Vector Calculus and Line Integrals

Reading: Textbook, §16.1 - 16.4

§1. Vector fields

As an abstraction of real life examples of winds, water currents, electric/magnectic fields etc, we make the following

Def. Let $D \subseteq \mathbb{R}^2$ be a planar region. A *vector field* on D is a function $\vec{F}$ that assigns to each point $(x,y) \in D$ a 2-dimensional vector $\vec{F}(x,y)$. Thus

$$\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}.$$

The components $P(x,y)$, $Q(x,y)$ are just functions (scalar fields) on D.

Likewise, a 3-dimensional vector field on a region $B \subseteq \mathbb{R}^3$ is a function $\vec{F}$ that assigns to each point $(x,y,z) \in B$ a 3-dimensional vector $\vec{F}(x,y,z)$. Thus

$$\vec{F}(x,y,z) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}.$$

Eg. Sketch the vector field $\vec{F}(x,y) = \dfrac{y\vec{i} - x\vec{j}}{\sqrt{x^2+y^2}}$.

Eg. Given a function $f(x,y)$, the gradient vector field of f is

$$(\vec{\nabla} f)(x,y) := f_x\vec{i} + f_y\vec{j}.$$

It is orthogonal everywhere to the level curves of f. Find the gradient vector field of $f(x,y) = 4x^2 + 9y^2$.

Vector fields on D can be thought of as families of vectors that vary according to positions in D. Thus you can

- $\|\vec{F}\|$: measuring the magnitude of each vector at any point in D (scalar field)
- $\vec{F} \cdot \vec{G}$: measuring the dot product of vectors at any point in D (scalar field)
- $\vec{F} \times \vec{G}$: measuring the cross product of vectors at any point in D (vector field)

Def A vector field $\vec{F}(x,y,z)$ ($\vec{F}(x,y)$) on a domain D is called conservative if the is a function $f(x,y,z)$ ($f(x,y)$) s.t.

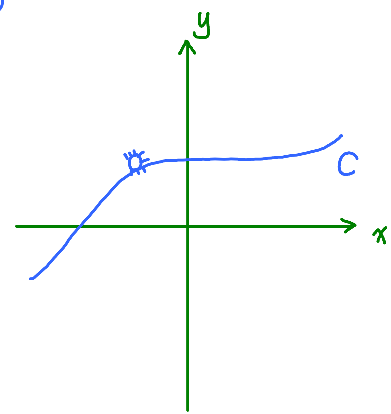
$$\vec{F}(x,y,z) = (\vec{\nabla} f)(x,y,z) \quad (\vec{F}(x,y) = (\vec{\nabla} f)(x,y))$$

The function f is called a potential for $\vec{F}$ (not unique if it exists).

§2 Line integrals

There are two types of line integrals: scalar and vector.

The scalar integral



It is the Riemann sum:

$$\int_C f(x,y)\,ds := \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i, y_i)\,\Delta s_i .$$

Since, infinitesimally,

$$\Delta s = \sqrt{\Delta x^2 + \Delta y^2} = \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}\ \Delta t$$

In the limit, we get, for C parametrized by $t \in [a,b]$.

$$\boxed{\int_C f(x,y)\,ds = \int_a^b f(x(t), y(t)) \cdot \sqrt{x'(t)^2 + y'(t)^2}\ dt}$$

Eg. Evaluate the integral $\int_C y^3 ds$, where C is given by

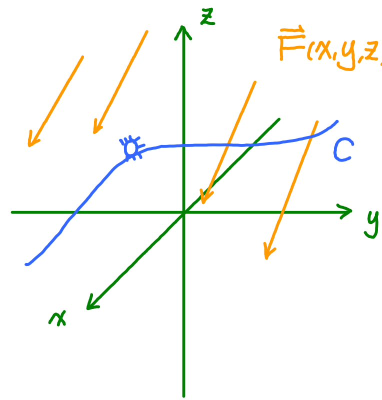
$$\vec{r}(t) = (t^3, t), \quad t \in [0,2].$$

The 3-dim'l case is similar: if C is a curve in $\mathbb{R}^3$ given by $\vec{r}(t) = (x(t), y(t), z(t))$, $t \in [a,b]$, then

$$\boxed{\int_C f(x,y,z)\,ds = \int_a^b f(x(t), y(t), z(t)) \cdot \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt}$$

Eg Find the integral $\int_C x e^{yz} ds$, where C is the line segment from $(0,0,0)$ to $(1,2,3)$.

Next, we investigate vector integrals :



Suppose $\vec{F}$ is a force field (wind) in $\mathbb{R}^3$, the work done by moving the particle along C against $\vec{F}$ is

$$\lim_{n\to\infty} \sum_{i=1}^{n} \vec{F}(x_i, y_i, z_i) \cdot \Delta\vec{r}_i =: \int_C \vec{F} \cdot d\vec{r}$$

If $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$, $d\vec{r} = (dx, dy, dz)$, then

$$\boxed{\vec{F} \cdot d\vec{r} = P\,dx + Q\,dy + R\,dz}$$

- How do we calculate $I = \int_C P\,dx + Q\,dy + R\,dz$?

If C is parametrized by $\vec{r} = (x(t), y(t), z(t))$, $t \in [a, b]$, then

$$dx(t) = x'(t)\,dt, \quad dy(t) = y'(t)\,dt, \quad dz(t) = z'(t)\,dt$$

and

$$I = \int_a^b P(x(t), y(t), z(t))\,x'(t)\,dt + \int_a^b Q(x(t), y(t), z(t))\,y'(t)\,dt$$
$$+ \int_a^b R(x(t), y(t), z(t))\,z'(t)\,dt.$$

Warning: A big difference between scalar and vector line integrals is that, if you reverse the direction / orientation of C,

- the scalar integral remains unchanged
- the vector integral picks up a "$-$" sign.

Eg. Evaluate $\int_C xye^{yz}\,dy$, where $C: \vec{r}(t) = (t, t^2, t^3)$ with $t \in [0,1]$.

Eg. Find the work done for moving a particle against the force field $\vec{F}(x,y,z) = \sin x\,\vec{i} + \cos y\,\vec{j} + xz\vec{k}$ along C:
$\vec{r}(t) = t^3\vec{i} - t^2\vec{j} + t\vec{k}$, $t \in [0,1]$.

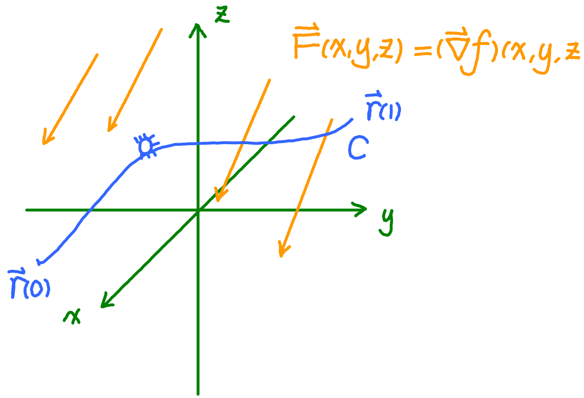
§3. The fundamental theorem of line integrals

The theorem states that, if you perform line integral for the total derivative

$$df = f_x\,dx + f_y\,dy + f_z\,dz$$

of a function along a curve $C: \vec{r} = \vec{r}(t)$, $t \in [0,1]$, ($\Longleftrightarrow$ vector integral of a conservative vector field), then, the outcome is just the difference between the ending and starting values of f.

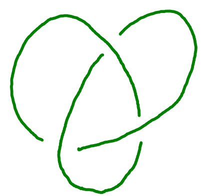
$$\boxed{\int_C df = \int_C f_x\,dx + f_y\,dy + f_z\,dz = f(\vec{r}(1)) - f(\vec{r}(0))}$$

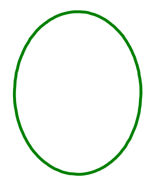


The proof reduces to the fundamental theorem of calculus in one variable, if you follow the definition of vector line integrals.

Eg. If $\vec{F}(x,y) = xy^2\vec{i} + x^2y\vec{j}$, find the integral $\int_C \vec{F}\cdot d\vec{r}$ where C : $\vec{r}(t) = (t\sin\frac{1}{2}\pi t,\ t+\cos\frac{1}{2}\pi t)$, $t\in[0,1]$.

The theorem implies that, if $\vec{F} = \vec{\nabla} f$ is conservative, then, along any closed (oriented) curve C, $\int_C \vec{F}\cdot d\vec{r} = 0$.

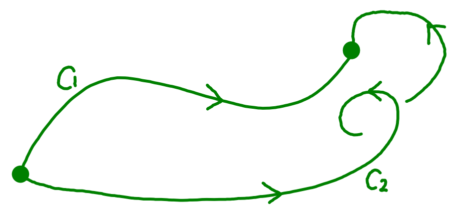




closed curves in $\mathbb{R}^3/\mathbb{R}^2$

- In turn, if C_1, C_2 are two curves in $\mathbb{R}^3$ starting and ending at the same points, and $\vec{F} = \vec{\nabla} f$ is conservative, then

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}.$$



This property is called paths independence.

Eg. Check if the following vector fields on their domains satisfy the path-independence property.

(1) $\vec{F}(x,y) = (xy^2,\ x^2y)$ (2) $\vec{F}(x,y) = \frac{(-y,\ x)}{\sqrt{x^2+y^2}}$

Thm. The integral $\int_C \vec{F}\cdot d\vec{r}$ is path-independent on a domain D if and only if $\int_L \vec{F}\cdot d\vec{r} = 0$ for any closed curve L in D.

Def. A region D is called connected if any two points in D can be joined by a curve in D. D is called open if for any point in D, there is a small enough disk/ball containing the point and contained in $\mathcal{D}$.

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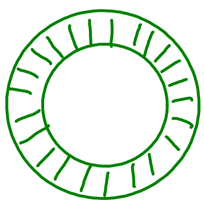
: open, connected domains



Non open ,

disconnected

Thm. Suppose $\vec{F}$ is a vector field on an open, connected region D. If $\int_C \vec{F} \cdot d\vec{r}$ is path-independent, then $\vec{F}$ is a conservative vector field on D.

This thm is a first step towards checking if a vector field $\vec{F}$ is conservative, but is hard to apply using integration along all paths.

Instead, we try to develop a differentiation criterion, at the cost of restricting the types of D.

Suppose now $D \subseteq \mathbb{R}^2$, $\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$.

Thm. If $\vec{F}$ is conservative, then $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$

Eg. Determine if the vector field $\vec{F}(x,y) = e^x \cos y\,\vec{i} + e^x \sin y\,\vec{j}$ is conservative or not.

- Q: Is the converse of the theorem above true?

Def. A region $D \subseteq \mathbb{R}^2$ is called simply-connected if any closed curve in D can continuously shrink to a point within D.

simply connected

non simply connected

Thm. Let $\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$ be a vector field on a simply-connected domain D. If $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ holds on D, then $\vec{F}$ is conservative.

Eg. Show that the integral $\int_C 2xe^{-y}dx + (2y - x^2e^{-y})dy$ has path-independence property. Evaluate the integral along a curve from $(1,0)$ to $(2,1)$.

What about $D \subseteq \mathbb{R}^3$, and $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$?

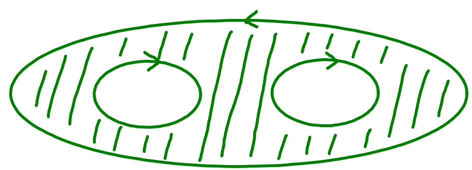
Eg. Show that, if $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is conservative on $D \subseteq \mathbb{R}^3$, then $P_y = Q_x$, $P_z = R_x$, $Q_z = R_y$.

The converse of this result will be deferred later.

§4. Green's Theorem

Green's Theorem relates two important concepts we have learnt so far: vector line integral and double integral. This is a second generalization of the fundamental theorem of calculus.

Thm. If $F(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$ is a vector field on $\mathbb{R}^2$, and D is a bounded closed region whose outer boundaries are oriented counter-clockwise and inner boundaries are oriented clockwise:



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Then:

$$\int_{\partial D} \vec{F}\cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Warning: $\vec{F}$ need only be well-defined in an neighborhood of D, but can not have singularities in D!

Eg. Evaluate the integral by two methods, directly and using Green's theorem.

$$\oint_C (x-y)dx + (x+y)dy$$

where C is the circle centered at $(0,0)$ of radius 2.

Eg. Use Green's theorem to evaluate the integral:

$$\oint_C (y + e^{\sqrt{x}})\,dx + (2x + \cos y^2)\,dy$$

where C is the boundary of the region enclosed by the parabolas $y = x^2$ and $x = y^2$.

Another example of a region with multiple boundary curves.

Eg. If $\vec{F}(x,y) = \frac{-y\vec{i} + x\vec{j}}{x^2+y^2}$, show that $\oint_C \vec{F}\cdot d\vec{r} = 2\pi$ for any simple closed curve encircling the origin.

§5. Differential forms

Let us give a simple way to help you remember Green's theorem algebraically.

Introduce the oriented infinitesimal symbols

$$dx, \quad dy, \quad dxdy$$

dx, dy measure the oriented infinitesimal displacement in the x, y directions, while $dxdy$ measures the oriented infinitesimal area on the xy-plane.

We have the Grassman rule for multiplying these symbols:

$$dx\,dx = 0\,, \quad dy\,dy = 0, \quad dx\,dy = -\,dy\,dx$$

Then,

$$\begin{aligned} d(P(x,y)dx + Q(x,y)dy) &= \left(\frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy\right)dx + \left(\frac{\partial Q}{\partial x}dx + \frac{\partial Q}{\partial y}dy\right)dy \\ &= \frac{\partial P}{\partial x}dxdx + \frac{\partial P}{\partial y}dydx + \frac{\partial Q}{\partial x}dxdy + \frac{\partial Q}{\partial y}dydy \\ &= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy \end{aligned}$$

Thus Green's Theorem $\Longleftrightarrow$

$$\boxed{\int_{\partial D} P(x,y)dx + Q(x,y)dy = \iint_D d(P(x,y)dx + Q(x,y)dy)}$$

Or, if $\omega = P(x,y)dx + Q(x,y)dy$ is a differential 1-form, and D is a 2-dim'l bounded region in $\mathbb{R}^2$:

$$\int_{\partial D} \omega = \int_D d\omega .$$

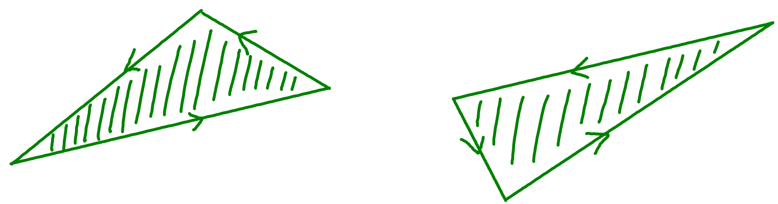
Similarly, the fundamental theorem of line integral applies to differential 0-forms (functions) and 1-dimensional regions (oriented curves) C in $\mathbb{R}^2/\mathbb{R}^3$:

$$\int_{\partial C} f = \int_C df$$

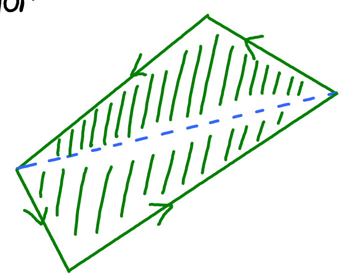
Here the l.h.s. means $f(b) - f(a)$, where a is the starting point of C and b is the end point of C.

§6. Sketch of proof of Green's Theorem

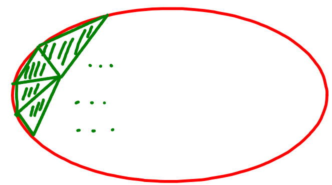
If GT holds for triangles:



then it holds for

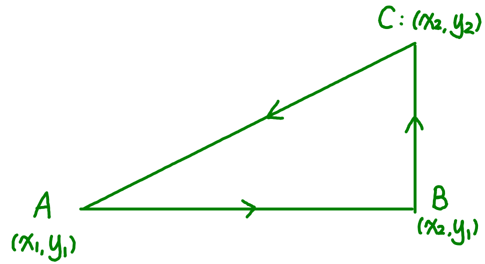


Thus we reduce GT to triangles by approximating D by many small triangles:



Zoom in on each of these triangles, and using the additivity of integrations, we just need to show

$$\int_{\partial T} P\,dx = -\iint_T \frac{\partial P}{\partial y}\,dx\,dy, \qquad \int_{\partial T} Q\,dy = \iint_T \frac{\partial Q}{\partial x}\,dx\,dy.$$



Then

$$\iint_T -\frac{\partial P}{\partial y}\,dx\,dy = -\int_{x_1}^{x_2} dx \int_{y_1}^{\frac{y_2-y_1}{x_2-x_1}(x-x_1)+y_1} \frac{\partial P}{\partial y}\,dy$$

$$= -\int_{x_1}^{x_2} dx \left(P\left(x, \frac{y_2-y_1}{x_2-x_1}(x-x_1)+y_1\right) - P(x, y_1)\right)$$

Next, parametrize AB, CA by their x-coordinates ($\int P\,dx$ on BC is 0!):

$$AB: \ \vec{r}_1(x) = (x, y_1), \qquad AC: \ \vec{r}_2(x) = \left(x, \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) + y_1\right)$$

with $x \in [x_1, x_2]$, then

$$\begin{aligned} \int_{\partial T} P\,dx &= \int_{AB} P\,dx - \int_{AC} P\,dx \\ &= \int_{x_1}^{x_2} P(x, y_1)\,dx - \int_{x_1}^{x_2} P\left(x, \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) + y_1\right) dx \end{aligned}$$

Comparing both sides gives us GT for T.

We will next move on towards establishing Green's Theorem for vector fields in $\mathbb{R}^3$.