Question 1 (Recurrence Relation Generating Function, 30 points)**.** *Consider the sequence a_n defined by:*

$$a_n = \begin{cases} 1 & \textit{if } n = 0 \\ 0 & \textit{if } n = 1 \\ 3a_{n-1} - a_{n-2} & \textit{otherwise.} \end{cases}$$

Give a closed form formula for the generating function $A(x) := \sum_{n=0}^{\infty} a_n x^n$.

Notice that

$$\begin{aligned} A(x) &= \sum_{n=0}^{\infty} a_n x^n \\ &= 1 + \sum_{n=2}^{\infty} a_n x^n \\ &= 1 + \sum_{n=2}^{\infty} (3a_{n-1} - a_{n-2}) x^n \\ &= 1 + 3\sum_{n=2}^{\infty} a_{n-1} x^n - \sum_{n=2}^{\infty} a_{n-2} x^n \\ &= 1 + 3\sum_{n=1}^{\infty} a_n x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+2} \\ &= 1 + 3x \left(\sum_{n=0}^{\infty} a_n x^n - 1 \right) - x^2 \sum_{n=0}^{\infty} a_n x^n \\ &= 1 + 3x(A(x) - 1) - x^2 A(x), \end{aligned}$$

where the fifth line is by shifting indices.

This means that

$$A(x)(x^2 - 3x + 1) = 1 - 3x,$$

or that

$$A(x) = \frac{1 - 3x}{1 - 3x + x^2}.$$

Question 2 (Committee Forming, 35 points)**.** *Let C_n be the number of ways of taking a group of n people, and breaking them up into an odd number of committees so that each person is in exactly one committee and so that each committee has a designated leader. Give a closed form expression for the exponential generating function $F(x) = \sum_{n=0}^{\infty} C_n x^n/n!$*

Putting the question in more mathematical terms, C_n is the number of set partitions of $[n]$ into an odd number of parts with one element selected from each part. The exponential generating function for the number of ways to do this with one only one part in the set partition is

$$\sum_{n=0}^{\infty} nx^n/n! = \sum_{n=1}^{\infty} x^n/(n-1)! = x\sum_{n=1}^{\infty} x^{n-1}/(n-1)! = xe^x.$$

The answer asked for is the number of ways to partition n things into an odd number of sets and apply this structure to each set. If we want to partition into k sets, the number of ways to do so will be given by the exponential generating function $(xe^x)^k/k!$. By summing over odd k, we get our answer as a composition of the generating functions $y = xe^x$ and

$$\sum_{k \text{ odd}} y^k/k! = \frac{1}{2}\left(\sum_{n=0}^{\infty} y^n/n! - \sum_{n=0}^{\infty}(-y)^n/n!\right) = \frac{e^y - e^{-y}}{2} = \sinh(y).$$

Thus the answer is

$$\sinh\left(xe^x\right) = \frac{e^{xe^x} - e^{-xe^x}}{2}.$$

Question 3 (Words Without Multiplicity Two Letters, 35 points)**.** *How many six letter words (by which I mean strings of 6 English letters) are there so that no letter occurs exactly twice in this word? Your answer should be a simple expression involving standard operations and standard combinatorial quantities like binomial coefficients and Stirling numbers.*

There are a total of 26^6 six letter words. We would like to exclude those with exactly two of any letter. If we let A_1 be the set of words, with exactly two 'A's, A_2 the set with exactly two 'B's, and so on, we find that our answer is $26^6 - |A_1 \cup A_2 \cup \ldots \cup A_{26}|$. To find the size of this union, we use inclusion-exclusion. In particular, it is

$$|A_1 \cup A_2 \cup \ldots \cup A_{26}| = \sum_{i=1}^{26} |A_i| - \sum_{1 \le i < j \le 26} |A_i \cap A_j| + \sum_{1 \le i < j < k \le 26} |A_i \cap A_j \cap A_k|.$$

Note that we can ignore that higher intersection terms, because it is impossible to have a 6 letter word with two copies of each of four different letters. Now for any i, we claim that

$$|A_i| = \binom{6}{2} 25^4.$$

This is because there are $\binom{6}{2}$ ways to select the locations of the two copies of the i^{th} letter, and the other 4 letters can be any of the 25 remaining possibilities. Similarly, for any $i < j$, we have

$$|A_i \cap A_j| = \binom{6}{2}\binom{4}{2} 24^2.$$

This is because there are $\binom{6}{2}$ ways to pick two locations for the i^{th} letter, $\binom{4}{2}$ ways to pick two remaining locations for the j^{th} letter, and 24 possibilities for each of the remaining two letters. By similar logic, for any $i < j < k$ we have that

$$|A_i \cap A_j \cap A_k| = \binom{6}{2}\binom{4}{2}\binom{2}{2}.$$

To get our final answer, we note that there are 26 terms in the first sum, $\binom{26}{2}$ terms in the second sum, and $\binom{26}{3}$ terms in the final sum. Therefore, the final answer is

$$26^6 - 26\binom{6}{2}25^4 + \binom{26}{2}\binom{6}{2}\binom{4}{2}24^2 - \binom{26}{3}\binom{6}{2}\binom{4}{2}\binom{2}{2} = 173,186,026.$$