Solutions to Midterm II of Math 103A, Fall 2018

(1) **(a)** Note that the orbits of σ are $\{1,3,4,5\}$, $\{2,6\}$, and $\{7\}$. So $\sigma = (1,3,4,5)(2,6)$ or $(1,3,4,5)(2,6)(7)$.

(b) By direct computation, we know $\sigma(1) = 2, \sigma(2) = 3, \sigma(3) = 4, \sigma(4) = 5, \sigma(5) = 6$, and $\sigma(6) = 1$. Thus $\sigma = (1,2,3,4,5,6)$.

(2) **(a)** Since σ is a cycle of length 5, the order of σ is 5. So $\sigma^{5k} = (1)$ for any integer k. As $2018 \equiv 3 (\text{mod } 5)$, i.e., $2018 = 5 \cdot 403 + 3$,

$$\sigma^{2018} = \sigma^{5\cdot 403+3} = (\sigma^5)^{403}\sigma^3 = \sigma^3 = (1,2,3,5,9)(1,3,9,2,5) = (1,5,2,9,3).$$

(b) Recall in HW7 (1-b), if $\gamma = (i_1, \dots, i_k)$ is a cycle and $\mu \in S_{10}$, then

$$\mu\gamma\mu^{-1} = (\mu(i_1), \cdots, \mu(i_k)).$$

Now we can find such a μ to transfrom τ to σ using the formula above. For example let

$$\mu = (1,4)(2,6)(3,7)(5,8)(9,10).$$

(3) First recall that

$$D_4 = \{(1), (1234), (13)(24), (1432), (12)(34), (13), (24), (14)(23)\},$$

and

$$A_4 = \{(1), (12)(34), (13)(24), (14)(23), (123), (132), (124), (142), (134), (143), (234), (243)\}.$$

(a)

$$H = D_4 \cap A_4 = \{(1), (12)(34), (14)(23), (13)(24)\}.$$

(b) First H is clearly a left coset. Then we pick some element in $A_4 \setminus H$, for example (123), and $(123)H$ is another left coset:

$$(123)H = \{(123), (134), (142), (243)\}.$$

Next, we pick another element in $A_4 \setminus (H \cup (123)H)$, e.g. (234). Then

$$(234)H = \{(234), (132), (124), (143)\}$$

is the only remaining left coset.

(4) Let $B_1, \cdots, B_r$ be the orbits of σ. $B_1, \cdots, B_r$ are disjoint since they are distinct equivalence classes under the equivalence relation

$$a, b \in \{1, \cdots, n\}, \quad a \sim b \iff a = \sigma^n(b) \text{ for some } n \in \mathbb{Z}.$$

Define

$$\mu_j(x) = \begin{cases} \sigma(x), & \text{if } x \in B_j; \\ x, & \text{if } x \notin B_j, \end{cases}$$

for $j = 1, \cdots, r$. Then clearly $\sigma = \mu_1 \cdots \mu_r$ and $\mu_1, \cdots, \mu_r$ are disjoint.

(5) Since the cyclic group $\langle g \rangle$ is a subgroup of G, by Lagrange's theorem, $|\langle g \rangle|$ which equals the order of g divides n. Thus $g^n = e$.

(6) **(a)** Let $g \in G$. If $g \in H$, then clearly $gH = H = Hg$ and thus $gHg^{-1} = H$. If $g \notin H, gH$ is the other left coset of H since $(G : H) = 2$. Since $G = H \cup (gH), H \cap gH = \emptyset$, we know $gH = G \setminus H$. Similarly, $Hg = G \setminus H$. Thus $gH = Hg$ and $gHg^{-1} = H$.

(b) Consider $G = S_3, H = \{(1), (12)\}$. Then $(G : H) = \frac{|S_3|}{2} = 3$. But if we choose $g = (13)$,

$$(23)(12)(23) = (13) \notin H.$$

Thus $gHg^{-1} \neq H$.

(7) (Bonus) Consider the set

$$\Sigma = \{gH, g^2H, \cdots, g^nH, g^{n+1}H\}.$$

Since $(G : H) = n, H$ has n left cosets and thus $|\Sigma| \leq n$. But the sequence

$$gH, g^2H, \cdots, g^nH, g^{n+1}H$$

is of length $n + 1$, which means there must be two integers $1 \leq s < t \leq n + 1$, such that $g^sH = g^tH$. (This is usually called pigeonhole principle.) Let $k = t - s$. Then $1 \leq k \leq n$ and $g^kH = H$. So $g^k \in H$.