

Math 184A Homework 6

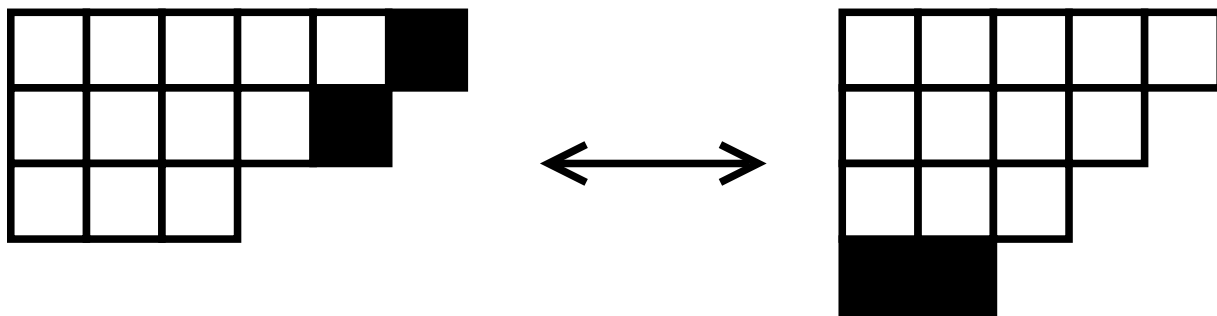
Fall 2015

This homework is due Monday November 23rd in discussion section. Remember to justify your work even if the problem does not explicitly say so. Writing your solutions in \LaTeX is recommended though not required. If you cannot solve one part of a problem, you may still use the results from it in later parts of the same problem.

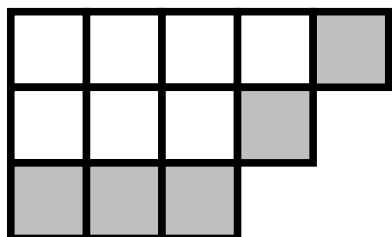
Optional Practice Problems: (do not turn in) Chapter 8 problems 1, 2, 7, 8, 9.

Question 1 (Pentagonal Number Theorem, 80 points). .

- (a) Let $F(x)$ be the function $(1-x)(1-x^2)(1-x^3)\cdots$. If $F(x) = \sum_{n=0}^{\infty} f_n x^n$ show that f_n equals the sum over integer partitions λ of n into parts of distinct sizes of $(-1)^{\text{Number of parts of } \lambda}$. [20 points]
- (b) It is actually possible to show pair up terms in the above sum in a way to show that most of them cancel out. In particular, consider the Ferrer's diagram. Suppose that the a largest parts all differ in size by 1 and that the smallest part has size b . If $a < b$, we can remove 1 from each of the a largest parts, creating a new part of size a . If $a \geq b$, we remove the smallest part and add 1 to each of the b largest parts (see figure below). Show that this method allows us to pair up most of partitions of n into distinct parts into



pairs where one element of each pair has an even number of parts and one has an odd number of parts. In particular, show that the only partitions that are not paired in this way are those where a equals the number of parts and is equal to either b or $b-1$, as shown below. [20 points]



- (c) Show that the pairings in part (b) allow us to cancel almost all the terms in the formula for f_n in part (a). In particular, show that

$$F(x) = \sum_{n=-\infty}^{\infty} (-1)^n x^{n(3n-1)/2} = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \dots$$

Note the surprising amount of cancellation showing up in this formula. [20 points]

(d) Use the above and the fact that the generating function for the partition numbers is given by

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{n=1}^{\infty} \frac{1}{1-x^n}$$

to prove the following recurrence for the partition numbers:

$$p(n) = \sum_{m \in \mathbb{Z}, m \neq 0} (-1)^{m+1} p(n - m(3m-1)/2) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + \dots$$

Note that this formula can be used to compute partition numbers fairly efficiently. In particular, if you have already computed $p(1), p(2), \dots, p(n)$, you can then compute $p(n+1)$ as a sum of approximately \sqrt{n} of these other terms. [20 points]

Question 2 (Generating Functions for Compositions, 20 points). Let $a_{n,k}$ be the number of compositions of n into k parts (recall that this is the number of ways to write $n = x_1 + x_2 + \dots + x_k$ where the x_i are any positive integers). Show that for any k we have the generating function identity

$$\sum_{n=0}^{\infty} a_{n,k} x^n = \left(\frac{x}{1-x} \right)^k.$$

Sum the above over k to find the generating function for a_n , the number of compositions of n into any number of parts. Provide a formula for a_n .

Question 3 (Extra credit, 1 point). Approximately how much time did you spend working on this homework?