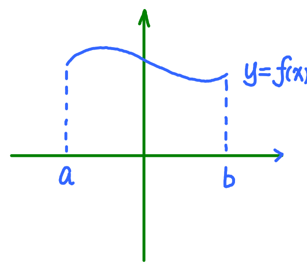
Double Integrals

Reading: Textbook, §15.1 - 15.4.

§1. Double integrals over rectangles

Recall in 1-variable calculus,

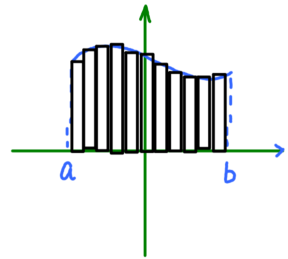
$$\int_a^b f(x)\,dx = \text{(signed) area under graph of } f.$$



This was first done by an estimate using "thin" rectangles:

$A(n) =$ sum of area of rectangles of width $\frac{b-a}{n}$.

$$\boxed{\int_a^b f(x)\,dx := \lim_{n\to\infty} A(n)}$$

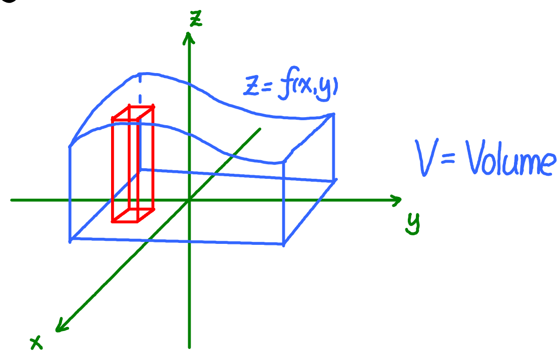


Riemann sum

We will adopt an analogous approach for multivariable functions.

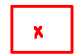
Here we start with the 2-variable case: double integral.

Consider the graph $z = f(x,y)$, where $f(x,y)$ is defined over a rectangular region $D = [a,b] \times [c,d]$.



Subdivide D into small rectangles of size $\Delta x \cdot \Delta y$, where $\Delta x = \frac{b-a}{n}$, $\Delta y = \frac{d-c}{m}$. The red column has volume

$f(x_i, y_j)\Delta x \Delta y$, where (x_i, y_j) is the middle point of the rectangle


(x_i, y_j)

Summing over all columns, we get an approximation (a double Riemann sum)

$$V(n,m) := \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i, y_j)\Delta x \Delta y$$

that becomes closer and closer to V.

Def. The double integral of $f(x,y)$ over D is

$$\iint_D f(x,y)\, dA := \lim_{n,m\to\infty} V(n,m)$$

Basic properties:

(1). Double integral is linear: $\forall a, b \in \mathbb{R}$, $f(x,y)$, $g(x,y)$ two functions over D:

$$\iint_D (af(x,y)+bg(x,y))\,dA = a\iint_D f(x,y)\,dA + b\iint_D g(x,y)\,dA.$$

(2). If $f(x,y) \geq g(x,y)$ at all points of D, then

$$\iint_D f(x,y)\,dA \geq \iint_D g(x,y)\,dA.$$

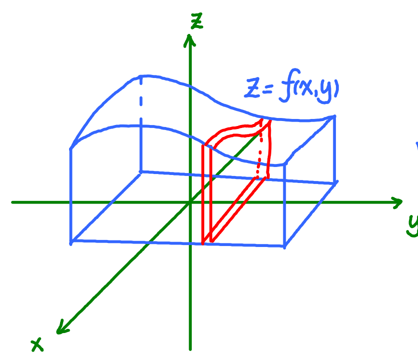
(3). If $D = D_1 \sqcup D_2$, then

$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$$

Eg. Find the integral $\iint_D (4-2y)\,dA$, where $D=[0,1]\times[0,1]$.

§2. Iterated integrals

Over a rectangle $D=[a,b]\times[c,d]$, when evaluating a double integral, we can first slice the "bread loaf" in the x-direction

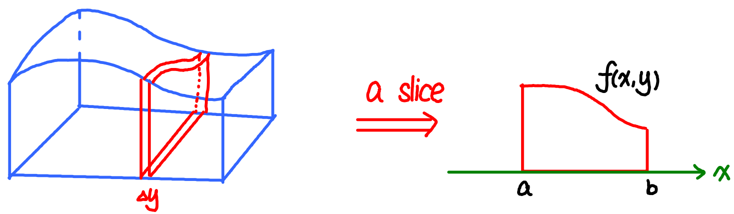


$V=$ Volume

and sum up the volume of the slices to get the total volume.

The area of the slice, for a fixed y-value, is given by

$$A(y) = \int_a^b f(x,y)\,dx$$



Then

$$V = \lim_{\Delta y \to 0} \sum A(y)\Delta y = \int_c^d A(y)\,dy = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$

Thus we obtain the iterated integral formula for a rectangular region $D = [a,b] \times [c,d]$.

$$\boxed{\iint_D f(x,y)\, dA = \int_c^d \left(\int_a^b f(x,y)\, dx \right) dy}$$

Eg. Find the iterated integral

$$\int_1^4 \left(\int_0^2 (6x^2y - 2x)\, dx \right) dy.$$

Clearly, the total volume of the "bread loaf" doesn't depend on how you slice it: you can also do it in the y-direction first!

$$\boxed{\iint_D f(x,y)\,dA = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx}$$

Sometimes slicing up in one direction might make a problem easier.

Eg. Evaluate the integral over $D = [0,1] \times [0,1]$.

$$\iint_D x(y+x^2)^4 \, dA$$

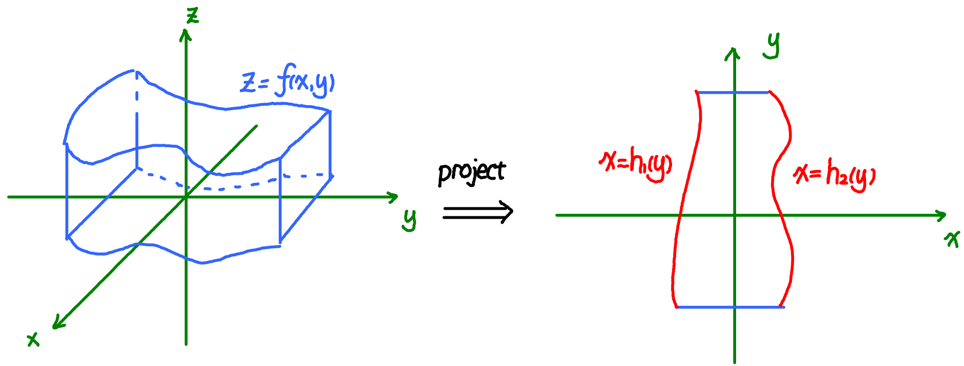
When you are asked about computing volume, think about converting the problem into a double integral:

Eg. Find the volume of the solid lying under the elliptic paraboloid $\frac{x^2}{4} + \frac{y^2}{9} + z = 1$ over the rectangle $[-1,1] \times [-2,2]$

§3. Double integrals over general regions

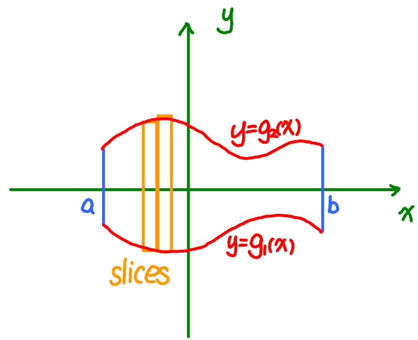
If your "bread loaf" is more irregularly shaped, you can still slice it up and calculate its volume in many cases.

For instance:



Type II regions: $\quad D_{II} = \{(x,y) \mid y \in [c,d],\ h_1(y) \le x \le h_2(y)\}$

Rotating such regions by 90°, we get Type I regions:



$$D_{\mathrm{I}} = \{(x,y) \mid x \in [a,b],\ g_1(x) \le y \le g_2(x)\}$$

Thus, slicing D_{I} / D_{II} type regions in the y/x directions first reduces the double integral into two single integrals.

Let's see this principle in action:

Eg. Evaluate the integral $\iint_D y^2\,dA$ where

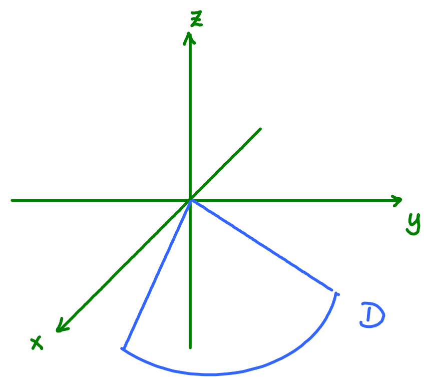
$$D = \{(x,y) \mid -1 \le y \le 1,\ -y-2 \le x \le y\}$$

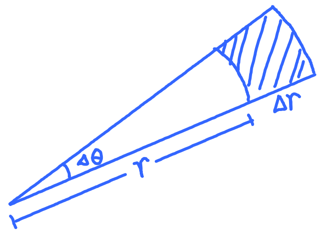
Eg. Evaluate the integral $\iint_D x\cos y\,dA$, where D is the region bounded by $y=0$, $y=x^2$ and $x=1$.

Eg. Find the volume of the solid under the plane $x-2y+z=1$ and above the region in the xy plane with boundary curves $x+y=1$ and $x^2+y=1$.

§3. Double integrals in polar coordinates

If your "bread loaf" lives over a region that resembles (part of) an annulus or disk, then using polar coordinates might be more convenient to describe the region and compute the integral.





$$\Delta A = r\Delta\theta \cdot \Delta r$$

$$\Longrightarrow \quad dA = r\,dr\,d\theta$$

Thus

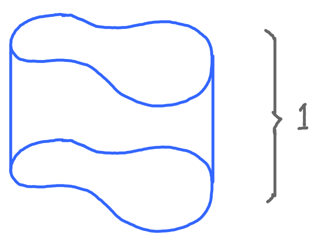
$$\iint_D f(x,y)\,dA = \iint_{D(r,\theta)} f(r,\theta)\,r\,dr\,d\theta.$$

region D described in polar coordinates

function f(x,y) described in polar coordinates

Eg. Find the integral $\iint_D e^{-x^2-y^2}\,dA$ where D is the region bounded by the semicircle $x=\sqrt{4-y^2}$ and the y-axis.

A solid of uniform height 1 has volume = area of its cross section:



$$V=\iint_D 1\cdot dA = \text{Area}(D).$$

Eg. Find the area of the region inside the circle $(x-1)^2+y^2=1$ and outside of circle $x^2+y^2=1$.

Finally, we compute some volume of solids using polar coordinates

Eg. Find the volume of the solid enclosed by the surface $z-x^2-y^2=1$ and the plane $z=2$.

Eg. Find the volume of the solid above the cone $z=\sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2=1$.