Gauss's Theorem

Reading: Textbook, §16.9-16.10

§1. The Theorem

Recall that, previously, we have defined the divergence of a vector field $\vec{F}(x,y,z) = (P(x,y,z),\ Q(x,y,z),\ R(x,y,z))$ as:

$$\mathrm{div}(\vec{F}) = \vec{\nabla}\cdot\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Also recall that, if $\vec{F}$ has a vector potential $\vec{G}$: $\vec{F} = \vec{\nabla}\times\vec{G}$, then $\vec{F}$ is divergence free:

$$\vec{\nabla}\cdot\vec{F} = \vec{\nabla}\cdot(\vec{\nabla}\times\vec{G}) = 0.$$

We will use these definitions and properties frequently in this section.

Thm. Let B be a simple solid region, and let Σ be the boundary surface oriented outwards. Suppose $\vec{F}(x,y,z)$ is a vector field defined near B which has continuous partial derivatives. Then

$$\iint_\Sigma \vec{F}\cdot d\vec{S} = \iiint_B (\vec{\nabla}\cdot\vec{F})\,dV$$

A proof of the theorem in some special cases is given in the textbook, which essentially reduces to the 1-variable fundamental theorem of calculus.

The theorem can be applied in both ways: using the volume integral of divergence to calculat $\iint_\Sigma \vec{F}\cdot d\vec{S}$, or vice versa. However, in practice, volume integrals are easier to compute, and we mostly use the theorem to calculate $\iint_\Sigma \vec{F}\cdot d\vec{S}$.

Eg. Suppose $\vec{F}(x,y,z) = (z, y, x)$. Compute $\iint_\Sigma \vec{F}\cdot d\vec{S}$ where Σ is the sphere $x^2+y^2+z^2 \le 16$.

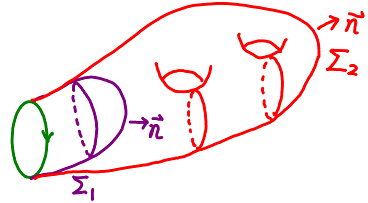
Eg. Compute the surface integral for $\vec{F}(x,y,z) = 3xy^2\vec{i} + xe^z\vec{j} + z^3\vec{k}$ and Σ is the surface of the solid bounded by the cylinder

$y^2+z^2=1$, $x=-1$ and $x=2$.

Do the same for the vector field $\vec{F}(x,y,z) = x^2\sin y\,\vec{i} + x\cos y\,\vec{j} - xz\sin y\,\vec{k}$ over the "fat sphere" $x^8+y^8+z^8=1$.

Eg. Prove the identity $\iint_\Sigma \vec{v}_0 \cdot d\vec{S} = 0$ for a constant vector field $\vec{F}(x,y,z) = \vec{v}_0$, and Σ any closed surface.

Eg. If $\vec{F} = \vec{\nabla}\times\vec{G}$, then the surface integral of $\vec{F}$ has the following surface independence property: if Σ_1 and Σ_2 are two oriented surfaces sharing the same oriented boundary curve C :



Then:

$$\iint_{\Sigma_1} \vec{F} \cdot d\vec{S} - \iint_{\Sigma_2} \vec{F} \cdot d\vec{S}$$

$$= \iint_{\Sigma_1 \cup \bar{\Sigma}_2} \vec{F} \cdot d\vec{S} \qquad (\bar{\Sigma}_2 : \Sigma_2 \text{ with the reverse orientation})$$

$$= \iiint_B (\operatorname{div} \vec{F})\, dV \qquad (B: \text{3-dim'l body enclosed by } \Sigma_1 \cup \bar{\Sigma}_2)$$

$$= \iiint_B 0 \cdot dV \qquad (\text{since } \operatorname{div} \vec{F} = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{G}) = 0)$$

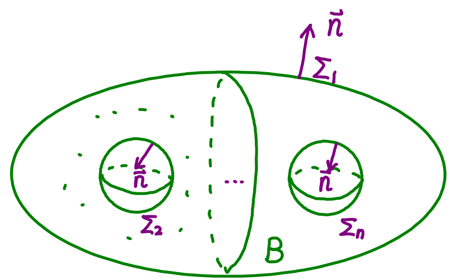
$$= 0$$

$$\Longrightarrow \quad \boxed{\iint_{\Sigma_1} \vec{F}\cdot d\vec{S} = \iint_{\Sigma_2} \vec{F}\cdot d\vec{S}} \quad \text{surface independence}$$

One can also compute $\iint_\Sigma \vec{F}\cdot d\vec{S}$ for some non-necessarily closed Σ.

Eg. Evaluate the integral $\iint_\Sigma \vec{F}\cdot d\vec{S}$. Here $\vec{F}$ is the vector field $\vec{F}(x,y,z) = \frac{x^2}{2}\vec{i} + y\vec{j} + z\vec{k}$ and Σ is the top of the cone $z = 1 - \sqrt{x^2+y^2}$ above the xy-plane, oriented upwards.

Gauss's Theorem affords a generalization to solids with several boundary components: Just make sure the boundary surfaces are oriented so that normal vectors are all pointing out of the solid.



$$\boxed{\sum_{i=1}^{n} \iint_{\Sigma_i} \vec{F} \cdot d\vec{S} = \iiint_B (\vec{\nabla} \cdot \vec{F})\, dV}$$

Eg. Evaluate the integral $\iint_\Sigma \vec{F} \cdot d\vec{S}$. Here $\vec{F}$ is the vector field

$$\vec{F}(x,y,z) = \frac{x}{(x^2+y^2+z^2)^{3/2}}\vec{i} + \frac{y}{(x^2+y^2+z^2)^{3/2}}\vec{j} + \frac{z}{(x^2+y^2+z^2)^{3/2}}\vec{k}$$

§2. Differential form version

Using differential forms, the two sides of Gauss's Theorem can be rewritten as

$$\vec{F}\cdot d\vec{S} = (P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k})\cdot(dydz\,\vec{i} + dzdx\,\vec{j} + dxdy\,\vec{k})$$
$$= P\,dydz + Q\,dzdx + R\,dxdy$$

$$\Longrightarrow\ d(\vec{F}\cdot d\vec{S}) = (P_x dx + P_y dy + P_z dz)\,dydz + (Q_x dx + Q_y dy + Q_z dz)\,dzdx$$
$$+ (R_x dx + R_y dy + R_z dz)\,dxdy$$
$$= P_x\,dxdydz + Q_y\,dydzdx + R_z\,dzdxdy$$

$$= (P_x + Q_y + R_z)\, dx\,dy\,dz$$
$$= (\vec{\nabla} \cdot \vec{F})\, dV$$

Thus, Gauss's Theorem $\Longleftrightarrow$

$$\boxed{\iint_{\partial B} \vec{F} \cdot d\vec{S} = \iiint_B d(\vec{F} \cdot d\vec{S})}$$

A much more general version of the Fundamental Theorem of Line Integrals, Green's Theorem, Stokes Theorem and Gauss's Theorem is given in the course "Calculus on Manifolds" at UVa!