REAL ANALYSIS GENERAL EXAM FALL 2022

Solve as many problems as you can. Full solutions on a smaller number of problems will be worth more than partial solutions on several problems.

Problem 1.

Let (X, μ) be a σ-finite measure space and $p \in [1, +\infty)$. Let $(f_n)_{n=1}^{\infty}$ be a sequence in $L^p(X, \mu)$ and suppose that $\|f_n\|_p \leq 1$, and that f_n converges a.e. to a measurable function f. Show that $\|f\|_p \leq 1$.

Fatou's lemma

Problem 2.

Let μ be a Borel probability measure on $\mathbb{R}$ without atoms. Suppose that $E \subseteq \mathbb{R}$ is a Borel set with $\mu(E) > 0$. Show that there is a $t \in \mathbb{R}$ with $\mu(E \cap (-\infty, t)) = \frac{1}{2}\mu(E)$.

Problem 3.

Let X be a set equipped with a σ-algebra of sets Σ. Suppose that $\mu, \nu \colon \Sigma \to [0, +\infty)$ are finite measures. Set $\lambda = \mu + \nu$. Let $f \colon X \to \mathbb{R}$ be any Σ-measurable function so that

$$\nu(E) = \int_E f \, d\lambda$$

for all $E \in \Sigma$.

(i) Show that $0 \leq f \leq 1$ λ-a.e.
(ii) If $F = \{x : f(x) = 1\}$, show that $\mu(F) = 0$.
(iii) If $A \subseteq \{x : 0 \leq f(x) < 1\}$ and $\mu(A) = 0$, show that $\nu(A) = 0$.

Problem 4.

Fix $p \in [1, +\infty)$. Let $W^p([0, 1])$ consist of all absolutely continuous functions $f \colon [0, 1] \to \mathbb{C}$ so that $f' \in L^p([0, 1])$. For $f \in W^p([0, 1])$ define

$$\|f\| = |f(0)| + \|f'\|_p.$$

Show that $\|\cdot\|$ is a norm which makes $W^p([0, 1])$ into a Banach space. (You are allowed to use that $L^p([0, 1])$ is a Banach space).

Problem 5.

Let m be Lebesgue measure on $\mathbb{R}$. Let $\Omega = \{1_E : E \subseteq \mathbb{R} \text{ is Borel and } m(E) < +\infty\}$ regarded as a subset of $L^1(\mathbb{R})$ (recall that we identify two elements of L^1 if they agree almost everywhere). Throughout this problem regard Ω as a metric space equipped with the L^1-distance.

(i) If $a < b$ are real numbers, show that the function $\Omega \to \mathbb{R}$ given by
$$1_E \mapsto m(E \cap [a, b])$$
is a continuous function.
(ii) If $a < b$ are real numbers, let $U_{a,b}$ be the subset of Ω consiting of all 1_E where $E \subseteq \mathbb{R}$ is Borel and
$$0 < m(E \cap [a, b]) < b - a.$$
Show that $U_{a,b}$ is open and dense in Ω.

(iii) Let D be the set of all 1_E where $E \subseteq \mathbb{R}$ is Borel and so that for every interval I of positive measure we have Use (ii)

$$0 < m(E \cap I) < m(I).$$

Show that there is a countable collection $\{U_j\}_{j \in J}$ of open and dense subsets of Ω with

$$D \supseteq \bigcap_{j \in J} U_j.$$

REAL ANALYSIS GENERAL EXAM FALL 2022

Solve as many problems as you can. Full solutions on a smaller number of problems will be worth more than partial solutions on several problems.

Problem 1.

Compute
$$\lim_{n\to\infty}\int_0^\infty \frac{n\sin(x/n)}{x(1+x^2)}\,dx.$$

Problem 2.

Fix $a<b$ in $\mathbb{R}$. Recall that $h\colon [a,b]\to\mathbb{C}$ is *absolutely continuous* if for every $\varepsilon>0$ there is a $\delta>0$ so that if $((a_j,b_j))_{j=1}^k$ are disjoint intervals in $[a,b]$ with $\sum_{j=1}^k(b_j-a_j)<\delta$, then $\sum_{j=1}^k(f(b_j)-f(a_j))<\varepsilon$. For a Lipschitz function $g\colon[a,b]\to\mathbb{C}$ we set
$$\|g\|_{Lip}=\sup_{x\neq y,x,y\in[a,b]}\frac{|g(x)-g(y)|}{|x-y|}.$$

(a) Show that $f\colon[a,b]\to\mathbb{C}$ is Lipschitz if and only if f is absolutely continuous and $f'\in L^\infty([a,b])$.
(b) If $f\colon[a,b]\to\mathbb{C}$ is Lipschitz, show that $\|f\|_{Lip}=\|f'\|_\infty$.

Problem 3.

Let (X,μ) be a σ-finite measure space. Show that if $f,g\in L^1(X,\mu)$ wih $0\leq f,g$ a.e., then
$$\|f-g\|_1=\int_0^\infty \mu(\{x: f(x)>t\}\Delta\{x: g(x)>t\})\,dt.$$
Here $E\Delta F=E\setminus F\cup F\setminus E$ for sets $E,F\subseteq X$. Suggestion: it might be helpful to first show that for $a,b\in[0,\infty)$ we have
$$|a-b|=\int_0^\infty |1_{(t,\infty)}(a)-1_{(t,\infty)}(b)|\,dt.$$

Note: for this problem you may take for granted that the function $X\times(0,\infty)\to\{0,1\}$ given by $(y,t)\mapsto 1_{\{x:f(x)>t\}}(y)$ and that the function $t\mapsto\mu(\{x: f(x)>t\}\Delta\{x: g(x)>t\})$ are measurable functions.

Problem 4.

Let (X,Σ) be a measurable space. Recall that if η is a signed measure on Σ, then $|\eta|=\eta_1+\eta_2$ where η_1,η_2 are the unique nonnegative measures with $\eta=\eta_1-\eta_2$ and $\eta_1\perp\eta_2$. Further, $\|\eta\|_{TV}=|\eta|(X)$. Suppose that μ,ν are signed measures on Σ, that $\|\mu\|_{TV},\|\nu\|_{TV}<+\infty$ and that $|\mu|,|\nu|$ are mutually singular.

(a) If $\mu=\mu_1-\mu_2$, $\nu=\nu_1-\nu_2$ with μ_i,ν_j nonnegative measures and $\mu_1\perp\mu_2$, $\nu_1\perp\nu_2$, show that $\mu_i\perp\nu_j$ for all $i,j\in\{1,2\}$.
(b) Show that
$$\|\mu+\nu\|_{TV}=\|\mu\|_{TV}+\|\nu\|_{TV}.$$

Problem 5.

(a) For $f \in L^1([0,1])$, set L_f be the set of $x \in [0,1]$ so that
$$\lim_{r \to 0} \frac{1}{2r} \int_{(x-r,x+r)} |f(y) - f(x)| \, dy = 0.$$
State the conclusion of the Lebesgue's differentiation theorem for L_f.

(b) For $n \in \mathbb{N}$, and $0 \leq j \leq 2^n - 1$, set $I_{n,j} = [j2^{-n}, (j+1)2^{-n})$. For $f \in L^1([0,1])$, define
$$E_n f = \sum_{j=0}^{2^n-1} \left(\frac{1}{m(I_{n,j})} \int_{I_{n,j}} f(t) \, dt \right) 1_{I_{n,j}}.$$
Show that
$$\lim_{n \to \infty} (E_n f)(x) = f(x) \text{ for almost every } x \in [0,1].$$

Instructions. *2 hours. Closed book examination. Be neat in your presentation. When invoking a theorem from previous courses, name the theorem and thoroughly check its hypotheses. You must solve a significant portion of each of the three problems in order to pass the exam.*

---

1. Let $(X, \mathscr{M}, \mu)$ be a measure space. Let (f_n) be a sequence of nonnegative functions functions in $L^1(X, \mathscr{M}, \mu)$ and let f be a nonnegative function in $L^1(X, \mathscr{M}, \mu)$. Suppose that

$$\int_X f_n \, \mathrm{d}\mu \longrightarrow \int_X f \, \mathrm{d}\mu$$

and that $f_n \to f$ pointwise. Prove that f_n converges to f in $L^1(X, \mathscr{M}, \mu)$. Hint: consider $g_n = \min(f, f_n)$.

2. Let $(X, \mathscr{M}, \mu)$ be a measure space. Let $p \in [1, \infty)$.

(a) Let (f_n) be a sequence of functions in $L^p(X, \mathscr{M}, \mu)$ and let f be a function in $L^p(X, \mathscr{M}, \mu)$. Suppose that f_n converges to f in $L^p(X, \mathscr{M}, \mu)$. Prove that there exists a subsequence (f_{n_k}) such that for μ-almost all x, $\lim_{k\to\infty} f_{n_k}(x) = f(x)$. Hint: remember the proof of completeness of L^p.

(b) Let h be a measurable function on X. Let

$$D = \{f \in L^p(X, \mathscr{M}, \mu) \mid hf \in L^p(X, \mathscr{M}, \mu)\} \, .$$

Let (f_n) be a sequence of elements of D, and let $f, g \in L^p(X, \mathscr{M}, \mu)$ be such that f_n converges to f in L^p, and hf_n converges to g in L^p. Show that $f \in D$ and $g = hf$.

3. For μ a Borel probability measure on $\mathbb{R}$, we will denote by $\widehat{\mu}$ the function $\mathbb{R} \to \mathbb{C}$ given by

$$\widehat{\mu}(t) = \int_{\mathbb{R}} e^{itx} \mathrm{d}\mu(x) \, .$$

We will also adopt the notational convention $\mathrm{sinc}(x) = \frac{\sin x}{x}$ if $x \neq 0$ and $\mathrm{sinc}(0) = 1$.

(a) Show that $\widehat{\mu}$ is a bounded continuous function.

(b) Let $\delta > 0$. Show that

$$\frac{1}{2\delta} \int_{-\delta}^{\delta} (1 - \mathrm{Re}(\widehat{\mu}(t))) \, \mathrm{d}t = \int_{\mathbb{R}} (1 - \mathrm{sinc}(\delta x)) \, \mathrm{d}\mu(x) \, .$$

(c) Show that for all $u \in \mathbb{R}$,

$$1 - \mathrm{sinc}(u) \geq \frac{1}{2} \, \chi_{(-\infty,-2)\cup(2,\infty)}(u) \, ,$$

and deduce that

$$\mu(\{x \in \mathbb{R} \mid |x| > 2\delta^{-1}\}) \ \leq \ \frac{1}{\delta}\int_{-\delta}^{\delta}(1 - \mathrm{Re}(\widehat{\mu}(t)))\ \mathrm{d}t\ .$$

(d) Let μ_n be a sequence of Borel probability measures on $\mathbb{R}$. Suppose that for all t, the limit $\Phi(t) = \lim_{n\to\infty}\widehat{\mu_n}(t)$ exists and that the resulting function $\Phi(t)$ is continuous at $t = 0$. Prove that for all $\epsilon > 0$, there exists a compact set K inside $\mathbb{R}$ such that, for all n, $\mu_n(K) \geq 1 - \epsilon$.

Real analysis Qualifying exam, January 2021

DO NOT WRITE YOUR NAME ON YOUR WORK

My cellphone in case of zoom disconnection: ***

In order to receive the full credit for a problem, a detailed argument (rather than a sketch of the proof) is needed. Whenever applying one of the standard theorems, please indicate that clearly. Full solutions on a smaller number of problems will be worth more than partial solutions on more problems.

1. Let f_n, $n \geq 1$, and f be measurable functions on a space $(\Omega, \mathcal{F}, \mu)$, such that $f_n \to f$ in measure. Does this imply that there exists a measurable set $A \subseteq \Omega$ with $\mu(\Omega \setminus A) = 0$ such that $f_n(x) \to f(x)$ for all $x \in A$?
If yes, prove this. If no, give a counterexample.

2. Let B be a measurable subset of the two-dimensional plane such that the intersection of B with every vertical line is finite or countable. Find $\mu(B)$, where μ is the two-dimensional Lebesgue measure. Justify your answer.

3. Let $(\Omega, \mathcal{F})$ be a measurable space, and μ, ν, ρ be three finite positive measures on $(\Omega, \mathcal{F})$ such that $\mu \ll \nu$ (i.e., μ is absolutely continuous with respect to ν). Show that there exists a measurable function f on Ω such that for all $E \in \mathcal{F}$ we have

$$\mu(E) = \int_E f \, d\nu + \int_E (f-1) \, d\rho.$$

(Hint: use Radon-Nikodym's Theorem)

4. Let f, g be nonnegative measurable functions on $[0,1]$, and $a, b, c, d \geq 0$ be arbitrary nonnegative numbers. Show that then

$$\left(ac + bd + \int_0^1 f(x) g(x) \, dx \right)^3 \leq \left(a^3 + b^3 + \int_0^1 (f(x))^3 \, dx \right) \left(c^{3/2} + d^{3/2} + \int_0^1 (g(x))^{3/2} \, dx \right)^2.$$

Partial credit is given for proving the inequality in the particular case $a = b = c = d = 0$.

5. Let $f(x)$ be a continuous function on $[0,1]$. Show that for every $\varepsilon > 0$ there exists $n \in \mathbb{Z}_{\geq 0}$ and constants $a_0, a_1, \ldots, a_n \in \mathbb{R}$ such that for the differential operator

$$D := \sum_{k=0}^{n} a_k \left(\frac{d}{dx} \right)^k = a_0 + a_1 \frac{d}{dx} + a_2 \left(\frac{d}{dx} \right)^2 + \ldots + a_n \left(\frac{d}{dx} \right)^n$$

we have $\left| f(x) - e^{x^2} (D e^{-x^2}) \right| < \varepsilon$ for all $x \in [0,1]$. Here $e^{x^2}(De^{-x^2})$ is the function obtained by applying D to e^{-x^2} and after that multiplying the result by e^{x^2}.

Real analysis Qualifying exam, August 2020

Make sure that you have signed the Honor Pledge on Collab.

In order to receive the full credit for a problem, a detailed argument (rather than a sketch of the proof) is needed. Whenever applying one of the standard theorems, please indicate that clearly.

Cantor function increasing and consinuous such that derivative is 0 a.e.

1. Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous, almost everywhere differentiable, and such that $f'(x) = 1$ almost everywhere. (Both "almost everywhere" properties are assumed with respect to the Lebesgue measure on $\mathbb{R}$.) Does this imply that $f(2) - f(1) = 1$?

If yes, prove this. If no, give a counterexample.

2. Is every open set in $\mathbb{R}^2$ a countable union of closed sets?

If yes, prove this. If no, give a counterexample.

3. Let $\mathcal{H}$ be a separable complex Hilbert space with basis (complete orthonormal system) $f_1, f_2, f_3, \ldots$. Define a linear operator P in $\mathcal{H}$ by setting

$$P(f_n) = f_{n+1}, \qquad n = 1, 2, \ldots.$$

(a) Find the adjoint P^* to P.

(b) Find the operators PP^* and P^*P.

4. Let $(X, \mathcal{F}, \mu)$ be a measure space with $\mu(X) = 1$. Let $f_n\colon X \to \mathbb{R}$ be measurable functions such that for all $t \in \mathbb{R}$,

$$\lim_{n \to +\infty} \mu\left(x\colon f_n(x) \le t\right) = \begin{cases} 0, & t < 0; \\ 1, & t \ge 0. \end{cases}$$

Show that $f_n \to 0$ in measure.

5. Show that the operator

$$(Tf)(x) := \int_0^\infty \frac{f(y)}{x+y}\, dy$$

is bounded in the space $L^p(\mathbb{R}_{\ge 0})$ for all $1 < p < +\infty$.

Real analysis general exam, January 2020

1. Let μ be the Lebesgue measure on $\mathbb{R}$. For a Lebesgue measurable set $A \subset [0,1]$, is it true that

   (a) $\mu(A) = \sup_{U \subset A,\, U \text{ open}} \mu(U)$? If true, prove this. If false, give a counterexample.

   (b) $\mu(A) = \inf_{U \supset A,\, U \text{ open}} \mu(U)$? If true, prove this. If false, give a counterexample.

2. Find a polynomial $P(x)$ of degree at most 3 such that $\int_{-1}^{1} |x^4 - P(x)|^2 \, dx$ is minimal.

3. Let X be a compact metric space, and $C(X)$ be the space of all real-valued continuous functions on X with the supremum norm. Assume that the subset $\mathcal{A} \subset C(X)$ satisfies the following properties:

   - (algebra) For all $f, g \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{R}$ we have $\alpha f + \beta g \in \mathcal{A}$ and $fg \in \mathcal{A}$.
   - (separates points) For any $x \neq y$ from X there exists a function $f \in \mathcal{A}$ such that $f(x) \neq f(y)$.

   This question has two parts:

   (a) Show by example that $\mathcal{A}$ need not be dense in $C(X)$, explicitly checking all the properties of your example $\mathcal{A}$.

   (b) In order to conclude that $\mathcal{A}$ is dense by Stone-Weierstrass Theorem, what additional condition(s) should be added?

4. Let μ be a measure on $(\mathbb{R}, \mathcal{B})$, where $\mathcal{B}$ is the Borel σ-algebra. Let $\mu(\mathbb{R}) = 1$. Next, let $\mathcal{F} \subset \mathcal{B}$ be the sub-σ-algebra of symmetric Borel sets, that is, $\mathcal{F}$ generated by all intervals of the form $(-a, a)$ with $a > 0$.

   Let $f \in L^1(\mathbb{R}, \mathcal{B}, \mu)$. Find a function g such that:

   (a) $g \in L^1(\mathbb{R}, \mathcal{F}, \mu)$ (in particular, g is $\mathcal{F}$-measurable).

   (b) For all $E \in \mathcal{F}$ we have $\int_E g \, d\mu = \int_E f \, d\mu$.

5. Let μ be a finite measure on some measurable space $(X, \mathcal{F})$.

   Show that a sequence of $\mathcal{F}$-measurable functions f_n converges to a function f in measure if and only if $\int_X \min\{1, |f_n - f|\} \, \mu(dx) \to 0$ as $n \to +\infty$.

Real analysis Qualifying exam, August 2019

1. Let $\mathcal{C}$ be the Cantor set on $[0,1]$. Recall that it is obtained by iteratively deleting the open middle third: $(\frac{1}{3}, \frac{2}{3})$, then $(\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9})$, and so on.

   (a) Show that $\mathcal{C} + \mathcal{C} := \{a + b \colon a, b \in \mathcal{C}\}$ is the full segment $[0, 2]$.

   (b) Find two sets $A, B \subset \mathbb{R}$, each of which is closed and has Lebesgue measure zero, such that $A + B = \{a + b \colon a \in A, \ b \in B\}$ is the full line $\mathbb{R}$.

2. Does there exist a measure space $(X, \mathcal{F}, \mu)$ with a finite measure μ, and a sequence of μ-measurable functions $\{f_n\}_{n=1,2,\ldots}$ on X such that:

   - $f_n(x) \geq 0$ for all n, x;
   - $f_n(x) \to 0$ as $n \to +\infty$ for all x;
   - $\int f_n(x) \mu(dx) \to 0$ as $n \to +\infty$;
   - $\Phi(x) := \sup_n f_n(x)$ has infinite integral?

   If yes, give an example of such a sequence $\{f_n\}$. If no, give a proof of nonexistence.

3. Let μ be a signed Borel measure on $\mathbb{R}^n$ which is bounded on bounded sets. Suppose that $\int f \, d\mu = 0$ for all continuous functions f with bounded support. Show that then $\mu = 0$.

4. Let $L^1(\mathbb{R})$ be the space of Lebesgue integrable functions on $\mathbb{R}$. For a positive function $f \in L^1(\mathbb{R})$ show that the function $\dfrac{1}{f(x)}$ does not belong to $L^1(\mathbb{R})$.

   (Hint: look at the function $1 = f^{1/2} f^{-1/2}$.)

5. Applying the Gram-Schmidt orthogonalization to $1, x, x^2, \ldots$ in the Hilbert space $L^2([-1, 1])$ (with Lebesgue measure), one gets the Legendre polynomials $L_n(x)$, $n = 0, 1, 2, \ldots$.

   (a) Show that the Legendre polynomials form a basis (= complete orthogonal system) in the Hilbert space $L^2([-1, 1])$

   (b) Show that the Legendre polynomials are given by the formula $L_n(x) = c_n \frac{d^n}{dx^n}(x^2 - 1)^n$ (you do not need to specify c_n).

   (Hint: employ integration by parts.)