Homework 1 Solutions

Question 1 (Fibonacci Numbers, 30 points). The Fibonacci Numbers is the sequence 1, 1, 2, 3, 5, 8, 13 … defined as follows:

$$F_1 = F_2 = 1$$
$$F_{n+2} = F_{n+1} + F_n \text{ for all } n \geq 1$$

(a) Prove that F_{n+2} counts the number of strings of length n using letters 'A' and 'B' so that there are no two consecutive A's. So for example "ABBAB" would be a valid string of length 5, but not "AABBA". [15 points]

Let S_n be the set of strings of length n using letters 'A' and 'B' so that there are no two consecutive A's, and let s_n be the number of such strings, i.e. the size of S_n. So the statement we are trying to prove is equivalent to "$s_n = F_{n+2}$ for all $n \geq 1$". For notational purposes, if we have two strings s_1 and s_2, let $s_1 s_2$ be the string with the letters of s_1 followed by the letters of s_2.

$S_1 = \{A, B\}$, so $s_1 = 2 = F_3$ and $S_2 = \{AB, BA, BB\}$, so $s_2 = 3 = F_4$. So then we have verified the equation for $n = 1$, and $n = 2$. For the inductive step, assume that the equation holds for $n-1$ and $n-2$, (i.e. $s_{n-1} = F_{n+1}, s_{n-2} = F_n$) and we'll show that it holds for n (i.e. $s_n = F_{n+2}$.) To do this, we'll show that $s_n = s_{n-1} + s_{n-2}$, since we can follow that with $s_{n-1} + s_{n-2} = F_{n+1} + F_n = F_{n+2}$ to show that the equation holds for n. We will create a bijection between $S_{n-1} \cup S_{n-2}$ and S_n, which will show that $S_{n-1} \cup S_{n-2}$ has the same size as S_n. Since the size of $S_{n-1} \cup S_{n-2}$ is $s_{n-1} + s_{n-2}$ and the size of S_n is s_n, this will give the equation $s_{n-1} + s_{n-2} = s_n$.

The bijection is fairly simple, $G : S_{n-1} \cup S_{n-2} \to S_n$ is given by

$$G(s) = \begin{cases} sB & \text{if } s \in S_{n-1} \\ sBA & \text{if } s \in S_{n-2} \end{cases}$$

In other words, if it's a string of length $n-1$, add a B to the end, and if it's a string of length $n-2$, add BA to the end. To prove this is a bijection, we show that it is one-to-one and onto.

To show it's one-to-one, given $s_1, s_2 \in S_{n-1} \cup S_{n-2}$ with $s_1 \neq s_2$, and want to show that $G(s_1) \neq G(s_2)$. Since strings in S_{n-1} map to strings ending in the letter B and strings in S_{n-2} map to strings ending in the letter A, if s_1 and s_2 are strings of different length, then $G(s_1)$ and $G(s_2)$ will have different letters at the end, and so will clearly be different. If s_1 and s_2 are strings of the same length, since $s_1 \neq s_2$ they differ in at least one letter. Assuming s_1 and s_2 differ at the i-th letter, $G(s_1)$ and $G(s_2)$ just adds B or BA to the end, which won't change the fact that they differ in position i. So then $G(s_1) \neq G(s_2)$, and hence G is one-to-one.

Now to show that G is surjective, (onto) we take some arbitrary string $w \in S_n$ and show that there's some $s \in S_{n-1} \cup S_{n-2}$ such that $G(s) = w$. To make things easier, let w_k be the first k letters of w. It's easy to see that w_k still has no 2 consecutive

A's, so $w_k \in S_k$. We have 2 cases: w ends in B, or it ends in A. If w ends in B, then $w = w_{n-1}B$ and hence G maps w_{n-1} to w–$G(w_{n-1}) = w_{n-1}B = w$. If w ends in A, since we know that w has no 2 consecutive A's, the letter in position $n-1$ *can't* be A, or we'd have 2 consecutive A's. So then the letter in position $n-1$ is a B. So then $w = w_{n-2}BA$, and so G maps w_{n-2} to w. So then w is mapped to regardless of whether it ends in an A or a B.

Since G is both one-to-one and onto, it's a bijection. Since G is a bijection, S_n and $S_{n-1} \cup S_{n-2}$ have the same size. Therefore $s_n = s_{n-1} + s_{n-2} = F_{n+1} + F_n = F_{n+2}$, and we've completed the inductive step.

(b) Prove by induction that

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

for all $n \geq 1$. Hint: when using induction you will need to handle the case of $n = 2$ as a second base case. [15 points]

To make this easier to write and read, let $a = \frac{1+\sqrt{5}}{2}$ and $b = \frac{1-\sqrt{5}}{2}$, making our hypothesized equation $F_n = \frac{1}{\sqrt{5}}(a^n - b^n)$. For the base cases $n = 1$ and $n = 2$, we'll need to calculate $a - b$, a^2, and b^2:

$$\begin{aligned}
a - b &= \frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2} = \frac{1+\sqrt{5}-1+\sqrt{5}}{2} = \frac{2\sqrt{5}}{2} = \sqrt{5} \\
a^2 &= \frac{\left(1+\sqrt{5}\right)^2}{2^2} = \frac{1^2 + 2\sqrt{5} + \sqrt{5}^2}{4} = \frac{6+2\sqrt{5}}{4} = \frac{3+\sqrt{5}}{2} = 1 + \frac{1+\sqrt{5}}{2} = 1 + a \\
b^2 &= \frac{\left(1-\sqrt{5}\right)^2}{2^2} = \frac{1^2 - 2\sqrt{5} + \sqrt{5}^2}{4} = \frac{6-2\sqrt{5}}{4} = \frac{3-\sqrt{5}}{2} = 1 + \frac{1-\sqrt{5}}{2} = 1 + b
\end{aligned}$$

Looking at base cases,

$$\begin{aligned}
F_1 &= \frac{1}{\sqrt{5}}(a-b) = \frac{1}{\sqrt{5}}\left(\sqrt{5}\right) = 1 \\
F_2 &= \frac{1}{\sqrt{5}}\left(a^2 - b^2\right) = \frac{1}{\sqrt{5}}((a+1)-(b+1)) = \frac{1}{\sqrt{5}}(a-b) = F_1 = 1
\end{aligned}$$

we can see that the equation works for $n = 1$ and $n = 2$.

Moving on to the inductive step, assume that the equation holds for $n-1$ and $n-2$, i.e. $F_m = \frac{1}{\sqrt{5}}(a^m - b^m)$ is true for $m = n-1, n-2$. We want to prove it holds for $m = n$. We start with the right-hand side and show it equals the left-hand side:

$$\begin{aligned}
\frac{1}{\sqrt{5}}\left(a^n - b^n\right) &= \frac{1}{\sqrt{5}}\left(a^2 a^{n-2} - b^2 b^{n-2}\right) \\
&= \frac{1}{\sqrt{5}}\left((a+1)\, a^{n-2} - (b+1)\, b^{n-2}\right) \\
&= \frac{1}{\sqrt{5}}\left(a^{n-1} + a^{n-2} - \left(b^{n-1} + b^{n-2}\right)\right) \\
&= \frac{1}{\sqrt{5}}\left(a^{n-1} - b^{n-1} + a^{n-2} - b^{n-2}\right) \\
&= \frac{1}{\sqrt{5}}\left(a^{n-1} - b^{n-1}\right) + \frac{1}{\sqrt{5}}\left(a^{n-2} - b^{n-2}\right) \\
&= F_{n-1} + F_{n-2} \\
&= F_n
\end{aligned}$$

proving the inductive step and completing our inductive proof.

Question 2 (Course Overlap, 30 points). In Binomial University, there are a total of 32 classes taught each term and each student takes exactly 5 classes each term. Show that for any group of 45 students that some two of them must share a pair of classes in common. *Hint: First show that there must be some class taken by at least 8 students.*

There are 45 students and each student is enrolled in 5 classes, so each student appears on 5 class lists. If we add up the names appearing on all of the class lists, we get $45 \cdot 5 = 225$ total names, or alternatively 225 student-class pairs. Since there are 32 classes (buckets) and $32 \cdot 7 = 224 < 225$, the Pigeonhole Principle tells us that at least one class has 8 names (students).

Given this class C with at least 8 students, consider the first 8 students and ignore all other students for now. Removing class C from consideration, each of these 8 students is also enrolled in 4 *other* classes among the 31 other classes, so there are 32 names (student-class pairs) of students enrolled in class C that are distributed among the remaining 31 other classes. Since there are 31 other classes and 32 names (student-class pairs) within the classes, using the Pigeonhole Principle again tells us at least one of the remaining classes C_2 has at least 2 names (student-class pairs), and since no student is enrolled in the same class twice, this class C_2 has at least 2 students in common with class C (since we only included the names of the first 8 students in class C, and determined that C_2 had at least 2 of them.) Then we know that at least 2 students share a pair of classes in common.

Question 3 (Counting Words, 40 points). For the purposes of this problem, a word is simply a sequence of letters from {a, b, ..., z}, the vowels are a, e, i, o, u. How many words are there of each of the following types (you can give answers as unexpanded expressions, like $\binom{26}{8}$, but you should still justify your answers) [5 points each]:

(a) With either exactly 4 or exactly 5 letters.

The number of words with exactly k letters is 26^k because you can choose the letters independently. So there are $26^4 + 26^5 = 26^4 \cdot 27$ words with either exactly 4 or exactly 5 leters.

(b) With at most 5 letters, all of which are the same.

Since we have at most 5 letters, we could have 1, 2, 3, 4, or 5 letters. If all of the letters are the same, then regardless of the length we only pick one letter (with 26 choices). So we get $26 + 26 + 26 + 26 + 26 = 26 \cdot 5$ total words.

(c) With exactly 5 letters, the first two of which are vowels.

If the first two letters are vowels, they each have only 5 choices. The other letters still each have 26 choices, so we get $5^2 \cdot 26^3$ words.

(d) With exactly 5 letters, all of which are distinct.

If all of the letters are distinct, we have 26 choices for the first letter chosen, 25 for the second, 24 for the third, 23 for the fourth, and 22 for the last letter chosen. So we get $26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 = (26)_5$ words.

(e) With exactly 5 letters, all distinct, appearing in alphabetical order.

Since we are given a particular order for the letters to appear in, we change our selection process. If we tried to count the number of words by again choosing letters independently, we would be overcounting by a large amount–for example, choosing 1, 2, 3, 4, then 5 would give the same word as choosing 1, 3, 2, 4, then 5.

Another way of thinking about words "with exactly 5 letters, all of which are distinct" is to choose 5 letters, ($\binom{26}{5}$ ways to do that) then count all of the ways the 5 letters can be ordered to create new words, (5! ways to order 5 letters) which gives $\binom{26}{5} \cdot 5!$.

In this problem, we can apply this idea to choose the 5 letters we want, then count the number of orderings of the letters that give words we want. Since there is exactly one ordering of the letters that gives the word we want, (the alphabetical ordering) we get that the number of words here is $\binom{26}{5} \cdot 1 = \binom{26}{5}$.

(f) With exactly 5 letters, at least one of which is a vowel.

The easier way to think about this problem is to think about which 5-letter words are *not* covered by this. The only words without at least one vowel are those with 5 non-vowels, (consonants) and there are 21^5 of those. So we take all of the 5-letter words and subtract the ones that don't fit the condition, giving $26^5 - 21^5$.

The harder way involves separating it into how many vowels there are, then for each picking the positions of the vowels before picking the vowels and non-vowels indpendently, which gives $\sum_{i=1}^{5} \binom{5}{i} 5^i 21^{5-i}$.

(g) With exactly 6 letters, exactly two of which are the same.

Notice that because exactly two letters are the same, the others are all distinct. Hence we have 5 different letters and one duplicate. First we pick which 2 positions in the word will have the same letter, getting $\binom{6}{2}$ possibilities for that. Then we simply pick the 5 distinct letters in order, (skipping the spot with the duplicate letter, or alternatively multiplying by 1 for the forced choice there) getting $26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 = (26)_5$ choices for the letters. Then we there are $\binom{6}{2}(26)_5$ total words that meet the condition.

(h) With exactly 5 letters and with all the vowels in the word coming before all of the non-vowels.

To make thinking about the problem easier, imagine that we are instead creating 2 different words, one with only vowels and one with only non-vowels, then sticking them together with the vowel-word first. The vowel-word has 5 choices for each letter in the word, all of which can be chosen independently. The non-vowel-word has 21 choices for each letter in the word, and they again can all be chosen independently.

We break up the problem into how many vowels appear in the word, which decides the length of the vowel-word, and from this we know exactly how many non-vowels will appear, (5 minus the number of vowels) and hence the length of the non-vowel-word. With 0 vowels, there are $5^0 \cdot 21^5$ words, with 1 vowel, there are $5^1 \cdot 21^4$ words, and so on, giving a total of $\sum_{i=0}^{5} 5^i \cdot 21^{5-i}$ words.