Integrable Systems of Calogero-Moser type on moduli spaces of flat connections

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1. Moduli spaces of flat connections over surfaces.

2. Poisson structure, symplectic leaves, examples

3. Poisson commuting functions H_C associated to nonintersecting cycles

4. Superintegrability of Hamiltonians H_C

Examples.

5. Chord diagrams and the "universal" integrable system on it.

6. Quantization

" .... Calogero-Moser type ... "

(Calogero-Moser-Sutherland-Olshanetsky-Perelomov)

- spin Calogero-Moser

$$L = \begin{bmatrix} p_1 & & \frac{\mu_{ij}}{sh\left(\frac{q_i-q_j}{2}\right)} \\ & \ddots & \\ \frac{\mu_{ji}}{sh\left(\frac{q_j-q_i}{2}\right)} & & p_n \end{bmatrix},$$

$$\{p_i, q_j\} = \delta_{ij}, \quad \{p_i, p_j\} = 0, \quad \{q_i, q_j\} = 0$$

$$\{p_i, \mu_{kl}\} = \{q_i, \mu_{kl}\} = 0$$

$$\{\mu_{ij}, \mu_{kl}\} = \delta_{jk}\mu_{il} - \delta_{il}\mu_{kj}$$

μ_{ij} - coord. functions on $\mathcal{O} \subset sl_n^*$

$$p_1 + \dots + p_n = 0, \quad q_1 + \dots + q_n = 0$$

$tr(\mu^k) = c_k$, $k = 2, \dots, n$ are fixed

$\mu_{ii} = 0$

only H-inv comb. μ_{ij}

Phase space: $T^*\mathbb{R}^{n-1} \times \mathcal{O}(c)$

Reduced phase space $S(\mathcal{O}) = \left(T^*\mathbb{R}^{n-1} \times \mathcal{O}(c) /\!/ H\right) S_n$

Poisson commuting Hamiltonians

$$H_k = \frac{1}{k} tr(L^k)$$

$$H_2 = \frac{1}{2}(p,p) + \sum_{i<j} \frac{\mu_{ij}\mu_{ji}}{sh^2\left(\frac{q_i - q_j}{2}\right)} , \quad \text{on } S(\mathcal{O})$$

1) Calogero - Moser - Sutherland when $rank(\mathcal{O}) = 1$

$$\mu_{ij} = \varphi_i \varphi_j^* - (\varphi, \varphi^*) \frac{\delta_{ij}}{n} , \quad tr(\mu^2) = c$$

$\mathcal{O}//H$ is a point and $\mu_{ij} = \frac{c}{n(n-1)}$, $\mu_{ii} = 0$

(Kazhdan - Kostant - Sternberg)

$$dim(S(\mathcal{O})) = 2(n-1)$$

$\#$(independent Poisson commuting integrals) $= n-1 = \frac{dim(S(\mathcal{O}))}{2}$

Liouville integrability

2) spin Calogero - Moser when $rank(\mathcal{O}) > 1$

$\#$(independent Poisson commuting integrals) $< \frac{dim(S(\mathcal{O}))}{2}$

integrability = ?

Superintegrable systems

Degenerate integrability, noncommutative integrability; Pauli; Fock; Smorodinsky, Winterniz.... Nekhoroshev, Mischenko-Fomenko.

(M_{2n}, ω) symplectic manifold

- $J_1, \dots, J_{2n-k}$ - independent functions

$$\{J_i, J_j\} = F_{ij}(J)$$

generate Poisson subalgebra J

- $I_1, .., I_k$ - independent functions

$$\{I_i, I_j\} = 0, \quad \{I_i, J_j\} = 0$$

$\{I_1, .., I_k\}$ - generate the Poisson center of J, $B = Z(J)$

Algebraically:

$$\begin{array}{ccccccc} & \mathcal{Z}(J) & & & & & \\ & \parallel & & & & & \\ H \in & B & \subset & J & \subset & A = C^\infty(M_{2n}) \\ & \cup & & \cup & & & \\ & I_1, \dots I_K & & J_1 \dots J_{2n-K} & & & \end{array}$$

Poisson subalgebra

Geometrically:

$$B_K \xleftarrow{p} \mathcal{P}_{2n-K} \xleftarrow{\pi} M_{2n}$$

Poisson surjective

$\cup$

$p^{-1}(b)$ connected components are symplectic leaves

Poisson surjective

trivial Poisson structure

Let H be : $\{H, I_i\} = 0$, i.e. $H \in \mathcal{Z}(J)$

Theorem.

i) Connected components of level surfaces of $J_1, \dots, J_{2n-K}$ are invariant with respect to the flow of H.

ii) Each nonsingular connected component of a level surface has an affine structure, i.e. an atlas such that transition functions are affine transformations of $\mathbb{R}^K$ (translations and linear transformations) such that flow lines of H are linear in these coordinates.

iii) Generic connected components are isomorphic to $\mathbb{T}^\ell \times \mathbb{R}^{K-\ell}$ for some $0 \le \ell \le K$.

<u>Remark</u> $(p \circ \pi)^{-1}(B) \subset M_{2n}$ is coisotropic fibered over symplectic $p^{-1}(B)$; fibers = Liouv. tori

<u>Refinement</u>

If we have

$$B_1 \hookrightarrow B_2 \hookrightarrow J_2 \hookrightarrow J_1 \hookrightarrow A = C^\infty(M_{2n})$$

such $B_1 \subset J_1 \subset A$ and $B_2 \subset J_2 \subset A$

are superintegrable systems we say

$B_1 \subset J_1 \subset A$ is a refinement of $B_2 \subset J_2 \subset A$

Equivalence

Two superintegrable systems

$$B_1 \subset J_1 \subset A, \quad B_2 \subset J_2 \subset A$$

$A = C^\infty(M_{2n})$ are equivalent

if $\exists\ \varphi : M_{2n} \circlearrowleft$ symplectomorphism

such that

$$\varphi^*(B_2) = B_1, \quad \varphi^*(J_2) = J_1$$

Back to spin Calogero-Moser system

Spin Calogero-Moser system for any simple Lie algebra:

G - simple Lie group (affine algebraic)
(or its real form...)

$\mathfrak{g} = \mathrm{Lie}(G)$

T^*G - symplectic, lift of conjugation action is Hamiltonian

$\mu : T^*G \to \mathfrak{g}^*$, the moment map

$$(x, g) \in \underbrace{T^*G \simeq}_{\text{by right translations}} \mathfrak{g}^* \times G, \quad \mu(x, g) = x - Ad^*_{g^{-1}}(x)$$

T^*G/G - Poisson, $\mathcal{O} \subset \mathfrak{g}^*$ coadjoint orbit

$\cup$

$S(\mathcal{O}) = \mu^{-1}(\mathcal{O})/G$, symplectic leaf

$T^*G \simeq \mathfrak{g}^* \times G \xrightarrow{\pi} \mathfrak{g}^*$ projection to the first factor

$\pi^*(C(\mathfrak{g}^*)^G) \subset C(S(\mathcal{O}))$

Poisson commutative subalgebra

normally $r < \dim(\mathcal{O})/2$

$$S(\mathcal{O}) \supset S_{reg}(\mathcal{O}) \simeq \left(T^*H \times \mu_0^{-1}(\mathcal{O})\right)/N(H)$$

$$\simeq \left(T^*H \times \mathcal{O}/\!/H\right)/W$$

(p,h) $\quad$ μ_α

$\mu_0 : \mathcal{O} \to \mathfrak{h}^*$
moment map for
Ad^* action of $H \subset G$

$$H = \frac{1}{2} C_2 = \frac{1}{2}(p,p) + \sum_{\alpha>0} \frac{\mu_\alpha \mu_{-\alpha}}{\left(h_{\frac{\alpha}{2}} - h_{-\frac{\alpha}{2}}\right)^2}$$

$$h_\alpha = e^{(q,\alpha)}$$

For any coadjoint orbit it is
a superintegrable system (R. 2002)

$$\begin{tikzcd}
T^*G/G \simeq (\mathfrak{g}^* \times G)/G \arrow[d, "\pi"] & \supset S(\mathcal{O}) \arrow[d, "\pi"] \\
(\mathfrak{g}^* \times_{\mathfrak{g}^*/G} \mathfrak{g}^*)/G \arrow[d, "p"] & \supset J(\mathcal{O}) \arrow[d, "p"] \\
\mathfrak{g}^*/G & \supset B(\mathcal{O})
\end{tikzcd}
\qquad
\begin{tikzcd}
G(x,g) \arrow[d, mapsto, "\mu_L \times \mu_R"] \\
G(x, -Ad^*_{g^{-1}}(x)) \arrow[d, mapsto] \\
Gx
\end{tikzcd}$$

$J(\mathcal{O}) = \pi(S(\mathcal{O}))$, $B(\mathcal{O}) = p(J(\mathcal{O}))$

$S(\mathcal{O}) = \mu^{-1}(\mathcal{O})/G$,

Superintegrability:

$$\dim B + \dim J = \dim S$$

<u>Ruijsenaars-Schneider (& spin R-S)</u>

$$\begin{array}{ccccc}
T^*G/G \simeq (\mathfrak{g}^* \times G)/G & \supset & S(\mathcal{O}) & \ni & G(x,g) \\
\downarrow \pi & & \downarrow \pi & & \downarrow \\
(\mathfrak{g}^* \times G)/G & \supset & J(\mathcal{O}) & \ni & (x - \mathrm{Ad}^*_{g^{-1}}(x), g) \\
\downarrow p & & \downarrow p & & \downarrow \\
G/G & \supset & B(\mathcal{O}) & \ni & G g
\end{array}$$

Superintegrability

$$\dim B + \dim J = \dim S$$

When $\mathcal{O} \subset SL_n$ has rank 1 this is RS.

<u>Duality</u>: sCM $\longleftrightarrow$ sRS

(action, angle) $\longleftrightarrow$ (angle, action)

Relativistic versions

$G \times G$, Poisson structure of the Heisenberg double of the standard Poisson Lie structure on G (nonlinear version of T^*G)

$$
\begin{array}{ccccc}
(G\times G)/G & \supset & S(C) & \ni & G(x,y) \\
\downarrow & & \downarrow & & \\
G \tilde{\times}_{G/G} G & \supset & J(C) & \ni & G(x, y^{-1}x^{-1}y) \\
\downarrow & & \downarrow & & \\
G/G & \supset & B(C) & \ni & Gx
\end{array}
$$

$$S(C) = \{(x,y) \in G\times G \mid xyx^{-1}y^{-1} \in C\}/G$$

$C \in G/\mathrm{Ad}_G$ Relativistic Calogero-Moser

Liouville integrability nonspin $G = SL_n$, C has rank 1 (Ruijsenaars, Fock & Rosly, 1992)

Superintegrability, spin (R., 2016)

Examples $\quad G = SL_n$

Thm (Fock, Rosly 1992 (ArXive 1998)) If $\operatorname{rank}(C) = 1$, then

(i) $\dim S(C) = 2(n-1)$

(ii) $\operatorname{tr}(x), \operatorname{tr}(x^2), \ldots, \operatorname{tr}(x^{n-1})$ Poisson commuting Hamiltonians of a Liouville integrable system.

In coordinates. Representative of $Ad_G(x)$:

$$x = \begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{pmatrix}, \quad x_i \neq x_j \text{ (regular)}$$

$$S(\mathcal{O})^{reg} \cong \Big(\underset{u}{(\mathbb{C}^*)^n} \times \underset{x}{(\mathbb{C}^*)^n_{reg}}\Big)/S_n \subset S(\mathcal{O}) \quad \text{open dense}$$

$$\operatorname{tr}_{\omega_2}(A) = \sum_{i<j} u_i u_j \prod_{\substack{a \in \{ij\} \\ b \in \{ij\}^\vee}} \frac{1 - q^{-1} x_a x_b^{-1}}{1 - x_a x_b^{-1}}, \quad (u \leftrightarrow y)$$

Similarly $\operatorname{tr}(y), \operatorname{tr}(y^2), \ldots, \operatorname{tr}(y^{n-1})$ also define a Liouville integrable system, Ruijsenaar-Schnider system.

Relativistic version of spin CM $\leftrightarrow$ spin RS duality:

$$\sigma : G \times G \circlearrowleft, \quad \sigma(x,y) = (y,x)$$

Note: $(G \times G)/G \simeq \pi_1(T \setminus pnt)$

and $\sigma \in$ Mapping class group of $T \setminus pnt$

Moduli space of flat connections

Σ - compact, oriented surface

G - compact simple Lie group

Conn^G_Σ - space of connections on a principle G-bundle over Σ

$\cup$

FConn^G_Σ - flat connections
locally $\Omega^1(M, \mathfrak{g})$

$$\mathcal{M}^G_\Sigma = \mathrm{FConn}^G_\Sigma \Big/ \{\text{Gauge group} = \mathrm{Maps}(\Sigma, G)\}$$

$$g : A \mapsto g^{-1} A g + g^{-1} dg$$

Finite dimensional, natural identification

$$\mathcal{M}^G_\Sigma = (\pi_1(\Sigma) \to G)/G$$

Finite dimensional model

$$\mathcal{M}^{G}_{\Sigma_{g,n}} \simeq \left(G^{\times 2g} \times G^{\times (n-1)}\right)/G$$

holonomies along fundamental cycles of Σ

holonomies around $(n-1)$ components of $\partial\Sigma$

$$x_1 y_1 x_1^{-1} y_1^{-1} \cdots x_g y_g x_g^{-1} y_g^{-1} z_1 \cdots z_n = 1$$

Poisson manifold (singular)

Poisson structure on $\mathrm{Hom}(\pi_1(\Sigma) \to G)$

(Fock, Rosly, 1992 (ArXiv 1998))

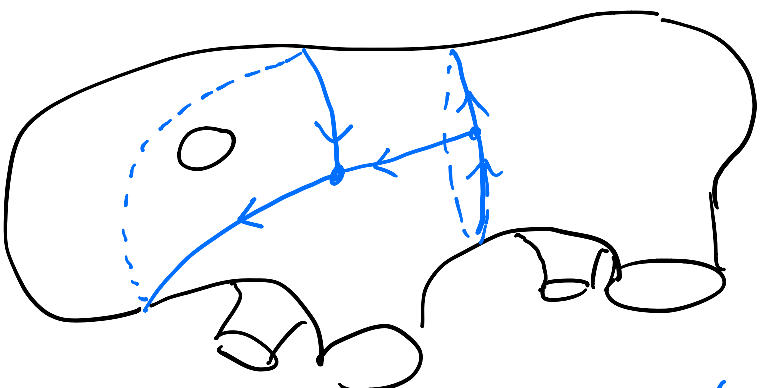
"Nonlinear" version of $(T^*G)^{\times g} \times (\mathfrak{g}^*)^{\times(n-1)}$

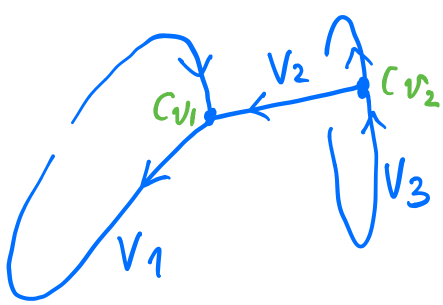
$\mathcal{M}^{G}_{\Sigma}$ = a "nonlinear" version of

$$(T^*G)^{\times g} \times (\mathfrak{g}^*)^{\times(n-1)}$$

Poisson structure on $\mathcal{M}^G_\Sigma$ in terms of graph functions

(Goldman 1982, Fock & Rosly 1992, Andersen, Mattes, R., 1996)





coloring: $e \mapsto V_e$

a finite dimensional G-module

$v \mapsto c_v$

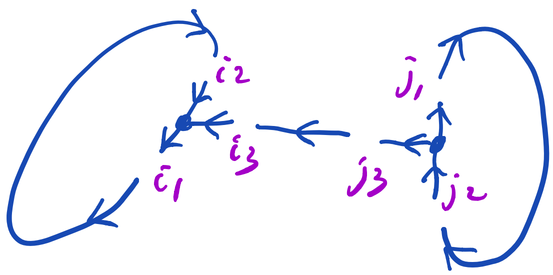
$c_v \in (V_{e_1}^{\pm} \otimes \dots \otimes V_{e_n}^{\pm})^G$

an invariant vector

$V^+ = V, \quad V^- = V^*$

choice of total ordering

Choose a basis in each V_e



$$F_{\Gamma, V, c}(A) =$$

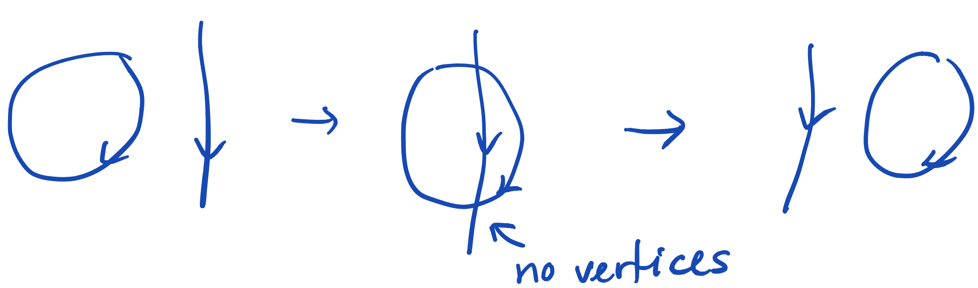
$$\sum_{\substack{\{i_1 i_2 i_3\} \\ \{j_1 j_2 j_3\}}} \left(hol^A_{e_1}\right)^{i_2}_{i_1} \left(c_{v_1}\right)^{i_1}_{i_2 i_3} \left(hol^A_{e_2}\right)^{i_3}_{j_3} \left(c_{v_2}\right)^{j_3 j_1}_{j_2} \left(hol^A_{e_3}\right)^{j_2}_{j_1}$$

is independent on total ordering of edges around vertices

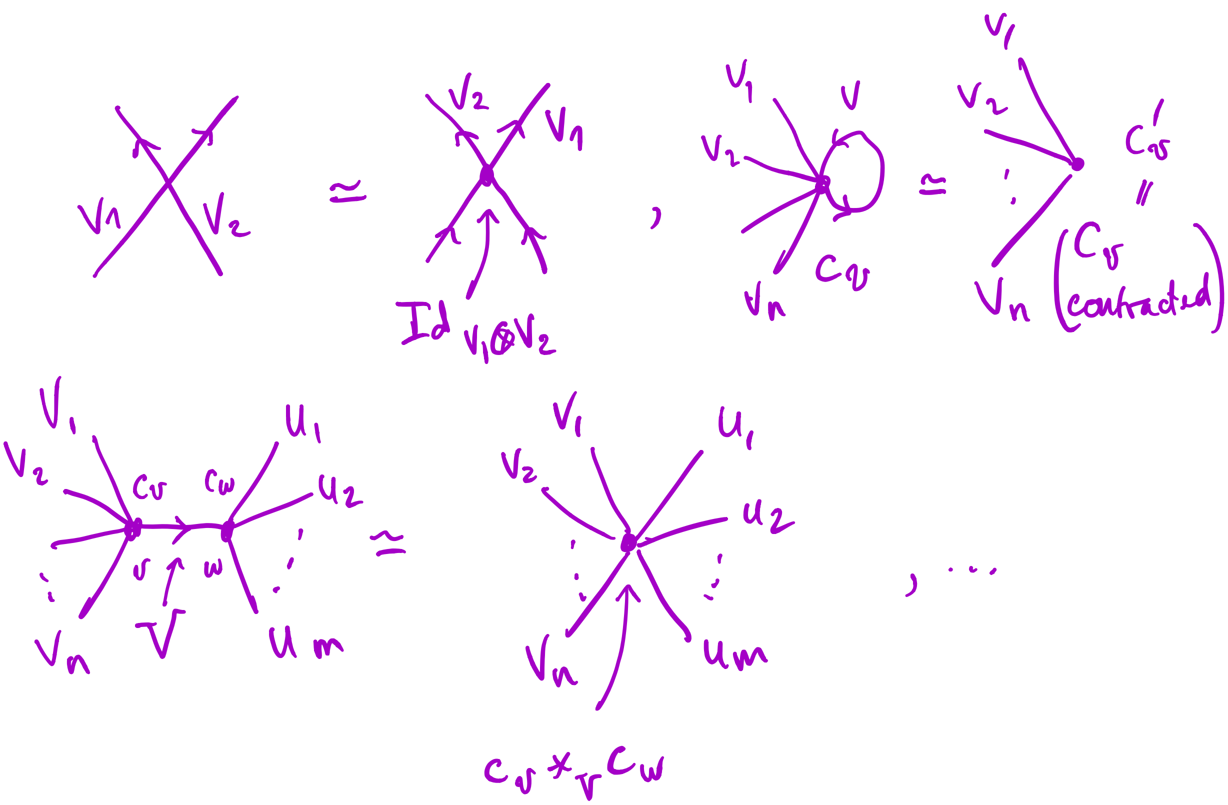
<u>Proposition</u> (i) $F_{\Gamma, V, c}(A^g) = F_{\Gamma, V, c}(A)$

(ii) $F_{\Gamma_1, V, c}(A) = F_{\Gamma_2, V, c}(A)$

if $\Gamma_1 \sim \Gamma_2$ homotop (can be continuosly deformed, intersections are allowed)



Equivalences between graph functions



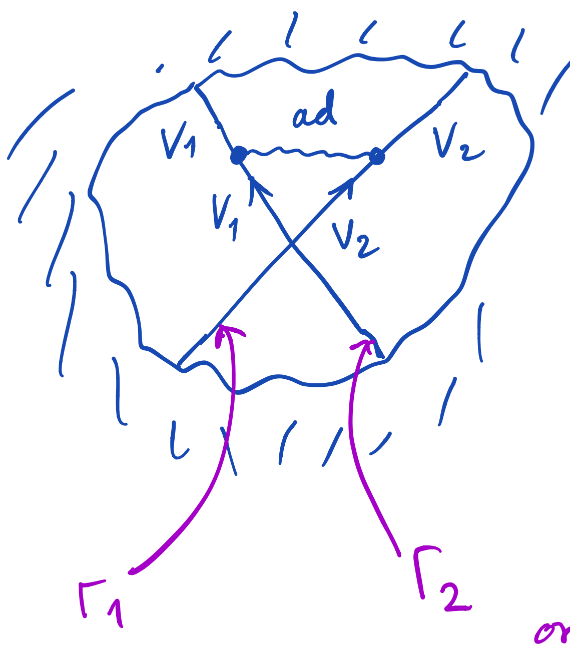
<u>Thm</u>. Graph functions span the ring of polynomial functions

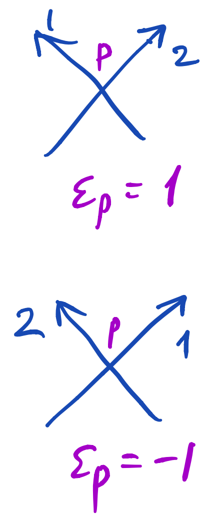
$$\mathcal{O}(\mathcal{M}^G_\Sigma) = \mathcal{O}(\pi_1(\Sigma) \to G)^G$$

$\Rightarrow$ we can describe the Poisson structure on $\mathcal{M}^G_\Sigma$ in terms of Poisson brackets between graph functions.

Poisson brackets:

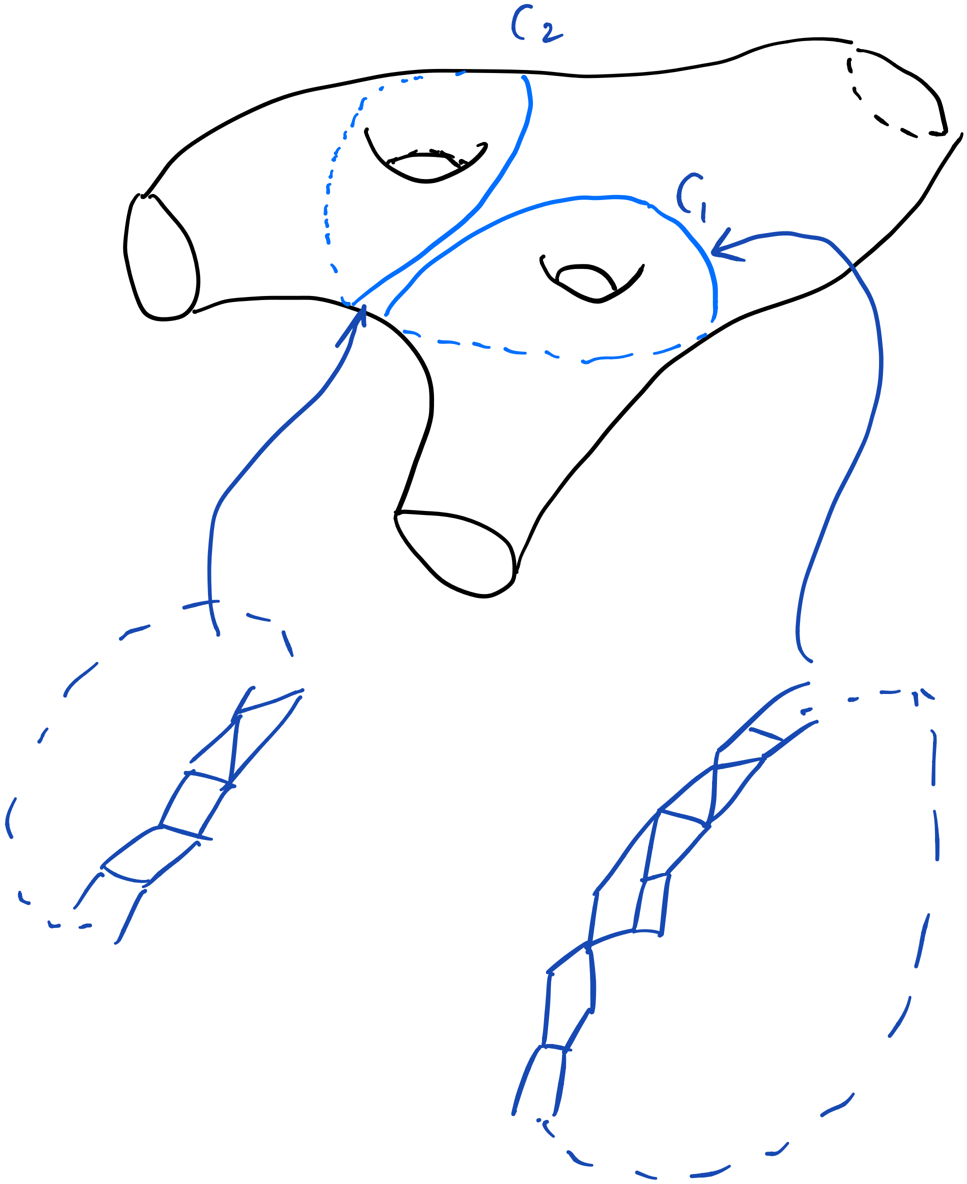
$$\{F_{\Gamma_1,V_1,c_1}, F_{\Gamma_2,V_2,c_2}\} = \sum_{p\in\Gamma_1\cap\Gamma_2} \varepsilon_p \, F_{\Gamma_1 *_p \Gamma_2,\, V_1,V_2,c_1,c_2}$$





Here we use the orientation of Σ and of Γ_1 & Γ_2.

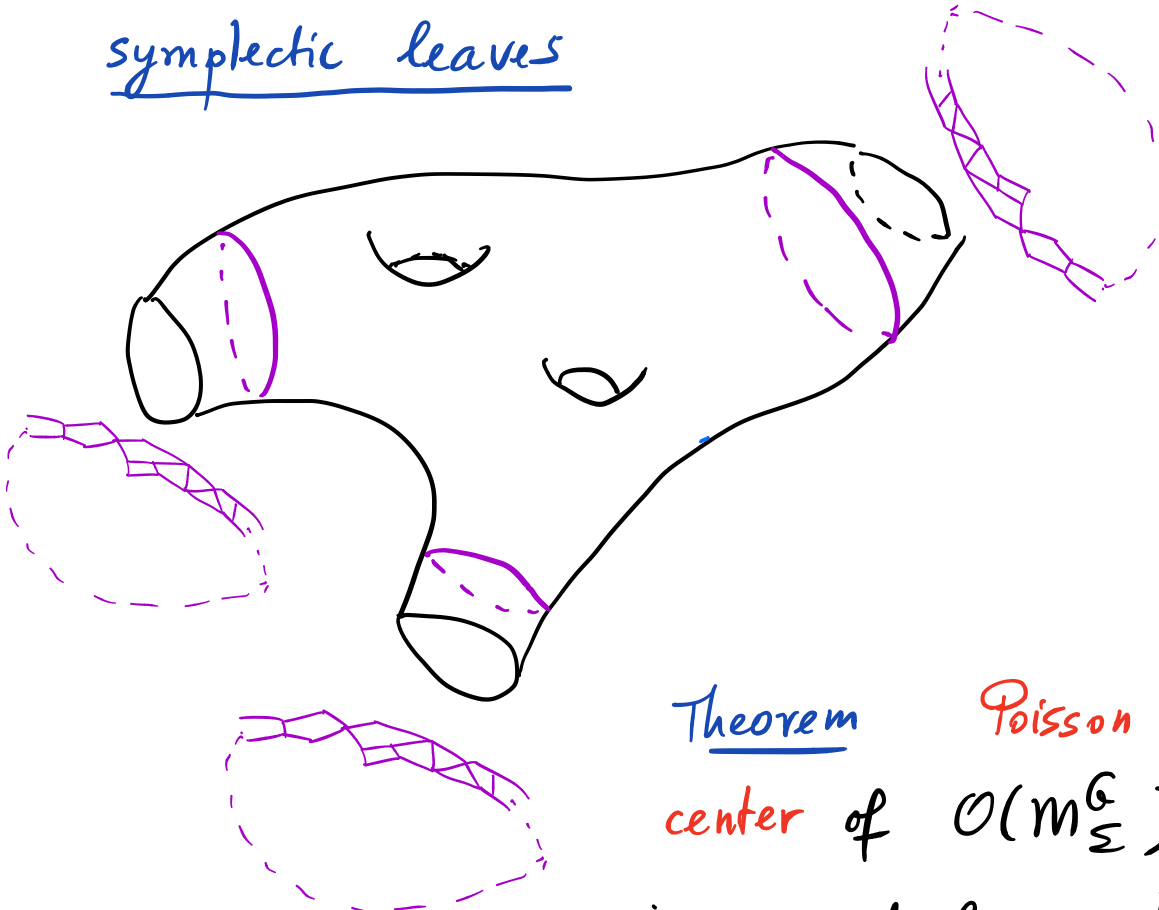
Poisson commutative subalgebras



B_{C_i} - Poisson commutative (any graph can be incerted)

$\{B_{C_1}, B_{C_2}\} = 0$

Casimir functions, Poisson center and symplectic leaves



Theorem Poisson center of $\mathcal{O}(\mathcal{M}_\Sigma^G)$ is spanned by graph functions contractible to boundary components of Σ.

Connected components of level curves are symplectic leaves $S \subset \mathcal{M}_\Sigma^G$

Equivalent to fixing conjugacy classes of holonomies around connected components of $\partial\Sigma$.

Superintegrable systems on $\mathcal{M}^G_\Sigma(C)$

Let $C_1, \dots, C_k$ be k nonintersecting nonselfintersecting curves on Σ.

Let $B_C \subset \mathcal{O}(\mathcal{M}^G_\Sigma)$ be the subalgebra generated by graph functions contractible to one of C_i.

Let $J_C \subset \mathcal{O}(\mathcal{M}^G_\Sigma)$ be the subalgebra generated by graph functions on $\Sigma \setminus C$

Theorem [Artamonov, R. 2019] Poisson embeddings

$$B_C \subset J_C \subset \mathcal{O}(\mathcal{M}^G_\Sigma)$$

is a superintegrable system.

i.e. #(of independent functions in B_C)+

+ #(of independent functions in J_C) =

= dim ($\mathcal{M}_\Sigma^G$)

If we write $B_C = C^\infty(\mathcal{B}_C)$, $J_C = C^\infty(\mathcal{J}_C)$

we have

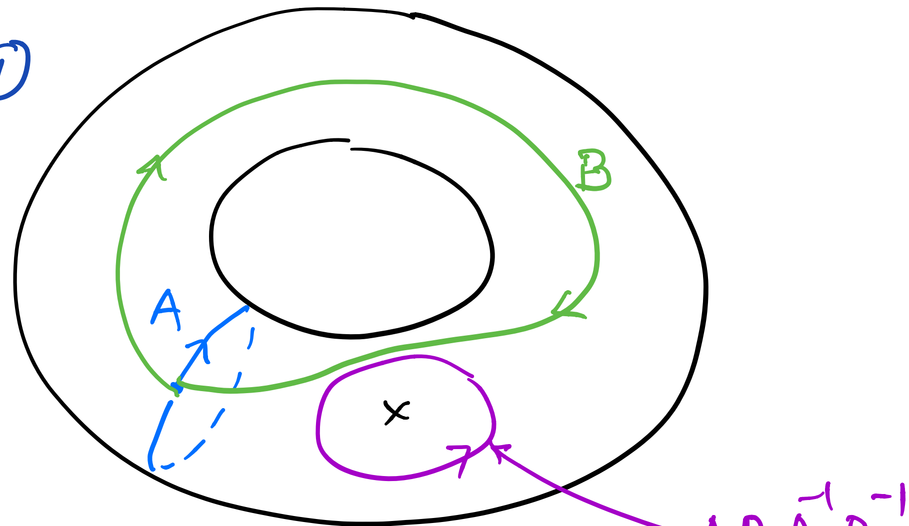
$$\dim(\mathcal{B}_C) + \dim(\mathcal{J}_C) = \dim(\mathcal{M}_\Sigma^G)$$

Mapping class group acts on superintegrable systems

$$B_C \subset J_C \subset A$$

making many of them equivalent.

Examples ①



$C = A$

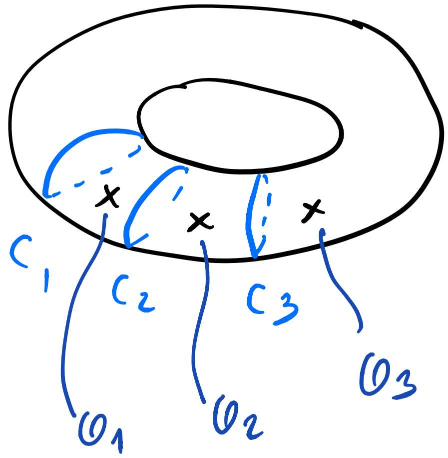
J_C = traces of holonomies along cycles nonintersecting C

$= \langle A, BA^{-1}B^{-1} \rangle^G$

Relativistic spin Calogero-Moser & spin Ruijsenaars-Schneider models

$CM \leftrightarrow SC \sim A \leftrightarrow B$, mapping class group

2



When $G = SL_n$ and $\mathcal{O}_i$ are conjugation orbits of rank 1

$\#$ of independent integrals $= k(n-1)$

$$\dim(\mathcal{M}^G_\Sigma) = 2k(n-1)$$

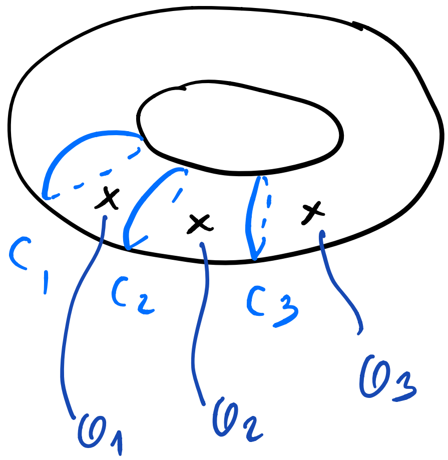
- If $C = C_1 \sqcup C_2 \sqcup \dots \sqcup C_k$ we have Liouville integrable system
- Choosing $B = B_{C_i}$ we have a superintegrable refinement

Chalykh, Fairon, 2018

Arutyunov, Olivecci, 2019

Coordinate description, similar to $k=1$ case

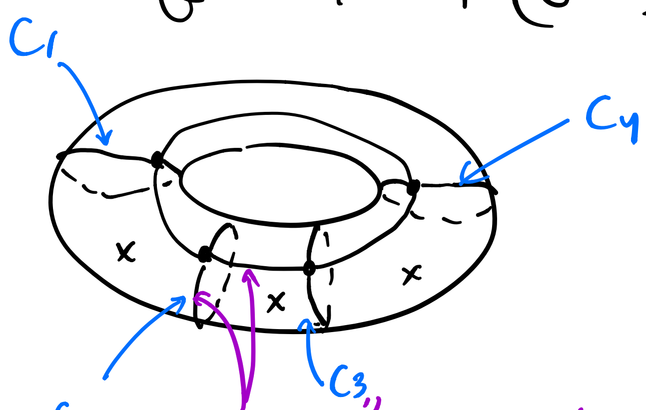
③



Classical limit of the dynamical qKZ (without spectral variables)

"Linearized" case:

$$G^{\times n} : T^*(G^{\times n}) \circlearrowleft$$



$C = (C_1, \dots, C_4)$

$\underset{x\quad g}{G\times G}$ "linearization" $\rightsquigarrow T^*G$

$$h(x, g) = \left(Ad^*_{h_n}(x_1),\ Ad^*_{h_1}(x_2), \dots, Ad^*_{h_{n-1}}(x_n);\ h_n g_1 h_1^{-1},\ h_1 g_2 h_2^{-1}, \dots, h_{n-1} g_n h_n^{-1} \right)$$

Hamiltonian with the moment map

$$\mu(x,g) = \left(x_2 - Ad^*_{g_1^{-1}}(x_1),\ x_3 - Ad^*_{g_2^{-1}}(x_2), \dots, \right.$$
$$\left. x_1 - Ad^*_{g_n^{-1}}(x_n) \right)$$

Symplectic leaf $S(\mathcal{O}) \subset T^*(G^{\times n})/G^{\times n}$

$$S(\mathcal{O}) = \mu^{-1}(\mathcal{O})/G^{\times n}$$

Superintegrable system corresponding to $C = (C_1, \dots, C_n)$:

$$\mathcal{P}(\mathcal{O}) = \{ (x_n, x_1, \dots, x_{n-1}), (y_1 \dots y_n) \mid G x_{i+1} = G y_i,$$
$$x_{i+1} - y_i \in \mathcal{O}_i \} / G^{\times n}$$

$$B(\mathcal{O}) = \mathcal{P}(\mathcal{O})/G^{\times n} \times G^{\times n} \simeq$$
$$\simeq \{ \underset{\substack{\parallel \\ Gx_1}}{\mathcal{O}^{(1)}}, \dots, \underset{\substack{\parallel \\ Gx_n}}{\mathcal{O}^{(n)}} \subset \mathfrak{g}^* \mid \mathcal{O}^{(i+1)} - \mathcal{O}^{(i)} \subset \mathcal{O}_i \}$$

Hamiltonians: $\quad H_K^{(i)} = c_K(x^{(i)})$

$$S(G) \supset S_{reg}(\mathcal{O}) \simeq \Big(\underset{\substack{\uparrow \\ p}}{T^*} \underset{\substack{\uparrow \\ h}}{H} \times \big(\underset{\substack{\uparrow \\ \mu^{(1)}}}{\mathcal{O}_1} \times \cdots \times \underset{\substack{\uparrow \\ \mu^{(n)}}}{\mathcal{O}_n} \big) /\!/ H \Big) \Big/ W$$

$$\mathcal{D}^{(i)} = \frac{1}{2}\left(H_2^{(i+1)} - H_2^{(i)} \right), \quad i = 1, \dots, n-1$$

$$\mathcal{D}^{(i)} = (\mu_0^{(i)}, p) + \sum_{j \neq i} r_{ij} \qquad ,$$

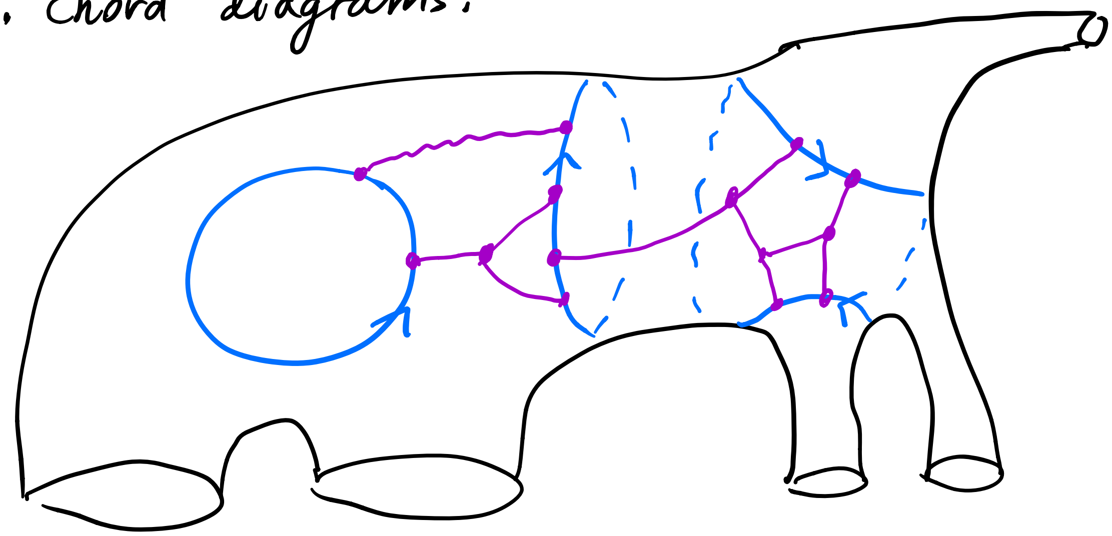
$$r_{ij} = \frac{1}{2}(\mu_0^{(i)}, \mu_0^{(j)}) + \sum_{\alpha} \frac{\mu_\alpha^{(i)} \mu_{-\alpha}^{(j)}}{h_\alpha - h_{-\alpha}}$$

and $H_2^{(n)}$ - spin CM "interacting" with $\mu^{(1)}, \dots, \mu^{(n)}$

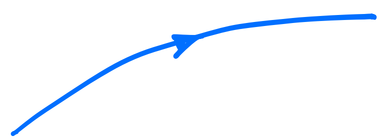
(part of an ongoing project with J. Stokman)

Chord diagrams and the "universal" superintegrable system

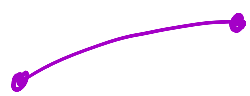
· chord diagrams:



Solid lines, oriented closed.

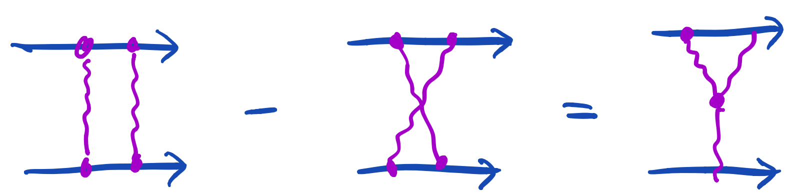


Chords, nonoriented



can end on a chord at a trivalent vertex or on a solid line.

Relations



• The space of chord diagrams

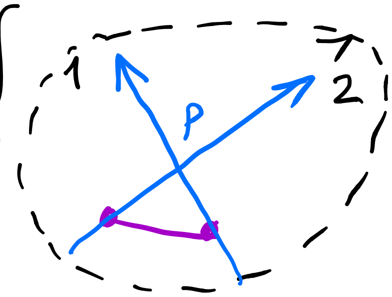
$$Ch_\Sigma = \left\{ \begin{array}{l} \text{linear combinations}/\mathbb{Z} \text{ of homotopy} \\ \text{classes of chord diagrams} \end{array} \right\} \Big/ \{\text{relations}\}$$

• Commutative algebra structure:

$$[D_1][D_2] = [D_1 \sqcup D_2]$$

• Poisson brackets:

$$\{[D_1],[D_2]\} = \sum_{p\in D_1\cap D_2} \varepsilon_p \left[D_1 *_p D_2\right]$$

$D_1 *_p D_2 = \Big\{$  $\Big\}$

G- colored chord diagrams

Universality:

$$\begin{tikzcd}
G\text{-colorde chord diagrams on } \Sigma \arrow[r] \arrow[d] \arrow[ddr] & \text{chord diagrams on } \Sigma \\
\mathcal{O}(\mathcal{M}^G_\Sigma) \arrow[dr] & \\
 & \mathcal{O}(\mathcal{M}^H_\Sigma) & H\text{-colored chord diagrams} \arrow[l] \arrow[uul]
\end{tikzcd}$$

"Universal" integrable system on Σ:

$$B_C \subset J_C \subset Ch_\Sigma$$

describe B_C as diagrams contractible to C

Quantization:

$\partial\Sigma = \emptyset$ for simplicity

$\mathcal{L}_\Sigma = \{\mathbb{Z}$-linear combination of links

in $\Sigma \times [0,1]\}$

Associative product:

$[L_1] * [L_2] = \{L_1$ "near" left end of $[0,1]$

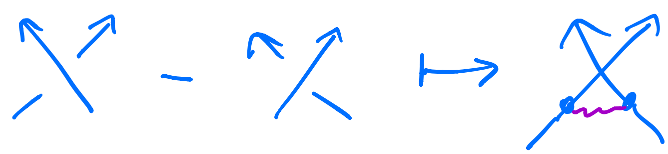
L_2 "near" the right end of $[0,1]\}$

Filtration:

$\deg\left(\sum_L c_L [L]\right) \le n$ if it has

$\le n$ combinations like

$$\begin{tikzcd} (Ch_\Sigma, \{\cdot,\cdot\}) \arrow[r] & \mathcal{L}_\Sigma \arrow[l, bend left] \end{tikzcd}$$

- overcrossing = undercrossings
- 

- $\mathcal{L}_\Sigma$ is the "universal quantization" of Ch_Σ

- universal quantization of B_C =
  $= B_C = \{$ linear combinations of links contractible to $C\}$
  commutative $B_C \subset \mathcal{L}_\Sigma$
- J_C = centralizer of $B_C \subset \mathcal{L}_\Sigma$

Finite dimensional representations of $\mathcal{L}_\Sigma \rightsquigarrow$ quantum Chern-Simons

Operators constructed in R.-Turaev 1992 quantize superintegrable systems on $\mathcal{M}_\Sigma^G$.