Math 184A Homework 3

Spring 2018

This homework is due on gradescope by Friday May 4th at 11:59pm. Remember to justify your work even if the problem does not explicitly say so. Writing your solutions in LaTeXis recommend though not required.

Question 1 (Summation Polynomials Redux, 20 points). *Recall that in the last homework we showed that*

$$\sum_{i=1}^{n} i^m = P_m(n)$$

where

$$P_m(x) = \sum_{k=0}^{m} k! S(m,k) \binom{x+1}{k+1}.$$

Suppose that we want to find the coefficients of the polynomial

$$P_m(x-1) = c_{m+1,m} x^{m+1} + c_{m,m} x^m + \ldots + c_{0,m}.$$

Show that there is a formula for the coefficients $c_{i,j}$ given as a summation involving Stirling numbers of the first and second kind.

Question 2 (Permutations Without 2-Cycles, 20 points). *Give a formula for the number of permutations of a set of $2n$ elements that have no cycles of length 2. Your formula may include a single summation.*

Question 3 (Stirling Number Identity, 20 points). *Prove that*

$$c(n,k) = \sum_{m=1}^{n} (n-1)_{m-1} c(n-m, k-1).$$

Question 4 (Average Number of Cycles, 40 points). .

(a) *For an ordering of the numbers from 1 to n, $a_1, a_2, \ldots, a_n$, let a record be a value i so that $a_i > a_j$ for all $j < i$. Show that the number of such orderings with exactly k records equals the number permutations of n with exactly k cycles. [20 points]*

(b) *Show that on average over permutations of $[n]$ of the number of cycles in the permutation is the harmonic number*

$$H_n = \sum_{i=1}^{n} \frac{1}{i}.$$

[20 points]

Question 5 (Extra credit, 1 point). *Approximately how much time did you spend on this homework?*