1. ① 设R为旋转矩阵

$$(RR^T)_{i,j} = \sum_{k=1}^{n} R_{i,k} R^T_{k,j} = \sum_{k=1}^{n} R_{i,k} R_{j,k} = R_{i,\cdot} \cdot R_{j,\cdot}$$

即$(RR)^T_{i,j}$为R中i行与j行的点乘

因R为正交矩阵，所以每行均为正交单位向量

于是 $R_{i,\cdot} \cdot R_{j,\cdot} = \begin{cases} 0 & 若\ i \neq j \\ 1 & 若\ i = j \end{cases}$

所以$(RR^T)_{i,j} = I$

② 旋转矩阵R把一组标准化正交基I转化为另一组正交基I'

所以$RI = I'$，即 $\det(RI) = \det(I') = 1$

$\det(R)\det(I) = 1$

$\det(R) = 1$

2. 一个四元数q有1个实部和3个虚部

所以$\vec{\varepsilon}$为3维，η为1维

3. $q_1 q_2 = \begin{pmatrix} \eta_1 \vec{\varepsilon}_2 + \vec{\varepsilon}_1 \eta_2 + \vec{\varepsilon}_1 \times \vec{\varepsilon}_2 \\ \eta_1 \eta_2 - \vec{\varepsilon}_1^T \vec{\varepsilon}_2 \end{pmatrix}$

$$q_1^+ q_2 = \begin{pmatrix} \eta_1 I + \vec{\varepsilon}_1^\times & \vec{\varepsilon}_1 \\ -\vec{\varepsilon}_1^T & \eta_1 \end{pmatrix} \begin{pmatrix} \vec{\varepsilon}_2 \\ \eta_2 \end{pmatrix}$$

$$= \begin{pmatrix} \eta_1 \vec{\varepsilon}_2 + \vec{\varepsilon}_1^\times \vec{\varepsilon}_2 + \eta_2 \vec{\varepsilon}_1 \\ -\vec{\varepsilon}_1^T \vec{\varepsilon}_2 + \eta_1 \eta_2 \end{pmatrix}$$

$$= \begin{pmatrix} \eta_1 \vec{\varepsilon}_2 + \vec{\varepsilon}_1 \times \vec{\varepsilon}_2 + \eta_2 \vec{\varepsilon}_1 \\ -\vec{\varepsilon}_1^T \vec{\varepsilon}_2 + \eta_1 \eta_2 \end{pmatrix}$$

$$q_2^\oplus q_1 = \begin{pmatrix} \eta_2 I - \vec{\varepsilon}_2^\times & \vec{\varepsilon}_2 \\ -\vec{\varepsilon}_2^T & \eta_2 \end{pmatrix} \begin{pmatrix} \vec{\varepsilon}_1 \\ \eta_1 \end{pmatrix}$$

$$= \begin{pmatrix} \eta_2 \vec{\varepsilon}_1 - \vec{\varepsilon}_2^\times \vec{\varepsilon}_1 + \eta_1 \vec{\varepsilon}_2 \\ -\vec{\varepsilon}_2^T \vec{\varepsilon}_1 + \eta_1 \eta_2 \end{pmatrix}$$

$$= \begin{pmatrix} \eta_2 \vec{\varepsilon}_1 + \vec{\varepsilon}_1 \times \vec{\varepsilon}_2 + \eta_1 \vec{\varepsilon}_2 \\ -\vec{\varepsilon}_1^T \vec{\varepsilon}_2 + \eta_1 \eta_2 \end{pmatrix}$$

$$q_1 q_2 = q_1^+ q_2 = q_2^\oplus q_1$$