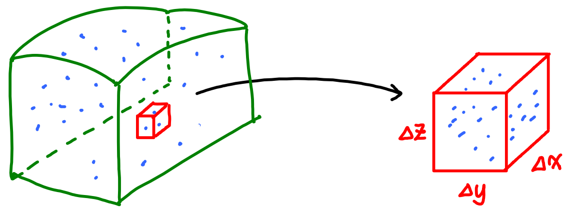
Triple Integrals

Reading: Textbook, §15.6 - 15.9

§1. Triple integrals: definition

Our next goal is to develop a 3-dim'l analogue of the Fundamental Theorem of Line Integrals, Green's Theorem and Stokes's Theorem. To do this we will first need to know how to integrate over 3-dim'l regions.

The triple integral is motivated from the following type of problem: if the salt concentration in a bread loaf B is given by $\rho(x,y,z)$ as a function of the position, how much salt in total is there in B?



Thus :

$$\text{Total Salt} \approx \sum_{\Delta x} \sum_{\Delta y} \sum_{\Delta z} \rho(x_0, y_0, z_0)\, \Delta x \Delta y \Delta z$$

Upon taking limit :

$$\boxed{\iiint_B \rho(x,y,z)\, dV := \lim_{\Delta x, \Delta y, \Delta z \to 0} \sum_{\Delta x} \sum_{\Delta y} \sum_{\Delta z} \rho(x_0, y_0, z_0)\, \Delta x \Delta y \Delta z}$$

This definition immediately gives the following way of computing triple integrals, over the simplest "bread loaf".

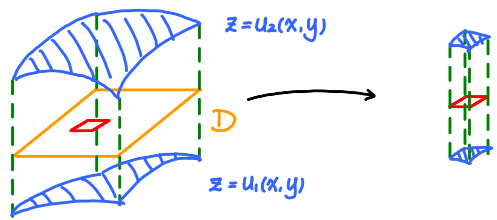
Thm. If $f(x,y,z)$ is a continuous function on the solid rectangular box $B: [a,b]\times[c,d]\times[r,s]$, then

$$\boxed{\iiint_B f(x,y,z)\,dV = \int_a^b\int_c^d\int_r^s f(x,y,z)\,dx\,dy\,dz}$$

In particular, the integral does not depend on the order the iterated integral is carried out.

In general, solid bodies are more complicated than solid rectangles. A slightly more general solid is bounded by two graphs living over the same region D in the xy-plane

$$B = \{(x,y,z) \mid (x,y) \in D,\ u_1(x,y) \le z \le u_2(x,y)\}$$



The total "salt" should be equal to the sum of salt in all small square "bread sticks" (area $\Delta x \cdot \Delta y$) over D:

$$\boxed{\iiint_B f(x,y,z)\,dV = \iint_D dx\,dy \int_{u_2(x,y)}^{u_1(x,y)} f(x,y,z)\,dz}$$

The function $\int_{u_2(x,y)}^{u_1(x,y)} f(x,y,z)\,dz$ now only depends on $(x,y)\in D$. Thus this reduces this triple integral into a double integral.

Eg. Find the triple integral $\iiint_E y\,dV$, where E is the region $\{(x,y,z) \mid 0\le x\le 3,\ 0\le y\le x,\ x-y\le z\le x+y\}$.

It's also important to be able to visualize the region that you are integrating over. In some cases, this allows you to change the order of integration.

Eg. Find the integral $\iiint_T x^2 \, dV$, where T is the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.

Eg. Sketch the solid region whose volume is given by the iterated integral

$$\int_0^1 \int_0^{1-x} \int_0^{2-2x-2z} dy \, dz \, dx$$

Note that, to find the volume of a solid body, you just need to do triple integral of the constant density function 1.

Eg. Find the volume of the solid enclosed by the cylinder $y = x^2$ and the planes $z = 0$ and $y + z = 1$

§2. Triple integrals in cylindrical coordinates

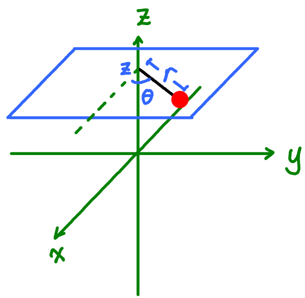
For many integrals, the usual x, y, z-coordinates may not necessarily be the most convenient to use. We will next introduce two other coordinate systems: the cylindrical and the spherical.

The cylindrical coordinates are very easy to use when integrating over a region that looks like a cylinder. In this coordinate, we can describe any point in $\mathbb{R}^3$ without the z-axis by 3 numbers:

(1). the height of the z-plane it's on (z-coordinate)

(2). on each fixed z-plane, the usual polar coordinates

(r, θ): $0 < r < \infty$, $0 \le \theta < 2\pi$ (up to shifts by 2π)



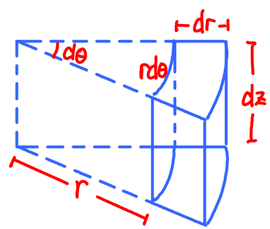
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \iff \begin{cases} \theta = \arctan y/x \\ r = \sqrt{x^2+y^2} \\ z = z \end{cases}$$

- In this coordinate, any point away from the z-axis has a unique description (r, θ, z).

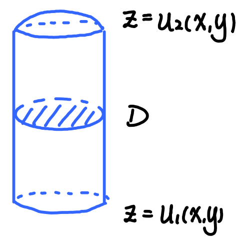
Eg. Rewrite the surface equation $x^2 - x + y^2 + z^2 = 1$ in cylindrical coordinates.

In cylindrical coordinates, the infinetesimal volume is given by

$$dV = r\,dr\,d\theta\,dz$$



Thus, if a cylindrical region lives over D in the xy plane is given by $E = \{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in D,\ u_1(x,y) \le z \le u_2(x,y)\}$, then



$$\boxed{\iiint_E f(x,y,z)\,dV = \iint_D \left(\int_{u_1(r\cos\theta,\, r\sin\theta)}^{u_2(r\cos\theta,\, r\sin\theta)} f\,dz \right) r\,dr\,d\theta}$$

$f = f(r\cos\theta, r\sin\theta, z)$

Eg. Sketch the solid in the integral, and evaluate the integral

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^2 \int_0^{r^2} r\,dz\,dr\,d\theta$$

Eg. Evaluate $\iiint_E \sqrt{x^2+y^2}\,dV$, where E is the region that lies inside the cylinder $x^2+y^2=16$ and between the planes $z=-5$, $z=4$.

Let us also practice converting rectangular triple integrals into cylindrical ones.

Eg. Evaluate the triple integral

$$\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{\sqrt{x^2+y^2}}^{2} xz\,dz\,dx\,dy$$

Eg. Evaluate the integral $\iiint_B z\,dV$, where B is the solid 3-dimensional cylinder bounded by the surfaces $(x-\frac{1}{2})^2+y^2=\frac{1}{4}$, $(x-1)^2+y^2=1$, $z=0$ and $z=1$.

§3. Triple integrals in spherical coordinates

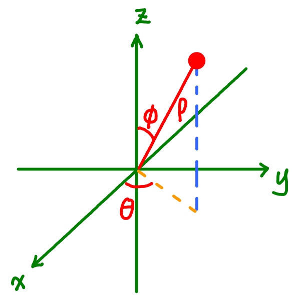
Another useful coordinate system is the spherical coordinates that is very convenient when describing (part of) a solid ball.

The main idea of this coordinate system is that, any point of $\mathbb{R}^3$ away from the origin can be uniquely described by

(1). its distance from the origin $\rho = \sqrt{x^2+y^2+z^2}$,

(2). its angle with respect to the z-axis : ϕ

(3). the angle its xy-projection makes with the x-axis : θ.



$0 < \rho < \infty$

$0 \le \phi \le \pi$

$0 \le \theta < 2\pi$

Thus

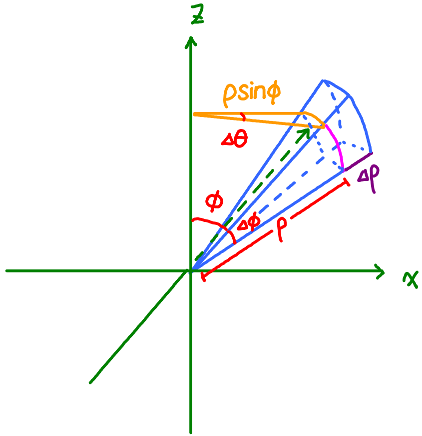
$$\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho \cos\phi \end{cases} \iff \begin{cases} \rho = \sqrt{x^2+y^2+z^2} \\ \phi = \tan^{-1} \dfrac{z}{\sqrt{x^2+y^2}} \\ \theta = \tan^{-1} \dfrac{y}{x} \end{cases}$$

Eg. Rewrite the surface $z^2 = x^2 + y^2$ in spherical coordinates.

Eg. Sketch the solid described by the equalities below:

$$2 \leq \rho \leq 4, \quad 0 \leq \phi \leq \pi/3, \quad 0 \leq \theta \leq \pi$$

We next investigate how to integrate in spherical coordinates.
First, we need to investigate how volume increases infinitesimally.



$\frown$: $\rho \sin\phi \, \Delta\theta$

$)$: $\rho \Delta\phi$

$-$: $\Delta\rho$

$$\Longrightarrow \Delta V = \rho^2 \sin\phi \, \Delta\rho \, \Delta\theta \, \Delta\phi$$

$$\boxed{dV = \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi}$$

Then :

$$\boxed{\iiint_B f(x,y,z)\,dV = \iiint_{B(\rho,\phi,\theta)} f(\rho\sin\phi\cos\theta,\ \rho\sin\phi\sin\theta,\ \rho\cos\phi)\ \rho^2\sin\phi\, d\rho\, d\theta\, d\phi}$$

Eg. Use spherical coordinates to evaluate

$$\iiint_B (x^2+y^2+z^2)^2\,dV$$

where B is the ball of radius 5 centered at the origin.

Eg. Evaluate the integral by converting to spherical coordinates:

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} xy \, dz \, dy \, dx$$

Eg. Show that

$$\iiint_{\mathbb{R}^3} \sqrt{x^2+y^2+z^2}\, e^{-x^2-y^2-z^2} \, dV = 2\pi .$$