Solutions to HW9 of Math 103A, Fall 2018

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(1) (P.133 2.) This is not a homomorphism. For example,

$$\phi(0.5) = 0, \quad \phi(0.5) + \phi(0.5) = 0, \quad \text{but } \phi(0.5 + 0.5) = \phi(1) = 1.$$

(2) (P.133 3.) ϕ is a homomorphism. Because for any $x, y \in \mathbb{R}^*$,

$$\phi(xy) = |xy| = |x||y| = \phi(x)\phi(y).$$

(3) (P.133 10.) ϕ is a homomorphism since for any continuous functions $f, g \in F$,

$$\phi(f + g) = \int_0^4 [f(x) + g(x)]dx = \int_0^4 f(x)dx + \int_0^4 g(x)dx = \phi(f) + \phi(g),$$

which is a result from calculus.

(4) (P.133 12.) ϕ is not a homomorphism. For example, let

$$A = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Then $A + B = I$ and

$$\det(A + B) = \det(I) = 1 \neq 0 = \det(A) + \det(B).$$

(5) (P.134 20.) We can compute $\phi(k)$ for any $k \in \mathbb{Z}_{10}$ using the fact that ϕ is a homomorphism:

$$\phi(2) = \phi(1 + 1) = \phi(1) + \phi(1) = 8 + 8 = 16,$$

$$\phi(3) = \phi(2+1) = \phi(2) + \phi(1) = 16 + 8 = 24 = 4 \quad \text{in } \mathbb{Z}_{20},$$

and so on. So $\phi(k) = 0$ for some integer k if and only if 20 divides $8k$. For $0 \leq k \leq 9$, only $0, 5$ satisfy this. Thus

$$\ker(\phi) = \{0, 5\}.$$

(6) (P.134 22.) For any integers r, s,

$$\phi(r, s) = r\phi(1, 0) + s\phi(0, 1) = 3r - 5s.$$

Then

$$\phi(-3, 2) = -9 - 10 = -19.$$

$(r, s) \in \ker(\phi)$ if and only if $3r - 5s = 0$. So

$$\ker(\phi) = \{(r, s) \in \mathbb{Z} \times \mathbb{Z} : 3r = 5s\}.$$

(7) (P.135 44.) Since G is finite, we can write $G = \{g_1, \cdots, g_m\}$. So $\phi[G] = \{\phi(g_1), \cdots, \phi(g_m)\}$ is also finite. Let $K = \ker(\phi)$ and Σ be the set of left cosets of K. We consider a mapping

$$\psi : \Sigma \to \phi[G], \quad \psi(gK) = \phi(g).$$

First we need to show ψ is well-defined. If $gK = g_1K$ for some $g, g_1 \in G$, then $g_1^{-1}g \in K = \ker(\phi)$, which means $\phi(g_1^{-1}g) = e$ and thus $\phi(g) = \phi(g_1)$. So ψ is well-define.
ψ is an injection: if $\phi(g) = \phi(h)$, then $\phi(h^{-1}g) = \phi(h)^{-1}\phi(g) = e$. So $h^{-1}g \in K$ and $gK = hK$.
ψ is onto: For any $\phi(g) \in \phi[G]$, clearly $\psi(gK) = \phi(g)$.
Therefore, ψ is a bijection and

$$|\phi[G]| = |\Sigma| = (G : K) = \frac{|G|}{|K|},$$

which is a divisor of $|G|$.

(8) (P.135 49.) For any $g_1, g_2 \in G$,

$$(\gamma\phi)(g_1g_2) = \gamma(\phi(g_1g_2)) = \gamma(\phi(g_1)\phi(g_2)) = \gamma(\phi(g_1))\gamma(\phi(g_2)) = (\gamma\phi)(g_1)(\gamma\phi)(g_2).$$

So $\gamma\phi$ is also a homomorphism.

(9) (P.135 50.)

$$\begin{aligned} & \phi[G] \text{ is abelian} \\ \iff & \phi(x)\phi(y) = \phi(y)\phi(x), \quad \forall x, y \in G, \\ \iff & \phi(xy) = \phi(yx), \quad \forall x, y \in G, \\ \iff & \phi(xy)\phi(yx)^{-1} = e, \quad \forall x, y \in G, \\ \iff & \phi(xy(yx)^{-1}) = e, \quad \forall x, y \in G, \\ \iff & \phi(xyx^{-1}y^{-1}) = e, \quad \forall x, y \in G, \\ \iff & xyx^{-1}y^{-1} \in \ker(\phi), \quad \forall x, y \in G. \end{aligned}$$

(10) (P.135 53.) Suppose ϕ is a homomorphism, we can try to derive some simple necessary conclusions. Since ϕ is a homomorphism,

$$h^m k^n h^p k^q = \phi(m,n)\phi(p,q) = \phi(m+p, n+q) = h^{m+p}k^{n+q} = h^m h^p k^n k^q,$$

for any integers m, n, p, q. We can then cancel h^m and k^q to get

$$k^n h^p = h^p k^n, \quad \forall n, p \in \mathbb{Z}.$$

So in particular,

$$kh = hk.$$

And we can show that this is an equivalence condition for ϕ to be a homomorphism. Now suppose $kh = hk$, then for any integers m, n, p, q,

$$\phi(m,n)\phi(p,q) = h^m k^n h^p k^q = h^m h^p k^n k^q = \phi(m+p, n+q).$$

So ϕ is a homomorphism and thus

$$kh = hk$$

is indeed an equivalence condition for ϕ to be a homomorphism.

(11) (P.135 55.) We claim that ϕ is a homomorphism if and only if $h^n = e$.

Proof. ($\Rightarrow$) Since ϕ is a homomorphism, in particular, it is a well-defined map. So

$$e = \phi(0) = \phi(n) = h^n.$$

($\Leftarrow$) Suppose $h^n = e$. We first need to check that ϕ is well-defined. Given any integers i, j such that $i \equiv j (\text{mod } n)$, we have $i = j + kn$ for some integer k. Then

$$h^i = h^{j+kn} = h^j (h^n) k = h^j.$$

So ϕ is well-defined. Then for any integers p, q,

$$\phi(p+q) = h^{p+q} = h^p h^q = \phi(p)\phi(q).$$

So ϕ is a homomorphism. (Note that ϕ is trivially a homomorphism if we can show that ϕ is well-defined.) $\square$

(12) (P.142 6.) The order of 4 in $\mathbb{Z}_{12}$ is 3 and the order of 3 in $\mathbb{Z}_{18}$ is 6. By Theorem 11.9, $|\langle(4,3)\rangle| = \text{lcm}(3,6) = 6$. So

$$|\mathbb{Z}_{12} \times \mathbb{Z}_{18}/\langle(4,3)\rangle| = \frac{|\mathbb{Z}_{12} \times \mathbb{Z}_{18}|}{|\langle(4,3)\rangle|} = \frac{12 \cdot 18}{6} = 36.$$

(13) (P.142 7.) The order of 1 in $\mathbb{Z}_2$ is 2 and the order of ρ_1 in S_3 is 3. So $|\langle(1,\rho_1)\rangle| = 6$ and

$$|\mathbb{Z}_2 \times S_3/\langle(1,\rho_1)\rangle| = \frac{6}{6} = 1.$$

(14) (P.142 12.)

$$2((3,1) + \langle(1,1)\rangle) = (2,2) + \langle(1,1)\rangle,$$

$$3((3,1) + \langle(1,1)\rangle) = (1,3) + \langle(1,1)\rangle,$$

$$4((3,1) + \langle(1,1)\rangle) = (0,0) + \langle(1,1)\rangle.$$

So the order of $(3,1) + \langle(1,1)\rangle)$ in $\mathbb{Z}_4 \times \mathbb{Z}_4/\langle(1,1)\rangle$ is 4.

(15) (P.142 22.) For any $x \in G/H$, there is $g \in G$ such that $x = gH$. Since G is a torsion group, there is integer $n > 0$ such that $g^n = e$. Then $x^n = (gH)^n = g^n H = H$ which is the identity of G/H. So G/H is also a torsion group.

(16) (P.143 24.) Recall from HW7 (2) that $\text{sgn} : S_n \to \mathbb{Z}_2$ is a homomorphism. Since for any odd permutation $\sigma, \text{sgn}(\sigma) = 1$. So sgn is onto. And note that $\ker(\text{sgn}) = A_n$. Thus A_n is a normal subgroup of S_n. By the fundamental homomorphism theorem,

$$S_n/A_n = S_n/\ker(\text{sgn}) \cong \text{im}(\text{sgn}) = \mathbb{Z}_2.$$

(17) (P.144 26.) T is defined to be the set consisting of elements in G of finite order. We first show that T is a subgroup of G. Clearly $e \in T$. For any $g \in T$, there is integer n such that $g^n = e$. Then $(g^{-1})^n = (g^n)^{-1} = e$. So $g^{-1} \in T$. For any $g, h \in T$, there are integers n, m such that $g^n = h^m = e$. Then

$$(gh)^{nm} = g^{nm}h^{nm} = (g^n)^m(h^m)^n = e.$$

So $gh \in T$. Thus T is a subgroup. Since G is abelian, T is also normal.

Suppose G/T is not torsion free, then there is $g \in G$, such that $gT \neq T$ and the order of gT is finite. So there is integer $k > 0$ such that $(gT)^k = T$. Then $g^k \in T$. But then there is l, such that that $g^{kl} = (g^k)^l = e$. So $g \in T$ and $gT = T$, which is a contradiction.

(18) (P.144 27.) We define

$$H \sim K \quad \iff \quad \exists\, g \in G, i_g[H] = K.$$

Since for any $H \leq G$, $eHe^{-1} = i_e[H] = H$,$H \sim H$.

Suppose $H \sim K$, then there is $g \in G$, such that $gHg^{-1} = K$. Then $g^{-1}Kg = H$ and thus $K \sim H$.

If $H \sim K, K \sim L$, then there are $a, b \in G$, such that

$$aHa^{-1} = K, \quad bKb^{-1} = L.$$

Then

$$(ba)H(ba)^{-1} = b(aHa^{-1})b^{-1} = bKb^{-1} = L.$$

Thus $H \sim L$. Therefore, $\sim$ is an equivalence relation.

(19) (P.143 32.) Consider

$$\Sigma = \{H | H \text{ is a normal subgroup of } G \text{ and } S \subseteq H\}.$$

Σ is nonempty since $G \in \Sigma$. Now we define

$$N = \cap_{H \in \Sigma} H.$$

N is nonempty since $e \in N$. By Exercise 31, N is also a normal subgroup. So N is the smallest normal subgroup containing S in the sense that if N_1 is another normal subgroup and $S \subseteq N_1$, then $N \subseteq N_1$.

(20) (P.143 33.) For any $a, b \in G$,

$$(aC)(bC)(aC)^{-1}(bC)^{-1} = aba^{-1}b^{-1}C = C.$$

So G/C is abelian.