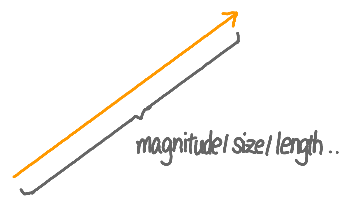
Geometry of Vectors in $\mathbb{R}^3$

Reading: Textbook, §12.1 - 12.5

§1. Vectors in $\mathbb{R}^3$

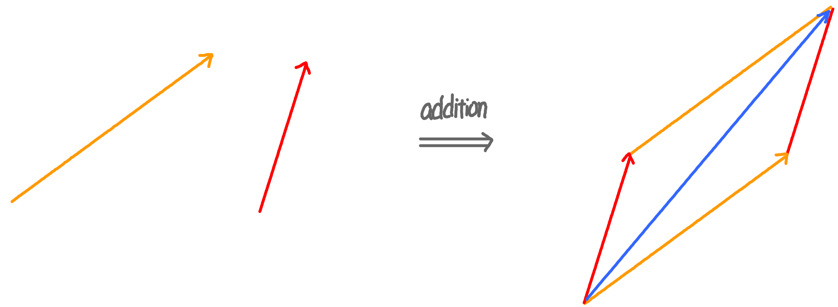
- A vector is a mathmatical quantity that has a magnitude and a direction.



- A vector can be rescaled by scalars (numbers in $\mathbb{R}$) to make new vectors, only affecting its size.

(What about rescaling by negative scalars?)

- Unlike scalars, vector addition obeys the parallelogram / triangle rule



(Q: How to perform subtraction ?)

- To mathematically describe vectors, we introduce Cartesian coordinates. Thus a vector in 3-dim'l space is described by 3 numbers: its projections onto the x, y, z - axis.

$$\vec{v} = (v_x, v_y, v_z)$$

Then: addition and scalar multiplication are done coordinatewise:

$\vec{v} = (v_x, v_y, v_z)$, $\vec{u} = (u_x, u_y, u_z)$, $a \in \mathbb{R}$

$$\Longrightarrow \begin{cases} \vec{v} + \vec{u} = (v_x + u_x, v_y + u_y, v_z + u_z) \\ a\vec{v} = (av_x, av_y, av_z) \end{cases}$$

Eg. How do we add the vectors $(4, 5, 6)$ and $(1, 2, 3)$?
How do we subtract $(4.5, 6)$ from $(1.2.3)$?

- It's convenient to give some standard vectors special names:

$$\vec{i} = (1,0,0), \qquad \vec{j} = (0,1,0), \qquad \vec{k} = (0,0,1)$$

Then any vector $\vec{v} = (a,b,c)$ can be rewritten as

$$\begin{aligned} \vec{v} &= a(1,0,0) + b(0,1,0) + c(0,0,1) \\ &= a\vec{i} + b\vec{j} + c\vec{k} \end{aligned}$$

- The Pythagorean Theorem tells us that the magnitude/length of a vector $\vec{v} = (a,b,c)$ is given by

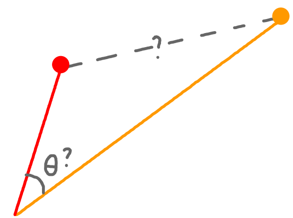
$$\boxed{\|\vec{v}\| = \sqrt{a^2+b^2+c^2}}$$

E.g. Draw all vectors of length 1 in $\mathbb{R}^3$.

E.g. What's the length of the vector $\frac{\vec{v}}{\|\vec{v}\|}$ if $\vec{v} \neq \vec{0}$?
(This vector is called the direction vector of $\vec{v}$, of length 1).

§2. Dot/inner product

The notion of length of a vector is closely related to distance between two points in $\mathbb{R}^3$ and angles between two vectors.



Def. If $\vec{u} = (u_x, u_y, u_z)$ and $\vec{v} = (v_x, v_y, v_z)$ are vectors in $\mathbb{R}^3$, then

$$\boxed{\vec{u} \cdot \vec{v} := u_x v_x + u_y v_y + u_z v_z \in \mathbb{R}}$$

Basic properties.

- For any vector $\vec{u} = (a, b, c) \in \mathbb{R}^3$

$$\vec{u} \cdot \vec{u} = a^2 + b^2 + c^2 = \left(\sqrt{a^2+b^2+c^2}\right)^2 = \|\vec{u}\|^2.$$

- $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$ (commutativity).

- $\vec{u} \cdot (a\vec{v} + b\vec{w}) = a\vec{u} \cdot \vec{v} + b\vec{u} \cdot \vec{w}$ (distributivity)

- Generalized Pythagorean Theorem : if $\vec{u}, \vec{v}$ make an angle θ :

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$$

$\Longrightarrow$ If $\vec{u}, \vec{v}$ are non-zero vectors making an angle θ,

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}.$$

(Here we usually assume $\theta \in (0, \pi)$. If $\theta = 0$ or π, the two vectors are called parallel).

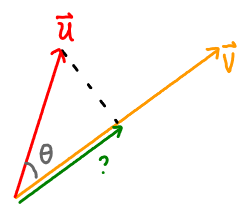
E.g. Show that the triangles with vertices $(1, -3, -2)$, $(2, 0, -4)$ and $(6, -2, -5)$ is right angled.

Using dot product, if $\vec{u} = (u_x, u_y, u_z)$ is a vector, then

$$u_x = \vec{u} \cdot \vec{i}\ , \qquad u_y = \vec{u} \cdot \vec{j}\ , \qquad u_z = \vec{u} \cdot \vec{k}$$

$$\Longrightarrow \qquad \boxed{\vec{u} = (\vec{u} \cdot \vec{i})\vec{i} + (\vec{u} \cdot \vec{j})\vec{j} + (\vec{u} \cdot \vec{k})\vec{k}}$$

Application: Project one vector $\vec{u}$ in the direction of another.



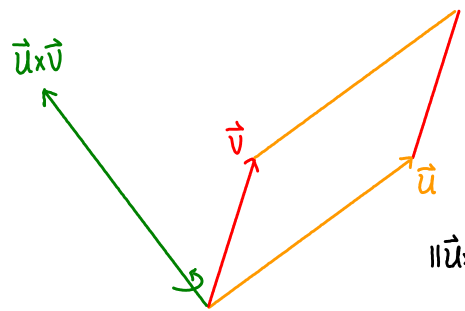
$$\mathrm{Proj}_{\vec{v}}(\vec{u}) = \|u\|\cos\theta \frac{\vec{v}}{\|\vec{v}\|} = \|u\| \cdot \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|} \frac{\vec{v}}{\|\vec{v}\|} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v}.$$

Eg. Find the projection of $(1,2,3)$ in the direction of the vector $(3,4,0)$, and the length of this projected vector.

§3. Cross product

Def. The cross product of two vectors $\vec{u}$ and $\vec{v}$ has

- magnitude = area of parallelogram formed by $\vec{u}$ and $\vec{v}$
- direction = perpendicular to $\vec{u}$ and $\vec{v}$ via the right hand rule.

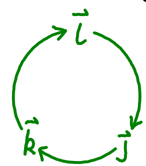


$\|\vec{u}\times\vec{v}\|$ = Area of parallelogram

- Helpful computational rule: the cross product is distributive and anti-commutative, and determined by

$$\vec{i}\times\vec{i} = \vec{j}\times\vec{j} = \vec{k}\times\vec{k} = 0$$

$$\vec{i}\times\vec{j} = \vec{k} = -\vec{j}\times\vec{i}, \quad \vec{j}\times\vec{k} = \vec{i} = -\vec{k}\times\vec{j}, \quad \vec{k}\times\vec{i} = \vec{j} = -\vec{i}\times\vec{k}$$



Eg. $(1,2,0)\times(0.4.6) = (1\vec{i}+2\vec{j})\times(4\vec{j}+6\vec{k})$

More generally, if $\vec{u}=(a_1, b_1, c_1)$, $\vec{v}=(a_2, b_2, c_2)$, then

$$\vec{u}\times\vec{v} = (a_1\vec{i} + b_1\vec{j} + c_1\vec{k})\times(a_2\vec{i} + b_2\vec{j} + c_2\vec{k})$$
$$= (b_1c_2 - b_2c_1)\vec{i} - (a_1c_2 - a_2c_1)\vec{j} + (a_1b_2 - a_2b_1)\vec{k}$$

A useful way to memorize the determinantal formula in linear algebra. if $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a 2×2 matrix, then

$$|A| = \left|\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right| = ad - bc.$$

So we can rewrite:

$$\vec{u} \times \vec{v} = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \vec{k}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

using the 3x3 determinantal formula.

Eg. Double check that $\vec{i} \times \vec{j} = \vec{k}$ etc.

E.g. Compute $\vec{u} \times \vec{v}$ for $\vec{u} = (1, 2, 3)$ and $\vec{v} = (4, 5, 6)$, and verify that $\vec{u}$, $\vec{v}$ are perpendicular to $\vec{u} \times \vec{v}$.

Thm (1) The vector $\vec{u} \times \vec{v}$ is perpendicular to both $\vec{u}$ & $\vec{v}$.

(2). The length of $\vec{u} \times \vec{v}$ is equal to $\|\vec{u}\| \|\vec{v}\| \sin\theta$, where $\theta \in [0, \pi]$ is the angle between $\vec{u}$ & $\vec{v}$.

E.g. Find a nonzero vector that is orthogonal to the plane containing the points $(1,0,1)$, $(-2,1,3)$ and $(4,2,5)$.

Thm. (Properties of $\times$). For any vectors $\vec{u},\vec{v},\vec{w}$, we have

1. $\vec{u}\times\vec{v} = -\vec{v}\times\vec{u}$ $\quad(\Longrightarrow \vec{u}\times\vec{u}=0)$.
2. $(r\vec{u})\times\vec{v} = r(\vec{u}\times\vec{v}) = \vec{u}\times(r\vec{v})$.
3. $\vec{u}\times(\vec{v}+\vec{w}) = \vec{u}\times\vec{v}+\vec{u}\times\vec{w}$, $\quad(\vec{v}+\vec{w})\times\vec{u} = \vec{v}\times\vec{u}+\vec{w}\times\vec{u}$.
4. $\vec{u}\cdot(\vec{v}\times\vec{w}) = \vec{v}\cdot(\vec{w}\times\vec{u}) = \vec{w}\cdot(\vec{u}\times\vec{v})$.
5. $\vec{u}\times(\vec{v}\times\vec{w}) = (\vec{u}\cdot\vec{w})\vec{v} - (\vec{u}\cdot\vec{v})\vec{w}$.

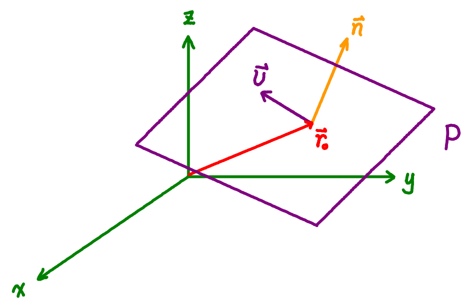
The quantity $\vec{u}\cdot(\vec{v}\times\vec{w})$ is also called the triple product of $\vec{u}$, $\vec{v}$ and $\vec{w}$. It is equal to the signed/oriented volume of the parallelpiped formed by $\vec{u}$, $\vec{v}$ & $\vec{w}$:

$$\vec{u}\cdot(\vec{v}\times\vec{w}) = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}.$$

E.g. Determine if the vectors $(2,8,-14)$, $(2,-1,4)$ and $(0,1,-2)$ are coplanar or not.

§4. Equations of planes and lines

As applications, we give several ways to describe lines in $\mathbb{R}^3$.



$$P = \{\vec{r}_0 + \vec{v} \mid \vec{v} \perp \vec{n}\} = \{\vec{r} \mid (\vec{r} - \vec{r}_0) \perp \vec{n}\} = \{\vec{r} \mid \vec{n} \cdot (\vec{r} - \vec{r}_0) = 0\}.$$

The vector $\vec{n}$ is called the normal direction of P.

In coordinates, if $\vec{n} = (a, b, c)$, $\vec{r}_0 = (x_0, y_0, z_0)$, then $(x, y, z) \in P$ means

$$0 = (\vec{r} - \vec{r}_0) \cdot \vec{n} = \big((x, y, z) - (x_0, y_0, z_0) \big) (a, b, c)$$

$$\Longrightarrow \quad \boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$$

Next, we describe lines. The easiest way to describe a line is to use the vector parametric form

$$\vec{v}(t) = \vec{r}_0 + t\vec{v} \qquad (t \in \mathbb{R})$$

Componentwise, if $\vec{v}(t) = (x(t), y(t), z(t))$, $\vec{r}_0 = (x_0, y_0, z_0)$ and $\vec{v} = (a, b, c)$, then

$$x(t) = x_0 + at, \quad y(t) = y_0 + bt, \quad z(t) = z_0 + ct.$$

Note that there are different ways to describe the same line by choosing different $\vec{r}_0$'s and (parallel) $\vec{v}$'s.

Assume $a \neq 0$, $b \neq 0$, $c \neq 0$ $\Longrightarrow$ symmetric form of the line

$$(t =) \quad \boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}}$$

This form is essentially describing the line as the intersection of 2 planes :

$$P_1 : \frac{x - x_0}{a} = \frac{y - y_0}{b} \qquad P_2 : \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

More generally, two non-parallel planes always intersect at a line.

E.g. What is the relationship between the planes

$$P_1 : 2x - 3y + 4z = 5 \qquad P_2 : x + 6y + 4z = 3$$

Find a parametric form of $P_1 \cap P_2$

E.g Can you give a condition for a line

$$L: \ x(t) = x_0 + at, \quad y(t) = y_0 + bt, \quad z(t) = z_0 + ct$$

to be parallel to a plane

$$P. \quad ux + vy + wz = d$$

Under what conditions will L and P intersect each other ?