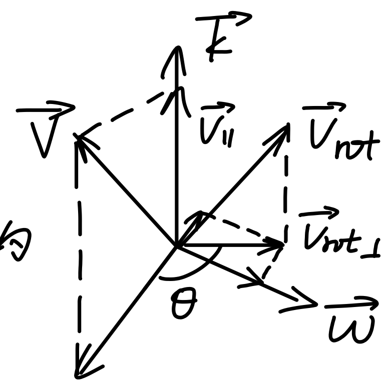
1. $\vec{v}$ 可分解为 $\vec{v}=\vec{v}_{\parallel}+\vec{v}_{\perp}$



$\vec{v}_{\parallel}=(\vec{v}\cdot\vec{k})\vec{k}$

$\vec{v}_{\perp}=\vec{v}-\vec{v}_{\parallel}$

旋转后跟旋转轴平行的向量不变

所以 $\vec{v}_{rot\parallel}=\vec{v}_{\parallel}$

同样 $\vec{v}_{rot}=\vec{v}_{rot\parallel}+\vec{v}_{rot\perp}$

$\vec{v}_{rot}$ 可分解为 $\vec{v}_{rot\perp}=\cos\theta\,\vec{v}_{\perp}+(\sin\theta\,\vec{k})\times\vec{v}$

于是 $\vec{v}_{rot}=\vec{v}_{rot\perp}+\vec{v}_{rot\parallel}$

$$
\begin{aligned}
&=\sin\theta\,\vec{k}\times\vec{v}+\cos\theta\,\vec{v}_{\perp}+(\vec{v}\cdot\vec{k})\vec{k}\\
&=\sin\theta\,\vec{k}\times\vec{v}+\cos\theta(\vec{v}-\vec{v}_{\parallel})+(\vec{v}\cdot\vec{k})\vec{k}\\
&=\sin\theta\,\vec{k}\times\vec{v}+\cos\theta(\vec{v}-(\vec{v}\cdot\vec{k})\vec{k})+(\vec{v}\cdot\vec{k})\vec{k}\\
&=\sin\theta\,\vec{k}\times\vec{v}+\cos\theta\,\vec{v}+(1-\cos\theta)(\vec{v}\cdot\vec{k})\vec{k}\\
&=\cos\theta\,\vec{v}+(1-\cos\theta)\vec{k}\vec{k}^T\vec{v}+\sin\theta\,\vec{k}^\wedge\vec{v}\\
&=(\cos\theta I+(1-\cos\theta)\vec{k}\vec{k}^T+\sin\theta\,\vec{k}^\wedge)\vec{v}
\end{aligned}
$$

所以 $R=\cos\theta I+(1-\cos\theta)\vec{k}\vec{k}^T+\sin\theta\,\vec{k}^\wedge$

2. $R^T=\cos\theta I+(1-\cos\theta)\vec{k}\vec{k}^T+\sin\theta(\vec{k}^\wedge)^T$

$$
\begin{aligned}
R^TR=&(\cos\theta I+(1-\cos\theta)\vec{k}\vec{k}^T+\sin\theta(\vec{k}^\wedge)^T)\cdot\\
&(\cos\theta I+(1-\cos\theta)\vec{k}\vec{k}^T+\sin\theta\,\vec{k}^\wedge)\\
=&\cos\theta^2 I+2\cos\theta(1-\cos\theta)\vec{k}\vec{k}^T+\cos\theta\sin\theta\,\vec{k}^\wedge\\
&+(1-\cos\theta)^2(\vec{k}\vec{k}^T)^2+\sin\theta(1-\cos\theta)\vec{k}\vec{k}^T\vec{k}^\wedge+\\
&\cos\theta\sin\theta(\vec{k}^\wedge)^T+\sin\theta(1-\cos\theta)(\vec{k}^\wedge)^T\vec{k}\vec{k}^T+\\
&\sin\theta^2(\vec{k}^\wedge)^T\vec{k}^\wedge\\
=&I
\end{aligned}
$$