Functions of several variables

Reading: Textbook, §14.1 - 14.4.

§1. Functions of several variables

Many rules in nature depend on more than one factor. Thus it is natural to investigate functions in several variables.

Def. A function of n variables $f = f(x_1, \cdots, x_n)$ $(n \geq 1)$ is a rule that assigns to each ordered pair of real numbers $(x_1, \cdots, x_n)$ in a set $D (\subseteq \mathbb{R}^n)$ a unique (real) number $f(x_1, \cdots, x_n)$

The set D is called the domain of f. The set of all values of f taken on D is called the range of f.

E.g. What is the domain and range of the function

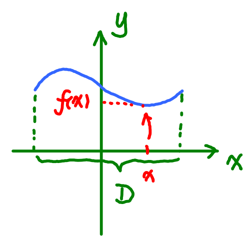
$$f(x,y) = \ln(9 - x^2 - 9y^2).$$

- Functions in 2 variables

In this case, we can describe the function by its graph.

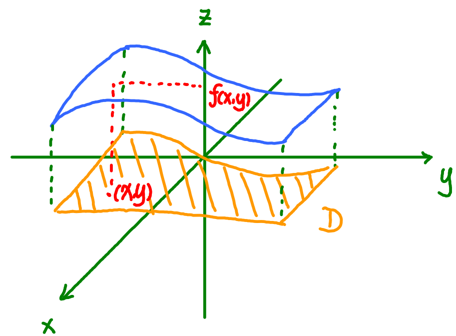
In the one variabe case $f(x)$, the graph is the curve

$\Gamma_f := \{(x,y) \mid y = f(x)\}$.



In the 2-variable case $f(x,y)$, the graph Γ_f live inside $D \times \mathbb{R}$ $(\subseteq \mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^3)$:

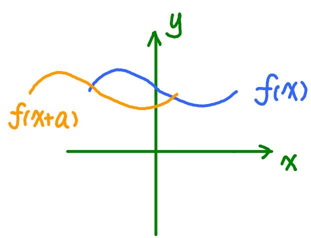
$$\Gamma_f = \{(x,y,z) \mid z = f(x,y),\ (x,y) \in D\}.$$

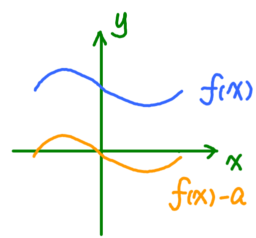


Eg. Plot the graph of $f(x,y) = ax + by + c$

Eg. Draw the graph of the function $f(x,y) = \sqrt{25 - x^2 - y^2}$. What is the domain and range of $f(x,y)$.

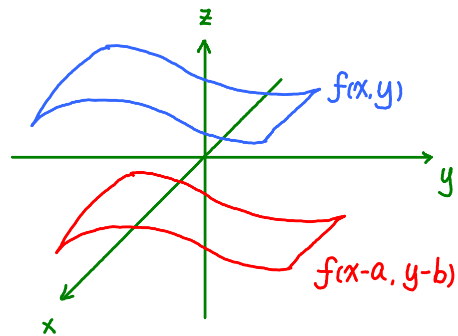
As in the 1-variable case, one can move/stretch the graph of a function :



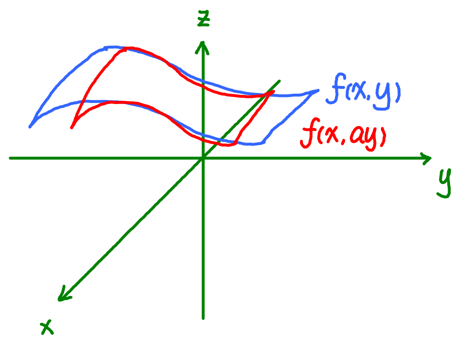


- If you want to shift the graph of $f(x,y)$ in the positive x-direction by a units, you change $f(x,y)$ into $f(x-a,y)$

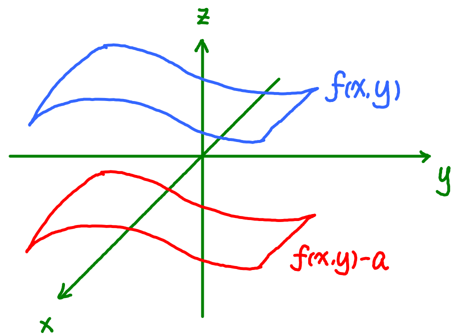
(y- direction) $f(x,y-a)$



- If you want to compress the graph of $f(x,y)$ in the x- (y-) direction by a factor of a, you change $f(x,y)$ into $f(ax,y)$ ($f(x,ay)$)



- If you want to shift the graph of a function up by a, you change $f(x,y)$ into $f(x,y)+a$



Eg. Discuss the graphs of the functions

(a) $f(x,y) = \sqrt{1-(x-1)^2-y^2}$.

(b) $g(x,y) = 4x^2+y^2$.

(c) $h(x,y) = 4-(x^2+y^2)$.

The graph approach doesn't work well for functions of 3 or more variables. Another way that works better is to draw the level curves of f. i.e., the set of points in the domain on which f takes a fixed value.

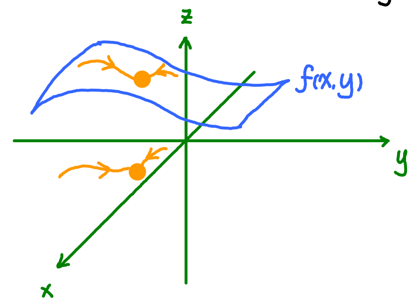
Eg. Find the level curves of the function $f(x,y) = x^2+y^2+3$. Compare it with the graph.

For functions of 3-variables, $f(x,y,z) = \text{const}$ is usually a surface, called the level surfaces.

Eg. Describe the level surfaces of $f(x,y,z) = x^2+y^2-z$.

§2. Limits and continuity

Intuitively, a 2-variable function $f(x,y)$ is continuous on D if its graph over D has no breaks or jumps.



Locally, no matter in which way you approach (x_0,y_0) in D, you arrive at the same height $f(x_0,y_0)$

Eg. Discuss the limit of the functions $f(x,y) = x^2+y^2$ and $g(x,y) = \frac{xy^2}{x^2+y^4}$

Def. A function of two variables is called continuous at (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b).$$

Likewise, a function of 3 (or more) variables is continuous at (a,b,c) if

$$\lim_{(x,y,z)\to(a,b,c)} f(x,y,z) = f(a,b,c).$$

f is a continuous function if it is continuous at every point in its domain.

Thus it is important to know if the limit of a multivariable function exists or not.

Prop. (1). If $f(x,y,z)$ and $g(x,y,z)$ have limits at (a,b,c), then so does $f(x,y,z)+g(x,y,z)$, with limit $f(a,b,c)+g(a,b,c)$.

(2). A multivariable polynomial function has limits everywhere.

(3). If $f(x,y,z)$ and $g(x,y,z)$ have limits at (a,b,c), and $g(a,b,c)\neq 0$, then so does $\frac{f(x,y,z)}{g(x,y,z)}$, with limit $\frac{f(a,b,c)}{g(a,b,c)}$.

(4). If $f(x,y,z)$ has limit at (a,b,c) and $g(x)$ has limit at $f(a,b,c)$, then $gf(x,y,z)$ has limit at (a,b,c).

Eg. Find the limit

$$\lim_{(x,y,z)\to(2,1,1)} \frac{4-xyz}{x^2+y^2+3z^2}$$

Eg. Find the limit

$$\lim_{(x,y,z)\to(0,0,0)} \ln(1-x^2-y^2-z^2)$$

Eg. Show that the limit does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{y^2 \sin^2 x}{x^4+y^4}$$

(Warning: Do not only test along coordinate axis!)

Thm (Squeeze Thm). If $f(x,y,z)$, $g(x,y,z)$ have the same limit at (a,b,c) and $f(x,y,z) \geq h(x,y,z) \geq g(x,y,z)$ on a domain containing (a,b,c), then $h(x,y,z)$ also has the same limit at (a,b,c).

Eg. Use polar coordinates to find the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^4+y^4}{x^2+y^2}.$$

Translating continuity at a point to a domain, we have

Prop. (1) If $f(x,y,z)$ and $g(x,y,z)$ are continuous on D then so is $f(x,y,z)+g(x,y,z)$.

(2). A multivariable polynomial function is continuous everywhere.

(3). If $f(x,y,z)$ and $g(x,y,z)$ are continuous on D, and $g(a,b,c) \neq 0$ on D, then so is $\frac{f(x,y,z)}{g(x,y,z)}$ continuous on D.

(4). If $f(x,y,z)$ is continuous and $g(x)$ is continuous on the range of $f(x,y,z)$, then so is $gf(x,y,z)$ continuous.

Eg. Determine the set of points on which

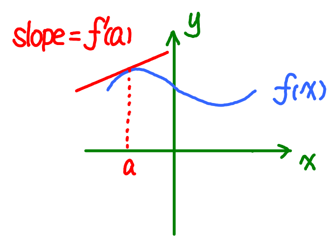
$$f(x,y,z) = \ln(1-x^2-y^2-z^2)$$

is continuous.

§3. Partial derivatives

We start with the 2-variable case. We will investigate how fast a function changes in any direction. In particular, how fast it changes along coordinate axis.

In the one-variable case, how fast $f(x)$ is changing is locally measured by its derivative

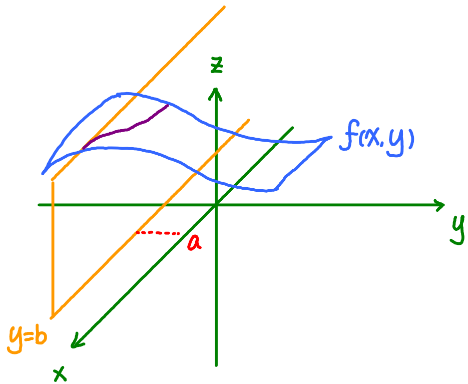


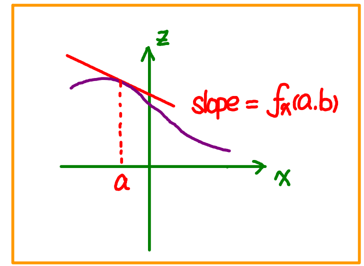
Def. If (a,b) is in the domain of $f(x,y)$, then the **partial derivatives** of $f(x,y)$ at (a,b) are defined by

$$\frac{\partial f}{\partial x}(a,b)\left(=f_x(a,b)\right) := \lim_{h\to 0}\frac{f(a+h,b)-f(a,b)}{h},$$

$$\frac{\partial f}{\partial y}(a,b)\left(=f_y(a,b)\right) := \lim_{h\to 0}\frac{f(a,b+h)-f(a,b)}{h}.$$

Comparing with the 1-variable case, one can see that $f_x(a,b)$ measures how fast the function $f(x,b)$ is changing at $x=a$.





Eg. Let $f(x,y) = e^{x/y}$. Compute its partial derivatives at the point $(0,1)$.

Similarly, for 3 (or more) variable functions

$$\frac{\partial f}{\partial x}(a,b,c) \ (= f_x(a,b,c)) := \lim_{h\to 0} \frac{f(a+h,b,c) - f(a,b,c)}{h} \quad \text{etc.}$$

If f_x, f_y etc are also functions that can be differentiated, we can define

$$f_{xx} := \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \quad f_{xy} := \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right), \quad f_{yx} := \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right),$$

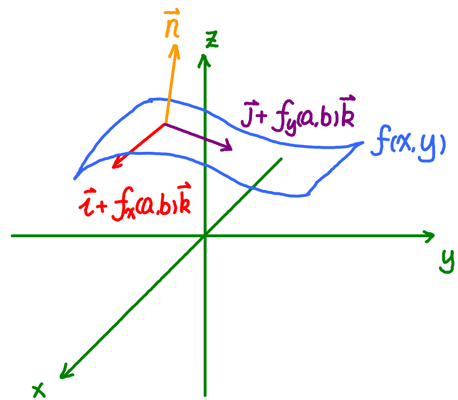
$$f_{yy} := \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) \cdots$$

Eg. Compute the mixed derivatives f_{xz} and f_{zx} for

$$f(x,y,z) = \cos(x^2yz).$$

Thm. (Clairaut) Suppose $f(x,y)$ is defined near (a,b), and suppose f_{xy}, f_{yx} are also continuous in that neighborhood. Then $f_{xy}(a,b) = f_{yx}(b,a)$.

Geometrically, we have found two tangent vectors of the graph $z = f(x,y)$.



Thus the normal vector $\vec{n}$ can be computed as

$$\vec{n} = (\vec{\imath} + f_x(a,b)\vec{k}) \times (\vec{\jmath} + f_y(a,b)\vec{k})$$
$$= \vec{\imath}\times\vec{\jmath} + f_x(a,b)\,\vec{k}\times\vec{\jmath} + f_y(a,b)\,\vec{\imath}\times\vec{k}$$
$$= \vec{k} - f_x(a,b)\,\vec{\imath} - f_y(a,b)\,\vec{\jmath}$$

The tangent plane to the graph at $(a, b, f(a,b))$ thus has equation

$$-f_x(a,b)(x-a) - f_y(a,b)(y-b) + (z - f(a,b)) = 0$$

$$\Longleftrightarrow \quad \boxed{z = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)}$$

Eg. Show that the tangent plane of the elliptic parabloid $\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ at a point (x_0, y_0, z_0) equals $\frac{z+z_0}{c} = \frac{2xx_0}{a^2} + \frac{2yy_0}{b^2}$.

Said differently, the tangent plane offers the best linear approximation for the graph $z = f(x,y)$ at $(a, b, f(a,b))$. Thus the best linear function that approximates $f(x,y)$ near (a,b) is just

$$f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)$$

Thus, if Δx, Δy are small, then

$$f(a+\Delta x, b+\Delta y) \approx f(a,b) + f_x(a,b)\Delta x + f_y(a,b)\Delta y.$$

As in the 1-variable case, we formalize Δx by dx, and let the total differential of $f(x,y)$ be

$$(df)(x,y) := f_x(x,y)\,dx + f_y(x,y)\,dy.$$

which measures $\Delta f(a,b) := f(a+\Delta x, b+\Delta y) - f(a,b)$ everywhere.

For functions of more variables:

$$(df)(x,y,z) := f_x(x,y,z)\,dx + f_y(x,y,z)\,dy + f_z(x,y,z)\,dz$$

$$\boxed{(df)(x_1,\cdots,x_n) := \sum_{i=1}^{n} f_{x_i}(x_1,\cdots,x_n)\,dx_i}$$

Eg. Find the total differential of $f(x,y,z) = \tan^{-1}(xy^2z)$.