Solutions to HW7 of Math 103A, Fall 2018

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(1) **(a)** Let

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 5 & 1 & 2 & 6 & 7 & 3 & 8 & 9 \end{pmatrix}$$

Then $\sigma_1 = \tau\sigma_2\tau^{-1}$.

(b) For indices k with $1 \leq k \leq l-1$,

$$\tau\sigma\tau^{-1}(\tau(i_k)) = \tau\sigma(i_k) = \tau(i_{k+1}).$$

Since

$$\tau\sigma\tau^{-1}(\tau(i_l)) = \tau\sigma(i_l) = \tau(i_1),$$

$(\tau(i_1), \cdots, \tau(i_l))$ is a cycle in the cycle decomposition of $\tau\sigma\tau^{-1}$. For number p different from $\tau(i_1), \cdots, \tau(i_l)$, $\tau^{-1}(p)$ is different from $i_1, \cdots, i_l$. Thus $\sigma\tau^{-1}(p) = \tau^{-1}(p)$ and

$$\tau\sigma\tau^{-1}(p) = \tau\tau^{-1}(p) = p,$$

which implies $\tau\sigma\tau^{-1}$ fixes p. Hence $\tau\sigma\tau^{-1} = (\tau(i_1), \cdots, \tau(i_l))$.

(2)

$$A_4 = \{(1), (12)(34), (13)(24), (14)(23), (123), (132), (124), (142), (134), (143), (234), (243)\}.$$

(3) sgn is well-defined by Theorem 9.15. For any $\sigma, \tau \in S_n$, suppose they can be written as

$$\sigma = \sigma_1 \cdots \sigma_r, \quad \tau = \tau_1 \cdots \tau_s,$$

where σ_i's and τ_j's are transpositions. By definition of sgn,

$$\mathrm{sgn}(\sigma) = r, \quad \mathrm{sgn}(\tau) = s$$

in $\mathbb{Z}_2$. Since $\sigma\tau$ can be written as

$$\sigma\tau = \sigma_1 \cdots \sigma_r \tau_1 \cdots \tau_s,$$

we have

$$\text{sgn}(\sigma\tau) = r + s = \text{sgn}(\sigma) + \text{sgn}(\tau).$$

Thus sgn is a homomorphism.

(4) **(a)** Let $\tau = (1234), \gamma = (567)$. We know the order of τ is 4 and the order of γ is 3. Since τ and γ are disjoint cycles, $\tau\gamma = \gamma\tau$ and

$$\sigma^2 = \tau^2\gamma^2, \quad \sigma^3 = \tau^3\gamma^3 = \tau^3, \quad \sigma^4 = \tau^4\gamma = \gamma, \cdots, \sigma^8 = \gamma^2, \cdots \sigma^{12} = \gamma^3 = \text{id}.$$

And 12 is the smallest number k such that $\sigma^k = \text{id}$, which means the order of σ is 12.

(b) We claim the order of σ is tm. Since τ, μ are disjoint cycles, $\tau\mu = \mu\tau$ and

$$\sigma^{tm} = \tau^{tm}\mu^{tm} = (\tau^t)^m(\mu^m)^t = \text{id}.$$

Let N be the order of σ. Then $N|tm$. Note that

$$\text{id} = \sigma^N = \tau^N\mu^N.$$

By the uniqueness of cycle decomposition, $\tau^N = \mu^N = \text{id}$. Thus $t|N, m|N$ and thus $tm = \text{l.c.m.}(t, m)|N$. Hence the order of σ is tm.

(5) Since part (a) is a special case of part (b) and the proof is not essentially simpler, we directly prove part (b).

Let G be the subgroup generated by $\{(12), (23), \cdots, (n-1, n)\}$. It suffices to prove G contains all the transpositions as every permutation can be written as a product of transpositions.

For transposition (i, j) in S_n with $1 \leq i < j \leq n$,

$$(i, j) = (j-1, j)(j-2, j-1)\cdots(i+1, i+2)(i, i+1)(i+1, i+2)\cdots(j-2, j-1)(j-1, j) \in G.$$

This computation follows by (1.b) of this homework and we consider consecutive conjugations of $(i, i+1)$. Thus G contains all the transpositions and $G = S_n$.

(6) **(a)** Clearly $e \in N_G(H)$. For $g \in N_G(H)$, $gHg^{-1} = H$, $Hg^{-1} = g^{-1}H, H = g^{-1}Hg$. Hence $g^{-1} \in N_G(H)$. We then show $N_G(H)$ is closed. For $g, h \in N_G(H)$,

$$(gh)H(gh)^{-1} = g(hHh^{-1})g^{-1} = gHg^{-1} = H.$$

So $gh \in N_G(H)$. Therefore, $N_G(H)$ is a subgroup.

(b) For $g_1, g_2 \in Z_G(H)$,

$$(g_1g_2)h = g_1g_2h = g_1hg_2 = hg_1g_2 = h(g_1g_2), \quad \forall h \in H.$$

Thus $g_1g_2 \in Z_G(H)$. Clearly $e \in Z_G(H)$. Suppose $g \in Z_G(H)$. Then $\forall h \in H$,

$$gh = hg, \quad h = g^{-1}hg, \quad hg^{-1} = g^{-1}h.$$

Thus $g^{-1} \in Z_G(H)$. Therefore $Z_G(H)$ is a subgroup.

(c)

$$\begin{aligned} g_1Hg_1^{-1} = g_2Hg_2^{-1} \quad &\iff \quad Hg_1^{-1} = g_1^{-1}g_2Hg_2^{-1} \\ &\iff \quad H = g_1^{-1}g_2Hg_2^{-1}g_1 \\ &\iff \quad g_1^{-1}g_2 \in N_G(H) \\ &\iff \quad g_1N_G(H) = g_2N_G(H). \end{aligned}$$

(7) P.101 3. All the cosets of $\langle 2 \rangle$ in $\mathbb{Z}_{12}$ are

$$\langle 2 \rangle = \{0, 2, 4, 6, 8, 10\}, \quad 1 + \langle 2 \rangle = \{1, 3, 5, 7, 9, 11\}.$$

(8) P.101 6. All left cosets of $H = \{\rho_0, \mu_2\}$ of D_4 are

$$H, \quad \rho_1H = \{\rho_1, \rho_1\mu_2 = \delta_2\}, \quad \rho_2H = \{\rho_2, \rho_2\mu_2 = \mu_1\}, \quad \rho_3H = \{\rho_3, \rho_3\mu_2 = \delta_1\}.$$

(9) P.102 15.

$$\sigma = (1254)(23) = (12354).$$

So the order of σ is 5. Since S_5 is finite,

$$(S_5 : \langle \sigma \rangle) = \frac{|S_5|}{|\sigma|} = \frac{120}{5} = 24.$$

(10) P.103 28. For every $h \in H$, $ghg^{-1} = (g^{-1})^{-1}hg^{-1} \in H$ by the assumption. So $ghg^{-1} = h_1$ for some h_1 in H. Then $gh = h_1g \in Hg$ and thus $gH \subseteq Hg$.

For every $h_2 \in H, g^{-1}h_2g \in H$ by the assumption. So $g^{-1}h_2g = h_3$ for some $h_3 \in H$. Then $h_2g = gh_3 \in gH$ and hence $Hg \subseteq gH$. Therefore $gH = Hg$.

(11) **P.103 29.** For all $g \in G$, by the assumption, gH must be equal to some right coset Hk for some $k \in G$. In particular,

$$k \in Hk = gH.$$

So there is $h_1 \in H, k = gh_1$. So $k^{-1}g = h_1^{-1} \in H$. Then

$$Hg = Hkk^{-1}g = gHk^{-1}g = gHh_1^{-1} = gH.$$

Thus $g^{-1}hg \in H, \forall h \in H$.

(12) **P.106 36.** Let G be an abelian group of order $2n$ where n is odd. By Exercise 29 of Section 4, there is $x \in G$ of order 2. Suppose there is another $y \in G$ of order 2 and $x \neq y$. Let $H = \{e, x, y, xy\}$. Since $x(xy) = y \in H, y(xy) = y^2x = x \in H, (xy)^2 = x^2y^2 = e$, we can see that H is closed. As $e \in H, x^{-1} = x \in H, y^{-1} = y \in H, (xy)^{-1} = xy \in H$, H is a subgroup of order 4. But by Lagrange theorem, $4 = |H||2n, 2|n$, which is a contradiction.

(13) **P.106 39.** Let $g \in G$. If $g \in H$, then $gH = H = Hg$. Now suppose $g \notin H$, then $gH \neq H$. Since $(G : H) = 2, G = H \cup gH$ as a disjoint union. So $gH = G \setminus H$. Similarly, $Hg = G \setminus H$. Thus $gH = G \setminus H = Hg$.