

1. 设 $p = (x, y, z, w)^T = (\vec{v}, 1)^T$ $q = (\eta, \vec{E})$

$$\begin{aligned}
 p' &= qpq^{-1} = q^+ p^+ q^{-1} = q^+ q^{-1} \oplus p \\
 &= \begin{pmatrix} \eta I + \vec{E}^x & \vec{E} \\ -\vec{E}^T & \eta \end{pmatrix} \begin{pmatrix} \eta I + \vec{E}^x - \vec{E} \\ \vec{E}^T & \eta \end{pmatrix} \begin{pmatrix} \vec{v} \\ 0 \end{pmatrix} \\
 &= \begin{pmatrix} (\eta I + \vec{E}^x)^2 + \vec{E} \vec{E}^T & -(\eta I + \vec{E}^x) \vec{E} + \eta \vec{E} \\ -\eta \vec{E}^T + \eta \vec{E}^T & \vec{E}^T \vec{E} + \eta^2 \end{pmatrix} \begin{pmatrix} \vec{v} \\ 0 \end{pmatrix} \\
 &= \begin{pmatrix} \eta^2 I + (\vec{E}^x)^2 + 2\eta \vec{E}^x + \vec{E} \vec{E}^T & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \vec{v} \\ 0 \end{pmatrix} \\
 &= \begin{pmatrix} (\eta^2 I + (\vec{E}^x)^2 + 2\eta \vec{E}^x + \vec{E} \vec{E}^T) \vec{v} \\ 0 \end{pmatrix}
 \end{aligned}$$

所以 p' 必为虚四元数

$$Q = \begin{pmatrix} \eta^2 I + (\vec{E}^x)^2 + 2\eta \vec{E}^x + \vec{E} \vec{E}^T & 0 \\ 0 & I \end{pmatrix}$$