More on Partial Derivatives

Reading: Textbook, §14.5 - 14.7

§1. The chain rule

We next investigate the analogue of the one-variable chain rule in the multivariable case.

Thm. If $z = f(x, y)$ is a differentiable function, and $x = x(t)$, $y = y(t)$ are differentiable functions in t, then for $z(t) = f(x(t), y(t))$

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Eg. Compute $\frac{dz}{dt}$ for $z = \sqrt{1 + x^2 + y^2}$ and $x = \sin t$, $y = \cos t$.

Using the theorem, one variable a time, we can prove

Thm. If $z=f(x,y)$ is a differentiable function, and $x=x(s,t)$, $y=y(s,t)$ are differentiable functions in s, t, then

$$\frac{\partial z}{\partial s}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial s},\quad \frac{\partial z}{\partial t}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial t}.$$

More generally, if $z=f(x_1,\cdots,x_n)$ is differentiable and $x_i=x_i(t_1,\cdots,t_k)$, then

$$\frac{\partial z}{\partial t_i}=\sum_{j=1}^{n}\frac{\partial f}{\partial x_j}\frac{\partial x_j}{\partial t_i}$$

Eg. Use the chain rule to find $\frac{\partial z}{\partial s}$, where

$$z = x^4 + x^2 y, \quad x = s + 2t - u, \quad y = stu^2.$$

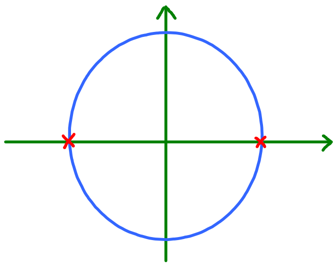
- Application: implicit function theorem

In certain cases, a function $y = y(x)$ is implicitly given as satisfying an equation of the form $F(x, y) = 0$. We can use partial derivatives to help us compute the ordinary derivative $\frac{dy}{dx}$. Regarded as a function of x,

$$0 = F(x, y(x)) \implies 0 = \frac{dF(x, y(x))}{dx} = \frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx}$$

$$\implies \frac{dy}{dx} = -\left(\frac{\partial F}{\partial x}\right) \Big/ \left(\frac{\partial F}{\partial y}\right).$$

Note that to do division, we need $\frac{\partial F}{\partial y} \neq 0$. This is equivalent to requiring the curve $F(x,y)=0$ is not, locally, parallel to the y-axis.



x: at these points $\frac{\partial F}{\partial y} = 0$

$\Longrightarrow$ $F=0$ can not be the graph of $y=y(x)$.

More generally, if $F(x,y,z)$ determines z implicity as a function $z=z(x,y)$, then

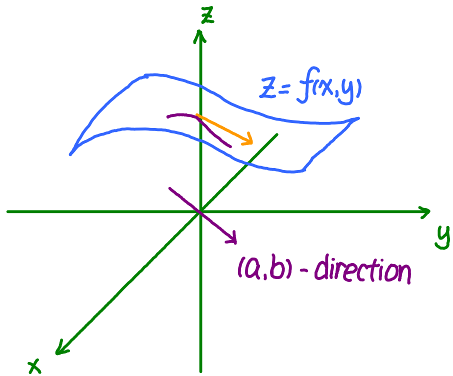
$$\frac{\partial z}{\partial x} = -\left(\frac{\partial F}{\partial x}\right) / \left(\frac{\partial F}{\partial z}\right), \qquad \frac{\partial z}{\partial y} = -\left(\frac{\partial F}{\partial y}\right) / \left(\frac{\partial F}{\partial z}\right).$$

Eg. Find $\frac{\partial z}{\partial x}$ at $(x,y) = (0,0)$ for the implicit function

$$x^2 + 4y^2 + z^2 = 1.$$

§2. Directional derivative

It turns out, for $f = f(x,y)$, once f_x and f_y are determined, you can find any directional derivative from them.



Let $\vec{u} = (a, b)$ be a unit vector (direction in xy-plane). A small displacement in the $\vec{u}$ direction is given by $\Delta t(a, b)$. The change of value of $f(x, y)$ along this small displacement is given by:

$$f(x_0 + \Delta t a, y_0 + \Delta t b) - f(x_0, y_0) \approx f_x(x_0, y_0)\Delta t a + f_y(x_0, y_0)\Delta t b$$

Dividing both sides by Δt and taking limit, we get

Thm The directional derivative along $\vec{u}$, denoted $D_{\vec{u}} f$, at a point (x_0, y_0) in the domain of f, is given by

$$(D_{\vec{u}} f)(x_0, y_0) = a f_x(x_0, y_0) + b f_y(x_0, y_0).$$

Eg. Find the directional derivative of $f(x,y) = ye^{-x}$ at the point $(0,4)$, in the unit direction $\vec{u}$ making an angle $2\pi/3$ with the positive x-axis.

Another way to describe the thm.

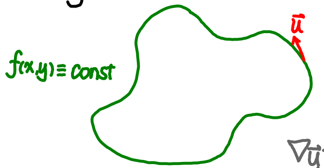
Def. Given (a,b) in the domain of $f(x,y)$, the gradient vector of f at (a,b), denoted $\nabla f(a,b)$, is the vector

$$\nabla f(a,b) := f_x(a,b)\vec{i} + f_y(a,b)\vec{j}.$$

Thus the thm

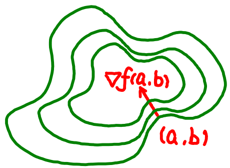
$$\Longleftrightarrow \quad \boxed{D_{\vec u} f = \nabla f \cdot \vec u}$$

- $\nabla f(a,b)$ is orthogonal to the level curves of f.



$$\nabla_{\vec u} f = \nabla f \cdot \vec u = 0 \Longrightarrow \nabla f \perp \vec u .$$

- $\nabla f(a,b)$ is the direction at (a,b) s.t. the value of f increases the fastest:



Eg. Find the gradient vector of the function $f(x,y) = x^2 - 4y^2$, and use it to find the directional derivative of $f(x,y)$ at $(2,1)$ in the direction $\vec{v} = \vec{i} + 3\vec{j}$. What is the maximal rate of change for f? In which direction does it happen?

- For functions in 3 variables (or more), we can define directional derivatives in a similar way. If $\vec{u} = (a, b, c)$ is a unit direction, then

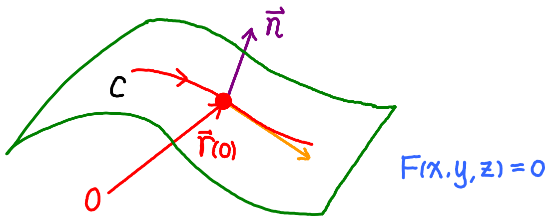
$$\nabla_{\vec{u}} f(x,y,z) := f_x a + f_y b + f_z c = \nabla f \cdot \vec{u},$$

where

$$\nabla f = f_x \vec{i} + f_y \vec{j} + f_z \vec{k}$$

is the gradient vector. This is also the direction in which f increases the fastest.

Application: tangent planes of surfaces in $\mathbb{R}^3$



If $C: \vec{r}(t) = (x(t), y(t), z(t))$ is a curve on $F(x,y,z)=0$, then

$$F(x(t), y(t), z(t)) = 0 \implies \left.\frac{dF}{dt}\right|_{t=0} = 0 \implies$$

$$\frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = \nabla F \cdot \vec{r}\,'(0) = 0$$

Since $\vec{r}\,'(0)$ is a tangent vector at $\vec{r}(0) = (x_0, y_0, z_0) \implies$

$\nabla F(x_0, y_0, z_0)$ is the normal direction to $F(x,y,z)=0$ at (x_0, y_0, z_0).

$\implies$ The tangent plane has equation

$$\boxed{F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0}$$

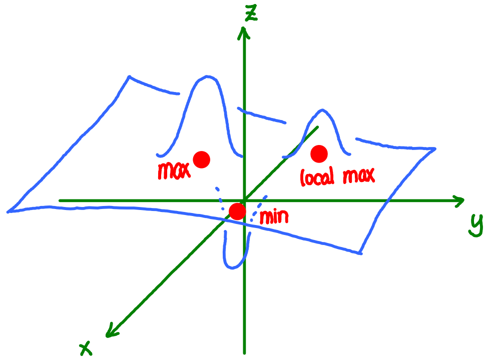
Eg. Find the equation of the tangent plane of the surface $x+y+z = e^{xyz}$ at the point $(0,0,1)$.

§3. Maximum and minimum values

Def. (1). A function f has a (local) maximum at P if in the domain D (a neighborhood V of P), $f(Q) \leq f(P)$, for any Q in D (V).

(2). A function f has a (local) minimum at P if in the domain D (a neighborhood V of P), $f(Q) \leq f(P)$, for any Q in D (V).

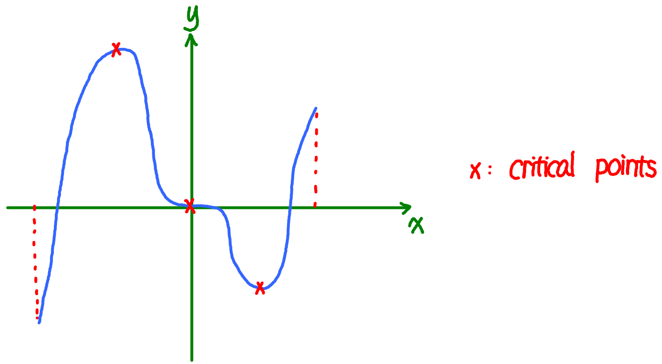
In the two-variable function case, the def. is more visual: (local) maximum corresponds to (local) summit on the graph.



In one variable case, we

(1). Find critical points by solving $f'(x) = 0$

(2). Perform a second order test.



local max : $f''(a) < 0$, so that graph locally looks like $\frac{f''(a)}{2}(x-a)^2$.

local min : $f''(a) > 0$, so that graph locally looks like $\frac{f''(a)}{2}(x-a)^2$.

inflection point : $f''(a) = 0$.

Now we will develope the analogue of these criteria for 2 (or more) variable functions.

Thm If f reaches local maximum or minimum at P in a neighborhood of P, and f is differentiable, then

$$\nabla f(P) = 0.$$

Def. The points where $\nabla f = 0$ are called *critical points* of f.

Eg. At which points can the function $f(x,y,z) = x^2+y^2+z^2-2y+4z+6$ attain an extreme value.

To determine whether a critical point is a local maximum/minimum, we need to resort to the second order information of f. For simplicity, we only consider the two-variable case.

Eg. Consider (a) $f(x,y) = x^2+y^2+2x$, (b) $g(x,y) = x^2-4y^2$. (c) $h(x,y) = 2-x^2-2y^2$. Draw their graphs and compute their Hessians $f_{xx}f_{yy} - f_{xy}^2$ at their critical points.

The above example models the local behavior of critical points.
It turns out we have the following.

Thm. Suppose $f(x,y)$ has continuous second order partial derivatives on a domain D, and (a,b) is a critical point of f. Let

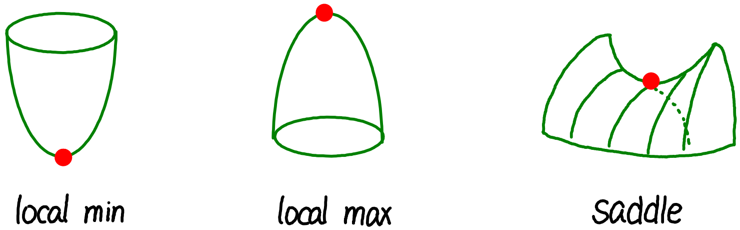
$$H(a,b) := f_{xx}(a,b) f_{yy}(a,b) - f_{xy}(a,b)^2.$$

Then, (a,b) is a

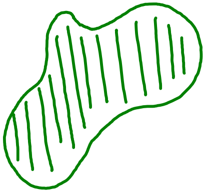
(1) local minimum if $f_{xx}(a,b) > 0$, $H(a,b) > 0$

(2) local maximum if $f_{xx}(a,b) < 0$, $H(a,b) > 0$

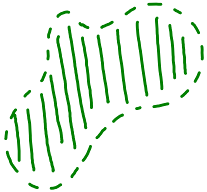
(3) saddle point if $H(a,b) < 0$.



The theorem allows us to find local max/min. But just as for 1-variable function, one also has to test the boundary behavior. This can be done if f is defined on a region D which is bounded and containing its boundary.



A bounded region containing its boundary



A bounded region not containing its boundary

Thm. If f is a continuous function defined on a bounded region D containg its boundary, then f attains its maximum/minimum on D.

Eg. Find the maximum and minimum of $f(x,y) = x^2+y^2-2x+1$ on the disk $D = \{(x,y) \mid x^2+y^2 \leq 4\}$.