Convex Analysis and Optimization

1 Introduction and Examples

1.1 What is Optimization?

Optimization problem:

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq b_i, \quad i=1,\ldots,m \end{array} \tag{1.1}$$

- optimization variables: $x=(x_1,\ldots,x_n)$
- objective function: $f_0:\mathbf{R}^n \to \mathbf{R}$
- constraint functions: $f_i:\mathbf{R}^n \to \mathbf{R}$ $i=1,\ldots,m$
- bounds for the constraints: constants $b_1,\ldots,b_m$
- optimal x^*: for any z with $f_1(z)\leq b_1,\ldots,f_m(z)\leq b_m$, we have $f_0(z)\geq f_0\left(x^*\right)$.

Remark 1.2. ***(linear programming vs. convex optimization)***

- ***linear*** *: if the objective and constraint functions $f_0,\ldots,f_m$ are linear, i.e., satisfy*
$$f_i(\alpha x+\beta y)=\alpha f_i(x)+\beta f_i(y)$$
for all $x,y\in\mathbf{R}^n$ and all $\alpha,\beta\in\mathbf{R}$.
- ***convex****: if the objective and constraint functions are convex, which means they satisfy the inequality*
$$f_i(\alpha x+\beta y)\leq \alpha f_i(x)+\beta f_i(y)$$
for all $x,y\in\mathbf{R}^n$ and all $\alpha,\beta\in\mathbf{R}$ with $\alpha+\beta=1,\alpha\geq 0,\beta\geq 0$.
- ***inear vs. convex:*** *convexity is more general than linearity: inequality replaces the more restrictive equality, and the inequality must hold only for certain values of α and β. convex optimization is a generalization of linear programming.*

1.2 Three Examples

1.2.1 Least-squares Problem

Definition 1.3. ***(Least-squares Problem)*** *A least-squares problem is an optimization problem with* ***no constraints*** *(i.e. $m=0$) and an objective which is a sum of squares of terms of the form $a_i^\top x-b_i$:*

$$\textit{minimize} \quad f_0(x)=\|Ax-b\|_2^2=\sum_{i=1}^{k}\left(a_i^\top x-b_i\right)^2 \tag{1.4}$$

Here $A\in\mathbf{R}^{k\times n}$ (with $k\geq n$), $a_i^\top$ are the rows of A, and the vector $x\in\mathbf{R}^n$ is the optimization variable.

Remark 1.5. ***(applications)*** *regression analysis, optimal control, and many parameter estimation and data fitting methods. It has a number of statistical interpretations, e.g., as maximum likelihood estimation of a vector x, given linear measurements corrupted by Gaussian measurement errors.*

Solution:

$$x = \left(A^\top A\right)^{-1} A^\top b$$

from converting (1.4) to $\left(A^\top A\right) x = A^\top b$.

Definition 1.6. ***(Weighted Least-squares)***

$$\sum_{i=1}^{k} w_i \left(a_i^\top x - b_i\right)^2 \tag{1.7}$$

where $w_1, \ldots, w_k$ are positive, is minimized.

Remark 1.8. ***(applications)*** *estimation of a vector x, given linear measurements corrupted by errors with unequal variances.*

Definition 1.9. ***(Regularization)***

$$\sum_{i=1}^{k} \left(a_i^\top x - b_i\right)^2 + \rho \sum_{i=1}^{n} x_i^2 \tag{1.10}$$

where $\rho > 0$.

Remark 1.11. *trade-off between making the original objective function $\sum_{i=1}^{k} \left(a_i^\top x - b_i\right)^2$ small, while keeping $\sum_{i=1}^{n} x_i^2$ not too big.*

Remark 1.12. ***(applicaiton)*** *in statistical estimation when the vector x to be estimated is given a prior distribution.*

Remark 1.13. *Both* (1.7) *and* (1.10) *can be cast and solved as a standard least-squares problem.*

1.2.2 Linear Programming

Definition 1.14. ***(Linear Programming)***

$$\begin{aligned} \textit{minimize} \quad & c^\top x \\ \textit{subject to} \quad & a_i^\top x \le b_i, \quad i = 1, \ldots, m \end{aligned}$$

Solution: no simple analytical formula for the solution of a linear program (as there is for a least-squares problem), but there are a variety of very effective methods for solving them, including Dantzig's simplex method, and the more recent interiorpoint methods

1.2.2.1 Chebyshev Approximation Problem

In many other cases the original optimization problem does not have a standard linear program form, but can be transformed to an equivalent linear program. As a simple example, consider the **Chebyshev approximation problem**:

$$\text{minimize} \max_{i=1,\ldots,k} \left|a_i^\top x - b_i\right|$$

Remark 1.15. ***(comparison of Chebyshev and least square)*** *both problems, the objective is a measure of the size of the terms $a_i^\top x - b_i$.*

- *In least-squares, we use the sum of squares of the terms as objective,*
- *In Chebyshev approximation, we use the maximum of the absolute values.*

Solution:

The Chebyshev approximation problem (1.6) can be solved by solving the linear program minimize t

$$\text{subject to} \quad \begin{aligned} & a_i^\top x - t \le b_i, \quad i = 1, \ldots, k \\ & -a_i^\top x - t \le -b_i, \quad i = 1, \ldots, k, \end{aligned}$$

with variables $x \in \mathbf{R}^n$ and $t \in \mathbf{R}$. (The details will be given in chapter 6.) Since linear programs are readily solved, the Chebyshev approximation problem is therefore readily solved.

1.2.3 Convex Optimization

We have introduced it in (1.1). The least-squares problem and linear programming problem are both special cases of the general convex optimization problem.

1.3 Notation

[may update]

- $\mathbf{R}$: set of real numbers
- $\mathbf{R}_+$: set of nonnegative real numbers
- $\mathbf{R}_{++}$: set of positive real numbers
- $\mathbf{R}^n$: set of real n-vectors
- $\mathbf{R}^{m\times n}$: set of real $m \times n$ matrices
- $\mathbf{S}^k$: set of symmetric $k \times k$ matrices
- $\mathbf{S}^k_+$: set of symmetric positive semidefinite $k \times k$ matrices
- $\mathbf{S}^k_{++}$: set of symmetric positive definite $k \times k$ matrices
- $\succeq(\succ)$: generalized inequality:
  * between vectors, it represents componentwise inequality
  * between symmetric matrices, it represents matrix inequality.
  * with a subscript, the symbol $\preceq_K$ (or $\prec_K$) denotes generalized inequality with respect to the cone K
- $f : \mathbf{R}^p \to \mathbf{R}^q$: f is an $\mathbf{R}^q$-valued function on some subset of $\mathbf{R}^p$
- $\operatorname{dom} f$: domain of f
- linear function $f : \mathbf{R}^n \to \mathbf{R}$: $f(x) = c^\top x$, where $c \in \mathbf{R}^n$.
- linear function $f : \mathbf{S}^n \to \mathbf{R}$: $f(X) = \operatorname{tr}(CX)$, where $C \in \mathbf{S}^n$

Remark 1.16. $\mathbf{R}^n$ *has* $\frac{k(k+1)}{2}$ *dimension with basis:*

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 \\ \vdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & \dots & 0 \\ 1 & 0 & 0 & 0 \\ \vdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \dots, \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 \\ \vdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We have $f(X) = \sum_{i=1}^{\frac{k(k+1)}{2}} a_i X_i = \operatorname{tr}(CX)$, *with* X_i *being the basis values similar as above and* C *being the matrix with* $a_i/2$ *in each position.*

1.4 Topological properties: relative interior, closure, etc.

1.4.1 Closure

Definition 1.17. ***(Closure)*** *The closure of* M *is:*

- ***inner*** *description: the set comprised of the limits of all converging sequences of elements of* M.
- ***outer*** *description: the smallest closed set containing* M

Example 1.18. *The closure of a set*

$$M = \left\{x \mid a_\alpha^\top x < b_\alpha, \alpha \in \mathcal{A}\right\}$$

given by strict linear inequalities: if such a set is ***nonempty****, then its closure is given by the nonstrict versions of the same inequalities:*

$$\operatorname{cl} M = \left\{x \mid a_\alpha^\top x \le b_\alpha, \alpha \in \mathcal{A}\right\}$$

Nonemptiness of M in the latter example is essential: the set M given by two strict inequalities

$$x < 0, \quad -x < 0$$

*in $\mathbf{R}$ clearly is empty, so that its closure also is empty; in contrast to this, applying formally the above rule, we would get **wrong** answer*

$$\operatorname{cl} M = \{x \mid x \le 0, x \ge 0\} = \{0\}$$

1.4.2 Interior

Definition 1.19. ***(Interior)*** *Let $M \subset \mathbf{R}^n$. We say that a point $x \in M$ is an interior for M, if some neighborhood of the point is contained in M, i.e.,:*

$$\exists\, r > 0 \quad B_r(x) \equiv \{y | \|y - x\| \le r\} \subset M$$

*The set of all interior points of M is called the **interior** of M, denoted as $\operatorname{int}(M)$.*

We have

$$\operatorname{int} M \subset M \subset \operatorname{cl} M$$

Example 1.20. *The interior of a polyhedral set $\{x \mid Ax \le b\}$ with matrix A **not containing zero rows** is the set $\{x \mid Ax < b\}$. ($0 \le 0$ is true while $0 < 0$ is false) Generally speaking, the latter statement is **not valid** for sets of solutions of **infinite** systems of linear inequalities. E.g., the system of inequalities*

$$x \le \frac{1}{n}, n = 1, 2, \ldots$$

in $\mathbf{R}$ has, as a solution set, the nonpositive ray $\mathbf{R}_- = \{x \le 0\}$; the interior of this ray is the negative ray $\{x < 0\}$. At the same time, strict versions of our inequalities

$$x < \frac{1}{n}, n = 1, 2, \ldots$$

define the same nonpositive ray, not the negative one.

1.4.3 Boundary

Definition 1.21. ***(Boundary)*** *The complement of the interior in the closure:*

$$\partial M := \operatorname{cl} M \backslash \operatorname{int} M$$

*is called the **boundary** of M, and the points of the boundary are called **boundary points** of M.*

Remark 1.22. ***(explanation)*** *These points not necessarily belong to M, since M can be less than $\operatorname{cl} M$; in fact, all boundary points belong to M **iff** $M = \operatorname{cl} M$, i.e., iff M is closed.*

The boundary is **closed**. From the definition of the boundary,

$$M \subset \operatorname{int} M \cup \partial M = \operatorname{cl} M \tag{1.23}$$

so that a point from M is either an interior, or a boundary point of M.

1.4.4 Relative Interior

Definition 1.24. ***(Relative Interior)*** *We say that a point $x \in M$ is relative interior for M, if M contains the intersection of a small enough ball centered at x with $\operatorname{Aff}(M)$:*

$$\exists\, r > 0 \quad B_r(x) \cap \operatorname{Aff}(M) \equiv \{y | y \in \operatorname{Aff}(M), \|y - x\| \le r\} \subset M$$

*The set of all relative interior points of M is called its **relative interior**, denoted as $\operatorname{ri}(M)$.*

Remark 1.25. *Note, [1, Theorem 2.30] may be useful in some cases. Here it is just the relative open (or relative ball) w.r.t the affine space.*

$$\operatorname{ri}(M) \textbf{\textit{ is relative open w.r.t. }} \operatorname{Aff}(M)$$

*Geometrically speaking, the relative interior is the interior when we regard M as a subset of its affine hull (the latter, geometrically, is nothing but $\mathbf{R}^k$, k being the affine dimension of $\operatorname{Aff}(M)$). **In proofs, we can just take $\mathbf{R}^k$ as the full space and the relative interior turns to the interior.***

Example 1.26.

- *the relative interior of a singleton is the singleton itself*
- *the relative interior of an affine set is the set itself*
- *the **interior** of a segment $[x, y](x \neq y)$ in $\mathbf{R}^n$ is empty whenever $n > 1$, in contrast to this, the **relative interior** is nonempty independently of n and is the interval (x, y), the segment with deleted endpoints.*

Since $\mathrm{Aff}(M)$, as any affine set, is closed and contains M, it contains also the smallest of closed sets containing M, i.e., $\mathrm{cl}\, M$. Therefore we have the followin::

$$\mathrm{ri}\, M \subset M \subset \mathrm{cl}\, M \subset \mathrm{Aff}(M)$$

1.4.5 Relative Boundary

Definition 1.27. ***(Relative Boundary)*** *Define the relative boundary $\partial_{\mathrm{ri}} M := \mathrm{cl}\, M \setminus \mathrm{ri}\, M$*

$\partial_{\mathrm{ri}} M$ is a **closed** set contained in Aff (M), and, as for the "actual" interior and boundary (1.23), we have

$$M \subset \mathrm{ri}\, M + \partial_{\mathrm{ri}} M = \mathrm{cl}\, M$$

Remark 1.28. *Of course, if $\mathrm{Aff}(M) = \mathbf{R}^n$ (i.e. when $\mathrm{int}\, M \neq \emptyset$), then the relative interior becomes the usual interior, and similarly for boundary.*

2 Convex Set

2.1 Definitions and Examples

2.1.1 Convex

Definition 2.1. ***(Convex Set and Segment)***

1. *Let x, y be two points in $\mathbf{R}^n$. The set*

$$[x, y] = \{z = \lambda x + (1 - \lambda) y \mid 0 \leq \lambda \leq 1\}$$

*is called a **segment with the endpoints** x, y.*

2. *A subset M of $\mathbf{R}^n$ is called **convex**, if $\forall\, x, y \in M$, M contains the entire segment $[x, y]$:*

$$\forall\, x, y \in M, 0 \leq \lambda \leq 1, \lambda x + (1 - \lambda) y \in M.$$

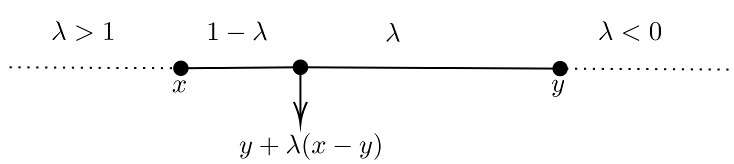


Fig. 1: Visualization of Convex Combination and Affine Combination, $\lambda x + (1 - \lambda) y$.

Example 2.2. ***(explanation)*** *See Fig. 1 for details, where please note $\lambda x + (1 - \lambda) y = y + \lambda(x - y)$. When $\lambda \in \mathbf{R}$, it is called the **affine (linear) combination**. See Section 2.1.2 for more details.*

Remark 2.3. ***(trivial examples)***

- *empty set*
- *singleton (set with single point)*
- *entire space $\mathbf{R}^n$*

Example 2.4. ***(polyhedral)*** *The solution set of an* ***arbitrary (possibly, infinite)*** *system of linear inequalities with n unknowns x:*

$$M = \left\{x \in \mathbf{R}^n \mid a_\alpha^\top x \leq b_\alpha, \alpha \in \mathcal{A}\right\} \tag{2.5}$$

is convex. In particular, the solution set of a ***finite*** *system*

$$Ax \preceq b$$

of m inequalities with n variables (A is $m \times n$ matrix) is convex; a set of this latter type is called ***polyhedral (polyhedron)****.*

Hint: let x, y be two solutions, and let $z = \lambda x + (1-\lambda)y$. For every $\alpha \in \mathcal{A}$ we have

$$a_\alpha^\top x \leq b_\alpha$$
$$a_\alpha^\top y \leq b_\alpha$$

whence, multiplying the inequalities by ***nonnegative*** *reals λ and $1-\lambda$ and taking sum, we have*

$$a_\alpha^\top z = \lambda a_\alpha^\top x + (1-\lambda) a_\alpha^\top y \leq \lambda b_\alpha + (1-\lambda) b_\alpha = b_\alpha,$$

Example 2.6. *What is a plane?*

- *Any* ***plane*** *in $\mathbf{R}^n$ (in particular, any linear subspace) is the set of all solutions to some system of linear* ***equations****.*
- *"A system of linear equations = a system of linear inequalities (a pair of opposite linear inequalities)"$\Rightarrow$* ***a plane is a polyhedral set (and convex).***

Example 2.7. ***(nonnegative orthant)*** *The nonnegative orthant is the set of points with nonnegative components, i.e.,*

$$\mathbf{R}_+^n := \{x \in \mathbf{R}^n \mid x_i \geq 0, i = 1, \dots, n\} = \{x \in \mathbf{R}^n \mid x \succeq 0\}$$

The nonnegative orthant is a polyhedron and a cone (and therefore called a polyhedral cone).

Definition 2.8. ***(Convex Combination)*** *A convex combination of* ***finitely many*** *vectors $y_1, \dots, y_m$ is their linear combination*

$$y = \sum_{i=1}^m \lambda_i y_i, \lambda_i \geq 0 \text{ and } \sum_{i=1}^m \lambda_i = 1$$

with ***nonnegative*** *coefficients with unit sum.*

Remark 2.9. ***(generalize to infinite sums)*** *The idea of a convex combination can be generalized to include* ***infinite sums, integrals, and, in the most general form, probability distributions****. Suppose $\lambda_1, \lambda_2, \dots$ satisfy*

$$\lambda_i \geq 0, \quad i = 1, 2, \dots, \quad \sum_{i=1}^\infty \lambda_i = 1$$

and $x_1, x_2, \dots \in C$, where $C \subseteq \mathbf{R}^n$ is ***convex****. Then*

$$\sum_{i=1}^\infty \lambda_i x_i \in C \tag{2.10}$$

if the series converges. More generally, suppose $p : \mathbf{R}^n \to \mathbf{R}$ satisfies $p(x) \geq 0$ for all $x \in C$ and $\int_C p(x)dx = 1$, where $C \subseteq \mathbf{R}^n$ is convex. Then

$$\int_C p(x)x dx \in C$$

if the integral exists. In the most general form, suppose $C \subseteq \mathbf{R}^n$ is convex and X is a random vector with $X \in C$ with probability one. Then $\mathbb{E}X \in C$. Indeed, this form includes all the others as special cases. For example, suppose $\mathbb{P}(X = x_1) = \theta$ and $\mathbb{P}(X = x_2) = 1-\theta$, where $0 \leq \theta \leq 1$. Then $\mathbb{E}X = \theta x_1 + (1-\theta)x_2$, and we are back to a simple convex combination of two points.

Hint: we just need to prove the infinite sums case (2.10) *since the general case can be then gotten from the DCT. I postpone the proof in Corollary 2.53.*

Lemma 2.11. ***(equivalence of convex set and convex combination)*** *M is convex $\Longleftrightarrow$ any convex combination of vectors from M again is a vector from M*

Remark 2.12. *convex set need the line segments between **two** points while convex combination is general **finite** summation.*

Proof. "$\Leftarrow$" : assume that M contains all convex combinations of the elements of M. Then, with any two points $x, y \in M$ and any $\lambda \in [0,1]$, M contains also the vector $\lambda x + (1-\lambda)y$ since it is a convex combination of x and y; thus, M is convex.

"$\Rightarrow$": assume that M is convex; we should prove that then M contains any convex combination

$$y = \sum_{i=1}^{m} \lambda_i y_i \tag{2.13}$$

of vectors $y_i \in M$. The proof is given by **induction** in m.

1. The case of $m = 1, 2$ is evident. Assume that we already know that any convex combination of $m-1$ vectors, $m \geq 3$, from M is again a vector from M.

2. Now prove the case for all convex combinations of m vectors from M. Let y be the combination in (2.13). We can assume that $1 > \lambda_m$, since otherwise it is obviously correct. Assuming $\lambda_m < 1$, we can write

   $$y = (1-\lambda_m)\left[\sum_{i=1}^{m-1} \frac{\lambda_i}{1-\lambda_m} y_i\right] + \lambda_m y_m$$

   What is in the brackets, clearly is a convex combination of $m-1$ points from M and therefore, by the inductive hypothesis, this is a point, let it be called z, from M; we have

   $$y = (1-\lambda_m)\, z + \lambda_m y_m$$

   with z and $y_m \in M$, and $y \in M$ by definition of a convex set M.

□

Lemma 2.14. ***(convexity of intersections)*** *Let $\{M_\alpha\}_\alpha$ be an arbitrary family of convex subsets of $\mathbf{R}^n$. Then the intersection*

$$M = \cap_\alpha M_\alpha$$

is convex.

Proof. easy □

Definition 2.15. ***(Convex Hull)*** *$\mathrm{Conv}(M)$ is the **smallest** convex set containing M, namely, the **intersection of all convex sets containing** M.*

Remark 2.16. *If M is convex, $\mathrm{Conv}(M) = M$.*

Corollary 2.17. ***(convex hull via convex combinations)*** *For a nonempty $M \subset \mathbf{R}^n$:*

$$\mathrm{Conv}(M) = M^* := \{\textit{the set of all convex combinations of vectors from } M\}$$

Proof. According to Lemma 2.11, $M^* \subset \mathrm{Conv}(M)$ since $\mathrm{Conv}(M)$ is a convex set containg M. We need show $\mathrm{Conv}(M) \subset M^*$. We need to prove M^* is convex. To prove that M^* is convex is the same as to prove that any convex combination $\nu x + (1-\nu)y$ of any two points $x = \sum_i \lambda_i x_i, y = \sum_i \mu_i x_i$ of M^* (if two different set $\{x_i\}$, just use the union) is again a convex combination of vectors from M. This is evident:

$$\nu x + (1-\nu)y = \nu \sum_i \lambda_i x_i + (1-\nu)\sum_i \mu_i x_i = \sum_i \xi_i x_i, \quad \xi_i = \nu\lambda_i + (1-\nu)\mu_i$$

and the coefficients ξ_i clearly are nonnegative with unit sum. □

Remark 2.18. ***(two descriptions of convex set)***

- ***inner ("worker's")*** *description: convex combination (hull) of all points (seeDefinition 2.8)*
- ***outer ("artist's")*** *description for* ***closed*** *convex set: In* (2.5)*, we show that* ***the intersection of closed half-spaces is closed convex****. The converse is also true: For every $x \notin M$, we can find the closed half-space $H_x = \left\{y \mid a_x^\top y \leq \alpha_x\right\}$ which contains M and does not contain x; consequently.*

  $$M = \bigcap_{x \notin M} H_x$$

  and therefore M ***equals intersection of all closed half-spaces which contain*** *M****.*** *(see Section 2.2.4. note it is solution set to nonstrict linear (possibly, infinite many) inequalities* (2.5)*)*

2.1.2 Affine

Definition 2.19. ***(Affine Set and Lines)***

1. *Let x, y be two points in $\mathbf{R}^n$. The set*

   $$\{z = \lambda x + (1-\lambda)y \mid \lambda \in \mathbf{R}\}$$

   is called a ***line*** *going through x, y.*

2. *A subset M of $\mathbf{R}^n$ is called* ***affine****, if $\forall\, x, y \in M$, M contains the line going through x, y:*

   $$\forall\, x, y \in M, \lambda \in \mathbf{R}, \lambda x + (1-\lambda)y \in M.$$

Definition 2.20. ***(Affine Combination)*** *A convex combination of* ***finitely many*** *vectors $y_1, \ldots, y_m$ is their linear combination*

$$y = \sum_{i=1}^{m} \lambda_i y_i, \lambda_i \in \mathbf{R} \text{ and } \sum_{i=1}^{m} \lambda_i = 1$$

with coefficients with unit sum.

Definition 2.21. ***(Affine Hull)*** $\operatorname{Aff}(M)$ *the smallest affine set containing M, namely, the intersection of all affine sets containing M.*

Remark 2.22. *Intuitively,* $\operatorname{Aff}(M)$ *is the "proper" space to look at M : we simply cut from $\mathbf{R}^n$ "useless" dimensions (such that the projection of M on these dimensions is a singleton.*

Corollary 2.23. ***(affine hull via affine combinations)*** *For a nonempty $M \subset \mathbf{R}^n$:*

$$\operatorname{Aff}(M) = \{\text{the set of all affine combinations of vectors from } M\}$$

Proof. Similar to the proof to Corollary 2.17. □

Remark 2.24. ***(Convex vs. Affine)*** *From the definitions, we have*

$$\text{affine} \Rightarrow \text{convex}$$

See also Remark 2.35 for more comparison.

2.1.2.1 Affine Set = Subspace + Translate

If C is an affine set and $x_0 \in C$, then the set

$$V = C - x_0 = \{x - x_0 \mid x \in C\}$$

is a subspace, i.e., closed under sums and scalar multiplication. To see this, suppose $v_1, v_2 \in V$ and $\alpha, \beta \in \mathbf{R}$. Then we have $v_1 + x_0 \in C$ and $v_2 + x_0 \in C$, and so

$$\alpha v_1 + \beta v_2 + x_0 = \alpha\left(v_1 + x_0\right) + \beta\left(v_2 + x_0\right) + (1-\alpha-\beta)x_0 \in C$$

since C is affine, and $\alpha + \beta + (1-\alpha-\beta) = 1$. We conclude that $\alpha v_1 + \beta v_2 \in V$.

Lemma 2.25. *Thus, the affine set C can be expressed as:*

$$C = V + x_0 = \{v + x_0 \mid v \in V\},$$

which is also called the ***translate of the subspace*** *V.*

Remark 2.26. *The subspace V associated with the affine set C does not depend on the choice of x_0, so x_0 can be chosen as any point in C.*

Definition 2.27. ***(Affine Dimension)*** *We define the dimension of an affine set C as the dimension of the subspace $V = C - x_0$, where x_0 is any element of C.*

Example 2.28. ***(solution set of linear equations)*** *The solution set of a system of linear equations, $C = \{x \mid Ax = b\}$, where $A \in \mathbf{R}^{m \times n}$ and $b \in \mathbf{R}^m$, is an* ***affine set****. To show this, suppose $x_1, x_2 \in C$, i.e., $Ax_1 = b, Ax_2 = b$. Then for any θ, we have*

$$\begin{aligned} A\left(\theta x_1 + (1-\theta)x_2\right) &= \theta A x_1 + (1-\theta) A x_2 \\ &= \theta b + (1-\theta) b \\ &= b \end{aligned}$$

which shows that the affine combination $\theta x_1 + (1-\theta)x_2$ is also in C. ***The subspace associated with the affine set C is the nullspace of A.***

We also have a ***converse: every affine set can be expressed as the solution set of a system of linear equations.*** *This is because every affine set can be expressed as translate to a subspace, and every subspace could be expressed as the null space of a linear mapping. One choice for the linear mapping is the projection as shown in [1, 229].*

affine set $\Longleftrightarrow$ solution set of a system of linear equations

2.1.2.2 Affine Independent

Definition 2.29. ***(Affine Independent)*** *Let $v_0, v_1 \dots v_k$ be points in $\mathbf{R}^n$. These points are called* ***affinely independent*** *if "$\sum_{i=0}^k \lambda_i v_i = 0$ and $\sum_{i=0}^k \lambda_i = 0 \Rightarrow \lambda_i = 0, \forall\, i$"*

Lemma 2.30. ***(Affine Independent and Linear Independent)***
"$v_0, v_1 \dots v_k$ are affinely independent" $\Longleftrightarrow$ "$v_1 - v_0, v_2 - v_0 \dots v_k - v_0$ are linearly independent"

Remark 2.31. *v_0 can be replaced by any v_i, which is called a* ***base point****.*

Proof. "$\Rightarrow$": Consider some $\lambda_1, \dots, \lambda_k$, such that:

$$\sum_{i=1}^{k} \lambda_i \left(v_i - v_0\right) = 0$$

Consider some $\lambda_0 \in \mathbf{R}$, such that $\sum_{i=0}^k \lambda_i = 0$ Also, we have:

$$\sum_{i=0}^{k} \lambda_i v_i = \sum_{i=1}^{k} \lambda_i \left(v_i - v_0\right) + \left(\sum_{i=0}^{k} \lambda_i\right) v_0 = 0$$

We get $\lambda_i = 0, \forall\, i$ due to affine independence, i.e. $v_i - v_0$ are linear independent.

"$\Leftarrow$": Consider some $\lambda_0, \lambda_1, \dots, \lambda_k$, such that: $\sum_{i=0}^k \lambda_i v_i = 0$ and $\sum_{i=0}^k \lambda_k = 0$ We have to show that all these coefficients must be zero under the condition of linear independence. We have $\sum_{i=1}^k \lambda_i \left(v_i - v_0\right) = 0$. Therefore, due to linear independence of $(v_i - v_0)$, we conclude that:

$$\lambda_1 = \lambda_2 = \ldots = \lambda_k = 0$$

Also, $\sum_{i=0}^k \lambda_k = 0 \Rightarrow \lambda_0 = 0$ This proves that they are affinely independent. □

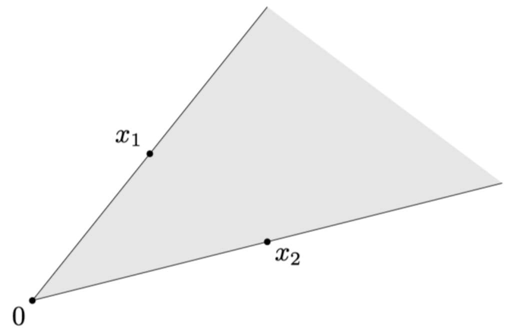


Fig. 2: Conic

2.1.3 Cone

Definition 2.32. ***(Conic)*** *A nonempty subset M of $\mathbf{R}^n$ is called **conic**, if it contains, along with every point $x \in M$, the entire ray $\mathbf{R}_x = \{tx \mid t \geq 0\}$ spanned by the point:*

$$\forall\, x \in M, t \geq 0, tx \in M$$

Definition 2.33. ***(Cone)*** *A **convex conic** set is called a **cone**[1]. Analogy to Definition 2.1, we have: A subset M of $\mathbf{R}^n$ is called **cone**, if $\forall\, x, y \in M$, M contains all conic combinations (see below for definition) of x, y:*

$$\forall\, x, y \in M, \lambda_1, \lambda_2 \geq 0, \lambda_1 x + \lambda_2 y \in M.$$

Remark 2.34. *See Fig. 2 for **conic** example. Note $\lambda_1 x + \lambda_2 y = \alpha(\beta x + (1-\beta)y)$ for some $\alpha, \beta \geq 0$. So it contains the rays going through any convex combination of x and y. The convex combination could be relaxed to sum (see Remark 2.39).*

Remark 2.35. ***(Convex vs. Affine vs. Cone)***

- *convex: $\lambda_i \geq 0$ and $\sum_i \lambda_i = 1$.*
- *affine: $\sum_i \lambda_i = 1$.*
- *cone: $\lambda_i \geq 0, \forall\, i$ ($\lambda \succeq 0$)*
- *so we have affine $\Rightarrow$ convex, and cone $\Rightarrow$ convex.*

Definition 2.36. ***(Conic Combination)*** *A conic combination of **finitely many** vectors $y_1, \ldots, y_m$ is their linear combination*

$$y = \sum_{i=1}^{m} \lambda_i y_i, \lambda_i \geq 0$$

*with **nonnegative** coefficients.*

Definition 2.37. ***(Conic Hull)*** *$\mathrm{Conic}(M)$ is the **smallest** cone set containing M, namely, the **intersection of all cone sets containing** M. This is because the intersection of cones is still a cone (analogous to Lemma 2.14)*

Corollary 2.38. ***(conic hull via conic combinations)*** *For a nonempty $M \subset \mathbf{R}^n$:*

$$\mathrm{Cone}(M) = \{\textit{the set of all conic combinations of vectors from } M\}$$

Proof. Similar to the proof to Corollary 2.17. □

Remark 2.39. ***(equivalent statement)*** *A nonempty subset M of $\mathbf{R}^n$ is a cone if and only if it possesses the following pair of properties:*

- *is conic: $\forall\, x \in M, t \geq 0, tx \in M$;*
- *contains sums of its elements: $\forall\, x, y \in M, x + y \in M$.*

[1]In [2], conic is called cone, while cone is called convex cone

2.1.3.1 Polyhedral Cone: inner and outer description

Example 2.40. *The solution set of an arbitrary (possibly, infinite) system,* $a_\alpha^\top x \leq 0, \alpha \in \mathcal{A}$*, of* ***homogeneous linear inequalities*** *with* n *unknowns* x*, namely, the set*

$$K = \left\{x \mid a_\alpha^\top x \leq 0, \forall \alpha \in \mathcal{A}\right\}$$

is a ***cone****. In particular, the solution set to a homogeneous* ***finite*** *system of* m *homogeneous linear inequalities*

$$Ax \preceq 0$$

(A *is* $m \times n$ *matrix) is a cone; a cone of this latter type is also a polyhedral as mentioned in Example 2.4, we call it the* ***polyhedral cone***

Remark 2.41. *Compare with Example 2.4,* 0 *on the right hand side is very important. Only* 0 *guarantees the cone.* $ab \leq b, \forall a \geq 0 \Rightarrow a = 0$.

Remark 2.42. *Note that the cones given by systems of linear homogeneous nonstrict inequalities necessarily are closed. Similar to Example 2.28 and Remark 2.18, we have*

closed cone $\Longleftrightarrow$ ***solution set for linear homogeneous nonstrict inequalities***

This is because closed cone are convex and thus could be expressed using the closed half-spaces in Remark 2.18. Since it is a conic, we then have the half-spaces $\left\{y \mid a_x^\top y \leq \alpha_x\right\}$ *need to be* ***homogeneous linear inequalities*** *(i.e. with* $\alpha_x = 0$*, similar to the reason mentioned in Remark 2.41)*

We therefore have the special case of the ***outer description of closed convex cone*** *(for the general case of closed convex set see Section 2.2.4):*

a closed cone is the intersection of all homogeneous halfspaces containing the cone.

Example 2.43. *In particular, the conic hull of a nonempty* ***finite*** *set* $M = \{u_1, \ldots, u_N\}$ *of vectors in* $\mathbf{R}^n$ *is the cone*

$$\mathrm{Cone}(M) = \left\{\sum_{i=1}^{N} \lambda_i u_i \mid \lambda_i \geq 0, i = 1, \ldots, N\right\}$$

Remark 2.44. *This is the* ***inner*** *description of a polyhedral cone while the set given by finitely many homogeneous linear inequalities in Example 2.40 is* ***outer*** *description.*

Lemma 2.45. ***(Weyl-Minkowski Lemma)***

"inner description with finite vectors" $\Leftrightarrow$ ***"outer with finite homogeneous linear inequalities"***

Proof. Let K be a finitely generated cone. Then for some matrix D we have

$$\begin{aligned} K &= \{x \in \mathbf{R}^n : x = Dy \text{ for some } y \succeq 0\} \\ &= \{x \in \mathbf{R}^n : \text{ the system } x - Dy = 0, y \succeq 0 \text{ is consistent in } x, y\} \\ &= \{x \in \mathbf{R}^n : Bx \succeq 0\} \quad \text{ for some matrix } B \\ &= \{x \in \mathbf{R}^n : -Bx \preceq 0\} \quad \text{ for some matrix } B \end{aligned}$$

by using Fourier-Motzkin elimination to eliminate y. □

2.1.3.2 Homogeneous Farkas Lemma

Theorem 2.46. ***(Homogeneous Farkas Lemma)*** *Let* $A \in \mathbf{R}^{n \times N}$ *and* $b \in \mathbf{R}^n$*. Then exactly one of the following two assertions is true:*

1. *There exists an* $\lambda \in \mathbf{R}^N$ *such that* $A\lambda = b$ *and* $\lambda \succeq 0$
2. *There exists a* $h \in \mathbf{R}^n$ *such that* $A^\top h \succeq 0$ *and* $b^\top h < 0$.

Remark 2.47. ***(explanation)*** *n: vector dimension; N: number of vectors. Consider the closed convex cone $C(A)$ spanned by the columns of A; that is,*

$$C(A) = \{A\lambda \mid \lambda \succeq 0\}$$

Observe that $C(A)$ is the set of the vectors b for which the first assertion in the statement of Farkas' lemma holds. On the other hand, the vector h ***in the second assertion is orthogonal to a hyperplane that separates*** *b* ***and*** *$C(A)$.*

The lemma follows from the observation that b belongs to $C(A)$ if and only if there is no hyperplane that separates it from $C(A)$. More precisely, let $a_1, \ldots, a_N \in \mathbf{R}^n$ denote the columns of A. In terms of these vectors, Farkas' lemma states that exactly one of the following two statements is true:

1. *There exist* ***nonnegative*** *coefficients $\lambda_1, \ldots, \lambda_N \in \mathbf{R}$ such that $b = \lambda_1 a_1 + \cdots + \lambda_n a_N$*
2. *There exists a vector $h \in \mathbf{R}^n$ such that $a_i^\top h \geq 0$ for $i = 1, \ldots, N$, and $b^\top h < 0$.*

Proof. We need to prove if $b \notin C(A)$, we have 2). From the separation of a convex set and a point outside of the set in Lemma 2.97, we know there exists a hyperplane that strongly separate b and $C(A)$, i.e.

$$b^\top h < \inf_{a \in C(A)} a^\top h$$

We only need to show

$$b^\top h < 0 \leq \inf_{a \in C(A)} a^\top h.$$

Select point $a = \mathbf{0} \in C(A)$, we get $b^\top y < 0$. If we can select one $a \in C(A)$ s.t. $a^\top h = -\epsilon$, where $\epsilon > 0$, for a large enough α, we have $\alpha a^\top h = -\alpha\epsilon < b^\top h$, a contradiction. We therefore have $b^\top h < 0 \leq \inf_{a \in C(A)} a^\top h$. □

Corollary 2.48. ***(Extended Farkas Lemma: sufficient and necessary)*** *Let $b, a_1, \ldots, a_N$ be vectors from $\mathbf{R}^n$.*
"b is a conic combination of a_i" $\Longleftrightarrow$ "h satisfies that $h^\top a_i \geq 0, i = 1, \ldots, N \Rightarrow h^\top b \geq 0$"

Proof. "$\Rightarrow$": This is a restatement of Theorem 2.46.

"$\Leftarrow$": Assume that every vector h satisfying $h^\top a_i \geq 0 \forall\, i$ satisfies also $h^\top b \geq 0$, and we need to prove that b is a conic combination of the vectors a_i.

From Remark 2.42, we know the set $\mathrm{Cone}(\{a_1, \ldots, a_N\})$ of all conic combinations of $a_1, \ldots, a_N$ is polyhedrally representable:

$$\mathrm{Cone}(\{a_1, \ldots, a_N\}) = \left\{x \mid p_j^\top x \geq 0, 1 \leq j \leq J\right\} \tag{2.49}$$

For every j, relation $p_j^\top a_i \geq 0$ for all i implies, by the premise of the statement we want to prove, that $p_j^\top b \geq 0$. We see that $p_j^\top a \geq 0$ for all j, meaning that b indeed belongs to the cone (2.49). □

Corollary 2.50. ***(Generalized Farkas Lemma)*** *Let $A \in \mathbf{R}^{n \times N}, b \in \mathbf{R}^n, S$ is a closed convex cone in $\mathbf{R}^N$, and the dual cone of S is $S^* = \left\{z \in \mathbf{R}^N \mid z^\top \lambda \geq 0, \forall\, \lambda \in S\right\}$. If convex cone $C(A) = \{A\lambda \mid \lambda \in S\}$ is closed, then exactly one of the following two statements is true:*

1. *There exists an $\lambda \in \mathbf{R}^N$ such that $A\lambda = b$ and $\lambda \in S$*
2. *There exists a $h \in R^n$ such that $A^\top h \in S^*$ and $b^\top h < 0$*

Remark 2.51. ***(explanation of generalized Farkas' lemma)*** *Either a vector is in a given closed convex cone, or there exists a hyperplane separating the vector from the cone; there are no other possibilities. See also Section 4.11.2 for* ***strong alternatives****.*

The closedness condition is necessary. For original Farkas' lemma, S is the nonnegative orthant $\mathbf{R}_+^N$, hence the closedness condition holds automatically. For polyhedral convex cone, the closedness condition holds automatically from the outer description.

Remark 2.52. ***(solvability of finite linear equalities)*** *By setting $S = \mathbf{R}^n$ and $S^* = \{0\}$ in generalized Farkas' lemma, we obtain the following corollary about the solvability for a finite system of linear equalities:*

Let $A \in \mathbf{R}^{n\times N}$ and $b \in \mathbf{R}^N$. Then exactly one of the following two statements is true:

1. *There exists an $\lambda \in \mathbf{R}^N$ such that $A\lambda = b$*
2. *There exists a $h \in \mathbf{R}^n$ such that $A^\top h = 0$ and $b^\top h \neq 0$. (note here $\neq$ is equivalent to $<$ because of the former $A^\top h = 0$)*

Corollary 2.53. ***(generalized convex combination Remark 2.9)*** *Suppose $\lambda_1, \lambda_2, \ldots$ satisfy*

$$\lambda_i \geq 0, \quad i = 1, 2, \ldots, \quad \sum_{i=1}^{\infty} \lambda_i = 1$$

*and $x_1, x_2, \ldots \in C$, where $C \subseteq \mathbf{R}^n$ is **convex**. Then*

$$\sum_{i=1}^{\infty} \lambda_i x_i \in C \tag{2.54}$$

if the series converges.

Proof. We can show by **induction on dimension** (the base case $n = 0$ being trivial). Let $x = \sum_{i=1}^{\infty} \lambda_i x_i$, and let

$$V \coloneqq \mathbf{R}_{>0}(C - x) = \{y \in \mathbf{R}^n \mid x + \alpha y \in C \text{ for some } \alpha > 0\} \tag{2.55}$$

Since C is convex, V is a convex cone. If $V = \mathbf{R}^n$ then $0 \in V$ implies $x \in C$, and we are done. Suppose $V \neq \mathbf{R}^n$. From Farkas lemma, V is contained in a half-space

$$H = \{y \in \mathbf{R}^n \mid h \cdot y \leq 0\}$$

where $h \in \mathbf{R}^n$ is nonzero. Hence $h \cdot c \leq \lambda$ for all $c \in C$ where $\lambda = h \cdot x$ from definition (2.55). In particular

$$\lambda = \sum_{i=1}^{\infty} \lambda_i (h \cdot x_i) \leq \left(\sum_{i=1}^{\infty} \lambda_i\right) \lambda = \lambda$$

so we must have $h \cdot x_i = \lambda$ for each i with $\lambda_i > 0$. Now the result follows from the inductive hypothesis applied to the hyperplane $\{y \in \mathbf{R}^n \mid h \cdot y = \lambda\}$, which has dimension $n - 1$. □

2.1.4 Operations Preserve Convexity

2.1.4.1 Intersection

It is Lemma 2.14. Let $\{M_\alpha\}_\alpha$ be an arbitrary family of convex subsets of $\mathbf{R}^n$. Then the intersection

$$M = \cap_\alpha M_\alpha$$

is convex.

- **Example 2.56.** ***(positive semidefinite cone)*** *The positive semidefinite cone S_+^n can be expressed as*

  $$\bigcap_{z \neq 0} \left\{X \in S^n \mid z^\top X z \geq 0\right\}.$$

  For each $z \neq 0$, $z^\top X z$ is a (not identically zero) linear function of X, so the sets

  $$\left\{X \in S^n \mid z^\top X z \geq 0\right\}$$

  are, in fact, halfspaces in S^n. Thus the positive semidefinite cone is the intersection of an infinite number of halfspaces, and so is convex.
- **Example 2.57.** *We consider the set*

  $$S = \{x \in \mathbf{R}^m || p(t) | \leq 1 \, \textit{for} \, |t| \leq \pi/3\}$$

  where $p(t) = \sum_{k=1}^{m} x_k \cos kt$. The set S can be expressed as the intersection of an infinite number of slabs: $S = \bigcap_{|t| \leq \pi/3} S_t$, where

  $$S_t = \left\{x \mid -1 \leq (\cos t, \ldots, \cos mt)^\top x \leq 1\right\}$$

  and so is convex.

2.1.4.2 Cartesian Product and Projection

• Cartesian product:

If S_1 and S_2 are convex, then so is the Cartesian product

$$S_1 \times S_2 = \{(x_1, x_2) \mid x_1 \in S_1, x_2 \in S_2\}$$

• Two projection:

- The projection of a convex set onto some of its coordinates is convex: if $S \subseteq \mathbb{R}^m \times \mathbb{R}^n$ is convex, then
$$T = \{x_1 \in \mathbb{R}^m \mid (x_1, x_2) \in S \text{ for some } x_2 \in \mathbb{R}^n\}$$
is convex.
- If $S \subseteq \mathbb{R}^m \times \mathbb{R}^n$ is convex, then for any constant h,
$$T = \{x_1 \in \mathbb{R}^m \mid (x_1, h) \in S\}$$
is convex or empty.

2.1.4.3 Affine Mapping

1). image under affine mapping:

If $M \subset \mathbf{R}^n$ is convex and $x \mapsto \mathcal{A}(x) \equiv Ax + b$ is an affine mapping from $\mathbf{R}^n$ into $\mathbf{R}^m$ (A is $m \times n$ matrix, b is m-dimensional vector), then the set

$$\mathcal{A}(M) = \{y = \mathcal{A}(x) \equiv Ax + b \mid x \in M\}$$

is a convex set in $\mathbf{R}^m$.

2). inverse image under affine mapping:

If $M \subset \mathbf{R}^n$ is convex and $y \mapsto Ay + b$ is an affine mapping from $\mathbf{R}^m$ to $\mathbf{R}^n$ (A is $n \times m$ matrix, b is n-dimensional vector), then the set

$$\mathcal{A}^{-1}(M) = \{y \in \mathbf{R}^m \mid \mathcal{A}(y) \in M\}$$

is a convex set in $\mathbf{R}^m$.

- **Example 2.58.** ***(polyhedron)*** *The polyhedron $\{x \mid Ax \preceq b, Cx = d\}$ ($Cx = d$ can be ignored, same as in Example 2.4. see Example 2.6) can be expressed as the inverse image of the Cartesian product of the nonnegative orthant (Example 2.7) and the origin under the affine function $f(x) = (b - Ax, d - Cx)$:*
$$\{x \mid Ax \preceq b, Cx = d\} = \left\{x \mid f(x) \in \mathbf{R}_+^m \times \{0\}\right\}$$
- **Example 2.59.** ***(solution set of linear matrix inequality)*** *The condition*
$$A(x) = x_1 A_1 + \cdots + x_N A_n \preceq B$$
where $B, A_i \in S^m$, is called a linear matrix inequality (LMI) *in x. (Note the similarity to an ordinary linear inequality,*
$$a^\top x = x_1 a_1 + \cdots + x_N a_N \leq b$$
with $b, a_i \in \mathbf{R}$.) The solution set of a linear matrix inequality, $\{x \mid A(x) \preceq B\}$, is convex. Indeed, it is the inverse image of the positive semidefinite cone under the affine function $f : \mathbf{R}^n \to S^m$ given by $f(x) = B - A(x)$

2.1.4.4 Arithmetic Summation and Multiplication by Reals:

If $M_1, \ldots, M_k$ are convex sets in $\mathbf{R}^n$ and $\lambda_1, \ldots, \lambda_k$ are **arbitrary reals (no need** ≥ 0), then the set

$$\lambda_1 M_1 + \ldots + \lambda_k M_k = \left\{\sum_{i=1}^{k} \lambda_i x_i \mid x_i \in M_i, i = 1, \ldots, k\right\}$$

is convex.

Remark 2.60. *We can take this as the affine mapping from $M_1 \times \ldots \times M_k \to \mathbf{R}$, so the conclusion holds.*

2.1.4.5 Linear-fractional and Perspective Function

1). image under perspective function:

Definition 2.61. ***(Perspective Function)***

$$\begin{aligned} P : \mathbf{R}^n \times \mathbf{R}_{++} &\to \mathbf{R}^n \\ (z,t) &\mapsto z/t \end{aligned}$$

Suppose that $x = (\tilde{x}, x_{n+1})\,, y = (\tilde{y}, y_{n+1}) \in \mathbf{R}^{n+1}$ with $x_{n+1} > 0$, $y_{n+1} > 0$. Then for $0 \leq \theta \leq 1$

$$P(\theta x + (1-\theta)y) = \frac{\theta \tilde{x} + (1-\theta)\tilde{y}}{\theta x_{n+1} + (1-\theta) y_{n+1}} = \mu P(x) + (1-\mu)P(y)$$

where

$$\mu = \frac{\theta x_{n+1}}{\theta x_{n+1} + (1-\theta) y_{n+1}} \in [0,1].$$

As θ varies between 0 and 1 (which sweeps out the line segment $[x,y]$), μ varies between 0 and 1 (which sweeps out the line segment $[P(x), P(y)]$). This shows that $P([x,y]) = [P(x), P(y)]$. We therefore get

$$P(C) \textbf{ is convex if } C \subseteq \operatorname{dom} P \textbf{ is convex} \tag{2.62}$$

Remark 2.63. *Geometrically, with each point z in $\mathbf{R}^n$ we associate the (open) ray $\mathcal{P}(z) = \{t(z,1) \mid t > 0\}$ in $\mathbf{R}^{n+1}$. That means we get*

1-1 matching between R^n and a set of rays in R^{n+1}

2). inverse image under perspective function:

If $C \subseteq \mathbf{R}^n$ is convex, then

$$P^{-1}(C) = \left\{(x,t) \in \mathbf{R}^{n+1} \mid x/t \in C, t > 0\right\}$$

is convex. To show this, suppose $(x,t) \in P^{-1}(C), (y,s) \in P^{-1}(C)$, and $0 \leq \theta \leq 1$. We need to show that

$$\theta(x,t) + (1-\theta)(y,s) \in P^{-1}(C)$$

i.e., that

$$\frac{\theta x + (1-\theta)y}{\theta t + (1-\theta)s} \in C$$

This follows from

$$\frac{\theta x + (1-\theta)y}{\theta t + (1-\theta)s} = \mu \frac{x}{t} + (1-\mu)\frac{y}{s}$$

where

$$\mu = \frac{\theta t}{\theta t + (1-\theta)s} \in [0,1].$$

We therefore have

$$P^{-1}(C) \textbf{ is convex if } C \subseteq \textbf{ is convex} \tag{2.64}$$

3) image and inverse image under linear-fractional functions:

A linear-fractional function is formed by composing the perspective function with an affine function. Suppose $g : \mathbf{R}^n \to \mathbf{R}^{m+1}$ is affine, i.e.,

$$g(x) = \begin{bmatrix} A \\ c^\top \end{bmatrix} x + \begin{bmatrix} b \\ d \end{bmatrix}$$

where $A \in \mathbf{R}^{m \times n}, b \in \mathbf{R}^m, c \in \mathbf{R}^n$, and $d \in \mathbf{R}$. The function $f : \mathbf{R}^n \to \mathbf{R}^m$ given by $f = P \circ g$, i.e.

$$f(x) = (Ax + b)/\left(c^\top x + d\right), \quad \operatorname{dom} f = \left\{x \mid c^\top x + d > 0\right\}$$

Remark 2.65. ***(affine, linear vs. linear-fractional)*** *If $c = 0$ and $d > 0$, the domain of f is $\mathbf{R}^n$, and f is an affine function. So we can think of affine and linear functions as special cases of linear-fractional functions.*

Remark 2.66. ***(interpretation)*** *It is often convenient to represent a linear fractional function as a matrix*

$$Q = \begin{bmatrix} A & b \\ c^\top & d \end{bmatrix} \in \mathbf{R}^{(m+1)\times(n+1)}$$

that acts on (multiplies) points of form $(x, 1)$, which yields $\left(Ax + b, c^\top x + d\right)$. This result is then scaled or normalized so that its last component is one, which yields $(f(x), 1)$.

The linear-fractional function can be expressed as

$$f(x) = P^{-1}(QP(x))$$

Like the perspective function, linear-fractional functions preserve convexity. If C is convex and lies in the domain of f (i.e., $c^\top x + d > 0$ for $x \in C$), then its image $f(C)$ is convex. This follows immediately from results above: the image of C under the affine mapping (2.12) is convex, and the image of the resulting set under the perspective function P, which yields $f(C)$, is convex. Similarly, if $C \subseteq \mathbf{R}^m$ is convex, then the inverse image $f^{-1}(C)$ is convex.

Example 2.67. ***(conditional probabilities)*** *Suppose u and v are random variables that take on values in $\{1, \ldots, n\}$ and $\{1, \ldots, m\}$, respectively, and let p_{ij} denote $\mathbb{P}(u = i, v = j)$. Then the conditional probability $f_{ij} = \mathbb{P}(u = i \mid v = j)$ is given by*

$$f_{ij} = \frac{p_{ij}}{\sum_{k=1}^n p_{kj}}$$

Thus f is obtained by a linear-fractional mapping from p. It follows that if C is a convex set of joint probabilities for (u, v), then the associated set of conditional probabilities of u given v is also convex.

2.1.5 Topological Properties of Convex Set

Lemma 2.68. *Let $x \in \operatorname{ri} M$ and $y \in \operatorname{cl} M$. Then all points from the half-segment $[x, y)$ belong to the* $\operatorname{ri} M$*:*

$$[x, y) = \{z = (1-\lambda)x + \lambda y \mid 0 \le \lambda < 1\} \subset \operatorname{ri} M$$

Proof. Let Aff $(M) = a + V, V$ being linear subspace; then

$$M \subset \operatorname{Aff}(M) = x + L$$

Let B be the unit ball in V :

$$B = \{h \in V \mid |h| \le 1\}$$

Since $x \in$ ri M, there exists radius $r > 0$ such that

$$x + rB \subset M$$

Since $y \in \operatorname{cl} M$, we have $y \in \operatorname{Aff}(M)$. Besides, for any $\epsilon > 0$ there exists $y' \in M$ such that $|y' - y| \le \epsilon$; since both y' and y belong to Aff (M), the vector $y - y'$ belongs to L and consequently to ϵB. Thus,

$$\forall \epsilon > 0, y \in M + \epsilon B$$

Let $z = (1-\lambda)x + \lambda y, 0 \le \lambda < 1$, we need to prove $\delta > 0$ s.t. $z + \delta B \subset M$.

For any $\delta > 0$, we have

$$z + \delta B = (1-\lambda)x + \lambda y + \delta B \subset (1-\lambda)x + \lambda[M + \delta B] + \delta B = (1-\lambda)\left[x + \frac{\lambda\delta}{1-\lambda}B + \frac{\delta}{1-\lambda}B\right] + \lambda M$$

for all $\delta > 0$. Now, for the centered at zero Euclidean ball B and nonnegative t', t'' one has

$$t'B + t''B \subset (t' + t'')\, B$$

by the triangle inequality. Given this inclusion, we get that

$$z + \delta B \subset (1-\lambda)\left[x + \frac{(1+\lambda)\delta}{1-\lambda}B\right] + \lambda M$$

for all $\delta > 0$. Setting δ small enough, we can make the coefficient at B in the right hand side less than r; for this choice of δ, we, in view of (1.1.4), have

$$x + \frac{(1+\lambda)\delta}{1-\lambda}B \subset M$$

and we come to

$$z + \delta B \subset (1-\lambda)M + \lambda M = M$$

□

Corollary 2.69. *Let M be a convex set. Given points $x_i \in \operatorname{cl} M$, we have*

$$\text{“convex combination } x = \textstyle\sum_i \lambda_i x_i, \text{ and } \exists\, i, \lambda_i > 0\text{”} \Longrightarrow \text{“}x \in \operatorname{ri} M\text{”}$$

Theorem 2.70. *Let M be a **convex** set in $\mathbf{R}^n$. Then*

1. $\operatorname{int} M$, $\operatorname{cl} M$, *and* $\operatorname{ri} M$ *are **convex**.*
2. *If M is nonempty, then the relative interior $\operatorname{ri} M$ of M is **nonempty**.*
3. *The closure of M is the same as the closure of its relative interior:*

$$\operatorname{cl} M = \operatorname{cl} \operatorname{ri} M$$

4. *The relative interior remains unchanged when we replace M with its closure:*

$$\operatorname{ri} M = \operatorname{ri} \operatorname{cl} M$$

Remark 2.71. *in particular, every point of $\operatorname{cl} M$ is the limit of a sequence of points from $\operatorname{ri} M$.*

Proof. **1).** For $\operatorname{cl} M$, for $x, y \in \operatorname{cl} M$, we just take the sequence $\lambda x_i + (1-\lambda)y_i$ with $x_i \to x, y_i \to y$ and both $\{x_i\}, \{y_i\} \subset M$. We then get $\lambda x + (1-\lambda)y \in \operatorname{cl} M$. For $\operatorname{int} M$ and $\operatorname{ri} M$, it suffices to only consider the case $\operatorname{int} M$ when $\operatorname{Aff}(M)$ is the entire space $\mathbf{R}^n$. Indeed, by translation of M we always may assume that $\operatorname{Aff}(M)$ contains 0, i.e., is a linear subspace.

For $x, y \in \operatorname{int} M$, we have $\exists\, \epsilon > 0$ s.t. $B_\epsilon(x) \subset M$ and $B_\epsilon(y) \subset M$. The set of all combinations $\{\lambda x_i + (1-\lambda)y_j \mid x_i \in B_\epsilon(x), y_j \in B_\epsilon(y)\}$ is a ball $B_\epsilon(\lambda x + (1-\lambda)y) \subset M$ for a fix λ (if we can vary λ, it is looks like a cylinder). We therefore have $\operatorname{int} M$ is convex.

2). Similar to the above, it suffices to consider the case when $\operatorname{Aff}(M)$ is the entire space $\mathbf{R}^n$. We should prove that the $\operatorname{int} M$ is nonempty.

$\operatorname{Aff}(M) = \mathbf{R}^n$ possesses an affine basis (of course they are affine independent, see Definition 2.29) $a_0, \ldots, a_n$ from M. Since $a_0, \ldots, a_n$ belong to M and M is convex, the entire convex hull of the vectors, i.e. the simplex Δ with the vertices $a_0, \ldots, a_n$, is contained in M. Consequently, an interior point of the simplex for sure is an interior point of M; thus, in order to prove that $\operatorname{int} M \neq \emptyset$, it suffices to prove that the interior of Δ is nonempty.

For x to be the affine combination, we need

$$\sum_{i=0}^{n} \lambda_i a_i = x; \quad \sum_{i=0}^{n} \lambda_i = 1$$

or, in the entrywise form:

$$\begin{aligned}
&a_{01}\lambda_0 + a_{11}\lambda_1 + \ldots + a_{n1}\lambda_n = x_1 \\
&a_{02}\lambda_0 + a_{12}\lambda_1 + \ldots + a_{n2}\lambda_n = x_2 \\
&a_{0n}\lambda_0 + a_{1n}\lambda_1 + \ldots + a_{nn}\lambda_n = x_n \\
&\lambda_0 + \quad \lambda_2 + \ldots + \quad \lambda_n = 1
\end{aligned}$$

(a_{pq} is q-th entry of vector a_p). Denote the coefficients as matrix A. From the definition of affine independence, we know "$Ax = 0 \Rightarrow x = 0$, which indicate A is nonsingular. Therefore, for any x, we have the unique solution $\{\lambda_i(x)\}$. Note also $\lambda_i(x)$ is continuous w.r.t. x.

Let us take any $x = x^*$ with $\lambda_i(x^*) > 0$, e.g., $x^* = (n+1)^{-1} \sum_{i=0}^{n} a_i$. Due to the continuity of $\lambda_i(\cdot)$, there is a neighborhood of $B_r(x^*)$ positive radius r s.t. λ_i still are positive:

$$x \in B_r(x^*) \Rightarrow \lambda_i(x) > 0, i = 0, \dots, n$$

It means that every $x \in B_r(x^*)$ is an affine combination of a_i with positive coefficients. Thus, Δ contains a neighborhood of x^*, so that x^* is an interior point of Δ.

3). Assume M is nonempty (otherwise all sets in question are empty and therefore coincide with each other). "$\operatorname{cl}\operatorname{ri} M \subset \operatorname{cl} M$" is clear. We need to show $\operatorname{cl} M \subset \operatorname{cl}\operatorname{ri} M$, i.e., to prove that every point $y \in \operatorname{cl} M$ is a limit of a sequence of points $\operatorname{ri} M$. From 2), there exists a point $x \in \operatorname{ri} M$. According to Lemma 2.68 , the half-segment $[x, y)$ belongs to $\operatorname{ri} M$, and y clearly is the limit of a sequence of points on this half-segment, e.g., the sequence $x_i = \frac{1}{n} x + \left(1 - \frac{1}{n}\right) y$.

4). Assume M is nonempty (otherwise all sets in question are empty and therefore coincide with each other). $\operatorname{ri} M \subset \operatorname{ri}\operatorname{cl} M$ is clear. We need to show $\operatorname{ri}\operatorname{cl} M \subset \operatorname{ri} M$. We have one $x \in \operatorname{ri} M$ from nonempty of 2). For $z \in \operatorname{ri}\operatorname{cl} M$, we can extend $[x, z)$ a little bit to point y with $y \in \operatorname{cl} M$ (see Fig. 3). We thus have $z \in \operatorname{ri} M$ from Lemma 2.68.

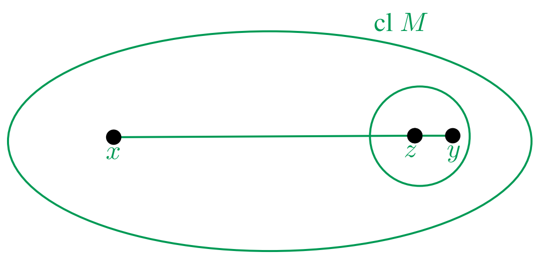


Fig. 3: Visualization of the proof

□

Proof. of **??**:

"$\Rightarrow$": obvious.

"$\Leftarrow$": this is exactly the proof 2) above.

□

2.1.6 More Examples of Convex

Definition 2.72. ***(Polyhedron)*** *See also Example 2.4 for details*

Example 2.73. ***(polytope: bounded polyhedron)*** *A polytope can also be represented as (inner description) the convex hull of a finite nonempty set in $\mathbf{R}^n$, i.e., the set of the form*

$$\operatorname{Conv}(\{u_1, \dots, u_N\}) = \left\{ \sum_{i=1}^{N} \lambda_i u_i \mid \lambda_i \geq 0, \sum_i \lambda_i = 1 \right\}$$

Example 2.74. ***(simplex)*** *An important case of a polytope is a simplex, the convex hull of $n+1$* ***affinely independent*** *points $v_1, \dots, v_{n+1}$ from $\mathbf{R}^n$:*

$$M = \operatorname{Conv}(\{v_1, \dots, v_{n+1}\}) = \left\{ \sum_{i=1}^{n+1} \lambda_i v_i \mid \lambda_i \geq 0, \sum_{i=1}^{n+1} \lambda_i = 1 \right\}$$

the points $v_1, \dots, v_{n+1}$ are called vertices of the simplex.

Example 2.75. ***(Euclidean balls and ellipsoids)*** *(see Example 3.26 for another proof of convex and closed)*

Euclidean balls:

$$B(x_c, r) \coloneqq \{x \mid \|x - x_c\|_2 \leq r\} = \left\{x \mid (x - x_c)^\top (x - x_c) \leq r^2\right\}$$

where $r > 0$, and $\|\cdot\|_2$ denotes the Euclidean norm

Remark 2.76. ***(equivalent form)***

$$B(x_c, r) = \{x_c + ru \mid \|u\|_2 \leq 1\}$$

Ellipsoids:

$$\mathcal{E} \coloneqq \left\{x \mid (x - x_c)^\top P^{-1} (x - x_c) \leq 1\right\}$$

where $P = P^\top \succ 0$, i.e., P is symmetric and positive definite.

Remark 2.77. ***(equivalent form)***

$$\mathcal{E} = \{x_c + Au \mid \|u\|_2 \leq 1\}$$

where A is square and nonsingular. In this representation we can assume without loss of generality that A is symmetric and positive definite by taking $A = P^{1/2}$. When the matrix A is symmetric positive semidefinite but singular, it is called a degenerate ellipsoid; its affine dimension is equal to the rank of A. Degenerate ellipsoids are also convex.

Example 2.78. ***(Norm Cone)*** *The norm cone associated with the any norm $\|\cdot\|$ is the set*

$$C \coloneqq \{(x, t) \mid \|x\| \leq t\} \subseteq \mathbf{R}^{n+1}$$

2.1.7 Generalized Inequalities

2.1.7.1 Proper Cones and Generalized Inequalities

Definition 2.79. ***(Proper Cone)*** *A convex cone $K \subseteq \mathbf{R}^n$ is called a proper cone if it satisfies the following:*

- *K is convex.*
- *K is closed.*
- *K is solid, which means it has nonempty interior.*
- *K is pointed, which means that it contains no line (or equivalently, $x \in K, -x \in K \Longrightarrow x = 0$)*

Definition 2.80. ***(Generalized Inequality)*** *We associate with the proper cone K the partial ordering on $\mathbf{R}^n$ defined by*

$$x \preceq_K y \Longleftrightarrow y - x \in K$$

We also write $x \succeq_K y$ for $y \preceq_K x$. Similarly, we define an associated strict partial ordering by

$$x \prec_K y \Longleftrightarrow y - x \in \operatorname{int} K$$

and write $x \succ_K y$ for $y \prec_K x$.

Properties of generalized inequalities:

A generalized inequality $\preceq_K$ satisfies many properties, such as

- $\preceq_K$ is preserved under addition: if $x \preceq_K y$ and $u \preceq_K v$, then $x + u \preceq_K y + v$.
- $\preceq_K$ is transitive: if $x \preceq_K y$ and $y \preceq_K z$ then $x \preceq_K z$.
- $\preceq_K$ is preserved under nonnegative scaling: if $x \preceq_K y$ and $\alpha \geq 0$ then $\alpha x \preceq_K \alpha y$
- $\preceq_K$ is reflexive: $x \preceq_K x$.

- $\preceq_K$ is antisymmetric: if $x \preceq_K y$ and $y \preceq_K x$, then $x = y$.
- $\preceq_K$ is preserved under limits: if $x_i \preceq_K y_i$ for $i = 1, 2, \ldots, x_i \to x$ and $y_i \to y$ as $i \to \infty$, then $x \preceq_K y$.

The corresponding strict generalized inequality $\prec_K$ satisfies, for example,

- if $x \prec_K y$ then $x \preceq_K y$.
- if $x \prec_K y$ and $u \preceq_K v$ then $x + u \prec_K y + v$
- if $x \prec_K y$ and $\alpha > 0$ then $\alpha x \prec_K \alpha y$.
- $x \nprec_K x$.
- if $x \prec_K y$, then for u and v small enough, $x + u \prec_K y + v$.

2.1.7.2 Minimum and Minimal Elements

Definition 2.81. ***(Minimum)*** *We say that $x \in S$ is the **minimum** element of S (with respect to the generalized inequality $\preceq_K$) if for every $y \in S$ we have $x \preceq_K y$. We define the **maximum** element of a set S, with respect to a generalized inequality, in a similar way. If a set has a minimum (maximum) element, then it is **unique**.*

Definition 2.82. ***(Minimal)*** *We say that $x \in S$ is a **minimal** element of S (with respect to the generalized inequality $\preceq_K$) if $y \in S, y \preceq_K x$ only if $y = x$. We define **maximal** element in a similar way. A set can have many different minimal (maximal) elements, so **not unique.***

$$x \in S \textbf{ is the minimum element of } S \Longleftrightarrow S \subseteq x + K$$

Here $x + K$ denotes all the points that are comparable to x and greater than or equal to x (according to $\preceq_K$).

$$x \in S \textbf{ is the minimal element of } S \Longleftrightarrow (x - K) \cap S = \{x\}$$

Here $x - K$ denotes all the points that are comparable to x and less than or equal to x (according to $\preceq_K$); the only point in common with S is x.

2.2 The Separation Theorem

In this section we answer the following question: assume we are given two convex sets S and T in $\mathbf{R}^n$, when can we separate them by a hyperplane.

Definition 2.83. ***(Hyperplane)*** *A hyperplane M in $\mathbf{R}^n$ (an affine set of dimension $n - 1$) is nothing but a level set of a nontrivial linear form:*

$$\exists\, a \in \mathbf{R}^n, b \in \mathbf{R}, a \neq 0 : \quad M = \left\{x \in \mathbf{R}^n \mid a^\top x = b\right\}$$

Definition 2.84. ***(Proper Separation)*** *We say that a hyperplane*

$$M = \left\{x \in \mathbf{R}^n \mid a^\top x = b\right\} \quad [a \neq 0]$$

properly separates (nonempty) convex sets S and T, if

(i) the sets belong to the opposite closed half-spaces into which M splits $\mathbf{R}^n$

(ii) at least one of the sets is not contained in M itself.

*We say that S and T can be **properly separated**, if there exists a hyperplane which properly separates S and T, i.e., if there exists $a \in \mathbf{R}^n$ such that*

$$\sup_{x \in S} a^\top x \leq \inf_{y \in T} a^\top y \text{ ($\Leftrightarrow$ condition (i))}$$

and

$$\inf_{x \in S} a^\top x < \sup_{y \in T} a^\top y \text{ ($\Leftrightarrow$ condition (ii))}$$

Definition 2.85. ***(Strong Separation)*** *We say that nonempty sets S and T in $\mathbf{R}^n$ can be strongly separated, if there exist two distinct parallel hyperplanes which separate S and T, i.e., if there exists $a \in \mathbf{R}^n$ such that*

$$\sup_{x\in S} a^\top x < \inf_{y\in T} a^\top y$$

Remark 2.86. *strong separation $\Rightarrow$ proper separation*

Theorem 2.87. ***(Proper Separation Theorem)*** *Two nonempty convex sets S and T in $\mathbf{R}^n$ can be properly separated if and only if their relative interiors do not intersect:*

$$\textbf{\textit{convex sets }} S \textbf{\textit{ and }} T \textbf{\textit{ can be properly separated}} \iff \operatorname{ri} S \cap \operatorname{ri} T = \emptyset$$

Example 2.88. ***(separation of an affine and a convex set)*** *Suppose C is convex and D is affine, i.e., $D = \{Fu + g \mid u \in \mathbf{R}^m\}$, where $F \in \mathbf{R}^{n\times m}$. Suppose C and D are disjoint, so by the separating hyperplane theorem there are $a \neq 0$ and μ such that $a^\top x \le \mu$ for all $x \in C$ and $a^\top x \ge \mu$ for all $x \in D$.*

Now $a^\top x \ge \mu$ for all $x \in D$ means $a^\top Fu \ge \mu - a^\top g$ for all $u \in \mathbf{R}^m$. But a linear function is bounded below on $\mathbf{R}^m$ only when it is zero, so we conclude $a^\top F = 0$ (and hence, $\mu \le a^\top g$).

Thus we conclude that $\exists\, a \neq 0$ such that $F^\top a = 0$ and $a^\top x \le \mu \le a^\top g$ for all $x \in C$.

Example 2.89. ***(alternatives for strict linear inequalities)*** *We derive the necessary conditions for solvability of a system of strict linear inequalities*

$$Ax \prec b \tag{2.90}$$

These inequalities are infeasible if and only if the (convex) sets

$$C = \{b - Ax \mid x \in \mathbf{R}^n\}, \quad D = \mathbf{R}^m_{++} = \{y \in \mathbf{R}^m \mid y \succ 0\}$$

do not intersect. The set D is open; C is an affine set. Hence by the result above, $\exists\, \mu$, s.t. $A^\top \lambda = 0$ and $\lambda^\top b \le \mu$. The second inequality means $\lambda^\top y \ge \mu$ for all $y \succ 0$. This implies $\mu \le 0$ and $\lambda \succeq 0, \lambda \neq 0$. (This is because if $\lambda_i < 0$, for large enough $y_i > 0$, the inequality is wrong. If $\mu > 0$, for small enough $\alpha > 0$, we have $\lambda^\top \alpha y < \mu$.)

Putting it all together, we find: (2.90) *is infeasible $\Longrightarrow$*

$$\exists\, \lambda \textit{ s.t. } \lambda \neq 0, \quad \lambda \succeq 0, \quad A^\top \lambda = 0, \quad \lambda^\top b \le 0 \tag{2.91}$$

This is also a system of linear inequalities and linear equations in the variable $\lambda \in \mathbf{R}^m$.

Remark 2.92. *Later in Theorem 2.128, we will show:*
(2.91)$\iff$ *"*(2.90) *is infeasible"*

2.2.1 Necessity of Proper Separation Theorem: "$\Rightarrow$"

We first introduce a lemma about linear function and its maximum/minimum:

Lemma 2.93. *"A linear function $f(x) = a^\top x$ can attain its maximum/minimum over a convex set M at a point $\bar{x} \in \operatorname{ri} M$" $\iff$ "the function is **constant** on M"*

Remark 2.94. *This lemma for the maximum will be generalized to constant of convex function in Theorem 3.141*

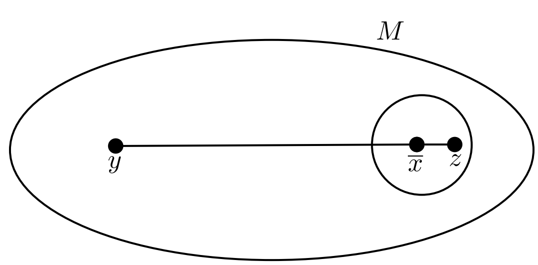


Fig. 4: Visualization of the proof

Proof. "$\Leftarrow$" part is evident. To prove the "$\Rightarrow$" part, let $\bar{x} \in \text{ri } M$ be, say, a minimizer of f over M, we need to prove that $f(\bar{x}) = f(y), \forall\, y \in M$ and $y \neq x$.

Since $\bar{x} \in \text{ri}\, M$, the segment $[y, \bar{x}]$, which is contained in M, can be extended a little bit through the point $\bar{x}$, not leaving M (since $\bar{x} \in \text{ri } M$), so that there exists $z \in M$ such that $\bar{x} \in (y, z)$, i.e., $\bar{x} = (1 - \lambda)y + \lambda z$ with certain $\lambda \in (0, 1)$. Since f is linear, we have

$$f(\bar{x}) = (1 - \lambda)f(y) + \lambda f(z). \tag{2.95}$$

Since $f(\bar{x}) \leq \min\{f(y), f(z)\}$ and $0 < \lambda < 1$, the about equality can be satisfied only when $f(\bar{x}) = f(y) = f(z)$. If $\bar{x}$ is a maximizer, the proof is similar. □

Assume that the sets can be properly separated with one a:

$$\sup_{x \in S} a^\top x \leq \inf_{y \in T} a^\top y; \quad \inf_{x \in S} a^\top x < \sup_{y \in T} a^\top y \tag{2.96}$$

Assume there exists a point $\bar{x} \in (\text{ri}\, S \cap \text{ri}\, T)$, then from the first inequality in (2.96) it is clear that $\bar{x}$ maximizes the linear function $f(x) = a^\top x$ on S and at the same time, it minimizes this form on T. According to Lemma 2.93, the second inequality cannot be correct in (2.96).

We therefore have "$\Rightarrow$" in Theorem 2.87.

2.2.2 Sufficiency of Proper Separation Theorem: "$\Leftarrow$"

2.2.2.1 Separate Closed Convex Set and Point Outside:

Lemma 2.97. ***(strong separation of closed convex set and point outside)*** *Let M be a nonempty and* ***closed convex*** *set in $\mathbf{R}^n$, and let x be a point outside $M (x \notin M)$. Consider the optimization program*

$$\min\{||x - y||_2 \mid y \in M\} \tag{2.98}$$

The program is solvable and has a unique solution y^*, and the linear form $a^\top h, a = x - y^*$ ***strongly separates*** *x and M :*

$$\sup_{y \in M} a^\top y = a^\top y^* = a^\top x - |a|^2 < a^\top x$$

The point y^* is called ***projection*** *of x on M.*

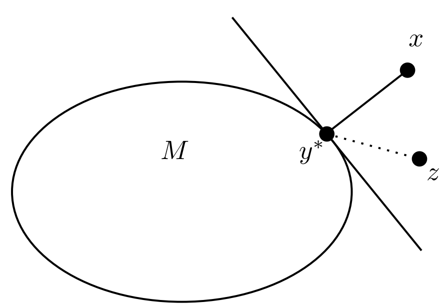


Fig. 5: Visualization of the proof

Proof. Sketch: We know (2.98) is bounded below with infimum $m > 0$ (> 0 if from outside closed set). Denote $f(y) := ||x - y||_2$. Set $N := \{y \mid f(y) \leq 2m\}$ is closed and bounded as therefore compact. $M \cap N$ is therefore compact. We can achieve the infimum m with $y = y^*$.

Uniqueness is from the strong convexity of norm function: if $f(y') = f(y^*)$, we have a point y'' in seg $[y', y^*]$ and $f(y'') < f(y')$.

We next prove $\sup_{y \in M} a^\top y = a^\top y^*$: if not true, we have a $z \in M$ s.t. $a^\top y^* < a^\top z$, i.e. $a^\top(y^* - z) < 0$, however this geometrically means $\exists\, y' \in$ segment (y^*, z) s.t. $f(y') < f(y^*)$. □

Let us consider an example of application of the above proposition:

Corollary 2.99. ***(support function and convex set)*** *Let M be a convex set. Consider a function*

$$\psi_M(x) = \sup\left\{y^\top x \mid y \in M\right\} \tag{2.100}$$

*Function $\psi_M(x)$ is called the **support function** of the set M. Now, let M_1 and M_2 be two closed convex sets.*

1. *If for any $x \in \operatorname{dom}\psi_{M_2}$ we have $\psi_{M_1}(x) \le \psi_{M_2}(x)$ then $M_1 \subset M_2$.*
2. *If $\operatorname{dom}\psi_{M_1} = \operatorname{dom}\psi_{M_2}$ and for any $x \in \operatorname{dom}\psi_{M_1}$ we have $\psi_{M_1}(x) = \psi_{M_2}(x)$. Then $M_1 \equiv M_2$.*

Remark 2.101. *The support function will be used in Section 3.4.2.*

Proof. For 1), assume, on the contrary, that there is $x_0 \in M_1$ and $x_0 \notin M_2$. From Lemma 2.97, we know there is $a \neq 0$ such that $a^\top x_0 > a^\top x$ for any $x \in M_2$. Hence, $a \in \operatorname{dom}\psi_{M_2}$ because $sup_x a^\top x$is bounded, and $\psi_{M_1}(a) > \psi_{M_2}(a)$, which is a contradiction. 2) is a direct conclusion from 1). □

Remark 2.102. *Supporting function $\psi_M(x)$ for any M (not necessarily convex) is positively homogenous of degree 1 :*

$$\psi_M(tx) = t\psi_M(x), \quad x \in \operatorname{dom}\psi_M, \quad t \ge 0$$

and if the set M is bounded then $\operatorname{dom}\psi_M = \mathbf{R}^n$

2.2.2.2 Separate Convex Set and Non-Interior Point

Now we consider a point outside the relative interior of a convex (**not necessarily closed**) set. In general, we do not have strong separation, but still have the proper separation.

Lemma 2.103. ***(Separation of a point and a set)*** *Let M be a convex set in $\mathbf{R}^n$, and let $x \notin \operatorname{ri} M$. Then x and M can be properly separated.*

Proof. $\operatorname{cl} M$ is a closed convex set with the same relative interior as that of M, and $x \notin \operatorname{ri}\operatorname{cl} M$.

If $x \notin \operatorname{cl} M$, then x and $\operatorname{cl} M$ (and also x and $M \subset \operatorname{cl} M$) can be strongly separated by Lemma 2.97.

If x is a point of relative boundary of $\operatorname{cl} M$: $x \in \partial_{\mathrm{ri}} \operatorname{cl} M \equiv \operatorname{cl} M \setminus \operatorname{ri}\operatorname{cl} M$. Without loss of generality we may assume that $x = 0$ (simply by translation). Let L be the linear span of M. Similar to the proof in Theorem 2.70, we may restrict our attention to the subspace L.

Since $x = 0$ is not in the relative interior of $\operatorname{cl} M$, there is a sequence of points $x_i \in L$ not belonging to $\operatorname{cl} M$ and converging to 0 and each x_i can be strongly separated from $\operatorname{cl} M$: there exists a linear form $a_i^\top x$ with the property

$$a_i^\top x_i < \inf_{y \in \operatorname{cl} M} a_i^\top y$$

By normalization, we can assume that all a_i belong to the unit ball. i.e., to a compact set. By resorting to a subsequence, we may assume that a_i themselves converge to certain vector $a \in L$. This vector is unit, since all a_i 's are unit. Now, we have for every fixed $y \in \operatorname{cl} M$:

$$a_i^\top x_i < a_i^\top y$$

As $i \to \infty$, we have $a_i \to a, x_i \to x = 0$, and passing to limit in our inequality, we get

$$a^\top x = 0 \le a^\top y, \quad \forall y \in \operatorname{cl} M$$

Thus, we have established the main part of the required statement: the linear form $f(u) = a^\top u$ separates $\{0\}$ and $\operatorname{cl} M$ (and thus $\{0\}$ and $M \subset \operatorname{cl} M$) . It remains to verify that this separation is proper, which in our case simply means that M is not contained in the hyperplane $\left\{u \mid a^\top u = 0\right\}$. But this is evident since $a \in L = \operatorname{Aff}(M)$. □

2.2.2.3 The General Case

Given two nonempty convex sets S and T with the non-intersecting relative interiors, and we should prove that the sets can be properly separated.

Let $S' = \operatorname{ri} S, T' = \operatorname{ri} T$; these are two nonempty convex sets and they do not intersect. Let $M = T' - S' = \{z = x - y \mid x \in T', y \in S'\}$ which is a nonempty convex (as a sum of two nonempty convex sets) set which does not contain 0. By Lemma 2.103, $\{0\}$ and M can be properly separated: there exists a such that

$$0 = a^\top 0 \leq \inf_{z\in M} a^\top z \equiv \inf_{y\in T', x\in S'} a^\top(y-x) = \left[\inf_{y\in T'} a^\top y\right] - \left[\sup_{x\in S'} a^\top x\right]$$

and

$$0 < \sup_{z\in M} a^\top z \equiv \sup_{y\in T', x\in S'} a^\top(y-x) = \left[\sup_{y\in T'} a^\top y\right] - \left[\inf_{x\in S'} a^\top x\right]$$

From Theorem 2.70, we have

$$\sup_{x\in S} a^\top x \leq \inf_{y\in T} a^\top y, \inf_{x\in S} a^\top x < \sup_{y\in T} a^\top y$$

which means proper separation of S and T.

2.2.3 Strong Separation

There is also a simple **necessary and sufficient** condition for two sets to be **strongly separated**:

Lemma 2.104. *Two nonempty convex sets S and T in $\mathbf{R}^n$ can be strongly separated if and only if:*

$$\rho(S,T) = \inf_{x\in S, y\in T} |x-y| > 0$$

One special case is one of the sets is compact, the other one is closed and the sets do not intersect.

Proof. "$\Leftarrow$": if S and T can be strongly separated, i.e., for certain a one has

$$\alpha \equiv \sup_{x\in S} a^\top x < \beta \equiv \inf_{y\in T} a^\top y$$

then from Cauchy's inequality for every pair (x,y) with $x\in S$ and $y \in T$ one has

$$|x-y| \geq \frac{\beta-\alpha}{|a|}$$

"$\Rightarrow$": consider the set $\Delta = S - T$. This is a convex set which clearly does not contain vectors of the length less than $\rho(S,T) > 0$; consequently, it does not intersect the ball B of a radius $r < \rho(S,T)$ centered at the origin. Consequently, by the Separation Theorem, Δ can be properly separated from B : there exists a such that

$$\inf_{z\in B} a^\top z \geq \sup_{x\in S, y\in T} a^\top(x-y), \qquad \sup_{z\in B} a^\top z > \inf_{x\in S, y\in T} a^\top(x-y)$$

From the second of these inequalities it follows that $a \neq 0$. Therefore $\inf_{z\in B} a^\top z < 0$, so that the first inequality in means that a strongly separates S and T. ☐

2.2.4 Outer Description of Closed Convex Set: supporting planes

We can now prove the "outer" characterization of a closed convex set announced in the beginning of this section:

Theorem 2.105. ***(outer description of closed convex set)*** *Any closed convex set M in $\mathbf{R}^n$ is the solution set of an (infinite) system of nonstrict linear inequalities.*

Geometrically: every closed convex set $M \subset \mathbf{R}^n$ which differs from the entire $\mathbf{R}^n$ is the intersection of all closed half-spaces which contain M.

Remark 2.106. *If M is convex but not necessarily closed, intersection of all closed half-spaces which contain M is $\operatorname{cl} M$.*

Proof. If M is empty, there is nothing to prove: an empty set is the intersection of two properly chosen closed half-spaces. If M is the entire space, there also is nothing to prove: this is the solution set to the empty system of linear inequalities. Now assume that M is convex, closed, nonempty and differs from the entire space. Let $x \notin M$; then x is at the positive distance from M since M is closed and therefore there exists a hyperplane strongly separating x and M:

$$\forall\, x \notin M \exists\, a_x : a_x^\top x > \alpha_x \equiv \sup_{y \in M} a_x^\top y$$

For every $x \notin M$ the closed half-space $H_x = \left\{ y \mid a_x^\top y \leq \alpha_x \right\}$ clearly contains M and does not contain x; consequently,

$$M = \cap_{x \notin M} H_x$$

□

Among the closed half-spaces which contain a closed convex and proper (i.e., nonempty and differing from the entire space) set M the most interesting are the "extreme" ones those with the boundary hyperplane touching M. The notion makes sense for an arbitrary (not necessary closed) convex set, but we will use it for closed sets only.

Definition 2.107. ***(Supporting Plane)*** *Let M be a convex closed set in $\mathbf{R}^n$, and let $x \in \partial_{\mathrm{ri}} M$. A hyperplane*

$$\Pi = \left\{ y \mid a^\top y = a^\top x \right\} \quad [a \neq 0]$$

*is called **supporting** to M at x, if it properly separates M and $\{x\}$, i.e., if*

$$a^\top x \geq \sup_{y \in M} a^\top y \quad \text{and} \quad a^\top x > \inf_{y \in M} a^\top y$$

Remark 2.108. ***(equivalent definition)*** *Note that since x is a point from the relative boundary of M and therefore belongs to $\operatorname{cl} M = M$, the first inequality is in fact is equality. Thus, an equivalent definition of a supporting plane is as follows:*

Let M be a closed convex set and $x \in \partial_{\mathrm{ri}} M$. The hyperplane $\left\{ y \mid a^\top y = a^\top x \right\}$ is called supporting to M at x, if the linear form $a(y) = a^\top y$:

*i). **attain the maximum** on M at the point x*

*ii). **nonconstant** on M*

The most important property of a supporting plane is its existence:

Lemma 2.109. ***(existence of supporting hyperplane)*** *Let M be a convex closed set in $\mathbf{R}^n$ and $x \in \partial_{\mathrm{ri}} M$. Then*

1. *There exists at least one hyperplane which is supporting to M at x;*
2. *If Π is supporting to M at x, then the intersection $M \cap \Pi$ is of affine dimension less than the affine dimension of M.*

Proof. 1). is easy from Proper Separation Theorem Theorem 2.87.

To prove 2)., note that if $\Pi = \left\{ y \mid a^\top y = a^\top x \right\}$ is supporting to M at $x \in \partial_{\mathrm{ri}} M$, then the set $M' = M \cap \Pi$ is nonempty (it contains x) convex set, and the linear form $a^\top y$ is constant on M' and therefore (why?) on the affine hull M'. At the same time, the form is nonconstant on M by definition of a supporting plane. Thus, Aff (M') is a proper (less than the entire Aff (M)) subset of Aff (M)), and therefore the affine dimension of M' is less than the affine dimension of M[2]. □

[2]we used the following fact: if $P \subset M$ are two affine sets, then the affine dimension of P is $\leq$ the one of M, with $\leq$ being $=$ if and only if $P = M$.

2.3 Minimal Representation of Convex Sets: extreme points

Extreme point of a convex set M is a point in M which cannot be obtained as a convex combination of other points of the set and set of all extreme points is the smallest set of points for which M is the convex hull.

Definition 2.110. ***(Extreme Points)*** *Let M be a nonempty convex set in $\mathbf{R}^n$. A point $x \in M$ is called an **extreme point** of M, if there is no nontrivial (of positive length) segment $[u, v] \in M$ for which x is an interior point, i.e., if we have*

$$\text{“}x = \lambda u + (1-\lambda)v\lambda \in (0,1), u, v \in M\text{”} \Rightarrow \text{“}u = v = x\text{”}$$

An equivalent statement of an extreme point is as follows:

Lemma 2.111. ***(Equivalence)*** *A point x in a convex set M is extreme if and only if the set $M \backslash \{x\}$ is convex.*

Proof. easy □

We start with 2 lemmas:

Lemma 2.112. *Let M be a closed convex set in $\mathbf{R}^n$. Assume that for some $\bar{x} \in M$ and $h \in \mathbf{R}^n M$ contains the ray*

$$\{\bar{x} + th \mid t \geq 0\}$$

starting at $\bar{x}$ with the direction h. Then M contains also all parallel rays starting at the points of M :

$$\forall x \in M : \{x + th \mid t \geq 0\} \subset M$$

In particular, if M contains certain line, then it contains also all parallel lines passing through the points of M.

Definition 2.113. ***(Recessive Direction)*** *For a convex set M, the directions h such that $x + th \in M$ for some (and thus for all) $x \in M$ and all $t \geq 0$ are called **recessive** for M. Later in Section 3, we need this concept.*

Proof. If $x \in M$ and $\bar{x} + th \in M$ for all $t \geq 0$ then, due to convexity, for any fixed $\tau \geq 0$ we have

$$\epsilon\left(\bar{x} + \frac{\tau}{\epsilon}h\right) + (1-\epsilon)x \in M$$

for all $\epsilon \in (0,1)$. As $\epsilon \to +0$, the left hand side tends to $x + \tau h$, and since M is closed, $x + \tau h \in M$ for every $\tau \geq 0$ □

Lemma 2.114. *Let M be a **closed** convex set, $\bar{x} \in \partial_{\text{ri}} M$ and Π be a hyperplane supporting to M at $\bar{x}$. Then all extreme points of the nonempty closed convex set $\Pi \cap M$ are extreme points of M.*

Proof. The set $\Pi \cap M$ is closed, convex and nonempty (contains $\bar{x}$). Let a be the linear form associated with Π :

$$\Pi = \left\{y \mid a^\top y = a^\top \bar{x}\right\}$$

so that

$$\inf_{x \in M} a^\top x < \sup_{x \in M} a^\top x = a^\top \bar{x}$$

from Definition 2.107. Assume that y is an extreme point of $\Pi \cap M$; what we should do is to prove that

$$y = \lambda u + (1-\lambda)v$$

for some $u, v \in M$ and $\lambda \in (0,1)$ is possible only if $y = u = v$. To this end it suffices to prove $u, v \in \Pi \cap M$ (or, which is the same, to prove that $u, v \in \Pi$, since the points are known to belong to M). Note that since $y \in \Pi$ we have

$$a^\top y = a^\top \bar{x} \geq \max\left\{a^\top u, a^\top v\right\}$$

On the other hand,

$$a^\top y = \lambda a^\top u + (1-\lambda) a^\top v$$

combining these observations and taking into account that $\lambda \in (0,1)$, we conclude that

$$a^\top y = a^\top u = a^\top v$$

But these equalities imply that $u, v \in \Pi$. □

Theorem 2.115. *Let M be a **closed** and nonempty convex set in $\mathbf{R}^n$. Then*

i). The set $\mathrm{Ext}(M)$ of extreme points of M is nonempty $\Longleftrightarrow$ M does not contain lines;

*ii). (KreinMilman Theorem) If M is **bounded**, then M is the convex hull of its extreme points:*

$$M = \mathrm{Conv}(\mathrm{Ext}(M))$$

so that every point of M is a convex combination of the points of $\mathrm{Ext}(M)$.

Proof. Proof of i).

"$\Rightarrow$": If M contains a line, then, by Lemma 2.112, there is a line in M passing through any given point of M, so that no point can be extreme.

"$\Leftarrow$": We use **induction on the dimension** of the convex set M. There is nothing to do if the dimension of M is zero, i.e., if M is a point. Now assume that we already have proved the nonemptiness of $\mathrm{Ext}(T)$ for all nonempty closed and not containing lines convex sets T of certain dimension k, and let us prove that the same statement is valid for the sets of dimension $k+1$. Let M be a closed convex nonempty and not containing lines set of dimension $k+1$. Since the dimension of M is positive $(k+1)$, $\partial_{\mathrm{ri}} M$ is not empty. Since M does not contain lines and is of positive dimension, it differs from entire Aff (M) and therefore it possesses a relative boundary point $\bar{x}$. Indeed, there exists $z \in \mathrm{Aff}(M) \backslash M$, so that the points

$$x_\lambda = x + \lambda(z - x),$$

where x is an arbitrary fixed point of M, do not belong to M for some $\lambda \geq 1$, while $x_0 = x$ belongs to M. The set of those $\lambda \geq 0$ for which $x_\lambda \in M$ is therefore nonempty and bounded from above; this set clearly is **closed** (since M is closed). Thus, there exists the largest $\lambda = \lambda^*$ for which $x_\lambda \in M$.

We claim that $x_{\lambda^*} \in \partial_{\mathrm{ri}} M$. Indeed, by construction this is a point from M. If $x \in \mathrm{ri}\, M$, all the points x_λ with close to λ^* and greater than λ^* values of λ would also belong to M, which contradicts the origin of λ^*.

According to Lemma 2.109 1), there exists hyperplane $\Pi = \left\{x \mid a^\top x = a^\top \bar{x}\right\}$ which supports M at $\bar{x}$:

$$\inf_{x \in M} a^\top x < \max_{x \in M} a^\top x = a^\top \bar{x}$$

By Lemma 2.109 2)., the set $T = \Pi \cap M$ (which is closed, convex and nonempty) is of affine dimension less than that of M, which clearly does not contain lines (since even the larger set M does not contain lines). By Inductive Hypothesis, T possesses extreme points, and by Lemma 2.114 all these points are extreme also for M The inductive step is complete, and (i) is proved.

Proof of ii).

What is immediately seen is that the right hand side set is contained in the left hand side one. Thus, all we need is to prove that any $x \in M$ is a convex combination of points from $\mathrm{Ext}(M)$. Here we again use **induction on the dimension** of M. The case of 0-dimensional set M (i.e., a point) is trivial. Assume that the statement in question is valid for all k dimensional convex closed and bounded sets, and let M be a convex closed and bounded set of dimension $k+1$.

Let $x \in M$. To represent x as a convex combination of points from $\mathrm{Ext}(M)$, let us pass through x an arbitrary line $l = \{x + \lambda h \mid \lambda \in \mathbf{R}\} (h \neq 0)$ in the $\mathrm{Aff}(M)$. Moving along this line from x in each of the two possible directions, we eventually leave M (since M is bounded); as it was explained in the proof of (i), it means that there exist nonnegative $\lambda_+$and λ_-such that the points

$$\bar{x}_\pm = x + \lambda_\pm h$$

both belong to the boundary of M. Let us verify that $\bar{x}_\pm$are convex combinations of the extreme points of M (this will then complete the proof). Indeed, M admits supporting at $\bar{x}_+$hyperplane Π; and the set $\Pi \cap M$ (which clearly is convex, closed and bounded) is of less dimension than that one of M; by the inductive hypothesis, the point $\bar{x}_+$of this set is a convex combination of extreme points of the set, and by Lemma 2.114 all these extreme points are extreme points of M as well. Thus, $\bar{x}_+$is a convex combination of extreme points of M. Similar reasoning is valid for $\bar{x}_-$. □

2.3.1 Application: extreme points of a polyhedral set

Consider a polyhedral set

$$K = \{x \in \mathbf{R}^n \mid Ax \leq b\}$$

A being a $m \times n$ matrix and b being a vector from $\mathbf{R}^m$. What are the extreme points of K? The answer is given by the following

Theorem 2.116. ***(extreme points of polyhedral set])*** *Let $x \in K$. The vector x is an extreme point of $K \iff$ some n **linearly independent** (i.e., with linearly independent vectors of coefficients) inequalities of the system $Ax \leq b$ are equalities at x.*

Proof. Let $a_i, i = 1, \ldots, m$, be the the rows of A.

"$\Rightarrow$" part: let x be an extreme point of K, and let I be the set of those indices i for which $a_i^\top x = b_i$; we should prove that the set F of vectors $\{a_i \mid i \in I\}$ contains n linearly independent vectors, or, which is the same, that linear span of F is $\mathbf{R}^n$. Assume that it is not the case; then the orthogonal complement to F contains a nonzero vector h. Consider the segment $\Delta_\epsilon = [x - \epsilon h, x + \epsilon h], \epsilon > 0$ being the parameter of our construction. Since h is orthogonal to the "active" vectors a_i, $i \in I$, all points y of this segment satisfy the relations $a_i^\top y = a_i^\top x = b_i$. Now, if i is a "nonactive" index with $a_i^\top x < b_i$, then $a_i^\top y \leq b_i$ for all $y \in \Delta_\epsilon$ provided that ϵ is small enough. Since there are finitely many nonactive indices, we can choose $\epsilon > 0$ in such a way that all $y \in \Delta_\epsilon$ will satisfy all "nonactive" inequalities $a_i^\top x \leq b_i, i \notin I$. Since $y \in \Delta_\epsilon$ satisfies, as we have seen, also all "active" inequalities, we conclude that with the above choice of ϵ we get $\Delta_\epsilon \subset K$, which is a contradiction: $\epsilon > 0$ and $h \neq 0$, so that Δ_ϵ is a nontrivial segment with the midpoint x, which contradicts with x is an extreme point of K

"$\Leftarrow$" part: assume that $x \in K$ is such that among the inequalities $a_i^\top x \leq b_i$ which are equalities at x there are n linearly independent, say, those with indices $1, \ldots, n$, and prove that x is an extreme point of K. This is immediate: assuming that x is not an extreme point, we would get the existence of a nonzero vector h such that $x \pm h \in K$. In other words, for $i = 1, \ldots, n$ we would have $b_i \pm a_i^\top h \equiv a_i^\top (x \pm h) \leq b_i$, which is possible only if $a_i^\top h = 0, i = 1, \ldots, n$. But the only vector which is orthogonal to n linearly independent vectors in $\mathbf{R}^n$ is the zero vector, and we get $h = 0$, which was assumed not to be the case. □

Corollary 2.117. *The set of extreme points of a polyhedral set is finite.*

Proof. The extreme points does not exceed the number C_m^n of $n \times n$ submatrices of the matrix A and is therefore finite. □

2.4 Dual Cones

2.4.1 Dual Cone

Definition 2.118. ***(Dual Cone)*** *Let K be a conic (not necessarily convex). The set*

$$K^* = \left\{y \mid x^\top y \geq 0 \, \textit{for all } x \in K\right\}$$

is called the dual cone of K.

Remark 2.119. *It is better to write $x^\top y$ as $\langle x, y\rangle$, since this is a general inner product. Other inner product besides $\mathbf{R}^n$ is also possible, see Example 2.125 for an example of $\mathbf{S}_+^n$.*

Remark 2.120. *Note if $x \in K$, we have $x^\top y \geq 0, \forall y \in K^{**}$, so we have $K \subseteq K^{**}$. Later we will show* $\operatorname{cl}\operatorname{Conv}(K) = K^{**}$

Remark 2.121. ***(equivalent statement)***
"$y \neq 0$ is the normal vector of a (homogeneous) halfspace containing K" $\Longleftrightarrow$ "$y \in K^*$"

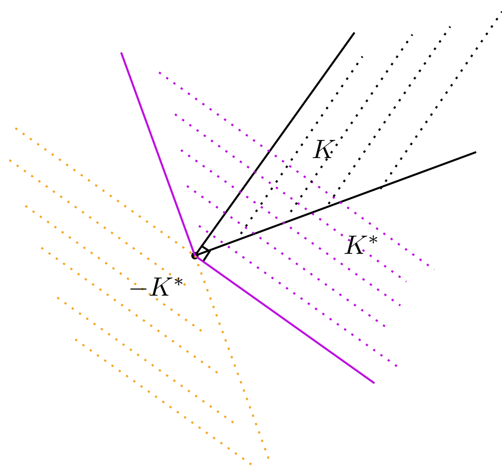


Fig. 6: Visualization of the dual cone. (Note in some books, $-K^*$ is defined as the dual cone.)

Remark 2.122. *As the name suggests, K^* is always a convex cone, even when the original cone K is just a conic but not convex.*

Example 2.123. ***(subspace)*** *The dual cone of a subspace $V \subseteq \mathbf{R}^n$ (which is a cone) is its orthogonal complement $V^\perp = \{y \mid v^\top y = 0 \text{ for all } v \in \bar{V}\}$. The equality is because $-y \in M$ if $y \in V$.*

Example 2.124. ***(nonnegative orthant)*** *The cone $\mathbf{R}_+^n$ is its own dual:*

$$y^\top x \geq 0 \textit{ for all } x \succeq 0 \Longleftrightarrow y \succeq 0$$

*We call such a cone **self-dual**.*

Example 2.125. ***(positive semidefinite cone)*** *On the set of symmetric $n \times n$ matrices $\mathbf{S}^n$, we use the standard inner product $\operatorname{tr}(XY) = \sum_{i,j=1}^n X_{ij} Y_{ij}$. The positive semidefinite cone $\mathbf{S}_+^n$ is self-dual, i.e., for $X, Y \in \mathbf{S}^n$*

$$\operatorname{tr}(XY) \geq 0 \textit{ for all } X \succeq 0 \Longleftrightarrow Y \succeq 0$$

We will show this fact:

"$\Rightarrow$": Suppose $Y \notin \mathbf{S}_+^n$. Then there exists $q \in \mathbf{R}^n$ with

$$q^\top Y q = \operatorname{tr}\left(q q^\top Y\right) < 0$$

Hence the positive semidefinite matrix $X = qq^\top$ satisfies $\operatorname{tr}(XY) < 0$; it follows that $Y \notin \left(\mathbf{S}_+^n\right)^*$.

"$\Leftarrow$": Now suppose $X, Y \in \mathbf{S}_+^n$. We can express X in terms of its eigenvalue decomposition as $X = \sum_{i=1}^n \lambda_i q_i q_i^\top$, where (the eigenvalues) $\lambda_i \geq 0, i = 1, \ldots, n$. Then we have

$$\operatorname{tr}(YX) = \operatorname{tr}\left(Y \sum_{i=1}^n \lambda_i q_i q_i^\top\right) = \sum_{i=1}^n \lambda_i q_i^\top Y q_i \geq 0$$

This shows that $Y \in \left(\mathbf{S}_+^n\right)^*$.

Example 2.126. ***(dual of a norm cone)*** *Let $\|\cdot\|$ be a norm on $\mathbf{R}^n$. The dual of the associated cone $K = \{(x,t) \in \mathbf{R}^{n+1} \mid \|x\| \leq t\}$ is the **cone defined by the dual norm**, i.e.,*

$$K^* = \left\{(u, v) \in \mathbf{R}^{n+1} \mid \|u\|_* \leq v\right\}$$

*where the **dual norm** is given by $\|u\|_* = \sup\left\{u^\top x \mid \|x\| \leq 1\right\}$ (see [3, P326]).*

To prove the result we have to show that

$$x^\top u + tv \geq 0 \textit{ whenever } \|x\| \leq t \Longleftrightarrow \|u\|_* \leq v$$

"$\Leftarrow$": Suppose $\|u\|_* \leq v$, and $\|x\| \leq t$ for some $t > 0$. (If $t = 0, x$ must be zero, so obviously $u^\top x + vt \geq 0$.) Applying the definition of the dual norm, and the fact that $\|-x/t\| \leq 1$, we have

$$u^\top(-x/t) \leq \|u\|_* \leq v$$

and therefore $u^\top x + vt \geq 0$

"$\Rightarrow$": Suppose $\|u\|_* > v$, i.e., that the righthand condition does not hold. Then by the definition of the dual norm, there exists an x with $\|x\| \leq 1$ and $x^\top u > v$ Taking $t = 1$, we have

$$u^\top(-x) + v < 0$$

which contradicts the lefthand condition.

2.4.1.1 Properties of Dual Cones

1. **K^* is closed and convex:**
   (K^* is the intersection of a set of halfspaces)
2. **$K_1 \subseteq K_2$ implies $K_2^* \subseteq K_1^*$.**
3. **If K has nonempty interior, then K^* is pointed:**
   (assume K^* is not pointed, it contains lines, i.e. $\exists\, x \neq 0$ s.t. $x \in K^*$ and $-x \in K^*$. For any $y \in K$ we have $x^\top y \geq 0$ and $-x^\top y \geq 0$. Therefore $x^\top y = 0$ for all $y \in K$. Note since K has nonempty interior, $\mathrm{Aff}(K)$ is the full space. This is not possible for $x^\top y = 0$ with $x \neq 0$.)
4. **If the closure of convex cone K is pointed then K^* has nonempty interior:**
   (assume K^* has empty interior, $\mathrm{Aff}(K^*)$ then has dimension $< n$: with possible translate, we have $\mathrm{Aff}(K^*)$ is the nullspace $H := \{x : Ax = 0\}$, and of course $K^* \subseteq H$. Then range $A = H^* \subseteq K^{**} = \mathrm{cl}\,\mathrm{Conv}(K)$ which is not pointed.)
5. $\mathrm{int}\, K^* = \{y \mid y^\top x > 0$ **for all** $x \in K, x \neq 0\}$:
   ("$\supset$": Let $H = \{x \mid ||x|| = 1, x \in K\}$. $y^\top x > 0, \forall\, x \in K \Rightarrow y^\top x > 0, \forall\, x \in H \Rightarrow (y+u)^\top x > 0$ for all $x \in H$ and all sufficiently small $||u|| < \delta \Rightarrow (y+u)^\top x > 0, \forall\, x \in K$ and all sufficiently small $||u|| < \delta \Rightarrow y \in \mathrm{int}\, K^*$.
   "$\subset$": if $y \in K^*$ and $y^\top x = 0$ for some $x \in K$, then $y \notin \mathrm{int}\, K^*$ because $(y - tx)^\top x < 0$ for all $t > 0$.)
6. **K^{**} is the closure of the convex hull of K;**
   Hence **if K is convex and closed, $K^{**} = K$:**
   (WLOG, assume K is convex. $y \neq 0$ is the normal vector of a (homogeneous) halfspace containing K if and only if $y \in K^*$. The intersection of all homogeneous halfspaces containing a convex cone K is the closure of K. Therefore the closure of K is

   $$\mathrm{cl}\, K = \bigcap_{y \in K^*} \{x \mid y^\top x \geq 0\} = \{x \mid y^\top x \geq 0 \text{ for all } y \in K^*\} = K^{**}$$

)
7. **If K is a proper cone, then so is its dual K^*, and moreover, that $K^{**} = K$**

2.4.2 Dual Generalized Inequalities

Now suppose that the convex cone K is proper, so it induces a generalized inequality $\preceq_K$. Then its dual cone K^* is also proper, and therefore induces a generalized inequality. We refer to the generalized inequality $\preceq_{K^*}$ as the dual of the generalized inequality $\preceq_K$.

2.4.2.1 Properties of Generalized Inequality and Dual

1. **$x \preceq_K y$ if and only if $\lambda^\top x \leq \lambda^\top y$ for all $\lambda \succeq_{K^*} 0$:**
   ($K^{**} = K$, so $y - x \in K \Longleftrightarrow y - x \in K^{**} \Longleftrightarrow \lambda^\top(y - x) \geq 0, \forall\, \lambda \in K^*$)

2. $x \prec_K y$ **if and only if** $\lambda^\top x < \lambda^\top y$ **for all** $\lambda \succeq_{K^*} 0, \lambda \neq 0$**:**
   ("$\Rightarrow$": from 1, if $\lambda^\top(y-x) = 0$, we have $\lambda = 0$ since $y - x \in \text{int } K$;
   "$\Leftarrow$": this is a restatement of Properties of Dual Cones 5.)

Remark 2.127. *Since $K = K^{**}$, the dual generalized inequality associated with $\preceq_{K^*}$ is $\preceq_K$, so these properties hold if the generalized inequality and its dual are swapped.*

Theorem 2.128. ***(alternatives for linear strict generalized inequalities)*** *Suppose $K \subseteq \mathbf{R}^m$ is a proper cone. Consider the strict generalized inequality*

$$Ax \prec_K b \tag{2.129}$$

where $x \in \mathbf{R}^n$. Consider also

$$\exists\, \lambda \in \mathbf{R}^m \text{ s.t } \lambda \neq 0, \quad \lambda \succeq_{K^*} 0, \quad A^\top \lambda = 0, \quad \lambda^\top b \leq 0 \tag{2.130}$$

We have

"(2.130) *is **feasible**" $\Longleftrightarrow$ "(2.129) *is **infeasible***"

Proof. "$\Leftarrow$":

Suppose (2.130) is infeasible, i.e., the affine set $\{b - Ax \mid x \in \mathbf{R}^n\}$ does not intersect the open convex set $\text{int } K$. Then there is a separating hyperplane, i.e., a nonzero $\lambda \in \mathbf{R}^m$ and $\mu \in \mathbf{R}$ such that $\lambda^\top(b - Ax) \leq \mu$ for all x, and $\lambda^\top y \geq \mu$ for all $y \in \text{int } K$. The first condition implies $A^\top \lambda = 0$ and $\lambda^\top b \leq \mu$. The second condition implies $\lambda^\top y \geq \mu$ for all $y \in K$, which can only happen if $\lambda \in K^*$ and $\mu \leq 0$. Putting it all together we get (2.130).

"$\Rightarrow$":

Suppose that both inequality systems hold. Then we have $\lambda^\top(b - Ax) > 0$, since $\lambda \neq 0, \lambda \succeq_{K^*} 0$, and $b - Ax \succ_K 0$. But using $A^\top \lambda = 0$ we find that $\lambda^\top(b - Ax) = \lambda^\top b \leq 0$, which is a contradiction.

□

Remark 2.131. ***strong alternatives*** *Thus, the inequality systems* (2.129) *and* (2.130) *are alternatives: for any data A, b, **exactly one of them is feasible**. This generalizes the alternatives Examples 2.88 and 2.89 for the special case $K = \mathbf{R}^m_+$.*

*This is quite similar to Farkas Lemma (see Corollary 2.50). Please compare. We state the general methodology in Section 4.11.2 called **strong alternatives**.*

2.4.2.2 Minimum and Minimal Elements via Dual Inequalities

We can use **dual generalized inequalities** to characterize minimum and minimal elements of a (possibly nonconvex) set $S \subseteq \mathbf{R}^m$ with respect to the generalized inequality induced by a proper cone K.

• dual characterization of minimum element

Lemma 2.132. *"x is the **minimum** element of S, w.r.t $\preceq_K$"$\Longleftrightarrow$ "**for all** $\lambda \succ_{K^*} 0, x$ is the unique minimizer of $\lambda^\top z$ over $z \in S$"*

Remark 2.133. ***(explanation)*** *Geometrically, this means that for any $\lambda \succ_{K^*} 0$, the hyperplane*

$$\left\{z \mid \lambda^\top(z - x) = 0\right\}$$

is a strict supporting hyperplane to S at x. (By strict supporting hyperplane, we mean that the hyperplane intersects S only at the point x.) Note that convexity of the set S is not required. See Fig. 7 for better understanding.

Proof. "$\Rightarrow$": suppose x is the minimum element of S, i.e., $x \preceq_K z$ for all $z \in S$, and let $\lambda \succ_{K^*} 0$. Let $z \in S, z \neq x$, we have $z - x \succeq_K 0$. From $\lambda \succ_{K^*} 0$ and $z - x \succeq_K 0, z - x \neq 0$, we conclude $\lambda^\top(z - x) > 0$. Since z is an arbitrary element of S, not equal to x, this shows that x is the unique minimizer of $\lambda^\top z$ over $z \in S$.

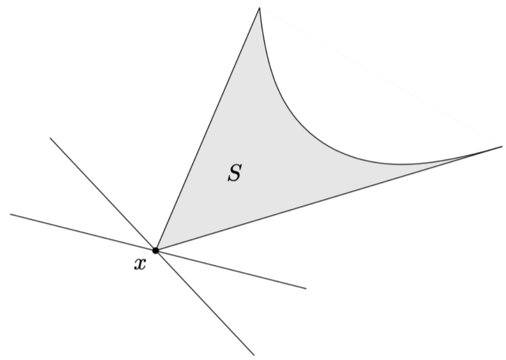


Fig. 7: Dual of minimum: x is minimum element of the set S w.r.t $\mathbf{R}_+^2$; for every $\lambda \succ 0$, the hyperplane $\left\{z \mid \lambda^\top (z-x) = 0\right\}$ strictly supports S at x, i.e., contains S on one side, and touches it only at x.

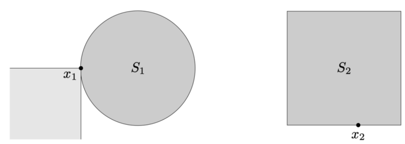


Fig. 8: The point $x_1 \in S_1$ is minimal, but is not a minimizer of $\lambda^\top z$ over S_1 for any $\lambda \succ 0$. (It does, however, minimize $\lambda^\top z$ over $z \in S_1$ for $\lambda = (1,0)$. Right. The point $x_2 \in S_2$ is not minimal, but it does minimize $\lambda^\top z$ over $z \in S_2$ for $\lambda = (0,1) \succeq 0$

"$\Leftarrow$": suppose that for all $\lambda \succ_{K^*} 0, x$ is the unique minimizer of $\lambda^\top z$ over $z \in S$, but x is not the minimum element of S. Then there exists $z \in S$ with $z \not\succeq_K x$. Since $z - x \not\succeq_K 0$, there exists $\tilde{\lambda} \succeq_K^* 0$ with $\tilde{\lambda}^\top (z-x) < 0$. Hence $\lambda^\top (z-x) < 0$ for $\lambda \succ_{K^*} 0$ in the neighborhood of $\tilde{\lambda}$. This contradicts the assumption that x is the unique minimizer of $\lambda^\top z$ over S. □

• dual characterization of minimal element

Lemma 2.134. ***(sufficiency)*** *If $\exists\, \lambda \succ_{K^*} 0$ and x minimizes $\lambda^\top z$ over $z \in S$, then x is minimal.*

Remark 2.135. *This is illustrated in Fig. 9. Please note $\succ_{K^*}$ cannot be replaced by $\succeq_{K^*}$. See Fig. 8*

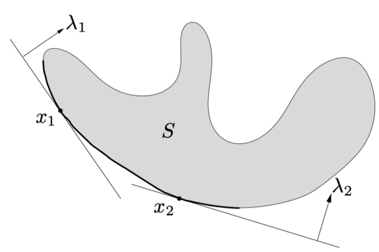


Fig. 9: A set $S \subseteq \mathbf{R}^2$. Its set of minimal points, w.r.t $\mathbf{R}_+^2$, is shown as the darker section of its (lower, left) boundary. The minimizer of $\lambda_1^\top z$ over S is x_1, and is minimal since $\lambda_1 \succ 0$. The minimizer of $\lambda_2^\top z$ over S is x_2, which is another minimal point of S, since $\lambda_2 \succ 0$

Proof. Suppose that $\lambda \succ_{K^*} 0$, and x minimizes $\lambda^\top z$ over S, but x is not minimal, i.e., there exists a $z \in S, z \neq x$, and $z \preceq_K x$. Then $\lambda^\top (x-z) > 0$, which contradicts our assumption that x is the minimizer of $\lambda^\top z$ over S. □

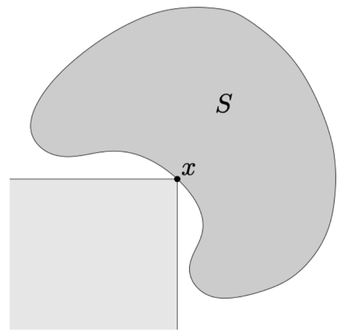


Fig. 10: The point x is a minimal element of $S \subseteq \mathbf{R}^2$ with respect to $\mathbf{R}_+^2$. However there exists no λ for which x minimizes $\lambda^\top z$ over $z \in S$.

However, the converse is in general false: a point x can be minimal in S, but not a minimizer of $\lambda^\top z$ over $z \in S$, for any λ, as shown in Fig. 10. We need **convexity**.

Lemma 2.136. ***(necessity)*** *Provided the set S is convex, we can say that* ***for any*** *minimal element x there exists a nonzero $\lambda \succeq_{K^*} 0$ such that x minimizes $\lambda^\top z$ over $z \in S$.*

Remark 2.137. *Please note $\succeq_{K^*}$ cannot be replaced by $\succ_{K^*}$. See Fig. 8.*

Proof. Suppose x is minimal, which means that $((x - K)\backslash\{x\}) \cap S = \emptyset$. Applying the separating hyperplane theorem to the convex sets $(x - K)\backslash\{x\}$ and S, we conclude that there is a $\lambda \neq 0$ and μ such that $\lambda^\top(x - y) \leq \mu$ for all $y \in K$ and $\lambda^\top z \geq \mu$ for all $z \in S$. From the first inequality we conclude $\lambda \succeq_{K^*} 0$. Since $x \in S$ and $x \in x - K$, we have $\lambda^\top x = \mu$, so the second inequality implies that μ is the minimum value of $\lambda^\top z$ over S. Therefore, x is a minimizer of $\lambda^\top z$ over S where $\lambda \neq 0, \lambda \succeq_{K^*} 0$ □

Example 2.138. ***(Pareto optimal production frontier)*** *With each production method, we associate a resource vector $x \in \mathbf{R}^n$, where x_i denotes the amount of resource i consumed by the method to manufacture the product. Pareto optimal is the minimal.*

We can find Pareto optimal production methods (i.e., minimal resource vectors) by minimizing

$$\lambda^\top x = \lambda_1 x_1 + \cdots + \lambda_n x_n$$

over the set P of production vectors, using any λ that satisfies $\lambda \succ 0$. However we this is the ***sufficient condition*** *shown in Lemma 2.134 and* ***cannot guarantee*** *we can find all Pareto optimal. See Fig. 11.*

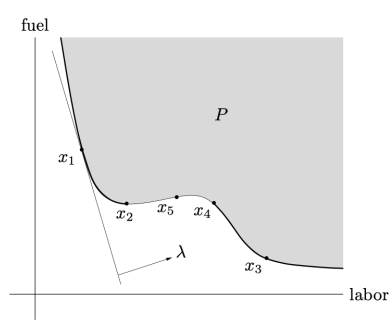


Fig. 11: The production set P, for a product that requires labor and fuel to produce, is shown shaded. The two dark curves show the efficient production frontier. The points x_1, x_2 and x_3 are efficient. The points x_4 and x_5 are not (since in particular, x_2 corresponds to a production method that uses no more fuel, and less labor). The point x_1 is also the minimum cost production method for the price vector λ (which is positive). The point x_2 is efficient, but **cannot be found by minimizing** the total cost $\lambda^\top x$ for any price vector $\lambda \succeq 0$.

3 Convex Function

3.1 Basic Definitions and Examples

Definition 3.1. ***(Convex Function)*** *A function $f : M \to \mathbf{R}$ defined on a nonempty subset M of $\mathbf{R}^n$ and taking real values is called* ***convex****, if*

- *the domain M of the function is* ***convex****;*
- *it satisfies*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y), \quad x, y \in M, \lambda \in [0,1] \tag{3.2}$$

If the above inequality is strict whenever $x \neq y$ and $0 < \lambda < 1$, f is called ***strictly convex****.*

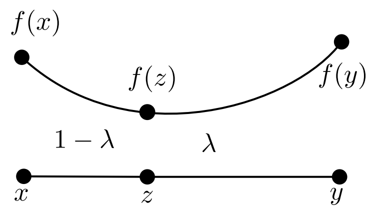


Fig. 12: Visualization of convex function.

Remark 3.3. ***(alternative form I)*** *With $z = y + \lambda(x-y)$ (3.2) sometimes is written as*

$$f(z) \leq f(y) + \lambda\left(f(x) - f(y)\right), \quad x, y \in M, \lambda \in [0,1] \tag{3.4}$$

or

$$f(z) - f(y) \leq \lambda\left(f(x) - f(y)\right), \quad x, y \in M, \lambda \in [0,1] \tag{3.5}$$

Remark 3.6. ***(alternative form II of univariate f)*** *In $\mathbf{R}$ space, f is defined in (a,b), for* ***any*** *three variable $a < x < z < y < b$, we have $z = x\frac{y-z}{y-x} + y\frac{z-x}{y-x}$. We sometimes write the convexity as follows:*

$$f(z) \leq \frac{y-z}{y-x}f(x) + \frac{z-x}{y-x}f(y), \forall\, a < x < z < y < b \Longleftrightarrow f \text{ is convex on } (a,b)$$

Equivalently we have

$$f(z) - f(x) \leq \frac{z-x}{y-x}(f(y) - f(x)), \forall\, a < x < z < y < b \Longleftrightarrow f \text{ is convex on } (a,b) \tag{3.7}$$

Or more generally, $\forall\, a < x < z < y < b$,

$$\textbf{any two of } \frac{f(z)-f(x)}{z-x} \leq \frac{f(y)-f(x)}{y-x} \leq \frac{f(y)-f(z)}{y-z} \Longleftrightarrow f \text{ is convex on } (a,b) \tag{3.8}$$

See also Theorem 3.62 for the nondecreasing property of f'.

Remark 3.9. ***(alternative form III)*** *Note, for general multivariate f, we can write $\|x-y\|_2$ instead of $x-y$, i.e.*

$$f(z) - f(x) \leq \frac{\|z-x\|_2}{\|y-x\|_2}(f(y) - f(x)), \forall\, x, z, y \text{ collinear with } z \text{ inside } [x,y] \Longleftrightarrow f \text{ is convex on } (a,b) \tag{3.10}$$

Definition 3.11. ***(Concave Function)*** *A function f such that $-f$ is convex is called* ***concave****:*

- *the domain M of the function is* ***convex****;*
- *it satisfies*

$$f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y), \quad x, y \in M, \lambda \in [0,1]$$

Example 3.12. ***(affine is convex and concave)*** *The simplest example of a convex function is an affine function*

$$f(x) = a^\top x + b$$

f ***both convex and concave on the entire space*** $\Longleftrightarrow$ f ***is an affine function.***

Theorem 3.13. ***(every norm is convex)*** *Let $\pi(x)$ be a real-valued function on $\mathbf{R}^n$ which is positively homogeneous of degree 1 :*

$$\pi(tx) = t\pi(x) \quad \forall\, x \in \mathbf{R}^n, t \geq 0$$

π is convex if and only if it is sub-additive:

$$\pi(x+y) \leq \pi(x) + \pi(y) \quad \forall\, x, y \in \mathbf{R}^n$$

In particular, a norm (which by definition is positively homogeneous of degree 1 and is subadditive) is convex.

Proof.

$$\begin{aligned} \pi(\lambda x + (1-\lambda)y) &\leq \pi(\lambda x) + \pi((1-\lambda)y) && \text{triangle inequality} \\ &= \lambda\pi(x) + (1-\lambda)\pi(y) && \text{homogeneity} \end{aligned}$$

for any $x, y \in \mathbf{R}^n$ and $0 \leq \lambda \leq 1$. □

3.1.1 Some Equivalent Convexity Form

Theorem 3.14. ***(equivalent condition for convex: I)***

f ***is convex*** $\Longleftrightarrow$ $g(t) = f(x+th)$ ***is convex*** $\forall\, x \in \operatorname{dom} f$ ***and*** h

Remark 3.15. *This property is very useful, since it allows us to check whether a function is convex by restricting it to a line. Here I mean $g(t)$ is convex on its domain: $\{t \mid x + tv \in \operatorname{dom} f\}$.*

Remark 3.16. *It is a speical case of "Composition with Affine Mapping" as shown in Section 3.2.1.*

Proof. "$\Rightarrow$": It is clear that $\operatorname{dom} g$ is convex. For any $t_1, t_2 \in \operatorname{dom} g$

$$\begin{aligned} g(\lambda t_1 + (1-\lambda)t_2) &= f(x + (\lambda t_1 + (1-\lambda)t_2)h)) \\ &= f(\lambda(t_1 h + x) + (1-\lambda)(t_2 h + x)) \\ &\leq \lambda g(t_1) + (1-\lambda)g(t_2) \end{aligned}$$

"$\Leftarrow$": If g is convex for any x and h. For any $x, y \in \operatorname{dom} f$, let $h = y - x$. We get convexity of f. □

Definition 3.17. ***(Epigraph)*** *Given a real-valued function f defined on a nonempty subset M of $\mathbf{R}^n$, we define its* ***epigraph*** *as the set*

$$\operatorname{Epi}(f) = \left\{(t, x) \in \mathbf{R}^{n+1} \mid x \in M, t \geq f(x)\right\} \tag{3.18}$$

Theorem 3.19. ***(equivalent condition for convex: II)***

f ***defined on*** $M \subseteq \mathbf{R}^n$ ***is convex*** $\Longleftrightarrow$ $\operatorname{Epi}(f)$ ***is a nonempty convex set in*** $\mathbf{R}^{n+1}$.

Remark 3.20. ***(what is a convex epigrah)*** $\operatorname{Epi}(f)$ *is* ***not*** *an arbitrary convex set in $\mathbf{R}^{n+1}$. It must contains one* ***recessive direction*** $(1, 0)$, $0 \in \mathbf{R}^n$, *see Definition 2.113. Conversely, for any convex set with one recessive direction, by possible rotation (coordinate transform), we may define a convex function with its epigraph being the convex set.*

Proof. "$\Rightarrow$": Since f is convex we have $\operatorname{dom} f$ is convex. Let $(t_1, x), (t_2, y) \in \operatorname{Epi}(f)$. $\lambda x + (1-\lambda)y \in \operatorname{dom} f$ and $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \leq \lambda t_1 + (1-\lambda)t_2$. We then have $(\lambda t_1 + (1-\lambda)t_2, \lambda x + (1-\lambda)y) \in \operatorname{Epi}(f)$.

"$\Leftarrow$": $\operatorname{dom} f$ is convex from the projection of $\operatorname{Epi}(f)$. $(f(x), x), (f(y), y) \in \operatorname{Epi}(f) \Leftarrow f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$. □

Example 3.21. ***(epigraph of matrix fractional function.)*** *The function* $f : \mathbf{R}^n \times \mathbf{S}^n \to \mathbf{R}$*, defined as*

$$f(x, Y) = x^\top Y^{-1} x$$

is convex on $\operatorname{dom} f = \mathbf{R}^n \times \mathbf{S}^n_{++}$*. (This generalizes the quadratic-over-linear function* $f(x, y) = x^2/y$*, with* $\operatorname{dom} f = \mathbf{R} \times \mathbf{R}_{++}$*.) One easy way to establish convexity of* f *is via its epigraph:*

$$\begin{aligned} \operatorname{Epi}(f) &= \left\{(x, Y, t) \mid Y \succ 0, x^\top Y^{-1} x \le t\right\} \\ &= \left\{(x, Y, t) \mid \begin{bmatrix} Y & x \\ x^\top & t \end{bmatrix} \succeq 0, Y \succ 0\right\} \end{aligned}$$

using the Schur complement condition for positive semidefiniteness of a block matrix. The last condition is a linear matrix inequality in (x, Y, t)*, and therefore* $\operatorname{Epi}(f)$ *is convex.*

Lemma 3.22. ***(Sufficient Convexity Condition for Continuous Function on Real Line)*** *Assume that* f *is a continuous real function defined in* (a, b) *such that*

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x) + f(y)}{2}$$

for all $x, y \in (a, b)$*. Prove that* f *is convex.*

Proof. See [1, page 101, ex. 24] □

3.1.2 Sublevel Set

Definition 3.23. ***(Sublevel Set)*** *Given a scalar* $c \in \mathbb{R}$ *and a function* $f : \mathbb{R}^n \to \mathbb{R}$*, a* ***sublevel set*** *of* f *associated with* c *is given by*

$$L_c(f) = \{x \in \operatorname{dom} f \mid f(x) \le c\} \tag{3.24}$$

Remark 3.25. ***(convex and sublevel set)*** *Every level set of a convex function is convex: if* $x, y \in L_\alpha(f)$ *and* $\lambda \in [0, 1]$*, then* $f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y) \le \lambda\alpha + (1-\lambda)\alpha = \alpha$*, so that* $\lambda x + (1-\lambda)y \in L_\alpha(f)$

Converse is ***false****: consider* $f(x) = -e^x$ *for* $x \in \mathbb{R}$*.*

Example 3.26. ***(norm ball and ellipsoid)*** *The unit ball of norm* $\|\cdot\|$ *- the set*

$$\{x \in \mathbf{R}^n \mid \|x\| \le 1\}$$

same as any other $\|\cdot\|$*-ball*

$$\{x \mid \|x - a\| \le r\}$$

($a \in \mathbf{R}^n$ *and* $r \ge 0$ *are fixed) is convex and closed (from continuity of* $||\cdot||$*).*

One special example is the ellipsoid $\mathcal{E} = \left\{x \mid (x - x_c)^\top P^{-1}(x - x_c) \le 1\right\}$ *with positive definite* P *is closed and norm because* $||x||_P = \sqrt{x^\top P x}$ *is also a norm.*

Example 3.27. ***(ϵ-neighborhood of a convex set)*** *Let* M *be a convex set in* $\mathbf{R}^n$*, and let* $\epsilon > 0$*. Then, for any norm* $\|\cdot\|$ *on* $\mathbf{R}^n$*, the* ϵ*-neighborhood of* M*, i.e., the set*

$$M_\epsilon = \left\{y \in \mathbf{R}^n \mid \operatorname{dist}_{\|\cdot\|}(y, M) := \inf_{x \in M} \|y - x\| \le \epsilon\right\}$$

is convex and closed:

Note $\operatorname{dist}_{\|\cdot\|}(\cdot, M)$ *is (uniform) continuous, closed is then clear. Note* $\operatorname{dist}_{\|\cdot\|}(y, M) = \inf_{x \in M} \|y - x\| = \inf_{x \in \operatorname{cl} M} \|y - x\|$*, we have* $M_\epsilon = \operatorname{cl} M + \{u \mid ||u|| \le \epsilon\}$ *is the sum of two convex set and therefore convex. Here we also use convex of function to prove sublevel set convex: use* $\|x, y\|$ *to denote* $\|x - y\|$*, we have* $\|\lambda x + (1-\lambda)y, z\| \le \lambda\|x, z\| + (1-\lambda)\|y, z\| \Rightarrow \operatorname{dist}(\lambda x + (1-\lambda)y, M) = \inf_{z \in M} \|\lambda x + (1-\lambda)y, z)\| \le \lambda\|x, z\| + (1-\lambda)\|y, z\| \Rightarrow \operatorname{dist}(\lambda x + (1-\lambda)y, M) \le \inf_{z \in M} \lambda\|x, z\| + \inf_{z \in M}(1-\lambda)\|y, z\| = \lambda \operatorname{dist}(x, M) + (1-\lambda)\operatorname{dist}(y, M)$

Remark 3.28. *Please also see Example 3.55 for an application of partial infimum of convex functions is still convex, this is just a restatement of epigraph intersection.*

3.1.3 Extended-value Extension

If f is convex we define its **extended-value extension** $\tilde{f}: \mathbf{R}^n \to \mathbf{R} \cup \{\infty\}$ by

$$\tilde{f}(x) = \begin{cases} f(x) & x \in \operatorname{dom} f \\ \infty & x \notin \operatorname{dom} f \end{cases}$$

The extension $\tilde{f}$ is defined on all $\mathbf{R}^n$, and takes values in $\mathbf{R} \cup \{\infty\}$. We can **recover the domain of the original function f from the extension $\tilde{f}$ as dom $f = \{x \mid \tilde{f}(x) < \infty\}$.**

Remark 3.29. ***(explanation)*** *The extension can simplify notation, since we do not need to explicitly describe the domain, or add the qualifier "for all $x \in$ dom f " every time we refer to $f(x)$.*

- *In terms of the extension $\tilde{f}$, we can express the* (3.2) *as: for* $0 < \lambda < 1$*,*

  $$\tilde{f}(\lambda x + (1-\lambda)y) \le \lambda \tilde{f}(x) + (1-\lambda)\tilde{f}(y)$$

  *for **any** x **and** y.:*

  For $\lambda = 0$ or $\lambda = 1$ the inequality always holds. For x and y both in $\operatorname{dom} f$*, this inequality coincides with* (3.2)*; if either is outside* $\operatorname{dom} f$*, then the righthand side is ∞, and the inequality therefore holds.*

- *Suppose f_1 and f_2 are two convex functions on $\mathbf{R}^n$. The pointwise sum $f = f_1 + f_2$ is the function with domain* $\operatorname{dom} f = \operatorname{dom} f_1 \cap$ *dom f_2, with $f(x) = f_1(x) + f_2(x)$ for any $x \in \operatorname{dom} f$. Using extended-value extensions we can simply say that for any x, $\tilde{\tilde{f}}(x) = \tilde{f}_1(x) + \tilde{f}_2(x)$. In this equation the domain of f has been automatically defined as* $\operatorname{dom} f = \operatorname{dom} f_1 \cap$ *dom f_2.*

In a similar way we can extend a concave function by defining it to be $-\infty$ outside its domain.

We will use the same symbol to denote a convex(concave) function and its extension, whenever there is no harm from the ambiguity.

Remark 3.30. ***The Epigraph of the extension is still defined by* (3.18) *with original*** $\operatorname{dom} f$***, and is a set in*** $\mathbf{R}^{n+1}$ ***(Not*** $\mathbf{R}^n \times \mathbf{R}_\infty$*).* (3.24) *keeps also the same for f and its extension from the definition.*

3.1.4 Closed Function and Lower-Semicontinuity

3.1.4.1 Convex, Closed and Continuous

Definition 3.31. ***(Lower-Semicontinuity)*** *A function f is lower-semicontinuous at a given vector x_0 if for **every sequence** $\{x_k\}$ converging to x_0, we have*

$$f(x_0) \le \liminf_{k \to \infty} f(x_k)$$

We say that f is lower-semicontinuous over a set X if f is lower-semicontinuous at every $x \in X$

Remark 3.32. ***(continuous vs. lower-semicontinuity)***

- *f is continuous with* $\operatorname{dom} f$ *is closed in $\mathbf{R}^n$, then f is closed.*
- *f is continuous with* $\operatorname{dom} f$ *is open in $\mathbf{R}^n$, then f is closed iff $f(x) \to \infty$ as $x_n \in \partial \operatorname{dom} f$.*
- *x_0 and $\{x_k\}$ does not need to be in $\{x \mid f(x) < \infty\}$*

Definition 3.33. ***(Closed Function)*** *A function f is **closed if*** $\operatorname{Epi}(f)$ ***is a closed set*** *in $\mathbb{R}^n \times \mathbb{R}$, i.e., for every sequence $\{(x_k, w_k)\} \subset \operatorname{Epi}(f)$ converging to some $(\widehat{x}, \widehat{w})$ we have $(\widehat{x}, \widehat{w}) \in \operatorname{Epi}(f)$.*

Remark 3.34. $\operatorname{dom} f$ *need not to be closed. See below examples and Example 3.49.*

Example 3.35.

- *$f(x) = \frac{1}{x}, x \in (0, +\infty)$ is closed. Note now* $\operatorname{dom} f$ *is not closed. If you assign any value to $x = 0$, the new function with* $\operatorname{dom} f = [0, +\infty)$ *is also closed.*

- $f(x) = \frac{1}{x+1}, x \in (0, +\infty)$ *is not closed. If you assign* $f(0) \leq 1$*, the new function with* $\operatorname{dom} f = [0, +\infty)$ *is closed. However if assign* $f(0)$ *to any value* > 1*, the new function with* $\operatorname{dom} f = [0, +\infty)$ *is not closed.*

Theorem 3.36. ***(equivalence of closed and lower-semicontinuous)*** *For a function* $f : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty, +\infty\}$*, the following statements are equivalent:*

1. *f is closed*
2. *Every sublevel set of f is closed*
3. *f is lower-semicontinuous (l.s.c.) over $\mathbb{R}^n$*

Remark 3.37. *For more strict continuous function we need* $f^{-1}(A)$ *is closed for any* A *while here for lower-semicontinuous we only need to ensure the closed when* $A = (-\infty, c]$ *for any* c*.*

Proof. "1) $\Rightarrow$ 2)": Here we use sequence limit to prove. Let c be any scalar and consider $L_c(f)$. If $L_c(f) = \emptyset$, then $L_c(f)$ is closed. Suppose now that $L_c(f) \neq \emptyset$. Pick $\{x_k\} \subset L_c(f)$ such that $x_k \to \bar{x}$ for some $\bar{x} \in \mathbb{R}^n$. We have $f(x_k) \leq c$ for all k, implying that $(x_k, c) \in \operatorname{Epi}(f)$ for all k. Since $(x_k, c) \to (\bar{x}, c)$ and $\operatorname{Epi}(f)$ is closed, it follows that $(\bar{x}, c) \in \operatorname{Epi}(f)$. Consequently $f(\bar{x}) \leq c$, showing that $\bar{x} \in L_c(f)$.

"2) $\Rightarrow$ 3)": Let $x_0 \in \mathbb{R}^n$ be arbitrary and let $\{x_k\}$ be a sequence such that $x_k \to x_0$. To arrive at a contradiction, assume that f is not l.s.c. at x_0, i.e., $\liminf_{k\to\infty} f(x_k) < f(x_0)$. Then, there exist a scalar γ and a subsequence $\{x_k\}_{\mathcal{K}} \subset \{x_k\}$ such that $f(x_k) \leq \gamma < f(x_0)$ for all $k \in \mathcal{K}$ yielding that $\{x_k\}_{\mathcal{K}} \subset L_\gamma(f)$. Since $x_k \to x_0$ and the set $L_\gamma(f)$ is closed, it follows that $x_0 \in L_\gamma(f)$. Hence, $f(x_0) \leq \gamma$, a contradiction. Thus, we must have

$$f(x_0) \leq \lim \inf_{k\to\infty} f(x_k)$$

"3) $\Rightarrow$ 1)": To arrive at a contradiction assume that $\operatorname{Epi}(f)$ is not closed. Then, there exists a sequence $\{(x_k, w_k)\} \subset \operatorname{Epi}(f)$ such that

$$(x_k, w_k) \to (\bar{x}, \bar{w}) \quad \text{and} \quad (\bar{x}, \bar{w}) \notin \operatorname{Epi}(f)$$

Since $(x_k, w_k) \in \operatorname{Epi}(f)$ for all k, we have

$$f(x_k) \leq w_k \quad \text{for all } k$$

Taking the limit inferior as $k \to \infty$, and using $w_k \to \bar{w}$, we obtain

$$\liminf_{k\to\infty} f(x_k) \leq \lim_{k\to\infty} w_k = \bar{w}$$

Since $(\bar{x}, \bar{w}) \notin \operatorname{Epi}(f)$, we have $f(\bar{x}) > \bar{w}$, implying that

$$\liminf_{k\to\infty} f(x_k) \leq \bar{w} < f(\bar{x})$$

On the other hand, because $x_k \to \bar{x}$, and f is l.s.c. at $\bar{x}$, we have

$$f(\bar{x}) \leq \liminf_{k\to\infty} f(x_k),$$

a contradiction. Hence, $\operatorname{Epi}(f)$ must be closed. □

Theorem 3.38. *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be convex and such that* $\operatorname{int}(\operatorname{dom} f) \neq \emptyset$*. Then,* f *is continuous over* $\operatorname{int}(\operatorname{dom} f)$*:*

$$\textit{convex} \Longrightarrow \textit{continuous}$$

Example 3.39. ***(extension)*** *We will later prove the Lipschitz continuity of convex functions over interior in Theorem 3.72.*

Proof. Using the translation if necessary, we may assume without loss of generality that the origin is in the interior of the domain of f. It is sufficient to show that f is continuous at the origin. By scaling the unit box if necessary, we may assume without loss of generality that the unit box $\{x \in \mathbb{R}^n \mid \|x\|_\infty \leq 1\}$ is contained in $\operatorname{dom} f$. Let $v_i, i \in \mathcal{I} = \{1, \ldots, 2^n\}$ be vertices of the unit box (i.e., each v_i has entries 1 or -1). The unit box can be viewed as a simplex generated by these vertices, i.e., every x with $\|x\|_\infty \le 1$ is a convex combination of vertices $v_i, i \in \mathcal{I}$ or equivalently: every x with $\|x\|_\infty \le 1$ is given by

$$x = \sum_{i \in \mathcal{I}} \alpha_i v_i \quad \text{ with } \alpha_i \ge 0 \text{ and } \sum_{i \in \mathcal{I}} \alpha_i = 1$$

Note that by convexity of f, we have

$$f(x) \le \max_{i \in \mathcal{I}} f(v_i) = M,$$

which means it is bounded. Let $x_k \to 0$ and assume that $x_k \ne 0$ for all k. We introduce $y_k = \frac{x_k}{\|x_k\|_\infty}$ and $z_k = \frac{-x_k}{\|x_k\|_\infty}$. Note that we can write 0 as a convex combination of y_k and z_k, as follows

$$0 = \frac{1}{\|x_k\|_\infty + 1} x_k + \frac{\|x_k\|_\infty}{\|x_k\|_\infty + 1} z_k \quad \text{ for all } k$$

By convexity of f it follows that

$$f(0) \le \frac{1}{\|x_k\|_\infty + 1} f(x_k) + \frac{\|x_k\|_\infty}{\|x_k\|_\infty + 1} f(z_k) \quad \text{ for all } k$$

By letting $k \to \infty$ and boundedness, we have

$$f(0) \le \liminf_{k \to 0} f(x_k)$$

Note that we can write $x_k = (1 - \|x_k\|_\infty) 0 + \|x_k\|_\infty y_k$ for all k. By using convexity, we obtain

$$f(x_k) \le (1 - \|x_k\|_\infty) f(0) + \|x_k\|_\infty f(y_k)$$

Taking the limsup as $k \to \infty$ and using boundedness, we see that

$$\limsup_{k \to \infty} f(x_k) \le f(0)$$

From this relation and Eq. (2), we see that $\lim_{k \to \infty} f(x_k) = f(0)$ showing that f is continuous at 0. $\square$

Remark 3.40. ***(convex $\not\Longrightarrow$ closed, closed $\not\Longrightarrow$ convex)***

We can only get continuity of convex function over $\operatorname{int} \operatorname{dom} f$*, but cannot get continuity over* $\operatorname{dom} f$*. For example Theorem 3.62, we know for univariate function* ***convex only implies upper semicontinuous for boundary points****.*

You may ask "what if a function is closed and convex, is it continuous? (since we have upper semicontinuous for boundary points and lower semicontinuous for boundary points from closed)." The answer is ***No*** *for generel* $\mathbf{R}^n$*,* $n \ge 2$*.*

Later in Section 3.6.2, we will study the closure of convex function whose epigraph is the closure orignal function: $\operatorname{Epi} f^* = \operatorname{cl} \operatorname{Epi} f$.

3.1.4.2 Operations Preserving Closedness

- **Positive Scaling**: for a closed function $f : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty, +\infty\}$ and $\lambda > 0$, the function $g(x) = \lambda f(x)$ is closed
- **Sum**: for closed functions $f_i : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty, +\infty\}, i = 1, \ldots, m$, the sum $g(x) = \sum_{i=1}^m f_i(x)$ is closed
- **Composition with Affine Mapping**: For an $m \times n$ matrix A, a vector $b \in \mathbb{R}^m$, and a closed function $f : \mathbb{R}^m \to \mathbb{R} \cup \{-\infty, +\infty\}$, the function $g(x) = f(Ax + b)$ is closed
- **Composition with Continuous Mapping**
- **Pointwise Supremum**: for a collection of closed functions $f_i : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty, +\infty\}$ over an arbitrary index set I, the function

$$g(x) = \sup_{i \in I} f_i(x) \quad \text{ is closed}$$

3.1.5 Jensen's Inequality

Theorem 3.41. ***(Jensen's Inequality)*** *Let f be convex. Then for any convex combination*

$$\sum_{i=1}^{N} \lambda_i x_i$$

one has

$$f\left(\sum_{i=1}^{N} \lambda_i x_i\right) \leq \sum_{i=1}^{N} \lambda_i f(x_i)$$

Proof. The points $(f(x_i), x_i)$ clearly belong to $\mathrm{Epi}(f)$; since f is convex, its epigraph is a convex set, so that the convex combination

$$\sum_{i=1}^{N} \lambda_i (f(x_i), x_i) = \left(\sum_{i=1}^{N} \lambda_i f(x_i), \sum_{i=1}^{N} \lambda_i x_i\right)$$

of the points also belongs to $\mathrm{Epi}(f)$. By definition of the epigraph, the latter means exactly that $\sum_{i=1}^{N} \lambda_i f(x_i) \geq f\left(\sum_{i=1}^{N} \lambda_i x_i\right)$. □

Corollary 3.42. *Let f be a convex function and let x be a convex combination of the points $x_1, \ldots, x_N$. Then*

$$f(x) \leq \max_{1 \leq i \leq N} f(x_i)$$

In other words, if Δ is a convex hull of $x_1, \ldots, x_N$, i.e.

$$\Delta = \mathrm{Conv}\{x_1, \ldots, x_N\} \equiv \left\{x \in \mathbf{R}^n \mid x = \sum_{i=1}^{N} \lambda_i x_i, \alpha \geq 0, \sum_{i=1}^{N} \alpha_i = 1\right\}$$

then $\max_{x \in \Delta} f(x) \leq \max_{1 \leq i \leq N} f(x_i)$

Remark 3.43. ***(some thinking)*** *We also have $\min_{x \in \Delta} f(x) \geq \min_{1 \leq i \leq N} f(x_i)$. If f attains the maximum in Δ with all $\lambda > 0$ (a interior), then $f(x)$ is a constant over Δ. This is analogy to the affine case Lemma 2.93, the difference is convex f can attains the minimum without being a constant.*

3.1.6 Examples

We have used the second-order conditions to detect convexity. The proof of this approach is defered to Section 3.2.2.

3.1.6.1 Familiar Functions

- Exponential. e^{ax} is convex on $\mathbf{R}$, for any $a \in \mathbf{R}$.
- Powers. x^a is convex on $\mathbf{R}_{++}$when $a \geq 1$ or $a \leq 0$, and concave for $0 \leq a \leq 1$.
- Powers of absolute value. $|x|^p$, for $p \geq 1$, is convex on $\mathbf{R}$.
- Logarithm. $\log x$ is concave on $\mathbf{R}_{++}$.

3.1.6.2 Negative Entropy

$f(x) = x \log x$ (either on $\mathbf{R}_{++}$, or on $\mathbf{R}_{+}$, defined as 0 for $x = 0$) is convex.

Remark 3.44. *Log sum inequality and convexity of relative entropy $D(p||q)$ can be proved using the negative entropy.*

3.1.6.3 Norms

Every norm on $\mathbf{R}^n$ is convex.

3.1.6.4 Max function

$$f(x) = \max\{x_1, \ldots, x_n\} \textbf{ is convex on } \mathbf{R}^n\textbf{:}$$

$$\begin{aligned} f(\theta x + (1-\theta)y) &= \max_i (\theta x_i + (1-\theta)y_i) \\ &\leq \theta \max_i x_i + (1-\theta) \max_i y_i \\ &= \theta f(x) + (1-\theta) f(y) \end{aligned}$$

3.1.6.5 Quadratic-over-linear function

$$f(x,y) = x^2/y\textbf{, with } \operatorname{dom} f = \mathbf{R} \times \mathbf{R}_{++} = \left\{(x,y) \in \mathbf{R}^2 \mid y > 0\right\} \textbf{ is convex:}$$

$$\nabla^2 f(x,y) = \frac{2}{y^3} \begin{bmatrix} y^2 & -xy \\ -xy & x^2 \end{bmatrix} = \frac{2}{y^3} \begin{bmatrix} y \\ -x \end{bmatrix} \begin{bmatrix} y \\ -x \end{bmatrix}^\top \succeq 0$$

3.1.6.6 Log-sum-exp

$$f(x) = \log\left(e^{x_1} + \cdots + e^{x_n}\right) \textbf{ is convex on } \mathbf{R}^n\textbf{:}$$

$$\nabla^2 f(x) = \frac{1}{\left(\mathbf{1}^\top z\right)^2} \left(\left(\mathbf{1}^\top z\right) \operatorname{diag}(z) - zz^\top\right)$$

where $z = \left(e^{x_1}, \ldots, e^{x_n}\right)$. To verify that $\nabla^2 f(x) \succeq 0$ we must show that for all v $v^\top \nabla^2 f(x) v \geq 0$, i.e.

$$v^\top \nabla^2 f(x) v = \frac{1}{\left(\mathbf{1}^\top z\right)^2} \left(\left(\sum_{i=1}^n z_i\right)\left(\sum_{i=1}^n v_i^2 z_i\right) - \left(\sum_{i=1}^n v_i z_i\right)^2\right) \geq 0$$

But this follows from the Cauchy-Schwarz inequality $\left(a^\top a\right)\left(b^\top b\right) \geq \left(a^\top b\right)^2$ applied to the vectors with components $a_i = v_i\sqrt{z_i}, b_i = \sqrt{z_i}$.

Remark 3.45. *This function can be interpreted as a differentiable (in fact, analytic) approximation of the max function, since*

$$\max\{x_1, \ldots, x_n\} \leq f(x) \leq \max\{x_1, \ldots, x_n\} + \log n$$

for all x. (The second inequality is tight when all components of x are equal.)

3.1.6.7 Geometric mean

$$f(x) = \left(\prod_{i=1}^n x_i\right)^{1/n} \textbf{ is concave on } \operatorname{dom} f = \mathbf{R}_{++}^n\textbf{:}$$

$$\frac{\partial^2 f(x)}{\partial x_k^2} = -(n-1)\frac{\left(\prod_{i=1}^n x_i\right)^{1/n}}{n^2 x_k^2}, \quad \frac{\partial^2 f(x)}{\partial x_k \partial x_l} = \frac{\left(\prod_{i=1}^n x_i\right)^{1/n}}{n^2 x_k x_l} \quad \text{for } k \neq l$$

and can be expressed as

$$\nabla^2 f(x) = -\frac{\prod_{i=1}^n x_i^{1/n}}{n^2}\left(n \operatorname{diag}\left(1/x_1^2, \ldots, 1/x_n^2\right) - qq^\top\right)$$

where $q_i = 1/x_i$. We must show that $\nabla^2 f(x) \preceq 0$, i.e., that

$$v^\top \nabla^2 f(x) v = -\frac{\prod_{i=1}^n x_i^{1/n}}{n^2} \left(n \sum_{i=1}^n v_i^2/x_i^2 - \left(\sum_{i=1}^n v_i/x_i \right)^2 \right) \leq 0$$

for all v. Again this follows from the Cauchy-Schwarz inequality $\left(a^\top a\right)\left(b^\top b\right) \geq \left(a^\top b\right)^2$, applied to the vectors $a = \mathbf{1}$ and $b_i = v_i/x_i$

3.1.6.8 Log-determinant

$f(X) = \log\det X$ **is concave on** $\operatorname{dom} f = \mathbf{S}^n_{++}$**:**

Define $g(t) = f(Z + tV)$, and restrict g to the interval of values of t for which $Z + tV \succ 0$ Without loss of generality, we can assume that $t = 0$ is inside this interval, i.e., $Z \succ 0$. We have

$$\begin{aligned} g(t) &= \log\det(Z + tV) \\ &= \log\det\left(Z^{1/2} \left(I + tZ^{-1/2} V Z^{-1/2} \right) Z^{1/2} \right) \\ &= \sum_{i=1}^n \log\left(1 + t\lambda_i\right) + \log\det Z \end{aligned}$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $Z^{-1/2} V Z^{-1/2}$. Therefore we have

$$g'(t) = \sum_{i=1}^n \frac{\lambda_i}{1 + t\lambda_i}, \quad g''(t) = -\sum_{i=1}^n \frac{\lambda_i^2}{\left(1 + t\lambda_i\right)^2}$$

Since $g''(t) \leq 0$, we conclude that f is concave.

3.2 How to detect convexity

Similar to continuous functions with compositions, here we should point out the list of operations which preserve convexity. A number of standard convex functions has already shown in Section 3.1.6. It suffices to demonstrate that the function can be obtained, in **finite** many steps, from standard functions by applying the combination rules which preserve convexity.

We also present the differential criteria of convexity in Section 3.2.2, which can be used to check functions convexity in general.

3.2.1 Operations Preserving Convexity of Functions

3.2.1.1 Nonnegative Weighted Sums

If f, g are convex (and closed) functions on $\mathbf{R}^n$ then their linear combination $\lambda f + \mu g$ with **nonnegative** coefficients again is convex (and closed), provided that it is finite at least at one point.

These properties extend to infinite sums and integrals. For example if $f(x, y)$ is convex in x for each $y \in \mathcal{A}$, and $w(y) \geq 0$ for each $y \in \mathcal{A}$, then the function g defined as

$$g(x) = \int_{\mathcal{A}} w(y) f(x, y) dy$$

is convex in x (provided the integral exists).

The fact that convexity is preserved under nonnegative scaling and addition is easily verified directly, or can be seen in terms of the associated epigraphs. For example, if $w \geq 0$ and f is convex, we have

$$\operatorname{Epi}(wf) = \begin{bmatrix} I & 0 \\ 0 & w \end{bmatrix} \operatorname{Epi}(f)$$

which is convex because the image of a convex set under a linear mapping is convex.

3.2.1.2 Composition with Affine Mapping

The superposition $\phi(x) = f(Ax + b)$ of a convex (and closed) function f on $\mathbf{R}^n$ and affine mapping $x \mapsto Ax + b$ from $\mathbf{R}^m$ into $\mathbf{R}^n$ is convex (and closed), provided that it is finite at least at one point:

Proof. Let x_1 and x_2 in $\mathbf{R}^m$ and $y_i = Ax_i + b$, $i = 1, 2$. Then for $0 \le \lambda \le 1$ we have:

$$\begin{aligned}\phi\left(\lambda x_1 + (1-\lambda)x_2\right) &= f\left(A\left(\lambda x_1 + (1-\lambda)x_2\right) + b\right) = f\left(\lambda y_1 + (1-\lambda)y_2\right)\\ &\le \lambda f\left(y_1\right) + (1-\lambda) f\left(y_2\right) = \lambda \phi\left(x_1\right) + (1-\lambda)\phi\left(x_2\right)\end{aligned}$$

The closeness of the epigraph of ϕ follows from the continuity of the affine mapping. □

3.2.1.3 Pointwise Sup

$f = \sup_\alpha f_\alpha(\cdot)$ of any family of convex (and closed) functions on $\mathbf{R}^n$ is convex (and closed), provided that this bound is finite at least at one point:

Proof. Note that the epigraph of the upper bound clearly is the intersection of epigraphs of the functions from the family. Convexity of $\operatorname{dom}\sup_\alpha f_\alpha(\cdot)$ is also clear: the projection of the epigraphs intersection. □

Example 3.46. *Supporting function $\psi_M(x) = \sup\left\{y^\top x \mid y \in M\right\}$ (see (2.100)) for **any** set M is convex.*

Example 3.47. ***(maximum eigenvalue of a symmetric matrix)*** *The function $f(X) = \lambda_{\max}(X)$, with $\operatorname{dom} f = \mathbf{S}^m$, is convex. To see this, we express f as*

$$f(X) = \sup\left\{y^\top X y \mid \|y\|_2 = 1\right\}$$

i.e., as the pointwise supremum of a family of linear functions of X (i.e., $y^\top X y\big)$ indexed by $y \in \mathbf{R}^m$

Example 3.48. ***(matrix norm)*** *Consider $f(X) = \|X\|_2$ with $\operatorname{dom} f = \mathbf{R}^{p\times q}$ where $\|\cdot\|_2$ denotes the spectral norm or maximum singular value. Convexity of f follows from*

$$f(X) = \sup\left\{u^\top X v \mid \|u\|_2 = 1, \|v\|_2 = 1\right\}$$

which shows it is the pointwise supremum of a family of linear functions of X. As a generalization suppose $\|\cdot\|_a$ and $\|\cdot\|_b$ are norms on $\mathbf{R}^p$ and $\mathbf{R}^q$, respectively. The induced norm of a matrix $X \in \mathbf{R}^{p\times q}$ is defined as

$$\|X\|_{a,b} = \sup_{v\neq 0} \frac{\|Xv\|_a}{\|v\|_b}$$

(This reduces to the spectral norm when both norms are Euclidean.) The induced norm can be expressed as

$$\begin{aligned}\|X\|_{a,b} &= \sup\left\{\|Xv\|_a \mid \|v\|_b = 1\right\}\\ &= \sup\left\{u^\top X v \mid \|u\|_{a*} = 1, \|v\|_b = 1\right\}\end{aligned}$$

where $\|\cdot\|_{a*}$ is the dual norm of $\|\cdot\|_a$, and we use the fact that

$$\|z\|_a = \sup\left\{u^\top z \mid \|u\|_{a*} = 1\right\}$$

Since we have expressed $\|X\|_{a,b}$ as a supremum of linear functions of X, it is a convex function.

Example 3.49. *Let us consider the function $\psi(x,\gamma) = \sup_{y\in M} \phi(y, x, \gamma)$, where*

$$\phi(y, x, \gamma) = y^\top x - \frac{\gamma}{2}|y|_2^2$$

This function is convex and closed. Let us look at its properties. If M is bounded then $\operatorname{dom}\psi = \mathbf{R}^n$. Consider the case $M = \mathbf{R}^n$. Clearly, $\operatorname{dom}\psi$ contains only points with $\gamma \ge 0$. If $\gamma = 0$, the only possible value of x is zero, since otherwise the function $\phi(y, x, 0)$ is unbounded. Finally, if $\gamma > 0$, *then the point maximizing $\phi(y, x, \gamma)$ with respect to y is $y^* = \frac{x}{\gamma}$ and $\psi(x, \gamma) = \frac{|x|_2^2}{2\gamma}$ When summing up,*

$$\psi(x, \gamma) = \begin{cases} 0, & \textit{if} \quad x = 0, \gamma = 0 \\ \frac{|x|_2^2}{2\gamma} & \textit{if}\, \gamma > 0 \end{cases}$$

*and the domain of ψ is the set $\mathbf{R}^n \times \{\gamma > 0\} \cup \{0, 0\}$. **This domain set is neither open nor closed, nevertheless, ψ is a closed convex function, since the epigraph is closed**. Note that this function is not continuous at the origin:*

$$\lim_{\gamma \downarrow 0} \psi(\sqrt{\gamma x}, \gamma) = \frac{1}{2}|x|_2^2$$

since $(\sqrt{\gamma x}, \gamma) \to (0, 0)$ for any fix x, while the the limit of ψ is $\frac{1}{2}|x|_2^2$ which is different for different x.

Remark 3.50. ***(converse)*** *Convex function can be expressed as the pointwise supremum of a family of affine functions. For example, if $f : \mathbf{R}^n \to \mathbf{R}$ is convex, with $\operatorname{dom} f = \mathbf{R}^n$, then we have*

$$f(x) = \sup\{g(x) \mid g \textit{ affine}, \, g(z) \leq f(z) \textit{ for all } z\}$$

We will prove a more general case with $\operatorname{dom} f$ not necessarily equals $\mathbf{R}^n$ in Theorem 3.147. The special case $\operatorname{dom} f = \mathbf{R}^n$ is proved in [2, Page 83].

3.2.1.4 Convex Monotone Composition

In this section we examine conditions on $h : \mathbf{R}^k \to \mathbf{R}$ and $g : \mathbf{R}^n \to \mathbf{R}^k$ that guarantee convexity or concavity of their composition $f = h \circ g : \mathbf{R}^n \to \mathbf{R}$, defined by

$$f(x) = h(g(x)), \quad \operatorname{dom} f = \{x \in \operatorname{dom} g \mid g(x) \in \operatorname{dom} h\}$$

• Scalar Composition:

We first consider the case $k = 1$, so $h : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R}^n \to \mathbf{R}$.

Assume $n = 1$, and h and g are **twice differentiable**, with $\operatorname{dom} g = \operatorname{dom} h = \mathbf{R}$. The second derivative of the composition function $f = h \circ g$ is given if

$$f''(x) = h''(g(x))g'(x)^2 + h'(g(x))g''(x) \geq 0$$

- f is convex if h is convex and nondecreasing, and g is convex,
- f is convex if h is convex and nonincreasing, and g is concave,
- f is concave if h is concave and nondecreasing, and g is concave,
- f is concave if h is concave and nonincreasing, and g is convex.

In the general case $n > 1$, without assuming differentiability of h and g, or that $\operatorname{dom} g = \mathbf{R}^n$ and $\operatorname{dom} h = \mathbf{R}$:

- f is convex if h is convex, $\tilde{h}$ is nondecreasing, and g is convex,
- f is convex if h is convex, $\tilde{h}$ is nonincreasing, and g is concave,
- f is concave if h is concave, $\tilde{h}$ is nondecreasing, and g is concave,
- f is concave if h is concave, $\tilde{h}$ is nonincreasing, and g is convex.

Here $\tilde{h}$ denotes the extended-value extension of the function h, which assigns the value $\infty(-\infty)$ to points not in $\operatorname{dom} h$ for h convex (concave). The only difference between the above two results is that we require that the extended value extension function $\tilde{h}$ be nonincreasing or nondecreasing, on all of $\mathbf{R}$.

Remark 3.51. *To say that $\tilde{h}$ is nondecreasing means that for any $x, y \in \mathbf{R}$, with $x < y$, we have $\tilde{h}(x) \leq \tilde{h}(y)$. In particular, this means that if $y \in \operatorname{dom} h$, then $x \in \operatorname{dom} h$.*

Example 3.52.

1. *The function $h(x) = \log x$, with $\operatorname{dom} h = \mathbf{R}_{++}$, is concave and satisfies $\tilde{h}$ nondecreasing.*
2. *The function $h(x) = x^{3/2}$, with $\operatorname{dom} h = \mathbf{R}_{+}$, is convex but* ***does not satisfy*** *the condition $\tilde{h}$ nondecreasing. For example, we have $\tilde{h}(-1) = \infty$, but $\tilde{h}(1) = 1$*

Proof. Here we prove if g is convex, h is convex, and $\tilde{h}$ is nondecreasing, then $f = h \circ g$ is convex: Assume that $x, y \in \operatorname{dom} f$, and $0 \leq \theta \leq 1$. Since $x, y \in \operatorname{dom} f$, we have that $x, y \in \operatorname{dom} g$ and $g(x), g(y) \in \operatorname{dom} h$. Since dom g is convex, we conclude that $\theta x + (1-\theta) y \in \operatorname{dom} g$, and from convexity of g, we have

$$g(\theta x + (1-\theta) y) \leq \theta g(x) + (1-\theta) g(y) \tag{3.53}$$

Since $g(x), g(y) \in \operatorname{dom} h$, we conclude that $\theta g(x) + (1-\theta) g(y) \in \operatorname{dom} h$. Now we use the assumption that $\tilde{h}$ is nondecreasing to get that the righthand side of (3.53) is in $\operatorname{dom} h$ the lefthand side $g(\theta x + (1-\theta) y) \in \operatorname{dom} h$. This means that $\theta x + (1-\theta) y \in \operatorname{dom} f$. At this point, we have shown that $\operatorname{dom} f$ is convex.

Now using the fact that $\tilde{h}$ is nondecreasing and the inequality (3.53), we get

$$h(g(\theta x + (1-\theta) y)) \leq h(\theta g(x) + (1-\theta) g(y))$$

From convexity of h, we have

$$h(\theta g(x) + (1-\theta) g(y)) \leq \theta h(g(x)) + (1-\theta) h(g(y))$$

□

• **Vector Composition:** We now turn to the more complicated case when $k \geq 1$. Suppose

$$f(x) = h(g(x)) = h\left(g_1(x), \ldots, g_k(x)\right)$$

with $h : \mathbf{R}^k \to \mathbf{R}$, $g_i : \mathbf{R}^n \to \mathbf{R}$.

- f is convex if h is convex, $\tilde{h}$ is nondecreasing in each argument, and g_i are convex,
- f is convex if h is convex, $\tilde{h}$ is nonincreasing in each argument, and g_i are concave,
- f is concave if h is concave, $\tilde{h}$ is nondecreasing in each argument, and g_i are concave.

3.2.1.5 Partial Minimization

If $f(x, y) : \mathbf{R}_x^n \times \mathbf{R}_y^m$ is **convex** (as a function of $z = (x, y)$; this is called joint convexity) and the function

$$g(x) = \inf_y f(x, y)$$

is proper, i.e., is $> -\infty$ everywhere and is finite at least at one point, then g is **convex**.

Proof. a We should prove that if $x, x' \in \operatorname{dom} g$ and $x'' = \lambda x + (1-\lambda) x'$ with $\lambda \in [0, 1]$, then $x'' \in \operatorname{dom} g$ and $g\left(x''\right) \leq \lambda g(x) + (1-\lambda) g\left(x'\right)$ Given positive ϵ, we can find y and y' such that $(x, y) \in \operatorname{dom} f$, $(x', y') \in \operatorname{dom} f$ and $g(x) + \epsilon \geq f(x, y)$, $g\left(x'\right) + \epsilon \geq f\left(x', y'\right)$. Taking weighted sum of these two inequalities, we get

$$\begin{aligned}
\lambda g(x) + (1-\lambda) g(x') + \epsilon &\geq \lambda f(x, y) + (1-\lambda) f\left(x', y'\right) \\
&\geq f\left(\lambda x + (1-\lambda) x', \lambda y + (1-\lambda) y'\right) = f\left(x'', \lambda y + (1-\lambda) y'\right) \\
&\geq g\left(x''\right)
\end{aligned}$$

We have $x'' \in \operatorname{dom} g$ and $g\left(x''\right) \leq \lambda g(x) + (1-\lambda) g\left(x'\right)$, as required.

Note the domain is convex can also be shown directly from $\operatorname{dom} g =$ projection of $\operatorname{dom} f$, and is therefore convex. □

Example 3.54. ***(Schur complement)*** *Suppose the quadratic function*

$$f(x,y) = x^\top Ax + 2x^\top By + y^\top Cy$$

(where A and C are symmetric) is convex in (x, y), which means

$$\begin{bmatrix} A & B \\ B^\top & C \end{bmatrix} \succeq 0$$

We can express $g(x) = \inf_y f(x, y)$ as

$$g(x) = x^\top \left(A - BC^\dagger B^\top\right) x$$

where $C^\dagger$ is the pseudo-inverse of C (see [4, A.5.4]). By the minimization rule, g is convex, so we conclude that $A - BC^\dagger B^\top \succeq 0$. If C is invertible, i.e., $C \succ 0$, then the matrix $A - BC^{-1}B^\top$ is called the $Schur$ complement of C in the matrix

$$\begin{bmatrix} A & B \\ B^\top & C \end{bmatrix}$$

Example 3.55. ***(distance to set)*** *Here we study again distance to a set (see Example 3.27). The distance of a point x to a set $S \subseteq \mathbf{R}^n$, in the norm $\|\cdot\|$, is defined as*

$$\operatorname{dist}(x, S) = \inf_{y \in S} \|x - y\|$$

The function $\|x - y\|$ is convex in (x, y), so if the set S is convex, the distance function $\operatorname{dist}(x, S)$ is a convex function of x

3.2.1.6 Perspective of Function

If $f : \mathbf{R}^n \to \mathbf{R}$, then the **perspective of** f is the function $g : \mathbf{R}^{n+1} \to \mathbf{R}$ defined by

$$g(x,t) = tf(x/t)$$

with domain

$$\operatorname{dom} g = \{(x,t) \mid x/t \in \operatorname{dom} f, t > 0\}$$

The perspective operation preserves convexity: If f is a convex function, then so is its perspective function g. Similarly, if f is concave, then so is g.

Proof. We give a short proof here using epigraphs and the perspective mapping on $\mathbf{R}^{n+1}$. For $t > 0$ we have

$$\begin{aligned} (x,t,s) \in \operatorname{Epi} g &\iff tf(x/t) \le s \\ &\iff f(x/t) \le s/t \\ &\iff (x/t, s/t) \in \operatorname{Epi} f \end{aligned}$$

Therefore $\operatorname{Epi} g$ is the inverse image of $\operatorname{Epi} f$ under the perspective mapping that takes (u, v, w) to $(u, w)/v$. It follows (see (2.62) and (2.64)) that $\operatorname{Epi} g$ is convex, so the function g is convex. □

Example 3.56. ***(Euclidean norm squared)*** *The perspective of the convex function $f(x) = x^\top x$ on $\mathbf{R}^n$ is*

$$g(x,t) = t(x/t)^\top(x/t) = \frac{x^\top x}{t}$$

which is convex in (x, t) for $t > 0$.

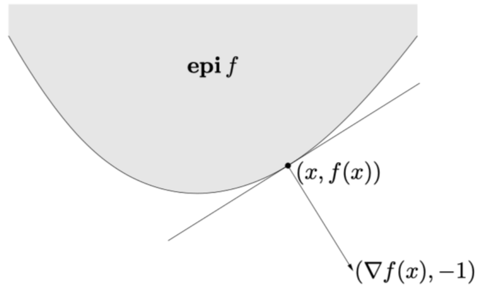


Fig. 13: For a **differentiable** convex function f, the vector $(\nabla f(x), -1)$ defines a supporting hyperplane to the epigraph of f at x.

3.2.2 Differential Criteria of Convexity

It follows from Theorem 3.14 that to detect convexity of a function, it, in principle, suffices to know how to **detect convexity of functions of one variable**.

Theorem 3.57. ***(convexity criterion for univariate smooth functions)*** *Let (a, b) be an interval on the real axis (we do not exclude the case of $a = -\infty$ and/or $b = +\infty$). Then*

1. *A **differentiable everywhere** on (a, b) function f is convex on $(a, b) \Longleftrightarrow$ its derivative f' is monotonically nondecreasing on (a, b);*
2. *A **twice differentiable everywhere** on (a, b) function f is convex on $(a, b) \Longleftrightarrow$ its second derivative f'' is nonnegative everywhere on (a, b).*

Proof. Proof of 1.

"$\Rightarrow$": from (3.7) and (3.8), if f is convex in (a, b) and if $a < s < t < u < b$, we have

$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t} \tag{3.58}$$

Since f is differentiable everywhere, we have $f'(s) \leq f'(u)$. So f' is monotonically nondecreasing.

"$\Leftarrow$": if f' is monotonically nondecreasing, from mean theorem we then have

$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(t)}{u - t}, \forall t < s < u \tag{3.59}$$

We then have $f(t) \leq \frac{t-s}{u-s} f(u) + \frac{u-t}{u-s} f(s)$ (This is from (3.8)). We have the convexity of f.

Proof of 2. This is directly from 1. $\square$

Remark 3.60. ***($\mathbf{R}^n$ extension)*** *Suppose f is **differentiable**, i.e. its gradient ∇f exists at each point in* $\operatorname{dom} f$ *which is **open**. Then*

$$f \textit{ \textbf{is convex}} \iff \operatorname{dom} f \textit{ \textbf{is convex and}} f(y) \geq f(x) + \nabla f(x)^\top (y - x)$$

holds for all $x, y \in \operatorname{dom} f$. This inequality is illustrated in Fig. 13:

$$(y, t) \in \operatorname{Epi} f \Longrightarrow \begin{bmatrix} \nabla f(x) \\ -1 \end{bmatrix}^\top \left(\begin{bmatrix} y \\ t \end{bmatrix} - \begin{bmatrix} x \\ f(x) \end{bmatrix} \right) \leq 0$$

*See [2, page 70] for one proof. We will provide another proof using general **subgradient** in Corollary 3.102 and Theorem 3.94. Note if **differentiable**, the **subdifferential** $\partial f(x) = \{\nabla f(x)\}$.*

*Under assumption f is **twice differentiable** with **open*** $\operatorname{dom} f$*, we have*

$$f \textit{ \textbf{is convex}} \iff \operatorname{dom} f \textit{ \textbf{is convex and}} \nabla^2 f(x) \succeq 0 \textit{ \textbf{for all}}\ x \in \operatorname{dom} f$$

For details, see Corollary 3.66 where Lemma 3.61 will be used to get an extension.

To deal with the **not open intervals (or more general not open sets)**, we can use the following lemma:

Lemma 3.61. ***(general convex set*** M***)*** *Let M be a convex set and f be a function with* $\operatorname{dom} f = M$. *Assume that f is convex on* $\operatorname{ri} M$ *and is continuous on M, i.e.,*

$$f(x_i) \to f(x), i \to \infty$$

whenever $x_i, x \in M$ and $x_i \to x$ as $i \to \infty$. Then f is convex on M.

Proof. For any $x, y \in M$, since $\operatorname{cl}\operatorname{ri} M = \operatorname{cl} M$, so we can select $\{x_i\} \subseteq \operatorname{ri} M$ and $\{y_i\} \subseteq \operatorname{ri} M$ converge to x, y respectively. Also $\lambda x_i + (1-\lambda) y_i \to \lambda x_i + (1-\lambda) y$. From continuity, we then have the conclusion. □

In fact, for functions of one variable there is a differential criterion of convexity which **does not assume any smoothness**, we have:

Theorem 3.62. ***(convexity criterion for univariate functions)*** *Let $g : \mathbf{R} \to \mathbf{R} \cup \{+\infty\}$ be a function. Let the domain $\Delta = \{t \mid g(t) < \infty\}$ of the function be a convex set which is not a singleton, i.e., let it be an interval (a, b) with possibly inclusion of one or both endpoints $(-\infty \le a < b \le \infty)$. g is convex if and only if it satisfies the following 3 requirements:*

1. *g is continuous on (a, b);*
2. *g is differentiable everywhere on (a, b), excluding, possibly, a countable set of points, and the derivative $g'(t)$ is nondecreasing on its domain;*
3. *at each endpoint u of the interval (a, b) which belongs to Δ, g is upper semicontinuous:*

$$g(u) \ge \lim \sup_{t \in (a,b), t \to u} g(t)$$

Proof. "⇒": g is convex then it is Lipschitz continuity over interior interval $[c, d]$ for **any** $a < c < d < b$ (see Theorem 3.72), and therefore absolutely continuous, and therefore differentiable almost surely w.r.t. Lebesgue measure with countable exceptions:

$$g = g(c) + \int_c^x g'(t)\,\mathrm{d}t \tag{3.63}$$

where $g'(t) \in \mathcal{L}^1(m_{c,d})$, according to [5, Theorem 5.4.3]. From Remark 3.6, we then get

$$\frac{g(j) - g(i)}{j - i} \le \frac{g(k) - g(j)}{k - j} \le \frac{g(l) - g(k)}{l - k} \tag{3.64}$$

for all $c \le i < j < k < l \le d$. With $i \to j$ and $k \to l$, we have $g'(j) \le g'(l)$ provided at j, l, g is differentiable.

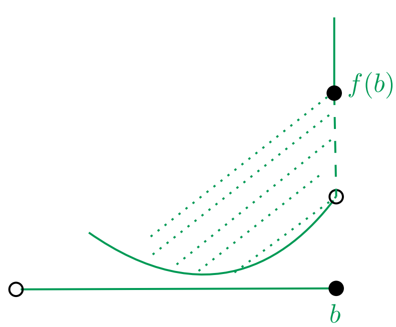


Fig. 14: Visualization of convex function.

1. and 2. are therefore proved. To prove 3. for sequence $\{x_i\} \subseteq (a, b)$ with $x_i \to b$, we have sequence $\{\lambda_i\}$ s.t. $x_i = \lambda_i c + (1 - \lambda_i) b$ (where $c \le \min x_i$). We have $\lambda_i \to 0$ since $x_i \to b$. From convex, we have $g(x_i) \le \lambda_i g(c) + (1 - \lambda_i) g(b)$. Take $\lim\sup$ from both sides, we have

$$\lim\sup g(x_i) \le g(b).$$

The other endpoint can be proved similarly. We therefore have proved 3.

"⇒": For any interior interval $[c, d]$ for **any** $a < c < d < b$ from 1. and 2., we get (3.63) (actually according to [5, Theorem 5.4.3], 1. is implied by 2.), and (3.64). And convex over any $[c, d]$ is achieved by Remark 3.6. In other words, convex over (c, d) is achieved. To get convex over $(c, d]$ if $g(d) < \infty$, for sequence $\{x_i\} \subseteq (a, b)$ with $x_i \to b$, if $x = \lambda c + (1-\lambda)b$, we have sequence $\{\lambda_i\}$ s.t. $\lambda_i \to \lambda$ and $x = \lambda_i c + (1-\lambda_i)x_i$. We then have $f(x) \leq \lambda_i f(c) + (1-\lambda_i)f(x_i)$. Take $\limsup$ from both sides, we have

$$f(x) \leq \lambda f(c) + (1-\lambda)f(b)$$

We therefore have proved g is convex over Δ with inclusion of one or both endpoints.

Remark 3.65. *Note here Lemma 3.61 cannot be used to prove the endpoint since we only have continuous over (a, b).*

□

Corollary 3.66. ***(convexity criterion for smooth functions on $\mathbf{R}^n$)*** *Let $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ be a function. Assume that the domain M of f is a convex set and that f is*

1. *continuous on M*
2. *twice differentiable on $\operatorname{ri} M$.*

Then f is convex $\Longleftrightarrow$ if its Hessian is positive semidefinite on $\operatorname{ri} M$:

$$h^\top f''(x)h \geq 0 \quad \forall\, x \in \operatorname{ri} M, \forall\, h \in \mathbf{R}^n$$

Proof. "⇒": if f is convex and $x \in \operatorname{ri} M$, then the function of one variable

$$g(t) = f(x + th) \tag{3.67}$$

(h is an arbitrary fixed direction in $\mathbf{R}^n$) is convex in certain neighborhood of the point $t = 0$ on the axis. Since f is twice differentiable in a neighborhood of x, g is twice differentiable in a neighborhood of $t = 0$, so that $g''(0) = h^\top f''(x)h \geq 0$ by Theorem 3.57.

"⇐": Let us first prove that f is convex on the relative interior M' of the domain M. First, clearly, $\operatorname{ri} M$ is a convex set. From Theorem 3.14, all we should prove is that every one-dimensional function

$$g(t) = f(x + t(y - x)), \quad 0 \leq t \leq 1$$

(x and y are from $\operatorname{ri} M$) is convex on the segment $0 \leq t \leq 1$. Since f is continuous on $M \supset \operatorname{ri} M$, g is continuous on the segment; and since f is twice continuously differentiable on $\operatorname{ri} M$, g is continuously differentiable on $(0, 1)$ with the second derivative

$$g''(t) = (y - x)^\top f''(x + t(y - x))(y - x) \geq 0$$

By using Lemma 3.61 twice, one is $(0, 1)$ to $[0, 1]$, the other is $\operatorname{ri} M$ to M, we have f is convex over entire M. □

3.3 Lipschitz Continuity of Convex Functions

3.3.1 Lipschitz Continuity

Convex functions possess very nice **local** properties. However, before showing this, we first show a **global** Lipschitz continuity of general **continuous (not necessarily convex)** functions with **strong assumptions of differentiability of f and boundedness of f'**:

Theorem 3.68. ***([1, Theorem 9.19])*** *Suppose **continuous** function f maps a **convex** open set $E \subset \mathbf{R}^n$ into $\mathbf{R}^m$, f is differentiable in E, and there is a real number L such that $\|f'(x)\|_2 \leq L$ Then*

$$|f(x) - f(y)| \leq L\|x - y\|_2$$

for all $x \in E, y \in E$.

Remark 3.69. *"open" is for differentiable at every point and can be replaced by any convex set E similar to Lemma 3.61 using sequence. In other words, Theorem 3.68 can be restated as: in $\operatorname{ri} E$, continuous function f is differentiable, and there is a real number L such that $\|f'(x)\| \leq L$, we then have $|f(x) - f(y)| \leq L\|x - y\|$ for all $x, y \in E$.*

Proof. We still defined $g(t)$ as in (3.67). We then have

$$g'(t) = f'(x + t(y - x))(y - x)$$

so that

$$g'(t) \leq L\|x - y\|$$

for all $t \in [0, 1]$, where convexity ensures $x + t(y - x) \in E$. By [1, Theorem 5.19], we have

$$|g(1) - g(0)| \leq L|b - a|$$

But $g(0) = f(x)$ and $g(1) = f(y)$. This completes the proof. □

Corollary 3.70. *If in addition, $f'(x) = 0$ for all $x \in E$, then f is constant (set $L = 0$).*

Remark 3.71. ***(explanation)*** *Convex of E is also necessary in this corollary, if only open, then f can be different constant on different disjoint subsets, convex here is essential to ensure the points on the lines between any pair points is in the domain to make sure constant over the total domain.*

Also please note here $f' = 0$ almost everywhere cannot guarantee constant, see cantor function. So that means continuous almost everywhere is not the same as continuous.

We next show a **local Lipschitz continuity** over a closed bounded set in relative interior of the domain without any assumption of the differential or boundedness of f':

Theorem 3.72. ***(boundedness and Lipschitz continuity of convex function)*** *Let f be a convex function and let K be a **closed and bounded set contained in the** $\operatorname{ri} \operatorname{dom} f$. Then f is Lipschitz continuous on K: there exists constant L for K, such that*

$$|f(x) - f(y)| \leq L\|x - y\|_2 \quad \forall x, y \in K$$

In particular, f is bounded on K.

Remark 3.73. *All three assumptions on K: ① closeness, ② boundedness and the assumption ③ $K \subset \operatorname{ri} \operatorname{dom} f$, are essential, as it is seen from the following examples:*

- $f(x) = 1/x$, $\operatorname{dom} f = (0, +\infty), K = (0, 1]$. *We have ② ③ and not ①; f is neither bounded, nor Lipschitz continuous on K.*
- $f(x) = x^2$, $\operatorname{dom} f = \mathbf{R}, K = \mathbf{R}$. *We have ① ③ and not ②; f is neither bounded nor Lipschitz continuous on K.*
- $f(x) = -\sqrt{x}$, $\operatorname{dom} f = [0, +\infty), K = [0, 1]$. *We have ① ② and not ③; f is not Lipschitz continuous on K, although is bounded: $\lim_{t \to +0} \frac{f(0) - f(t)}{t} = \lim_{t \to +0} t^{-1/2} = +\infty$, while for a Lipschitz continuous f, the ratios $t^{-1}(f(0) - f(t))$ should be bounded.*

Proof. We will start with the following a local version of the theorem.

Lemma 3.74. *Let f be a convex function, and let $\bar{x} \in \operatorname{ri} \operatorname{dom} f$. Then*

1). f is bounded around $\bar{x}$: there exists a positive r such that f is bounded in the r-neighborhood $U_r(\bar{x})$ of $\bar{x}$ in the $\operatorname{Aff} \operatorname{dom} f$:

$$\exists\, r > 0, C : \quad |f(x)| \leq C \quad \forall\, x \in U_r(\bar{x}) = \{x \mid x \in \operatorname{Aff}(\operatorname{dom} f), \|x - \bar{x}\|_2 \leq r\}$$

2). f is Lipschitz continuous around $\bar{x}$: there exists a positive ρ and a constant L such that

$$|f(x) - f(x')| \leq L\|x - x'\|_2, \forall\, x, x' \in U_\rho(\bar{x})$$

Proof. of lem:llll

Proof of 1).

Since $\bar{x} \in \operatorname{ri} \operatorname{dom} f$, there exists a neighborhood $U_{\bar{r}}(\bar{x}) \subseteq \operatorname{dom} f$. Now, we can find a small simplex Δ of the dimension $m = \dim \operatorname{Aff}(\operatorname{dom} f)$ with the vertices $x_0, \ldots, x_m$ in $U_{\bar{r}}(\bar{x})$ in such a way that $\bar{x}$ will be a convex combination of the vectors x_i with positive coefficients, even with the coefficients $1/(m+1)$:

$$\bar{x} = \sum_{i=0}^{m} \frac{1}{m+1} x_i$$

We know that $\bar{x} \in \operatorname{ri} \Delta$ from Corollary 2.69. Since Δ spans the same affine set as $\operatorname{dom} f$, it means that Δ contains $U_r(\bar{x})$ with certain $r > 0$. Now, from Corollary 3.42, in

$$\Delta = \left\{ \sum_{i=0}^{m} \lambda_i x_i \mid \lambda_i \geq 0, \sum_i \lambda_i = 1 \right\}$$

f is bounded from above by the quantity $\max_{0 \leq i \leq m} f(x_i)$ by Jensen's inequality:

$$f\left(\sum_{i=0}^{m} \lambda_i x_i\right) \leq \sum_{i=0}^{m} \lambda_i f(x_i) \leq \max_i f(x_i)$$

Consequently, f is **bounded from above**, by the same quantity, in $U_r(\bar{x})$.

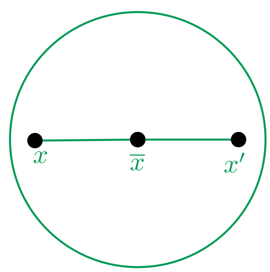


Fig. 15: Visualization of proof.

Now let us prove that if f is above bounded, by some W, in $U_r(\bar{x})$, then it in fact is **below bounded** in this neighborhood (and, consequently, is bounded in U_r).

Indeed, let $x \in U_r(\bar{x})$ s.t. $|x - \bar{x}| \leq r$. Setting $x' = \bar{x} - [x - \bar{x}] = 2\bar{x} - x$, we get $|x' - \bar{x}| = |x - \bar{x}| \leq r$ and $x' \in \operatorname{Aff}(\operatorname{dom} f)$, so that $x' \in U_r(\bar{x})$. Since $\bar{x} = \frac{1}{2}[x + x']$ we have

$$2f(\bar{x}) \leq f(x) + f(x')$$

whence

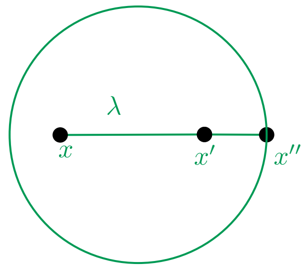
$$f(x) \geq 2f(\bar{x}) - f(x') \geq 2f(\bar{x}) - W, \quad x \in U_r(\bar{x})$$

and f indeed is below bounded in $U_r(\bar{x})$.

Proof of 2).

2). is an immediate consequence of 1). Indeed, let us prove that f is Lipschitz continuous in the neighborhood $U_{r/2}(\bar{x})$, where $r > 0$ is such that f is bounded in $U_r(\bar{x})$ from 1). Let $|f| \leq C$ in $U_r(\bar{x})$, and let $x, x' \in U_{r/2}(\bar{x}), x \neq x'$. Let us extend the segment $[x, x']$ through the point x' until it reaches, at certain point x'', the (relative) boundary of U_r; then we will get

$$x' \in (x, x'') ; \quad |x'' - \bar{x}| = r$$



By the convexity of f we have for $\lambda = \frac{\|x'-x\|_2}{\|x''-x\|_2} \in (0,1)$,

$$f(x') - f(x) \leq \lambda(f(x'') - f(x)) = \|x' - x\|_2 \frac{f(x'') - f(x)}{\|x'' - x\|_2}$$

The second factor in the right hand side does not exceed the quantity $(2C)/(r/2) = 4C/r$. Thus, we have

$$f(x') - f(x) \leq (4C/r)\|x' - x\|_2, \forall x, x' \in U_{r/2}$$

swapping x and x', we come to

$$f(x) - f(x') \leq (4C/r)\|x' - x\|_2$$

whence

$$|f(x) - f(x')| \leq (4C/r)\|x - x'\|_2, x, x' \in U_{r/2}$$

as required in 2). □

After Lemma 3.74, we next prove Theorem 3.72: Assume, on contrary, that f is not Lipschitz continuous on K; then for every integer i there exists a pair of points $x_i, y_i \in K$ such that

$$f(x_i) - f(y_i) \geq i\|x_i - y_i\|_2 \tag{3.75}$$

Since K is **compact**, passing to a subsequence we can ensure that $x_i \to x \in K$ and $y_i \to y \in K$.

By Lemma 3.74, the case $x = y$ is impossible since we have shown f is Lipschitz continuous in a neighborhood of $x = y$ and the ratios $(f(x_i) - f(y_i))/\|x_i - y_i\|_2$ form a bounded sequence, which we know is not the case. Thus, the case $x = y$ is impossible.

The case $x \neq y$ is also impossible. From Lemma 3.74, we know f is continuous on $\operatorname{ri}\operatorname{dom} f$ at both the points x and y so that we would have $f(x_i) \to f(x)$ and $f(y_i) \to f(y)$ as $i \to \infty$. Thus, the left hand side in (3.75) remains bounded as $i \to \infty$ while the right hand side tends to ∞ as $i \to \infty$; this is the desired contradiction. □

3.3.2 Directional Derivative and Bounded Below

In what follows in this section, for the sake of conciseness, WLOG, we will assume that the the domain of f is full dimensional, meaning that $\operatorname{Aff}(\operatorname{dom} f) = \mathbf{R}^n$. We can get the general case by translating the corresponding statements for the case when $\operatorname{Aff}(\operatorname{dom} f)$ is smaller than the whole $\mathbf{R}^n$: interior becomes relative interior, $h \in \mathbf{R}^n$ becomes $h \in V$, such that $\operatorname{Aff}(\operatorname{dom} f) = x_0 + V$, etc.

3.3.2.1 Directional Derivative

Definition 3.76. ***(Directional Derivative)*** *Let $x \in \operatorname{dom} f$. We call the function f **differentiable in the direction** h at x if the following limit **exists**:*

$$f_h'(x) = \lim_{t \downarrow 0} \frac{f(x + ht) - f(x)}{t}$$

Remark 3.77. *If f' exists, $f_h'(x) = f'h = \nabla f^\top h$, see [1, Page 217 (39)]. Please note here we need the **right-hand limit** $\lim_{t\downarrow 0}$, not $\lim_{t\to 0}$.*

Theorem 3.78. ***(directional differentiability of convex functions)***

*Convex function f is differentiable in **any direction** $h \in \mathbf{R}^n$ at **any point** $x \in \operatorname{int} \operatorname{dom} f$.*

Remark 3.79. ***(explanation)*** *It means for any $x \in \operatorname{ri} \operatorname{dom} f$, the domain of function $s_x(h) := f'_h(x)$ is $\mathbf{R}^n$.*

Proof. Let $x \in \operatorname{int} \operatorname{dom} f$. Consider the function of one variable

$$g(t) := f(x + th). \tag{3.80}$$

and the function

$$\phi(t) := \frac{g(t) - g(0)}{t}, t > 0 \tag{3.81}$$

For small enough t, we have $x + th$ is inside $\operatorname{dom} f$ since $x \in \operatorname{int} \operatorname{dom} f$. We then have $g(t)$ is **convex** on a small neighbour $[-r, r]$. $g(t)$ is a univariate function, so from (3.8), we know $\phi(t)$ is **increasing** for $t \in (0, r)$. From Theorem 3.72, we know $g(t)$ is Lipschitz continuous on $[-r, r]$ which implies $\phi(t)$ is bounded for $t \in (0, r]$. We then have the existence of right-hand limit $\phi(0+)$ from [1, Theorem 4.29], which equals $g'(0+) = f'_h(x)$. $\square$

3.3.2.2 Below Boundedness

Theorem 3.72 says that a convex function f is **bounded on every compact subset of** $\operatorname{ri} \operatorname{dom} f$. In fact there is much stronger statement on the **below boundedness** of f: f is below bounded on **any bounded subset** of $\mathbf{R}^n$. This results is an immediate consequence of the following lower bound:

Lemma 3.82. ***(global lower bound)*** *Let f be a convex function and $x \in \operatorname{int} \operatorname{dom} f$. Then $f'_h(x)$ is **convex positive homogenous (of degree 1) function** of h, and for any $y \in \operatorname{dom} f$*

$$f(y) \geq f(x) + f'_{y-x}(x) \tag{3.83}$$

Remark 3.84. *With $h = y - x$, in terms of $g(t)$ defined in* (3.80)*, it says $g(1) \geq g(0) + g'(0+)$. Intuitively, this is true, from the increasing of $\phi(t)$.*

Proof. Let us prove first that the directional derivative is homogenous. Indeed, for any $h \in \mathbf{R}^n$ and $\tau > 0$

$$f'_{\tau h}(x) = \lim_{t \downarrow 0} \frac{f(x + \tau h t) - f(x)}{t} = \tau \lim_{\alpha \downarrow 0} \frac{f(x + h\alpha) - f(x)}{\alpha} = \tau f'_h(x)$$

Further, for any $h_1, h_2 \in \mathbf{R}^n$, and $\lambda \in [0, 1]$, by the convexity of f we get

$$\begin{aligned} f'_{\lambda h_1 + (1-\lambda) h_2}(x) &= \lim_{t \downarrow 0} \frac{1}{t} \left[f\left(x + (\lambda h_1 + (1-\lambda) h_2) t\right) - f(x) \right] \\ &\leq \lim_{t \downarrow 0} \frac{1}{t} \left\{ \lambda \left[f\left(x + t h_1\right) - f(x) \right] + (1 - \lambda) \left[f\left(x + t h_2\right) - f(x) \right] \right\} \\ &= \lambda f'_{h_1}(x) + (1 - \lambda) f'_{h_2}(x) \end{aligned}$$

Thus $f'_h(x)$ is convex in h. Finally, let $t \in (0, 1], y \in \operatorname{dom} f$ and

$$y_t = x + t(y - x) = (1 - t)x + ty$$

Then

$$f(y) = f\left(y_t + \frac{1}{t}(1 - t)\left(y_t - x\right)\right) \geq f\left(y_t\right) + \frac{1 - t}{t}\left[f\left(y_t\right) - f(x)\right]$$

(the latter inequality is nothing but the convexity property of $f : f\left(y_t\right) \leq t f(y) + (1 - t) f(x)$) and we conclude (3.83) taking the limit as $t \downarrow 0$. $\square$

Corollary 3.85. ***(a restatement)*** *$s_x(h) := f'_h(x)$ is convex in h and positive homogenous (of degree 1), with $\operatorname{dom} s_x(h) = \mathbf{R}^n$.*

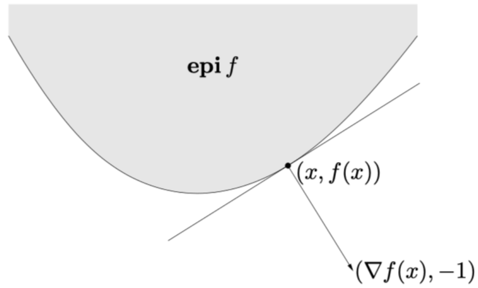


Fig. 16: if **differentiable**, $(\nabla f(x), -1)$ defines a supporting hyperplane to the epigraph.

Corollary 3.86. ***(below boundedness)***

*f is below bounded on **any bounded subset** of $\mathbf{R}^n$*

Remark 3.87. *We will give a another simple proof in Corollary 3.154, which will not use the directional derivative.*

Proof. As in Remark 3.79, $s_x(h)$ has domain $\mathbf{R}^n$. For any **bounded subset** $K \subseteq \operatorname{dom} f$, $s_x(y-x)$ is then bounded since $s_x(h)$ is convex and $K - x$ is bounded closed (and in interior of $\mathbf{R}^n$ of course) as required by Theorem 3.72. Below bounded is then from (3.83). $\square$

Corollary 3.88. *Convex function f can achieve the infimum on a **bounded closed** set B which intersects $\operatorname{dom} f$ (not necessarily $\subseteq \operatorname{dom} f$).*

Remark 3.89. *For examples, relative entropy $D(\cdot||Q)$ can achieve the infimum on any closed subset of $\Delta \cap \operatorname{dom} D(\cdot||Q)$, where Δ is the probability simplex.*

Proof. Since f is below bounded on B, $\inf_B f$ exist. For $\epsilon > 0$, $C := \{x \mid f(x) \leq \inf_B f + \epsilon\}$ is the sublevel set of f and is convex and closed (because of continuous Theorem 3.36). For $B \cap C \subseteq \operatorname{dom} f$ which is compact, f is continuous, we have that f can achieve $\inf_B f$. $\square$

3.4 Subgradients of Convex Functions

3.4.1 Definitions and Theorems

We are now ready to introduce a "good surrogate" of the notion of the gradient for a convex function.

Definition 3.90. ***(Subgradient of Convex Function)*** *Let f be function (not necessarily convex). A vector h is called **subgradient** of function f at a point $x \in \operatorname{dom} f$ if for **any** $y \in \operatorname{dom} f$ we have*

$$f(y) \geq f(x) + h^\top (y - x) \tag{3.91}$$

*The set $\partial f(x)$ of all subgradients of f at x is called the **subdifferential** of f at the point x.*

Remark 3.92. ***(explanation)*** *This definition says: for a subgradient h, it means there exists an **affine minorant** $h^\top x - a$ of f which **coincides** with f at x :*

$$f(y) \geq h^\top y - a, \forall y, \text{ and } f(x) = h^\top x - a$$

In the following, we will also see (3.83) $f(y) \geq f(x) + f'_{y-x}(x)$ is just an inequality with a special h in (3.91) with h being selected in (3.95).

Remark 3.93. *Affine function $\alpha y = h^\top x - a$ must have $\alpha \neq 0$, $y \in \mathbf{R}$ and $x, h \in \mathbf{R}^n$. It is a **non-vertical** hyperplane in $\mathbf{R}^{n+1}$, i.e. cannot be be a hyperplane $0 = h^\top x - a$.*

The most elementary properties of the subgradients are summarized in the following theorem:

Theorem 3.94. ***(subgradient of convex functions)*** *Let f be a convex function and x be a point from* $\operatorname{int} \operatorname{dom} f$. *Then*

*1). $\partial f(x)$ is a **closed convex set** which for sure is **nonempty and bounded**.*

2). for any $d \in \mathbf{R}^n$

$$f'_m(x) = \max\left\{h^\top m \mid h \in \partial f(x)\right\} \tag{3.95}$$

*In other words, the directional derivative is nothing but the **support function** of the set ∂f.*

3). If f is differentiable at x, then $\partial f(x) = \{\nabla f(x)\}$, a singleton.

Remark 3.96. *∂f at x **may be empty if** x **is a boundary point of the domain**. See Example 3.101.*

Proof. Proof of 1).

Closeness and convexity of $\partial f(x)$ are evident: (3.91) is an infinite system of nonstrict linear inequalities with respect to h, the inequalities being indexed by $y \in \mathbf{R}^n$.

Nonemptiness of $\partial f(x)$ for the case when $x \in \operatorname{int} \operatorname{dom} f$: Point $(f(x), x)$ belongs to the boundary of $\operatorname{Epi}(f)$. Hence, from separation theorem there is a linear form $(-\alpha, h)$ which properly separates x and $\operatorname{Epi}(f)$

$$h^\top y - \alpha\tau \leq h^\top x - \alpha f(x) \tag{3.97}$$

for any $(\tau, y) \in \operatorname{Epi}(f)$, i.e. $\tau \geq f(y)$. Note that we can take

$$\|h\|_2^2 + \alpha^2 = 1 \tag{3.98}$$

and such that $(-\alpha, h) \in \operatorname{Aff} \operatorname{Epi}(f))$. And since for any $\tau \geq f(x)$ the point (τ, x) belongs to the $\operatorname{Epi}(f)$, we conclude that $\alpha \geq 0$.

Now recall that a convex function is locally Lipschitz continuous on the interior of its domain. This means that there exist some $\epsilon > 0$ and $M > 0$ such that the ball of the radius ϵ, centered at x, belongs to $\operatorname{int} \operatorname{dom} f$ (since $x \in \operatorname{int} \operatorname{dom} f$) and for any y in this ball

$$f(y) - f(x) \leq L\|y - x\|_2$$

Thus, for any y in the ball

$$h^\top(y - x) \leq \alpha(f(y) - f(x)) \leq \alpha L\|y - x\|_2$$

When choosing $y = x + \epsilon h$ we get $\|h\|_2^2 \leq L\alpha\|h\|_2$ and together with the normalizing equation (3.98), we have

$$\alpha \geq \frac{1}{\sqrt{1 + L^2}}$$

The vertical hyper-plane is therefore not possible. Now from (3.97), with τ being $f(y)$, we can choose $\bar{h} = h/\alpha$ to get

$$f(y) \geq f(x) + \bar{h}^\top(y - x)$$

Boundedness: if $h \in \partial f(x), h \neq 0$, then by choosing $y = x + \epsilon h/\|h\|_2$, we obtain:

$$\epsilon\|h\|_2 = h^\top(y - x) \leq f(y) - f(x) \leq L\|y - x\|_2 = L\epsilon$$

what implies the **boundedness of $\partial f(x)$: $\|h\|_2 \leq L$.**

Proof of 2).

Note that, as $f'_0(x) = 0$, we have

$$f'_m(x) - f'_0(x) = f'_m(x) = \lim_{t \downarrow 0} \frac{f(x + mt) - f(x)}{t} \geq h^\top m \tag{3.99}$$

for any vector h from $\partial f(x)$. Therefore, the subdifferential of the function $s_x(m) := f'_m(x)$ at $m = 0$ exists and

$$\partial f(x) \subset \partial s_x(0)$$

As $s_x(m)$ is convex in m (cf. Corollary 3.85), from definition of subgradient, we know

$$s_x(y-x) = s_x(y-x) - s_x(0) \geq h^\top(y-x)$$

for any $h \in \partial s_x(0)$, and by Lemma 3.82 we have for any $y \in \operatorname{dom} f$

$$f(y) \geq f(x) + f'_{y-x}(x) \geq f(x) + h^\top(y-x) \text{ for } h \in \partial s_x(0)$$

We conclude that $\partial s_x(0) \subset \partial f(x)$. Therefore we have

$$\partial_m f'_0(x) = \partial s_x(0) \equiv \partial f(x).$$

Let now $d_m \in \partial s_x(m)$ (which is not empty by 1). since $s_x(m)$ is convex and has domain $\mathbf{R}^n$). **We prove this is the h that achieves the maximum for the support function.**

We need first to prove $d_m \in \partial s_x(0)$, i.e. $d_m \in \partial f(x)$. For any $v \in \mathbf{R}^n$ and $\tau > 0$ we have

$$\tau f'_v(x) = f'_{\tau v}(x) \geq f'_m(x) + d_m^\top(\tau v - m)$$

so that when $\tau \to \infty$ we obtain $f'_v(x) \geq d_m^\top v$ what means that $d_m \in \partial s_x(0)$.

Next, when $\tau \to 0$ we get $f'_m(x) - d_m^\top m \leq 0$ and by (3.99) we conclude that $d_m^\top m = f'_m(x)$, what implies 2). and d_h is the one that achieves the maximum for the support function.

Proof of 3).

If $x \in \operatorname{int} \operatorname{dom} f$ and f is differentiable at x, then $\nabla f(x) \in \partial f(x)$ by Remark 3.77 and Lemma 3.82. To prove that $\nabla f(x)$ is the **only subgradient** of f at x, note that if $h \in \partial f(x)$, then, by definition,

$$f(y) - f(x) \geq h^\top(y-x) \quad \forall y$$

Substituting $y - x = td$, d being a fixed direction and t being > 0, dividing both sides of the resulting inequality by t and passing to limit as $t \to +0$, we get

$$d^\top \nabla f(x) \geq h^\top d$$

This inequality should be valid for all $d \in \mathbf{R}^n$, which is possible if and only if $h = \nabla f(x)$. $\square$

Example 3.100.

$$f(x) = |x|$$

on the axis. We have

$$\partial|x| = \begin{cases} \{-1\}, & x < 0 \\ [-1,1], & x = 0 \\ \{+1\}, & x > 0 \end{cases}$$

Example 3.101. *Note also that if x is a boundary point of the domain of a convex function, ∂f at x may be empty, as it is the case with the function*

$$f(y) = \begin{cases} -\sqrt{y}, & y \geq 0 \\ +\infty, & y < 0 \end{cases}$$

it is clear that there is no non-vertical supporting line to the epigraph of the function at the point $(0, f(0))$, and, consequently, there is no affine minorant of the function which is exact at $x = 0$

Corollary 3.102. ***(converse of Theorem 3.94)*** *Let a continuous function f be such that for **any** $x \in \operatorname{int} \operatorname{dom} f$ the subdifferential $\partial f(x)$ is **not empty**. Then f is convex.*

Proof. Let $x, y \in \operatorname{int} \operatorname{dom} f, 0 \leq \lambda \leq 1$ and $z = x + \lambda(y-x) (\in \operatorname{int} \operatorname{dom} f)$. Let $h \in \partial f(z)$, then

$$f(y) \geq f(z) + h^\top(y-z) = f(z) + (1-\lambda)h^\top(y-x)$$
$$f(x) \geq f(z) + h^\top(x-z) = f(z) - \lambda h^\top(y-x)$$

and we can get

$$\lambda f(y) + (1-\lambda) f(x) \geq f(z)$$

the proof is completed using Lemma 3.61. $\square$

3.4.2 Subgradient Calculus

As we already know from Theorem 3.94, the directional derivative $f'_h(x)$ is the support function of the subdifferential $\partial f(x)$. **This basic observation is the basis of our future developments.** We show the rule of computing subgradients of "composite" functions, like sums, superpositions, maxima, etc..

3.4.2.1 Weighted Sums

If f, g are convex functions on $\mathbf{R}^n$ and $\lambda, \mu > 0$ then the subgradient of the function $h(x) = \lambda f(x) + \mu g(x)$ satisfies

$$\partial h(x) = \lambda \partial f(x) + \mu \partial g(x) \tag{3.103}$$

for any $x \in \operatorname{int} \operatorname{dom} h$:

Proof. Let $x \in \operatorname{int} \operatorname{dom} f \cap \operatorname{int} \operatorname{dom} g$. Then for any $h \in \mathbf{R}^n$ we have by Theorem 3.94:

$$\begin{aligned} f'_h(x) &= \lambda f'_h(x) + \mu g'_h(x) \\ &= \max\left\{\lambda h^\top d_1 \mid d_1 \in \partial f(x)\right\} + \max\left\{\mu h^\top d_2 \mid d_2 \in \partial g(x)\right\} \\ &= \max\left\{h^\top (\lambda d_1 + \mu d_2) \mid d_1 \in \partial f(x), d_2 \in \partial g(x)\right\} \\ &= \max\left\{h^\top d \mid d \in \lambda \partial f(x) + \mu \partial g(x)\right\} \end{aligned}$$

Using Corollary 2.99, together with $s_x(h) := f'_h(x)$ has domain $\mathbf{R}^n$, we obtain (3.103). □

3.4.2.2 Affine Substitution

Let the function $f(y)$ be convex with $\operatorname{dom} f \subset \mathbf{R}^m$. Consider the affine operator

$$\begin{aligned} \mathcal{A} : \mathbf{R}^n &\to \mathbf{R}^m \\ x &\mapsto Ax + b \end{aligned}$$

and $\phi(x) = f(\mathcal{A}(x))$. Then for any $x \in \operatorname{dom} \phi = \{x \in \mathbf{R}^n \mid \mathcal{A}(x) \in \operatorname{dom} f\}$

$$\partial \phi(x) = A^\top \partial f(\mathcal{A}(x)) \tag{3.104}$$

Proof. if $y = \mathcal{A}(x)$ then for any $h \in \mathbf{R}^n$ we have

$$\phi'_h(x) = f'_{Ah}(y) = \max\left\{d^\top Ah \mid d \in \partial f(y)\right\} = \max\left\{d^\top h \mid d \in A^\top \partial f(y)\right\}$$

Now by Theorem 3.94 and Corollary 2.99, we get $\partial \phi(x) = A^\top \partial f(\mathcal{A}(x))$. □

3.4.2.3 Pointwise Sup

• Finite Supreme:

We first state a special case with supreme taking over **finite functions.**

Let $f = \sup_i f_i$ of a **finite** family of convex functions on $\mathbf{R}^n$. Then its subgradient at any $x \in \operatorname{int} \operatorname{dom} f$ satisfies

$$\partial f = \operatorname{Conv}\{\partial f_i \mid i \in I(x)\}$$

where $I(x) = \{i \mid f_i(x) = f(x)\}$ is the set of functions f_i which are **active** at x.

Proof. Unfortunately, the above rule does not have a "closed form". However, it allows to compute elements of the subgradient. □

Proof. let $x \in \cap_i \operatorname{int}\operatorname{dom} f_i$, and assume that $I(x) = 1, .., k$. Then for any $h \in \mathbf{R}^n$ we have by Theorem 3.94 and the definition of directional derivative:

$$f'_h(x) = \lim_{t \downarrow 0} \frac{\max f_i(x+ht) - f(x)}{t} = \max_{1 \le i \le k} f'_{i,h}(x) = \max_{1 \le i \le k} \max \left\{ h^\top d_i \mid d_i \in \partial f_i(x) \right\}$$

Note that for any numbers $a_1, \dots, a_k$, we have

$$\max a_i = \max \left\{ \sum_i \lambda_i a_i \mid \lambda \in \Delta_k \right\}$$

where $\Delta_k = \{\lambda \ge 0, \sum_i \lambda_i = 1\}$ is the standard simplex in $\mathbf{R}^k$. Thus

$$\begin{aligned} f'_h(x) &= \max_{\lambda \in \Delta_k} \sum_{i=1}^k \lambda_i \max \left\{ h^\top d_i \mid d_i \in \partial f_i(x) \right\} \\ &= \max \left\{ h^\top \left(\sum_{i=1}^k \lambda_i d_i \right) \mid d_i \in \partial f_i(x), \lambda \in \Delta_k \right\} \\ &= \max \left\{ h^\top d \mid d = \sum_{i=1}^k \lambda_i d_i, d_i \in \partial f_i(x), \lambda \in \Delta_k \right\} \\ &= \max \left\{ h^\top d \mid d \in \operatorname{Conv} \left\{ \partial f_i(x), 1 \le i \le k \right\}. \right] \end{aligned}$$

□

• **General Supreme:**

We state the general case without proof:

Lemma 3.105. *Let $f = \sup_{\alpha \in \mathcal{F}} f_\alpha$ of an **arbitrary** family $\mathcal{F}$ of convex and closed functions. Then for any x from $\operatorname{dom} f = \cap_\alpha \operatorname{dom} f_\alpha$, we have*

$$\partial f(x) \supset \operatorname{Conv} \{\partial f_\alpha(x) \mid \alpha \in \alpha(x)\}$$

*where $\alpha(x) = \{\alpha \mid f_\alpha(x) = f(x)\}$. Furthermore, if the set $\mathcal{F}$ is **compact (in some metric) and the function** $\alpha \to f_\alpha(x)$ **is closed,** then*

$$\partial f(x) = \operatorname{Conv} \{\partial f_\alpha(x) \mid \alpha \in \alpha(x)\}$$

3.4.2.4 Examples

Example 3.106. *Consider the function*

$$f(x) = \sum_{i=1}^m \left| a_i^\top x - b_i \right|$$

Let for $x \in \mathbf{R}^n$,

$$\begin{aligned} I_-(x) &= \left\{ i : a_i^\top x - b_i < 0 \right\} \\ I_+(x) &= \left\{ i : a_i^\top x - b_i > 0 \right\} \\ I_0(x) &= \left\{ i : a_i^\top x - b_i = 0 \right\} \end{aligned}$$

Then

$$\partial f(x) = - \sum_{i \in I_-(x)} a_i + \sum_{i \in I_+(x)} a_i + \sum_{i \in I_0(x)} [-a_i, a_i]$$

Note here for set I_0, each $[-a_i, a_i]$ could also be viewed as $\operatorname{Conv}\{-a_i, a_i\}$ because $|x|$ is equivalent to $\max(x, -x)$.

Example 3.107. *Consider the function*

$$f(x) = \max_{1\le i\le n} x^{(i)},$$

where $x^{(i)}$ are the components of x and denote $I(x) = \left\{i : x^{(i)} = f(x)\right\}$. Then

$$\partial f(x) = \mathrm{Conv}\left\{e_i \mid i \in I(x)\right\},$$

where e_i are the orths of the canonical basis of $\mathbf{R}^n$.

In particular, for $x = 0$ the subdifferential is the standard simplex, the convex hull of the origin and canonical orths: $\partial f(x) = \mathrm{Conv}\left\{e_i \mid 1 \le i \le n\right\}$.

• subdifferential of several vector norms:

Recall the dual norm form of norm: $f(x) = \sup_{f^D(y)\le 1} x^\top y$, where f is the norm and f^D is the dual norm. Lemma 3.105 will be used to get the subdifferential.

Example 3.108. ***(vector** l_2**-norm)*** *For the Euclidean norm*

$$f(x) = \|x\|_2 = \sup_{\|y\|_2\le 1} x^\top y,$$

we have

$$\partial f(x) = \begin{cases} \left\{\frac{x}{\|x\|_2}\right\} & \text{for } x \neq 0 \\ B_2(0,1) := \left\{x \in \mathbf{R}^n \mid \|x\|_2 \le 1\right\} & \text{for } x = 0 \end{cases}$$

Note at $x = 0$, all $y \in B_2(0,1)$ is active; while if $x \neq 0$, only one $y \in B_2(0,1)$ is active.

Example 3.109. ***(vector** l_∞**-norm)*** *For the l_∞-norm*

$$f(x) = \|x\|_\infty = \max_{1\le i\le n}\left|x^{(i)}\right| = \sup_{\|y\|_1\le 1} x^\top y$$

we have

$$\partial f(x) = \mathrm{Conv}\left\{\left\{e_i \mid i \in I_+(x)\right\} \cup \left\{-e_j \mid j \in I_-(x)\right\}\right\}$$

where $I_+(x) = \left\{i : x^{(i)} = |x|_\infty\right\}, I_-(x) = \left\{i : -x^{(i)} = |x|_\infty\right\}$. In particular, $\partial f(0) = B_1(0,1) = \left\{x \in \mathbf{R}^n \mid \|x\|_1 \le 1\right\}$

Example 3.110. ***(vector** l_1**-norm)*** *For the l_1-norm*

$$f(x) = \|x\|_1 = \sum_{i=1}^n \left|x^{(i)}\right| = \sup_{\|y\|_\infty\le 1} x^\top y$$

we have

$$\partial f(x) = \sum_{i\in I_+} e_i - \sum_{i\in I_-} e_i + \sum_{i\in I_0} [-e_i, e_i]$$

where $I_+(x) = \left\{i : x^{(i)} > 0\right\}, I_-(x) = \left\{i : x^{(i)} < 0\right\}$ and $I_0(x) = \left\{i : x^{(i)} = 0\right\}$. In particular, $\partial f(0) = B_\infty(0,1) = \left\{x \in \mathbf{R}^n \mid \|x\|_\infty \le 1\right\}$.

• matrix function:

Example 3.111. *Maximum eigenvalue of a symmetric matrix:*

$$f(x) = \lambda_{\max}(A(x)),$$

where

$$A(x) = A_0 + x_1 A_1 + \ldots + x_n A_n$$

with $m \times m$ symmetric matrices $A_1, \ldots, A_n m \times m$ and $x \in \mathbf{R}^n$. Here $\lambda_{\max}(A)$ stands for the maximal eigenvalue of A.

*We can express f as the pointwise supremum of convex functions, using **Rayleigh's variational definition of the maximal eigenvalue of a symmetric matrix** (cf. cite[Sec 4.2]horn2012matrix):*

$$\lambda_{\max}(A(x)) = \sup_{\|y\|_2=1} y^\top A(x) y.$$

*Each of the functions $f_y(x) = y^\top A(x) y$ is **affine in x for fixed y**, as can be easily seen from*

$$y^\top A(x) y = y^\top A_0 y + x_1 y^\top A_1 y + \ldots + x_n y^\top A_n y$$

so it is differentiable with gradient $\nabla f_y(x) = \left(y^\top A_1 y, \ldots, y^\top A_n y\right)$. The active functions $y^\top A(x) y$ are those associated with the eigenvectors y corresponding to the maximum eigenvalue. We have

$$\partial f(x) = \operatorname{Conv}\left\{\nabla f_y(x) \mid A(x) y = \lambda_{\max}(A(x)) y, \|y\|_2 = 1\right\}$$

3.5 Optimality Conditions

3.5.1 Minimum

3.5.1.1 General Case

Theorem 3.112. ***(unimodality: local mim $\Rightarrow$ global min)*** *Let f be a convex function on a convex set $M \subset \mathbf{R}^n$, and let $x^* \in M \cap \operatorname{dom} f$ be a **local minimizer** of f on M :*

$$(\exists\, r > 0): \quad f(y) \geq f\left(x^*\right) \quad \forall\, y \in M, \|y - x\|_2 < r$$

Then

*1). x^* is a **global minimizer** of f on M :*

$$f(y) \geq f\left(x^*\right) \quad \forall\, y \in M$$

*2). the set $\arg\min_M f$ of all local ($\equiv$ global) minimizers of f on M is **convex**.*

*3). If f is **strictly convex**, then the $\arg\min_M f$ is either **empty or is a singleton**.*

Remark 3.113. $\arg\min_M f$ *can be empty, e.g. M is open and the minimizers are on the boundary.*

Proof. Proof of 1).

Let x^* be a local minimizer of f on M and $y \in M, y \neq x^*$; we should prove that $f(y) \geq f\left(x^*\right)$. There is nothing to prove if $f(y) = +\infty$, so that we may assume that $y \in \operatorname{dom} f$. By the convexity of f, for all $\lambda \in (0,1)$ we have for $x_\lambda = \lambda y + (1-\lambda) x^*$,

$$f\left(x_\lambda\right) - f\left(x^*\right) \leq \lambda\left(f(y) - f\left(x^*\right)\right)$$

Since x^* is a local minimizer of f, the left hand side in this inequality is nonnegative for all small enough values of $\lambda > 0$. We conclude that the right hand side is nonnegative, i.e., $f(y) \geq f\left(x^*\right), \forall\, y \in \operatorname{dom} f$.

Proof of 2).

To prove convexity of $\arg\max_Q f$, note that $\arg\max_M f$ is nothing but the sublevel set $L_\alpha(f)$ of f with $\alpha = \min_M f$.

Proof of 3).

To prove that the set $\arg\max_M f$ associated with a strictly convex f is, if nonempty, a singleton, note that if there were two distinct minimizers x', x'', then, from strict convexity, we would have

$$f\left(\frac{1}{2} x' + \frac{1}{2} x''\right) < \frac{1}{2}\left[f\left(x'\right) + f\left(x''\right)\right] = \min_M f$$

which clearly is impossible since $\frac{1}{2} x' + \frac{1}{2} x'' \in M$. $\square$

Another pleasant fact is the following

Theorem 3.114. ***(necessary and sufficient condition of optimality:*** $0 \in \partial f(x^*)$***)***

$$\text{“}x^* \in \operatorname{dom} f \textit{ is the minimizer of convex function } f(x)\text{”} \Longleftrightarrow \text{“}0 \in \partial f(x^*)\text{”}$$

Proof. “$\Leftarrow$”: When $0 \in \partial f(x^*)$, by the definition of the subgradient,

$$f(x) \geq f(x^*) + 0^\top (x - x^*) = f(x^*)$$

for any $x \in \operatorname{dom} f$.

“$\Rightarrow$”: if $f(x) \geq f(x^*)$ for any $x \in \operatorname{dom} f$, then by the definition of the subgradient, $0 \in \partial f(x^*)$. □

3.5.1.2 Special Case I: Interior and Differentiable

Corollary 3.115. ***(Fermat's theorem)*** *From Theorem 3.94, for one* $x \in \operatorname{int}\operatorname{dom} f$*, and convex* f *is differentiable at* x^**, we have* $\partial f(x) = \{\nabla f(x)\}$*. We therefore have that for* ***convex*** *function* f*:*

$$x^* \in \operatorname{int}\operatorname{dom} f \textit{ is the minimizer of } f(x) \textit{ and } f \textit{ is differentiable at } x^* \Longleftrightarrow \nabla f(x^*) = 0$$

3.5.1.3 Special Case II: Differentiable (not necessarily interior)

A natural question is what happens if x^*in Corollary 3.115 is **not necessarily an interior point** of $M = \operatorname{dom} f$?

We only consider the case of convex function f on $M \subset \mathbf{R}^n$, which is **differentiable** at $x^* \in M$.

To continue we need to define a new object:

Definition 3.116. ***(Tangent Cone)*** *Let* M *be a (nonempty) convex set, and let* $x^* \in M$*. The tangent cone of* M *at* x^* *is the cone*

$$T_M(x^*) = \{h \in \mathbf{R}^n \mid x^* + th \in M \quad \forall \textit{ small enough } t > 0\}$$

Remark 3.117. ***(explanation)*** *Geometrically, this is the set of all directions leading from* x^* *inside* M*, so that a* ***small enough positive step*** *from* x^* *along the direction keeps the point in* M*. From the convexity of* M *it immediately follows that the tangent cone indeed is a (convex) cone (not necessary closed).*

Example 3.118. *If* $x^* \in \operatorname{int} M$*, we have* $T_M(x^*) = \mathbf{R}^n$*.*

Example 3.119. ***(tangent cone to a polyhedral set)*** *Let polyhedral be*

$$M = \{x \mid Ax \leq b\} = \left\{x \mid a_i^\top x \leq b_i, i = 1, \ldots, m\right\}$$

For $x^* \in M$*, the corresponding tangent cone clearly is the* ***polyhedral cone***

$$T_M(x^*) = \left\{h \mid a_i^\top h \leq 0, \forall i : a_i^\top x^* = b_i\right\} \tag{3.120}$$

where a_i *is active at* x^* *(i.e.* $a_i^\top x^* = b_i$*). Note, for strict inequality* $a_j^\top x^* < b_j$*, we have* $\mathbf{R}^n$ *satisfies this* j*.*

Theorem 3.121. ***(necessary and sufficient condition for optimality of differentiable function)***

$$x^* \in \operatorname{dom} f \textit{ is the minimizer of } f(x) \textit{ and } f \textit{ is differentiable at } x^* \Longleftrightarrow h^\top \nabla f(x^*) \geq 0, \forall h \in T_M(x^*)$$

Proof. “$\Rightarrow$”: This is an evident fact which has nothing in common with convexity. Assume that x^* is a local minimizer of f on M, we note that if there were $h \in T_M(x^*)$ with $h^\top \nabla f(x^*) < 0$, then from Definition 3.76 we would have

$$f(x^* + th) < f(x^*)$$

for all small enough positive t. On the other hand, $x^* + th \in M$ for all small enough positive t due to $h \in T_M(x^*)$. We conclude that in every neighborhood of x^* there are points from M with strictly smaller than $f(x^*)$; this contradicts the assumption that x^* is a local minimizer of f on M.

“$\Leftarrow$”: If $x \in M$, then $h = x - x^* \in T_M(x^*)$. From Theorem 3.94 and (3.91), we have

$$f(x) \geq f(x^*) + (x - x^*)\nabla f(x^*) \geq f(x^*)$$

□

Corollary 3.122. ***(a restatement of Theorem 3.121)*** *Recall the definition of dual cone in Definition 2.118, we have*

$$\begin{aligned}
x^* \in \operatorname{dom} f \textbf{\textit{ is the minimizer of f(x) and }} f \textbf{\textit{ is differentiable at }} x^* \\
\Longleftrightarrow h^\top \nabla f(x^*) \geq 0, \forall h \in T_M(x^*) \\
\Longleftrightarrow \nabla f(x^*) \in \textbf{\textit{ dual cone of }} T_M(x^*)
\end{aligned}$$

Remark 3.123. ***(a restatement of Fermat's theorem)*** *If x^* is an interior point of M, we know from Example 3.118 that $T_M(x^*) = \mathbf{R}^n$, Corollary 3.115 is a direction result from the fact:*

$$h^\top \nabla f(x^*) \mathbf{R}^n \geq 0, \forall h \in \mathbf{R}^n \Longleftrightarrow \nabla f(x^*) = 0.$$

3.5.1.4 Special Case III: domain is the polyhedral set

When M is the polyhedral set in Example 3.119, the tangent cone is

$$T_M(x^*) = \left\{ h \mid a_i^\top h \leq 0, \forall i : a_i^\top x^* = b_i \right\} \tag{3.124}$$

The dual is comprised of all vectors which have nonnegative inner products with all these directions, i.e., of all vectors a such that " $h^\top a_i \leq 0, \forall i \in I(x^*) \equiv \left\{ i \mid a_i^\top x^* = b_i \right\} \Rightarrow$ "$h^\top a \geq 0$"

From the Homogeneous Farkas Corollary 2.48, we conclude that the dual cone of $T_M(x^*)$ is simply the conic hull of the vectors $-a_i, i \in I(x^*)$:

$$\textbf{dual cone of } T_M(x^*) = \left\{ z \in \mathbf{R}^n \mid z = -\sum_{i \in I(x^*)} \lambda_i a_i, \lambda_i \geq 0 \right\}$$

Thus, from Corollary 3.122 we have

Corollary 3.125. *Let M is the polyhedral set in Example 3.119, we have:*

$$\begin{aligned}
x^* \in M \textbf{\textit{ is the minimizer of f(x) and }} f \textbf{\textit{ is differentiable at }} x^* \\
\Longleftrightarrow \nabla f(x^*) \in \textbf{\textit{ dual cone of }} T_M(x^*) \\
\Longleftrightarrow \exists\, \lambda_i^* \geq 0, \nabla f(x^*) + \sum_{i \in I(x^*)} \lambda_i^* a_i = 0
\end{aligned}$$

3.5.1.5 A Summary

Sufficient and necessary condition for $x^* \in \operatorname{dom} f$ is the minimizer of convex function $f(x)$:

- **in general:**

$$0 \in \partial f(x^*)$$

  - **if differentiable:**

$$h^\top \nabla f(x^*) \geq 0, \forall h \in T_M(x^*)$$
$$\text{i.e., } \nabla f(x^*) \in \text{ dual cone of } T_M(x^*)$$

    - **if differentiable and interior:**

$$\nabla f(x^*) = 0 \text{ (since } T_M(x^*) = \mathbf{R}^n\text{)}$$

      - **if differentiable, interior, domain is polyhedral set:**

$$\lambda_i^* \geq 0, \nabla f(x^*) + \sum_{i \in I(x^*)} \lambda_i^* a_i = 0, \text{ where } I(x^*) \equiv \left\{ i \mid a_i^\top x^* = b_i \right\} \tag{3.126}$$

3.5.1.6 Support Vectors for Sublevel Sets

The next result is of main importance in optimization, it is the basis of the cutting plane scheme we consider in the sequel.

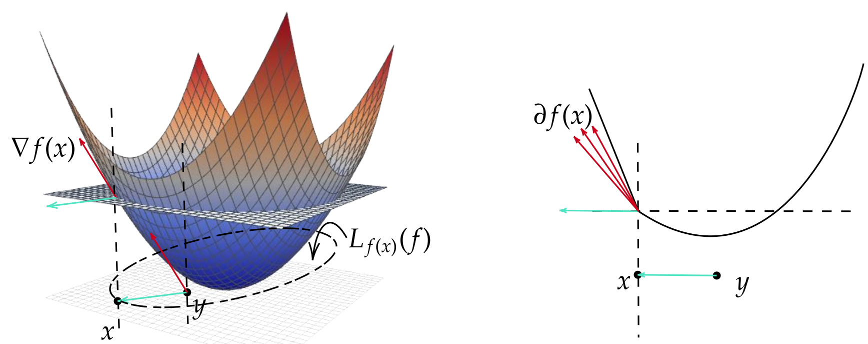
Theorem 3.127. ***(support vectors for level sets)*** *For any $x \in \operatorname{dom} f$, all vectors $d \in \partial f(x)$ satisfy*

$$d^\top (x - y) \geq 0, \quad \textit{for any } y \in L_{f(x)}(f)$$

where sublevel set $L_{f(x)}(f) = \{y \in \operatorname{dom} f \mid f(y) \leq f(x)\}$.

Remark 3.128. *We say that such vectors d are **supporting to** the set $L_{f(x)}(f)$ at x.*

Proof. If $f(y) \leq f(x)$ and $d \in \partial f(x)$, then $f(x) + d^\top (y - x) \leq f(y) \leq f(x)$. □



(a). 3-D case (b). 2-D case

Fig. 17: Visualization of Theorem 3.127.

Corollary 3.129. *Let $M \subset \operatorname{dom} f$ be a convex closed set, $x \in M$ and*

$$x^* = \underset{x \in M}{\operatorname{argmin}} f(x)$$

Then since x^* must in all $L_{f(x)}(f)$, we have:

$$d^\top (x - x^*) \geq 0, \forall x \in M, d \in \partial f(x)$$

3.5.1.7 Karush-Kuhn-Tucker: constrained minimum

Theorem 3.130. ***(Karush-Kuhn-Tucker)*** *Let $f, g_i, i = 1, \ldots, m$ be **differentiable** convex functions (implies open domains from definition). Suppose that there exists a point $\bar{x}$ (open implies interior point) such that*

$$\textbf{\textit{Slater's condition:}}\ g_i(\bar{x}) < 0, \forall i = 1, \ldots, m$$

*A point x^* is a solution to the **convex optimization problem***

$$\begin{aligned} &\operatorname{minimize} f(x) \\ &\textit{subject to } g_i(x) \leq 0, \quad i = 1, \ldots, m \end{aligned} \tag{3.131}$$

if and only if ① *there exist **nonnegative real** $\lambda_i, i = 1, \ldots, m$ such that*

$$\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x^*) = 0 \tag{3.132}$$

and

$$② \textbf{\textit{ complementary slackness:}} \quad \lambda_i g_i(x^*) = 0, \quad i = 1, \ldots, m \tag{3.133}$$

Remark 3.134. ***(equivalent restatement)*** *Let $I^* = \{1 \le i \le m \mid g_i(x^*) = 0\}$ be the active constraints, from complementary slackness* (3.133)*, we have that **if** g_i **not active,** $\lambda_i = 0$.*

① and ② can be written as:

$$\nabla f(x^*) + \sum_{i \in I^*} \lambda_i \nabla g_i(x^*) = 0, \lambda_i \ge 0, \forall i \in I^* \tag{3.135}$$

Remark 3.136. ***(a new explanation of polyhedral set Corollary 3.125)*** (3.126) *is just the KKT of optimization problem* (3.131) *with g_i being the outer description of a polyhedral set.*

Here we have a question, in (3.126)*, domain is the polyhedral set M in Example 3.119, however here domain is $M \cap \operatorname{dom} f$. It seems*

- *if the constraint g_i are **linear(affine)** functions, the shape of* $\operatorname{dom} f$ *does not matter.*
- *from weak Slater's condition* (4.48)*, we know if all g_i are **affine**, no Slater's condition is needed.*

Remark 3.137. (3.131) *is a special case of our general* (4.2) *defined later. Also the KKT theorem is a special case of the general case Section 4.7.3. However, KKT theorem shown here has presented full methodology.*

- *Here we directly construct the following function $\phi(x)$ and prove:*

  Under "Slater's condition + convex": "x^*, λ_i^* satisfies KKT" $\Longleftrightarrow$ "x^* is optimal". (3.138)

- *In Sections 4.5.2 and 4.7.3, we follows this line: "Slater's condition + convex" $\Longrightarrow$ "strong duality" $\Longrightarrow$ KKT. Conversely, we also show "KKT + convex" $\Longrightarrow$ "strong duality".*

  Of course we achieve conclusion (3.138) *again.*

Proof. We reduce the constrained convex optimization problem (3.131) to an **unconstrained though not smooth optimization problem** $\phi(x)$ below in (3.140). In this proof we implicitly constraints x to the domain of (3.131), i.e. $\cap_i \operatorname{dom} g_i \cap \operatorname{dom} f$.

Consider the parametric (with parameter x) function with max over $m+1$ values:

$$m(t;x) = \max\{f(x) - t, g_i(x), i = 1, \ldots, m\}$$

and the function with min over all x:

$$h(t) = \min_x m(t;x).$$

Lemma 3.139. *Let f^* be the optimal value of the optimization problem* (3.131)*. Then*

$$h(t) \le 0 \text{ for all } t \ge f^*$$
$$h(t) > 0 \text{ for all } t < f^*$$

Proof. of Lemma 3.139

Let x^* be an optimal solution to (3.131). If $t \ge f^* = f(x^*)$ then

$$h(t) \le m(t;x^*) = \max\{f(x^*) - t; g_i(x^*)\} \le 0$$

On the other hand, if $t < f^*$, assume $h(t) \le 0$ then there exists $x \in \mathbf{R}^n$ such that

$$f(x) \le t < f^*, \text{ and } g_i(x) \le 0, \quad i = 1, \ldots, m$$

Thus, f^* cannot be the optimal value of (3.131), a contradition. □

After Lemma 3.139, we next turn back to prove Theorem 3.130. Define the **convex** function

$$\phi(x) = \max\{f(x) - f^*; g_i(x), i = 1, \ldots, m\} \tag{3.140}$$

From Lemma 3.139, we know

$$x^* \text{ is an optimal solution of (3.131)} \Longleftrightarrow x^* \text{ is a global minimizer of } \phi(x)$$

"$\Rightarrow$": $\forall\, y \neq x^*$, $f(y) - f(x^*) \geq 0$, $g(y) \leq 0$. We therefore have $\phi(y) \geq 0$. Since $\phi(x^*) = 0$. We have x^* is the minimizer of $\phi(x)$.

"$\Leftarrow$": from Lemma 3.139, we have $h(f^*) \leq 0$, i.e. $\min_x m(f^*, x) = \min_x \phi(x) = \phi(x^*) \leq 0$. So $f(x^*) \leq f^*$.

We have from Theorem 3.114 that this is the case if and only if $0 \in \partial\phi\,(x^*)$. From pointwise sup of subgradient calculus in Section 3.4.2, we have

$$0 \in \operatorname{Conv}\{\nabla f, \nabla g_i \mid i \in I(x^*)\},$$

which means

$$\mu_0 \nabla f\,(x^*) + \sum_{i\in I^*} \mu_i \nabla g_i\,(x^*) = 0, \quad \mu \geq 0, \quad \mu_0 + \sum_{i\in I^*} \mu_i = 1$$

where $I^* = \{1 \leq i \leq m \mid g_i\,(x^*) = 0\}$.

We next exclude the case $\mu_0 = 0$. Assume if we had $\mu_0 = 0$ then $\sum_{i\in I^*} \mu_i \nabla g_i\,(x^*) = 0$ and

$$\sum_{i\in I^*} \mu_i g_i(\bar{x}) \geq \sum_{i\in I^*} \mu_i \left[g_i\,(x^*) + \nabla g_i\,(x^*)^\top (\bar{x} - x^*) \right] = 0$$

what is a contradiction.

Therefore, $\mu_0 \neq 0$ and we can set $\lambda_i = \mu_i/\mu_0$ for $i \in I$ to get (3.135), which is equivalent to 1 and 2 as mentioned in Remark 3.134. $\square$

3.5.2 Maximum

Theorem 3.112 demonstrate that the fact that a point $x^* \in \operatorname{dom} f$ is a global minimizer of a convex function f depends only on the local behavior of f at x^*. This is not the case with maximizers of a convex function. In fact, such a maximizer, if exists, in all nontrivial cases should **belong to the boundary of the domain of the function**:

Theorem 3.141. ***(interior is maximizer $\Leftrightarrow$ f is constant)*** *Let f be convex, and let M be the domain of f. Assume that f attains its maximum on M at a point x^* from the relative interior of M. Then f is constant on M.*

Proof. The proof is similar to the proof of Lemma 2.93. We use the notation there, with $\bar{x}$ being the maximizer. (2.95) is written as

$$f(\bar{x}) \leq (1-\lambda) f(y) + \lambda f(z). \tag{3.142}$$

For $0 < \lambda < 1$, we would get $f(\bar{x}) < \max\{f(y), f(z)\}$ if $f(y) < f(\bar{x})$ or $f(z) < f(\bar{x})$, which is not possible. See also Remark 3.43 for the a similar reason. $\square$

The next two theorems give further information on **maxima** of convex functions:

Theorem 3.143. ***(max over set = max over its convex hull)*** *Let f be a convex function on $\mathbf{R}^n$ and E be a subset of $\mathbf{R}^n$. Then*

$$\sup_{\operatorname{Conv} E} f = \sup_E f \tag{3.144}$$

In particular, if $S \subset \mathbf{R}^n$ is convex and compact set, then the upper bound of f on S is equal to the upper bound of f on the set of extreme points of S :

$$\sup_S f = \sup_{\operatorname{Ext}(S)} f \tag{3.145}$$

Proof. To prove (3.144), we only need to prove "$\leq$". Let $x \in \operatorname{Conv} E$, so that x is a convex combination of points from E:

$$x = \sum_i \lambda_i x_i \quad \left[x_i \in E, \lambda_i \geq 0, \sum_i \lambda_i = 1\right]$$

Applying Jensen's inequality, we get

$$f(x) \leq \sum_i \lambda_i f(x_i) \leq \sum_i \lambda_i \sup_E f = \sup_E f.$$

We therefore have $\sup_{\operatorname{Conv} E} f \leq \sup_E f$.

(3.145) follows from (3.144) and KreinMilman Theorem (Theorem 2.115). $\square$

Theorem 3.146. ***(maxmizer "is" extreme point)*** *Let f be a convex function such that the domain M of f is* ***closed and does not contain lines.*** *If the set $\arg\max_M f(x)$ of global maximizers of f is* ***nonempty****, then it intersects the set $\operatorname{Ext}(M)$ of the extreme points of M. In other words,* ***at least one of the maximizers of f is an extreme point of M.***

Proof. We will prove this statement by **induction on the dimension of** M. The base $\dim M = 0$, i.e., the case of a singleton M, is trivial, since here $M = \operatorname{Ext} M = \arg\max_M f$. Now assume that the statement is valid for the case of $\dim M \leq p$. Let us prove that it is valid also for the case of $\dim M = p+1$.

We first verify that the set $\arg\max_M f \cap \partial_{\mathrm{ri}} M \neq \emptyset$. Indeed, let $x \in \arg\max_M f$. There is nothing to prove if x itself is a relative boundary point of M; and if x is not a boundary point, then f is constant on M, so that $\arg\max_M f = M$. Since M is closed, any relative boundary point of M (Theorem 2.115 such a point does exist, since M does not contain lines and is of positive dimension) is a maximizer of f on M, so that here again $\arg\max_M f$ intersects $\partial_{\mathrm{ri}} M$.

We next prove $\operatorname{Ext}(M) \cap \arg\max_M f \neq \emptyset$**.** Select one $x \in \arg\max_M f \cap \partial_{\mathrm{ri}} M$. Let H be the hyperplane which properly supports M at x, and let $M' = M \cap H$. The set M' is closed and convex (since M and H are), nonempty (it contains x) and does not contain lines (since M does not). We have $\max_M f = f(x) = \max_{M'} f$ (note that $M' \subset M$), therefore

$$\emptyset \neq \arg\max_{M'} f \subset \arg\max_M f$$

Same as in the proof of the Krein-Milman Theorem (Theorem 2.115), we have $\dim M' < \dim M$. In view of this inequality we can apply to f and M' our inductive hypothesis to get

$$\operatorname{Ext}(M') \cap \arg\max_{M'} f \neq \emptyset$$

Since $\operatorname{Ext}(M') \subset \operatorname{Ext}(M)$ by Lemma 2.114 and, as we just have seen, $\arg\max_{M'} f \subset \arg\max_M f$, we conclude that the set $\operatorname{Ext}(M) \cap \arg\max_M f$ is not smaller than $\operatorname{Ext}(M') \cap \arg\max_{M'} f$ and is therefore nonempty, as required. $\square$

3.6 Legendre transformation and Proper functions

3.6.1 Convex vs. Supremum of Affine Functions

We have mentioned partially in Remark 3.50 that convex function can be expressed as the pointwise supremum of a family of affine functions (i.e.non-vertical hyperplane). In this section, we statement the general form with $\operatorname{dom} f$ not necessarily equals $\mathbf{R}^n$.

Theorem 3.147. ***(convex $f \approx \sup$ of affine functions)*** *Let $f : \mathbf{R}^n \to \mathbf{R}$ be a convex function. Define $\tilde{f} : \mathbf{R}^n \to \mathbf{R}$ as the pointwise supremum of all* ***affine functions that are global underestimators (i.e. affine minorants)*** *of f :*

$$\tilde{f}(x) = \sup\{g(x) \mid g \text{ is affine, } g(z) \leq f(z) \text{ for all } z\} \tag{3.148}$$

1). $f(x) = \tilde{f}(x)$ for $x \in \operatorname{int} \operatorname{dom} f$.

2). $f = \tilde{f}$ if f is closed.

Remark 3.149. ***(convex** $f = \sup$ **of affine functions?)*** *2). is saying convex $f = \sup$ of affine minorants iff f is* ***closed****.*

Remark 3.150. ***(subgradient: new view)*** *Now we can know subgradient in (3.91) is also just affine minorant, but it is speical one which touch the $\mathrm{Epi}\, f$ with $(f(x), x)$.*

Proof. Proof of 1).

The point $(x, f(x))$ is in the boundary of $\mathrm{Epi}\, f$. We know there is a supporting hyperplane to $\mathrm{Epi}\, f$ at $(x, f(x))$, i.e., $\exists\, a \in \mathbf{R}^n, b \in \mathbf{R}$ such that

$$a^\top z + bt \geq a^\top x + bf(x) \text{ for all } (z, t) \in \mathrm{Epi}\, f$$

Since t can be arbitrarily large if $(z, t) \in \mathrm{Epi}\, f$, we conclude that $b \geq 0$.

Suppose $b = 0$. Then

$$a^\top z \geq a^\top x \text{ for all } z \in \mathrm{dom}\, f$$

which contradicts $x \in \mathrm{int\,dom}\, f$. **Therefore $b > 0$, and we exclude the vertical hyperplanes.**

Dividing the above inequality by b yields

$$t \geq f(x) + (a/b)^\top (x - z) \text{ for all } (z, t) \in \mathrm{Epi}\, f$$

Therefore the affine function

$$g(z) = f(x) + (a/b)^\top (x - z)$$

is an affine global underestimator of f, and hence by definition of $\tilde{f}$

$$f(x) \geq \tilde{f}(x) \geq g(x).$$

However $g(x) = f(x)$, so we must have $f(x) = \tilde{f}(x)$.

Proof of 2).

A closed convex set is the intersection of all halfspaces that contain it . We will apply this result to $\mathrm{Epi}\, f$. Define

$$H = \left\{ (a, b, c) \in \mathbf{R}^{n+2} \mid (a, b) \neq 0, \inf_{(x,t) \in \mathrm{Epi}\, f} \left(a^\top x + bt \right) \geq c \right\}$$

Loosely speaking, H is the set of all halfspaces **(inclusing vertical ones)** that contain $\mathrm{Epi}\, f$. By the result in chapter 2 ,

$$\mathrm{Epi}\, f = \bigcap_{(a,b,c) \in H} \left\{ (x, t) \mid a^\top x + bt \geq c \right\}$$

It is clear that all elements of H satisfy $b \geq 0$. **We need to prove that**

$$\mathrm{Epi}\, f = \bigcap_{(a,b,c) \in H, b > 0} \left\{ (x, t) \mid a^\top x + bt \geq c \right\}$$

(In words, $\mathrm{Epi}\, f$ is the intersection of all "non-vertical" halfspaces that contain $\mathrm{Epi}\, f$.) Note that H may contain elements with $b = 0$, so this does not immediately follow from 1). We will show that

$$\bigcap_{(a,b,c) \in H, b > 0} \left\{ (x, t) \mid a^\top x + bt \geq c \right\} = \bigcap_{(a,b,c) \in H} \left\{ (x, t) \mid a^\top x + bt \geq c \right\}$$

It is obvious of "$\supseteq$". To show that they are identical, assume $(\bar{x}, \bar{t})$ lies in the set on the left, i.e.,

$$a^\top \bar{x} + b\bar{t} \geq c$$

for all halfspaces $a^\top x + bt \geq c$ that are nonvertical (i.e., $b > 0$) and contain $\mathrm{Epi}\, f$. Assume that $(\bar{x}, \bar{t})$ is not in the set on the right, i.e., there exist $(\tilde{a}, \tilde{b}, \tilde{c}) \in H$ (necessarily with $\tilde{b} = 0$), such that

$$\tilde{a}^\top \bar{x} < \tilde{c}$$

H contains at least one element (a_0, b_0, c_0) with $b_0 > 0$. (Otherwise $\operatorname{Epi} f$ would be an intersection of vertical halfspaces.) Consider the halfspace defined by $(\tilde{a}, 0, \tilde{c}) + \epsilon\,(a_0, b_0, c_0)$ for small positive ϵ. This halfspace is nonvertical and it contains $\operatorname{Epi} f$:

$$(\tilde{a} + \epsilon a_0)^\top x + \epsilon b_0 t = \tilde{a}^\top x + \epsilon\left(a_0^\top x + b_0 t\right) \geq \tilde{c} + \epsilon c_0$$

for all $(x, t) \in \operatorname{Epi} f$, because the halfspaces $\tilde{a}^\top x \geq \tilde{c}$ and $a_0^\top x + b_0 t \geq c_0$ both contain $\operatorname{Epi} f$. However,

$$(\tilde{a} + \epsilon a_0)^\top \bar{x} + \epsilon b_0 \bar{t} = \tilde{a}^\top \bar{x} + \epsilon\left(a_0^\top \bar{x} + b_0 \bar{t}\right) < \tilde{c} + \epsilon c_0$$

for small ϵ, so the halfspace does not contain $(\bar{x}, \bar{t})$. This contradicts our assumption that $(\bar{x}, \bar{t})$ is in the intersection of all nonvertical halfspaces containing $\operatorname{Epi} f$. We have completed the proof. □

3.6.2 Closure of Convex Function

Definition 3.151. ***(Proper Function)*** *A function f is **proper** if $\operatorname{Epi} f$ is nonempty, closed and convex, i.e. f is convex and closed.*

From Theorem 3.147 2)., we got a nice result on the **"outer description"** of a proper convex function: it is the upper bound of a family of affine functions. Note that, vice versa, the upper bound of every family of affine functions is a proper function, provided that this upper bound is finite at least at one point since we know superme of lower semicontinuous functions (e.g., affine ones) is still lower semicontinuous, i.e. closed.

If a convex function f is not proper (i.e., its epigraph is not closed), we can "correct" the function by replacing it with a new function with the epigraph being the closure of $\operatorname{Epi} f$

Definition 3.152. ***(Closure of Convex Function)*** *For convex function f, we define the **closure of f**, denoted as $\operatorname{cl} f$, as the new proper function s.t. $\operatorname{Epi} \operatorname{cl} f = \operatorname{cl} \operatorname{Epi} f$.*

Remark 3.153. ***(justification)*** *We should be sure that the closure of the epigraph of a convex function is also an epigraph of such a function. To see it, it suffices to note that a set G in $\mathbf{R}^{n+1}$ is the epigraph of a function taking values in $\mathbf{R} \cup \{+\infty\}$ if and only if the intersection of G with every vertical line $\{x = const,\ t \in \mathbf{R}\}$ is either 1 empty, or is 2 a closed ray of the form $\{x = const,\ t \geq \bar{t} > -\infty\}$.*

Now, it is absolutely evident that if G is the closure of the epigraph of a function f, that its intersection with a vertical line is either 1 empty, or is 2 a closed ray, or is 3 the entire line (the last case indeed can take place, e.g. look at the closure of the epigraph of the function equal to $-\frac{1}{x}$ for $x > 0$ and $+\infty$ for $x \leq 0$).

We see that in order to justify our idea of "proper correction" of a convex function we should prove that if f is convex, then the last case 3 never happens. This fact is evident from the following corollary.

Corollary 3.154. ***(below boundedness: repeat of Corollary 3.86)***
*f is below bounded on **any bounded subset** of $\mathbf{R}^n$*

Proof. Without loss of generality we may assume that the domain of the function f is full-dimensional and that $0 \in \operatorname{int} \operatorname{dom} f$. Theorem 3.72 says that a convex function f is **bounded on every compact subset of** $\operatorname{ri} \operatorname{dom} f$.

So here, for some radius $r > 0$, f is bounded from above by some C over the ball $B_r(0)$. Now, if $R > 0$ is arbitrary and x is an arbitrary point with $|x| \leq R$, then the point

$$y = -\frac{r}{R} x$$

belongs to $B_r(0)$, and we have

$$0 = \frac{r}{r+R} x + \frac{R}{r+R} y$$

since f is convex, we conclude that

$$f(0) \leq \frac{r}{r+R} f(x) + \frac{R}{r+R} f(y) \leq \frac{r}{r+R} f(x) + \frac{R}{r+R} C$$

and we get the lower bound

$$f(x) \geq \frac{r+R}{r} f(0) - \frac{r}{R} C$$

for the values of f in the centered at 0 ball of radius R. $\square$

The following statement gives direct description of $\operatorname{cl} f$ in terms of f:

Corollary 3.155. *Let f be a convex function and $\operatorname{cl} f$ be its closure. Then*

1). *For every x one has*

$$\operatorname{cl} f(x) = \lim_{r \to 0+} \inf_{x' : \|x' - x\|_2 \leq r} f(x')$$

In particular,

$$f(x) \geq \operatorname{cl} f(x), \forall x \in \mathbf{R}^n$$

We have

$$f(x) = \operatorname{cl} f(x), \forall x \in \operatorname{ri} \operatorname{dom} f \text{ and } \forall x \notin \operatorname{cl} \operatorname{dom} f.$$

and $\operatorname{cl} f$ may vary f only at the points from the relative boundary of $\operatorname{dom} f$. We also have

$$\operatorname{dom} f \subset \operatorname{dom} \operatorname{cl} f \subset \operatorname{cl} \operatorname{dom} f$$

and

$$\operatorname{ri} \operatorname{dom} f = \operatorname{ri} \operatorname{dom} \operatorname{cl} f$$

2). *The family of affine minorants of $\operatorname{cl} f$ is exactly the family of affine minorants of f, so that*

$$\operatorname{cl} f(x) = \tilde{f}(x) := \sup\{g(x) : g \text{ is an affine minorant of } f\}$$

and the sup in the right hand side can be replaced with max whenever $x \in \operatorname{ri} \operatorname{dom} \operatorname{cl} f = \operatorname{ri} \operatorname{dom} f$.

Proof. 1). is obvious. We only prove that 2).:

$$\operatorname{cl} f(x) = \tilde{f}(x).$$

We first note that $\operatorname{Epi} \tilde{f}$ is closed and containing $\operatorname{Epi} f$, and therefore contains $\operatorname{cl} f$ which is the smallest closed set that containing $\operatorname{Epi} f$. That is to say all affine minorants of f are still affine minorants of $\operatorname{cl} f$. All affine minorants of $\operatorname{cl} f$ is clearly affine minorants of f (since $f \geq \operatorname{cl} f$, i.e. $\operatorname{cl} \operatorname{Epi} f \supseteq \operatorname{Epi} f$). We therefore have $\operatorname{cl} f = \widetilde{\operatorname{cl} f} = \tilde{f}$ from Theorem 3.147 2). since $\operatorname{cl} f$ is closed. $\square$

3.6.3 Legendre Transformation (Conjugate)

3.6.3.1 Definitions

If the slope d of an affine function $d^\top x - a$ is fixed, we have

$$\begin{aligned}
&\text{An affine function } d^\top x - a \text{ is an affine minorant of } f \\
&\Longleftrightarrow f(x) \geq d^\top x - a \text{ for all } x \\
&\Longleftrightarrow a \geq d^\top x - f(x) \text{ for all } x \\
&\Longleftrightarrow a \geq \sup_{x \in \operatorname{dom} f} \left[d^\top x - f(x)\right] \qquad (3.156)
\end{aligned}$$

Definition 3.157. ***(Legendre Transformation)*** *The supremum in the right hand side of the latter relation is certain function of d; this function is called the **Legendre transformation (conjugate)** of f (not necessarily convex) and is denoted f^* :*

$$f^*(d) := \sup_{x \in \operatorname{dom} f} \left[d^\top x - f(x)\right] \qquad (3.158)$$

Remark 3.159. ***(explanation)*** *Geometrically, the Legendre transformation answers the following question: given a slope d of an affine function, i.e., given the hyperplane $t = d^\top x$ in $\mathbf{R}^{n+1}$, what is the* ***minimal "shift down"*** *of the hyperplane which places it below the graph of f?*

*$f^*(d)$ is ① **convex and** ② **closed**. and ③ **nonempty**.*

*If f is **convex**, we have a equivalent form*

$$\sup_{x\in\mathbf{R}^n}\left[d^\top x - f(x)\right] = \sup_{x\in\operatorname{dom} f}\left[d^\top x - f(x)\right] \tag{3.160}$$

Note, in general Legendre Transformation is defined even when f is not convex, note also f^* is always convex.

The most elementary (and the most fundamental) fact about the Legendre transformation is its symmetry:

3.6.3.2 Legendre Transformation Dual Theorem

Theorem 3.161. ***(Legendre Transformation Dual Theorem)*** *Let f be a convex function. Then twice taken Legendre transformation of f is the closure cl f of f :*

$$(f^*)^* = \mathrm{cl} f$$

In particular, if f is closed, then it is the Legendre transformation of its Legendre transformation (which also is closed).

Proof. The Legendre transformation of f^* at the point x is, by definition,

$$\sup_{d\in\mathbf{R}^n}\left[x^\top d - f^*(d)\right] = \sup_{d\in\mathbf{R}^n, a\geq f^*(d)}\left[d^\top x - a\right] \tag{3.162}$$

the second sup here is exactly the supremum of all affine minorants of f: this is (3.156) of the Legendre transformation, $a \geq f^*(d)$ if and only if the affine form $d^\top x - a$ is a minorant of f. And we already know that the upper bound of all affine minorants of f is the closure of f from Corollary 3.155 2). □

3.6.3.3 Conclusions and Applications

The Legendre transformation a "global" transformation, so that **local properties of** f^* correspond to **global properties of** f:

- $d = 0$ belongs to the domain of $f^* \iff f$ is below bounded. If it is the case, then $f^*(0) = -\inf f$.
- if f is closed, then the subgradient of f^* at 0 are exactly the minimizers of f on $\mathbf{R}^n$: "Since $f^*(d) \geq f^*(0) + s^\top d$, we have $\inf f = -f^*(0) \geq s^\top d - f^*(d)$. Then $\inf f \geq \sup_d s^\top d - f^*(d) = f(s)$ from (3.162)."
- $\operatorname{dom} f^*$ is the entire $\mathbf{R}^n$ if and only if $f(x)$ grows, as $|x| \to \infty$, faster than $|x|$: i.e. there exists a function $r(t) \to \infty$, as $t \to \infty$ such that

$$f(x) \geq r(|x|) \quad \forall\, x$$

Thus, whenever we can compute explicitly the Legendre transformation of f, we get a lot of "global" information on f.

Several simple facts and examples:

Corollary 3.163. ***(Fenchel's inequality)*** *From the definition of Legendre transformation,*

$$f(x) + f^*(d) \geq x^\top d \quad \forall\, x, d$$

Specifying here f and f^*, we get certain inequality, e.g., the following one:

Corollary 3.164. ***(Young's Inequality)*** *If p and q are positive reals such that $\frac{1}{p}+\frac{1}{q}=1$, then*

$$\frac{|x|^p}{p}+\frac{|d|^q}{q}\geq xd \quad \forall x,d\in\mathbf{R}$$

Proof. The Legendre transformation of the function $|x|^p/p$ is $|d|^q/q$:

$$dx-\frac{x^p}{p}\Longrightarrow d=x^{p-1}\Longrightarrow f^*(d)=d^{\frac{p}{p-1}}\frac{p-1}{p}=\frac{d^q}{q} \tag{3.165}$$

□

Very simple-looking Young's inequality gives rise to a very nice and useful Hölder inequality:

Corollary 3.166. ***(Hölder inequality)*** *Let $1\leq p\leq\infty$ and let q be such $\frac{1}{p}+\frac{1}{q}=1(p=1\Rightarrow q=\infty,p=\infty\Rightarrow q=1)$. For every two vectors $x,y\in\mathbf{R}^n$ one has*

$$\sum_{i=1}^{n}|x_iy_i|\leq\|x\|_p\|y\|_q \tag{3.167}$$

Proof. If p or q is ∞, the inequality becomes the evident relation

$$\sum_i|x_iy_i|\leq\left(\max_i|x_i|\right)\left(\sum_i|y_i|\right)$$

Now let $1<p<\infty$, so that also $1<q<\infty$. In this case we should prove that

$$\sum_i|x_iy_i|\leq\left(\sum_i|x_i|^p\right)^{1/p}\left(\sum_i|y_i|^q\right)^{1/q}$$

If one of the factors in the right hand side vanishes; thus, we can assume that $x\neq0$ and $y\neq0$. Now, both sides of the inequality are of homogeneity degree 1 with respect to x (when we multiply x by t, both sides are multiplied by $|t|$), and similarly with respect to y. Multiplying x and y by appropriate reals, we can make both factors in the right hand side equal to $1:\|x\|_p=\|y\|_p=1$.

Now we should prove that under this **normalization** the left hand side in the inequality is ≤1, which is immediately given by the Young inequality:

$$\sum_i|x_iy_i|\leq\sum_i\left(|x_i|^p/p+|y_i|^q/q\right)=1/p+1/q=1$$

□

With $p=q=2$, we have

Corollary 3.168. ***(Cauchy inequality)***

$$\left|x^\top y\right|\leq\|x\|_2\|y\|_2$$

Definition 3.169. ***(Dual Norm)*** *For any norm $\|\cdot\|$, the **dual norm** $\|d\|_*$ is defined as*

$$\|d\|_*=\sup\left\{d^\top x\mid\|x\|\leq1\right\}$$

Remark 3.170. *For the general verification that the dual norm is a norm, see [3]. Below we give proof for the special case p-norm $\|\cdot\|_p$ in Corollary 3.171.*

Corollary 3.171. *To get equality in* (3.167)*, we have $\forall\,x$, $\exists\,y$ with $\|y\|_q=1$ such that*

$$x^\top y=\|x\|_p \quad \left(=\|x\|_p\|y\|_q\right).$$

From (3.165)*, we know it suffices to take*

$$y_i=|x|_p^{1-p}\,|x_i|^{p-1}\operatorname{sign}(x_i)$$

(here $x\neq0$; the case of $x=0$ is trivial: y can be an arbitrary vector with $\|y\|_q=1$)

*With $\frac{1}{p}+\frac{1}{q}=1$, we come to an extremely important fact for **dual norm**:*

$$\|x\|_p=\max\left\{y^\top x\mid\|y\|_q\leq1\right\}$$

Corollary 3.172. ***(Legendre transformation vs. dual norm)*** *For any norm $f(x) = \|x\|$, the Legendre transformation is*

$$f^*(d) = \begin{cases} 0, & \text{if } \|d\|_* \le 1. \\ +\infty, & \text{otherwise}. \end{cases} \tag{3.173}$$

Proof. Using the fact that $\|x\| \equiv \sup_{y\in\mathbb{R}^n, \|y\|_*\le 1} x^\top y$, you immediately get

$$\begin{aligned} \sup_{x\in\mathbb{R}^n} x^\top d - \|x\| = \sup_{x\in\mathbb{R}^n} x^\top d - \sup_{y\in\mathbb{R}^n, \|y\|_*\le 1} x^\top y &= \inf_{y\in\mathbb{R}^n, \|y\|_*\le 1} \sup_{x\in\mathbb{R}^n} x^\top (d-y) \\ &= \inf_{y\in\mathbb{R}^n, \|y\|_*\le 1} \begin{cases} 0, & \text{if } y = d \\ +\infty, & \text{otherwise} \end{cases} \\ &= \begin{cases} 0, & \text{if } \|d\|_* \le 1 \\ +\infty, & \text{otherwise} \end{cases} \end{aligned}$$

where the second equality follows from Sion's minimax theorem. □

Lemma 3.174. ***(Sion's minimax theorem)*** *Let X be a compact convex subset of a linear topological space and Y a convex subset of a linear topological space. If f is a real-valued function on $X \times Y$ with*

- *$f(x, \cdot)$ upper semicontinuous and quasi-concave on $Y, \forall\, x \in X$,*
- *$f(\cdot, y)$ lower semicontinuous and quasi-convex on X, $\forall\, y \in Y$*

then,

$$\min_{x\in X} \sup_{y\in Y} f(x, y) = \sup_{y\in Y} \min_{x\in X} f(x, y)$$

3.6.3.4 Examples

Example 3.175. ***(constant or affine)*** *For $f(x) \equiv -a$, the Legendre transformation is*

$$f^*(d) = \begin{cases} a, & \text{if } d = 0. \\ +\infty, & \text{otherwise}. \end{cases} \tag{3.176}$$

For affine function $f(x) = \bar{d}^\top x - a$, the Legendre transformation is

$$f^*(d) = \begin{cases} a, & \text{if } d = \bar{d}. \\ +\infty, & \text{otherwise}. \end{cases} \tag{3.177}$$

Example 3.178. ***(negative logarithm)*** *$f(x) = -\log x$, with $\operatorname{dom} f = \mathbf{R}_{++}$. The function $xy + \log x$ is unbounded above if $y \ge 0$ and reaches its maximum at $x = -1/y$ otherwise.*

In summary, the Legendre transformation is

$$f^*(d) = \begin{cases} -\log(-d) - 1, & \text{if } d < 0. \\ +\infty, & \text{otherwise}. \end{cases} \tag{3.179}$$

Example 3.180. ***(exponential)*** *For $f(x) = e^x$, since $y - e^x$ is unbounded if $y < 0$. For $y > 0, xy - e^x$ reaches its maximum at $x = \log y$, so we have $f^*(y) = y \log y - y$. For $y = 0$, $f^*(y) = \sup_x -e^x = 0$.*

In summary, the Legendre transformation is

$$f^*(d) = \begin{cases} d \log d - d, & \text{if } d \ge 0. \\ +\infty, & \text{otherwise}. \end{cases} \tag{3.181}$$

Example 3.182. ***(negative entropy)*** *$f(x) = x \log x$, with $\operatorname{dom} f = \mathbf{R}_+$(and $f(0) = 0$). The function $xy - x \log x$ is bounded above on $\mathbf{R}_+$for all y, hence $\operatorname{dom} f^* = \mathbf{R}$. It attains its maximum at $x = e^{y-1}$,.*

In summary, the Legendre transformation is

$$f^*(y) = e^{y-1}$$

Example 3.183. ***(inverse)*** *$f(x) = 1/x$ on $\mathbf{R}_{++}$. For $y > 0, yx - 1/x$ is unbounded above. For $y = 0$ this function has supremum 0; for $y < 0$ the supremum is attained at $x = (-y)^{-1/2}$.*

In summary, the Legendre transformation is

$$f^*(d) = \begin{cases} -2(-d)^{1/2}, & \text{if } d \leq 0. \\ +\infty, & \text{otherwise}. \end{cases} \tag{3.184}$$

Example 3.185. ***(strictly convex quadratic function)*** *For strictly convex quadratic function*

$$f(x) = \frac{1}{2} x^\top A x$$

(A is positive definite symmetric matrix) the Legendre transformation of the

$$f^*(d) = \frac{1}{2} d^\top A^{-1} d$$

Example 3.186. ***(log-determinant)*** *We consider $f(X) = \log\det X^{-1}$ on $\mathbf{S}^n_{++}$. The conjugate function is defined as*

$$f^*(Y) = \sup_{X \succ 0} (\operatorname{tr}(YX) + \log\det X)$$

since $\operatorname{tr}(YX)$ is the standard inner product on $\mathbf{S}^n$. We first show that $\operatorname{tr}(YX) +$ $\log\det X$ is unbounded above unless $Y \prec 0$. If $Y \not\prec 0$, then Y has an eigenvector v with $\|v\|_2 = 1$, and eigenvalue $\lambda \geq 0$. Taking $X = I + tvv^\top$ we find that

$$\operatorname{tr}(YX) + \log\det X = \operatorname{tr} Y + t\lambda + \log\det\left(I + tvv^\top\right) = \operatorname{tr} Y + t\lambda + \log(1+t)$$

which is unbounded above as $t \to \infty$. Now consider the case $Y \prec 0$. We can find the maximizing X by setting the gradient with respect to X equal to zero:

$$\nabla_X(\operatorname{tr}(YX) + \log\det X) = Y + X^{-1} = 0,$$

which yields $X = -Y^{-1}$ (which is, indeed, positive definite).

In summary, the Legendre transformation is

$$f^*(Y) = \begin{cases} \log\det(-Y)^{-1} - n, & \text{if } Y \prec 0. \\ +\infty, & \text{otherwise}. \end{cases} \tag{3.187}$$

Example 3.188. ***(indicator function)*** *Let I_S be the indicator function of a (not necessarily convex) set $S \subseteq \mathbf{R}^n$, i.e., $I_S(x) = 0$ on $\operatorname{dom} I_S = S$.*

The Legendre transformation is

$$I_S^*(y) = \sup_{x \in S} y^\top x$$

which is the support function of the set S.

Example 3.189. ***(log-sum-exp function)*** *To derive the conjugate of the log-sum-exp function*

$$f(x) = \log\left(\sum_{i=1}^n e^{x_i}\right),$$

we first determine the values of y for which the maximum over x of $y^\top x - f(x)$ is attained. By setting the gradient with respect to x equal to zero, we obtain the condition

$$y_i = \frac{e^{x_i}}{\sum_{j=1}^n e^{x_j}}, \quad i = 1, \ldots, n$$

These equations are solvable for x if and only if $y \succ 0$ and $\mathbf{1}^\top y = 1$. By substituting the expression for y_i into $y^\top x - f(x)$ we obtain $f^*(y) = \sum_{i=1}^n y_i \log y_i$. This expression for f^* is still correct if some components of y are zero, as long as $y \succeq 0$ and $\mathbf{1}^\top y = 1$, and we interpret $0 \log 0$ as 0 . In fact the domain of f^* is exactly given by $\mathbf{1}^\top y = 1, y \succeq 0$. To show this, suppose that a component of y is *negative, say, $y_k < 0$. Then we can show that $y^\top x - f(x)$ is unbounded above by choosing $x_k = -t$, and $x_i = 0, i \neq k$, and letting t go to infinity. If $y \succeq 0$ but $\mathbf{1}^\top y \neq 1$, we choose $x = t\mathbf{1}$, so that*

$$y^\top x - f(x) = t\mathbf{1}^\top y - t - \log n$$

If $\mathbf{1}^\top y > 1$, this grows unboundedly as $t \to \infty$; if $\mathbf{1}^\top y < 1$, it grows unboundedly as $t \to -\infty$.

In summary, the Legendre transformation is

$$f^*(y) = \begin{cases} \sum_{i=1}^n y_i \log y_i & \textit{if } y \succeq 0 \textit{ and } \mathbf{1}^\top y = 1 \\ \infty & \textit{otherwise} \end{cases}$$

In other words, the conjugate of the log-sum-exp function is the negative entropy function, restricted to the probability simplex.

4 Convex Optimization and Duality

4.1 Convex Optimization

4.1.1 Definition of Optimization: basic terminology

Definition 4.1. ***(Optimization Standard Form)*** *We consider an optimization (**not necessarily convex**) problem in the **standard form**:*

$$\begin{array}{rll} \text{minimize} & f_0(x) & \\ \textit{subject to} & f_i(x) \leq 0, & i = 1, \ldots, m \\ & h_i(x) = 0, & i = 1, \ldots, p \end{array} \tag{4.2}$$

with variable $x \in \mathbf{R}^n$.

- *We assume its **domain** $\mathcal{D} = \bigcap_{i=0}^m \operatorname{dom} f_i \cap \bigcap_{i=1}^p \operatorname{dom} h_i$ is **nonempty***
- *We denote the **optimal value** of (4.2) by p^*.*
- *We call $x \in \mathbf{R}^n$ the **optimization variable** and the function $f_0 : \mathbf{R}^n \to \mathbf{R}$ the **objective function or cost function.***
- *The inequalities $f_i(x) \leq 0$ are called **inequality constraints**.*
- *The equations $h_i(x) = 0$ are called the **equality constraints.***
- *If there are no constraints (i.e., $m = p = 0$) we say (4.2) is **unconstrained.***
- *A point $x \in \mathcal{D}$ is **feasible if it satisfies the constraints** $f_i(x) \leq 0, i = 1, \ldots, m$, and $h_i(x) = 0, i = 1, \ldots, p$.*
- *The problem (4.2) is said to be **feasible** if there exists at least one feasible point, and infeasible otherwise. The set of all feasible points is called the **feasible set** or the constraint set.*
- *We allow p^* to take on the extended values $\pm\infty$. If the problem is infeasible, we have $p^* = \infty$ (following the standard convention that the **infimum of the empty set is** ∞).*
- *The **optimal value** p^* of the problem (4.2) is defined as*

  $$p^* = \inf \{f_0(x) \mid f_i(x) \leq 0, i = 1, \ldots, m, h_i(x) = 0, i = 1, \ldots, p\}$$

- *If there are feasible points x_k with $f_0(x_k) \to -\infty$ as $k \to \infty$, then $p^* = -\infty$, and we say the problem (4.2) is **unbounded below.***
- *If x is feasible and $f_i(x) = 0$, we say the i-th inequality constraint $f_i(x) \leq 0$ is **active** at x. If $f_i(x) < 0$, we say the constraint $f_i(x) \leq 0$ is inactive.*
- *The equality constraints are always **active at all feasible points.***
- *We say that a constraint is **redundant** if deleting it does not change the feasible set.*

- (4.2) *is called* ***convex optimization problem*** *if it has the form*

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \textit{subject to} & f_i(x) \leq 0, \quad i=1,\dots,m \\ & Ax = b. \end{array} \tag{4.3}$$

  In other words.

  - *the objective function must be* ***convex****, i.e., f_0 is convex*
  - *the inequality constraint functions must be* ***convex****, i.e., all $f_i, i=1,...,m$ convex.*
  - *the equality constraint functions must be* ***affine****, i.e. and $h_i(x) = a_i^\top x - b_i$, $i=m,..,p$*

Remark 4.4. *For* ***convex optimization problem****, the domain $\mathcal{D} = \bigcap_{i=0}^m \operatorname{dom} f_i$ is a convex set. The* ***feasible set is also convex*** *since it is the intersection of $\mathcal{D}$ and level set of f_i. Our previous* (3.131) *is a special case of convex optimization with no equality constraint.*

4.1.2 Optimal and Locally Optimal Points

Definition 4.5. ***(Optimal Point)*** *We say x^* is an* ***optimal point****, or solves the problem* (4.2)*, if x^* is feasible and $f_0(x^*) = p^*$. The set of all optimal points is the optimal set, denoted*

$$X_{\text{opt}} = \{x \mid f_i(x) \leq 0, i=1,\dots,m, h_i(x) = 0, i=1,\dots,p, f_0(x) = p^*\}$$

Remark 4.6. *If there exists an optimal point for the problem* (4.2)*, we say the optimal value is* ***attained or achieved,*** *(e.g. compact set) and the problem is solvable. If X_{opt} is empty, we say the optimal value is not attained or not achieved. When unbounded, of course the optimal value is not attained. Some case* $\inf$ *is finite, but we cannot have x that achieves it.*

When the problem is unbounded below, we can see the optimal value is not attained.

Definition 4.7. ***(ϵ-suboptimal)*** *A* ***feasible point*** *x with $f_0(x) \leq p^* + \epsilon$ (where $\epsilon > 0$) is called* ***ϵ-suboptimal,*** *and the set of all ϵ-suboptimal points is called the* ***ϵ-suboptimal set*** *for the problem* (4.2).

In Theorem 3.112, we have shown unimodality of convex function: local mim $\Rightarrow$ global min. Here we definite the locally optimal for standard optimization problem (4.2):

Definition 4.8. ***(Locally Optimal)*** *We say a feasible point x is locally optimal if $\exists\, r > 0$ such that*

$$\begin{aligned} f_0(x) = \inf \{ f_0(z) \mid\ & f_i(z) \leq 0, i=1,\dots,m \\ & h_i(z) = 0, i=1,\dots,p, \|z-x\|_2 \leq r \} \end{aligned}$$

or, in other words, x solves the optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(z) \\ \textit{subject to} & f_i(z) \leq 0, \quad i=1,\dots,m \\ & h_i(z) = 0, \quad i=1,\dots,p \\ & \|z-x\|_2 \leq r \end{array}$$

with variable z.

Remark 4.9. *The term "globally optimal" is sometimes used for "optimal" to distinguish between "locally optimal" and "optimal".*

4.1.3 Equivalent problems

We call two problems **equivalent** if from a solution of one, a solution of the other is readily found, and vice versa.

Example 4.10. *As a simple example, consider the problem*

$$\begin{array}{ll} \text{minimize} & \tilde{f}(x) = \alpha_0 f_0(x) \\ \textit{subject to} & \tilde{f}_i(x) = \alpha_i f_i(x) \leq 0, \quad i=1,\dots,m \\ & \tilde{h}_i(x) = \beta_i h_i(x) = 0, \quad i=1,\dots,p \end{array} \tag{4.11}$$

where $\alpha_i > 0, i=0,\dots,m$, and $\beta_i \neq 0, i=1,\dots,p$. A point x is optimal for the original problem (4.2) *if and only if it is optimal for the scaled problem* (4.11)*, so we say the two problems are equivalent.*

4.1.3.1 Change of Variables

Suppose $\phi : \mathbf{R}^n \to \mathbf{R}^n$ satisfies:

1. ϕ is one-to-one
2. Image of ϕ covers the problem domain $\mathcal{D}$ i.e., $\phi(\operatorname{dom}\phi) \supseteq \mathcal{D}$.

We define functions $\tilde{f}_i$ and $\tilde{h}_i$ as

$$\tilde{f}_i(z) = f_i(\phi(z)), \quad i = 0, \ldots, m, \quad \tilde{h}_i(z) = h_i(\phi(z)), \quad i = 1, \ldots, p$$

Now consider the problem

$$\begin{array}{ll} \text{minimize} & \tilde{f}_0(z) \\ \text{subject to} & \tilde{f}_i(z) \leq 0, \quad i = 1, \ldots, m \\ & \tilde{h}_i(z) = 0, \quad i = 1, \ldots, p \end{array} \tag{4.12}$$

with variable z.

The two problems are clearly equivalent: if x solves the problem (4.2), then $z = \phi^{-1}(x)$ solves the problem (4.12); if z solves the problem (4.12), then $x = \phi(z)$ solves the problem (4.2).

4.1.3.2 Transformation of Objective and Constraint Functions

Suppose that

1. $\psi_0 : \mathbf{R} \to \mathbf{R}$ is monotone increasing,
2. $\psi_1, \ldots, \psi_m : \mathbf{R} \to \mathbf{R}$ satisfy $\psi_i(u) \leq 0$ if and only if $u \leq 0$, and
3. $\psi_{m+1}, \ldots, \psi_{m+p} : \mathbf{R} \to \mathbf{R}$ satisfy $\psi_i(u) = 0$ if and only if $u = 0$.

We define functions $\tilde{f}_i$ and $\tilde{h}_i$ as the compositions

$$\tilde{f}_i(x) = \psi_i\left(f_i(x)\right), \quad i = 0, \ldots, m, \quad \tilde{h}_i(x) = \psi_{m+i}\left(h_i(x)\right), \quad i = 1, \ldots, p$$

Evidently the associated problem

$$\begin{array}{ll} \text{minimize} & \tilde{f}_0(x) \\ \text{subject to} & \tilde{f}_i(x) \leq 0, \quad i = 1, \ldots, m \\ & \tilde{h}_i(x) = 0, \quad i = 1, \ldots, p \end{array} \tag{4.13}$$

and the standard form problem (4.2) are equivalent; indeed, the feasible sets are identical, and the optimal points are identical. The example (4.11) is a special case with all ψ_i are linear.

Example 4.14. ***(least-norm and least-norm-squared problems)*** *Consider the unconstrained Euclidean norm minimization problem*

$$\text{minimize} \quad \|Ax - b\|_2$$

with variable $x \in \mathbf{R}^n$. It is equivalent to

$$\text{minimize} \quad \|Ax - b\|_2^2 = (Ax - b)^\top (Ax - b)$$

4.1.3.3 Slack Variables

One simple transformation is based on the observation that $f_i(x) \leq 0$ **if and only if there is an** $s_i \geq 0$ **that satisfies** $f_i(x) + s_i = 0$. Using this transformation we obtain the problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & s_i \geq 0, \quad i = 1, \ldots, m \\ & f_i(x) + s_i = 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p \end{array} \tag{4.15}$$

where the variables are $x \in \mathbf{R}^n$ and $s \in \mathbf{R}^m$. This problem has $n+m$ variables, m inequality constraints (the nonnegativity constraints on s_i), and $m+p$ equality constraints.

The new variable s_i is called the **slack variable** associated with the original inequality constraint $f_i(x) \leq 0$. Introducing slack variables replaces each inequality constraint with an equality constraint, and a nonnegativity constraint.

The problem (4.15) is equivalent to the original standard form problem (4.2). Indeed, if (x, s) is feasible for the problem (4.15), then x is feasible for the original problem, since $s_i = -f_i(x) \geq 0$. Conversely, if x is feasible for the original problem, then (x, s) is feasible for the problem (4.7), where we take $s_i = -f_i(x)$. Similarly, x is optimal for the original problem (4.2) if and only if (x, s) is optimal for the problem (4.7), where $s_i = -f_i(x)$.

4.1.3.4 Eliminating Equality Constraints

If we can explicitly parametrize all solutions of the equality constraints:

$$h_i(x) = 0, \forall\, i = 1, \ldots, p \Longleftrightarrow x = \phi(z)$$

using some parameter $z \in \mathbf{R}^k$, the optimization problem

$$\begin{array}{ll} \text{minimize} & \tilde{f}_0(z) = f_0(\phi(z)) \\ \text{subject to} & \tilde{f}_i(z) = f_i(\phi(z)) \leq 0, \quad i = 1, \ldots, m \end{array}$$

is then equivalent to the original problem (4.2) without equality constraints.

This transformed problem has variable $z \in \mathbf{R}^k$, m inequality constraints, and no equality constraints. If z is optimal for the transformed problem, then $x = \phi(z)$ is optimal for the original problem. Conversely, if x is optimal for the original problem, then (since x is feasible) there is at least one z such that $x = \phi(z)$. Any such z is optimal for the transformed problem.

4.1.3.5 Eliminating Linear Equality Constraints: a special case of above

The process of eliminating variables can be described more explicitly, and easily carried out numerically, when the equality constraints are all linear, i.e., have the form $Ax = b$. If $Ax = b$ is inconsistent, i.e., $b \notin \mathcal{R}(A)$, then the original problem is infeasible.

Assuming this is not the case, let x_0 denote any solution of the equality constraints. Let $F \in \mathbf{R}^{n \times k}$ be any matrix with $\mathcal{R}(F) = \mathcal{N}(A)$, **so the general solution of the linear equations $Ax = b$ is given by $Fz + x_0$, where $z \in \mathbf{R}^k$.** (We can choose F to be full rank, in which case we have $k = n - \operatorname{rank} A$.) Substituting $x = Fz + x_0$ into the original problem yields the problem

$$\begin{array}{ll} \text{minimize} & f_0\left(Fz + x_0\right) \\ \text{subject to} & f_i\left(Fz + x_0\right) \leq 0, \quad i = 1, \ldots, m \end{array}$$

with variable z, which is equivalent to the original problem, has no equality constraints, and $\operatorname{rank} A$ fewer variables.

4.1.3.6 Introducing Equality Constraints

We can also **introduce equality constraints and new variables into a problem.**

Instead of describing the general case, which is complicated and not very illuminating, we give a typical example that will be useful later. Consider the problem

$$\begin{array}{ll} \text{minimize} & f_0\left(A_0 x + b_0\right) \\ \text{subject to} & f_i\left(A_i x + b_i\right) \leq 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p \end{array}$$

where $x \in \mathbf{R}^n, A_i \in \mathbf{R}^{k_i \times n}$, and $f_i : \mathbf{R}^{k_i} \to \mathbf{R}$. In this problem the objective and constraint functions are given as compositions of the functions f_i with affine transformations defined by $A_i x + b_i$

We **introduce new variables** $y_i \in \mathbf{R}^{k_i}$**, as well as new equality constraints** $y_i = A_i x + b_i$**,** for $i = 0, \ldots, m$, and form the equivalent problem

$$\begin{array}{ll} \text{minimize} & f_0(y_0) \\ \text{subject to} & f_i(y_i) \leq 0, \quad i = 1, \ldots, m \\ & y_i = A_i x + b_i, \quad i = 0, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p \end{array}$$

This problem has $k_0 + \cdots + k_m$ new variables,

$$y_0 \in \mathbf{R}^{k_0}, \quad \ldots, \quad y_m \in \mathbf{R}^{k_m}$$

and $k_0 + \cdots + k_m$ new equality constraints,

$$y_0 = A_0 x + b_0, \quad \ldots, \quad y_m = A_m x + b_m$$

The objective and inequality constraints in this problem are **independent, i.e., involve different optimization variables.**

4.1.3.7 Optimizing over Some Variables

We always have

$$\inf_{x,y} f(x, y) = \inf_x \tilde{f}(x)$$

where $\tilde{f}(x) = \inf_y f(x, y)$. This simple and general principle can be used to transform problems into equivalent forms. The general case is cumbersome to describe and not illuminating, so we describe instead an example.

Suppose the variable $x \in \mathbf{R}^n$ is partitioned as $x = (x_1, x_2)$, with $x_1 \in \mathbf{R}^{n_1}$ $x_2 \in \mathbf{R}^{n_2}$, and $n_1 + n_2 = n$. We consider the problem

$$\begin{array}{ll} \text{minimize} & f_0(x_1, x_2) \\ \text{subject to} & f_i(x_1) \leq 0, \quad i = 1, \ldots, m_1 \\ & \tilde{f}_i(x_2) \leq 0, \quad i = 1, \ldots, m_2 \end{array}$$

in which the constraints are independent, in the sense that each constraint function depends on x_1 or x_2. We first minimize over x_2. Define the function $\tilde{f}_0$ of x_1 by

$$\tilde{f}_0(x_1) = \inf \left\{ f_0(x_1, z) \mid \tilde{f}_i(z) \leq 0, i = 1, \ldots, m_2 \right\}$$

The problem (4.9) is then equivalent to

$$\begin{array}{ll} \text{minimize} & \tilde{f}_0(x_1) \\ \text{subject to} & f_i(x_1) \leq 0, \quad i = 1, \ldots, m_1 \end{array}$$

Example 4.16. ***(minimizing a quadratic function with constraints on some variables)*** *Consider a problem with strictly convex quadratic objective, with some of the variables unconstrained:*

$$\begin{array}{ll} \textit{minimize} & x_1^\top P_{11} x_1 + 2 x_1^\top P_{12} x_2 + x_2^\top P_{22} x_2 \\ \textit{subject to} & f_i(x_1) \leq 0, \quad i = 1, \ldots, m \end{array}$$

where P_{11} and P_{22} are symmetric. Here we can analytically minimize over x_2 :

$$\inf_{x_2} \left(x_1^\top P_{11} x_1 + 2 x_1^\top P_{12} x_2 + x_2^\top P_{22} x_2 \right) = x_1^\top \left(P_{11} - P_{12} P_{22}^{-1} P_{12}^\top \right) x_1$$

(see § A.5.5). Therefore the original problem is equivalent to

$$\begin{array}{ll} \textit{minimize} & x_1^\top \left(P_{11} - P_{12} P_{22}^{-1} P_{12}^\top \right) x_1 \\ \textit{subject to} & f_i(x_1) \leq 0, \quad i = 1, \ldots, m \end{array}$$

4.1.3.8 Epigraph Problem Form

The epigraph form of the standard problem (4.2) is the problem

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & f_0(x) - t \leq 0 \\ & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p \end{array} \tag{4.17}$$

with variables $x \in \mathbf{R}^n$ and $t \in \mathbf{R}$.

We can easily see that it is equivalent to the original problem: (x, t) is optimal for (4.17) if and only if x is optimal for (4.2) and $t = f_0(x)$. Note that the objective function of the epigraph form problem is a linear function of the variables x, t.

The epigraph form problem (4.17) can be interpreted geometrically as an optimization problem in the "graph space" (x, t) : we minimize t over the epigraph of f_0, subject to the constraints on x. This is illustrated in figure 4.1.

4.1.3.9 Implicit and Explicit Constraints

The standard form problem (4.2) can be expressed as the unconstrained problem

$$\text{minimize} F(x) \tag{4.18}$$

where we define the function F as f_0, but **with domain restricted to the feasible set**:

$$\operatorname{dom} F = \{x \in \operatorname{dom} f_0 \mid f_i(x) \leq 0, i = 1, \ldots, m, h_i(x) = 0, i = 1, \ldots, p\}$$

and $F(x) = f_0(x)$ for $x \in \operatorname{dom} F$. (Equivalently, we can define $F(x)$ to have value ∞ for x not feasible.)

Of course this transformation is nothing more than a notational trick. Making the constraints implicit has not made the problem any easier to analyze or solve, even though the problem (4.18) is, at least nominally, unconstrained. In some ways the transformation makes the problem more difficult. Suppose, for example, that the objective f_0 in the original problem is differentiable, so in particular its domain is open. The restricted objective function F is probably not differentiable, since its domain is likely not to be open.

4.2 The Lagrange Dual Function

4.2.1 The Lagrangian

Definition 4.19. ***(Lagrangian)*** *We define the* ***Lagrangian*** $L : \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^p \to \mathbf{R}$ *associated with the problem* (4.2) *as*

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)$$

with $\operatorname{dom} L = \mathcal{D} \times \mathbf{R}^m \times \mathbf{R}^p$.

The vectors λ and ν are called the ***dual variables or Lagrange multiplier vectors*** *associated with the problem* (4.2).

- *We refer to λ_i as the Lagrange multiplier associated with the i-th inequality constraint $f_i(x) \leq 0$;*
- *We refer to ν_i as the Lagrange multiplier associated with the i-th equality constraint $h_i(x) = 0$.*

4.2.2 The Lagrange Dual Function

Definition 4.20. ***(Lagrange dual function)*** *We define the* ***Lagrange dual function*** *(or just dual function) $g : \mathbf{R}^m \times \mathbf{R}^p \to \mathbf{R}$ as the* ***minimum value of the Lagrangian over*** *x: for $\lambda \in \mathbf{R}^m, \nu \in \mathbf{R}^p$*

$$g(\lambda, \nu) = \inf_{x \in \mathcal{D}} L(x, \lambda, \nu) = \inf_{x \in \mathcal{D}} \left(f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \right)$$

Remark 4.21. *From the definition, we have*

- *Similar to* (3.160)*, if f is convex, the dual function can also be written as:*
$$g(\lambda, \nu) = \inf_{x \in \mathbf{R}^n} L(x, \lambda, \nu)$$
- *When the Lagrangian is unbounded below in x, the dual function takes on the value $-\infty$.*
- *Since the dual function is the pointwise infimum of a family of affine functions of (λ, ν), it is always **concave**, even when* (4.2) *is not convex.*

4.2.3 Lower Bounds on Optimal Value

Lemma 4.22. *The dual function yields **lower bounds** on the optimal value p^* of* (4.2)*: **for any** $\lambda \succeq 0$ **and any** ν we have*

$$g(\lambda, \nu) \leq p^* \tag{4.23}$$

Proof. Suppose $\tilde{x}$ is a feasible point for the problem (4.2), i.e., $f_i(\tilde{x}) \leq 0$ and $h_i(\tilde{x}) = 0$, and $\lambda \succeq 0$. Then we have

$$\sum_{i=1}^{m} \lambda_i f_i(\tilde{x}) + \sum_{i=1}^{p} \nu_i h_i(\tilde{x}) \leq 0$$

since each term in the first sum is nonpositive, and each term in the second sum is zero, and therefore

$$L(\tilde{x}, \lambda, \nu) = f_0(\tilde{x}) + \sum_{i=1}^{m} \lambda_i f_i(\tilde{x}) + \sum_{i=1}^{p} \nu_i h_i(\tilde{x}) \leq f_0(\tilde{x})$$

Hence

$$g(\lambda, \nu) = \inf_{x \in \mathcal{D}} L(x, \lambda, \nu) \leq L(\tilde{x}, \lambda, \nu) \leq f_0(\tilde{x})$$

(4.23) then follows. □

Remark 4.24. *The inequality* (4.23) *holds, but is meaningless, when $g(\lambda, \nu) = -\infty$. The dual function gives a nontrivial lower bound on p^* **only when** $\lambda \succeq 0$ **and** $(\lambda, \nu) \in \operatorname{dom} g$ **(i.e.,** $g(\lambda, \nu) > -\infty$).*

Definition 4.25. ***(Dual Feasible)*** *We refer to a pair (λ, ν) with $\lambda \succeq 0$ and $(\lambda, \nu) \in \operatorname{dom} g$ as **dual feasible**.*

4.2.4 Linear Approximation Interpretation

The Lagrangian and lower bound property can be given a simple interpretation, based on a linear approximation of the indicator functions of the sets $\{0\}$ and $-\mathbf{R}_+$.

We first rewrite the original problem (4.2) as an unconstrained problem,

$$\text{minimize} \quad f_0(x) + \sum_{i=1}^{m} I_-\left(f_i(x)\right) + \sum_{i=1}^{p} I_0\left(h_i(x)\right) \tag{4.26}$$

where $I_- : \mathbf{R} \to \mathbf{R}$ is the indicator function for the nonpositive reals,

$$I_-(u) = \begin{cases} 0 & u \leq 0 \\ \infty & u > 0 \end{cases}$$

and similarly, I_0 is the indicator function of $\{0\}$.

Now suppose in the formulation (4.26), we **replace the function $I_-(u)$ with the linear function $\lambda_i u$, where $\lambda_i \geq 0$, and the function $I_0(u)$ with $\nu_i u$.** The objective becomes the Lagrangian function $L(x, \lambda, \nu)$, and the dual function value $g(\lambda, \nu)$ is the optimal value of the problem

$$\text{minimize} \quad L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)$$

In this formulation, we use a linear or "soft" displeasure function in place of I_- and I_0. Clearly the approximation of the indicator function $I_-(u)$ with a linear function $\lambda_i u$ is rather poor. But the linear function is at least an underestimator of the indicator function: since $\lambda_i u \leq I_-(u)$ and $\nu_i u \leq I_0(u)$ for all u, we see immediately that the dual function yields a lower bound on the optimal value of the original problem.

4.2.5 Examples

In this section we give some examples for which we can derive an analytical expression for the Lagrange dual function.

4.2.5.1 Least-squares Solution of Linear Equations

• **Problem**:

$$\begin{array}{ll} \text{minimize} & x^\top x \\ \text{subject to} & Ax = b \end{array}$$

where $A \in \mathbf{R}^{p\times n}$. This problem has no inequality constraints and p (linear) equality constraints.

• **Lagrangian:**

$$L(x, \nu) = x^\top x + \nu^\top (Ax - b)$$

with domain $\mathbf{R}^n \times \mathbf{R}^p$.

• **Lagrangian dual function**:

$$g(\nu) = \inf_x L(x, \nu).$$

Since $L(x, \nu)$ is a convex quadratic function of x, we can find the minimizing x from the optimality condition

$$\nabla_x L(x, \nu) = 2x + A^\top \nu = 0$$

which yields $x = -(1/2)A^\top \nu$. Therefore the dual function is

$$g(\nu) = L\left(-(1/2)A^\top \nu, \nu\right) = -(1/4)\nu^\top AA^\top \nu - b^\top \nu$$

which is a concave quadratic function, with domain $\mathbf{R}^p$.

• **Lower bound property**:

The lower bound property (4.23) states that for any $\nu \in \mathbf{R}^p$, we have

$$-(1/4)\nu^\top AA^\top \nu - b^\top \nu \leq \inf\left\{x^\top x \mid Ax = b\right\}$$

4.2.5.2 Standard form LP

• **Problem**:

$$\begin{array}{ll} \text{minimize} & c^\top x \\ \text{subject to} & Ax = b \\ & x \succeq 0 \end{array}$$

which has inequality constraint functions $f_i(x) = -x_i, i = 1, \ldots, n.$

• **Lagrangian:**

$$L(x, \lambda, \nu) = c^\top x - \sum_{i=1}^{n} \lambda_i x_i + \nu^\top (Ax - b) = -b^\top \nu + \left(c + A^\top \nu - \lambda\right)^\top x$$

• **Lagrangian dual function**:

$$g(\lambda, \nu) = \inf_x L(x, \lambda, \nu) = -b^\top \nu + \inf_x \left(c + A^\top \nu - \lambda\right)^\top x \tag{4.27}$$

which is easily determined analytically, since a linear function is bounded below only when it is identically zero. Thus, $g(\lambda, \nu) = -\infty$ except when $c + A^\top \nu - \lambda = 0$, in which case it is $-b^\top \nu$:

$$g(\lambda, \nu) = \begin{cases} -b^\top \nu & A^\top \nu - \lambda + c = 0 \\ -\infty & \text{otherwise} \end{cases}$$

Note that the dual function g is finite only on a proper affine subset of $\mathbf{R}^m \times \mathbf{R}^p$. **We will see that this is a common occurrence.**

• **Lower bound property**:

The lower bound property (4.23) is nontrivial only when λ and ν satisfy $\lambda \succeq 0$ and $A^\top \nu - \lambda + c = 0$. When this occurs, $-b^\top \nu$ is a lower bound on the optimal value of the LP (5.6).

4.2.5.3 Two-way Partitioning Problem

• **Problem**:

$$\begin{array}{ll} \text{minimize} & x^\top W x \\ \text{subject to} & x_i^2 = 1, \quad i = 1, \ldots, n \end{array} \tag{4.28}$$

where $W \in \mathbf{S}^n$. It is (nonconvex) . The constraints restrict the values of x_i to 1 or -1, so the problem is equivalent to finding the vector with components ± 1 that minimizes $x^\top W x$. The feasible set here is finite (it contains 2^n points) so this problem can in principle be solved by simply checking the objective value of each feasible point. Since the number of feasible points **grows exponentially**, however, this is possible only for small problems (say, with $n \leq 30$). In general (and for n larger than, say, 50) the problem (4.28) is very difficult to solve.

We can interpret the problem (4.28) as a two-way partitioning problem on a set of n elements, say, $\{1, \ldots, n\}$: A feasible x corresponds to the partition

$$\{1, \ldots, n\} = \{i \mid x_i = -1\} \cup \{i \mid x_i = 1\}$$

The matrix coefficient W_{ij} can be interpreted as the cost of having the elements i and j in the same partition, and $-W_{ij}$ is the cost of having i and j in different partitions. The objective in (4.28) is the total cost, over all pairs of elements, and the problem (4.28) is to find the partition with least total cost.

• **Lagrangian:**

$$\begin{aligned} L(x, \nu) &= x^\top W x + \sum_{i=1}^n \nu_i \left(x_i^2 - 1\right) \\ &= x^\top (W + \operatorname{diag}(\nu)) x - \mathbf{1}^\top \nu \end{aligned}$$

• **Lagrangian dual function**:

$$\begin{aligned} g(\nu) &= \inf_x x^\top (W + \operatorname{diag}(\nu)) x - \mathbf{1}^\top \nu \\ &= \begin{cases} -\mathbf{1}^\top \nu & W + \operatorname{diag}(\nu) \succeq 0 \\ -\infty & \text{otherwise} \end{cases} \end{aligned}$$

where we use the fact that the infimum of a quadratic form is either zero (if the form is positive semidefinite) or $-\infty$ (if the form is not positive semidefinite).

• **Lower bound property**:

This dual function provides lower bounds on the optimal value of the difficult problem (4.28). For example, we can take the specific value of the dual variable

$$\nu = -\lambda_{\min}(W) \mathbf{1}$$

which is dual feasible, since

$$W + \operatorname{diag}(\nu) = W - \lambda_{\min}(W) I \succeq 0$$

This yields the bound on the optimal value p^*

$$p^* \geq -\mathbf{1}^\top \nu = n \lambda_{\min}(W)$$

Remark 4.29. *This lower bound on p^* can also be obtained without using the Lagrange dual function and using the modified problem*

$$\begin{array}{ll} \text{minimize} & x^\top W x \\ \textit{subject to} & \sum_{i=1}^n x_i^2 = n \end{array}$$

4.2.6 The Lagrange Dual Function and Legendre Transformation

The Legendre transformation (conjugate) function (3.158) and Lagrange dual function are closely related. To see one simple connection, consider the problem (which is not very interesting, and solvable by inspection)

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x = 0 \end{array}$$

(which is not very interesting, and solvable by inspection). This problem has Lagrangian $L(x,\nu) = f(x) + \nu^\top x$, and dual function

$$g(\nu) = \inf_x \left(f(x) + \nu^\top x\right) = -\sup_x \left((-\nu)^\top x - f(x)\right) = -f^*(-\nu)$$

More generally (and more usefully), consider:

• **Optimization with LINEAR inequality and equality constraints,**:

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & Ax \preceq b \\ & Cx = d \end{array}$$

Using the conjugate of f_0 we can write the **Lagrange dual** as

$$\begin{aligned} g(\lambda,\nu) &= \inf_x \left(f_0(x) + \lambda^\top (Ax - b) + \nu^\top (Cx - d)\right) \\ &= -b^\top \lambda - d^\top \nu + \inf_x \left(f_0(x) + \left(A^\top \lambda + C^\top \nu\right)^\top x\right) \\ &= -b^\top \lambda - d^\top \nu - f_0^*\left(-A^\top \lambda - C^\top \nu\right) \end{aligned} \tag{4.30}$$

The domain of g follows from the domain of f_0^* :

$$\operatorname{dom} g = \left\{(\lambda,\nu) \mid -A^\top \lambda - C^\top \nu \in \operatorname{dom} f_0^*\right\}$$

Let us illustrate this with a few examples:

4.2.6.1 Equality Constrained Norm Minimization

Consider the problem

$$\begin{array}{ll} \text{minimize} & \|x\| \\ \text{subject to} & Ax = b \end{array}$$

where $\|\cdot\|$ is any norm. Recall that the conjugate of $f_0 = \|\cdot\|$ is given by

$$f_0^*(y) = \begin{cases} 0 & \|y\|_* \le 1 \\ \infty & \text{otherwise} \end{cases}$$

the indicator function of the dual norm unit ball.

Using the result (4.30) above, the **Lagrange dual** function is given by

$$g(\nu) = -b^\top \nu - f_0^*\left(-A^\top \nu\right) = \begin{cases} -b^\top \nu & \left\|A^\top \nu\right\|_* \le 1 \\ -\infty & \text{otherwise} \end{cases}$$

4.2.6.2 Entropy Maximization

Consider the entropy maximization problem

$$\begin{array}{ll} \text{minimize} & f_0(x) = \sum_{i=1}^n x_i \log x_i \\ \text{subject to} & Ax \preceq b \\ & \mathbf{1}^\top x = 1 \end{array}$$

where $\operatorname{dom} f_0 = \mathbf{R}^n_{++}$. The conjugate of the negative entropy function $u \log u$ with scalar variable u, is e^{v-1}. Since f_0 is a sum of negative entropy functions of different variables, we conclude that its conjugate is

$$f_0^*(y) = \sum_{i=1}^n e^{y_i - 1}$$

with $\operatorname{dom} f_0^* = \mathbf{R}^n$.

Using the result (4.30) above, the **Lagrange dual** function is given by

$$g(\lambda, \nu) = -b^\top \lambda - \nu - \sum_{i=1}^n e^{-a_i^\top \lambda - \nu - 1} = -b^\top \lambda - \nu - e^{-\nu - 1} \sum_{i=1}^n e^{-a_i^\top \lambda}$$

where a_i is the i th column of A.

4.2.6.3 Minimum Volume Covering Ellipsoid

Consider the problem with variable $X \in \mathbf{S}^n$,

$$\begin{array}{ll} \text{minimize} & f_0(X) = \log\det X^{-1} \\ \text{subject to} & a_i^\top X a_i \leq 1, \quad i = 1, \ldots, m \end{array} \tag{4.31}$$

where $\operatorname{dom} f_0 = \mathbf{S}^n_{++}$. The problem (4.31) has a simple geometric interpretation. With each $X \in \mathbf{S}^n_{++}$ we associate the ellipsoid, centered at the origin,

$$\mathcal{E}_X = \left\{ z \mid z^\top X z \leq 1 \right\}$$

The volume of this ellipsoid is proportional to $\left(\det X^{-1}\right)^{1/2}$, so the objective of (4.31) is, except for a constant and a factor of two, the logarithm of the volume of $\mathcal{E}_X$. The constraints of the problem (4.31) are that $a_i \in \mathcal{E}_X$. Thus the problem (4.31) is to **determine the minimum volume ellipsoid, centered at the origin, that includes the points** $a_1, \ldots, a_m$**.**

The inequality constraints in problem (4.31) are affine; they can be expressed as

$$\operatorname{tr}\left(\left(a_i a_i^\top\right) X\right) \leq 1$$

From Example 3.186, we have the conjugate of f_0 is

$$f_0^*(Y) = \log\det(-Y)^{-1} - n$$

with $\operatorname{dom} f_0^* = -\mathbf{S}^n_{++}$.

Applying the result (4.30) above, the **Lagrange dual** function for the problem (4.31) is given by

$$g(\lambda) = \begin{cases} \log\det\left(\sum_{i=1}^m \lambda_i a_i a_i^\top\right) - \mathbf{1}^\top \lambda + n & \sum_{i=1}^m \lambda_i a_i a_i^\top \succ 0 \\ -\infty & \text{otherwise.} \end{cases}$$

Thus, for any $\lambda \succeq 0$ with $\sum_{i=1}^m \lambda_i a_i a_i^\top \succ 0$, the number

$$\log\det\left(\sum_{i=1}^m \lambda_i a_i a_i^\top\right) - \mathbf{1}^\top \lambda + n$$

is a lower bound on the optimal value of the problem (4.31).

4.3 The Lagrange Dual Problem

• **Lagrange Dual Problem:** We get the best lower bound as shown in Lemma 4.22, we need the **concave** optimization problem

$$\begin{array}{ll} \text{maximize} & g(\lambda, \nu) \\ \text{subject to} & \lambda \succeq 0. \end{array} \tag{4.32}$$

This problem is called the **Lagrange dual problem** associated with the problem (4.2).

- **Dual feasible**, defined in Definition 4.25 to describe a pair (λ, ν) with $\lambda \succeq 0$ and $g(\lambda, \nu) > -\infty$, means that (λ, ν) is feasible for the dual problem (4.32).
- The Lagrange dual problem (4.32) is **always a convex optimization problem** whether or not the primal problem (4.2) is convex.
- We refer to (λ^*, ν^*) as **dual optimal or optimal Lagrange multipliers** if they are optimal for the problem (4.32).

• **Primal Problem:**

In this context the original problem (4.2) is sometimes called the primal problem.

Remark 4.33. ***(Lagrange dual problem vs. Lagrange dual function)*** *Please don't be confused:*

- *Note the Lagrange dual problem is the optimization for Lagrange dual function* $g(\lambda, \nu)$.
- *Lagrange dual function* $g(\lambda, \nu)$ *is the* $\inf$ *of the Lagrangian* $L(x, \lambda, \nu)$.

4.3.1 Making Dual Constraints Explicit

The examples above show that it is not uncommon for the domain of the dual function,

$$\operatorname{dom} g = \{(\lambda, \nu) \mid g(\lambda, \nu) > -\infty\}$$

to have dimension smaller than $m + p$. In many cases we can identify the affine hull of $\operatorname{dom} g$, and describe it as a set of linear equality constraints.

4.3.1.1 Lagrange Dual of Standard Form LP

In (4.27), we show the Lagrange dual function for the standard form LP:

• **Standard form LP:**

$$\begin{array}{ll} \text{minimize} & c^\top x \\ \text{subject to} & Ax = b \\ & x \succeq 0 \end{array} \tag{4.34}$$

• **Lagrange dual function:**

$$g(\lambda, \nu) = \begin{cases} -b^\top \nu & A^\top \nu - \lambda + c = 0 \\ -\infty & \text{otherwise} \end{cases}$$

• **Lagrange dual problem:**

$$\begin{array}{ll} \text{maximize} & g(\lambda, \nu) = \begin{cases} -b^\top \nu & A^\top \nu - \lambda + c = 0 \\ -\infty & \text{otherwise} \end{cases} \\ \text{subject to} & \lambda \succeq 0 \end{array} \tag{4.35}$$

• **Equivalent Lagrange dual problem:**

Here g is finite only when $A^\top \nu - \lambda + c = 0$. We can form an equivalent problem by making these equality constraints explicit: using the slack technique shown in (4.15):

$$\begin{array}{ll} \text{maximize} & -b^\top \nu \\ \text{subject to} & A^\top \nu + c \succeq 0 \end{array} \tag{4.36}$$

which is **an LP in inequality form.**

Remark 4.37. *With some abuse of terminology, we refer to the problem* (4.36) *or the problem* (4.35) *as the Lagrange dual of the standard form LP.*

4.3.1.2 Lagrange Dual of Inequality Form LP

• **LP in inequality form:**

$$\begin{array}{ll} \text{minimize} & c^\top x \\ \text{subject to} & Ax \preceq b \end{array} \tag{4.38}$$

• **Lagrangian:**

$$L(x,\lambda) = c^\top x + \lambda^\top (Ax - b) = -b^\top \lambda + \left(A^\top \lambda + c\right)^\top x$$

• **Lagrange dual function:**

$$g(\lambda) = \inf_x L(x,\lambda) = -b^\top \lambda + \inf_x \left(A^\top \lambda + c\right)^\top x$$

The infimum of a linear function is $-\infty$, except in the special case when it is identically zero, so the dual function is

$$g(\lambda) = \begin{cases} -b^\top \lambda & A^\top \lambda + c = 0 \\ -\infty & \text{otherwise} \end{cases}$$

• **Lagrange dual problem:**

$$\begin{array}{ll} \text{maximize} & g(\lambda) = \begin{cases} -b^\top \lambda & A^\top \lambda + c = 0 \\ -\infty & \text{otherwise} \end{cases} \\ \text{subject to} & \lambda \succeq 0 \end{array} \tag{4.39}$$

• **Equivalent Lagrange dual problem:**

$$\begin{array}{ll} \text{maximize} & -b^\top \lambda \\ \text{subject to} & A^\top \lambda + c = 0 \\ & \lambda \succeq 0 \end{array} \tag{4.40}$$

which is an **LP in standard form.**

4.3.1.3 Summary of Standard Form LP and Inequality Form LP

- Note the interesting symmetry between the standard and inequality form LPs and their duals: The dual of a standard form LP is an LP with only inequality constraints, and vice versa.
- Dual of dual is the original primary problem:
  1. One can also verify that the Lagrange dual problem of (4.40) is (equivalent to) the primal problem (4.38).
  2. One can also verify that the Lagrange dual problem of (4.36) is (equivalent to) the primal problem (4.34).

4.3.2 Weak Duality

Lemma 4.41. ***(weak duality)*** *Denote d^* as the* ***optimal value of the Lagrange dual problem****. This is the best lower bound on p^* that can be obtained from the Lagrange dual function. We have*

$$d^* \le p^* \tag{4.42}$$

which ***holds even if the original problem is not convex.***

Remark 4.43. ***(infinite case)*** *The weak duality inequality* (4.42) *holds when d^* and p^* are infinite: for example,*

- *if the primal problem is unbounded below, so that $p^* = -\infty$, we must have $d^* = -\infty$, i.e., the Lagrange dual problem is infeasible.*
- *if the dual problem is unbounded above, so that $d^* = \infty$, we must have $p^* = \infty$, i.e., the primal problem is infeasible.*

Definition 4.44. ***(Optimal Duality Gap)*** *We refer to the difference $p^* - d^*$ as the* ***optimal duality gap*** *of the original problem. The optimal duality gap is always nonnegative.*

Remark 4.45. *The bound* (4.42) *can sometimes be used to find a lower bound on the optimal value of a problem that is difficult to solve, since the dual problem is always convex, and in many cases can be solved efficiently, to find d^*.*

4.3.3 Strong Duality and Slater's Constraint Qualification

Definition 4.46. ***(Strong Duality)***

$$d^* = p^*$$

Strong duality does not, in general, hold. But if the primal problem (4.2) is convex, i.e., of the form

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array} \tag{4.47}$$

with convex f_i, we usually (but not alwavs) have strong duality:

• **Constraint qualifications**:

There are many results that establish conditions, called **constraint qualifications**, beyond convexity, under which strong duality holds. One simple constraint qualification has already been shown in Theorem 3.130 which is a special case of the general form:

• **Slater's condition: a special constraint qualification**

There exists an $x \in \operatorname{ri} \mathcal{D}$ such that

$$f_i(x) < 0, \quad i = 1, \ldots, m, \quad Ax = b$$

- Such a point is sometimes called **strictly feasible**, since the inequality constraints hold with strict inequalities.
- Slater's theorem states that if Slater's condition holds (and the problem is convex), strong duality holds, and dual optimal point (λ^*, ν^*) is attained if $d^* > -\infty$.

• **Weak Slater's condition**:

Slater's condition can be refined when **some of the inequality constraint functions f_i are affine.** If the first k constraint functions $f_1, \ldots, f_k$ are affine, then strong duality holds provided the following **weaker condition** holds:

There exists an $x \in \operatorname{int} \mathcal{D}$ with

$$f_i(x) \leq 0, \quad i = 1, \ldots, k, \quad f_i(x) < 0, \quad i = k+1, \ldots, m, \quad Ax = b \tag{4.48}$$

Remark 4.49. ***(all linear case)*** *Note that weak Slater condition* (4.48) *reduces to feasibility when the constraints are all linear equalities and inequalities, and* $\operatorname{dom} f_0$ *is open, which is the also discussed in Remark 3.136.*

Remark 4.50. *Slater's condition not only implies strong duality for convex problems. It also implies that the dual optimal value is* ***attained when*** $d^* > -\infty$*, i.e., there exists a dual feasible* (λ^*, ν^*) *with* $g(\lambda^*, \nu^*) = d^* = p^*$*. We will prove Slater's theorem in Section 4.5.2*

4.3.4 Examples

4.3.4.1 Least-squares Solution of Linear Equations

• **Problem**:

$$\begin{array}{ll} \text{minimize} & x^\top x \\ \text{subject to} & Ax = b \end{array}$$

where $A \in \mathbf{R}^{p \times n}$.

• **Lagrange dual problem:**

$$\text{maximize} \quad -(1/4)\nu^\top A A^\top \nu - b^\top \nu$$

which is an unconstrained concave quadratic maximization problem.

• **Strong Duality:**

From Remark 4.49, we know Slater's condition is simply that the primal problem is **feasible**:

$$b \in \mathcal{R}(A), \text{ i.e., } p^* < \infty$$

In fact for this problem we **always have strong duality, even when** $p^* = \infty$**.** This is the case when $b \notin \mathcal{R}(A)$, so there is a z with $A^\top z = 0, b^\top z \neq 0$. It follows that the dual function is unbounded above along the line $\{tz \mid t \in \mathbf{R}\}$, so $d^* = \infty$ as well.

4.3.4.2 Lagrange Dual of LP

By the weaker form of Slater's condition, we find that strong duality holds for any LP (in standard or inequality form) provided the primal problem is **feasible**.

Applying this result to the duals, we conclude that strong duality holds for LPs if the dual is feasible (c.f. section 4.3.1.3). This leaves **only one possible situation in which strong duality for LPs can fail: both the primal and dual problems are infeasible.**

4.3.4.3 Lagrange Dual of QCQP

• **Problem**:

$$\begin{array}{ll} \text{minimize} & (1/2)x^\top P_0 x + q_0^\top x + r_0 \\ \text{subject to} & (1/2)x^\top P_i x + q_i^\top x + r_i \le 0, \quad i = 1, \ldots, m \end{array} \tag{4.51}$$

with $P_0 \in \mathbf{S}^n_{++}$, and $P_i \in \mathbf{S}^n_{+}, i = 1, \ldots, m$.

• **Lagrangian:**

$$L(x, \lambda) = (1/2)x^\top P(\lambda)x + q(\lambda)^\top x + r(\lambda)$$

where

$$P(\lambda) = P_0 + \sum_{i=1}^m \lambda_i P_i, \quad q(\lambda) = q_0 + \sum_{i=1}^m \lambda_i q_i, \quad r(\lambda) = r_0 + \sum_{i=1}^m \lambda_i r_i$$

• **Lagrangian dual function**:

It is possible to derive an expression for $g(\lambda)$ for general λ, but it is quite complicated. If $\lambda \succeq 0$, however, we have $P(\lambda) \succ 0$ and

$$g(\lambda) = \inf_x L(x, \lambda) = -(1/2)q(\lambda)^\top P(\lambda)^{-1} q(\lambda) + r(\lambda)$$

• **Lagrange dual problem:**

$$\begin{array}{ll} \text{maximize} & -(1/2)q(\lambda)^\top P(\lambda)^{-1} q(\lambda) + r(\lambda) \\ \text{subject to} & \lambda \succeq 0 \end{array} \tag{4.52}$$

• **Strong Duality:**

The Slater condition says that strong duality between (4.51) and (4.52) holds if the quadratic inequality constraints are strictly feasible, i.e., there exists an x with

$$(1/2)x^\top P_i x + q_i^\top x + r_i < 0, \quad i = 1, \ldots, m$$

4.3.4.4 Entropy Maximization

• **Problem**:

$$\begin{array}{ll} \underset{i=1}{\text{minimize}} & \sum_{i=1}^n x_i \log x_i \\ \text{subject to} & Ax \preceq b \\ & \mathbf{1}^\top x = 1 \end{array}$$

with domain $\mathcal{D} = \mathbf{R}^n_+$.

• **Lagrange dual problem:**

$$\begin{array}{ll} \text{maximize} & -b^\top \lambda - \nu - e^{-\nu-1} \sum_{i=1}^n e^{-a_i^\top \lambda} \\ \text{subject to} & \lambda \succeq 0 \end{array} \tag{4.53}$$

with variables $\lambda \in \mathbf{R}^m, \nu \in \mathbf{R}$.

• **Strong Duality:**

The (weaker) Slater condition tells us that the optimal duality gap is zero if there exists an $x \succ 0$ with $Ax \preceq b$ and $\mathbf{1}^\top x = 1$

Remark 4.54. ***(equivalent dual problem)*** *We can simplify the dual problem* (4.53) *by maximizing over the dual variable ν analytically. For fixed λ, the objective function is maximized when the derivative with respect to ν is zero, i.e.,*

$$\nu = \log \sum_{i=1}^{n} e^{-a_i^\top \lambda} - 1$$

Substituting this optimal value of ν into the dual problem gives

$$\begin{array}{ll} \text{maximize} & -b^\top \lambda - \log\left(\sum_{i=1}^n e^{-a_i^\top \lambda}\right) \\ \textit{subject to} & \lambda \succeq 0 \end{array}$$

which is a geometric program (in convex form) with nonnegativity constraints.

4.3.4.5 Minimum Volume Covering Ellipsoid

• **Problem**:

$$\begin{array}{ll} \text{minimize} & \log\det X^{-1} \\ \text{subject to} & a_i^\top X a_i \leq 1, \quad i = 1, \dots, m \end{array} \tag{4.55}$$

with domain $\mathcal{D} = \mathbf{S}^n_{++}$.

• **Lagrange dual problem:**

$$\begin{array}{ll} \text{maximize} & \log\det\left(\sum_{i=1}^m \lambda_i a_i a_i^\top\right) - \mathbf{1}^\top \lambda + n \\ \text{subject to} & \lambda \succeq 0 \end{array}$$

where we take $\log\det X = -\infty$ if $X \not\succ 0$

• **Strong Duality:**

The (weaker) Slater condition for the problem (4.55) is that there exists an $X \in \mathbf{S}^n_{++}$ with $a_i^\top X a_i \leq 1$, for $i = 1, \dots, m$. This is always satisfied, so strong duality **always obtains.**

4.3.4.6 A Nonconvex Quadratic Problem with Strong Duality

On rare occasions strong duality obtains for a nonconvex problem. As an important example, we consider the problem of minimizing a nonconvex quadratic function over the unit ball:

• **Problem**:

$$\begin{array}{ll} \text{minimize} & x^\top A x + 2b^\top x \\ \text{subject to} & x^\top x \leq 1 \end{array} \tag{4.56}$$

where $A \in \mathbf{S}^n$, $A \not\succeq 0$, and $b \in \mathbf{R}^n$. Since $A \not\succeq 0$, this is not a convex problem. This problem is sometimes called the trust region problem, and arises in minimizing a second-order approximation of a function over the unit ball, which is the region in which the approximation is assumed to be approximately valid.

• **Lagrangian:**

$$L(x,\lambda) = x^\top A x + 2b^\top x + \lambda\left(x^\top x - 1\right) = x^\top (A + \lambda I)x + 2b^\top x - \lambda$$

• **Lagrangian dual function**:

$$g(\lambda) = \begin{cases} -b^\top (A+\lambda I)^\dagger b - \lambda & A + \lambda I \succeq 0, \quad b \in \mathcal{R}(A + \lambda I) \\ -\infty & \text{otherwise} \end{cases}$$

where $(A + \lambda I)^\dagger$ is the pseudo-inverse of $A + \lambda I$.

• **Lagrange dual problem:**

$$\begin{array}{ll} \text{maximize} & -b^\top (A+\lambda I)^\dagger b - \lambda \\ \text{subject to} & A + \lambda I \succeq 0, \quad b \in \mathcal{R}(A + \lambda I) \end{array}$$

with variable $\lambda \in \mathbf{R}$. This is a **convex** optimization problem:

In fact, it is readily solved since it can be expressed as

$$\begin{array}{ll} \text{maximize} & -\sum_{i=1}^{n} \left(q_i^\top b\right)^2 / (\lambda_i + \lambda) - \lambda \\ \text{subject to} & \lambda \geq -\lambda_{\min}(A) \end{array}$$

where λ_i and q_i are the eigenvalues and corresponding (orthonormal) eigenvectors of A, and we interpret $\left(q_i^\top b\right)^2/0$ as 0 if $q_i^\top b = 0$ and as ∞ otherwise.

• **Strong Duality:**

Despite the fact that the original problem (4.56) is not convex, we always have zero optimal duality gap for this problem.

Remark 4.57. *In fact, a more general result holds: strong duality holds for any optimization problem with* ***quadratic objective and one quadratic inequality constraint****, provided Slater's condition holds; see [2, B.1.]*

4.4 Random Strategies for Matrix Games

In this section we use strong duality to derive a basic result for a random zero-sum matrix game:

- Player 1 makes a choice (or move) $k \in \{1, \dots, n\}$,
- Player 2 makes a choice $l \in \{1, \dots, m\}$.
- Player 1 then makes a payment of P_{kl} to player 2 , where $P \in \mathbf{R}^{n \times m}$ is the payoff matrix for the game.
- The random choice for each player is **independent** with some distribution u and v:
$$\mathbb{P}(k = i) = u_i, \quad i = 1, \dots, n, \quad \mathbb{P}(l = i) = v_i, \quad i = 1, \dots, m$$
- The goal of player 1 is to choose u to minimize the expected payoff from player 1 to player 2:
$$\sum_{k=1}^{n} \sum_{l=1}^{m} u_k v_l P_{kl} = u^\top P v,$$
while the goal of player 2 is to choose v to maximize it.

• **Assuming player 1 strategy u is known to player 2:**

This clearly gives an advantage to player 2. Player 2 will choose v to maximize $u^\top P v$, which results in the expected payoff

$$\sup \left\{u^\top P v \mid v \succeq 0, \mathbf{1}^\top v = 1\right\} = \max_{i=1,\dots,m} \left(P^\top u\right)_i$$

The best thing player 1 can do is to choose u to **minimize this worst-case payoff** to player 2, i.e., to choose a strategy u that solves the problem

$$\begin{array}{ll} \text{minimize} & \max_{i=1,\dots,m} \left(P^\top u\right)_i \\ \text{subject to} & u \succeq 0, \quad \mathbf{1}^\top u = 1 \end{array} \tag{4.58}$$

which is a piecewise-linear **convex optimization problem.** We will denote the optimal value of this problem as p_1^*.

• **Assuming player 2 strategy u is known to player 1:**

This gives an advantage to player 1. In this case player 1 chooses u to minimize $u^\top P v$, which results in an expected payoff of

$$\inf \left\{u^\top P v \mid u \succeq 0, \mathbf{1}^\top u = 1\right\} = \min_{i=1,\dots,n} (Pv)_i$$

Player 2 chooses v to **maximize this**, i.e., chooses a strategy v that solves the problem

$$\begin{array}{ll} \text{maximize} & \min_{i=1,\dots,n} (Pv)_i \\ \text{subject to} & v \succeq 0, \quad \mathbf{1}^\top v = 1 \end{array} \tag{4.59}$$

which is another **convex optimization** problem, with piecewise-linear **(concave)** objective. We will denote the optimal value of this problem as p_2^*.

Remark 4.60. *($p_1^* \geq p_2^*$) It is intuitive that knowing your opponent's strategy gives an advantage, we therefore have:*

$$p_1^* \geq \text{the expected pay in case the strategies are not revealed to both players} \geq p_2^*$$

*We can interpret the difference, $p_1^* - p_2^*$, which is nonnegative, as the **advantage** conferred on a player by knowing the opponent's strategy.*

• **Show $p_1^* = p_2^*$ using Lagrange dual:**

We will establish a surprising result, $p_1^* = p_2^*$, by showing that the two problems (4.58) and (4.59) are Lagrange dual problems, for which strong duality obtains. In other words, in a matrix game with random strategies, **there is no advantage to knowing your opponent's strategy.**

We start by formulating (4.58) as an LP:

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & u \succeq 0, \quad \mathbf{1}^\top u = 1 \\ & P^\top u \preceq t\mathbf{1} \end{array}$$

with extra variable $t \in \mathbf{R}$.

Introducing the multiplier λ for $P^\top u \preceq t\mathbf{1}$, μ for $u \succeq 0$, and ν for $\mathbf{1}^\top u = 1$, the **Lagrangian** is

$$t + \lambda^\top \left(P^\top u - t\mathbf{1}\right) - \mu^\top u + \nu\left(1 - \mathbf{1}^\top u\right) = \nu + \left(1 - \mathbf{1}^\top \lambda\right) t + (P\lambda - \nu\mathbf{1} - \mu)^\top u$$

Lagrange dual function:

$$g(\lambda, \mu, \nu) = \begin{cases} \nu & \mathbf{1}^\top \lambda = 1, \quad P\lambda - \nu\mathbf{1} = \mu \\ -\infty & \text{otherwise} \end{cases}$$

Lagrange dual problem:

$$\begin{array}{ll} \text{maximize} & \nu \\ \text{subject to} & \lambda \succeq 0, \quad \mathbf{1}^\top \lambda = 1, \quad \mu \succeq 0 \\ & P\lambda - \nu\mathbf{1} = \mu \end{array}$$

Equivalent Lagrange dual problem:

Eliminating μ we obtain the following Lagrange dual of (4.58):

$$\begin{array}{ll} \text{maximize} & \nu \\ \text{subject to} & \lambda \succeq 0, \quad \mathbf{1}^\top \lambda = 1 \\ & P\lambda \succeq \nu\mathbf{1} \end{array}$$

with variables λ, ν. But this is clearly equivalent to (4.59).

Since the LPs are feasible, we have strong duality (see section 4.3.4.2), and the optimal values of (4.58) and (4.59) are equal.

4.5 Geometric interpretation

4.5.1 Weak and Strong Duality via Set of Values

Define the set of values taken on by the constraint and objective functions:

$$\mathcal{G} = \{(f_1(x), \ldots, f_m(x), h_1(x), \ldots, h_p(x), f_0(x)) \in \mathbf{R}^m \times \mathbf{R}^p \times \mathbf{R} \mid x \in \mathcal{D}\}$$

The optimal value p^* of (4.2) is easily expressed in terms of $\mathcal{G}$ as:

• **Problem**:

$$p^* = \inf\{t \mid (u, v, t) \in \mathcal{G}, u \preceq 0, v = 0\}$$

Note the optimization variable is $(u, v, t) \in \mathcal{G}$.

• **Lagrangian:** the affine function

$$L = (\lambda, \nu, 1)^\top (u, v, t) = \sum_{i=1}^m \lambda_i u_i + \sum_{i=1}^p \nu_i v_i + t$$

• Lagrangian dual function:

Minimizing over $(u, v, t) \in \mathcal{G}$, we have

$$g(\lambda, \nu) = \inf \left\{ (\lambda, \nu, 1)^\top (u, v, t) \mid (u, v, t) \in \mathcal{G} \right\}$$

• Supporting hyperplane:

If $g(\lambda, \nu)$ is finite, then the inequality

$$(\lambda, \nu, 1)^\top (u, v, t) \geq g(\lambda, \nu)$$

defines a supporting hyperplane to $\mathcal{G}$. It is **nonvertical** supporting hyperplane, because the last component of the normal vector is nonzero.

• Lower bound and weak duality:

Now **suppose** $\lambda \succeq 0$**.** Then, obviously, $t \geq (\lambda, \nu, 1)^\top (u, v, t)$ if $u \preceq 0$ and $v = 0$ Therefore

$$\begin{aligned}
p^* &= \inf\{t \mid (u, v, t) \in \mathcal{G}, u \preceq 0, v = 0\} \\
&\geq \inf \left\{ (\lambda, \nu, 1)^\top (u, v, t) \mid (u, v, t) \in \mathcal{G}, u \preceq 0, v = 0 \right\} \\
&\geq \inf \left\{ (\lambda, \nu, 1)^\top (u, v, t) \mid (u, v, t) \in \mathcal{G} \right\} \\
&= g(\lambda, \nu) \qquad (4.61)
\end{aligned}$$

This is the lower bound property as described in Lemma 4.22. Maximize over $\lambda \succeq 0$, we have **weak duality:** $p^* \geq d^* = \max_{\lambda \succeq 0} g(\lambda, \nu)$.

• Figure illustration:

This interpretation is illustrated in figures Fig. 18 and Fig. 19, for a simple problem with one inequality constraint.

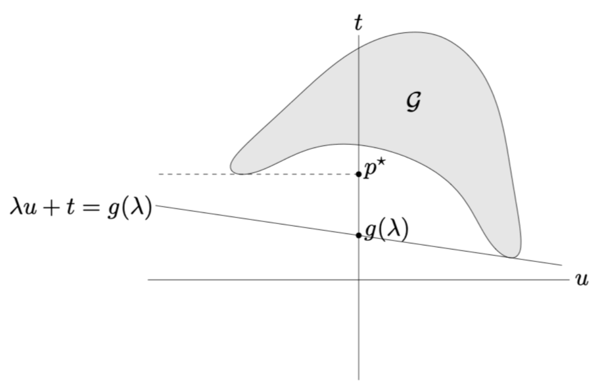


Fig. 18: Geometric interpretation of dual function and lower bound $g(\lambda) \leq p^*$, for a problem with one (inequality) constraint. Given λ, we minimize $(\lambda, 1)^\top (u, t)$ over $\mathcal{G} = \{(f_1(x), f_0(x)) \mid x \in \mathcal{D}\}$. This yields a supporting hyperplane with slope $-\lambda$. The intersection of this hyperplane with the $u = 0$ axis gives $g(\lambda)$.

4.5.1.1 Epigraph variation

We describe a variation on the geometric interpretation of duality in terms of $\mathcal{G}$, which explains why strong duality obtains for (most) convex problems.

We define the set $\mathcal{A} \subseteq \mathbf{R}^m \times \mathbf{R}^p \times \mathbf{R}$ as

$$\mathcal{A} = \mathcal{G} + \left(\mathbf{R}_+^m \times \{0\} \times \mathbf{R}_+ \right)$$

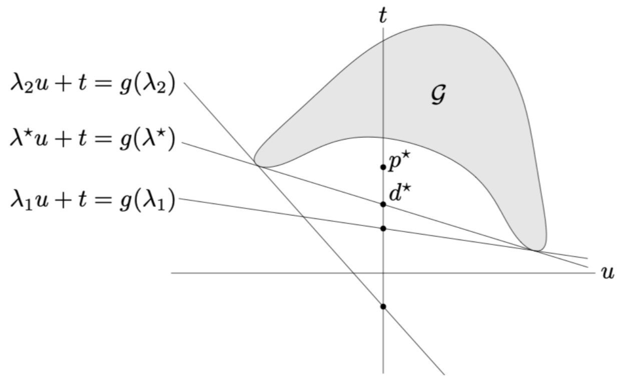


Fig. 19: Supporting hyperplanes corresponding to three dual feasible values of λ, including the optimum λ^*. Strong duality does not hold; the optimal duality gap $p^* - d^*$ is positive.

or, more explicitly,

$$\begin{aligned}\mathcal{A} = \{(u, v, t) \mid \exists\, x \in \mathcal{D}, f_i(x) \leq u_i, i = 1, \ldots, m \\ h_i(x) = v_i, i = 1, \ldots, p, f_0(x) \leq t\}\end{aligned}$$

We can think of $\mathcal{A}$ as **a sort of epigraph form** of $\mathcal{G}$, since $\mathcal{A}$ includes all the points in $\mathcal{G}$, as well as points that are **"worse"**, i.e., those with larger objective or inequality constraint function values.

We can express the equations and values in terms of $\mathcal{A}$ instead of $\mathcal{G}$:

• **Problem**:

$$p^* = \inf\{t \mid (0, 0, t) \in \mathcal{A}\}$$

• **Lagrangian:** the affine function

$$L = (\lambda, \nu, 1)^\top (u, v, t)$$

• **Lagrangian dual function:**

Minimizing over $(u, v, t) \in \mathcal{A}$, we have

$$g(\lambda, \nu) = \inf \left\{(\lambda, \nu, 1)^\top (u, v, t) \mid (u, v, t) \in \mathcal{A}\right\}$$

• **Supporting hyperplane:**

If $g(\lambda, \nu)$ is **finite**, then the inequality

$$(\lambda, \nu, 1)^\top (u, v, t) \geq g(\lambda, \nu)$$

defines a **nonvertical** supporting hyperplane to $\mathcal{A}$.

Note here

$$g(\lambda, \nu) \text{ is } \textbf{finite} \text{ has already implies } \lambda \succeq 0 \tag{4.62}$$

, this is unlike in (4.61) where we need **explicitly** assume $\lambda \succeq 0$.

• **Lower bound and weak duality:**

In particular, since $(0, 0, p^*) \in \partial\mathcal{A}$, we have

$$p^* = (\lambda, \nu, 1)^\top (0, 0, p^*) \geq g(\lambda, \nu) \tag{4.63}$$

This is the lower bound property as described in Lemma 4.22. Maximize over $\lambda \succeq 0$, we have **weak duality:** $p^* \geq d^* = \max_{\lambda \succeq 0} g(\lambda, \nu)$.

Strong duality holds if and only if we have equality in (4.63) for some dual feasible (λ, ν)**, i.e., there exists a nonvertical supporting hyperplane to** $\mathcal{A}$ **at its boundary point** $(0, 0, p^*)$**.**

• **Figure illustration:**

This second interpretation is illustrated in Fig. 20.

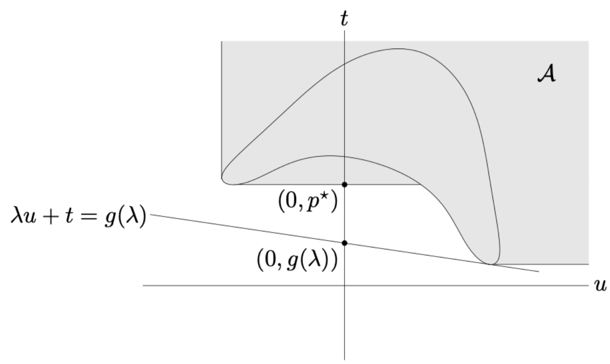


Fig. 20: Geometric interpretation of dual function and lower bound $g(\lambda) \leq p^*$, for a problem with one (inequality) constraint. Given λ, we minimize $(\lambda, 1)^\top(u, t)$ over $\mathcal{A} = \{(u, t) \mid \exists\, x \in \mathcal{D}, f_0(x) \leq t, f_1(x) \leq u\}$. This yields a supporting hyperplane with slope $-\lambda$. The intersection of this hyperplane with the $u = 0$ axis gives $g(\lambda)$.

4.5.2 Proof of Strong Duality under Constraint Qualfication (Slater's constraint)

In this section we prove that **Slater's constraint qualification guarantees strong duality** (and that the dual optimum is attained if $d^* > -\infty$) for a convex problem.

• **Convex primal problem**

We consider (4.2):

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array} \tag{4.64}$$

with $f_0, \ldots, f_m$ convex.

• **Slater's condition**:

There exists an $\tilde{x} \in \operatorname{ri} \mathcal{D}$ such that

$$f_i(\tilde{x}) < 0, \quad i = 1, \ldots, m, \quad A\tilde{x} = b$$

• **Goal:** prove strong duality.

In order to simplify the proof, we make **two additional assumptions:**

1. $\mathcal{D}$ has nonempty interior (hence, $\operatorname{ri} \mathcal{D} = \operatorname{int} \mathcal{D}$)
2. $\operatorname{rank} A = p$.

• **Proof:**

We can assume that p^* is finite. (Since there is a feasible point, we can only have $p^* = -\infty$ or p^* finite; if $p^* = -\infty$, then $d^* = -\infty$ by weak duality.)

It is easy to show that the set $\mathcal{A}$ defined in section 4.5.1.1 is shown to be convex if the underlying problem is convex (similar to Remark 3.20). We define a second convex set $\mathcal{B}$ as

$$\mathcal{B} = \{(0, 0, s) \in \mathbf{R}^m \times \mathbf{R}^p \times \mathbf{R} \mid s < p^*\}$$

The sets $\mathcal{A}$ and $\mathcal{B}$ do not intersect. To see this, suppose $(u, v, t) \in \mathcal{A} \cap \mathcal{B}$. Since $(u, v, t) \in \mathcal{B}$ we have $u = 0, v = 0$, and $t < p^*$. Since $(u, v, t) \in \mathcal{A}$, there exists an x with $f_i(x) \leq 0, i = 1, \ldots, m, Ax - b = 0$, and $f_0(x) \leq t < p^*$, which is impossible since p^* is the optimal value of the primal problem.

By the separating hyperplane theorem, there exists $(\tilde{\lambda}, \tilde{\nu}, \mu) \neq 0$ and α such that

$$(u, v, t) \in \mathcal{A} \Longrightarrow \tilde{\lambda}^\top u + \tilde{\nu}^\top v + \mu t \geq \alpha \tag{4.65}$$

and

$$(u, v, t) \in \mathcal{B} \Longrightarrow \tilde{\lambda}^\top u + \tilde{\nu}^\top v + \mu t \leq \alpha \tag{4.66}$$

From (4.65) we conclude that $\tilde{\lambda} \succeq 0$ and $\mu \geq 0$ (similar to the reason in (4.62)). The condition (4.66) simply means that $\mu t \leq \alpha$ for all $t < p^*$, and hence, $\mu p^* \leq \alpha$. Together with (4.65) we conclude that for any $x \in \mathcal{D}$ (since $\mathcal{D} \subseteq \mathcal{A}$)

$$\sum_{i=1}^m \tilde{\lambda}_i f_i(x) + \tilde{\nu}^\top (Ax - b) + \mu f_0(x) \geq \alpha \geq \mu p^* \tag{4.67}$$

If that $\mu > 0$ (nonvertical case). In that case we can divide (4.67) by μ to obtain

$$L(x, \tilde{\lambda}/\mu, \tilde{\nu}/\mu) \geq p^*$$

for all $x \in \mathcal{D}$, from which it follows, by minimizing over x, that $g(\lambda, \nu) \geq p^*$, where we define

$$\lambda = \tilde{\lambda}/\mu, \quad \nu = \tilde{\nu}/\mu$$

By weak duality we have $g(\lambda, \nu) \leq p^*$, so in fact $g(\lambda, \nu) = p^*$. **This shows that strong duality holds, and that the dual optimum is attained if $\mu > 0$.**

We now exclude the case $\mu = 0$ (vertical case). From (4.67), we conclude that for all $x \in \mathcal{D}$,

$$\sum_{i=1}^m \tilde{\lambda}_i f_i(x) + \tilde{\nu}^\top (Ax - b) \geq 0$$

Applying this to the point $\tilde{x}$ that satisfies the Slater condition, we have

$$\sum_{i=1}^m \tilde{\lambda}_i f_i(\tilde{x}) \geq 0$$

Since $f_i(\tilde{x}) < 0$ and $\tilde{\lambda}_i \geq 0$, we conclude that $\tilde{\lambda} = 0$. From $(\tilde{\lambda}, \tilde{\nu}, \mu) \neq 0$ and $\tilde{\lambda} = 0, \mu = 0$, we conclude that $\tilde{\nu} \neq 0$. Then (4.67) implies that for all $x \in \mathcal{D}$, $\tilde{\nu}^\top (Ax - b) \geq 0$. But $\tilde{x}$ satisfies $\tilde{\nu}^\top (A\tilde{x} - b) = 0$, and since $\tilde{x} \in \operatorname{int} \mathcal{D}$, there are points in $\mathcal{D}$ with $\tilde{\nu}^\top (Ax - b) < 0$ unless $A^\top \tilde{\nu} = 0$. This, of course, contradicts our assumption that $\operatorname{rank} A = p$.

• **Figure illustration:**

The geometric idea behind the proof is illustrated in Fig. 21, for a simple problem with one inequality constraint. The hyperplane separating $\mathcal{A}$ and $\mathcal{B}$ defines a supporting hyperplane to $\mathcal{A}$ at $(0, p^*)$. **Slater's constraint qualification is used to establish that the hyperplane must be nonvertical** (i.e., has a normal vector of the form $(\lambda^*, 1)$).

4.5.3 Multicriterion interpretation

• **Lagrange duality for a problem without equality constraints**

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \end{array} \tag{4.68}$$

• **scalarization method for the (unconstrained) multicriterion problem**

In scalarization, to scalarize the multicriterion problem:

$$\text{minimize (w.r.t. } \mathbf{R}_+^{m+1}) \quad F(x) = (f_1(x), \ldots, f_m(x), f_0(x))$$

we choose a **positive** vector $\tilde{\lambda} \succ 0$ and minimize the scalar function

$$\tilde{\lambda}^\top F(x)$$

From section 2.4.2.2, we know any minimizer is guaranteed to be Pareto optimal.

Since we can scale $\tilde{\lambda}$ by a positive constant, without affecting the minimizers, we can, without loss of generality, take $\tilde{\lambda} = (\lambda, 1)$. Thus, **in scalarization we minimize the function**

$$\tilde{\lambda}^\top F(x) = f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) \tag{4.69}$$

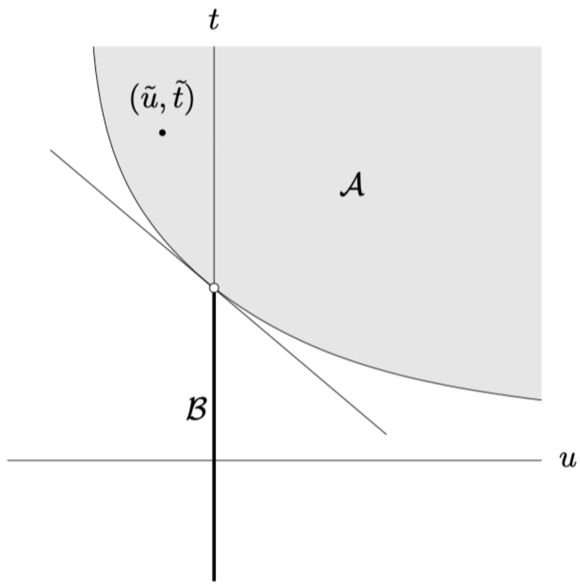


Fig. 21: Illustration of strong duality proof, for a convex problem that satisfies Slater's constraint qualification. The set $\mathcal{A}$ is shown shaded, and the set $\mathcal{B}$ is the thick vertical line segment, not including the point $(0, p^*)$, shown as a small open circle. The two sets are convex and do not intersect, so they can be separated by a hyperplane. Slater's constraint qualification guarantees that any separating hyperplane must be nonvertical, since it must pass to the left of the point $(\tilde{u}, \tilde{t}) = (f_1(\tilde{x}), f_0(\tilde{x}))$, where $\tilde{x}$ is strictly feasible.

• **Relation: "sufficiency"**

For any **positive vector** $\lambda \succ 0$, the scalarization (4.69) is exactly the Lagrangian for the problem (4.68).

• **Relation: "necessity (convex)"**

We need to prove:

"In **convex** multicriterion problem , $\forall x$ minimizes the multicriterion function (i.e. every Pareto optimal point), there exists a **nonzero** $\tilde{\lambda} \succeq 0$ so that it x minimizer $\tilde{\lambda}^\top F(x)$."

We considered the set $\mathcal{A}$, defined in section 4.5.1.1

$$\mathcal{A} = \left\{ t \in \mathbf{R}^{m+1} \mid \exists\, x \in \mathcal{D}, f_i(x) \le t_i, i = 0, \dots, m \right\}$$

The set $\mathcal{O}$ of all Pareto optimal point does not need to convex even the problem is convex (see Fig. 8 the bottom edge of the square). However $\mathcal{A}$ is convex if the problem is convex and the minimal elements of $\mathcal{A}$ are the same as the minimal points of $\mathcal{O}$ (c.f. [2, ex 4.53]). From Lemma 2.136, we know we can have one $\tilde{\lambda} \succeq 0$ so that it x minimizes $\tilde{\lambda}^\top x$ over $\mathcal{A}$. Therefore this x is also a minimizer of $\tilde{\lambda}^\top F(x)$ (i.e. x also minimizes of $\tilde{\lambda}^\top x$ over $\mathcal{O}$).

4.6 Saddle-Point Interpretation

In this section we give several interpretations of Lagrange duality.

4.6.1 Max-min Characterization of Weak and Strong Duality

It is possible to express the primal and the dual optimization problems in a form that is more symmetric. To simplify the discussion we assume there are no equality constraints; the results are easily extended to cover them.

First note that

$$\sup_{\lambda\succeq 0} L(x,\lambda) = \sup_{\lambda\succeq 0}\left(f_0(x)+\sum_{i=1}^m \lambda_i f_i(x)\right)$$
$$= \begin{cases} f_0(x) & f_i(x)\le 0, \quad i=1,\dots,m \\ \infty & \text{otherwise}\end{cases}$$

This means that we can express the optimal value of the primal problem as

$$p^* = \inf_x \sup_{\lambda\succeq 0} L(x,\lambda)$$

By the definition of the dual function, we also have

$$d^* = \sup_{\lambda\succeq 0}\inf_x L(x,\lambda)$$

Thus, **weak duality** can be expressed as the inequality

$$\sup_{\lambda\succeq 0}\inf_x L(x,\lambda) \le \inf_x \sup_{\lambda\succeq 0} L(x,\lambda) \tag{4.70}$$

and **strong duality** as the equality

$$\sup_{\lambda\succeq 0}\inf_x L(x,\lambda) = \inf_x \sup_{\lambda\succeq 0} L(x,\lambda) \tag{4.71}$$

• **max-min inequality:**

(4.70) is **always** true from the "max-min inequality":

$$\sup_{z\in Z}\inf_{w\in W} f(w,z) \le \inf_{w\in W}\sup_{z\in Z} f(w,z) \tag{4.72}$$

for any $f:\mathbf{R}^n\times\mathbf{R}^m\to\mathbf{R}$ (and any $W\subseteq\mathbf{R}^n$ and $Z\subseteq\mathbf{R}^m$).

• **strong max-min property:**

We say that f (and W and Z) satisfy the **strong max-min property or the saddlepoint property** if

$$\sup_{z\in Z}\inf_{w\in W} f(w,z) = \inf_{w\in W}\sup_{z\in Z} f(w,z) \tag{4.73}$$

Of course the strong max-min property holds only in special cases, for example:

- when $f:\mathbf{R}^n\times\mathbf{R}^m\to\mathbf{R}$ is the Lagrangian of a problem for which strong duality obtains, $W=\mathbf{R}^n$, and $Z=\mathbf{R}^m_+$. **In other words, it is (4.73).**
- Another speical case is Sion's minimax theorem Lemma 3.174

4.6.2 Saddle-point Interpretation

Definition 4.74. ***(Saddle-point)*** *We refer to a pair $\tilde{w}\in W, \tilde{z}\in Z$ as a* ***saddle-point*** *for f(and W and Z) if*

$$f(\tilde{w},z)\le f(\tilde{w},\tilde{z})\le f(w,\tilde{z})$$

for all $w\in W$ and $z\in Z$.

In other words,

- *$\tilde{w}$ minimizes $f(w,\tilde{z})$ (over $w\in W$): $f(\tilde{w},\tilde{z})=\inf_{w\in W} f(w,\tilde{z})$,*
- *$\tilde{z}$ maximizes $f(\tilde{w},z)$ (over $z\in Z$): $f(\tilde{w},\tilde{z})=\sup_{z\in Z} f(\tilde{w},z)$*

• **Saddle-point $\Rightarrow$ strong max-min property**

This implies that the **strong max-min property** (4.73) holds, and that the common value is $f(\tilde{w},\tilde{z})$:

$$\inf_w \sup_z f(w,z)\le \sup_z f(\tilde{w},z) = f(\tilde{w},\tilde{z}) = \inf_w f(w,\tilde{z}) \le \sup_z \inf_w f(x,z)$$

• **Strong duality vs. Saddle-point**

- "$\Rightarrow$": if x^* and λ^* are primal and dual optimal points for a problem in which strong duality obtains, they form a saddle-point for the Lagrangian.
- "$\Leftarrow$": if (x,λ) is a saddle-point of the Lagrangian, then x is primal optimal, λ is dual optimal, and the optimal duality gap is zero.

4.6.3 Game Interpretation

Section 4.4 shows that for a random zero-sum **discrete** matrix game strong duality holds. In this section, we show in general any strong duality can be interpreted as a **continuous** zero-sum game.

- Player 1 makes a choice (or move) $w \in W$.
- Player 2 makes a choice $z \in Z$.
- Player 1 then makes a payment of $f(w, z)$ to player 2.
- The goal of player 1 is to choose u to **minimize** the payoff f, while the goal of player 2 is to choose v to **maximize** f.

If the first player chooses $w \in W$, and the second player selects $z \in Z$, then player 1 pays an amount $f(w, z)$ to player 2. Player 1 therefore wants to minimize f, while player 2 wants to maximize f. (The game is called continuous since the choices are vectors, and not discrete.)

• Assuming player 1 strategy w is known to player 2:

Suppose that player 1 makes his choice first, and then player 2, after learning the choice of player 1, makes her selection. Player 2 wants to maximize the payoff $f(w, z)$, and so will choose $z \in Z$ to maximize $f(w, z)$. The resulting payoff will be

$$\sup_{z \in Z} f(w, z)$$

which depends on w, the choice of the first player. (We assume here that the supremum is achieved; if not the optimal payoff can be arbitrarily close to $\sup_{z \in Z} f(w, z)$.)

The best thing player 1 can do is to choose $w \in W$ to **minimize this worst-case payoff** to player 2:

$$\inf_{w \in W} \sup_{z \in Z} f(w, z)$$

• Assuming player 2 strategy z is known to player 2:

Suppose that player 2 makes his choice first, and then player 1, after learning the choice of player 2, makes her selection. Player 1 must choose

$$\inf_{w \in W} f(w, z)$$

The best thing player 2 can do is to choose $z \in Z$ to **maximize this payoff**:

$$\sup_{z \in Z} \inf_{w \in W} f(w, z)$$

• Saddle-point:

- The max-min inequality (4.72) states, similar to Remark 4.60, that there is some advantage if knowing your opponent's strategy.
- When the saddle-point property (4.73) holds, there is no advantage to playing second.

If $(\tilde{w}, \tilde{z})$ is a saddle-point for f (and W and Z), then it is called a **solution of the game**; $\tilde{w}$ is called the optimal choice or strategy for player 1 , and $\tilde{z}$ is called the optimal choice or strategy for player 2. In this case there is no advantage to playing second.

• Lagrange duality vs. Game:

Consider the special case where the payoff function is the Lagrangian, $W = \mathbf{R}^n$ and $Z = \mathbf{R}^m_+$.

- advantage if knowing your opponent's strategy corresponds to weak duality
- no advantage if knowing your opponent's strategy corresponds to strong duality

4.7 Optimality Conditions

We remind the reader that we do not assume the problem (4.2) is convex, unless explicitly stated.

4.7.1 Certificate of Suboptimality and Stopping Criteria

From lower bound property Lemma 4.22, we know if x is primal feasible and (λ, ν) is dual feasible, we then have

$$f_0(x) - p^* \leq f_0(x) - g(\lambda, \nu)$$

In particular, this establishes that x is ϵ-suboptimal (c.f. Definition 4.7), with $\epsilon = f_0(x) - g(\lambda, \nu)$. (It also establishes that (λ, ν) is ϵ-suboptimal for the dual problem.)

Definition 4.75. ***(Duality Gap)*** *We refer to the gap between primal and dual objectives,*

$$f_0(x) - g(\lambda, \nu)$$

as the duality gap associated with the primal feasible point x and dual feasible point (λ, ν).

Remark 4.76. *Note Definition 4.44 refers $p^* - d^*$ as the* ***optimal duality gap***

• **Bounds:**

A primal dual feasible pair $x, (\lambda, \nu)$ localizes the optimal value of the primal (and dual) problems to an interval:

$$p^* \in [g(\lambda, \nu), f_0(x)], \quad d^* \in [g(\lambda, \nu), f_0(x)]$$

the width of which is the duality gap.

If the duality gap of the primal dual feasible pair $x, (\lambda, \nu)$ is zero, i.e., $f_0(x) = g(\lambda, \nu)$, then x is primal optimal and (λ, ν) is dual optimal. We can think of (λ, ν) as a **certificate that proves x is optimal** (and, similarly, we can think of x as a certificate that proves (λ, ν) is dual optimal).

• **Stopping criteria:**

These observations can be used in optimization algorithms to provide **nonheuristic stopping criteria.** Suppose an algorithm produces a sequence of primal feasible $x^{(k)}$ and dual feasible $\left(\lambda^{(k)}, \nu^{(k)}\right)$, for $k = 1, 2, \ldots$, and $\epsilon_{\text{abs}} > 0$ is a given required absolute accuracy. Then the stopping criterion

$$f_0\left(x^{(k)}\right) - g\left(\lambda^{(k)}, \nu^{(k)}\right) \leq \epsilon_{\text{abs}}$$

guarantees that when the algorithm terminates, $x^{(k)}$ is ϵ_{abs} -suboptimal.

Indeed, $\left(\lambda^{(k)}, \nu^{(k)}\right)$ is a certificate that proves it. (Of course strong duality must hold if this method is to work for arbitrarily small tolerances ϵ_{abs} .)

A similar condition can be used to guarantee a given relative accuracy $\epsilon_{\text{rel}} > 0$. If

$$g\left(\lambda^{(k)}, \nu^{(k)}\right) > 0, \quad \frac{f_0\left(x^{(k)}\right) - g\left(\lambda^{(k)}, \nu^{(k)}\right)}{g\left(\lambda^{(k)}, \nu^{(k)}\right)} \leq \epsilon_{\text{rel}}$$

holds, or

$$f_0\left(x^{(k)}\right) < 0, \quad \frac{f_0\left(x^{(k)}\right) - g\left(\lambda^{(k)}, \nu^{(k)}\right)}{-f_0\left(x^{(k)}\right)} \leq \epsilon_{\text{rel}}$$

holds, then $p^* \neq 0$ and the relative error

$$\frac{f_0\left(x^{(k)}\right) - p^*}{|p^*|}$$

is guaranteed to be less than or equal to ϵ_{rel} .

4.7.2 Complementary Slackness: one of KKT conditions

• **Assumption: strong duality holds (and primal and dual optimal values are attained)**

Let x^* be a primal optimal and (λ^*, ν^*) be a dual optimal point. This means that

$$\begin{aligned} f_0(x^*) &= g(\lambda^*, \nu^*) \\ &= \inf_x \left(f_0(x) + \sum_{i=1}^m \lambda_i^* f_i(x) + \sum_{i=1}^p \nu_i^* h_i(x) \right) \\ &\le f_0(x^*) + \sum_{i=1}^m \lambda_i^* f_i(x^*) + \sum_{i=1}^p \nu_i^* h_i(x^*) \\ &\le f_0(x^*) \end{aligned}$$

- The first line states that the optimal duality gap is zero,
- The second line is the definition of the dual function.
- The third line follows since the infimum of the Lagrangian over x is less than or equal to its value at $x = x^*$.
- The last inequality follows from $\lambda_i^* \ge 0, f_i(x^*) \le 0, i = 1, \ldots, m$, and $h_i(x^*) = 0, i = 1, \ldots, p$. We conclude that the two inequalities in this chain hold with equality.

• **Conclusions:**

1. Since the inequality in the third line is an equality, we conclude that x^* minimizes $L(x, \lambda^*, \nu^*)$ over x. (The Lagrangian $L(x, \lambda^*, \nu^*)$ can have other minimizers, not unique.)
2. **"complementary slackness:"**
   We have
   $$\sum_{i=1}^m \lambda_i^* f_i(x^*) = 0$$
   Since each term in this sum is nonpositive, we get **complementary slackness in 3 equivalent form:**
   $$\begin{aligned} &1: \quad \lambda_i^* f_i(x^*) = 0, \quad i = 1, \ldots, m \\ &2: \quad \lambda_i^* > 0 \Longrightarrow f_i(x^*) = 0, \quad i = 1, \ldots, m \\ &3: \quad f_i(x^*) < 0 \Longrightarrow \lambda_i^* = 0, \quad i = 1, \ldots, m \end{aligned} \tag{4.77}$$

Remark 4.78. ***(explanation)*** *Simliar to the complementary slackness in Remark 3.134, roughly speaking, this means the i-th optimal Lagrange multiplier is zero unless the i-th constraint is* ***active*** *at the x^*.*

4.7.3 KKT optimality conditions

We now assume that the functions $f_0, \ldots, f_m, h_1, \ldots, h_p$ are **differentiable (and therefore have open domains)**, but we make no assumptions yet about convexity.

4.7.3.1 KKT Conditions for Nonconvex Problems: "strong dual $\Longrightarrow$ KKT"

• **Assumption: strong duality holds (and primal and dual optimal values are attained)**

Let x^* be a primal optimal and (λ^*, ν^*) be a dual optimal point. Since x^* minimizes $L(x, \lambda^*, \nu^*)$ over x, it follows that its gradient must vanish at x^*, i.e.,

$$\nabla f_0(x^*) + \sum_{i=1}^m \lambda_i^* \nabla f_i(x^*) + \sum_{i=1}^p \nu_i^* \nabla h_i(x^*) = 0$$

Thus we have **"Karush-Kuhn-Tucker (KKT) conditions"**:

$$\begin{aligned} f_i(x^*) &\le 0, \quad i=1,\ldots,m \\ h_i(x^*) &= 0, \quad i=1,\ldots,p \\ \lambda_i^* &\ge 0, \quad i=1,\ldots,m \\ \lambda_i^* f_i(x^*) &= 0, \quad i=1,\ldots,m \\ \nabla f_0(x^*) + \sum_{i=1}^m \lambda_i^* \nabla f_i(x^*) + \sum_{i=1}^p \nu_i^* \nabla h_i(x^*) &= 0, \end{aligned}$$

4.7.3.2 KKT Conditions for Convex Problems: " KKT + convex $\Longrightarrow$ strong dual"

• Assumption: convex + KKT:

If f_i are convex and h_i are affine, and $\tilde{x}, \tilde{\lambda}, \tilde{\nu}$ are any points that satisfy the **KKT conditions**

$$\begin{aligned} f_i(\tilde{x}) &\le 0, \quad i=1,\ldots,m \\ h_i(\tilde{x}) &= 0, \quad i=1,\ldots,p \\ \tilde{\lambda}_i &\ge 0, \quad i=1,\ldots,m \\ \tilde{\lambda}_i f_i(\tilde{x}) &= 0, \quad i=1,\ldots,m \\ \nabla f_0(\tilde{x}) + \sum_{i=1}^m \tilde{\lambda}_i \nabla f_i(\tilde{x}) + \sum_{i=1}^p \tilde{\nu}_i \nabla h_i(\tilde{x}) &= 0 \end{aligned}$$

then we can **strong duality: $\tilde{x}$ and $(\tilde{\lambda}, \tilde{\nu})$ are primal and dual optimal, with zero duality gap.**

To see this, note that the first two conditions state that $\tilde{x}$ is primal feasible. Since $\tilde{\lambda}_i \ge 0, L(x, \tilde{\lambda}, \tilde{\nu})$ is convex in x; the last KKT condition states that its gradient with respect to x vanishes at $x = \tilde{x}$, so it follows that $\tilde{x}$ minimizes $L(x, \tilde{\lambda}, \tilde{\nu})$ over x. From this we conclude that

$$\begin{aligned} g(\tilde{\lambda}, \tilde{\nu}) &= L(\tilde{x}, \tilde{\lambda}, \tilde{\nu}) \\ &= f_0(\tilde{x}) + \sum_{i=1}^m \tilde{\lambda}_i f_i(\tilde{x}) + \sum_{i=1}^p \tilde{\nu}_i h_i(\tilde{x}) \\ &= f_0(\tilde{x}) \end{aligned}$$

This shows that $\tilde{x}$ and $(\tilde{\lambda}, \tilde{\nu})$ have zero duality gap, and therefore are primal and dual optimal with zero optimal duality gap.

4.7.3.3 Summary

Under "Slater's condition + convex + differentiable objective and constraint functions":

$$\text{"}x, \lambda_i, \nu_i \text{ satisfies KKT"} \Longleftrightarrow \text{"strong dual with } x \text{ is optimal, } \lambda_i, \nu_i \text{ are dual optimal".} \tag{4.79}$$

4.7.3.4 Examples:

The KKT conditions play an important role in optimization. Many algorithms for convex optimization can be interpreted as solving the KKT conditions.

Example 4.80. ***(Equality constrained convex quadratic minimization)*** *We consider the problem*

$$\begin{array}{ll} \textit{minimize} & (1/2)x^\top P x + q^\top x + r \\ \textit{subject to} & Ax = b \end{array} \tag{4.81}$$

where $P \in \mathbf{S}_+^n$. The KKT conditions for this problem are

$$Ax^* = b, \quad Px^* + q + A^\top \nu^* = 0$$

which we can write as

$$\begin{bmatrix} P & A^\top \\ A & 0 \end{bmatrix} \begin{bmatrix} x^* \\ \nu^* \end{bmatrix} = \begin{bmatrix} -q \\ b \end{bmatrix}$$

Solving this set of $m+n$ equations in the $m+n$ variables x^*, ν^* gives the optimal primal and dual variables for (4.81).

Example 4.82. ***(Water-filling)*** *We consider the convex optimization problem*

$$\begin{array}{rl} \text{minimize} & -\sum_{i=1}^n \log(\alpha_i + x_i) \\ \textit{subject to} & x \succeq 0, \quad \mathbf{1}^\top x = 1 \end{array}$$

*where $\alpha_i > 0$. This problem arises in **information theory**, in allocating power to a set of n communication channels. The variable x_i represents the transmitter power allocated to the i-th channel, and $\log(\alpha_i + x_i)$ gives the capacity or communication rate of the channel, so the problem is to allocate a total power of one to the channels, in order to maximize the total communication rate.*

Introducing Lagrange multipliers $\lambda^* \in \mathbf{R}^n$ for the inequality constraints $x^* \succeq 0$, and a multiplier $\nu^* \in \mathbf{R}$ for the equality constraint $\mathbf{1}^\top x = 1$, we obtain the KKT conditions

$$\begin{gathered} x^* \succeq 0, \quad \mathbf{1}^\top x^* = 1, \quad \lambda^* \succeq 0, \quad \lambda_i^* x_i^* = 0, \quad i = 1, \dots, n \\ -1/(\alpha_i + x_i^*) - \lambda_i^* + \nu^* = 0, \quad i = 1, \dots, n \end{gathered}$$

We can directly solve these equations to find x^*, λ^*, and ν^*. We start by noting that λ^* acts as a slack variable in the last equation, so it can be eliminated, leaving

$$\begin{aligned} x^* \succeq 0, \quad \mathbf{1}^\top x^* = 1, \quad x_i^*\left(\nu^* - 1/(\alpha_i + x_i^*)\right) = 0, \quad i = 1, \dots, n \\ \nu^* \geq 1/(\alpha_i + x_i^*), \quad i = 1, \dots, n \end{aligned}$$

If $\nu^* < 1/\alpha_i$, this last condition can only hold if $x_i^* > 0$, which by the third condition implies that $\nu^* = 1/(\alpha_i + x_i^*)$. Solving for x_i^*, we conclude that $x_i^* = 1/\nu^* - \alpha_i$ if $\nu^* < 1/\alpha_i$. If $\nu^* \geq 1/\alpha_i$, then $x_i^* > 0$ is impossible, because it would imply $\nu^* \geq 1/\alpha_i > 1/(\alpha_i + x_i^*)$, which violates the complementary slackness condition. Therefore, $x_i^* = 0$ if $\nu^* \geq 1/\alpha_i$. Thus we have

$$x_i^* = \begin{cases} 1/\nu^* - \alpha_i & \nu^* < 1/\alpha_i \\ 0 & \nu^* \geq 1/\alpha_i \end{cases}$$

or, put more simply, $x_i^* = \max\{0, 1/\nu^* - \alpha_i\}$. Substituting this expression for x_i^* into the condition $\mathbf{1}^\top x^* = 1$ we obtain

$$\sum_{i=1}^n \max\{0, 1/\nu^* - \alpha_i\} = 1$$

The lefthand side is a piecewise-linear increasing function of $1/\nu^*$, with breakpoints at α_i, so the equation has a unique solution which is readily determined.

This solution method is called water-filling for the following reason. We think of α_i as the ground level above patch i, and then flood the region with water to a depth $1/\nu$. The total amount of water used is then $\sum_{i=1}^n \max\{0, 1/\nu^* - \alpha_i\}$. We then increase the flood level until we have used a total amount of water equal to one. The depth of water above patch i is then the optimal value x_i^*

4.7.4 Solving the Primal Problem via the Dual

We mentioned in Section 4.7.2 that: if strong duality holds and a dual optimal solution (λ^*, ν^*) exists, then any primal optimal point is also a minimizer of $L(x, \lambda^*, \nu^*)$.

Suppose we have strong duality and an optimal (λ^*, ν^*) is known. Suppose that the minimizer of $L(x, \lambda^*, \nu^*)$, i.e., the solution of

$$\text{minimize} \quad f_0(x) + \sum_{i=1}^m \lambda_i^* f_i(x) + \sum_{i=1}^p \nu_i^* h_i(x) \tag{4.83}$$

is unique.

Then

- if the solution of Section 4.7.4 is primal feasible, it must be **primal optimal**;
- if it is not primal feasible, then no primal optimal point can exist, i.e., we can conclude that the primal optimum is not attained.

Example 4.84. ***(Entropy Maximization)***

$$\begin{array}{ll} \text{minimize} & f_0(x) = \sum_{i=1}^n x_i \log x_i \\ \textit{subject to} & Ax \preceq b \\ & \mathbf{1}^\top x = 1 \end{array}$$

with domain $\mathbf{R}^n_{++}$, *and its dual problem*

$$\begin{array}{ll} \textit{maximize} & -b^\top \lambda - \nu - e^{-\nu-1} \sum_{i=1}^n e^{-a_i^\top \lambda} \\ \textit{subject to} & \lambda \succeq 0 \end{array}$$

where a_i *are the columns of* A. *We assume that the weak form of Slater's condition holds, i.e., there exists an* $x \succ 0$ *with* $Ax \preceq b$ *and* $\mathbf{1}^\top x = 1$, *so strong duality holds and an optimal solution* (λ^*, ν^*) *exists. Suppose we have solved the dual problem. The Lagrangian at* (λ^*, ν^*) *is*

$$L(x, \lambda^*, \nu^*) = \sum_{i=1}^n x_i \log x_i + \lambda^{*\top}(Ax - b) + \nu^* \left(\mathbf{1}^\top x - 1\right)$$

which is strictly convex on $\mathcal{D}$ *and bounded below, so it has a unique solution* x^*, *given by*

$$x_i^* = 1/\exp\left(a_i^\top \lambda^* + \nu^* + 1\right), \quad i = 1, \ldots, n$$

If x^* *is primal feasible, it must be the optimal solution of the primal problem. If* x^* *is not primal feasible, then we can conclude that the primal optimum is not attained.*

Example 4.85. ***(Minimizing a separable function subject to an equality constraint.)*** *We consider the problem*

$$\begin{array}{ll} \text{minimize} & f_0(x) = \sum_{i=1}^n f_i(x_i) \\ \textit{subject to} & a^\top x = b \end{array}$$

where $a \in \mathbf{R}^n, b \in \mathbf{R}$, *and* $f_i : \mathbf{R} \to \mathbf{R}$ *are differentiable and strictly convex. The objective function is called separable since it is a sum of functions of the individual variables* $x_1, \ldots, x_n$. *We assume that the domain of* f_0 *intersects the constraint set, i.e., there exists a point* $x_0 \in \operatorname{dom} f_0$ *with* $a^\top x_0 = b$. *This implies the problem has a unique optimal point* x^*. *The Lagrangian is*

$$L(x, \nu) = \sum_{i=1}^n f_i(x_i) + \nu\left(a^\top x - b\right) = -b\nu + \sum_{i=1}^n \left(f_i(x_i) + \nu a_i x_i\right)$$

which is also separable, so the dual function is

$$\begin{aligned} g(\nu) &= -b\nu + \inf_x \left(\sum_{i=1}^n \left(f_i(x_i) + \nu a_i x_i\right)\right) \\ &= -b\nu + \sum_{i=1}^n \inf_{x_i} \left(f_i(x_i) + \nu a_i x_i\right) \\ &= -b\nu - \sum_{i=1}^n f_i^*(-\nu a_i) \end{aligned}$$

The dual problem is thus

$$\textit{maximize} \quad -b\nu - \sum_{i=1}^n f_i^*(-\nu a_i)$$

with (scalar) variable $\nu \in \mathbf{R}$. *Now suppose we have found an optimal dual variable* ν^*. *(There are several simple methods for solving a convex problem with one scalar variable, such as the bisection method.) Since each* f_i *is strictly convex, the function* $L(x, \nu^*)$ *is strictly convex in* x, *and so has a unique minimizer* $\tilde{x}$. *But we also know that* x^* *minimizes* $L(x, \nu^*)$, *so we must have* $\tilde{x} = x^*$. *We can recover* x^* *from* $\nabla_x L(x, \nu^*) = 0$, *i.e., by solving the equations* $f_i'(x_i^*) = -\nu^* a_i$

4.8 Perturbation and Sensitivity Analysis

When strong duality obtains, the optimal dual variables give very useful information about the sensitivity of the optimal value with respect to **perturbations of the constraints.**

4.8.1 The Perturbed Problem

We consider the following perturbed version of the original optimization problem (4.2):

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq u_i, \quad i = 1, \ldots, m \\ & h_i(x) = v_i, \quad i = 1, \ldots, p \end{array} \tag{4.86}$$

with variable $x \in \mathbf{R}^n$.

1. We define $p^*(u, v)$ as the optimal value of the perturbed problem (4.86) :

$$p^*(u,v) = \inf \{ f_0(x) \mid \exists\, x \in \mathcal{D}, f_i(x) \leq u_i, i = 1, \ldots, m \\ h_i(x) = v_i, i = 1, \ldots, p \}$$

2. The original problem (4.2) is $u = 0, v = 0$, and $p^*(0, 0) = p^*$
3. The perturbed problem (4.86) results from the original problem (4.2) by tightening or relaxing each inequality constraint by u_i, and changing the righthand side of the equality constraints by v_i.
4. When the original problem is convex, the function $p^*(u, v)$ is convex; indeed, its epigraph is precisely the closure of the set $\mathcal{A}$ defined in section 4.5.1.1.

4.8.2 A Global Inequality

• Assumption: strong duality holds (and primal and dual optimal values are attained)

Let (λ^*, ν^*) be optimal for the dual (4.32) of the unperturbed problem (4.86) .

• Lower bound:

Then for all u and v we have

$$p^*(u, v) \geq p^*(0, 0) - {\lambda^*}^\top u - {\nu^*}^\top v \tag{4.87}$$

Proof. To establish this inequality, suppose that x is any feasible point for the perturbed problem, i.e., $f_i(x) \leq u_i$ for $i = 1, \ldots, m$, and $h_i(x) = v_i$ for $i = 1, \ldots, p$. Then we have, by strong duality,

$$\begin{aligned} p^*(0, 0) = g(\lambda^*, \nu^*) &\leq f_0(x) + \sum_{i=1}^{m} \lambda_i^* f_i(x) + \sum_{i=1}^{p} \nu_i^* h_i(x) \\ &\leq f_0(x) + {\lambda^*}^\top u + {\nu^*}^\top v \end{aligned}$$

(The first inequality follows from the definition of $g(\lambda^*, \nu^*)$; the second follows since $\lambda^* \succeq 0$.) We conclude that for any x feasible for the perturbed problem, we have

$$f_0(x) \geq p^*(0, 0) - {\lambda^*}^\top u - {\nu^*}^\top v$$

from which (4.87) follows. □

• Sensitivity interpretations

When strong duality holds, various sensitivity interpretations of the optimal Lagrange variables follow directly from the inequality (4.87) . Some of the conclusions are:

1. If λ_i^* is large and we tighten the i th constraint (i.e., choose $u_i < 0$), then the optimal value $p^*(u, v)$ is guaranteed to increase greatly.
2. If ν_i^* is large and positive and we take $v_i < 0$, or if ν_i^* is large and negative and we take $v_i > 0$, then the optimal value $p^*(u, v)$ is guaranteed to increase greatly.

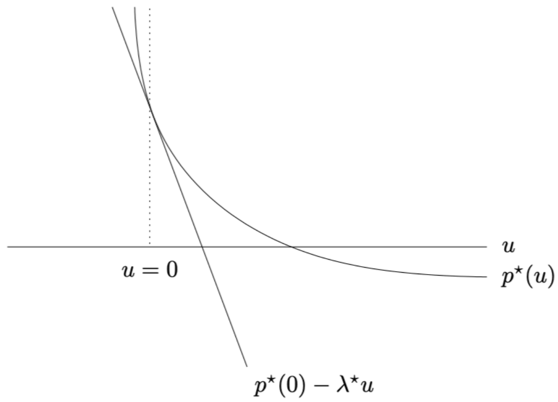


Fig. 22: Optimal value $p^*(u)$ of a convex problem with one constraint $f_1(x) \leq u$, as a function of u. For $u = 0$, we have the original unperturbed problem; for $u < 0$ the constraint is tightened, and for $u > 0$ the constraint is loosened. The affine function $p^*(0) - \lambda^* u$ is a lower bound on p^*.

3. If λ_i^* is small, and we loosen the i th constraint $(u_i > 0)$, then the optimal value $p^*(u, v)$ will not decrease too much.
4. If ν_i^* is small and positive, and $v_i > 0$, or if ν_i^* is small and negative and $v_i < 0$, then the optimal value $p^*(u, v)$ will not decrease too much.

• Figure illustration:

The inequality (4.87) is illustrated in Fig. 22 for a convex problem with one inequality constraint. The inequality states that the affine function $p^*(0) - \lambda^* u$ is a lower bound on the convex function p^*.

4.9 Local Sensitivity Analysis

• Assumption: strong duality + differentiable:

Suppose now that $p^*(u, v)$ is differentiable at $u = 0, v = 0$. Then, provided strong duality holds, the optimal dual variables λ^*, ν^* are related to the gradient of p^* at $u = 0, v = 0$:

$$\lambda_i^* = -\frac{\partial p^*(0,0)}{\partial u_i}, \quad \nu_i^* = -\frac{\partial p^*(0,0)}{\partial v_i} \tag{4.88}$$

Proof. To show (4.88), suppose $p^*(u, v)$ is differentiable and strong duality holds. For the perturbation $u = te_i, v = 0$, where e_i is the i th unit vector, we have

$$\lim_{t \to 0} \frac{p^*\left(te_i, 0\right) - p^*}{t} = \frac{\partial p^*(0,0)}{\partial u_i}$$

The inequality (4.87) states that for $t > 0$,

$$\frac{p^*\left(te_i, 0\right) - p^*}{t} \geq -\lambda_i^*$$

while for $t < 0$ we have the opposite inequality. Taking the limit $t \to 0$, with $t > 0$, yields

$$\frac{\partial p^*(0,0)}{\partial u_i} \geq -\lambda_i^*$$

while taking the limit with $t < 0$ yields the opposite inequality, so we conclude that

$$\frac{\partial p^*(0,0)}{\partial u_i} = -\lambda_i^*$$

The same method can be used to establish

$$\frac{\partial p^*(0,0)}{\partial v_i} = -\nu_i^*$$

□

• Interpretion: local sensitivities = optimal Lagrange multipliers

This property can be seen in the example shown in Fig. 22, where $-\lambda^*$ is the slope of p^* near $u = 0$

Thus, when $p^*(u, v)$ is differentiable at $u = 0, v = 0$, and strong duality holds, the **optimal Lagrange multipliers are exactly the local sensitivities of the optimal value with respect to constraint perturbations.**

- **non-active**: From complementary slackness (4.77), if $f_i(x^*) < 0$, then the constraint is inactive, and $\lambda_i^* = 0$. It follows that the constraint can be tightened or loosened a small amount without affecting the optimal value.
- **active**: From complementary slackness (4.77), if $f_i(x^*) < 0$, then the constraint is active. λ_i **tells us how active the constraint is**: If λ_i^* is small, it means that the constraint can be loosened or tightened a bit without much effect on the optimal value; if λ_i^* is large, it means that if the constraint is loosened or tightened a bit, the effect on the optimal value will be great.

4.10 Examples

In this section we show by example that **simple equivalent reformulations of a problem can lead to very different dual problems.** We consider the following types of reformulations:

- Introducing new variables and associated equality constraints section 4.1.3.6.
- Replacing the objective with an increasing function of the original objective section 4.1.3.2.
- Making explicit constraints implicit, i.e., incorporating them into the domain of the objective section 4.1.3.9.

4.10.1 Introducing New Variables and Equality Constraints

4.10.1.1 Unconstrained Problem

• Original problem:

Consider an unconstrained problem of the form

$$\text{minimize} \quad f_0(Ax + b)$$

• Original Lagrange dual function:

Its Lagrange dual function is the constant $g = p^*$. So while we do have strong duality, i.e., $p^* = d^*$, the Lagrangian dual is neither useful nor interesting.

• Reformulated problem:

$$\begin{array}{ll} \text{minimize} & f_0(y) \\ \text{subject to} & Ax + b = y \end{array} \tag{4.89}$$

Here we have introduced new variables y, as well as new equality constraints $Ax + b = y$.

• New Lagrange dual problem of reformulated problem:

We can use (4.30) and (4.34) to get the Lagrange dual problem is:

$$\begin{array}{ll} \text{maximize} & b^\top \nu - f_0^*(\nu) \\ \text{subject to} & A^\top \nu = 0 \end{array}$$

Thus, the dual of the reformulated problem (4.89) is considerably more useful than the dual of the original problem.

Example 4.90. ***(Unconstrained geometric program)*** *Consider the unconstrained geometric program*

$$\text{minimize} \quad \log\left(\sum_{i=1}^m \exp\left(a_i^\top x + b_i\right)\right)$$

We first reformulate it by introducing new variables and equality constraints:

$$\begin{array}{ll} \text{minimize} & f_0(y) = \log\left(\sum_{i=1}^m \exp y_i\right) \\ \textit{subject to} & Ax + b = y \end{array}$$

where $a_i^\top$ *are the rows of* A*. The conjugate of the log-sum-exp function (c.f. Example 3.189) is*

$$f_0^*(\nu) = \begin{cases} \sum_{i=1}^m \nu_i \log \nu_i & \nu \succeq 0, \mathbf{1}^\top \nu = 1 \\ \infty & \textit{otherwise} \end{cases}$$

so the dual of the reformulated problem can be expressed as which is an entropy maximization problem.

Example 4.91. ***(Norm approximation problem)*** *We consider the unconstrained norm approximation problem*

$$\textit{minimize } \|Ax - b\|$$

where $\|\cdot\|$ *is any norm. Here too the Lagrange dual function is constant, equal to the optimal value, and therefore not useful. Once again we reformulate the problem as*

$$\begin{array}{ll} \text{minimize} & \|y\| \\ \textit{subject to} & Ax - b = y \end{array}$$

The Lagrange dual problem is (c.f. Corollary 3.171):

$$\begin{array}{ll} \text{maximize} & b^\top \nu \\ \textit{subject to} & \|\nu\|_* \leq 1 \\ & A^\top \nu = 0 \end{array}$$

4.10.1.2 Constrained Problem

• **Original problem**:

$$\begin{array}{ll} \text{minimize} & f_0\left(A_0 x + b_0\right) \\ \text{subject to} & f_i\left(A_i x + b_i\right) \leq 0, \quad i = 1, \ldots, m \end{array} \tag{4.92}$$

where $A_i \in \mathbf{R}^{k_i \times n}$ and $f_i : \mathbf{R}^{k_i} \to \mathbf{R}$ are convex. (For simplicity we do not include equality constraints here.)

• **Reformulated problem:**

We introduce a new variable $y_i \in \mathbf{R}^{k_i}$, for $i = 0, \ldots, m$, and reformulate the problem as

$$\begin{array}{ll} \text{minimize} & f_0\left(y_0\right) \\ \text{subject to} & f_i\left(y_i\right) \leq 0, \quad i = 1, \ldots, m \\ & A_i x + b_i = y_i, \quad i = 0, \ldots, m \end{array}$$

• **New Lagrangian of reformulated problem:**

The Lagrangian for this problem is

$$L\left(x, y_0, \ldots, y_m, \lambda, \nu_0, \ldots, \nu_m\right) = f_0\left(y_0\right) + \sum_{i=1}^m \lambda_i f_i\left(y_i\right) + \sum_{i=0}^m \nu_i^\top\left(A_i x + b_i - y_i\right)$$

• **New Lagrange dual function of reformulated problem:**

To find the dual function we minimize over x and y_i. The minimum over x is $-\infty$ unless

$$\sum_{i=0}^m A_i^\top \nu_i = 0$$

in which case we have, for $\lambda \succ 0$,

$$
\begin{aligned}
& g\left(\lambda, \nu_0, \ldots, \nu_m\right) \\
& \quad = \sum_{i=0}^m \nu_i^\top b_i + \inf_{y_0, \ldots, y_m} \left(f_0\left(y_0\right) + \sum_{i=1}^m \lambda_i f_i\left(y_i\right) - \sum_{i=0}^m \nu_i^\top y_i \right) \\
& \quad = \sum_{i=0}^m \nu_i^\top b_i + \inf_{y_0} \left(f_0\left(y_0\right) - \nu_0^\top y_0 \right) + \sum_{i=1}^m \lambda_i \inf_{y_i} \left(f_i\left(y_i\right) - \left(\nu_i/\lambda_i\right)^\top y_i \right) \\
& \quad = \sum_{i=0}^m \nu_i^\top b_i - f_0^*\left(\nu_0\right) - \sum_{i=1}^m \lambda_i f_i^*\left(\nu_i/\lambda_i\right)
\end{aligned}
$$

The last expression involves the perspective of the conjugate function, and is therefore concave in the dual variables. Finally, we address the question of what happens when $\lambda \succeq 0$, but some λ_i are zero. If $\lambda_i = 0$ and $\nu_i \neq 0$, then the dual function is $-\infty$. If $\lambda_i = 0$ and $\nu_i = 0$, however, the terms involving y_i, ν_i, and λ_i are all zero. Thus, the expression above for g is valid for all $\lambda \succeq 0$, if we take $\lambda_i f_i^*\left(\nu_i/\lambda_i\right) = 0$ when $\lambda_i = 0$ and $\nu_i = 0$, and $\lambda_i f_i^*\left(\nu_i/\lambda_i\right) = \infty$ when $\lambda_i = 0$ and $\nu_i \neq 0$.

• **New Lagrange dual problem of reformulated problem:**

Therefore we can express the dual of the problem as:

$$
\begin{array}{ll}
\text{maximize} & \sum_{i=0}^m \nu_i^\top b_i - f_0^*\left(\nu_0\right) - \sum_{i=1}^m \lambda_i f_i^*\left(\nu_i/\lambda_i\right) \\
\text{subject to} & \lambda \succeq 0 \\
& \sum_{i=0}^m A_i^\top \nu_i = 0
\end{array}
$$

Example 4.93. ***(Inequality constrained geometric program)*** *The inequality constrained geometric program*

$$
\begin{array}{ll}
\text{minimize} & \log\left(\sum_{k=1}^{K_0} e^{a_{0k}^\top x + b_{0k}}\right) \\
\textit{subject to} & \log\left(\sum_{k=1}^{K_i} e^{a_{ik}^\top x + b_{ik}}\right) \leq 0, \quad i = 1, \ldots, m
\end{array}
$$

is of the form (4.92) *with* $f_i : \mathbf{R}^{K_i} \to \mathbf{R}$ *given by* $f_i(y) = \log\left(\sum_{k=1}^{K_i} e^{y_k}\right)$. *The conjugate of this function is*

$$
f_i^*(\nu) = \begin{cases} \sum_{k=1}^{K_i} \nu_k \log \nu_k & \nu \succeq 0, \quad \mathbf{1}^\top \nu = 1 \\ \infty & \textit{otherwise} \end{cases}
$$

We can immediately write down the dual problem as

$$
\begin{array}{ll}
\text{maximize} & b_0^\top \nu_0 - \sum_{k\bar{1}}^{K_0} \nu_{0k} \log \nu_{0k} + \sum_{i=1}^m \left(b_i^\top \nu_i - \sum_{k=1}^{K_i} \nu_{ik} \log\left(\nu_{ik}/\lambda_i\right) \right) \\
\textit{subject to} & \nu_0 \succeq 0, \quad \mathbf{1}^\top \nu_0 = 1 \\
& \nu_i \succeq 0, \quad \mathbf{1}^\top \nu_i = \lambda_i, \quad i = 1, \ldots, m \\
& \lambda_i \geq 0, \quad i = 1, \ldots, m \\
& \sum_{i=0}^m A_i^\top \nu_i = 0
\end{array}
$$

which further simplifies to

$$
\begin{array}{ll}
\text{maximize} & b_0^\top \nu_0 - \sum_{k=1}^{K_0} \nu_{0k} \log \nu_{0k} + \sum_{i=1}^m \left(b_i^\top \nu_i - \sum_{k=1}^{K_i} \nu_{ik} \log\left(\nu_{ik}/\mathbf{1}^\top \nu_i\right) \right) \\
\textit{subject to} & \nu_i \succeq 0, \quad i = 0, \ldots, m \\
& \mathbf{1}^\top \nu_0 = 1 \\
& \sum_{i=0}^m A_i^\top \nu_i = 0
\end{array}
$$

4.10.2 Transforming the Objective

If we replace the objective f_0 by an increasing function of f_0, the resulting problem is clearly equivalent. The dual of this equivalent problem, however, can be very different from the dual of the original problem.

Example 4.94. *We consider again the minimum norm problem*

$$\textit{minimize } \|Ax - b\|$$

where $\|\cdot\|$ is some norm. We reformulate this problem as

$$\begin{array}{ll} \text{minimize} & (1/2)\|y\|^2 \\ \textit{subject to} & Ax - b = y \end{array}$$

Here we have introduced new variables, and replaced the objective by half its square. Evidently it is equivalent to the original problem. The dual of the reformulated problem is

$$\begin{array}{ll} \textit{maximize} & -(1/2)\|\nu\|_*^2 + b^\top \nu \\ \textit{subject to} & A^\top \nu = 0 \end{array}$$

where we use the fact that the conjugate of $(1/2)\|\cdot\|^2$ is $(1/2)\|\cdot\|_*^2$.

*Note that this dual problem is **not the same as the dual problem Example 4.91** derived earlier.*

4.10.3 Implicit Constraints

The next simple reformulation we study is to include some of the constraints in the objective function, by modifying the objective function to be infinite when the constraint is violated.

Example 4.95. ***(Linear program with box constraints)*** *We consider the linear program*

$$\begin{array}{ll} \text{minimize} & c^\top x \\ \textit{subject to} & Ax = b \\ & l \preceq x \preceq u \end{array}$$

where $A \in \mathbf{R}^{p \times n}$ and $l \prec u$. The constraints $l \preceq x \preceq u$ are sometimes called box constraints or variable bounds.

We can, of course, derive the dual of this linear program. The dual will have a Lagrange multiplier ν associated with the equality constraint, λ_1 associated with the inequality constraint $x \preceq u$, and λ_2 associated with the inequality constraint $l \preceq x$. The dual is

$$\begin{array}{ll} \text{maximize} & -b^\top \nu - \lambda_1^\top u + \lambda_2^\top l \\ \textit{subject to} & A^\top \nu + \lambda_1 - \lambda_2 + c = 0 \\ & \lambda_1 \succeq 0, \quad \lambda_2 \succeq 0 \end{array}$$

*Instead, let us first **reformulate the problem** as*

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \textit{subject to} & Ax = b \end{array}$$

where we define

$$f_0(x) = \begin{cases} c^\top x & l \preceq x \preceq u \\ \infty & \textit{otherwise} \end{cases} \tag{4.96}$$

The dual function for the problem (4.96) *is*

$$\begin{aligned} g(\nu) &= \inf_{l \leq x \leq u} \left(c^\top x + \nu^\top (Ax - b)\right) \\ &= -b^\top \nu - u^\top \left(A^\top \nu + c\right)^- + l^\top \left(A^\top \nu + c\right)^+ \end{aligned}$$

where $y_i^+ = \max\{y_i, 0\}$, $y_i^- = \max\{-y_i, 0\}$. So here we are able to derive an analytical formula for g, which is a concave piecewise-linear function. The dual problem is the unconstrained problem

$$\textit{maximize} \quad -b^\top \nu - u^\top \left(A^\top \nu + c\right)^- + l^\top \left(A^\top \nu + c\right)^+$$

which has a quite different form from the dual of the original problem.

4.11 Theorems of Alternatives

4.11.1 Weak Alternatives via the Dual Function

In this section we apply Lagrange duality theory to the problem of determining feasibility of a system of inequalities and equalities

$$f_i(x) \leq 0, \quad i = 1, \dots, m, \quad h_i(x) = 0, \quad i = 1, \dots, p \tag{4.97}$$

Definition 4.98. ***(Feasibility Problems)*** *If the objective function is identically zero, the optimal value is either zero (if the feasible set is nonempty) or ∞ (if the feasible set is empty). We call this the **feasibility problem**, and will sometimes write it as*

$$\begin{array}{rl} \text{minimize} & 0 \\ \textit{subject to} & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & h_i(x) = 0, \quad i = 1, \dots, p \end{array} \tag{4.99}$$

The feasibility problem is thus to determine whether the constraints are consistent, and if so, find a point that satisfies them.

This problem has optimal value

$$p^* = \begin{cases} 0 & \text{(4.97) is feasible} \\ \infty & \text{(4.97) is infeasible} \end{cases} \tag{4.100}$$

so solving the optimization problem (4.99) is the same as solving the inequality system (4.97).

4.11.1.1 The Dual Function

We associate with the inequality system (4.97) the dual function

$$g(\lambda, \nu) = \inf_{x \in \mathcal{D}} \left(\sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \right)$$

Since $f_0 = 0$, the dual function is positive homogeneous in (λ, ν) : For $\alpha > 0, g(\alpha\lambda, \alpha\nu) = \alpha g(\lambda, \nu)$.

The dual problem associated with (4.99) is to maximize $g(\lambda, \nu)$ subject to $\lambda \succeq 0$. Since g is homogeneous, the optimal value of this dual problem is given by

$$d^* = \begin{cases} \infty & \lambda \succeq 0, g(\lambda, \nu) > 0 \text{ is feasible} \\ 0 & \lambda \succeq 0, g(\lambda, \nu) > 0 \text{ is infeasible} \end{cases} \tag{4.101}$$

Weak duality tells us that $d^* \leq p^*$. Combining this fact with (4.100) and (4.101) yields the following:

$$\lambda \succeq 0, \quad g(\lambda, \nu) > 0 \tag{4.102}$$

(4.102) is feasible $\implies$ (4.97) is infeasible
(4.97) is feasible $\implies$ (4.102) is infeasible

Indeed, we can interpret any solution (λ, ν) of the inequalities (4.102) as a proof or certificate of infeasibility of the system (4.97). And an x which satisfies (4.97) as a certificate establishing infeasibility of the inequality system (4.102).

Definition 4.103. ***(Weak Alternatives)*** *Two systems of inequalities (and equalities) are called **weak alternatives if at most one of the two is feasible.***

Remark 4.104. *In weak alternatives, **both can be infeasible but can not be both feasible.***

Example 4.105. *The systems* (4.97) *and* (4.102) *are weak alternatives. This is true whether or not the inequalities* (4.97) *are convex (i.e., f_i convex, h_i affine); moreover, the alternative inequality system* (4.102) *is always convex (i.e., g is concave and the constraints $\lambda_i \geq 0$ are convex).*

4.11.1.2 Strict Inequalities

We can also study feasibility of the strict inequality system

$$f_i(x) < 0, \quad i = 1, \ldots, m, \quad h_i(x) = 0, \quad i = 1, \ldots, p \tag{4.106}$$

With g defined as for the nonstrict inequality system, we have the alternative inequality system

$$\lambda \succeq 0, \quad \lambda \neq 0, \quad g(\lambda, \nu) \geq 0 \tag{4.107}$$

We can show directly that (4.106) and (4.107) are weak alternatives. Suppose there exists an $\tilde{x}$ with $f_i(\tilde{x}) < 0, h_i(\tilde{x}) = 0$. Then for any $\lambda \succeq 0, \lambda \neq 0$, and ν,

$$\lambda_1 f_1(\tilde{x}) + \cdots + \lambda_m f_m(\tilde{x}) + \nu_1 h_1(\tilde{x}) + \cdots + \nu_p h_p(\tilde{x}) < 0$$

It follows that

$$\begin{aligned} g(\lambda, \nu) &= \inf_{x \in \mathcal{D}} \left(\sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \right) \\ &\leq \sum_{i=1}^{m} \lambda_i f_i(\tilde{x}) + \sum_{i=1}^{p} \nu_i h_i(\tilde{x}) \\ &< 0 \end{aligned}$$

Therefore, feasibility of (4.106) implies that there does not exist (λ, ν) satisfying (4.107).

Thus, we can prove infeasibility of (4.106) by producing a solution of the system (4.107), we can prove infeasibility of (4.107) by producing a solution of the system (4.106).

4.11.2 Strong Alternative

Compare to weak alternative where at most one of the two is feasible. Here we define a new concept:

Definition 4.108. ***(Strong Alternatives)*** *Two systems of inequalities (and equalities) are called* ***weak alternatives if exactly one of the two alternatives holds.*** *In other words, each of the inequality systems is feasible if and only if the other is infeasible.*

Remark 4.109. ***(weak vs. strong)*** *weak alternative is one direction while strong alternative is a two direction:*

- *weak:* **(4.102)** ***is feasible*** $\implies$ **(4.97)** ***is infeasible***
- *strong:* **(4.111)** ***is feasible*** $\iff$ **(4.112)** ***is infeasible***

In this section we assume that f_i are convex and h_i are affine, so the inequality system (4.97) can be expressed as

$$f_i(x) \leq 0, \quad i = 1, \ldots, m, \quad Ax = b \tag{4.110}$$

where $A \in \mathbf{R}^{p \times n}$.

4.11.2.1 Strict Inequalities

We first study the strict inequality system

$$f_i(x) < 0, \quad i = 1, \ldots, m, \quad Ax = b \tag{4.111}$$

and its alternative

$$\lambda \succeq 0, \quad \lambda \neq 0, \quad g(\lambda, \nu) \geq 0 \tag{4.112}$$

Assumption: There exists an $x \in \operatorname{ri} \mathcal{D}$ with $Ax = b$.

In other words we not only assume that the linear equality constraints are consistent, but also that they have a solution in $\operatorname{ri} \mathcal{D}$. (Very often $\mathcal{D} = \mathbf{R}^n$, so the condition is satisfied if the equality constraints are consistent.)

Conclusion: (4.111) and (4.112) are strong alternatives.

Proof. We will establish this result by reformulation.

Reformulated problem:

$$\begin{array}{ll} \text{minimize} & s \\ \text{subject to} & f_i(x) - s \leq 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array} \tag{4.113}$$

with variables x, s, and domain $\mathcal{D} \times \mathbf{R}$. The optimal value p^* of this problem is negative if and only if there exists a solution to the strict inequality system (4.111).

Lagrange dual function:

$$\inf_{x \in \mathcal{D}, s} \left(s + \sum_{i=1}^{m} \lambda_i \left(f_i(x) - s \right) + \nu^\top (Ax - b) \right) = \begin{cases} g(\lambda, \nu) & \mathbf{1}^\top \lambda = 1 \\ -\infty & \text{otherwise} \end{cases}$$

Lagrange dual problem:

$$\begin{array}{ll} \text{maximize} & g(\lambda, \nu) \\ \text{subject to} & \lambda \succeq 0, \quad \mathbf{1}^\top \lambda = 1 \end{array}$$

Now we observe that Slater's condition holds for the problem (4.113): By the hypothesis there exists an $\tilde{x} \in \operatorname{ri} \mathcal{D}$ with $A\tilde{x} = b$. Choosing any $\tilde{s} > \max_i f_i(\tilde{x})$ yields a point $(\tilde{x}, \tilde{s})$ which is strictly feasible for (4.113)

So we have $d^* = p^*$, and the dual optimum d^* is attained if $d^* > -\infty$. In other words, there exist (λ^*, ν^*) such that

$$g\left(\lambda^*, \nu^*\right) = p^*, \quad \lambda^* \succeq 0, \quad \mathbf{1}^\top \lambda^* = 1 \tag{4.114}$$

Now suppose that the strict inequality system (4.111) is infeasible, which means that $p^* \geq 0$. Then (λ^*, ν^*) from (4.114) satisfy the alternate inequality system (4.112). Similarly, if the alternate inequality system (4.112) is feasible, then $d^* = p^* \geq 0$, which shows that the strict inequality system (4.111) is infeasible. Thus, the inequality systems (4.111) and (4.112) are strong alternatives; each is feasible if and only if the other is not. □

4.11.2.2 Nonstrict Inequalities

We now consider the nonstrict inequality system

$$f_i(x) \leq 0, \quad i = 1, \ldots, m, \quad Ax = b \tag{4.115}$$

and its alternative

$$\lambda \succeq 0, \quad g(\lambda, \nu) > 0 \tag{4.116}$$

Assumption: There exists an $x \in \operatorname{ri} \mathcal{D}$ with $Ax = b$ and p^* of (4.113) is attained.

Conclusion: (4.115) and (4.116) are strong alternatives.

Proof. With these assumptions we have, as in the strict case, that $p^* = d^*$, and that both the primal and dual optimal values are attained. Now suppose that the nonstrict inequality system (4.115) is infeasible, which means that $p^* > 0$. (Here we use the assumption that the primal optimal value is attained.) Then (λ^*, ν^*) from (4.114) satisfy the alternate inequality system (4.116). The other direction is from (4.102). □

4.11.3 Examples

4.11.3.1 Linear Inequalities

Consider the system of linear inequalities $Ax \preceq b$. The dual function is

$$g(\lambda) = \inf_x \lambda^\top (Ax - b) = \begin{cases} -b^\top \lambda & A^\top \lambda = 0 \\ -\infty & \text{otherwise} \end{cases}$$

The alternative inequality system is therefore

$$\lambda \succeq 0, \quad A^\top \lambda = 0, \quad b^\top \lambda < 0$$

These are strong alternatives from last section.

This follows since the optimum in the related problem (5.81) is achieved, unless it is unbounded below.

We now consider the system of strict linear inequalities $Ax \prec b$, which has the strong alternative system

$$\lambda \succeq 0, \quad \lambda \neq 0, \quad A^\top \lambda = 0, \quad b^\top \lambda \leq 0$$

These are strong alternatives from Theorem 2.128.

4.11.3.2 Intersection of Ellipsoids

We consider m ellipsoids, described as

$$\mathcal{E}_i = \{x \mid f_i(x) \leq 0\}$$

with $f_i(x) = x^\top A_i x + 2b_i^\top x + c_i, i = 1, \ldots, m$, where $A_i \in \mathbf{S}_{++}^n$. We ask when the intersection of these ellipsoids has nonempty interior. This is equivalent to feasibility of the set of strict quadratic inequalities

$$f_i(x) = x^\top A_i x + 2b_i^\top x + c_i < 0, \quad i = 1, \ldots, m \tag{4.117}$$

The dual function g is

$$\begin{aligned} g(\lambda) &= \inf_x \left(x^\top A(\lambda)x + 2b(\lambda)^\top x + c(\lambda)\right) \\ &= \begin{cases} -b(\lambda)^\top A(\lambda)^\dagger b(\lambda) + c(\lambda) & A(\lambda) \succeq 0, \quad b(\lambda) \in \mathcal{R}(A(\lambda)) \\ -\infty & \text{otherwise} \end{cases} \end{aligned}$$

where

$$A(\lambda) = \sum_{i=1}^m \lambda_i A_i, \quad b(\lambda) = \sum_{i=1}^m \lambda_i b_i, \quad c(\lambda) = \sum_{i=1}^m \lambda_i c_i$$

Note that for $\lambda \succeq 0, \lambda \neq 0$, we have $A(\lambda) \succ 0$, so we can simplify the expression for the dual function as

$$g(\lambda) = -b(\lambda)^\top A(\lambda)^{-1} b(\lambda) + c(\lambda)$$

The strong alternative of the system (4.117) is therefore

$$\lambda \succeq 0, \quad \lambda \neq 0, \quad -b(\lambda)^\top A(\lambda)^{-1} b(\lambda) + c(\lambda) \geq 0 \tag{4.118}$$

We can give a simple geometric interpretation of this pair of strong alternatives. For any nonzero $\lambda \succeq 0$, the (possibly empty) ellipsoid

$$\mathcal{E}_\lambda = \left\{x \mid x^\top A(\lambda)x + 2b(\lambda)^\top x + c(\lambda) \leq 0\right\}$$

contains $\mathcal{E}_1 \cap \cdots \cap \mathcal{E}_m$, since $f_i(x) \leq 0$ implies $\sum_{i=1}^m \lambda_i f_i(x) \leq 0$.

Now, $\mathcal{E}_\lambda$ has empty interior if and only if

$$\inf_x \left(x^\top A(\lambda)x + 2b(\lambda)^\top x + c(\lambda)\right) = -b(\lambda)^\top A(\lambda)^{-1} b(\lambda) + c(\lambda) \geq 0$$

Therefore the alternative system (4.118) means that $\mathcal{E}_\lambda$ has empty interior.

Weak duality is obvious: If (4.118) holds, then $\mathcal{E}_\lambda$ contains the intersection $\mathcal{E}_1 \cap \cdots \cap \mathcal{E}_m$, and has empty interior, so naturally the intersection has empty interior. The fact that these are strong alternatives states the (not obvious) fact that if the intersection $\mathcal{E}_1 \cap \cdots \cap \mathcal{E}_m$ has empty interior, then we can construct an ellipsoid $\mathcal{E}_\lambda$ that contains the intersection and has empty interior.

4.11.3.3 Farkas' Lemma

We have shown it in Theorem 2.46. Here we use dual problem to prove:

$$Ax \preceq 0, \quad c^\top x < 0 \tag{4.119}$$

where $A \in \mathbf{R}^{m \times n}$ and $c \in \mathbf{R}^n$, and the system of equalities and inequalities

$$A^\top y + c = 0, \quad y \succeq 0 \tag{4.120}$$

are strong alternatives.

Proof. We can prove Farkas' lemma directly, using LP duality. Consider the LP

$$\begin{array}{ll} \text{minimize} & c^\top x \\ \text{subject to} & Ax \preceq 0 \end{array} \tag{4.121}$$

and its dual

$$\begin{array}{ll} \text{maximize} & 0 \\ \text{subject to} & A^\top y + c = 0 \\ & y \succeq 0 \end{array} \tag{4.122}$$

The primal LP (4.121) is homogeneous, and so has optimal value 0 if (4.119) is not feasible, and optimal value $-\infty$ if (4.120) is feasible. The dual LP (4.122) has optimal value 0 if (4.120) is feasible, and optimal value $-\infty$ if (4.120) is infeasible. Since $x = 0$ is feasible in (4.121), we can rule out the one case in which strong duality can fail for LPs, so we must have $p^* = d^*$. Combined with the remarks above, this show strong alternatives. □

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