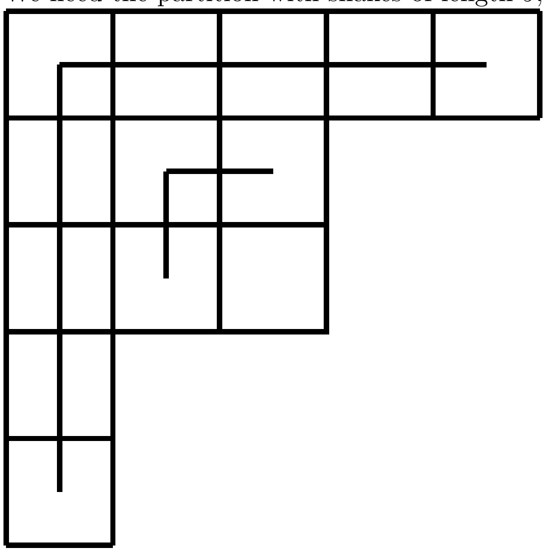
Question 1 (Self-Conjugate Partitions, 30 points). *Consider the partition of 13 into distinct odd parts given by* $9 + 3 + 1 = 13$. *What is the corresponding self-conjugate partition of 13?*

We need the partition with snakes of length $9, 3$, and 1. It is shown below.



The partition is $5 + 3 + 3 + 1 + 1$.

Question 2 (Multiple Colliding Pairs, 35 points). *Suppose that n pigeons are placed into m holes with $n \geq m-1+2k$ for some positive integer k. Show that one can find k disjoint pairs of pigeons $(p_1, q_1), (p_2, q_2), \ldots, (p_k, q_k)$ so that p_i and q_i were placed in the same hole for each i. Hint: use induction on k.*

We proceed by induction on k. If $k = 1$, this is just the pigeonhole principle. Assuming that this statement holds for a given value of k, we need to show it for $k+1$. Assume that $n \geq m-1+2(k+1)$. Since $n > m$, the pigeonhole principle implies that there is some pair of pigeons p, q placed in the same hole. Call these pigeons p_{k+1}, q_{k+1}. Removing these pigeons from the collection we have $n-2 \geq m-1+2k$ pigeons remaining. By the inductive hypothesis there are k disjoint pairs of these pigeons (p_i, q_i), with each pair landing in the same hole. Adding (p_{k+1}, q_{k+1}) gives $k+1$ pairs of pigeons from our original collection. This completes the inductive step and proves our result.

Question 3 (Restricted Compositions, 35 points). *In how many ways can one select integers* a_1, a_2, a_3, a_4 *with* $1 \leq a_i \leq 9$ *so that* $a_1 + a_2 + a_3 + a_4 = 15$*? You may leave your solution in terms of a closed formula and need not compute the numerical answer.*

If we do not impose the upper bound on the a_i, we want the number of compositions of 15 into 4 parts, which is $\binom{15-1}{4-1} = \binom{14}{3}$. We need to remove from this the number of quadruples with $a_i \geq 10$ for some i. Note that at most one a_i is this large (otherwise the sum would be at least 20). Thus, it is enough to subtract the sum over i of the number of quadruples with $a_i \geq 10$. Subtracting 9 from a_i yields an arbitrary composition of $15 - 9 = 6$ into 4 parts. Therefore the number of such compositions with $a_i \geq 10$ is $\binom{6-1}{4-1} = \binom{5}{3}$. Thus, the final answer is $\binom{14}{3} - 4\binom{5}{3}$.