Math 184A HW 2 Solution

Question 1 (Equal Sum Subsets, 20 points). *Let S be a set of 10 positive integers each at most* 100*. Show that there exist two different subsets $A \subseteq S$ and $B \subseteq S$ so that the sum of the elements of A equals the sum of the elements of B.*

Answer.

S is a set of 10 elements, then

$$\# \text{ of subsets of } S = 2^{10} = 1024.$$

(Since when we are choosing a subset $A \subseteq S$, every element $s_i \in S$ can be either chosen or not chosen, that is two choices for every element. In total there are $2^{10} = 1024$ ways to choose a subset.)

For any subset $A \subseteq S$, since every element is ≤ 100 and the number of elements is ≤ 10, the sum of the elements of A

$$\sum_{a \in A} a \leq 100 \times 10 = 1000.$$

Since $0 \leq \sum_{a \in A} a$, the sum of the elements of a subset of S can only be $0, 1, 2, 3, \cdots, 1000$. 1001 choices in total. $1024 > 1001$, so by Pigeonhole Principle, there must exist two different subsets $A \subseteq S$ and $B \subseteq S$ so that the sum of the elements of A equals the sum of the elements of B.

Question 2 (Counting Poker Hands, 80 points). *Recall that a standard deck of cards has cards with* 13 *different ranks $2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A$ and* 4 *different suits $\clubsuit, \diamondsuit, \heartsuit, \spadesuit$ for a total of $4 \times 13 = 52$ total cards. We say that a k-card hand is an (unordered) collection of k cards from this deck. How many hands are there of each of the following types?*

For each part you should justify your answer in addition to giving a number. The number is allowed to be given in terms of standard operations and binomial coefficients (for example, if you got an answer of $\binom{13}{6} \cdot 10 - 4^7$, you may leave it like that rather than computing the actual value of this number). Each part is worth 10 points.

(a) *How many hands are there with at most* 5 *cards?*

(b) *How many* 5 *card hands have no two cards of the same rank?*

(c) *How many* 5 *card hands are a* flush*, that is have all cards of the same suit?*

(d) *How many* 5 *card hands are a* full house*, that is have three cards of one rank and two cards of another rank?*

(e) *How many* 5 *card hands are a* straight*, that is have all five cards with five consecutive ranks (for this purpose consider the ranks to be ordered in a* 13*-long sequence)?*

(f) *How many* 6 *card hands have* 3 pairs*, that is have three different ranks with two cards in the hand of each rank?*

(g) *How many* 4 *card hands are a* flash*, that is have four cards of different suits?*

(h) *How many* 4 *card hands are a* 2 pair, flash*, that is have four cards of different suits and have two ranks each with two cards?*

Answer.

(a) We need to count the number of hands of 1 card, 2 cards, $\cdots$, 5 card and then add them up. The answer is

$$\binom{52}{1} + \binom{52}{2} + \binom{52}{3} + \binom{52}{4} + \binom{52}{5}.$$

(b) The number of ways to choose 5 different ranks is $\binom{13}{5}$,and then for every rank, there are 4 ways to choose a suit for it, so the ways to choose such 5 card hands is

$$\binom{13}{5} \times 4^5.$$

(c) There are 4 ways to choose a suit, then after we fix the suit, we have $\binom{13}{5}$ ways to choose 5 cards from the suit. So the number of flushes is

$$4 \times \binom{13}{5}.$$

(d) There are 13 ways to choose a rank for the triple, and then we have 12 ways to choose a rank for the pair. After we fix the ranks, we have $\binom{4}{3}$ ways to choose three cards for the first rank and $\binom{4}{2}$ ways to choose two cards for the second rank, so the number of full houses is

$$13 \times 12 \times \binom{4}{3} \times \binom{4}{2}.$$

(e) There are 9 ways to choose ranks for the straight (23456,34567,$\cdots$,10JQKA), and then for every rank we have 4 ways to choose a suit. The number of straights is

$$9 \times 4^5.$$

(f) There are $\binom{13}{3}$ ways to choose ranks for the three pairs, then for every rank there are $\binom{4}{2}$ ways to choose two cards to form a pair. The answer is

$$\binom{13}{3} \times \binom{4}{2}^3.$$

(g) There are 13 ways to choose a rank for every suit, so the number of flashes is

$$13^4.$$

(h) There are $\binom{13}{2}$ ways to choose ranks for the two pairs, namely $r_1, r_2 (r_1 < r_2)$. There are $\binom{4}{2}$ ways to choose two suits for the first pair with rank r_1, and when the suits are fixed for the first pair, the suits for the second pair are also fixed in order to keep this hand a flash. So, the number of such hands is

$$\binom{13}{2} \times \binom{4}{2}.$$