Math 184A Homework 5 Solution

Question 1 (Combinatorial Identity, 20 points)**.** *Come up with a combinatorial proof of the following identity for $n \geq 2m > 0$:*

$$\sum_{k=m}^{n-m} \binom{n}{k} c(k,m)c(n-k,m) = \binom{2m}{m} c(n,2m).$$

Solution. We count the same number twice to prove this identity. We color the numbers in $[n]$ with red or blue. We say a cycle is red(blue) if all numbers in the cycle are red(blue). We want to count the number of colored permutations with $2m$ cycles, m of which are red while the rest m cycles are blue. We count in the following two ways:

(1) suppose k of the numbers are red, then $n-k$ are blue. we have $\binom{n}{k}$ ways to pick such k numbers out of $[n]$ and color them red(the rest are automatically blue). Then, in each coloring, we have $c(k,m)$ ways to distribute the k red numbers into m cycles and $c(n-k,m)$ ways to distribute the $n-k$ blue numbers into m cycles. Note that we have at least m red numbers and m blue numbers, so totally we have $\sum_{k=m}^{n-m} \binom{n}{k} c(k,m)c(n-k,m)$ such colored permutations.

(2) We first distribute the set $[n]$ into $2m$ cycles, in $c(n,2m)$ possible ways. Then, in each way, we color m of the cycles red, in $\binom{2m}{m}$ ways. Then, the rest cycles are automatically blue. Totally we have $\binom{2m}{m} c(n,2m)$ such colored permutations.

We can say now the number of colored permutations of $[n]$ into $2m$ cycles with m cycles red and m cycles blue is

$$\sum_{k=m}^{n-m} \binom{n}{k} c(k,m)c(n-k,m) = \binom{2m}{m} c(n,2m).$$

□

Question 2 (Generating Functions, 50 points)**.** .

(a) *Consider the sequence defined by the recurrence, $a_0 = 0, a_1 = 3$ and*

$$a_{n+2} = a_{n+1} + 2a_n - 6$$

for $n \geq 0$. Find a formula for the generating function $A(x) = \sum_{n=0}^{\infty} a_n x^n$. [10 points]

(b) *Using this generating function find a formula for a_n (you will want to find a partial fractions decomposition). [10 points]*

(c) *Consider the sequence defined by the recurrence, $b_0 = 0$ and*

$$b_n = n + \frac{2}{n}\sum_{i=0}^{n-1} b_i.$$

Find a differential equation satisfied by the generating function $B(x) = \sum_{n=0}^{\infty} b_n x^n$ (you do not have to solve it). You may need to use the identity that

$$\sum_{n=0}^{\infty} (n+1)^2 x^n = \frac{1+x}{(1-x)^3}.$$

Note: For those of you interested in computer science, b_n is related to the runtime of the quicksort algorithm. [15 points]

(d) *It turns out that the generating function above is given by*

$$B(x) = \frac{2\log\left(\frac{1}{1-x}\right) - x}{(1-x)^2}.$$

Use this to give a formula for b_n. You may need to use the harmonic numbers $H_k = \sum_{n=1}^{k} \frac{1}{n} \approx \log(k)$ to express your answer. Recall that $\log(1/(1-x)) = \sum_{n=1}^{\infty} x^n/n$. [15 points]

Solution.

(a)

$$\begin{aligned}
A(x) &= \sum_{n=0}^{\infty} a_n x^n = 0 + 3x + \sum_{n=2}^{\infty} a_n x^n \\
&= 3x + \sum_{n=2}^{\infty} (a_{n-1} + 2a_{n-2} - 6)x^n \\
&= 3x + \sum_{n=2}^{\infty} a_{n-1}x^n + \sum_{n=2}^{\infty} 2a_{n-2}x^n + \sum_{n=2}^{\infty} -6x^n \\
&= 3x + \sum_{n=1}^{\infty} a_n x^{n+1} + \sum_{n=0}^{\infty} 2a_n x^{n+2} - \frac{6x^2}{1-x} \\
&= 3x + x\sum_{n=1}^{\infty} a_n x^n + x^2 \sum_{n=0}^{\infty} 2a_n x^n - \frac{6x^2}{1-x} \\
&= 3x + (x+2x^2)\sum_{n=0}^{\infty} a_n x^n - \frac{6x^2}{1-x} \\
&= 3x + (x+2x^2)A(x) - \frac{6x^2}{1-x},
\end{aligned}$$

so we have

$$A(x) = \frac{\frac{6x^2}{1-x} - 3x}{2x^2 + x - 1} = \frac{3x - 9x^2}{(1-x)(1+x)(1-2x)}.$$

(b) Writing $A(x)$ in partial fraction expansion(recall your Math 20B), we have

$$\begin{aligned}
A(x) &= \sum_{n=0}^{\infty} a_n x^n \\
&= \frac{3}{1-x} - \frac{2}{x+1} - \frac{1}{1-2x} \\
&= 3\sum_{n=0}^{\infty} x^n - \sum_{n=0}^{\infty} (-x)^n - \sum_{n=0}^{\infty} (2x)^n \\
&= \sum_{n=0}^{\infty} (3 - 2(-1)^n - 2^n)x^n,
\end{aligned}$$

so

$$a_n = 3 - 2(-1)^n - 2^n.$$

(c) The identity is equivalent to

$$nb_n = n^2 + 2\sum_{i=0}^{n-1} b_i.$$

We multiply both sides by x^{n-1} to obtain

$$nb_n x^{n-1} = n^2 x^{n-1} + 2(\sum_{i=0}^{n-1} b_i)x^{n-1}.$$

Sum both sides for n from 1 to ∞, we have

$$\sum_{n=1}^{\infty} nb_n x^{n-1} = \sum_{n=1}^{\infty} n^2 x^{n-1} + 2(\sum_{i=0}^{n-1} b_i)x^{n-1}, \text{ or}$$

$$\sum_{n=1}^{\infty} nb_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)^2 x^n + 2\sum_{n=0}^{\infty}(\sum_{i=0}^{n} b_i)x^n.$$

We have two identities for generating function which help us to simplify the equation above. The first one is

$$B'(x) = (\sum_{n=0}^{\infty} b_n x^n)' = \sum_{n=0}^{\infty} (b_n x^n)' = \sum_{n=1}^{\infty} nb_n x^{n-1},$$

the second one is

$$\frac{B(x)}{1-x} = B(x) \cdot \frac{1}{1-x} = (\sum_{n=0}^{\infty} b_n x^n) \cdot (\sum_{n=0}^{\infty} x^n) = \sum_{n=0}^{\infty}(\sum_{i=0}^{n} b_i)x^n.$$

Based on these two identities, and also the identity given in the question that

$$\sum_{n=0}^{\infty} (n+1)^2 x^n = \frac{1+x}{(1-x)^3},$$

our equation becomes a differential equation about $B(x)$ that

$$B'(x) = \frac{1+x}{(1-x)^3} + \frac{2B(x)}{1-x}.$$

(d) We need the following 2 identities:

$$\frac{1}{(1-x)^2} = \left(\frac{1}{1-x}\right)' = (\sum_{i=0}^{\infty} x^n)' = \sum_{i=0}^{\infty} nx^{n-1} = \sum_{i=0}^{\infty} (n+1)x^n$$

$$\text{and } \log(\frac{1}{1-x}) = \sum_{n=1}^{\infty} \frac{x^n}{n}.$$

Now, let's write $B(x)$ in power series:

$$\begin{aligned}
B(x) &= (2\log\left(\frac{1}{1-x}\right) - x) \cdot \frac{1}{(1-x)^2} \\
&= ((\sum_{n=1}^{\infty} \frac{x^n}{n}) - x)(\sum_{i=0}^{\infty} (n+1)x^n) \\
&= \sum_{n=1}^{\infty} ((\sum_{k=1}^{n} \frac{2}{k}(n-k+1)) - n)x^n \\
&= \sum_{n=1}^{\infty} ((2(n+1)\sum_{k=1}^{n} \frac{1}{k}) - 2n - n)x^n \\
&= \sum_{n=1}^{\infty} ((2(n+1)H_n - 3n)x^n.
\end{aligned}$$

So, $b_n = 2(n+1)H_n - 3n$.

$\square$

Question 3 (Partition Generating Functions, 30 points)**.** *(a) Let a_n be the number of integer partitions of n into distinct parts. Show that this sequence has the generating function*

$$\sum a_n x^n = (1+x)(1+x^2)(1+x^3)\cdots = \prod_{n=1}^{\infty}(1+x^n).$$

[10 points]

(b) Let b_n be the number of integer partitions of n into odd parts. Show that this sequence has the generating function

$$\sum b_n x^n = \frac{1}{(1-x)(1-x^3)(1-x^5)\cdots} = \prod_{n=1}^{\infty}\frac{1}{1-x^{2n-1}}.$$

[10 points]

(c) Show directly that the above generating functions are equal. [10 points]

Solution.

(a) Like the solution of **Example 8.9** in the text book, we solve this question by analyze the coefficient of x^n of both sides. The RHS is a infinite product. We can take it as a sum of infinite products. In a infinite product term contributing to x^n, if x^{j_i} is taken in the j_i^{th} parenthesis for $i = 1, \ldots, k$ and 1 is taken in all the other parentheses, we will get $\sum_{i=1}^{k} j_i = n$, thus we get a partition, $(j_k, j_{k-1}, \ldots, j_1)$, of n into k distinct parts. Conversely, every partition of n into distinct parts can be associated to a product on the RHS, meaning that the coefficient of x^n, a_n, is the number of partitions of n into distinct parts.

(b) We want to show that

$$\sum b_n x^n = \prod_{n=1}^{\infty}(1 + x^{2n-1} + (x^{2n-1})^2 + (x^{2n-1})^3 + \cdots).$$

The RHS is also a sum of infinite products. In a infinite product term contributing to x^n, if $(x^{2i-1})^{\alpha_i}$ is taken in the i^{th} parenthesis, we will get $\sum_{i=1}^{\infty}(2i-1)\alpha_i = n$ and a partition $(\prod_{i=1}^{\infty}(2i-1)^{\alpha_i})$ of n into odd parts. Conversely, every partition of n into odd parts can be associated to a product on the RHS(if the partition contains α_i parts of size $(2i-1)$, we take $(x^{2i-1})^{\alpha_i}$ in the i^{th} parenthesis), meaning that the coefficient of x^n, a_n, is the number of partitions of n into odd parts.

(c) Using direct computation,

$$\begin{aligned}
\sum a_n x^n &= \prod_{n=1}^{\infty}(1+x^n) \\
&= \prod_{n=1}^{\infty}\frac{1-x^{2n}}{1-x^n} \\
&= \frac{\prod_{n=1}^{\infty} 1-x^{2n}}{\prod_{n=1}^{\infty} 1-x^n} \\
&= \frac{\prod_{n>0\ even} 1-x^n}{\prod_{n>0\ even\ and\ odd} 1-x^n} \\
&= \frac{1}{\prod_{n>0\ odd} 1-x^n} \\
&= \sum b_n x^n,
\end{aligned}$$

so that the two generating functions are equal.

$\square$