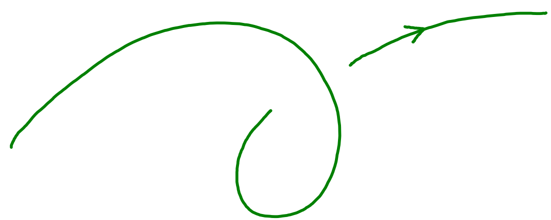
Vector valued functions

Reading: Textbook, §13.1 - 13.4

§1. One-variable vector valued functions

As we have seen, a line can be described as the trajectory of a point moving in $\mathbb{R}^3$. More generally, the trajectory doesn't need to be a line but rather a curve:



Mathematically, this is described by

$$x = x(t), \qquad y = y(t), \qquad z = z(t). \quad \text{(smooth functions)}$$

$$\Longleftrightarrow \qquad \vec{r} = \vec{r}(t) = (x(t),\ y(t),\ z(t)).$$

Eg. What is the space curve like?

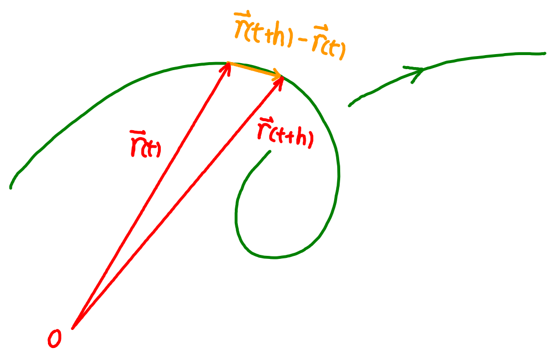
$$\vec{r}(t) = (\cos(2\pi t),\ 2\sin(2\pi t),\ 2t).$$

Eg. Find the space curve form of the intersection of the two surfaces $z = 4x^2 + y^2$ and $y = x^2$?

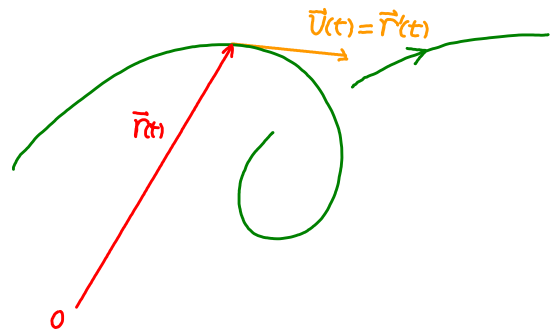
§2. Derivatives of Vector-valued functions

As with the single-variable case, the derivative of a continuous vector-valued function is defined as

$$\vec{r}'(t) := \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$



$$\vec{r}'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$



- If $\vec{r} = \vec{r}(t)$ describes the position vector of a particle moving in space, then $\vec{r}'(t)$ is just the instantaneous velocity vector at time t (position $\vec{r}(t)$).

Thm. If $\vec{r}(t) = (x(t), y(t), z(t))$ is a vector-valued function, then

$$\vec{r}'(t) = (x'(t), y'(t), z'(t)).$$

Eg. Find the derivatives of the vector-valued function

$$\vec{r}(t) = (\cos t, \sin t, t)$$

Thm. If $\vec{u}(t)$ and $\vec{v}(t)$ are two vector-valued functions, then their derivatives satisfy:

(1) $(a\vec{u}(t)+b\vec{v}(t))' = a\vec{u}'(t)+b\vec{v}'(t)$ for any $a, b \in \mathbb{R}$.

(2). If $f(t)$ is a scalar function, then

$$(f\vec{u}(t))' = f'(t)\vec{u}(t) + f(t)\vec{u}'(t) \qquad \text{(Leibnitz rule)}$$

(3). $(\vec{u}(t)\times\vec{v}(t))' = \vec{u}'(t)\times\vec{v}(t) + \vec{u}(t)\times\vec{v}'(t)$.

(4). $\vec{u}(f(t))' = \vec{u}'(f(t))\cdot f'(t)$. (chain rule)

Further, the derivative of the scalar function $\vec{u}(t)\cdot\vec{v}(t)$ also satisfies the Leibnitz rule

$$(\vec{u}(t)\cdot\vec{v}(t))' = \vec{u}'(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\vec{v}'(t).$$

- Similarly, integration of a vector-valued function is also done componentwise:

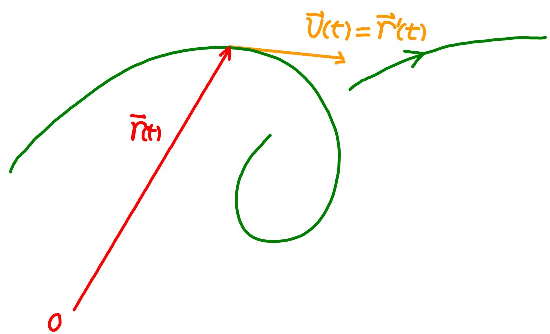
Thm. If $\vec{r}(t)$ is a vector-valued function, then

$$\int_a^b \vec{r}(t)\,dt = \left(\int_a^b x(t)\,dt,\ \int_a^b y(t)\,dt,\ \int_a^b z(t)\,dt \right).$$

- If $\vec{r}(t)$ is the velocity vector of a point moving in space, then $\int_a^b \vec{r}(t)\,dt$ computes the displacement vector from time a to b.

Eg. If the velocity vector of a point moving in space is given by $\vec{v}(t) = (2-t,\ t, 5t)$ What is the displacement of the point after 5 seconds from time zero ?

§3. Arc length formula



- The instantaneous velocity is a vector (valued function), its magnitude is a scalar function called the speed.

- If a point is moving in space with velocity vector

$$\vec{U}(t) = (x(t), y(t), z(t))$$

the total distance travelled from time a to b equals:

$$\int_a^b \|\vec{U}(t)\| \, dt = \int_a^b \sqrt{x(t)^2 + y(t)^2 + z(t)^2} \, dt$$

Thm. (*Arc length formula*) If $\vec{r}(t) = (x(t), y(t), z(t))$, $t \in [a, b]$, describes a space curve, then its total length is equal to

$$\ell = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt .$$

If $\vec{r}(t) = (x(t), y(t))$ is a plane curve, we can regard it as a space curve $(x(t), y(t), 0)$, and thus its length is given by

$$\ell = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + 0^2}\, dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt.$$

Eg. If a point moves from (a_0, b_0, c_0) to (a_1, b_1, c_1) in a straight line, what's is the total displacement?

- Independence of parametrization: If $\vec{u}(s) = \vec{v}(t(s))$ where $t(s)$ is a monotone increasing function, then

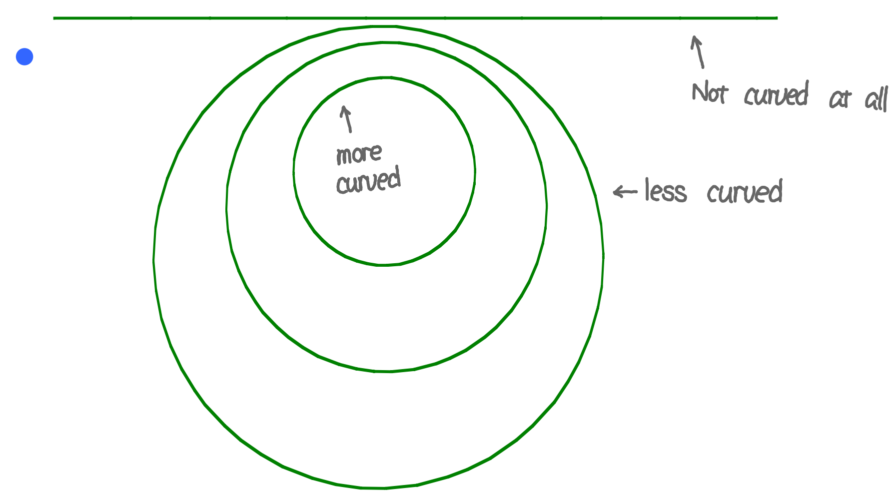
$$\begin{aligned}
\int_a^b \|\vec{u}'(s)\|\, ds &= \int_a^b \sqrt{\vec{u}'(s)\cdot\vec{u}'(s)}\; ds && \text{(def. of norm)} \\
&= \int_a^b \sqrt{(t'(s)\,\vec{v}'(t(s)))\cdot(t'(s)\,\vec{v}'(t(s)))}\; ds && \text{(chain rule)} \\
&= \int_a^b \sqrt{\vec{v}'(t(s))\cdot\vec{v}'(t(s))\;(t'(s))^2}\; ds \\
&= \int_a^b \|\vec{v}'(t(s))\|\; t'(s)\, ds && (t'(s) \geq 0 \text{ everywhere})
\end{aligned}$$

$$= \int_{t(a)}^{t(b)} \|\vec{v}'(t)\| \, dt \quad \text{(1-variable chain rule)}$$

$$\Longrightarrow \quad \boxed{\int_a^b \|\vec{u}'(s)\| \, ds = \int_{t(a)}^{t(b)} \|\vec{v}'(t)\|}$$

§4. Curvature

Given a smooth curve $\vec{r} = \vec{r}(t)$ we want to measure how "curved" it is at any place



In the above pictures, more "curved" is measured by measuring how fast the tangent vector is changing.

The unit tangent vector of the curve is

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}.$$

- The measurement must be done by experimenting with uniform speed (faster traveller experience faster directional changes).

Thus: parametrize $\vec{u} = \vec{u}(s)$, s: arc-length parameter.

$$s(t) = \int_0^t \|\vec{r}'(t)\|\,dt \quad \text{and} \quad \vec{r}(t) = \vec{u}(s(t)).$$

Using the arc-length parametrization, we have

$$\left\| \frac{d\vec{u}(s)}{ds} \right\| = \left\| \frac{\frac{d\vec{u}(s(t))}{dt}}{\frac{ds(t)}{dt}} \right\| = \left\| \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} \right\| = \frac{\|\vec{r}'(t)\|}{\|\vec{r}'(t)\|} = 1$$

i.e. speed is always 1.

- The curvature of $\vec{r} = \vec{r}(s)$ is

$$\boxed{\kappa(s) = \left\| \frac{d\vec{T}(s)}{ds} \right\|}$$

Using the original parametrization $\vec{r}(t) = \vec{u}(s(t))$:

$$\kappa(t) = \left\| \frac{d\vec{T}(s)}{ds} \right\| = \left\| \frac{\frac{d\vec{u}'(s(t))}{dt}}{\frac{ds}{dt}} \right\| = \left\| \frac{\vec{u}''(s(t))s'(t)}{s'(t)} \right\| = \|\vec{u}''(s(t))\|$$

Eg. Compute the curvature of a circle of radius r.

Thm. The curvature of $\vec{r}(t)$ is given by

$$\kappa(t) = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}.$$

Reason : $\vec{u}(s)\cdot\vec{u}'(s)=1 \Longrightarrow \vec{u}'(s)\cdot\vec{u}''(s)=0 \Longrightarrow \vec{u}''(s)\perp\vec{u}'(s)$

$$\Longrightarrow \|\vec{u}''(s)\| = \|\vec{u}'(s)\times\vec{u}''(s)\|$$

Now $\quad \vec{u}(s(t))=\vec{r}(t) \Longrightarrow \vec{u}'(s)\cdot s'(t)=\vec{r}'(t) \qquad (1)$

$$\Longrightarrow \vec{u}''(s)(s'(t))^2+\vec{u}'(s)\cdot s''(t)=\vec{r}''(t) \qquad (2)$$

Now $(1)\times(2) \Longrightarrow \vec{r}'(t)\times\vec{r}''(t)=(\vec{u}'(s)\times\vec{u}''(s))(s'(t))^3$, and $s'(t)=\|\vec{r}'(t)\|$, we get

$$\kappa(t)=\|\vec{u}''(s(t))\|=\|\vec{u}'(s(t))\times\vec{u}''(s(t))\|=\frac{\|\vec{r}'(t)\times\vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}.$$

§5. Motion in space

We have seen that if a particle is moving in space with its position trajectory described by $\vec{r} = \vec{r}(t)$, then the instantaneous velocity vector

$$\vec{v}(t) = \vec{r}\,'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

is a vector-valued function.

In contrast, the speed is the scalar function

$$s(t) = \|\vec{v}(t)\|$$

The acceleration vector is the instantaneous rate of changing of velocity :

$$\vec{a}(t) = \vec{v}'(t) = \lim_{h \to 0} \frac{\vec{v}(t+h) - \vec{v}(t)}{h}$$

Newton's second law of motion. If $\vec{F}(t)$ is the force exerted on a particle of mass m, then

$$\boxed{\vec{F}(t) = m\,\vec{a}(t)}$$

Eg. Find the velocity, speed and acceleration of a particle in space with position vector

$$\vec{r}(t) = e^{t}(\cos t,\ \sin t,\ t)$$

If the particle has mass m, what is the force $\vec{F}(t)$ exerted on it at time t?