1. 设 $p=(x,y,z,w)^T=(\vec{v},0)^T$ $\quad q=(\eta,\vec{\varepsilon})$

$$p'=qpq^{-1}=q^{+}p^{+}q^{-1}=q^{+}q^{-1\oplus}p$$

$$=\begin{pmatrix}\eta I+\vec{\varepsilon}^{\times} & \vec{\varepsilon}\\ -\vec{\varepsilon}^{T} & \eta\end{pmatrix}\begin{pmatrix}\eta I+\vec{\varepsilon}^{\times} & -\vec{\varepsilon}\\ \vec{\varepsilon}^{T} & \eta\end{pmatrix}\begin{pmatrix}\vec{v}\\ 0\end{pmatrix}$$

$$=\begin{pmatrix}(\eta I+\vec{\varepsilon}^{\times})^2+\vec{\varepsilon}\vec{\varepsilon}^{T} & -(\eta I+\vec{\varepsilon}^{\times})\vec{\varepsilon}+\eta\vec{\varepsilon}\\ -\eta\vec{\varepsilon}^{T}+\eta\vec{\varepsilon}^{T} & \vec{\varepsilon}^{T}\vec{\varepsilon}+\eta^2\end{pmatrix}\begin{pmatrix}\vec{v}\\ 0\end{pmatrix}$$

$$=\begin{pmatrix}\eta^2 I+(\vec{\varepsilon}^{\times})^2+2\eta\vec{\varepsilon}^{\times}+\vec{\varepsilon}\vec{\varepsilon}^{T} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\vec{v}\\ 0\end{pmatrix}$$

$$=\begin{pmatrix}(\eta^2 I+(\vec{\varepsilon}^{\times})^2+2\eta\vec{\varepsilon}+\vec{\varepsilon}\vec{\varepsilon}^{T})\vec{v}\\ 0\end{pmatrix}$$

所以 p' 必为虚四元数

$$Q=\begin{pmatrix}\eta^2 I+(\vec{\varepsilon}^{\times})^2+2\eta\vec{\varepsilon}^{\times}+\vec{\varepsilon}\vec{\varepsilon}^{T} & 0\\ 0 & 1\end{pmatrix}$$