}^{\infty} A_n x^n$ is given by a rational function in x (namely a ratio of polynomials in x). [15 points]*

Solution. $F(x) = \sum_{n=0}^{\infty} A_n x^n = \sum_{n=k}^{\infty} A_n x^n + \sum_{n=0}^{k} A_n x^n = \sum_{n=k}^{\infty}(C_1 A_{n-1} + C_2 A_{n-2} + \cdots + C_k A_{n-k}) + \sum_{n=0}^{k} A_n x^n = C_1 x F(x) + C_2 x^2 F(x) + \ldots + C_k x^k F(x) + \sum_{n=0}^{k} A_n x^n = (\sum_{n=1}^{k} C_n x^n) F(x) + \sum_{n=0}^{k} A_n x^n$
let $P(x) = \sum_{n=1}^{k} C_n x^n$ *and* $Q(x) = \sum_{n=0}^{k} A_n x^n$
Thus we have $F(x) = P(x)F(x) + Q(x) \Rightarrow F(x) = \frac{Q(x)}{1-P(x)}$, *which proves that $F(x)$ is given by a rational function in x.*

(b) *Given that partial fraction decompositions, allow you to write any rational function as a polynomial plus a linear combination of terms of the form $1/(1-b_i x)^{a_i}$, show that there's a formula expressing A_n as some linear combination of terms of the form $n^{k_i} b_i^n$ for all sufficiently large n. [15 points]*

Solution. *Given the partial fraction decomposition, there exists some polynomial $P(x)$, and $\{a_i\}_{i=1}^{m}$, $\{b_i\}_{i=1}^{m}$, and $\{c_i\}_{i=1}^{m}$ s.t.*

$$F(x) = P(x) + \sum_{i=1}^{m} \frac{c_i}{(1-b_i x)^{a_i}}$$

For each i, we have

$$\frac{c_i}{(1-b_i x)^{a_i}} = c_i(1 + (b_i x) + (b_i x)^2 + (b_i x)^2 + \ldots)^{a_i}$$

by product of generating functions

$$= c_i \sum_{k=0}^{\infty} \binom{k + a_i - 1}{k} {b_i}^k x^k$$

hence

$$F(x) = P(x) + \sum_{i=1}^{m} \frac{c_i}{(1-b_i x)^{a_i}} = P(x) + c_i \sum_{i=1}^{m} \sum_{k=0}^{\infty} \binom{k + a_i - 1}{k} {b_i}^k x^k = P(x) + \sum_{k=0}^{\infty} [c_i \sum_{i=1}^{m} \binom{k + a_i - 1}{k} {b_i}^k] x^k$$

now $A_n =$ *the coefficient of* x^n *in* $F(x) = c_i \sum_{i=1}^{m} \binom{k+a_i-1}{k} {b_i}^k$ *for* $n > deg(P(x))$, *which is some linear combination of terms of the form $n^{k_i} b_i^n$ for all sufficiently large n.*

Question 5 (Extra credit, 1 point). *Approximately how much time did you spend on this homework?*