HW 6 Solutions

1.a

$$\frac{2x}{1-3x^2} = 2x(1+3x^2+(3x^2)^2+(3x^2)^3+\dots) = 0+2x+0x^2+6x^3+0x^4+18x^5+0x^6+54x^7+\dots$$

1.b

$$(1-3x)^{\frac{1}{3}} = 1+\binom{\frac{1}{3}}{1}(-3x)+\binom{\frac{1}{3}}{2}(-3x)^2+\binom{\frac{1}{3}}{3}(-3x)^3+\binom{\frac{1}{3}}{4}(-3x)^4+\dots =$$

$$1-x-x^2-\frac{5}{3}x^3-\frac{10}{3}x^4+\dots$$

1.c.

$$\frac{1}{2-x} = \frac{1}{2}\cdot\frac{1}{1-\frac{1}{2}x} = \frac{1}{2}(1+\frac{x}{2}+(\frac{x}{2})^2+(\frac{x}{2})^3+(\frac{x}{2})^4+\dots) =$$

$$\frac{1}{2}+\frac{x}{4}+\frac{x^2}{8}+\frac{x^3}{16}+\frac{x^4}{32}+\dots$$

1.d.

$$\frac{\log(1+x^2)}{1-x} = -\log(\frac{1}{1+x^2})\cdot\frac{1}{1-x} = -(-x^2+\frac{x^4}{2}+\dots)(1+x+x^2+x^3+x^4+\dots) =$$

$$0+0x+x^2+x^3+\frac{1}{2}x^4+\frac{1}{2}x^5+\dots$$

1.e.

$$(1+x^2+x^3)^{\frac{1}{2}} = (1+(x^2+x^3))^{\frac{1}{2}} = 1+\binom{\frac{1}{2}}{1}(x^2+x^3)+\binom{\frac{1}{2}}{2}(x^2+x^3)^2+\binom{\frac{1}{2}}{3}(x^2+x^3)^3+\dots =$$

$$1+\frac{1}{2}(x^2+x^3)-\frac{1}{8}(x^4+2x^5+x^6)+\frac{1}{16}(x^6+3x^7+3x^8+x^9)+\dots = 1+\frac{1}{2}x^2+\frac{1}{2}x^3-\frac{1}{8}x^4-\frac{1}{4}x^5-\frac{1}{16}x^6+\dots$$

1.f. To solve this we compare the right and left sides of this equation, the left side is

$$a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+a_6x^6+a_7x^7+a_8x^8+a_9x^9+\dots,$$

while the right side expands to

$$1+a_0x+0x^2+a_1x^3+0x^4+a_2x^5+0x^6+a_3x^7+0x^8+a_4x^9+\dots,$$

setting the coefficients of each term equal gives

$$a_0 = 1$$
$$a_1 = a_0 = 1$$
$$a_2 = 0$$
$$a_3 = a_1 = 1$$
$$a_4 = 0$$
$$a_5 = a_2 = 0$$
$$a_6 = 0$$
$$a_7 = a_3 = 1$$

$$a_8 = 0$$
$$a_9 = a_4 = 0$$

This gives that

$$A(x) = 1 + x + 0x^2 + x^3 + 0x^4 + 0x^5 + 0x^6 + x^7 + 0x^8 + 0x^9 + \ldots$$

1.g.

$$(\sum_{n=0}^{\infty} x^{n^2})^3 = (1+x+x^4+x^9+\ldots)^3 = 1+3x+3x^3+x^3+3x^4+6x^5+3x^6+0x^7+3x^8+6x^9+\ldots$$

The combinatorial interpretation of the mth coefficient of this generating function is the number of weak composition of m into 3 squares.

1.h.

$$\frac{e^x + e^{-x}}{2} = \frac{1}{2}(e^x+e^{-x}) = \frac{1}{2}[(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots)+(1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots)] =$$
$$(1 + 0x + \frac{x^2}{2!} + 0\frac{x^3}{3!} + \frac{x^4}{4!} + \ldots)$$

1.i.

$$\frac{e^x}{1-x} = (1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\ldots)(1+x+x^2+x^3+x^4+\ldots) = 1+2x+5\frac{x^2}{2!}+16\frac{x^3}{3!}+65\frac{x^4}{4!}+\ldots$$

1.j.

$$e^{\frac{x^2}{2}} = 1+\frac{x^2}{2}+(\frac{x^2}{2})^2\cdot\frac{1}{2!}+(\frac{x^2}{2})^3\cdot\frac{1}{3!}+\ldots = 1+0x+\frac{x^2}{2!}+0\frac{x^3}{3!}+3\frac{x^4}{4!}+0\frac{x^5}{5!}+15\frac{x^6}{6!}+\ldots$$

The combinatorial interpretation of the mth coefficients is the number of permutations of n into cycles of length 2. This can be seen a result of the composition formula for exponential generating functions since $\frac{x^2}{2}$ is the exponential generating function for the number of ways of putting n elements into exactly one 2-cycle (there is one way if n=2, and zeros ways otherwise) and e^x is the exponential generating function for the number of ways of putting n two-cycles into permutation (there is exactly one for all n). The composition of these two functions is $e^{\frac{x^2}{2}}$ which gives the result.

2.a. To make our computation easier we start by shifting the recurrence relation we are trying to prove to:

$$S(n+1, k+1) = S(n,k) + (k+1)S(n,k+1).$$

We then consider the following calculations:

$$\frac{\partial}{\partial y}\sum_{n,k} S(n,k)\frac{x^n}{n!}y^k = \sum_{n,k} kS(n,k)\frac{x^n}{n!}y^{k-1} = \sum_{n,k}(k+1)S(n,k+1)\frac{x^n}{n!}y^k$$

and

$$\frac{\partial}{\partial x}(\frac{1}{y}(\sum_{n,k} S(n,k)\frac{x^n}{n!}y^k) - 1) = \frac{\partial}{\partial x}\sum_{n,k} S(n,k)\frac{x^n}{n!}y^{k-1} =$$

$$\sum_{n,k} S(n,k)\frac{x^{n-1}}{(n-1)!}y^{k-1} = \sum_{n,k} S(n+1,k+1)\frac{x^n}{n!}y^k$$

We then consider multiplying the reurrence relation by $\frac{x^n}{n!}y^k$, and summing over n, k, using the above calculations and the generating function for $S(n,k)$ gives that the LHS simplifies to

$$\frac{\partial}{\partial x}\frac{1}{y}(e^{y(e^x-1)}-1) = e^{y(e^x-1)}e^x$$

The RHS simplifies to

$$e^{y(e^x-1)} + \frac{\partial}{\partial y}e^{y(e^x-1)} = e^{y(e^x-1)} + e^{y(e^x-1)}(e^x-1) = e^{y(e^x-1)}e^x$$

Thus the generating function for both sides of the relation are the same, so that the relation holds.

2.b. The exponential generating function for $n!$ is

$$\sum_n n!\frac{x^n}{n!} = \sum_n x^n = \frac{1}{1-x}.$$

By the formula for the product of exponential generating functions, the exponential generating function for $\sum_k \binom{n}{k} D_{n-k}$ is the product of the exponential generating functions of D_n and the exponential generating function for 1. The exponential generating function for 1 is $\sum_n \frac{x^n}{n!} = e^x$, and thus the product is

$$\frac{e^{-x}}{1-x}\cdot e^x = \frac{1}{1-x}.$$

Thus the exponential generating functions for both sides of the identity are equal, and thus the identity holds.