

Math 184A Homework 2

Fall 2016

This homework is due on gradescope by Friday October 14th at 11:59pm. Remember to justify your work even if the problem does not explicitly say so. Writing your solutions in L^AT_EX is recommended though not required.

Question 1 (Colored Balls in Bins, 30 points). *Consider the problem of placing unlabelled red and blue colored balls into labelled bins. As the balls are unlabelled you cannot distinguish one from the other except by color. You can distinguish two arrangements based on the number of red and blue balls in each bin.*

(a) *Show that the number of ways to put n red and m blue balls into k bins is*

$$\binom{n+k-1}{k-1} \binom{m+k-1}{k-1}.$$

[10 points]

(b) *Suppose that you have a total of N balls that can be colored either color. Show that the number of distinct ways to color these balls and put them into k bins is*

$$\binom{N+2k-1}{2k-1}.$$

Hint: Think of breaking each bin into two sub-bins each storing the balls of only one color. [10 points]

(c) *Show that*

$$\binom{N+2k-1}{2k-1} = \sum_{n=0}^N \binom{n+k-1}{k-1} \binom{N-n+k-1}{k-1}.$$

[10 points]

Question 2 (Bell Numbers and Restricted Growth Sequences, 20 points). *Call a sequence a_1, a_2, \dots, a_n of positive integers a restricted growth sequence if $a_1 = 1$ and for each $i > 0$*

$$a_i \leq \max_{j < i} (a_j) + 1.$$

In particular the i^{th} term of the sequence is at most one more than the largest previous term. Show that $B(n)$ counts the number of restricted growth sequences of length n . Hint: Show that these sequences are in bijection to set partitions. In particular, think of a_i as the label for the part containing the element i .

Question 3 (Stirling Number Formula, 20 points). *Find a closed form formula for $S(n, n-2)$ for all $n \geq 2$. Hint: think about which parts do not consist of single elements.*

Question 4 (Compositions and Partitions, 30 points). *Recall that the number of ways to place n unlabelled balls into k non-empty bins is given by $\binom{n-1}{k-1}$. The number of ways of putting n unlabelled balls into k non-empty unlabelled bins is $p_k(n)$. Show that*

$$\binom{n-1}{k-1} \geq p_k(n) \geq \binom{n-1}{k-1} / k!$$

Hint: Each partition gives rise to several possible compositions. Try to prove bounds on how many compositions per partition there are.

Question 5 (Extra credit, 1 point). *Approximately how much time did you spend working on this homework?*