Curl, div and parametric surfaces

Reading: Textbook, §16.5-16.6

§1. Curl and divergence

Our next goal is to extend Green's Theorem to vector fields in $\mathbb{R}^3$. One key quantity we need to generalize is $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ for vector fields having 3 components:

$$\vec{F}(x,y,z) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}.$$

Def. If $\vec{F}(x,y,z)$ is a vector field as above, we define its **curl** as the vector field (also written as $\text{curl}\,\vec{F}$):

$$\boxed{\vec{\nabla}\times\vec{F} := \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}}$$

Here we have formally identified $\vec{\nabla} = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$ as a vector-valued differential operator.

- Grassmann algebra in $\mathbb{R}^3$.

We extend our earlier usage of dx, dy, $dxdy$ into dx, dy, dz, satisfying

$$dxdx = dydy = dzdz = 0, \quad dxdy = -dydx, \quad dxdz = -dzdx,$$
$$dydz = -dzdx, \quad dxdydz = dydzdx = dzdxdy.$$

Then $\vec{F}\cdot d\vec{r} = Pdx + Qdy + Rdz$, and

$$\begin{aligned} d(\vec{F}\cdot d\vec{r}) &= d(Pdx + Qdy + Rdz) \\ &= (R_y - Q_z)\,dydz - (R_x - P_z)\,dzdx + (Q_x - P_y)\,dxdy. \end{aligned}$$

Eg. Find the curl of $\vec{F}(x,y,z) = (xy,\ xz,\ xyz^2)$.

Eg. Where on $\mathbb{R}^3$ is the vector field $\vec{F}(x,y,z) = \dfrac{x\vec{i} + y\vec{j} + z\vec{k}}{\sqrt{(x^2+y^2+z^2)^3}}$ defined? Find its curl.

Thm. If $\vec{F}$ is conservative, then $\vec{\nabla}\times\vec{F} = 0$.

Using the Grassmannian language, $\vec{F}$ being conservative $\Longleftrightarrow$ $\vec{F}\cdot d\vec{r} = df$ for some function f on the domain of $\vec{F}$.
Then $\vec{\nabla}\times\vec{F}$ has components computed by $d(\vec{F}\cdot d\vec{r}) = d^2 f$:

$$
\begin{aligned}
d^2 f &= d(f_x\,dx + f_y\,dy + f_z\,dz) \\
&= (f_{xx}\,dx + f_{xy}\,dy + f_{xz}\,dz)\,dx + (f_{yx}\,dx + f_{yy}\,dy + f_{yz}\,dz)\,dy \\
&\quad + (f_{zx}\,dx + f_{zy}\,dy + f_{zz}\,dz)\,dz \\
&= (f_{zy} - f_{yz})\,dy\,dz - (f_{zx} - f_{xz})\,dz\,dx + (f_{yx} - f_{xy})\,dx\,dy \\
&= 0 \qquad (\text{Clairaut's thm})
\end{aligned}
$$

Eg. Determine if the vector field

$$\vec{F}(x,y,z) = y^2z^3\,\vec{i} + 2xyz^3\,\vec{j} + 3xyz^2\,\vec{k}$$

is conservative or not. If it is, find a potential function.

Def If $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is a vector field as before, then its **divergence** is the function (also written as $\operatorname{div}\vec{F}$)

$$\boxed{\vec{\nabla}\cdot\vec{F} := \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}}$$

- In terms of Grassmann algebra, introduce the **oriented infinitesimal area**

$$d\vec{S} := dydz\,\vec{i} + dzdx\,\vec{j} + dxdy\,\vec{k}$$

Then $\vec{F}\cdot d\vec{S} = P\,dydz + Q\,dzdx + R\,dxdy \Longrightarrow$

$$d(\vec{F}\cdot d\vec{S}) = (P_x + Q_y + R_z)\,dxdydz = (\vec{\nabla}\cdot\vec{F})\,dxdydz.$$

We will also run into $d\vec{S}$ again when talking about vector integral over parametric surfaces.

Thm. If $\vec{F}(x,y,z) = \vec{\nabla} \times \vec{G}(x,y,z)$ on a domain $D \subseteq \mathbb{R}^3$, then $\vec{\nabla} \cdot \vec{F} = 0$ on D.

In fact, $\vec{F} = \vec{\nabla} \times \vec{G} \iff \vec{F} \cdot d\vec{S} = d(\vec{G} \cdot d\vec{r})$. Thus

$$(\vec{\nabla} \cdot \vec{F})\, dx\,dy\,dz = d(d\vec{F} \cdot d\vec{S}) = d^2(\vec{G} \cdot d\vec{r}) = 0$$

again by Clairaut's Theorem.

- Such $\vec{G}$ that $\vec{\nabla}\times\vec{G} = \vec{F}$ is also called a vector potential.

Eg. Does the vector field $\vec{F}(x,y,z) = (x\sin y,\ \cos y,\ z - xy)$ have a vector potential ?

Warning. Do not confuse $\vec{\nabla}\cdot(\vec{\nabla}f)$ with $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{F}) \equiv 0$. The first quantity equals

$$\vec{\nabla}\cdot(\vec{\nabla}f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \quad (=: \Delta f)$$

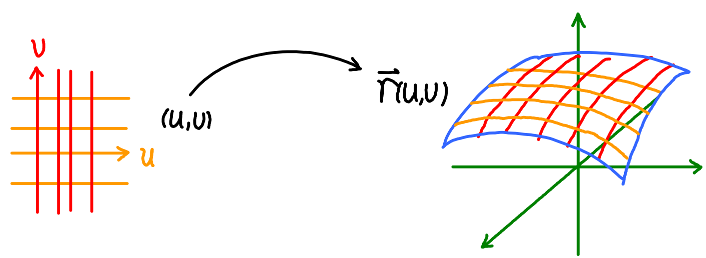
Solutions to $\Delta f \equiv 0$ on $\mathbb{R}^3$ are called harmonic functions.

Eg. Prove the equality:

$$\vec{\nabla}\cdot(f\vec{F}) = (\vec{\nabla}f)\cdot\vec{F} + f(\vec{\nabla}\cdot\vec{F}).$$

§2. Parametric surfaces and their areas

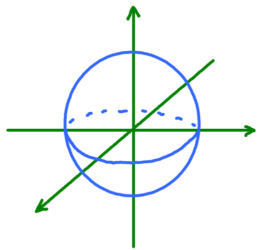
Previously, we have described space/plane curves as a one-parameter family of vectors. Similarly, to describe a surface in $\mathbb{R}^3$, we need a two-parameter family of vectors.



Eg. We have already seen some special parametric surfaces in $\mathbb{R}^3$. For instance, the graph of a function $f(x,y)$ can be regarded as

$$\vec{r}(x,y) = x\vec{i} + y\vec{j} + f(x,y)\vec{k} = (x, y, f(x,y)).$$

Eg. As another example, the sphere $x^2+y^2+z^2=1$ is not the graph of any function, but near any point, it can be written as a parametric surface. Say, near $(-1,0,0)$

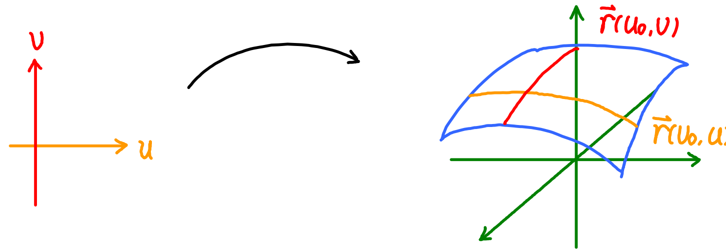


The "back" hemisphere

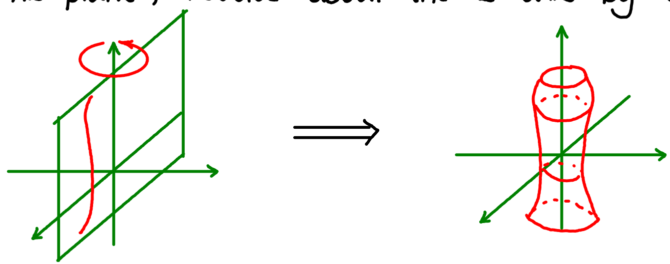
$$\vec{r}(u,v) = -\sqrt{1-u^2-v^2}\,\vec{i} + u\vec{j} + v\vec{k}.$$

$$(u,v) \in D = \{(u,v) \mid u^2+v^2 \le 1\}$$

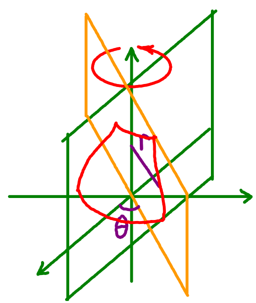
On parametric surfaces, there are many natural parametric curves: fixing either u or v and let the other variable change, we obtain the grid curves on $\vec{r} = \vec{r}(u,v)$.



An important class of parametric surfaces is given by surfaces of revolution: if you start with a curve on the xz-plane, revolve about the z-axis by $360°$:



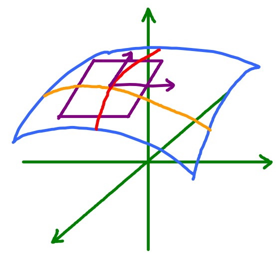
In special case, the curve on the xy-plane is given by the graph of a function $z = f(x)$. When the positive x-coordinate is rotated, distance on it becomes a parameter r (distance from the z-axis). Another natural parameter is the angle θ



$$\vec{r}(r,\theta) = r\cos\theta\,\vec{i} + r\sin\theta\,\vec{j} + f(r)\,\vec{k}$$

$$r \in [a, b], \quad \theta \in [0, 2\pi].$$

- At each point $\vec{r}(u_0, v_0)$ of a parametric surface, the surface is best linearly approximated by its tangent plane at $\vec{r}(u_0, v_0)$:



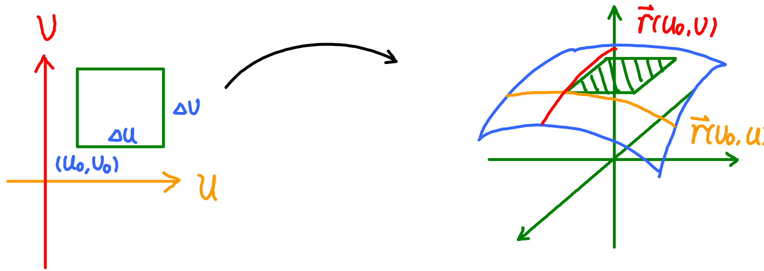
To find the plane, we first find the tangent directions to the grid lines through $\vec{r}(u_0, v_0)$: $\vec{r}_u(u_0, v_0)$, $\vec{r}_v(u_0, v_0)$.

Thus the normal vector . $\vec{n}(u_0,v_0) = \vec{r}_u(u_0,v_0) \times \vec{r}_v(u_0,v_0)$, and the tangent plane .

$$\boxed{(\vec{r} - \vec{r}(u_0,v_0)) \cdot (\vec{r}_u(u_0,v_0) \times \vec{r}_v(u_0,v_0)) = 0}$$

Eg. Let $\vec{r}(u,v) = (u-v)\vec{i} + (u^2-2v)\vec{j} + v\vec{k}$. Find the tangent plane at the point $(0,-1,1)$.

The tangent plane helps us with local approximation of the area of a surface. In turn this helps with finding the area of a parametric surface.



The parallelogram is formed by the vectors $\vec{r}(u_0, v_0)$

$$\vec{r}(u_0+\Delta u, v_0) \approx \vec{r}(u_0, v_0) + \vec{r}_u(u_0, v_0)\Delta u$$

$$\vec{r}(u_0, v_0+\Delta v) \approx \vec{r}(u_0, v_0) + \vec{r}_v(u_0, v_0)\Delta v,$$

$$\vec{r}(u_0+\Delta u, v_0+\Delta v) \approx \vec{r}(u_0, v_0) + \vec{r}_u(u_0, v_0)\Delta u + \vec{r}_v(u_0, v_0)\Delta v$$

Thus the side vectors are given by

$$\vec{r}_u(u_0,v_0)\Delta u, \qquad \vec{r}_v(u_0,v_0)\Delta v$$

and the small area equals

$$\Delta A = \|\vec{r}_u(u_0,v_0)\times\vec{r}_v(u_0,v_0)\|\Delta u\Delta v$$

Summing up ΔA's and taking limit, we get

Thm. If a smooth parametric surface S is given by $\vec{r}=\vec{r}(u,v)$ where $(u,v)\in D$, then

$$\boxed{A(S) = \iint_D \|\vec{r}_u\times\vec{r}_v\|\,du\,dv}$$

Eg. If S is the graph $z=f(x,y)$ for a smooth function $f(x,y)$ defined over D, then $\vec{r}=\vec{r}(x,y)=(x,\ y,\ f(x,y))$

$$\vec{r}_x \times \vec{r}_y = (\vec{i}+f_x\vec{k})\times(\vec{j}+f_y\vec{k}) = -f_x\vec{i}-f_y\vec{j}+\vec{k}.$$

$$\Longrightarrow \qquad A(S)=\iint_D \sqrt{f_x^2+f_y^2+1}\,dxdy.$$

Eg. Find the area of the sphere $x^2+y^2+z^2=R^2$.

Eg. Find the area of the surface $z=xy$ that lies inside the cylinder $x^2+y^2=1$.