Surface Integral and Stokes' Theorem

Note Title 1/4/2022

Reading: Textbook, §16.7-16.8

§1. Surface integrals

Just as for line integrals, there are two types of surface integrals: scalar and vector.

• Surface scalar integral

If $f(x,y,z)$ is a smooth function defined in a neighborhood of a parametric surface $S: \vec{r} = \vec{r}(u,v)$, $(u,v) \in D$, then the surface integral of $f(x,y,z)$ is, by definition,

$$\iint_S f(x,y,z)\,dA := \iint_D f(x(u,v), y(u,v), z(u,v))\, \|\vec{r}_u \times \vec{r}_v\|\, du\,dv.$$

In particular, surface integral of $f(x,y,z) \equiv 1$ recovers the area formula of a surface.

Eg. If the surface is given as the graph of $z = g(x,y)$, show that the surface scalar integral is given by

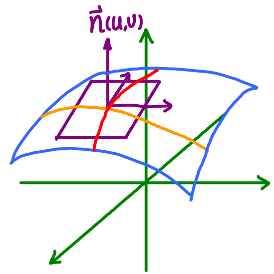
$$\iint_D f(x,y,g(x,y))\sqrt{g_x^2+g_y^2+1}\; dxdy.$$

Eg Evaluate the integral $\iint_S x^2yz\,dA$, where S is the part of the plane $2x+3y-z=-1$ over the rectangle $[0,3]\times[0,2]$.

- Surface vector integral

Analogous to line vector integrals, surface vector integral depends on the orientation of the surface.

The surface $\vec{r} = \vec{r}(u,v)$ is oriented by consistently choosing a unital normal direction $\vec{n}(u,v)$ that varies continuously as (u,v) varies in the domain. This normal direction $\vec{n}(u,v)$ is just a unit normal direction to the tangent plane of S at $\vec{r}(u,v)$



$$\boxed{\vec{n}(u,v) = \frac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}}$$

Warning: Not all surfaces have a consistent choice of continuously varying unit normal vectors. A famous example is the Mobius surface. Surfaces that do have a consistent choice of $\vec{n}(u,v)$ are called orientable, and the choice of $\vec{n}(u,v)$ at each $\vec{r}(u,v)$ is called an orientation.

Def. Let $S: \vec{r} = \vec{r}(u,v)$ be a parametric surface where $(u,v) \in D$. Suppose $\vec{F}(x,y,z)$ is a vector field defined near S. The *vector integral* of $\vec{F}(x,y,z)$ on S is

$$\boxed{\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(x(u,v), y(u,v), z(u,v)) \cdot \vec{n}(u,v)\, dA}$$

This is also called the *flux* of $\vec{F}(x,y,z)$ through S.

Note that one choice of $\vec{n}(u,v)$ is (u first, v second).

$$\vec{n}(u,v) = \frac{\vec{r}_u(u,v) \times \vec{r}_v(u,v)}{\| \vec{r}_u(u,v) \times \vec{r}_v(u,v) \|}.$$

Taking $-\dfrac{\vec{r}_u(u,v) \times \vec{r}_v(u,v)}{\| \vec{r}_u(u,v) \times \vec{r}_v(u,v) \|}$ is the opposite choice (v first, u second).

Then

$$\vec{n}(u,v)\,dA = \frac{\vec{r}_u(u,v) \times \vec{r}_v(u,v)}{\|\vec{r}_u(u,v) \times \vec{r}_v(u,v)\|} \cdot \|\vec{r}_u(u,v) \times \vec{r}_v(u,v)\|\,du\,dv$$

$$= \vec{r}_u(u,v) \times \vec{r}_v(u,v)\,du\,dv$$

and the vector integral reduces to

$$\boxed{\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(x(u,v), y(u,v), z(u,v)) \cdot (\vec{r}_u(u,v) \times \vec{r}_v(u,v))\,du\,dv}$$

Warning: reversing the orientation for the surface S results in the negative integral value.

Eg. Evaluate the vector integral $\iint_S \vec{F} \cdot d\vec{S}$ for the vector field $\vec{F}(x,y,z) = ze^{xy}\vec{i} - 3ze^{xy}\vec{j} + xy\vec{k}$ along the oriented parametric surface $\vec{r}(u,v) = (u+v,\ u-v,\ 1+2u+v)$, $0 \le u \le 2$, $0 \le v \le 1$.

- Differential forms approach

The oriented infinitesimal surface area is given by

$$d\vec{S} = dydz\vec{i} + dzdx\vec{j} + dxdy\vec{k}$$

Then if a surface is parametrized as

$$\vec{r}(u,v) = (x(u,v),\ y(u,v),\ z(u,v))$$

then

$$
\begin{aligned}
d\vec{S}(u,v) &= dy(u,v)\,dz(u,v)\,\vec{i} + dz(u,v)\,dx(u,v)\,\vec{j} + dx(u,v)\,dy(u,v)\,\vec{k} \\
&= (y_u du + y_v dv)(z_u du + z_v dv)\,\vec{i} + (z_u du + z_v dv)(x_u du + x_v dv)\,\vec{j} \\
&\quad + (x_u du + x_v dv)(y_u du + y_v dv)\,\vec{k} \\
&= (y_u z_v - y_v z_u)\,du\,dv\,\vec{i} + (z_u x_v - z_v x_u)\,\vec{j} + (x_u y_v - x_v y_u)\,\vec{k} \\
&= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} du\,dv \\
&= (\vec{r}_u \times \vec{r}_v)\,du\,dv
\end{aligned}
$$

$$
\Longrightarrow \boxed{\vec{F}\cdot d\vec{S} = \vec{F}(x(u,v), y(u,v), z(u,v)) \cdot (\vec{r}_u(u,v) \times \vec{r}_v(u,v))\,du\,dv}
$$

The left-hand-side is just a differential 2-form, for a vector field $\vec{F}(x,y,z) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$:

$$\boxed{\vec{F}\cdot d\vec{S} = P(x,y,z)\,dydz + Q(x,y,z)\,dzdx + R(x,y,z)\,dxdy}$$

To calculate its integration along an oriented surface, you just need to plug in $x = x(u,v)$, $y = y(u,v)$, $z = z(u,v)$ from the surface parametrization, and remember $du^2 = 0$, $dv^2 = 0$,

$$dudv = -dvdu \quad (\text{preferred oriented area})$$

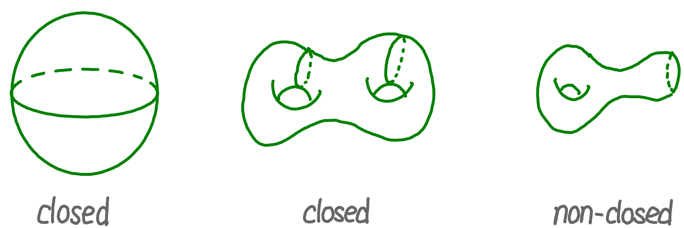
to convert the problem into a double integral.

Eg. Evaluate the vector integral $\iint_S \vec{F} \cdot d\vec{S}$, where

$$\vec{F}(x,y,z) = xy\,\vec{i} + yz\,\vec{j} + zx\,\vec{k}$$

and S is part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the square $0 \le x \le 1$, $0 \le y \le 1$.

A surface S is closed if it has no boundaries:



For a closed surface with outward orientation, $\iint_S \vec{F} \cdot d\vec{S}$ calculates how much the vector field is "flowing out" of the surface S.

- For an electric field $\vec{E}(x,y,z)$, Gauss's Law tells us that the amount of electric field flowing out is equal to the electric charge inside the closed surface:

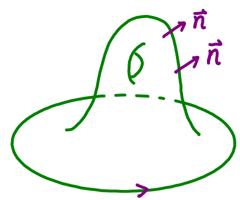
$$\iint_S \vec{E} \cdot d\vec{S} = \varepsilon_0 Q,$$

where Q is the total electric charge enclosed inside S.

Eg. Use Gauss's Law to find the electric charge inside the solid hemisphere $x^2+y^2+z^2 \le a^2$, $z \ge 0$, if the electric field is given by $\vec{E}(x,y,z) = x\vec{i} + y\vec{j} + 2z\vec{k}$.

§2. Stokes's Theorem

The theorem will be a bridge between vector line integrals and vector surface integrals on an oriented surface with boundaries:



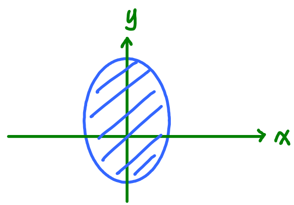
Note: Both integrals depend on orientations!

Thm (**Stokes**) Let S be an oriented (piecewise) smooth surface that is bounded by a simple closed (piecewise) smooth curve C with the induced right-hand orientation. Let $\vec{F}(x,y,z)$ be a smooth vector field defined near S. Then:

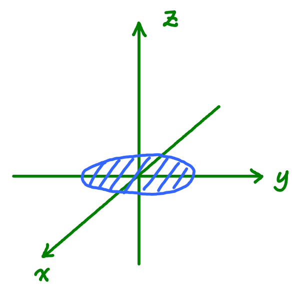
$$\boxed{\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\vec{\nabla} \times \vec{F})\, d\vec{S}}$$

A proof of the theorem in the special case when S is the graph of a function $z = f(x,y)$ can be found in the textbook.

- Special case If $F(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$ is a 2-dim'l vector field and S is a planar region with boundary:



$\Longrightarrow$



we can regard $\vec{F}$ as a 3-dim'l vector field

$$\vec{F}(x,y,z) = P(x,y)\vec{i} + Q(x,y)\vec{j} + 0\vec{k}$$

and S as a surface in $\mathbb{R}^3$

Then

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \vec{k},$$

and $(\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$. Thus Stokes's Theorem recovers Green's Theorem

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C P(x,y)\,dx + Q(x,y)\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$

Eg. Use Stokes's Theorem to evaluate $\int_C \vec{F}\cdot d\vec{r}$, where C is the circle $x^2+y^2=16$, $z=5$ and $\vec{F}(x,y,z) = yz\,\vec{i} + 2xz\,\vec{j} + e^{xy}\,\vec{k}$

Eg. A hemisphere $H: x^2+y^2+z^2=4$, $z \ge 0$ and a part of a paraboloid $P: z = 4-x^2-y^2$, $z \ge 0$ are placed in $\mathbb{R}^3$, both with upward orientation. Explain why

$$\iint_H (\vec{\nabla}\times\vec{F})\cdot d\vec{S} = \iint_P (\vec{\nabla}\times\vec{F})\cdot d\vec{S}$$

for any vector field $\vec{F}$ on $\mathbb{R}^3$.

Eg. Evaluate the integral $\iint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S}$, where

$$\vec{F}(x,y,z) = xyz\,\vec{i} + xy\,\vec{j} + x^2yz\,\vec{k}$$

and S consists all but the bottom square of the cube boundary with vertices at $(\pm 1, \pm 1, \pm 1)$.

- **Differential form of Stokes's Theorem**

Just as for Green's Theorem, Stokes's Theorem can be written in a more aesthetic way via differential forms.

Recall that, if $F(x,y,z) = P\vec{i} + Q\vec{j} + R\vec{k}$, then

$$(\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy\,dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dz\,dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$= d(P\,dx + Q\,dy + R\,dz)$$

$$= d(\vec{F} \cdot d\vec{r}).$$

Thus Stokes's Theorem becomes

$$\boxed{\int_{\partial S} \vec{F} \cdot d\vec{r} = \iint_S d(\vec{F} \cdot d\vec{r})}$$

where ∂S means taking boundary of the oriented surface S with the induced orientation on the boundary.