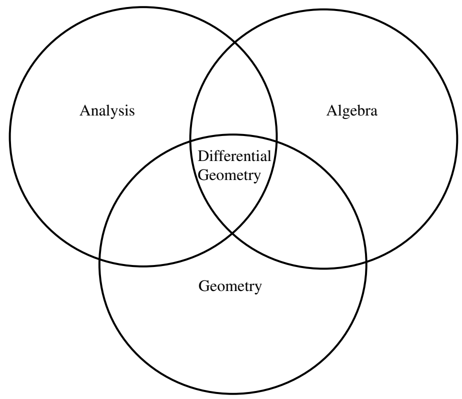
Summary of Differential Geometry

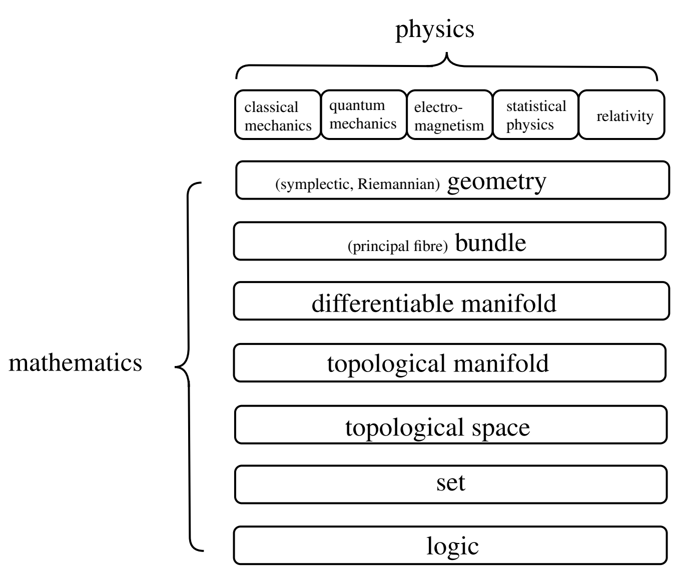


1 Introduction

Theoretical physics is all about casting our concepts about the real world into rigorous mathematical form, for better or worse. It does so in order to fully explore the implications of our concepts about the real world. While from the point of view of logic and mathematics $A \Leftrightarrow B$ may be a tautology, psychologically, in terms of our understanding of A, it may be very useful to have a reformulation of A in terms of B. With the understanding that mathematics just gives us a language for what we want to do, the idea of this course is to **provide proper language, namely differential geometry, for theoretical physics**. In particular, we will provide the proper mathematical language for classical mechanics, electromagnetism, quantum mechanics, and statistical physics.

In the notes, I roughly use magenta to highlight very important concepts, orange to highlight important concepts, and other color like cyan, green and purple for other highlights. Note, the text highlighted using the same color (except cyan which is used everywhere) used in one section may have some correspondence relation. The colors are not used randomly. The importance is judged just from my personal perspective. Please modify them freely according to your own knowledge base since, for example, you may not be familiar with some concepts that I am familiar with. Unimportant parts are in tiny font sizes.

Course structure:



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2 Propositional Logic

1. **propositional logic:** A proposition p is a variable (i.e., a formal expression, with no extra structure assumed) that can take the values true (T) or false (F), and **no others**.
   - It is **not** the task of propositional logic to decide whether a complex statement of the form “there is extraterrestrial life” is true or not.
   - More explanation of $p \Rightarrow q$[1]. I like the following two explanations:
     (a) From Herbert Enderton’s "A Mathematical Introduction to Logic" page 21:
     For example, we might translate the English sentence, "If you’re telling the truth then I’m a monkey’s uncle," by the formula $(V \to M)$. We assign this formula the value T whenever you are fibbing. In assigning the value T, we are certainly not assigning any causal connection between your veracity and any simian features of my nephews or nieces. The sentence in question is a conditional statement. It makes an assertion about my relatives provided a certain condition - that you are telling the truth - is met. Whenever that condition fails, the statement is *vacuously true*.
     Very roughly, we can think of a conditional formula $(p \to q)$ as expressing a **promise** that if a certain condition is met (viz., that p is true), then q is true. If the condition p turns out not to be met, then the **promise stands unbroken**, regardless of q.
     (b) "implies" means the same as "subset" in set theory: the empty set is a subset of any set, and a false statement implies any statement.
2. **predicate logic:** A predicate is (informally) a proposition-valued function of some variable or variables. In particular, a predicate of two variables is called a **relation**.
   - Examples: $P(x)$, $Q(x, y)$.
   - We want to only later define the notion of **set**, using the language of propositional and predicate logic. So now, we do not ask where x comes from and leave it completely open. It can be anything. (Note, a set cannot contain anything, i.e. **there does not exist a universal set.**)
   - **unique existential quantifier** definition using logic language $\exists\, !x : P(x) :\Leftrightarrow (\exists\, x : \forall\, y : P(y) \Leftrightarrow x = y)$
3. Some notations:
   (a) Let M be an assignment of atoms (like the axioms). A formula A is said to be true under M if $v_M(A) = \mathrm{T}$, and false under M if $v_M(A) = \mathrm{F}$.
   (b) A set of formulas S is said to be **satisfiable** if there exists an assignment M which satisfies S, i.e., $v_M(A) = \mathrm{T}$ for all $A \in S$.
   (c) Let S be a set of formulas. A formula B is said to be a **logical consequence** of S if it is true under **all** assignments which satisfy S.
   (d) A formula B is said to be **logically valid** (or a **tautology**) if B is true under all assignments. Equivalently, B is a logical consequence of the empty set.
   (e) B is a **logical consequence** of $A_1, \dots, A_n$ if and only if
   $$(A_1 \wedge \cdots \wedge A_n) \Rightarrow B$$
   is **logically valid**. B is logically valid if and only if $\neg B$ is not satisfiable.
4. **axiomatic systems and theory of proofs:** See my notes of logic proof and number theory. Roughly speaking, an axiomatic system is a finite sequence of propositions $a_1, a_2, \dots, a_N$. Proof is a finite sequence of propositions where each proposition is from **axioms (or definitions, assumptions) (A)**, **tautology (T)**, or **Modus Ponens (M)**.
   - **axiomatic system:** An **axiomatic system** is a **finite sequence of propositions** $a_1, a_2, \dots, a_N$, which are called the **axioms** of the system.
   - **proof:** A **proof** of a proposition p within an axiomatic system $a_1, a_2, \dots, a_N$ is a **finite sequence of propositions** $q_1, q_2, \dots, q_M$ such that $q_M = p$ and for any $1 \le j \le M$ one of the following is satisfied:
     (A) q_j is a proposition from the list of axioms;

[1] https://math.stackexchange.com/questions/48161/in-classical-logic-why-is-p-rightarrow-q-true-if-both-p-and-q-are-false

(T) q_j is a tautology;

(M) $\exists\ 1 \leq m, n < j : (q_m \wedge q_n \Rightarrow q_j)$ is true.

- **notation:** If p can be proven within an axiomatic system $a_1, a_2, \ldots, a_N$, we write:

$$a_1, a_2, \ldots, a_N \vdash p$$

  and we read "$a_1, a_2, \ldots, a_N$ proves p".
- Some important remarks:
  * Proof is try to construct a sequence such that "$(A_1 \wedge \cdots \wedge A_n) \Rightarrow B$ is tautology" is then shown where A_i are the axioms (or definitions, assumptions).
  * Note, Modus Ponens is just a Tautology in the form of $P \Rightarrow Q$. So in some sense, proof is just **a sequence of tautologies**. And proof is the flow of "true" from the start to the end as discussed in the notes.
  * If no givens are from hypotheses (axioms), we can only prove a tautology B (something that is always true). This is because for $p \Rightarrow q$, we know only have tautology p, so q must be a tautology.
  * An axiomatic system is **consistent** if there exists a proposition q which cannot be proven from the axioms.
  * Any axiomatic system powerful enough to encode elementary arithmetic is either inconsistent or contains an undecidable proposition, i.e. a proposition that can be neither proven nor disproven within the system. See the screenshots.

3 Axioms of Set Theory

There will be no definition of what $\in$ is, or of what a set is. Instead, we will have nine axioms concerning $\in$ and sets, and it is **only in terms of these nine axioms that $\in$ and sets are defined at all**. We will establish Zermelo-Fraenkel axioms of set theory which can be remembered as

E E P U R P I C F

Using the $\in$-relation we can immediately define the following relations:

1. $x \notin y :\Leftrightarrow \neg(x \in y)$
2. $x \subseteq y :\Leftrightarrow \forall\, a : (a \in x \Rightarrow a \in y)$
3. $x = y :\Leftrightarrow (x \subseteq y) \wedge (y \subseteq x)$
4. $x \subset y :\Leftrightarrow (x \subseteq y) \wedge \neg(x = y)$

Zermelo-Fraenkel axioms:

1. **"E": axiom on the $\in$-relation:** The expression $x \in y$ is a proposition (exclusively true or false, and no others) if, and only if, both x and y are sets. In symbols:

$$\forall\, x : \forall\, y : (x \in y) \veebar \neg(x \in y)$$

   - $x \in y$ is a relation. There is no concept of element here. All variables considered here are set. But if $x \in y$, we may now call x is an **element** of y in human language.
   - **Russell's paradox:** if u contains all the sets that are not elements of themselves and no others. u is not a set. **So there is no universal set.**
2. **"E": axiom on the existence of an empty set:** There exists a set that contains no elements. In symbols:

$$\exists\, y : \forall\, x : x \notin y.$$

   - There is only one empty set, and we denote it by $\varnothing$. So we can say "axiom on the existence of **the** empty set".
3. **"P" axiom on pair sets:** Let x and y be sets. Then there exists a set that contains as its elements precisely x and y. In symbols:

$$\forall\, x : \forall\, y : \exists\, m : \forall\, u : (u \in m \Leftrightarrow (u = x \vee u = y)).$$

   The set m is called the pair set of x and y, and it is denoted by $\{x, y\}$.

- We can show $\{x, y\} = \{y, x\}$, so it is an **unordered** pair.
- The defining property of an ordered pair is the following:

$$(x, y) = (a, b) \Leftrightarrow x = a \wedge y = b.$$

  One candidate which satisfies this property is $(x, y) := \{x, \{x, y\}\}$, which is a set by the axiom on pair sets.
- If x is a set, then we can define $\{x\} := \{x, x\}$, called a **singleton set**.
- Note, up to now, we cannot define $\{x, y, z\}$ three-element set. We need the following axiom on union sets.

4. **"U" axiom on union sets:** Let x be a set. Then there exists a set whose elements are precisely the *elements of the elements* of x. In symbols:

$$\forall\, x : \exists\, u : \forall\, y : (y \in u \Leftrightarrow \exists\, s : (y \in s \wedge s \in x))$$

   The set u is denoted by $\bigcup x$.
   - The union set axiom is really needed to construct sets with more than 2 elements.
   - So, we can take **at most "set" many unions**, e.g. finite or infinite many unions as long as they can be put in a set. It is not possible to take the union of all the sets that do not contain themselves.

5. **"R" axiom of replacement:** Let R be a **functional relation** and let m be a set. Then the **image** of m under R, denoted by $\mathrm{im}_R(m)$, is again a set.
   - Functional is just our traditional definition of a function. Formally, a relation R is said to be **functional** if:

$$\forall\, x : \exists\, !y : R(x, y).$$

   - Let m be a set and let R be a functional relation. The **image** of m under R consists of all those y for which there is an $x \in m$ such that $R(x, y)$.
   - **principle of restricted comprehension** (implied by "R"): Let $P(x)$ be a **predicate** and let m be a set. Then the elements $y \in m$ such that $P(y)$ is true constitute a set, which we denote by:

$$\{y \in m \mid P(y)\}.$$

     Note, "principle" of **universal** comprehension which states that $\{y \mid P(y)\}$ is a set for any predicate was shown to be inconsistent by Russell.
     * Observe that the $y \in m$ condition makes it so that $\{y \in m \mid P(y)\}$ cannot have more elements than m itself.
     * Now, we can define **intersection:** Let x be a set, the intersection of x is

$$\bigcap x := \{a \in \bigcup x \mid \forall\, b \in x : a \in b\}.$$

     * Let u and m be sets such that $u \subseteq m$. Then the **complement** of u relative to m is defined as:

$$m \backslash u := \{x \in m \mid x \notin u\}.$$

       These are both sets by the principle of restricted comprehension, which is ultimately due to the axiom of replacement.

6. **"P" axiom on the existence of power sets:** Let m be a set. Then there exists a set, denoted by $\mathcal{P}(m)$, whose elements are precisely the subsets of m. In symbols:

$$\forall\, x : \exists\, y : \forall\, a : (a \in y \Leftrightarrow a \subseteq x).$$

   - Why we need this? In naive set theory, the principle of universal comprehension was thought to be needed:

$$\mathcal{P}(m) := \{y \mid y \subseteq m\}.$$

     We therefore need a "bigger" (or biggest) set where the elements of the power set come from. However, this is not possible (See above) and there is no other choice but to dedicate an additional axiom for the existence of power sets.

- If one defines $(a,b) := \{a, \{a,b\}\}$, then the cartesian product $x \times y$ of two sets x and y, which informally is the set of all ordered pairs of elements of x and y, satisfies:
$$x \times y \subseteq \mathcal{P}(\mathcal{P}(\bigcup\{x,y\})).$$
Hence, the existence of $x \times y$ as a set follows from the axioms on unions, pair sets, power sets, and the principle of restricted comprehension.

7. **"I" axiom of infinity:** There exists a set that contains the empty set and, together with every other element y, it also contains the set $\{y\}$ as an element. In symbols:
$$\exists\, x : \varnothing \in x \wedge \forall\, y : (y \in x \Rightarrow \{y\} \in x).$$
   - $\mathbb{N} := x$ is then a set: $\varnothing \in x$ and hence $\{\varnothing\} \in x$. Thus, we also have $\{\{\varnothing\}\} \in x$ and so on. Therefore:
$$x = \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\{\{\varnothing\}\}\}, \ldots\}.$$
We can introduce the following notation for the elements of x :
$$0 := \varnothing, \quad 1 := \{\varnothing\}, \quad 2 := \{\{\varnothing\}\}, \quad 3 := \{\{\{\varnothing\}\}\}, \quad \cdots$$
     * A modern version of the axiom and set natural number: *There exists a set that contains the empty set and, together with every other element y, it also contains the set $y \cup \{y\}$ as an element.* (Here we used the notation: $x \cup y := \bigcup\{x,y\}$.) With this formulation, the natural numbers look like:
$$\mathbb{N} := \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}, \ldots\}$$
It is much nicer for two reasons. First, the natural number n is represented by an n-element set rather than a one-element set. Second, it generalizes much more naturally to the system of transfinite ordinal numbers where the successor operation $s(x) = x \cup \{x\}$ applies to transfinite ordinals as well as natural numbers (I don't know this, just ignore it currently). Moreover, the natural numbers have the same defining property as the ordinals: they are transitive sets strictly well-ordered by the $\in$-relation.
   - We can define $\mathbb{R} := \mathcal{P}(\mathbb{N})$, which is a set by the axiom on power sets.

8. **"C" axiom of choice:** Let x be a set whose elements are non-empty and mutually disjoint. Then there exists a set y which contains exactly one element of each element of x. In symbols:
$$\forall\, x : P(x) \Rightarrow \exists\, y : \forall\, a \in x : \exists\, !b \in a : a \in y,$$
where $P(x) \Leftrightarrow (\exists\, a : a \in x) \wedge (\forall\, a : \forall\, b : (a \in x \wedge b \in x) \Rightarrow \bigcap\{a,b\} = \varnothing)$.
   - The axiom of choice is independent of the other 8 axioms, which means that one could have a set theory with or without the axiom of choice. However, standard mathematics uses the axiom of choice, and hence so will we. There is a number of theorems that can only be proved by using the axiom of choice, like
     * every vector space has a basis;
     * there exists a complete system of representatives of an equivalence relation.

9. **"F" axiom of foundation (regularity):** Every non-empty set x contains an element y that has none of its elements in common with x. In symbols:
$$\forall\, x(x \neq \varnothing \rightarrow \exists\, y(y \in x \wedge y \cap x = \varnothing)).$$
   - An immediate consequence of this axiom is that there is no set that contains itself as an element.
   - I think of the axiom of regularity along with the axiom of extensionality as formalizing what I mean by "set". Once upon a time, before paradoxes, one could think of sets as just any collection of things. Unfortunately, axioms based on that picture, in particular the unrestricted comprehension axiom, led to contradictions, so it became clear that the original, contradictory notion of "set" must be replaced by something clearer. (People might have thought the original notion was perfectly clear, but the paradoxes show that it isn't.) The clearer picture that emerged is of **a cumulative hierarchy**, in which sets are obtained as follows.
Begin with some non-set entities called atoms ("some" could be "none" if you want a world consisting exclusively of sets), then form all sets of these, then all sets whose elements are atoms or sets of atoms, etc. This "etc." means to build more and more levels of sets, where a set at any level has elements only from earlier levels (and the atoms constitute the lowest level). This iterative construction can be continued transfinitely, through arbitrarily long well-ordered sequences of levels.
This so-called cumulative hierarchy is what I (and most set theorists) mean when we talk about sets. A set is anything that is formed at some level of this hierarchy. This meaning of "set" has replaced older meanings.
The axiom of regularity is clearly true with this understanding of what a set is. It expresses the idea that the stages of the cumulative hierarchy come in a well-ordered sequence. (Without well-ordering, the instructions for each level, namely "form all sets whose elements are at earlier levels," would not be an inductive definition but a circularity.)
Although there are set theories that contradict regularity, I would say that any such theory (and also any theory that contradicts extensionality) is not about sets but about some different (though presumably similar) entities.

4 Classification of Sets

A recurrent theme in mathematics is the **classification of spaces** by means of **structure preserving maps** between them. A space is usually meant to be some set equipped with some structure, which is usually some other set. We will define each instance of space precisely when we will need them:

$$\begin{tikzcd}
\text{set} \arrow[r] & \text{topological space} \arrow[r] & \text{topological manifold} \arrow[r] & \text{differentiable manifold} \arrow[r] \arrow[d] & \text{principal fibre bundle} \arrow[r] & \text{associated fibre bundles} \\
 & & & \text{Lie group} \arrow[ur] \arrow[d, shift left] & & \\
 & & & \text{Lie algebra} \arrow[u, shift left] & &
\end{tikzcd}$$

For manifold with bundle, we also have other structures. In this section we focus on set. Bijections are the "structure-preserving" maps for sets.

A summary:

- We give the definition of **map**.
- We use **map** to **classify sets** as finite and infinite.
- We introduce **composition** of maps and **diagram commute**.
- We define **equivalence relations** and **quotient set**.
- Finally we use equivalence to **construct** $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ **and** $\mathbb{R}$ step by step.

1. Let A, B be sets. A **map** $\phi : A \to B$ is a relation such that for each $a \in A$ there exists exactly one $b \in B$ such that $\phi(a, b)$. The standard notation for a map is:
$$\begin{aligned} \phi : A &\to B \\ a &\mapsto \phi(a) \end{aligned}$$
   - Note, here we have written $\phi(a, b)$ as $b = \phi(a)$.
   - Note, map has specified set while functional does not. See Section 3. No other difference.
2. **set isomorphic:** Two sets A and B are called (set-theoretic) isomorphic if there exists a **bijection** $\phi : A \to B$. In this case, we write $A \cong_{\text{set}} B$.
   - If there is any bijection $A \to B$ then generally there are many. This is knowledge from the permutation group.
3. **classification of sets:** A set A is:
   (a) **infinite** if there exists a **proper** subset $B \subsetneqq A$ such that $B \cong_{\text{set}} A$. In particular, if A is infinite, we further define A to be:
      i). **countably infinite** if $A \cong_{\text{set}} \mathbb{N}$;
      ii). **uncountably infinite** otherwise.
   (b) **finite** if it is not infinite. In this case, we have $A \cong_{\text{set}} \{1, 2, \ldots, N\}$ for some $N \in \mathbb{N}$ and we say that the **cardinality** of A, denoted by $|A|$, is N.
4. **composition and commute:** Given two maps $\phi : A \to B$ and $\psi : B \to C$, we can construct a third map, called the composition of ϕ and ψ, denoted by $\psi \circ \phi$(read "psi after phi"), defined by:
$$\begin{aligned} \psi \circ \phi : A &\to C \\ a &\mapsto \psi(\phi(a)). \end{aligned}$$
   This is often represented by drawing the following diagram. We say the diagram **commutes:**
$$\begin{tikzcd}
 & B \arrow[dr, "\psi"] & \\
A \arrow[ur, "\phi"] \arrow[rr, "\psi \circ \phi"'] & & C
\end{tikzcd}$$

- Composition of maps is **associative:** $\xi \circ (\psi \circ \phi) = (\xi \circ \psi) \circ \phi$.
- See algebra for the bijective and identity mapping relations:

$$\begin{tikzcd} A \arrow[loop left, "\mathrm{id}_A"] \arrow[rr, bend left, "\phi"] & & B \arrow[ll, bend left, "\phi^{-1}"] \arrow[loop right, "\mathrm{id}_B"] \end{tikzcd}$$

5. **preimage** Let $\phi : A \to B$ be a map and let $V \subseteq B$. Then we define the set:

$$\mathrm{preim}_\phi(V) := \{a \in A \mid \phi(a) \in V\}$$

called the **pre-image** of V under ϕ.

- Let $\phi : A \to B$ be a map, let $U, V \subseteq B$ and $C = \{C_j \mid j \in J\} \subseteq \mathcal{P}(B)$. Then:
  (a) $\mathrm{preim}_\phi(\varnothing) = \varnothing$ and $\mathrm{preim}_\phi(B) = A$;
  (b) $\mathrm{preim}_\phi(U \backslash V) = \mathrm{preim}_\phi(U) \backslash \mathrm{preim}_\phi(V)$;
  (c) $\mathrm{preim}_\phi(\bigcup C) = \bigcup_{j \in J} \mathrm{preim}_\phi(C_j)$ and $\mathrm{preim}_\phi(\bigcap C) = \bigcap_{j \in J} \mathrm{preim}_\phi(C_j)$.

6. **equivalence relations and quotient set:** We omit the definition of equivalence relation. Let $\sim$ be an equivalence relation on M. Then we define the **quotient set** of M by $\sim$ as:

$$M/\sim := \{[m] \mid m \in M\}.$$

- This is indeed a set since $[m] \subseteq \mathcal{P}(M)$ and hence we can write more precisely:

$$M/\sim := \{[m] \in \mathcal{P}(M) \mid m \in M\}.$$

  Then clearly $M/\sim$ is a set by the power set axiom and the **principle of restricted comprehension**.
- Due to the axiom of choice, there exists a complete system of representatives for $\sim$, i.e. a set R such that $R \cong$ set $M/\sim$.

7. **construction of $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$:** Recall that, invoking the axiom of infinity, we defined:

$$\mathbb{N} := \{0, 1, 2, 3, \ldots\}$$

where:

$$0 := \varnothing, \quad 1 := \{\varnothing\}, \quad 2 := \{\{\varnothing\}\}, \quad 3 := \{\{\{\varnothing\}\}\}, \quad \ldots$$

We would now like to define an addition operation on $\mathbb{N}$ by using the axioms of set theory. We will need some preliminary definitions.

(a) The **successor** map S on $\mathbb{N}$ is defined by:

$$\begin{aligned} S : \mathbb{N} &\to \mathbb{N} \\ n &\mapsto \{n\}. \end{aligned}$$

(b) The **predecessor** map P on $\mathbb{N}^* := \mathbb{N} \backslash \{\varnothing\}$ is defined by:

$$\begin{aligned} P : \mathbb{N}^* &\to \mathbb{N} \\ n &\mapsto m \text{ such that } m \in n. \end{aligned}$$

(c) Let $n \in \mathbb{N}$. The **n-th power** of S, denoted S^n, is defined recursively by:

$$\begin{aligned} S^n &:= S \circ S^{P(n)} \quad \text{if } n \in \mathbb{N}^* \\ S^0 &:= \mathrm{id}_\mathbb{N}. \end{aligned}$$

(d) The **addition operation** on $\mathbb{N}$ is defined as a map:

$$\begin{aligned} + : \mathbb{N} \times \mathbb{N} &\to \mathbb{N} \\ (m, n) &\mapsto m + n := S^n(m) \end{aligned}$$

- We can show that $+$ is commutative and associative.
- The **multiplication** can be defined similarly, we do not construct it here. But we will assume it can used to construct $\mathbb{Z}$ and $\mathbb{Q}$.

(e) Let $\sim$ be the relation on $\mathbb{N} \times \mathbb{N}$ defined by:

$$(m, n) \sim (p, q) :\Leftrightarrow m + q = p + n.$$

We have the **set of integers** $\mathbb{Z} : = (\mathbb{N} \times \mathbb{N}) / \sim$.

- $\mathbb{N} \subseteq \mathbb{Z}$ is in the sense that $\mathbb{N}$ can be embedded into $\mathbb{Z}$, i.e. there exists an **inclusion map** ι, given by:

$$\begin{aligned} \iota : \mathbb{N} &\hookrightarrow \mathbb{Z} \\ n &\mapsto [(n, 0)] \end{aligned}$$

(f) Let $n := [(n, 0)] \in \mathbb{Z}$. Then we define the **inverse** of n to be $-n := [(0, n)]$.

(g) We define the **addition of integers** $+ : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ by:

$$[(m, n)] +_{\mathbb{Z}} [(p, q)] := [(m + p, n + q)]$$

- We can then define $\times$ on $\mathbb{Z}$ similarly.

(h) In a similar fashion, we define the **set of rational numbers** by:

$$\mathbb{Q} : = (\mathbb{Z} \times \mathbb{Z}^{*}) \, / \sim,$$

where $\mathbb{Z}^{*:} = \mathbb{Z} \backslash \{0\}$ and $\sim$ is a relation on $\mathbb{Z} \times \mathbb{Z}^{*}$ given by:

$$(p, q) \sim (r, s) :\Leftrightarrow ps = qr,$$

assuming that a **multiplication** operation on the integers has already been defined.

- $\mathbb{Z} \subseteq \mathbb{Q}$ is in the sense of canonical embedding of $\mathbb{Z}$ into $\mathbb{Q}$:

$$\begin{aligned} \iota : \mathbb{Z} &\mapsto \mathbb{Q} \\ p &\mapsto [(p, 1)] \end{aligned}$$

(i) We define the **addition of rational numbers** $+\mathbb{Q}$:$\mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ by:

$$[(p, q)] +_{\mathbb{Q}} [(r, s)] := [(ps + rq, qs)]$$

(j) The **multiplication of rational numbers** is given by:

$$[(p, q)] \cdot_{\mathbb{Q}} [(r, s)] := [(pr, qs)],$$

where the operations of addition and multiplication that appear on the right hand sides are the ones defined on $\mathbb{Z}$.

(k) There are many ways to construct the reals from the rationals. One is to define a set $\mathscr{A}$ of **almost homomorphisms**[2] on $\mathbb{Z}$ and hence define:

$$\mathbb{R} : = \mathscr{A} \, / \sim,$$

where $\sim$ is a "suitable" equivalence relation on $\mathscr{A}$.

5 Topological Space: Basic

$$\begin{tikzcd}
\text{set} \arrow[r, red] & \text{topological space} \arrow[r] & \text{topological manifold} \arrow[r] & \text{differentiable manifold} \arrow[r] \arrow[d] & \text{principal fibre bundle} \arrow[r] & \text{associated fibre bundles} \\
& & & \text{Lie group} \arrow[ur] \arrow[d, shift left] & & \\
& & & \text{Lie algebra} \arrow[u, shift left] & &
\end{tikzcd}$$

As we will see, a topology on a set provides the weakest structure in order to define the two very important notions of convergence of sequences to points in a set, and of continuity of maps between two sets. Homeomorphism is the "structure-preserving" (i.e. topologically isomorphic) maps for topological spaces. A summary:

[2]Do not need to know in this course. Personally, I do not know this concept, just copy-paste.

- ► We give the definition of **topology**.
- ► We construct new topologies from given ones using **induced (subset) topology**, **quotient (finest, or called final) topology** and **product (weak, or called initial) topology**.
- ► We define **continuous map** and **homeomorphism**.

1. **topology:** Let M be a set. A **topology** on M is a set $\mathcal{O} \subseteq \mathcal{P}(M)$ such that:
   (a) $\varnothing \in \mathcal{O}$ and $M \in \mathcal{O}$;
   (b) $\{U, V\} \subseteq \mathcal{O} \Rightarrow \bigcap\{U, V\} \in \mathcal{O}$ i.e. **finite intersection**
   (c) $C \subseteq \mathcal{O} \Rightarrow \bigcup C \in \mathcal{O}$. i.e. **arbitrary union** (how many? under the set of power set)

   The pair $(M, \mathcal{O})$ is called a **topological space**.
   - Unless $|M| = 1$, there are (usually many) different topologies $\mathcal{O}$ that one can choose on the set M.

$\lvert M\rvert$	Number of topologies
1	1
2	4
3	29
4	355
5	6,942
6	209,527
7	9,535,241

   - I omit lots of things here, Just see the analysis notes.
2. **construction of new topologies from given ones:**
   (a) **induced (subset) topology:** Let $(M, \mathcal{O})$ be a topological space and let $N \subset M$. Then:
   $$\mathcal{O}|_{N:} = \{U \cap N \mid U \in \mathcal{O}\} \subseteq \mathcal{P}(N)$$
   is a topology on N called the induced (subset) topology.
   (b) **quotient topology:** Let $(M, \mathcal{O})$ be a topological space and let $\sim$ be an equivalence relation on M. Then, the quotient set:
   $$M/\sim = \{[m] \in \mathcal{P}(M) \mid m \in M\}$$
   can be equipped with the quotient topology $\mathcal{O}_{M/\sim}$ defined by:
   $$\mathcal{O}_{M/\sim:} = \left\{ U \subseteq M/\sim \;\middle|\; \bigcup U = \bigcup_{[a]\in U} [a] \in \mathcal{O} \right\}$$
   - An equivalent definition of the quotient topology is as follows. Let $q : M \to M/\sim$ be the map:
   $$\begin{aligned} q : M &\to M/\sim \\ m &\mapsto [m] \end{aligned}$$
   $M/\sim$ is then equipped with the **final (finest) topology**[3] for q (to make q continuous):
   $$\mathcal{O}_{M/\sim:} = \left\{ U \subseteq M/\sim \mid \mathrm{preim}_q(U) \in \mathcal{O} \right\}.$$
   - It has relations to **quotient map, fibers, saturated set and surjection**[4].

[3] Think about why finest and coarsest topology are "dual". For $f : X \to Y$, to make it continuous: if we know topology on X, we then define the finest topology on Y (the coarsest one is trivial); if we know topology on Y, we then need the coarsest topology on X

[4] See `https://en.wikipedia.org/wiki/Quotient_space_(topology)`

    * **quotient map:** $f : X \to Y$ is a **quotient map** if it is **onto** and Y is equipped with the **final topology** with respect to (w.r.t.) f.
    * Quotient map (strong condition because of the finest topology) is continuous, while the inverse is not true generally (if X has some saturated open subset S such that $f(S)$ is not open in Y).
    * **saturated sets:** A subset S of X is called **saturated** (w.r.t. f) if it is of the form $S = f^{-1}(T)$ for some set T, which is true if and only if $f^{-1}(f(S)) = S$. The assignment $T \mapsto f^{-1}(T)$ establishes a **one-to-one correspondence** (whose inverse is $S \mapsto f(S)$) between subsets T of Y and saturated subsets of X. With this terminology, we have

    "a surjection $f : X \to Y$ is a **quotient map**"

    $$\Updownarrow$$

    "for every saturated subset S of X, S is open in X if and only if $f(S)$ is open in Y."

    In particular, open subsets of X that are not saturated have no impact on whether or not the function f is a quotient map; non-saturated subsets are irrelevant to the definition of "quotient map" just as they are irrelevant to the open-set definition of continuity (because a function $f : X \to Y$ is continuous if and only if for every saturated subset S of X, $f(S)$ being open in $f(X)$ implies S is open in X). Indeed, if τ is a topology on X and $f : X \to Y$ is any map then set τ_f of all $U \in \tau$ that are open saturated subsets of X forms a topology on X. If Y is also a topological space then $f : (X, \tau) \to Y$ is a quotient map (respectively, continuous) if and only if the same is true of $f : (X, \tau_f) \to Y$. **Continuous open surjection map** is therefore a quotient map.
    * Every surjection induces an equivalence relation. Roughly speaking, if the surjection is equipped with the final topology for Y, we get a quotient topology on Y. (Strictly speaking, we need to distinguish the singleton set and its unique element. See wiki.)
  - Example: let $\sim$ be the equivalence relation on $\mathbb{R}$ defined by:

    $$x \sim y :\Leftrightarrow \exists\, n \in \mathbb{Z} : x = y + 2\pi n.$$

    Then the circle can be defined as the set $S^{1:} = \mathbb{R}/\sim$ equipped with the quotient topology.

  (c) **product topology:** it is **coarsest topology** for which all the projections are continuous.

3. **convergence:** It has two net-based convergence and sequence-based convergence. I omit them here.
   - Consider the topological space $(M, \mathcal{P}(M))$. Then only definitely constant sequences converge, where a sequence is definitely constant with value $c \in M$ if:

     $$\exists\, N \in \mathbb{N} : \forall\, n > N : q(n) = c.$$

     This is immediate from the definition of convergence since in the discrete topology all singleton sets (i.e. one-element sets) are open.
4. **continuous:** Let $(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$ be topological spaces and let $\phi : M \to N$ be a map. Then, ϕ is said to be continuous (w.r.t. the topologies $\mathcal{O}_M$ and $\mathcal{O}_N$) if:

   $$\forall\, S \in \mathcal{O}_N, \mathrm{preim}_\phi(S) \in \mathcal{O}_M,$$

5. **homeomorphism:** Let $(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$ be topological spaces. A bijection $\phi : M \to N$ is called a homeomorphism if both $\phi : M \to N$ and $\phi^{-1} : N \to M$ are continuous.
   - Homeomorphisms are the structure-preserving maps in topology.
   - If there exists a homeomorphism between two topological spaces $(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$, we say that the two spaces are homeomorphic or topologically isomorphic and we write $(M, \mathcal{O}_M) \cong_{\text{top}} (N, \mathcal{O}_N)$.
   - If $(M, \mathcal{O}_M) \cong_{\text{top}} (N, \mathcal{O}_N)$, then $M \cong_{\text{set}} N$.

6 Topological Spaces: Invariant Properties

A property of a topological space is called an invariant if any two homeomorphic spaces share the property. A classification of topological spaces would be a list of topological invariants such that any two spaces which share these invariants are homeomorphic. As of now, **no** such list is known. A summary:

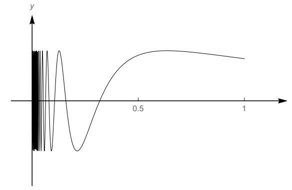
- ▶ We give the definition of **separation**.
- ▶ We give definitions of **cover**, from which concepts like **locally finite, refinement, compactness and paracompactness** can be defined.
- ▶ Partition of unity (a set of continuous maps) can be defined using **locally finite cover**. Give an existing cover C, partition of unity is said to be subordinate if the set interiors of supports is a refinement of C.
- ▶ We then compare **connected** and **path-connected**.
- ▶ We give the definition of **homotopic** of curves which can be regarded as an **equivalence relation** (able of continuously deformate between each other).
- ▶ Finally, we are able to give the definition of **fundamental group**, a "group-valued" property of a topological space.

1. **separation properties:** T_1, T_2, ... classification. It is complicated, and each space has many useful properties. I choose to omit them here. Please see my topology notes.
2. **cover, compactness and paracompactness:** I'm familiar with compact and its properties. I choose to omit them. Please see my topology notes. The following mainly focuses on others. Paracompact is weaker than compact.
   (a) **cover:** Let $(M, \mathcal{O})$ be a topological space. A set $C \subseteq \mathcal{P}(M)$ is called a *cover* (of M) if:
   $$\bigcup C = M.$$
   Additionally, it is said to be
   - **open** if $C \subseteq \mathcal{O}$
   - **locally finite** if for any $p \in M$ there exists a neighbourhood $U(p)$ such that the set:
   $$\{U \in C \mid U \cap U(p) \neq \varnothing\}$$
   is finite as a set.

   (b) **refinement:** A refinement of a cover C of a topological space X is **a new cover** D of X such that every set in D is contained in some set in C. Formally, $D = \{V_\beta\}_{\beta \in B}$ is a refinement of $C = \{U_\alpha\}_{\alpha \in A}$
   if for all $\beta \in B$ there exists $\alpha \in A$ such that $V_\beta \subseteq U_\alpha$.
   - Any subcover of a cover is a refinement of that cover, but the converse may be false.
   - A refinement R is said to be **open** or **locally finite** if R is open or locally finite.

   (c) **paracompact:** A topological space $(M, \mathcal{O})$ is said to be paracompact if every open cover has an open refinement that is locally finite.
   - If a topological space is compact, then it is also paracompact.
   - A topological space $(M, \mathcal{O})$ is said to be **metrisable** if there exists a metric d such that the topology induced by d is precisely $\mathcal{O}$, i.e. $\mathcal{O}_d = \mathcal{O}$. **Every metrisable space is paracompact.** So R^d with the standard topology is not compact but paracompact.
   - Example: $\mathbb{R}$ is equivalent to $\mathbb{Z} \times [0, 1)$. The long line L is defined analogously as $L : \omega_1 \times [0, 1)$, where ω_1 is an uncountably infinite set. The resulting space L is not paracompact.
   - **product of paracompact is paracompact:** Let $(M, \mathcal{O}_M)$ be a paracompact space and let $(N_i, \mathcal{O}_{N_i})$ be compact spaces for every $1 \leq i \leq n$. Then $M \times N_1 \times \cdots \times N_n$ is paracompact.

   (d) **partition of unity**: Let $(M, \mathcal{O}_M)$ be a topological space. A partition of unity of M is a set $\mathcal{F}$ of continuous maps from M to the interval $[0, 1]$ such that for each $p \in M$ the following conditions hold:
   i). there exists $U(p)$ such that the set $\{f \in \mathcal{F} \mid \forall x \in U(p) : f(x) \neq 0\}$ is **finite**; In other words, the set $\left\{\operatorname{supp}(f)^\circ : f \in \mathcal{F}\right\}$, i.e. interiors of supports, is a **locally finite cover** of M.
   ii). $\sum_{f \in \mathcal{F}} f(p) = 1$.



- **subordinate** If C is an open cover, then partition of unity $\mathcal{F}$ is said to be subordinate to the cover C if the set $\{\mathrm{supp}\,(f)^{\circ} : f \in \mathcal{F}\}$, i.e., interiors of supports, is a **refinement** of C. Namely, if

$$\forall\, f \in \mathcal{F} : \exists\, U \in C : f(x) \neq 0 \Rightarrow x \in U$$

- Let $(M, \mathcal{O}_M)$ be a **Hausdorff** topological space. Then $(M, \mathcal{O}_M)$ is paracompact if, and only if, every open cover admits a partition of unity subordinate to that cover.

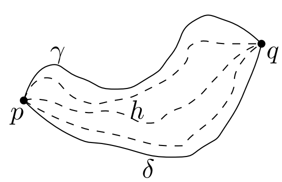
3. **connected:** A topological space $(M, \mathcal{O})$ is said to be **connected** unless there exist two non-empty, non-intersecting **open** sets A and B such that $M = A \cup B$.
   - A topological space $(M, \mathcal{O})$ is connected if, and only if, the **only** subsets that are both open and closed are $\varnothing$ and M.
4. **path-connected :** A topological space $(M, \mathcal{O})$ is said to be path-connected if for every pair of points $p, q \in M$ there exists a continuous curve $\gamma : [0,1] \to M$ such that $\gamma(0) = p$ and $\gamma(1) = q$.
   - Example: Let $S := \{(x, \sin(\frac{1}{x})) \mid x \in (0,1]\} \cup \{(0,0)\}$ be equipped with the subset topology inherited from $\mathbb{R}^2$. The space $(S, \mathcal{O}_{\mathrm{std}}|_S)$ is connected but not path-connected.
   - path-connected $\Rightarrow$ connected.
5. **homotopic:** Two curves are homotopic if they can be continuously deformed into one another. Formally, let $(M, \mathcal{O})$ be a topological space. Two curves $\gamma, \delta : [0,1] \to M$ such that:

$$\gamma(0) = \delta(0) \quad \text{and} \quad \gamma(1) = \delta(1)$$

are said to be homotopic if there exists a continuous map $h : [0,1] \times [0,1] \to M$ such that for all $\lambda \in [0,1]$:

$$h(0, \lambda) = \gamma(\lambda) \quad \text{and} \quad h(1, \lambda) = \delta(\lambda).$$

   - Let $\gamma \sim \delta :\Leftrightarrow$ "γ and δ are homotopic". Then, $\sim$ is an **equivalence relation**.



6. **fundamental group**[5]**:** It is a group with the element being the equivalent loop and "+" being the concatenation.
   - All the previously discussed topological properties are "boolean-valued", i.e. a topological space is either connected or not connected, and so on. The fundamental group is a "group-valued" property.
   - **loop:** Let $(M, \mathcal{O})$ be a topological space. Then, for every $p \in M$, we define the space of loops at p by: $\mathscr{L}_p := \{\gamma : [0,1] \to M \mid \gamma \text{ is continuous and } \gamma(0) = \gamma(1)\}$
   - **concatenation of loop:** Let $\mathscr{L}_p$ be the space of loops at $p \in M$. We define the concatenation operation $* : \mathscr{L}_p \times \mathscr{L}_p \to \mathscr{L}_p$ by:

$$(\gamma * \delta)(\lambda) := \begin{cases} \gamma(2\lambda) & \text{if } 0 \leq \lambda \leq \frac{1}{2} \\ \delta(2\lambda - 1) & \text{if } \frac{1}{2} \leq \lambda \leq 1 \end{cases}$$

[5] If there exists a group isomorphism between $(G, \bullet)$ and $(H, \circ)$, we write $G \cong_{\mathrm{grp}} H$.

- Let $(M, \mathcal{O})$ be a topological space. The **fundamental group** $\pi_1(p)$ of $(M, \mathcal{O})$ at $p \in M$ is the set:

$$\pi_1(p) := \mathscr{L}_p / \sim = \{[\gamma] \mid \gamma \in \mathscr{L}_p\},$$

  where $\sim$ is the homotopy equivalence relation, together with the map

$$\begin{aligned} \bullet : \pi_1(p) \times \pi_1(p) &\to \pi_1(p) \\ (\gamma, \delta) &\mapsto [\gamma] \bullet [\delta] := [\gamma * \delta]. \end{aligned}$$

  (a) group operator is **associative**, this is from "composition of maps is associative".
  (b) group neutral element is (the equivalence class of) the constant curve γ_e defined by:

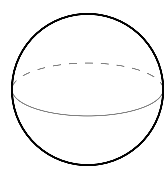
$$\begin{aligned} \gamma_e : [0,1] &\to M \\ \lambda &\mapsto \gamma_e(0) = p \end{aligned}$$

  (c) **inverse:** for each $[\gamma] \in \pi_1(p)$, the inverse under $\bullet$ is the element $[-\gamma]$, where $-\gamma$ is defined by:

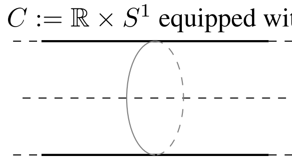
$$\begin{aligned} -\gamma : [0,1] &\to M \\ \lambda &\mapsto \gamma(1-\lambda) \end{aligned}$$

- Examples:
  † The **sphere** has the property that all the loops at any point are homotopic, hence the fundamental group (at every point) of the sphere is the trivial group:

$$\forall\, p \in S^2 : \pi_1(p) = 1 := \{[\gamma_e]\}.$$



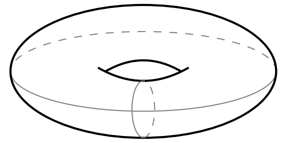
  † The **cylinder** is defined as $C := \mathbb{R} \times S^1$ equipped with the product topology.



  A loop in C can either go around the cylinder (i.e. around its central axis) or not. If it does not, then it can be continuously deformed to a point (the identity loop). If it does, then it cannot be deformed to the identity loop (intuitively because the cylinder is infinitely long) and hence it is a homotopically different loop. The number of times a loop winds around the cylinder is called the **winding number**. Loops with different winding numbers are not homotopic. Moreover, loops with different *orientations* are also not homotopic. We have:

$$\forall\, p \in C : (\pi_1(p), \bullet) \cong_{\mathrm{grp}} (\mathbb{Z}, +).$$

  † The 2-torus is defined as the set $T^2 := S^1 \times S^1$ equipped with the product topology.



  A loop in T^2 can intuitively wind around the cylinder-like part of the torus as well as around the hole of the torus. That is, there are two independent winding numbers and hence:

$$\forall\, p \in T^2 : \pi_1(p) \cong_{\mathrm{grp}} \mathbb{Z} \times \mathbb{Z},$$

  where $\mathbb{Z} \times \mathbb{Z}$ is understood as a group under pairwise addition.

7 Topological Manifolds and Bundles

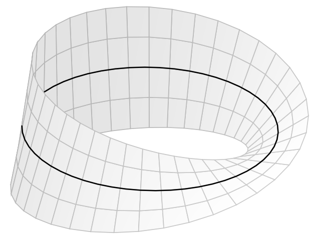
$$\begin{tikzcd}
\text{set} \arrow[r] & \text{topological space} \arrow[r, red] & \text{topological manifold} \arrow[r] & \text{differentiable manifold} \arrow[r] \arrow[d] & \text{principal fibre bundle} \arrow[r] & \text{associated fibre bundles} \\
 & & & \text{Lie group} \arrow[ur] \arrow[d, shift right] & & \\
 & & & \text{Lie algebra} \arrow[u, shift right] & &
\end{tikzcd}$$

A summary:

- We give definitions for **topological manifold**, **submanifold** and **product manifold**.
- A **bundle** consisting **total manifold space**, **base manifold space** and the **projection operator**.
- The concepts like **fibre**, **section**, **sub-bundle**, **bundle morphism** and **locally trivial** are then given.
- We then further define **fibre bundle** which locally is a product bundle with a fixed fibre F for all points.
- Finally, we view manifolds from **local charts** and give the definition of **atlases**.

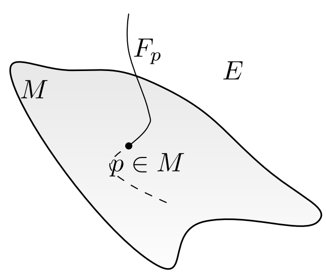
1. **topological manifold**: A paracompact, Hausdorff, topological space $(M, \mathcal{O})$ is called a d-dimensional (topological) manifold if for every point $p \in M$ there exist a neighbourhood $U(p)$ and a **homeomorphism** $x : U(p) \to x(U(p)) \subseteq \mathbb{R}^d$. [6] We also write $\dim M = d$.
   - Let M be a d-dimensional manifold and let $U, V \subseteq M$ be open, with $U \cap V \neq \varnothing$. If x and y are two homeomorphisms
   $$x : U \to x(U) \subseteq \mathbb{R}^d \quad \text{and} \quad y : V \to y(V) \subseteq \mathbb{R}^{d'}$$
   then $d = d'$. This ensures that the concept of **dimension is indeed well-defined**, i.e. it is the same at every point, at least on *each connected component* of the manifold.
   - Examples: $\mathbb{R}^d$ is a d-dimensional manifold for any $d \geq 1$. The space S^1 is a 1-dimensional manifold while the spaces S^2, cylinder C and torus T^2 are 2-dimensional manifolds.
2. **submanifold:** Let $(M, \mathcal{O})$ be a topological manifold and let $N \subseteq M$. Then $(N, \mathcal{O}|_N)$ is called a **submanifold** of $(M, \mathcal{O})$ if it is a manifold in its own right.
   - Examples: The space S^1 is a submanifold of $\mathbb{R}^2$ while the spaces S^2, C and T^2 are submanifolds of $\mathbb{R}^3$.
3. **product manifold:** Let $(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$ be topological manifolds of dimension m and n, respectively. Then, $(M \times N, \mathcal{O}_{M \times N})$ is a topological manifold of dimension $m + n$ called the **product manifold**.
   - Examples: $T^2 = S^1 \times S^1$; or more general n-dimensional manifold $T^n := \underbrace{S^1 \times S^1 \times \cdots \times S^1}_{n \text{ times}}$. The cylinder $C = S^1 \times \mathbb{R}$ is a 2-dimensional product manifold.
   - But Möbius strip is **not** a product manifold. It is only a 2-dimensional manifold. To understand it, we need the concept **fibre bundle** that is locally a product space.

[6] To define complex topological manifolds, we require that the map x be a homeomorphism onto an open subset of $\mathbb{C}^d$.

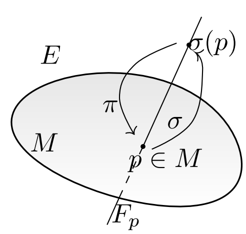


4. **bundle:** A bundle (of topological manifolds) is a triple (E, π, M) where E and M are topological manifolds called the **total space** and the **base space** respectively, and π is a **continuous, surjective**[7] map $\pi : E \to M$ called the **projection map**.
    - Bundle can be defined for the general topological space without assuming E and M are manifold. But, in this course, we focus on manifold.
    - **notation:** We will often denote the bundle (E, π, M) by $E \xrightarrow{\pi} M$.
    - **fibre:** Let $E \xrightarrow{\pi} M$ be a bundle and let $p \in M$. Then, $F_p := \text{preim}_\pi(\{p\})$ is called the fibre at the point p. However, F_p may vary topologically from point to point.
    - **product bundle:** Let M and N be manifolds. Then, the triple $(M \times N, \pi, M)$, where:
    $$\pi : M \times N \to M$$
    $$(p, q) \mapsto p$$
    is a **bundle** since (one can easily check) π is a *continuous open surjective map*. Similarly, $(M \times N, \pi, N)$ with the appropriate π, is also a **bundle**. Later we will see it is a **fiber bundle**.

    Intuitively, the fibre at the point $p \in M$ is a set of points in E (represented below as a line) attached to the point p. The projection map sends all the points in the fibre F_p to the point p.



5. **(cross-) section**: Let $E \xrightarrow{\pi} M$ be a bundle (**not restrict to fibre bundle**). A (cross-) section of the bundle is a **continuous** map $\sigma : M \to E$ such that $\pi \circ \sigma = \text{id}_M$.
    - Intuitively, a section is a map σ which sends each point $p \in M$ to some point $\sigma(p)$ in its fibre F_p, so that the projection map π takes $\sigma(p) \in F_p \subseteq E$ back to the point $p \in M$.



[7] recall the quotient map, but here we do not use it until "fibre bundle"

- Example: Let $(M \times F, \pi, M)$ be a **product bundle**. Then, a section of this bundle is a map:

$$\begin{aligned} \sigma : M &\to M \times F \\ p &\mapsto (p, s(p)) \end{aligned}$$

where $s : M \to F$ is continuous map.
- Note, sections may **not** exist due the requirement of continuous. See below fibre bundles.

6. **sub-bundle:** A **sub-bundle** of a bundle (E, π, M) is a triple (E', π', M') where $E' \subseteq E$ and $M' \subseteq M$ are submanifolds and $\pi' := \pi|_{E'}$.
7. **restricted bundle:** Let (E, π, M) be a bundle and let $N \subseteq M$ be a submanifold. The *restricted bundle* (to N) is the triple (E, π', N) where:

$$\pi' := \pi|_{\text{preim}_\pi(N)}$$

8. **bundle morphism:** Let $E \xrightarrow{\pi} M$ and $E' \xrightarrow{\pi'} M'$ be bundles and let $u : E \to E'$ and $v : M \to M'$ be maps. Then (u, v) is called a **bundle morphism** if the following diagram commutes:

$$\begin{tikzcd} E \arrow[r, "u"] \arrow[d, "\pi"] & E' \arrow[d, "\pi'"] \\ M \arrow[r, "v"] & M' \end{tikzcd}$$

- **uniquness:** If (u, v) and (u, v') are both bundle morphisms, then $v = v'$. That is, given u, if there exists v such that (u, v) is a bundle morphism, then v is **unique**.
- Later we will see for principal fiber bundles with structure group G and whose total spaces are (right) G-spaces, bundle morphisms are also required to be G-equivariant on the fibers.

9. **bundle isomorphic:** Two bundles $E \xrightarrow{\pi} M$ and $E' \xrightarrow{\pi'} M'$ are said to be **isomorphic (as bundles)** if there exist bundle morphisms (u, v) and (u^{-1}, v^{-1}) satisfying:

$$\begin{tikzcd} E \arrow[r, shift left, "u"] \arrow[d, "\pi"'] & E' \arrow[l, shift left, "u^{-1}"] \arrow[d, "\pi'"] \\ M \arrow[r, shift left, "v"] & M' \arrow[l, shift left, "v^{-1}"] \end{tikzcd}$$

Such a (u, v) is called a *bundle isomorphism*

- Bundle isomorphisms are the structure-preserving maps for bundles.
- **notation:** We write $E \xrightarrow{\pi} M \cong_{\text{bdl}} E' \xrightarrow{\pi'} M'$.

10. **locally isomorphic:** A bundle $E \xrightarrow{\pi} M$ is said to be locally isomorphic (as a bundle) to a bundle $E' \xrightarrow{\pi'} M'$ if for all $p \in M$ there exists a neighbourhood $U(p)$ such that the restricted bundle:

$$\text{preim}_\pi(U(p)) \xrightarrow{\pi|_{\text{preim}_\pi(U(p))}} U(p)$$

is isomorphic to the bundle $E' \xrightarrow{\pi'} M'$.

11. **trivial; locally trivial:** A bundle $E \xrightarrow{\pi} M$ is said to be:
    - *trivial* if it is isomorphic to a product bundle;
    - *locally trivial* if it is locally isomorphic to a product bundle (It is a fibre bundle. See below for more details).
    - Examples:
      † The cylinder C is trivial as a bundle, and hence also locally trivial.
      † The Möbius strip is not trivial but it is locally trivial.

12. **pull-back bundle:** Let $E \xrightarrow{\pi} M$ be a bundle and let $f : M' \to M$ be a map from some manifold M'. The **pull-back bundle** of $E \xrightarrow{\pi} M$ *induced by* f is defined as $E' \xrightarrow{\pi'} M'$, where:
$$E' := \{(m', e) \in M' \times E \mid f(m') = \pi(e)\}$$
and $\pi'(m', e) := m'$.

    - If $E' \xrightarrow{\pi'} M'$ is the pull-back bundle of $E \xrightarrow{\pi} M$ induced by f, then one can easily construct a **bundle morphism** by defining:
    $$\begin{aligned} u :& E' \to E \\ & (m', e) \mapsto e \end{aligned}$$
    This corresponds to the diagram:
    $$\begin{tikzcd} E' \arrow[r, "u"] \arrow[d, "\pi'"] & E \arrow[d, "\pi"] \\ M' \arrow[r, "f"] & M \end{tikzcd}$$

    - **pull back of section:** Let $E' \xrightarrow{\pi'} M'$ be the pull-back bundle of $E \xrightarrow{\pi} M$ induced by f.
    $$\begin{tikzcd} E' \arrow[d, shift left, "\pi'"] & & E \arrow[d, shift left, "\pi"] \\ M' \arrow[u, shift left, "\sigma'"] \arrow[urr, "\sigma \circ f"] \arrow[rr, "f"] & & M \arrow[u, shift left, "\sigma"] \end{tikzcd}$$
    If σ is a section of $E \xrightarrow{\pi} M$, then $\sigma \circ f$ determines a map from M' to E which sends each $m' \in M'$ to $\sigma(f(m')) \in E$. However, since σ is a section, we have:
    $$\pi(\sigma(f(m')) = (\pi \circ \sigma \circ f)(m') = (\mathrm{id}_M \circ f)(m') = f(m')$$
    and hence $(m', (\sigma \circ f)(m')) \in E'$ by definition of E'. Moreover:
    $$\pi'(m', (\sigma \circ f)(m')) = m'$$
    and hence the map:
    $$\begin{aligned} \sigma' :& M' \to E' \\ & m' \mapsto (m', (\sigma \circ f)(m')) \end{aligned}$$
    satisfies $\pi' \circ \sigma' = \mathrm{id}_{M'}$ and it is thus **a section on the pull-back bundle** $E' \xrightarrow{\pi'} M'$.

13. **fibre bundle:** Let $E \xrightarrow{\pi} M$ be a bundle and let F be a manifold. Then, $E \xrightarrow{\pi} M$ is called a fibre bundle, with (typical) **fibre** F, if we have, roughly speaking,
$$\forall p \in M : F_p := \mathrm{preim}_\pi(\{p\}) \cong_{\text{top}} F.$$
More strictly speaking, if it satisfies the local triviality condition:

    - We require that for every $p \in M$, there is an open neighborhood $U \subseteq M$ of p (which will be called a **trivializing neighborhood**) such that there is a **homeomorphism** $\varphi : \pi^{-1}(U) \to U \times F$ (where $\pi^{-1}(U)$ is given the subspace topology, and $U \times F$ is the product space) in such a way that π agrees with the projection onto the first factor. That is, the following diagram should **commute** (i.e. it is a **local isomorphism**):
    $$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "\pi"'] & U \times F \arrow[dl, "\mathrm{proj}_1"] \\ U & \end{tikzcd}$$

where $\text{proj}_1 : U \times F \to U$ is the **natural projection** and $\varphi : \pi^{-1}(U) \to U \times F$ is a **homeomorphism**. The set of all $\{(U_i, \varphi_i)\}$ is called a **local trivialization** of the bundle.

- Thus, for any $p \in B$, the preimage $\pi^{-1}(\{p\})$ is **homeomorphic** to F (since this is true of $\text{proj}_1^{-1}(\{p\})$). Every fiber bundle $\pi : E \to M$ is an **continuous open map**, since projections of products are open maps and φ is a homeomorphism. Therefore M carries the **quotient topology** determined by the map π.
- **notation:** A fiber bundle is often denoted

$$F \longrightarrow E \xrightarrow{\pi} M$$

- A **smooth fiber bundle** is a fiber bundle in the category of smooth[8] manifolds. That is, E, M, and F are required to be smooth manifolds and all the functions above are required to be smooth maps.
- **product bundle is a fiber bundle** since (one can easily check) π is a *continuous open surjective map*. Similarly, $(M \times N, \pi, N)$ with the appropriate π, is also a **fiber bundle**.

  "product bundle (i.e. manifold)" $\subseteq$ fibre bundle $\subseteq$ bundle
- Example I: The Möbius strip is a fibre bundle $F \longrightarrow E \xrightarrow{\pi} S^1$, with fibre $F := [0, 1]$, where $E \neq S^1 \times [0, 1]$, i.e. the Möbius strip is not a product bundle.
- Example II: This hairbrush is like a fiber bundle in which the base space is a cylinder and the fibers (bristles) are line segments. The mapping $\pi : E \to B$ would take a point on any bristle and map it to its root on the cylinder.

- **important clarification:** Fiber bundles do **not** in general have such **global sections**. But locally, since we have the locally trivial, we can always define a **local section** of a fiber bundle. Locally it is a continuous map $s : U \to E$ where U is an open set in B and $\pi(s(x)) = x$ for all x in U. If (U, φ) is a local trivialization of E, where φ is a homeomorphism from $\pi^{-1}(U)$ to $U \times F$ (where F is the fiber), then local sections always exist over U in bijective correspondence with continuous maps from U to F.
  - $*$ What is hidden behind is that there is **no canonical way to identify the fibers.** We only have each F_p **topologically equals** F, i.e., $\forall\, p \in M : \text{preim}_\pi(\{p\}) \cong_{\text{top}} F$. In the very definition of fiber bundle, close fibers are identified to each other thanks to **local trivializations.** But there is **no global identification.**
  - $*$ To define a continuous function, we only need to prescribe its behavior on an open covering, if we make sure that the definitions coincide with each other on the intersections(that is the glueing). Now, if we have a fiber bundle, we are given an open cover of B, say $B = \cup_i U_i$, and isomorphisms $\phi_i : \pi^{-1}(U_i) \to U_i \times F$. So when you are checking the compatibility condition on $U_i \cap U_j$, you have to remember that you used the ϕ_i to define your section. That means you have used **different trivial neighbouhoods** covering B.
  - $*$ Example: the fiber bundle over S^1 with fiber $F = \mathbb{R}\backslash\{0\}$ obtained by taking the Möbius bundle and removing the zero section do not in general have global sections.
  - $*$ So, locally, any section on fibre bundle can be represented as a map from base space to the fibre F. Globally it may not be always possible unless, for example, it is a **product product.**

[8] Definition of smooth is listed in Section 8

* A principal bundle has a global section if and only if it is trivial (See Section 20). On the other hand, a vector bundle (i.e., an associated bundle with each fibre carrying a vector space structure, e.g. the tensor bundle, see Section 24) always has a global section, namely the **zero section**. However, it only admits a nowhere vanishing section if its Euler class is zero. See also **hairy ball theorem** in Section 12.

14. **viewing manifolds from atlases:**

(a) **chart:** Let $(M, \mathcal{O})$ be a d-dimensional manifold. Then, a pair (U, x) where $U \in \mathcal{O}$ and $x : U \to x(U) \subseteq \mathbb{R}^d$ is a homeomorphism, is said to be a *chart* of the manifold.

(b) **component functions:**
The **component functions (or maps)** of $x : U \to x(U) \subseteq \mathbb{R}^d$ are the maps:

$$\begin{aligned} x^i : & U \to \mathbb{R} \\ & p \mapsto \mathrm{proj}_i(x(p)) \end{aligned}$$

for $1 \leq i \leq d$, where $\mathrm{proj}_i(x(p))$ is the i-th component of $x(p) \in \mathbb{R}^d$. The $x^i(p)$ are called the *coordinates* of the point $p \in U$ w.r.t. the chart (U, x).

(c) **atlas** An *atlas* of a manifold M is a collection $\mathscr{A} := \{(U_\alpha, x_\alpha) \mid \alpha \in \mathcal{A}\}$ of charts such that:

$$\bigcup_{\alpha \in \mathcal{A}} U_\alpha = M.$$

(d) **chart transition map:** For two charts (U, x) and (V, y) are said to be $\mathcal{C}^0$-*compatible* if $U \cap V = \varnothing$

$$y \circ x^{-1} : x(U \cap V) \to y(U \cap V)$$

is continuous.
Note that $y \circ x^{-1}$ is a map from a subset of $\mathbb{R}^d$ to a subset of $\mathbb{R}^d$.

$$\begin{tikzcd} & U \cap V \subseteq M \arrow[dl, "x"'] \arrow[dr, "y"] & \\ x(U \cap V) \subseteq \mathbb{R}^d \arrow[rr, "y \circ x^{-1}"'] & & y(U \cap V) \subseteq \mathbb{R}^d \end{tikzcd}$$

(e) The map $y \circ x^{-1}$ (and its inverse $x \circ y^{-1}$) is called the *coordinate change map* or *chart transition map*.

(f) **various compatible:** See my riemannian manifold notes and next Section 8. In this section we only consider $\mathcal{C}^0$-compatibility.

(g) **maximal $\mathcal{C}^0$ atlas:** A $\mathcal{C}^0$-atlas $\mathscr{A}$ is said to be a **maximal atlas** if for every $(U, x) \in \mathscr{A}$, we have (V, y) must be in $\mathscr{A}$ for all (V, y) charts that are $\mathcal{C}^0$-compatible with (U, x).

(h) **two point of view:** We can now look at "objects on" topological manifolds from two points of view. For instance, consider a curve on a d-dimensional manifold M, i.e. a map $\gamma : \mathbb{R} \to M$. We now ask whether this curve is continuous, as it should be if models the trajectory of a particle on the "physical space" M.

i). **from topology:** A first answer is that $\gamma : \mathbb{R} \to M$ is continuous if it is continuous as a map between the **topological spaces** $\mathbb{R}$ and M.

ii). **from local representation:** Secondly, we consider only a portion (open subset U) of the physical space M and, instead of studying the map $\gamma : \mathrm{preim}_\gamma(U) \to U$ directly, we study the map:

$$x \circ \gamma : \mathrm{preim}_\gamma(U) \to x(U) \subseteq \mathbb{R}^d,$$

where (U, x) is a chart of M. More likely, you would be checking the continuity of the coordinate maps $x^i \circ \gamma$, which would then imply the continuity of the "real" curve $\gamma : \mathrm{preim}_\gamma(U) \to U$ (real, as opposed to its coordinate representation).

- You may chose a different chart (U, y) and then study the coordinate map $y \circ \gamma$. Notice that some results (e.g. the continuity of γ) obtained in the previous chart (U, x) can be immediately "transported" to the new chart (U, y) via the chart transition map $y \circ x^{-1}$. Intuitively speaking, the map $y \circ x^{-1}$ allows us to just use representation to study the real world without worries.

$$
\begin{tikzcd}
& y(U) \subseteq \mathbb{R}^d \\
\mathrm{preim}_\gamma(U) \subseteq \mathbb{R} \arrow[ur, "y\circ\gamma"] \arrow[r, "\gamma"] \arrow[dr, "x\circ\gamma"'] & U \subseteq M \arrow[u, "y"'] \arrow[d, "x"] \\
& x(U) \subseteq \mathbb{R}^d \arrow[uu, bend right=60, "y\circ x^{-1}"']
\end{tikzcd}
$$

8 Differentiable Structures: Definition and Classification

$$
\begin{tikzcd}
\text{set} \arrow[r] & \text{topological space} \arrow[r] & \text{topological manifold} \arrow[r, red] & \text{differentiable manifold} \arrow[r] \arrow[d] & \text{principal fibre bundle} \arrow[r] & \text{associated fibre bundles} \\
& & & \text{Lie group} \arrow[ur] \arrow[d, shift left] & & \\
& & & \text{Lie algebra} \arrow[u, shift left] & &
\end{tikzcd}
$$

We now add more structures to vanilla topological manifold and define the **differentiable manifold**.

1. **adding ❀-compatibility by refining the (maximal) $\mathcal{C}^0$ atlas:** An atlas $\mathscr{A}$ for a topological manifold is called a **❀-atlas** if any two charts $(U, x), (V, y) \in \mathscr{A}$ are ❀-compatible. In other words, either $U \cap V = \varnothing$ or if $U \cap V \neq \varnothing$, then the transition map $y \circ x^{-1}$ from $x(U \cap V)$ to $y(U \cap V)$ must be ❀.

$$
\begin{tikzcd}
& U \cap V \subseteq M \arrow[dl, "x"'] \arrow[dr, "y"] & \\
x(U \cap V) \subseteq \mathbb{R}^{\dim M} \arrow[rr, "y\circ x^{-1}"'] & & y(U \cap V) \subseteq \mathbb{R}^{\dim M}
\end{tikzcd}
$$

   The symbol ❀ is being used as a placeholder for any of the following:

   - ❀ $= \mathcal{C}^0$: this just reduces to the previous definition;
   - ❀ $= \mathcal{C}^k$: the transition maps are k-times continuously differentiable as maps between open subsets of $\mathbb{R}^{\dim M}$;
   - ❀ $= \mathcal{C}^\infty$: the transition maps are **smooth**; equivalently, the atlas is $\mathcal{C}^k$ for all $k \geq 0$;
   - ❀ $= \mathcal{C}^\omega$: the transition maps are **(real) analytic**, which is stronger than being smooth;
   - ❀ $=$ complex: if $\dim M$ is even, M is a **complex manifold** if the transition maps are continuous and satisfy the Cauchy-Riemann equations.

2. **differentiable manifold:** When we say differentiable manifold, we mean a d-dimensional (topological) manifold together with a **maximal differentiable atlas**[9] on it.

   - Note differentiable manifolds are equipped **maximal** differentiable atlas.
   - **$\mathcal{C}^k$-manifold:** A $\mathcal{C}^k$*-manifold* is a triple $(M, \mathcal{O}, \mathscr{A})$, where $(M, \mathcal{O})$ is a topological manifold and $\mathscr{A}$ is a **maximal** $\mathcal{C}^k$-atlas.
   - **smooth manifold:** it has a **maximal** $\mathcal{C}^\infty$-atlas.

[9] "maximal" is defined similar to Section 7 with ❀-compatibility requirement between any two charts.

3. **Whitney theorem:** Any maximal $\mathcal{C}^k$-atlas, with $k \geq 1$, contains a $\mathcal{C}^\infty$-atlas. Moreover, any two maximal $\mathcal{C}^k$-atlases that contain the same $\mathcal{C}^\infty$-atlas are identical.

   - An immediate implication is that if we can find a $\mathcal{C}^1$-atlas for a manifold, then we can also assume the existence of a $\mathcal{C}^\infty$-atlas for that manifold simply by removing charts, keeping only the ones which are $\mathcal{C}^\infty$-compatible..
   - This is not the case for topological manifolds in general: a space with a $\mathcal{C}^0$-atlas may not admit any $\mathcal{C}^1$-atlas.
   - For the purposes of this, in this course we will not distinguish between $\mathcal{C}^k$ $(k \geq 1)$ and $\mathcal{C}^\infty$-manifolds in the above sense.
   - But two different $\mathcal{C}^k$ maximal atlas must carry different $\mathcal{C}^\infty$-atlas. See below for the existence of two different (i.e. incompatible) $\mathcal{C}^k$ maximal atlas.

4. **compatible atlases:** Two ❀-atlases $\mathscr{A}$, $\mathscr{B}$ are **compatible** if their union $\mathscr{A} \cup \mathscr{B}$ is again a ❀-atlas, and are incompatible otherwise. Alternatively, we can define the compatibility of two atlases in terms of the compatibility of any pair of charts, one from each atlas.

   - A given topological manifold can **carry different incompatible atlases.**
   - Example: Let $(M, \mathcal{O}) = (\mathbb{R}, \mathcal{O}_{\text{std}})$. Consider the two atlases $\mathscr{A} = \{(\mathbb{R}, \text{id}_\mathbb{R})\}$ and $\mathscr{B} = \{(\mathbb{R}, x)\}$, where $x : a \mapsto \sqrt[3]{a}$. Since they both contain a single chart, the compatibility condition on the transition maps is easily seen to hold (in both cases, the only transition map is $\text{id}_\mathbb{R}$). Hence they are both $\mathcal{C}^\infty$-atlases.
     Consider now $\mathscr{A} \cup \mathscr{B}$. The transition map $\text{id}_\mathbb{R} \circ x^{-1}$ is the map $a \mapsto a^3$, which is smooth. However, the other transition map, $x \circ \text{id}_\mathbb{R}^{-1}$, is the map x, which is not even differentiable once (the first derivative at 0 does not exist). Consequently, $\mathscr{A}$ and $\mathscr{B}$ are not even $\mathcal{C}^1$-compatible. But later we will see they are the same **up to diffeomorphism**

5. **differentiable (or even smooth) map between manifolds:** Let $\phi : M \to N$ be a map, where $(M, \mathcal{O}_M, \mathscr{A}_M)$ and $(N, \mathcal{O}_N, \mathscr{A}_N)$ are $\mathcal{C}^k$ (or even $\mathcal{C}^\infty$)-manifolds. Then ϕ is said to be ($\mathcal{C}^k$-)*differentiable at* $p \in M$ if for some charts $(U, x) \in \mathscr{A}_M$ with $p \in U$ and $(V, y) \in \mathscr{A}_N$ with $\phi(p) \in V$, the map $y \circ \phi \circ x^{-1}$ is k-times continuously differentiable at $x(p) \in x(U) \subseteq \mathbb{R}^{\dim M}$ in the usual sense.

$$\begin{tikzcd}
U \subseteq M \arrow[r, "\phi"] \arrow[d, "x"] & V \subseteq N \arrow[d, "y"] \\
x(U) \subseteq \mathbb{R}^{\dim M} \arrow[r, "y \circ \phi \circ x^{-1}"] & y(V) \subseteq \mathbb{R}^{\dim N}
\end{tikzcd}$$

   - The definition of differentiability is well-defined by using the following diagram:

$$\begin{tikzcd}
\widetilde{x}(U \cap \widetilde{U}) \subseteq \mathbb{R}^{\dim M} \arrow[r, "\widetilde{y} \circ \phi \circ \widetilde{x}^{-1}"] & \widetilde{y}(V \cap \widetilde{V}) \subseteq \mathbb{R}^{\dim N} \\
U \cap \widetilde{U} \subseteq M \arrow[r, "\phi"] \arrow[u, "\widetilde{x}"] \arrow[d, "x"] & V \cap \widetilde{V} \subseteq N \arrow[u, "\widetilde{y}"'] \arrow[d, "y"'] \\
x(U \cap \widetilde{U}) \subseteq \mathbb{R}^{\dim M} \arrow[r, "y \circ \phi \circ x^{-1}"] \arrow[uu, bend left=60, "\widetilde{x} \circ x^{-1}"] & y(V \cap \widetilde{V}) \subseteq \mathbb{R}^{\dim N} \arrow[uu, bend right=60, "\widetilde{y} \circ y^{-1}"']
\end{tikzcd}$$

- **Special examples:** Let $(M, \mathcal{O}, \mathscr{A})$ be a d-dimensional smooth manifold and let $(U, x) \in \mathscr{A}$. Then $x : U \to x(U) \subseteq \mathbb{R}^d$ is smooth map (between manifolds):

$$\begin{array}{ccc} U & \xrightarrow{\quad x \quad} & x(U) \\ \Big\downarrow y & & \Big\downarrow \mathrm{id}_{x(U)} \\ x(U) \subseteq \mathbb{R}^d & \xrightarrow{\mathrm{id}_{x(U)} \circ x \circ y^{-1}} & x(U) \subseteq \mathbb{R}^d \end{array}$$

Note, here we view both U and $x(U)$ as the submanifold of M and $\mathbb{R}^d$ respectively. $x : U \to x(U)$ is smooth since the map $\mathrm{id}_{x(U)} \circ x \circ y^{-1}$ is smooth. Note here y is a chart that overlaps with x. Similarly, the coordinate maps $x^i := \mathrm{proj}_i \circ x : U \to \mathbb{R}$ are also smooth from U to $\mathbb{R}$.

6. **diffeomorphism:** Let $\phi : M \to N$ be a **bijective** map between **smooth** manifolds. If both ϕ and ϕ^{-1} are smooth, then ϕ is said to be a **diffeomorphism**.
   - Diffeomorphisms are the structure preserving maps between smooth manifolds.
   - **notation:** Two manifolds $(M, \mathcal{O}_M, \mathscr{A}_M)$, $(N, \mathcal{O}_N, \mathscr{A}_N)$ are said to be *diffeomorphic* if there exists a diffeomorphism $\phi : M \to N$ between them. We write $M \cong_{\text{diff}} N$.
   - It is customary to consider diffeomorphic manifolds to be **the same** from the point of view of differential geometry. Being diffeomorphic is an equivalence relation. This is similar to the situation with topological spaces, where we consider homeomorphic spaces to be the same from the point of view of topology. This is typical of all structure preserving maps.
7. **classification of differentiable structures**: How many smooth structures on a given topological space are there, **up to diffeomorphism**? It depends on the dimension of the manifold!
   - Let M be a manifold with $\dim M = 1, 2$, or 3. Then there is a unique smooth structure on M up to diffeomorphism.
     * Recall we showed that we can equip $(\mathbb{R}, \mathcal{O}_{\text{std}})$ with two incompatible atlases $\mathscr{A}$ and $\mathscr{B}$. Let $\mathscr{A}_{\max}$ and $\mathscr{B}_{\max}$ be their extensions to maximal atlases. Clearly, these are different manifolds, because the atlases are different, but since $\dim \mathbb{R} = 1$, they must be diffeomorphic.
   - **surgery theory:** This is a collection of tools and techniques in topology with which one obtains a new manifold from given ones by performing surgery on them, i.e. by cutting, replacing and gluing parts in such a way as to control topological invariants like the fundamental group. There are only **finitely many** smooth manifolds (up to diffeomorphism) one can make from a topological manifold if $\dim M > 4$.
     * For $\dim M = 4$ there are infinitely many: If M is a non-compact topological manifold, then there are uncountably many non-diffeomorphic smooth structures; In the compact case there are partial results to give infinitely many.

9 Tensor Space Theory I: Over A Field

A summary:

- ▶ We give the definition of **vector space** and **linear (isomorphism) map**.
- ▶ We definite special nonlinear maps called **bilinear map** and more general **multilinear map** over Cartesian product of vector spaces.
- ▶ **Tensor** and **tensor space** (a vector space) is then defined.
- ▶ We give the definition of **Hamel basis** and **dimension** for a general vector space.
- ▶ Special **finite-dimensional** vector space is discussed and some properties are concluded, e.g. **dual basis**.
- ▶ We then discuss **Einstein's summation convention** and matrix or vector representation of tensor.
- ▶ We give **change of basis** rules.

- ▶ We define **determinant** which is only applicable for **endomorphisms**. n**-form** (a special $(0, n)$-tensor) and volume (top) form is given.
- ▶ We give the definition of **pseudo inner product** on vector space.

1. **vector space:** Let $(K, +, \cdot)$ be a field. A *K-vector space*, or *vector space over K* is a triple $(V, \oplus, \odot)$, where V is a set and

$$\oplus : V \times V \to V$$
$$\odot : K \times V \to V$$

are maps satisfying the following axioms:

- $(V, \oplus)$ is an abelian group;
- the map $\odot$ is an *action* of K on $(V, \oplus)$:
  (a) $\forall\, \lambda \in K : \forall\, v, w \in V : \lambda \odot (v \oplus w) = (\lambda \odot v) \oplus (\lambda \odot w)$;
  (b) $\forall\, \lambda, \mu \in K : \forall\, v \in V : (\lambda + \mu) \odot v = (\lambda \odot v) \oplus (\mu \odot v)$;
  (c) $\forall\, \lambda, \mu \in K : \forall\, v \in V : (\lambda \cdot \mu) \odot v = \lambda \odot (\mu \odot v)$;
  (d) $\forall\, v \in V : 1 \odot v = v$.

2. **vector subspace:** Let $(V, \oplus, \odot)$ be a vector space over K and let $U \subseteq V$ be non-empty. Then we say that $(U, \oplus|_{U\times U}, \odot|_{K\times U})$ is a *vector subspace* of $(V, \oplus, \odot)$ if $\forall\, u_1, u_2 \in U : \forall\, \lambda \in K : (\lambda \odot u_1) \oplus u_2 \in U$.

3. **linear map:** Let $(V, \oplus, \odot)$, $(W, \boxplus, \boxdot)$ be vector spaces over the same field K and let $f : V \to W$ be a map. We say that f is a **linear map** if for all $v_1, v_2 \in V$ and all $\lambda \in K$

$$f((\lambda \odot v_1) \oplus v_2) = (\lambda \boxdot f(v_1)) \boxplus f(v_2).$$

- **notation:** From now on, we will drop the special notation for the vector space operations and suppress the dot for scalar multiplication.

4. **linear isomorphism**: A **bijective linear map** is called a **linear isomorphism** of vector spaces. Two vector spaces are said to be *isomorphic* is there exists a linear isomorphism between them. It is the structure-preserving maps between vector spaces[10]

- **notation:** We write $V \cong_{\text{vec}} W$.
- **hom-set:** Let V and W be vector spaces over the same field K. Define the set

$$\mathrm{Hom}(V, W) := \{f \mid f : V \xrightarrow{\sim} W\},$$

where the notation $f : V \xrightarrow{\sim} W$ stands for "f is a linear map from V to W". Note, for the set, we have implicitly used the principle of restricted comprehension, and it is better to write it as $\{f \in \mathcal{P}(A \times B) \mid f : A \to B \text{ and } p(f)\}$, where p is some properties of f, here it is "linear"

- **hom-set is a vector space:** The hom-set $\mathrm{Hom}(V, W)$ can itself be made into a vector space over K by defining the operations pointwisely on the original vector space[11]:

$$\begin{aligned} \mathbin{\diamond\!\!\!\!+} :\ & \mathrm{Hom}(V, W) \times \mathrm{Hom}(V, W) \to \mathrm{Hom}(V, W) \\ & (f, g) \mapsto f \mathbin{\diamond\!\!\!\!+} g \end{aligned}$$

where

$$\begin{aligned} f \mathbin{\diamond\!\!\!\!+} g :\ & V \xrightarrow{\sim} W \\ & v \mapsto (f \mathbin{\diamond\!\!\!\!+} g)(v) := f(v) + g(v), \end{aligned}$$

and

$$\begin{aligned} \odot :\ & K \times \mathrm{Hom}(V, W) \to \mathrm{Hom}(V, W) \\ & (\lambda, f) \mapsto \lambda \odot f \end{aligned}$$

where

$$\begin{aligned} \lambda \odot f :\ & V \xrightarrow{\sim} W \\ & v \mapsto (\lambda \odot f)(v) := \lambda f(v). \end{aligned}$$

[10] Note, the inverse of a bijective linear map is automatically linear.
[11] This will be used repeatedly in the course

- check $\lambda \diamond f \in \mathrm{Hom}(V, W)$:

$$\begin{aligned}
(\lambda \diamond f)(\mu v_1 + v_2) &= \lambda f(\mu v_1 + v_2) && \text{(by definition)}\\
&= \lambda(\mu f(v_1) + f(v_2)) && \text{(since } f \text{ is linear)}\\
&= \lambda\mu f(v_1) + \lambda f(v_2) \\
&= \mu\lambda f(v_1) + \lambda f(v_2) && \text{(since } K \text{ is a field)}\\
&= \mu(\lambda \diamond f)(v_1) + (\lambda \diamond f)(v_2)
\end{aligned}$$

Note, in the definition of vector space, none of the axioms require that K necessarily be a field. In fact, just a ring[12] would suffice. Vector spaces over rings, called **modules** over a ring. See Section 12 for details. However, for modules V and W, $\mathrm{Hom}(V, W)$ is not a module, since the multiplication in a ring *may* be **not commutative.** Later in Section 12, we however use a commutative ring $\mathcal{C}^\infty(M)$, so it is still a module.

- **endomorphism:** Let V be a vector space. An **endomorphism** of V is a linear map $V \to V$. We write $\mathrm{End}(V) := \mathrm{Hom}(V, V)$.
- **automorphism:** Let V be a vector space. An **automorphism** of V is a linear isomorphism $V \to V$. We write $\mathrm{Aut}(V) := \{f \in \mathrm{End}(V) \mid f \text{ is an isomorphism}\}$.

5. **dual (vector) space:** Let V be a vector space over K. The **dual** vector space to V is

$$V^* := \mathrm{Hom}(V, K),$$

where K is considered as a vector space over itself.

- The linear maps in the dual vector space are variously called **linear functionals**, **covectors**, or **one-forms** on V.

6. **bilinear:** Let V, W, Z be vector spaces over K. A map $f : V \times W \to Z$ is said to be *bilinear* if

(a) $\forall\, w \in W : \forall\, v_1, v_2 \in V : \forall\, \lambda \in K : f(\lambda v_1 + v_2, w) = \lambda f(v_1, w) + f(v_2, w)$;
(b) $\forall\, v \in V : \forall\, w_1, w_2 \in W : \forall\, \lambda \in K : f(v, \lambda w_1 + w_2) = \lambda f(v, w_1) + f(v, w_2)$;

- Compare this with the definition of a linear map $f : V \times W \xrightarrow{\sim} Z$ (note here view $V \times W$ as a vector space):

$$\forall\, x, y \in V \times W : \forall\, \lambda \in K : f(\lambda x + y) = \lambda f(x) + f(y).$$

More explicitly, if $x = (v_1, w_1)$ and $y = (v_2, w_2)$, then:

$$f(\lambda(v_1, w_1) + (v_2, w_2)) = \lambda f((v_1, w_1)) + f((v_2, w_2)).$$

A bilinear map out of $V \times W$ is *not* the same as a linear map out of $V \times W$. In fact, bilinearity is just a special kind of non-linearity.

- Examples: The map $f : \mathbb{R}^2 \to \mathbb{R}$ given by $(x, y) \mapsto x + y$ is linear but not bilinear, while the map $(x, y) \mapsto xy$ is bilinear but not linear.

7. **multilinear map:** similar to the above, linearly change w.r.t. one variable when others are fixed in a Cartesian product of vector spaces.

8. **tensor (multilinear map) and tensor space (vector space):** Let V be a vector space over K. A (p, q)-*tensor* T on V is a **multilinear** map

$$T : \underbrace{V^* \times \cdots \times V^*}_{p \text{ copies}} \times \underbrace{V \times \cdots \times V}_{q \text{ copies}} \to K.$$

We write

$$T^p_q V := \underbrace{V \otimes \cdots \otimes V}_{p \text{ copies}} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{q \text{ copies}} := \{T \mid T \text{ is a } (p, q)\text{-tensor on } V\}.$$

- Note, here $\otimes$ is a definition of tensor space, **not** definition for circle product, but later we will see that it can be viewed as a **tensor product**.
- **covariant tensor:** A type $(p, 0)$ tensor is called a **covariant** p**-tensor**, while

[12] In this course, it call the ring a unital ring.

- **contravariant tensor:** a tensor of type $(0,q)$ is called a **contravariant q-tensor**.
- By convention, a $(0,0)$ on V is just an element of K, and hence $T^0_0 V = K$.
- The set $T^p_q V$ can be equipped with a K**-vector space** structure by defining the operations pointwisely on the original vector space:

$$\begin{aligned} \oplus :\ T^p_q V \times T^p_q V &\to T^p_q V \\ (T,S) &\mapsto T \oplus S \end{aligned}$$

and

$$\begin{aligned} \odot :\ K \times T^p_q V &\to T^p_q V \\ (\lambda,T) &\mapsto \lambda \odot T, \end{aligned}$$

where $T \oplus S$ and $\lambda \odot T$ are defined **pointwise**, as we did with $\mathrm{Hom}(V,W)$.

9. **tensor product:** Let $T \in T^p_q V$ and $S \in T^r_s V$. The **tensor product** of T and S is the tensor $T \otimes S \in T^{p+r}_{q+s} V$ defined by:

$$\begin{aligned} &(T \otimes S)(\omega_1,\dots,\omega_p,\omega_{p+1},\dots,\omega_{p+r},v_1,\dots,v_q,v_{q+1},\dots,v_{q+s}) \\ &\qquad\qquad := T(\omega_1,\dots,\omega_p,v_1,\dots,v_q)\, S(\omega_{p+1},\dots,\omega_{p+r},v_{q+1},\dots,v_{q+s}), \end{aligned}$$

with $\omega_i \in V^*$ and $v_i \in V$.

- Examples I (any dimension):
  † $T^0_1 V := \{T \mid T :\ V \xrightarrow{\sim} K\} = \mathrm{Hom}(V,K) =: V^*$. It is linear since the maps only have one argument.
  † $T^1_1 V \equiv V \otimes V^* := \{T \mid T \text{ is a bilinear map } V^* \times V \to K\}$. We claim:

$$T^1_1 V \cong_{\text{vec}} \mathrm{End}(V^*).$$

  ∗ Given $T \in V \otimes V^*$, we can construct $\widehat{T} \in \mathrm{End}(V^*)$ as follows:

$$\begin{aligned} \widehat{T} :\ V^* &\xrightarrow{\sim} V^* \\ \omega &\mapsto T(\omega,-) \end{aligned}$$

  ∗ Reconstruct T from $\widehat{T}$:

$$\begin{aligned} T :\ V \times V^* &\to K \\ (v,\omega) &\mapsto T(v,\omega) := (\widehat{T}(\omega))(v). \end{aligned}$$

- Examples II(**finite** dimension). See below for reasons:
  † $T^0_1 V \cong_{\text{vec}} V$
  † $T^1_1 V \cong_{\text{vec}} \mathrm{End}(V)$
  † $(V^*)^* \cong_{\text{vec}} V$

10. **Hamel basis:** Let $(V,+,\cdot)$ be a vector space over K. A subset $\mathcal{B} \subseteq V$ is called a **Hamel basis** for V if

(a) every finite subset $\{b_1,\dots,b_N\}$ of $\mathcal{B}$ is linearly independent, i.e.

$$\sum_{i=1}^{N} \lambda^i b_i = 0 \ \Rightarrow\ \lambda^1 = \cdots = \lambda^N = 0;$$

(b) $\mathcal{B}$ is a *generating* or *spanning set* of V, i.e.

$$\forall\ v \in V : \exists\ v^1,\dots,v^M \in K : \exists\ b_1,\dots,b_M \in \mathcal{B} : v = \sum_{i=1}^{M} v^i b_i.$$

Some remarks:

- **span:** The second condition more succinctly by defining

$$\mathrm{span}_K(\mathcal{B}) := \left\{ \sum_{i=1}^{n} \lambda^i b_i \,\middle|\, \lambda^i \in K \wedge b_i \in \mathcal{B} \wedge n \geq 1 \right\}$$

and thus writing $V = \mathrm{span}_K(\mathcal{B})$.

- Let V be a vector space (for any dimension) and $\mathcal{B}$ a Hamel basis of V. Then $\mathcal{B}$ is a **minimal spanning** and **maximal independent subset** of V, i.e., if $S \subseteq V$, then
  i). $\text{span}(S) = V \;\Rightarrow\; |S| \geq |\mathcal{B}|$;
  ii). S is linearly independent $\Rightarrow\; |S| \leq |\mathcal{B}|$.

11. **dimension:** Let V be a vector space. The **dimension** of V is $\dim V := |\mathcal{B}|$, where $\mathcal{B}$ is a Hamel basis for V.
    - If $\dim V < \infty$ and $S \subseteq V$, then we have the following:
      (a) If $\text{span}_K(S) = V$ and $|S| = \dim V$, then S is a Hamel basis of V;
      (b) If S is linearly independent and $|S| = \dim V$, then S is a Hamel basis of V.
    - If $\dim V < \infty$, then $(V^*)^* \cong_{\text{vec}} V$. If $\dim V = \infty$, see `https://mathoverflow.net/questions/13322/slick-proof-a-vector-space-has-the-same-dimension-as-its-dual-if-and-only-if-i`
12. "A gentleman only chooses a basis if he must." While a choice of basis often simplifies things, when defining new objects it is important to do so without making reference to a basis. Otherwise, we need that the thing is well-defined.
13. **dual basis:** Let V be a **finite-dimensional** vector space with basis $\mathcal{B} = \{e_1, \ldots, e_{\dim V}\}$. The **dual basis** to $\mathcal{B}$ is the unique basis $\mathcal{B}' = \{f^1, \ldots, f^{\dim V}\}$ of V^* such that

$$\forall\, 1 \leq i, j \leq \dim V: \quad f^i(e_j) = \delta^i_j := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

    - If V is finite-dimensional, then V is isomorphic to both V^* and $(V^*)^*$. In the case of V^*, an isomorphism is given by sending each element of a basis $\mathcal{B}$ of V to a different element of the dual basis $\mathcal{B}'$, and then extending linearly to V.
    - **canonically isomorphic:** from category theory. A vector space is **canonically isomorphic** to its double dual, but **not** canonically isomorphic to its dual, because an arbitrary choice of basis on V is necessary in order to provide an isomorphism. The proper treatment of this matter falls within the scope of *category theory*, and the relevant notion is called **natural isomorphism**.
14. **component:** Let V be a finite-dimensional vector space over K with **basis** $\mathcal{B} = \{e_1, \ldots, e_{\dim V}\}$ and let $T \in T^p_q V$. We define the **components** of T in the basis $\mathcal{B}$ to be the numbers

$$T^{a_1 \ldots a_p}{}_{b_1 \ldots b_q} := T(f^{a_1}, \ldots, f^{a_p}, e_{b_1}, \ldots, e_{b_q}) \in K,$$

    where $1 \leq a_i, b_j \leq \dim V$ and $\{f^1, \ldots, f^{\dim V}\}$ is the **dual basis** to $\mathcal{B}$.
    - The components completely determine the tensor, given the components, we reconstruct as the following:

$$T = \underbrace{\sum_{a_1=1}^{\dim V} \cdots \sum_{b_q=1}^{\dim V}}_{p+q \text{ sums}} T^{a_1 \ldots a_p}{}_{b_1 \ldots b_q} e_{a_1} \otimes \cdots \otimes e_{a_p} \otimes f^{b_1} \otimes \cdots \otimes f^{b_q},$$

15. **Einstein's summation convention:** The Einstein's summation convention should only be used when dealing with linear spaces and multilinear maps. If in any term the same index name appears twice, as both an upper and a lower index, that term is assumed to be summed over all possible values of that index (usually from 1 to the dimension of the space). The convention is:
    - vectors—lower indices; covectors—upper indices
    - vectors components—upper indices; covectors components—lower indices.
16. **matrix or vector representation:** We may use matrix or (row or column) vector to represent the tensor. But, try your best **not** to think of vectors, covectors and tensors as arrays of numbers. Instead, always try to understand them from the abstract, intrinsic, component-free point of view. One reason:
    - For $\phi \in T^1_1 V$, we have a matrix

$$\phi = \phi^i{}_j\, e_i \otimes f^j \quad \leftrightsquigarrow \quad \phi \mathrel{\hat{=}} \begin{pmatrix} \phi^1{}_1 & \phi^1{}_2 & \cdots & \phi^1{}_d \\ \phi^2{}_1 & \phi^2{}_2 & \cdots & \phi^2{}_d \\ \vdots & \vdots & \ddots & \vdots \\ \phi^d{}_1 & \phi^d{}_2 & \cdots & \phi^d{}_d \end{pmatrix}$$

- For $g \in T^0_2 V$, we have a matrix

$$g = g_{ij}\, f^i \otimes f^j \qquad \leftrightsquigarrow \qquad g \mathrel{\hat{=}} \begin{pmatrix} g_{11} & g_{12} & \cdots & g_{1d} \\ g_{21} & g_{22} & \cdots & g_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ g^d_{d1} & g^d_{d2} & \cdots & g^d_{dd} \end{pmatrix}$$

But they are totally different and will behave differently:
  - $*$ ϕ is an **endomorphism** of V; the first index in $\phi^a{}_b$ transforms like a vector index, while the second index transforms like a covector index;
  - $*$ g is a **bilinear form** on V; both indices in g_{ab} transform like covector indices. We cannot define determinant for g.
  - $*$ under change of basis:

$$\phi \to A^{-1}\phi A \qquad \text{and} \qquad g \to A^T g A,$$

17. **change of basis:**
    - **change of basis in vector space:** For a finite dimensional vector space V, if $\{e_a\}$ and $\{\widetilde{e}_a\}$ are two basis, we have

$$\widetilde{e}_a = A^b{}_a e_b \qquad \text{and} \qquad e_a = B^m{}_a \widetilde{e}_m,$$

    with $A^{-1} = B$.
    - **change of basis in tensor space:** Let $T \in T^p_q V$. Then:

$$T^{a_1 \ldots a_p}{}_{b_1 \ldots b_q} = A^{a_1}{}_{m_1} \cdots A^{a_p}{}_{m_p} B^{n_1}{}_{b_1} \cdots B^{n_q}{}_{b_q} \widetilde{T}^{m_1 \ldots m_p}{}_{n_1 \ldots n_q},$$

    i.e. the upstair indices transform like vector indices, and the downstair indices transform like covector indices.

18. **determinants:** The notion of determinant is only defined for **endomorphisms.** So above g does not have determinant. We first state some definitions:
    - **permutation:** Let M be a set. A **permutation** of M is a bijection $M \to M$.
    - **symmetric group:** The **symmetric group** of order n, denoted S_n, is the set of permutations of $\{1, \ldots, n\}$ under the operation of functional composition.
    - **transposition:** A **transposition** is a permutation which exchanges two elements, keeping all other elements fixed.
      - $*$ Every permutation $\pi \in S_n$ can be written as a product (composition) of transpositions in S_n.
      - $*$ **sign:** While this decomposition is not unique, for each given $\pi \in S_n$, the number of transpositions in its decomposition is always either **even or odd**. Hence, we can define the **sign** (or *signature*) of $\pi \in S_n$ as:

$$\operatorname{sgn}(\pi) := \begin{cases} +1 & \text{if } \pi \text{ is the product of an even number of transpositions} \\ -1 & \text{if } \pi \text{ is the product of an odd number of transpositions.} \end{cases}$$

    - **n-form:** Let V be a d-dimensional vector space. An **n-form** on V is a $(0, n)$**-tensor** ω that is **totally antisymmetric**, i.e.

$$\forall\ \pi \in S_n :\ \omega(v_1, v_2, \ldots, v_n) = \operatorname{sgn}(\pi)\, \omega(v_{\pi(1)}, v_{\pi(2)}, \ldots, v_{\pi(n)}).$$

      - $*$ n-form is special **contravariant n-tensor**.
      - $*$ Examples: A 0-form is a scalar, and a 1-form is a covector. A d-form is also called a **top form**.
      - $*$ **equivalent definition of n-form:** A $(0, n)$-tensor ω is an n-form if, and only if, $\omega(v_1, \ldots, v_n) = 0$ whenever $\{v_1, \ldots, v_n\}$ is linearly dependent. (Antisymmetric then comes from linearity.)
      - $*$ $T^0_n V$ is certainly non-empty when $n > d$; However, any n-form with $n > d$ must be identically zero because a collection of more than d vectors from a d-dimensional vector space is necessarily linearly dependent.

* **dimension and basis:** Denote by $\Lambda^n V$ the vector space of n-forms on V. Then we have
$$\dim \Lambda^n V = \begin{cases} \binom{d}{n} & \text{if } 1 \leq n \leq d \\ 0 & \text{if } n > d, \end{cases}$$
where $\binom{d}{n} = \frac{d!}{n!(d-n)!}$ is the binomial coefficient, read as "d choose n".
* In particular, $\dim \Lambda^d V = 1$. This means that
$$\forall\, \omega, \omega' \in \Lambda^d V : \exists\, c \in K : \ \omega = c\,\omega',$$
i.e. there is essentially only one top form on V, up to a scalar factor.

- **volume form:** A choice of top form on V is called a choice of **volume form** on V. A vector space with a chosen volume form is then called a **vector space with volume**.
  * **volume:** Let $\dim V = d$ and let $\omega \in \Lambda^d V$ be a volume form on V. Given $v_1, \ldots, v_d \in V$, the *volume* spanned by $v_1, \ldots, v_d$ is
$$\mathrm{vol}(v_1, \ldots, v_d) := \omega(v_1, \ldots, v_d).$$
  * Intuitively, whenever the set $\{v_1, \ldots, v_d\}$ is not linearly independent, they only span a $(d-1)$-dimensional hypersurface in V at most, which should have 0 volume.
- **determinant** Let V be a d-dimensional vector space and let $\phi \in \mathrm{End}(V) \cong_{\text{vec}} T^1_1 V$. The **determinant** of ϕ is
$$\det \phi := \frac{\omega(\phi(e_1), \ldots, \phi(e_d))}{\omega(e_1, \ldots, e_d)}$$
for some volume form $\omega \in \Lambda^d V$ and some basis $\{e_1, \ldots, e_d\}$ of V.
  * This is well-defined: $\det \phi$ is *independent of the choice of* ω is clear, since if $\omega, \omega' \in \Lambda^d V$, then there is a $c \in K$ such that $\omega = c\,\omega'$, and hence
$$\frac{\omega(\phi(e_1), \ldots, \phi(e_d))}{\omega(e_1, \ldots, e_d)} = \frac{\cancel{c}\,\omega'(\phi(e_1), \ldots, \phi(e_d))}{\cancel{c}\,\omega'(e_1, \ldots, e_d)}.$$
The *independence from the choice of basis* is more cumbersome to show, but it does hold, and thus $\det \phi$ is well-defined.
  * ϕ needs to be an **endomorphism** because we need to apply ω to $\phi(e_1), \ldots, \phi(e_d)$, and thus ϕ needs to output a vector.
  * See my analysis notes for the definition comparison between rudin and this course.
  * As we have mentioned, we cannot define the determinant for g even it can be viewed as a matrix. The determinant of g then transforms as
$$\det(A^T g A) = \det(A^T)\det(g)\det(A) = (\det A)^2 \det(g)$$
i.e. it **not** invariant under a change of basis. It is **not** a well-defined object[13]. But later we can defined a **scalar density of weight 2** for it!
$$X \to \frac{1}{(\det A)^2} X$$
We will have to introduce **principal fibre bundles**. Using them, we will be able to give a **bundle definition of tensor** and of **tensor densities** which are, loosely speaking, quantities that transform with powers of $\det A$ under a change of basis.

19. **pseudo inner product:** Let V be an n-dimensional $\mathbb{R}$-vector space. A **pseudo inner product** on V is an **bilinear map** $(-,-) : V \times V \to \mathbb{R}$ satisfying
    (a) **symmetry:** $\forall\, v, w \in V : \ (v, w) = (w, v)$;
    (b) **non-degeneracy:** $(\forall\, w \in V : (v, w) = 0) \Rightarrow v = 0$.
    - Ordinary **inner products** satisfy a **stronger** condition than non-degeneracy, called **positive definiteness**, which is $\forall\, v \in V : (v, v) \geq 0$ and $(v, v) = 0 \Rightarrow v = 0$.
    - **signature:** Given a **symmetric bilinear map** $(-,-)$ on V, there is always a basis $\{e_a\}$ of V such that $(e_a, e_a) = \pm 1$ and zero otherwise. If **non-degeneracy**, we will get p-many 1s and q-many -1s with $p + q = n$ (i.e., **no** zeros), then the pair (p, q) is called the **signature** of the **pseudo inner product**.

[13] For endomorphism ϕ, $\det(A^{-1}\phi A) = \det(A^{-1})\det(\phi)\det(A) = \det(A^{-1}A)\det(\phi) = \det(\phi)$

* $*$ Positive definiteness is the requirement that the signature be $(n, 0)$,
  * $*$ In relativity we require the signature to be $(n-1, 1)$.
  * $*$ A theorem states that there are (up to isomorphism) only as many pseudo inner products on V as there are different signatures.
* **symmetric w.r.t. the pseudo inner product:** A linear map $\phi: V \xrightarrow{\sim} V$ is said to be **symmetric** w.r.t. the pseudo inner product $(-,-)$ on V if

$$\forall\, v, w \in V: \ (\phi(v), w) = (v, \phi(w)).$$

If, instead, we have

$$\forall\, v, w \in V: \ (\phi(v), w) = -(v, \phi(w)),$$

then ϕ is said to be **antisymmetric** w.r.t. $(-,-)$.

10 Tangent Vector Space, Algebras and Derivations

From now on, whenever we say "manifold", we mean a (real) d-dimensional smooth manifold M. Note, in the following, sometimes we could just define a local object like $\mathcal{C}^\infty(U)$ with $U \subseteq M$ instead of $\mathcal{C}^\infty(M)$. However, for simplicity, we only use $\mathcal{C}^\infty(M)$ with understanding of feasible generalization.

A summary:

- ▶ We give definition of **tangent vector** and **tangent vector space** at one point $p \in M$. Note we totally give **4 equivalent definitions** of tangent space.
- ▶ We first give definition for **a special vector space called algebra** with one additional operator $\bullet$, and then **a special algebra called Lie algebra** with constraints on $\bullet$.
- ▶ We give the definition **derivation** for algebra. We then **construct derivation given Lie algebra**, and vice versa (i.e. **construct Lie algebra given derivation**). (Tangent space can be defined using **derivation**).
- ▶ Finally we give a **chart induced basis** for the tangent space and show the change of basis rule.

1. **smooth function $\mathbb{R}$-vector space:** Let M be a manifold. We define the infinite-dimensional vector space over $\mathbb{R}$ with underlying set

$$\mathcal{C}^\infty(M) := \{f: M \to \mathbb{R} \mid f \text{ is smooth}\}$$

and operations defined pointwise, i.e. for any $p \in M$,

$$(f+g)(p) := f(p) + g(p)$$
$$(\lambda f)(p) := \lambda f(p).$$

   * We can similarly define $\mathcal{C}^\infty(U)$, with U an open subset of M.

2. **smooth curve:** A **smooth curve** on M is a smooth map $\gamma: \mathbb{R} \to M$ (where $\mathbb{R}$ is understood as a 1-dimensional manifold). This definition also applies to smooth maps $I \to M$ for an open interval $I \subseteq \mathbb{R}$.
3. **tangent vector:** It is a **directional derivative operator**: Let $\gamma: \mathbb{R} \to M$ be a smooth curve through $p \in M$; w.l.o.g. let $\gamma(0) = p$. The **directional derivative operator** at p along γ is the linear map

$$\begin{aligned} X_{\gamma,p}: \ \mathcal{C}^\infty(M) &\xrightarrow{\sim} \mathbb{R} \\ f &\mapsto (f \circ \gamma)'(0), \end{aligned}$$

where $\mathbb{R}$ is understood as a 1-dimensional vector space over the field $\mathbb{R}$.

   * $f \circ \gamma$ is a map $\mathbb{R} \to \mathbb{R}$, hence we can calculate the usual derivative and evaluate it at 0.
   * Note, it then satisfies **Leibniz rule** for fg from undergraduate derivative.
   * Intuitively, $X_{\gamma,p}$ is the **velocity** of γ at p.
   * $X_{\gamma,p}$ is a **linear map**.

4. **tangent space (equivalent definition I):** Let M be a manifold and $p \in M$. The **tangent space** to M at p is the **vector space** over $\mathbb{R}$ with underlying set
$$T_pM := \{X_{\gamma,p} \mid \gamma \text{ is a smooth curve through } p\},$$
with the addition
$$\begin{aligned}\oplus :\ T_pM \times T_pM &\to T_pM\\ (X_{\gamma,p}, X_{\delta,p}) &\mapsto X_{\gamma,p} \oplus X_{\delta,p},\end{aligned}$$
and scalar multiplication
$$\begin{aligned}\odot :\ \mathbb{R} \times T_pM &\to T_pM\\ (\lambda, X_{\gamma,p}) &\mapsto \lambda \odot X_{\gamma,p},\end{aligned}$$
both defined pointwise (here "point" is $f \in \mathcal{C}^\infty(M)$), i.e. for any $f \in \mathcal{C}^\infty(M)$,
$$\begin{aligned}(X_{\gamma,p} \oplus X_{\delta,p})(f) &:= X_{\gamma,p}(f) + X_{\delta,p}(f)\\ (\lambda \odot X_{\gamma,p})(f) &:= \lambda X_{\gamma,p}(f).\end{aligned}$$
   - Check well-defined: Let $X_{\gamma,p}, X_{\delta,p} \in T_pM$ and $\lambda \in \mathbb{R}$. Then, we have $X_{\gamma,p} \oplus X_{\delta,p} \in T_pM$ and $\lambda \odot X_{\gamma,p} \in T_pM$. (See my riemannian manifold notes for proof.)
   - Derivative is a **local**[14] concept, it is only the behaviour of curves near p that matters. If two curves γ and δ agree on a **neighbourhood** of p, then $X_{\gamma,p}$ and $X_{\delta,p}$ are the same element of T_pM. Hence, we can work locally by using a **chart** on M.
   - **tangent space equivalent definition II:** Consider the set of smooth curves
   $$S = \{\gamma :\ I \to M \mid \text{with } I \subseteq \mathbb{R} \text{ open, } 0 \in I \text{ and } \gamma(0) = p\}$$
   and define the **equivalence relation** $\sim$ on S
   $$\gamma \sim \delta \ :\Leftrightarrow\ (x \circ \gamma)'(0) = (x \circ \delta)'(0)$$
   for some (and hence every) chart (U, x) containing p. Then, we can define
   $$T_pM := S/\sim .$$

5. **algebra:** An **algebra** over a field K is a quadruple $(A, +, \cdot, \bullet)$, where $(A, +, \cdot)$ is a K**-vector space** and $\bullet$ is a **product** on A, i.e. a **(**K**-)bilinear map** $\bullet :\ A \times A \to A$.
   - Algebra is a ring from definition where the ring addition is $+$ and multiplication is $\bullet$.
   - Example: Define a product on $\mathcal{C}^\infty(M)$ by
   $$\begin{aligned}\bullet :\ \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) &\to \mathcal{C}^\infty(M)\\ (f, g) &\mapsto f \bullet g,\end{aligned}$$
   where $f \bullet g$ is defined pointwise on M. Then $(\mathcal{C}^\infty(M), +, \cdot, \bullet)$ is an algebra over $\mathbb{R}$. $(\mathcal{C}^\infty(M), +, \cdot, \bullet)$ is an associative, unital, commutative algebra (so it is also a ring, which will be used on Section 12). See below.
   - An algebra $(A, +, \cdot, \bullet)$ is said to be
     * **associative** if $\forall\ v, w, z \in A : v \bullet (w \bullet z) = (v \bullet w) \bullet z$;
     * **unital** if $\exists\ \mathbf{1} \in A : \forall\ v \in V : \mathbf{1} \bullet v = v \bullet \mathbf{1} = v$;
     * **commutative** or **abelian** if $\forall\ v, w \in A : v \bullet w = w \bullet v$.
   - **algebra homomorphism:** If A and B are algebras over a field (or commutative ring) K, **algebra homomorphism** is a function $F : A \to B$ such that for all k in K and x, y in A:
   (a) $F(kx) = kF(x)$
   (b) $F(x + y) = F(x) + F(y)$
   (c) $F(xy) = F(x)F(y)$
   In other words, F is **a** K**-linear map** (or K module homomorphism if K is a commutative ring), and F is a (non-unital) **ring homomorphism.**
   If F is **bijective**, F is said to be an **algebra isomorphism** between A and B.

[14]This local is different from the covariant derivative locality in [1][Lemma 4.1 and Proposition 4.3]. We still need neighbourhood, similar to Lie derivative.

6. **Lie algebra:** It is a special algebra with the product $v \bullet w$ be written as $[v, w]$. A **Lie algebra** A is an algebra whose product $[-,-]$, called **Lie bracket**, satisfies
   (a) **alternativity:** $\forall\ v \in A : [v, v] = 0$;
   (b) **Jacobi identity:** $\forall\ v, w, z \in A : [v, [w, z]] + [w, [z, v]] + [z, [v, w]] = 0$.
   (c) **antisymmetry:** $[v, w] = -[w, v]$ for all $v, w \in A$ (implied from alternative and the bilinear).
   - Note that the zeros above is the additive identity element in A, **not** the zero scalar in K.
   - From antisymmetry, a (non-trivial) Lie algebra cannot be unital.
7. **from algebra to construct Lie algebra:** Given an associative algebra $(A, +, \cdot, \bullet)$, if we define
$$[v, w] := v \bullet w - w \bullet v,$$
then $(A, +, \cdot, [-,-])$ is a Lie algebra. The Lie bracket is typically called the **commutator**.
   - Example: Let V be a vector space over K. Then $(\mathrm{End}(V), +, \cdot, \circ)$ is an associative, unital, non-commutative algebra over K. Define
$$\begin{aligned}[-,-]:\ \mathrm{End}(V) \times \mathrm{End}(V) &\to \mathrm{End}(V)\\ (\phi, \psi) &\mapsto [\phi, \psi] := \phi \circ \psi - \psi \circ \phi.\end{aligned}$$
8. **derivation:** Let A be an algebra. A **derivation** on A is a K**-linear map** $D :\ A \xrightarrow{\sim} A$ satisfying the **Leibniz rule**
$$D(v \bullet w) = D(v) \bullet w + v \bullet D(w)$$
for all $v, w \in A$.
   - Example I: derivation on $\mathcal{C}^\infty(\mathbb{R})$ for the algebra of smooth **real** functions, since it is linear and satisfies the Leibniz rule:
$$D(f \bullet_A g) = D(f) \bullet g + f \bullet D(g),$$
where $\bullet$ is the pointwise multiplication on $\mathbb{R}$.
   - The second derivative operator is not a derivation on $\mathcal{C}^\infty(\mathbb{R})$, since it does not satisfy the Leibniz rule. This shows that the composition of derivations need not be a derivation.
   - **notation:** We denote by $\mathrm{Der}_K(A)$ the set of derivations on a K-algebra $(A, +, \cdot, \bullet)$.
   - $\mathrm{Der}_K(A)$ can be endowed with a K-vector space structure by defining the vector space operations $+$ and $\cdot$ pointwise on A. But it cannot be made into an algebra under composition of derivations.
   - **extension to maps $A \to B$:** consider two algebras $(A, +_A, \cdot_A, \bullet_A)$, $(B, +_B, \cdot_B, \bullet_B)$, and require $D :\ A \xrightarrow{\sim} B$ to satisfy
$$D(v \bullet_A w) = D(v) \bullet_B w +_B v \bullet_B D(w).$$
However, this is meaningless as it stands since $\bullet_B :\ B \times B \to B$, but on the right hand side $\bullet_B$ acts on elements from A too. We need B is a (ring) A**-bimodule** where please note **algebra is a also ring** from definition.
     * **notation:** We denote by $\mathrm{Der}_K(A, B)$ the set of derivations.
     * Example II: At point p with a neighbor U, we can define derivation $\mathcal{C}^\infty(U) \to \mathbb{R}$:
$$D(f \bullet_A g) = D(f) \bullet_B g + f \bullet_B D(g),$$
where $D(f) \bullet_B g$ is defined as $D(f) \cdot g(p)$ at point p while $\bullet_A$ is pointwise multiplication on U. Please compare with above Example I derivation on algebra of smooth **real** functions
9. **from Lie algebra to construct derivation:** Consider again the Lie algebra $(A, +, \cdot, [-,-])$ and fix $\xi \in A$. If we define
$$\begin{aligned}D_\xi := [\xi, -] :\ A &\xrightarrow{\sim} A\\ \phi &\mapsto [\xi, \phi],\end{aligned}$$
then D_ξ is a **derivation** on $(A, +, \cdot, [-,-])$ since it is linear and
$$\begin{aligned}D_\xi([\phi, \psi]) &:= [\xi, [\phi, \psi]]\\ &= -[\psi, [\xi, \phi]] - [\phi, [\psi, \xi]] &&\text{(by the Jacobi identity)}\\ &= [[\xi, \phi], \psi] + [\phi, [\xi, \psi]] &&\text{(by antisymmetry)}\\ &=: [D_\xi(\phi), \psi] + [\phi, D_\xi(\psi)].\end{aligned}$$

- Later in Section 15, this is called **adjoint map**, which is important in Lie algebra.

10. **from a derivation construct a Lie algebra:** From above, composition of derivation does not make it as an algebra product. However, derivations are maps, so we can still compose them as maps and define

$$\begin{aligned}[-,-]:\ \mathrm{Der}_K(A)\times\mathrm{Der}_K(A)&\to\mathrm{Der}_K(A)\\ (D_1,D_2)&\mapsto[D_1,D_2]:=D_1\circ D_2-D_2\circ D_1.\end{aligned}$$

The map $[D_1,D_2]$ is (perhaps surprisingly) a **derivation**, since it is linear and

$$\begin{aligned}[D_1,D_2](v\bullet w) &:= (D_1\circ D_2-D_2\circ D_1)(v\bullet w)\\ &= D_1(D_2(v\bullet w))-D_2(D_1(v\bullet w))\\ &= D_1(D_2(v)\bullet w+v\bullet D_2(w))-D_2(D_1(v)\bullet w+v\bullet D_1(w))\\ &= D_1(D_2(v)\bullet w)+D_1(v\bullet D_2(w))-D_2(D_1(v)\bullet w)-D_2(v\bullet D_1(w))\\ &= D_1(D_2(v))\bullet w+\cancel{D_2(v)\bullet D_1(w)}+\cancel{D_1(v)\bullet D_2(w)}+v\bullet D_1(D_2(w))\\ &\quad -D_2(D_1(v))\bullet w-\cancel{D_1(v)\bullet D_2(w)}-\cancel{D_2(v)\bullet D_1(w)}-v\bullet D_2(D_1(w))\\ &= (D_1(D_2(v))-D_2(D_1(v)))\bullet w+v\bullet(D_1(D_2(w))-D_2(D_1(w)))\\ &= [D_1,D_2](v)\bullet w+v\bullet[D_1,D_2](w)\end{aligned}$$

Then $(\mathrm{Der}_K(A),+,\cdot,[-,-])$ is a **Lie algebra** over K.

- Compare with **from algebra to construct Lie algebra**, here even $\mathrm{Der}_K(A)$ is not an algebra, we could get a **Lie algebra**.
- Do be confused about the subscript. The subscripts in $\mathrm{Der}_K(A)$ and $\mathrm{Der}_p(A)$ have different indication.

11. **derivation on manifold:** Let M be a manifold and let $p\in U\subseteq M$, where U is open. A **derivation on** U **at** p is an $\mathbb{R}$**-linear map** $D:\ \mathcal{C}^\infty(U)\xrightarrow{\sim}\mathbb{R}$ satisfying the Leibniz rule

$$D(fg)=D(f)g(p)+f(p)D(g).$$

We denote by $\mathrm{Der}_p(U)$ the $\mathbb{R}$-vector space of derivations on U at p, with operations defined pointwise.

- Note here again it is a **bimodule**. See also above Example II. There we focus on one point, so we need the bimodule. However, in Section 12, we extend to $\mathcal{C}^\infty(M)$, i.e. $\mathrm{Der}_{\mathbb{R}}(\mathcal{C}^\infty(M))$ and do not need bimodule again.
- **tangent space equivalent definition III:** The tangent vector $X_{\gamma,p}$ is a derivation on $U\subseteq M$ at p, where U is any neighbourhood of p. We have

$$T_pM:=\mathrm{Der}_p(U),$$

for some open U containing p. This does not depend on which neighbourhood U of p we pick.

12. **(chart induced) basis for the tangent space:** Let (U,x) be a chart for d-dimensional smooth manifold M and let $\gamma:\mathbb{R}\to M$ be a curve that passes through point $p\in U$ as $\gamma(0)=p$. Now we have the calculation

$$\begin{aligned}X_{\gamma,p}(f)&:=(f\circ\gamma)'(0)\\ &=(f\circ x^{-1}\circ x\circ\gamma)'(0)\\ &=(x^i\circ\gamma)'(0)\cdot\partial_i\big(f\circ x^{-1}\big)\big|_{x(p)}\end{aligned}$$

We introduce some new notation in order to simplify it: we define

$$\left(\frac{\partial f}{\partial x^i}\right)_p:=\partial_i\big(f\circ x^{-1}\big)\big|_{x(p)},\qquad\text{and}\qquad\dot\gamma^i_x(0):=(x^i\circ\gamma)'(0),$$

Then

$$X_{\gamma,p}(f)=\dot\gamma^i_x(0)\cdot\left(\frac{\partial}{\partial x^i}\right)_p(f),$$

or, as a *map*, we can write

$$X_{\gamma,p}=\dot\gamma^i_x(0)\cdot\left(\frac{\partial}{\partial x^i}\right)_p.$$

- Note later we will use the following notation for vector field

$$\left(\frac{\partial}{\partial x^a}\right)(f) = \partial_a(f \circ x^{-1}) \circ x.$$

  Please remember that at each point $p \in M$, we are computing $\partial_a(f \circ x^{-1})|_{x(p)}$.
- The symbol $\left(\frac{\partial}{\partial x^a}\right)_p$ is **just notation**. It looks like a partial derivative, however strictly it is something **completely different**.
  * But it is **notationally consistent** with partial derivative: Let $M = \mathbb{R}^d$, $(U, x) = (\mathbb{R}^d, \mathrm{id}_{\mathbb{R}^d})$ and let $\left(\frac{\partial}{\partial x^a}\right)_p \in T_p\mathbb{R}^d$. If $f \in \mathcal{C}^\infty(\mathbb{R}^d)$, then

$$\left(\frac{\partial}{\partial x^a}\right)_p (f) = \partial_a(f \circ x^{-1})(x(p)) = \partial_a f(p),$$

    since $x = x^{-1} = \mathrm{id}_{\mathbb{R}^d}$. Moreover, we have $\mathrm{proj}_a = x^a$.
  * It will posses all of the properties we'd want from a partial derivative like **symmetric of second order derivative**.
  * **notation:** So we often write $\left(\frac{\partial}{\partial x^a}\right)_p (f)$ as $\partial_a f$.
  * Thus, we can think of $x^1, \ldots, x^d$ as the independent variables of f, and we can then write

$$\left(\frac{\partial}{\partial x^a}\right)_p (f) = \frac{\partial f}{\partial x^a}(p).$$

- **basis:** The set

$$\left\{\left(\frac{\partial}{\partial x^1}\right)_p, \ldots, \left(\frac{\partial}{\partial x^d}\right)_p\right\}$$

  constitutes a basis for the tangent space T_pM, and it's known as the **chart induced basis** or **coordinate basis**.
- The dimension of the tangent space is equal to the dimension of the manifold

$$\dim T_pM = \dim M.$$

- **change of basis:** Let (U, x) and (V, y) be overlapping charts for a smooth manifold.

$$X = X^i_{(x)}\left(\frac{\partial}{\partial x^i}\right)_p \qquad \text{and} \qquad X = X^i_{(y)}\left(\frac{\partial}{\partial y^i}\right)_p$$

  To study how these relate, consider the following

$$\begin{aligned}
\left(\frac{\partial}{\partial x^i}\right)_p (f) &:= \partial_i\big(f \circ x^{-1}\big)\big|_{x(p)} \\
&= \partial_i\big(f \circ y^{-1} \circ y \circ x^{-1}\big)\big|_{x(p)} \\
&= \partial_i\big(y^j \circ x^{-1}\big)\big|_{x(p)} \cdot \partial_j\big(f \circ y^{-1}\big)\big|_{y(p)} \\
&= \left(\frac{\partial y^j}{\partial x^i}\right)_p \cdot \left(\frac{\partial}{\partial y^j}\right)_p (f) \\
\implies \left(\frac{\partial}{\partial x^i}\right)_p &= \left(\frac{\partial y^j}{\partial x^i}\right)_p \left(\frac{\partial}{\partial y^j}\right)_p
\end{aligned}$$

  For the components, we have

$$X^i_{(x)} = X^j_{(y)}\left(\frac{\partial x^i}{\partial y^j}\right)_p.$$

  Note $\left[\left(\frac{\partial y^j}{\partial x^i}\right)_p\right]$ and $\left[\left(\frac{\partial x^i}{\partial y^j}\right)_p\right]$ are matrix inverse of each other.

* **relation to Jacobian**: This is a general fact: if $\{*\}$ is a singleton (we let $*$ denote its unique element) and $x: \{*\} \to A$, $y: A \to B$ are maps, then $y \circ x$ is the same as the map y with independent variable x. Intuitively, x just "chooses" an element of A. The function $y = y(x)$ expresses the new coordinates, and $\left[\left(\frac{\partial y^j}{\partial x^i}\right)_p\right]$ is the **Jacobian** matrix of this map, evaluated at $x(p)$.
* **tangent space equivalent definition IV:** Let $\mathscr{A}_p : \{(U, x) \in \mathscr{A} \mid p \in U\}$ be the set of charts on M containing p. A **tangent vector** v at p is a map

$$v: \mathscr{A}_p \to \mathbb{R}^d$$

satisfying

$$v((V, y)) = A\, v((U, x))$$

where A is the Jacobian matrix of $y \circ x^{-1}: \mathbb{R}^d \to \mathbb{R}^d$ at $x(p)$. In components, we have

$$[v((V, y))]^b = \frac{\partial y^b}{\partial x^a}(x(p))\,[v((U, x))]^a.$$

The tangent space T_pM is then defined to be the set of all tangent vectors at p, endowed with the appropriate vector space structure.
* **a natural isomorphism from tangent vector space to $\mathbb{R}^d$:** There is a **natural (basis-independent) isomorphism** $T_pM \cong_{\text{vec}} \mathbb{R}^d$, obtained by associating a vector $X \in T_pM$ with the directional derivative

$$Xf = \left.\frac{d}{dt}\right|_{t=0} f(p + tX).$$

In terms of chart induced basis, we have

$$\begin{aligned} \iota_d: T_p\mathbb{R}^d &\to \mathbb{R}^d \\ X &\mapsto (X(x^1), \dots, X(x^d)) = (X^1, \dots, X^d) \end{aligned}$$

In case $M = \mathbb{R}^d$, then the isomorphism is an **identification**.

11 Cotangent Vector Space and Tangent Bundle

We continue from last Section 10 (this section and Section 10 can be put into one section but it is too long). Note, we still only consider (real) finite-dimensional differentiable manifolds in this section.

A summary:

- We give definition of **cotangent vector, cotangent space**, and correspondingly the general **tensor spaces**.
- We give a thorough comparison between **differential and gradient**. Differential is a push-forward map.
- We then discuss **push-forward of vector** and **pull-back of covector at one point** which are always possible. Please note the **global** push-forward for **vector fields** needs additional condition **diffeomorphism**. However **global** pull-back for **covector fields** is still always possible. We also need **diffeomorphism** to define **push-forward of covector** and **pull-back of vector**.
- We compare **immersions**, **submersions** and **embeddings**.
- We construct **tangent bundle** and show why it is a manifold fibre bundle by construct the atlas for the total space and projection and section map. The **cotangent and more general tensor bundle** follows analogously. The bundles will be used to construct (vector, covector, tensor) fields in the next Section 12 using **smooth sections**.

1. **cotangent space:** Let M be a manifold and $p \in M$. The **cotangent space** to M at p is

$$T_p^*M := (T_pM)^*.$$

- From Section 9, we know for finite dimensional vector space, $T_pM \cong_{\text{vec}} T_p^*M$.

2. **tensor spaces:** Once we have the cotangent space, we can define the **tensor spaces**. Let M be a manifold and $p \in M$. The **tensor space** $(T_s^r)_pM$ is defined as

$$(T_s^r)_pM := T_s^r(T_pM) = \underbrace{T_pM \otimes \cdots \otimes T_pM}_{r \text{ copies}} \otimes \underbrace{T_p^*M \otimes \cdots \otimes T_p^*M}_{s \text{ copies}}.$$

3. **differential (derivative) and gradient**[15]**:** We compare the differential, the general differential and gradient.
   - **(general) differential of differentiable map φ between manifolds - pushforward:** Let $\varphi : M \to N$ be a smooth map of smooth manifolds. Given $x \in M$, the **differential** of φ at p is the **pushforward**, a **linear map** between two tangent spaces T_pM and $T_{\varphi(p)}N$:

$$\begin{aligned} \mathrm{d}\varphi_p : T_pM &\to T_{\varphi(p)}N \\ X &\mapsto d\varphi_p(X)(f) := X(f \circ \varphi), \end{aligned}$$

     * So $d\varphi_p(X)$ becomes a velocity (i.e. directional derivative operator) at $\varphi(p)$ and acts for the function f in N.
     * We give more explanation of **pushforward** later. See below.
     * If $M = \mathbb{R}^d$ and $N = \mathbb{R}^{d'}$, then the differential of $\varphi : \mathbb{R}^d \to \mathbb{R}^{d'}$ at $p \in \mathbb{R}^d$

$$\mathrm{d}_p\varphi : T_p\mathbb{R}^d \cong_{\text{vec}} \mathbb{R}^d \to T_{\varphi(p)}\mathbb{R}^{d'} \cong_{\text{vec}} \mathbb{R}^{d'}$$

       is just the **Jacobian** of f at p.
   - **(special) differential of $f \in \mathcal{C}^\infty(M)$ at p is covector:** We consider a special case where a function f is in $\mathcal{C}^\infty(M)$, i.e. f is a smooth map from manifold M to manifold $\mathbb{R}$.

$$\begin{aligned} \mathrm{d}_pf : T_pM &\xrightarrow{\sim} T_{f(p)}\mathbb{R} \cong_{\text{vec}} \mathbb{R} \\ X &\mapsto \mathrm{d}_pf(X) := X(f). \end{aligned}$$

     * Here we have abused the notation. It will be more clear if we write $\mathrm{d}_pf(X)g := X(g \circ f)$ where g has domain of one point, i.e. g is a real number. We take g as id_R to define $\mathrm{d}_pf(X) := X(\mathrm{id}_\mathbb{R} \circ f)$. Why taking g as $\mathrm{id}_\mathbb{R}$? Okay, you can take any real number, but here it is definition! (Or think that we are using the natural isomorphism $X(\mathrm{id}_\mathbb{R})$ introduced in Section 10)
     * You can forget the abuse clarification, just remember **differential d_pf is covector.** It takes in a tangent vector X and returns the real number $X(f)$, in a linear fashion.
     * **dual basis of cotangent space:** If $\{(\frac{\partial}{\partial x^a})_p\}$ is the basis of T_pM induced by some chart (U, x), then the **dual basis of T_p^*M** is the set

$$\left\{ \left(\mathrm{d}x^1\right)_p, \ldots, \left(\mathrm{d}x^{\dim M}\right)_p \right\}.$$

       where, we have used, by definition, $(\mathrm{d}_px^a)\left(\left(\frac{\partial}{\partial x^b}\right)_p\right) = \left(\left(\frac{\partial}{\partial x^b}\right)_p\right)x^a = \delta_b^a$, and the convection of **notation**:

$$(\mathrm{d}x^a)_p = \mathrm{d}_px^a, \qquad 1 \le a \le \dim M.$$

     * **differential operator:** We can define the **differential operator at** $p \in M$ as the $\mathbb{R}$-linear map

$$\begin{aligned} \mathrm{d}_p : \mathcal{C}^\infty(M) &\xrightarrow{\sim} T_p^*M \\ f &\mapsto \mathrm{d}_pf, \end{aligned}$$

       with $p \in M$.

[15] Here we do not follow the video

- **gradient:** If g is a Riemannian metric on M and $f : M \to \mathbb{R}$ is a smooth function, the **gradient of f is the vector field** $\operatorname{grad} f = (\mathrm{d}f)^\sharp$ obtained from $\mathrm{d}f$ by **raising an index**. Unwinding the definitions, we see that $\operatorname{grad} f$ is characterized by the fact that

$$\mathrm{d}f_p(X) = \left\langle \operatorname{grad} f|_p \, , X \right\rangle \qquad \text{for all } p \in M, X \in T_pM$$

and has the local basis expression

$$\operatorname{grad} f = \left(g^{ij} E_i(f)\right) E_j$$

where E_i and ε^i is the **local chart induced (co)-frame basis** (at each point p, it degenerates to the (co)-tangent space basis, see [2] and Section 12). Note that then

$$\begin{aligned} \left\langle \operatorname{grad} f|_p \, , X \right\rangle &= \left\langle \left(g^{ij} E_i(f)\right) E_j, \varepsilon^i(X) E_i \right\rangle \\ &= E_i(f)\varepsilon^i(X) \\ &= \mathrm{d}_p f(X) \end{aligned}$$

  - Gradient is just raising an index from differential. The operation value is the same.

4. **push-forward:** Let $\phi : M \to N$ be a smooth map between smooth manifolds. The **push-forward** of ϕ at $p \in M$ is the **linear map**:

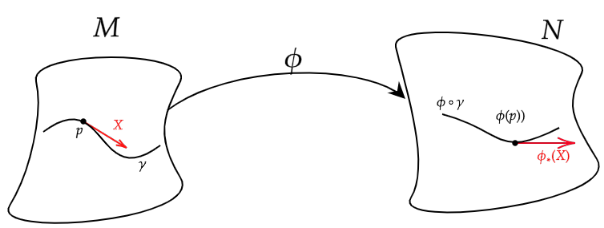
$$\begin{aligned} (\phi_*)_p : \; T_pM &\xrightarrow{\sim} T_{\phi(p)}N \\ X &\mapsto (\phi_*)_p(X) := X(- \circ \phi). \end{aligned}$$

- As we have mention above, the **differential (derivative)** of differentiable map between manifolds is constructed using **pushforward**.
- **notation:** $(\phi_*)_p$ and $d\varphi_p$ can both be used.
- **curve push-forward:** If $\gamma : \mathbb{R} \to M$ is a smooth curve on M and $\phi : M \to N$ is smooth, then $\phi \circ \gamma : \mathbb{R} \to N$ is a smooth curve on N. More specifically, we have

$$\begin{aligned} (\phi_*)_p(X_{\gamma,p})(f) &= (X_{\gamma,p})(f \circ \phi) \\ &= ((f \circ \phi) \circ \gamma)'(0) \\ &= (f \circ (\phi \circ \gamma))'(0) \\ &= X_{\phi\circ\gamma,\phi(p)}(f) \end{aligned}$$

We conclude $X_{\gamma,p} \in T_pM$ is pushed forward to $X_{\phi\circ\gamma,\phi(p)} \in T_{\phi(p)}N$, i.e.

$$(\phi_*)_p(X_{\gamma,p}) = X_{\phi\circ\gamma,\phi(p)}.$$



- **local representation of push-forward:** with chart (U, x) defined on M and (V, y) defined on N such that $p \in U$ and $\phi(p) \in V$, if $X = X^i_{(x)} \left(\frac{\partial}{\partial x^i}\right)_p$, we have

$$\begin{aligned} X_{\phi\circ\gamma,\phi(p)}(f) &= (f \circ (\phi \circ \gamma))'(0) \\ &= (f \circ y^{-1} \circ y \circ \phi \circ \gamma)'(0) \\ &= (y^i \circ \phi \circ \gamma)'(0) \cdot \partial_i\left(f \circ y^{-1}\right)\big|_{y(\phi(p))} \\ &= (y^i \circ \phi \circ x^{-1} \circ x \circ \gamma)'(0) \cdot \partial_i\left(f \circ y^{-1}\right)\big|_{y(\phi(p))} \\ &= \partial_j(y^i \circ \phi \circ x^{-1})|_{x(p)} X^j_{(x)} \left(\frac{\partial}{\partial y^i}\right)_{\phi(p)} \end{aligned}$$

So $(\phi_*)_p$ has mapped

$$X = X^i_{(x)}\left(\frac{\partial}{\partial x^i}\right)_p \Rightarrow (\phi_*)_p(X) = \partial_j(y^i \circ \phi \circ x^{-1})|_{x(p)} X^j_{(x)}\left(\frac{\partial}{\partial y^i}\right)_{\phi(p)}$$

5. **pull-back:** Let $\phi:\ M \to N$ be a smooth map between smooth manifolds. The **pull-back** of ϕ at $p \in M$ is the **linear map**:

$$\begin{aligned}(\phi^*)_p:\ T^*_{\phi(p)}N &\xrightarrow{\sim} T^*_pM\\ \omega &\mapsto (\phi^*)_p(\omega),\end{aligned}$$

where $(\phi^*)_p(\omega)$ is defined as

$$\begin{aligned}(\phi^*)_p(\omega):\ T_pM &\xrightarrow{\sim} \mathbb{R}\\ X &\mapsto \omega((\phi_*)_p(X)),\end{aligned}$$

- **one convention:** Given a smooth map $\phi:\ M \to N$, if $f \in \mathcal{C}^\infty(N)$, sometimes $f \circ \phi$ is often called the **pull-back of f along ϕ.**
- **local representation of pull-back:** with chart (U, x) defined on M and (V, y) defined on N such that $p \in U$ and $\phi(p) \in V$, if $\omega = \omega_{k(y)}(\mathrm{d}y^k)_{\phi(p)}$, we then use dual basis:

$$\begin{aligned}\omega((\phi_*)_p(\left(\frac{\partial}{\partial x^m}\right)_p)) &= \omega_{k(y)}(\mathrm{d}y^k)_{\phi(p)}\left(\partial_m(y^i \circ \phi \circ x^{-1})|_{x(p)}\left(\frac{\partial}{\partial y^i}\right)_{\phi(p)}\right)\\ &= \omega_{k(y)}\partial_m(y^k \circ \phi \circ x^{-1})|_{x(p)}\end{aligned}$$

So $(\phi^*)_p$ has mapped

$$\omega = \omega_{k(y)}(\mathrm{d}y^k)_{\phi(p)} \Rightarrow (\phi^*)_p(\omega) = \omega_{k(y)}\partial_m(y^k \circ \phi \circ x^{-1})|_{x(p)}(\mathrm{d}x^m)_p$$

Note, here $\omega_{k(y)}$ is the component at $\phi(p)$!

6. **discussion of push-forward and pull-back:**
- In most cases, **vectors are pushed forward; covectors are pulled back.** They are always well-defined and are $\mathbb{R}$**-linear maps**.

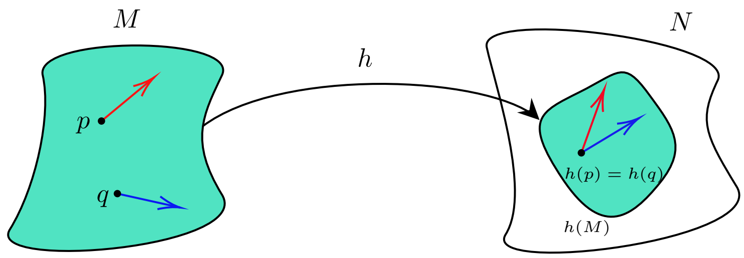
$$\begin{tikzcd}
\mathcal{C}^\infty(M) \arrow[d, "X"] & \mathcal{C}^\infty(N) \arrow[l, "-\circ\phi"'] \arrow[dl, "(\phi_*)_p(X)"] \\
\mathbb{R} &
\end{tikzcd}
\qquad
\begin{tikzcd}
T_pM \arrow[r, "(\phi_*)_p"] \arrow[dr, "(\phi^*)_p(\omega)"'] & T_{\phi(p)}N \arrow[d, "\omega"] \\
& \mathbb{R}
\end{tikzcd}$$

  $*$ Generally speaking, any tensor **at one point** can be push-forward and pull-back.
- If $\phi:\ M \to N$ is a **diffeomorphism**, **by using** ϕ^{-1} then we can also **pull a vector** $Y \in T_{\phi(p)}N$ **back** to a vector $(\phi^*)_p(Y) \in T_pM$, and **push a covector** $\eta \in T^*_pM$ **forward** to a covector $(\phi_*)_p(\eta) \in T^*_{\phi(p)}N$:

$$\begin{aligned}(\phi^*)_p(Y) &:= ((\phi^{-1})_*)_{\phi(p)}(Y)\\ (\phi_*)_p(\eta) &:= ((\phi^{-1})^*)_{\phi(p)}(\eta).\end{aligned}$$

$$\begin{tikzcd}
\mathcal{C}^\infty(M) \arrow[r, "-\circ\phi^{-1}"] \arrow[dr, "(\phi^*)_p(Y)"'] & \mathcal{C}^\infty(N) \arrow[d, "Y"] \\
& \mathbb{R}
\end{tikzcd}
\qquad
\begin{tikzcd}
T_pM \arrow[d, "\eta"'] & T_{\phi(p)}N \arrow[l, "((\phi^{-1})_*)_{\phi(p)}"'] \arrow[dl, "(\phi_*)_p(\eta)"] \\
\mathbb{R} &
\end{tikzcd}$$

- We have that **global push-forward for vector fields** needs **diffeomorphism**. However **global pull-back for covector fields** is still always well-defined. See Section 12 for more details.



7. **immersion:** A smooth map $\phi:\ M \to N$ is said to be an **immersion** of M into N if the differential
$$d\phi_p:\ T_pM \xrightarrow{\sim} T_{\phi(p)}N$$
is **injective**, for all $p \in M$. The manifold M is said to be an **immersed submanifold** of N.
   - For $\phi:\ M \to N$ to be an immersion, we must have $\dim M \leq \dim N$.

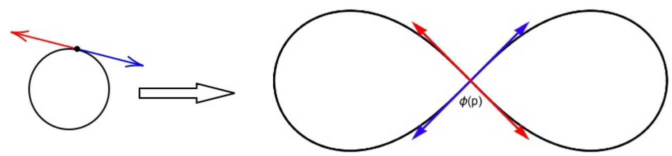


Fig. 1: Immersion

   - Example: The map ϕ is not injective. However, the maps $(\phi_*)_p$ and $(\phi_*)_q$ are both injective. Hence, the map ϕ is immersion.

8. **submersion:** A smooth map $\phi : M \to N$ is a **submersion** at a point $p \in M$ if its differential
$$d\phi_p : T_pM \xrightarrow{\sim} T_{\phi(p)}N$$
is a **surjective** linear map. A differentiable map ϕ that is a **submersion** at each point $p \in M$ is called a submersion.
   - Equivalently, ϕ is a submersion if its differential $d\phi_p$ has constant rank equal to the dimension of N.
   - **regular and critical point:** In the case of submersion, p is called a **regular point** of the map ϕ, otherwise, p is a **critical point**. A point $q \in N$ is a regular value of ϕ if all points p in the preimage $\phi^{-1}(q)$ are regular points.
   - For $\phi:\ M \to N$ to be an submersion, we must have $\dim M \geq \dim N$.

9. **immersion vs. submersion:** **The Rank Theorem** may provide some insight into these concepts. Suppose M and N are smooth manifolds of dimensions m and n, respectively, and $F : M \to N$ is a smooth map with constant rank r. For each $p \in M$ there exist smooth charts (U, φ) for M centered at p and (V, ψ) for N centered at $F(p)$ such that $F(U) \subseteq V$, in which F has a coordinate representation of the form
$$\hat{F}\left(x^1, \ldots, x^r, x^{r+1}, \ldots, x^m\right) = \left(x^1, \ldots, x^r, 0, \ldots, 0\right).$$
In particular, if F is a smooth **submersion**, this becomes
$$\hat{F}\left(x^1, \ldots, x^n, x^{n+1}, \ldots, x^m\right) = \left(x^1, \ldots, x^n\right),$$
and if F is a smooth **immersion**, it is
$$\hat{F}\left(x^1, \ldots, x^m\right) = \left(x^1, \ldots, x^m, 0, \ldots, 0\right).$$
   - A **submersion** locally looks like a **projection** $\mathbb{R}^n \times \mathbb{R}^{m-n} \to \mathbb{R}^n$,
   - An **immersion** locally looks like an **inclusion** $\mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}^{n-m}$.

10. **embedding:** A smooth map $\phi:\ M \to N$ is said to be a **(smooth) embedding** of M into N if
   - $\phi:\ M \to N$ is an **immersion**;
   - $M \cong_{\text{top}} \phi(M) \subseteq N$, where $\phi(M)$ carries the **subset topology** inherited from N.

The manifold M is said to be an **embedded submanifold** of N.

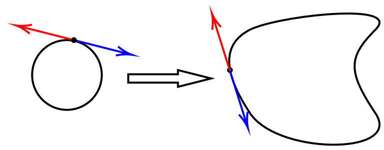


Fig. 2: Embedding

- embedding is called **extrinsic view**, while in this course we mainly focus on **intrinsic view** where the tangent space are defined without using any ambient space.
- Example: See the figure.

11. **Whitney's theorem:** Any smooth manifold M can be
    (a) embedded in $\mathbb{R}^{2\dim M}$;
    (b) immersed in $\mathbb{R}^{2\dim M-1}$.

    - Whitney's theorem states that **extrinsic view** and **intrinsic view** are essentially the same.
    - Whitney's theorem states the loosest bound.
    - Example: The Klein bottle can be embedded in $\mathbb{R}^4$ but not in $\mathbb{R}^3$. It can, however, be immersed in $\mathbb{R}^3$. The intersection is still a issue in $\mathbb{R}^3$, similar to the intersection issue in Fig. 2.
    - **improved version:** Any smooth manifold can be immersed in $\mathbb{R}^{2\dim M-a(\dim M)}$, where $a(n)$ is the number of 1s in a binary expansion of $n \in \mathbb{N}$. If $\dim M = 3$, then as $3_{10} = (1 \times 2^1 + 1 \times 2^0)_{10} = 11_2$, we have $a(\dim M) = 2$,

12. **tangent bundle:** Given a smooth manifold M, the **tangent bundle** of M is the **disjoint union** of all the tangent spaces to M, i.e.

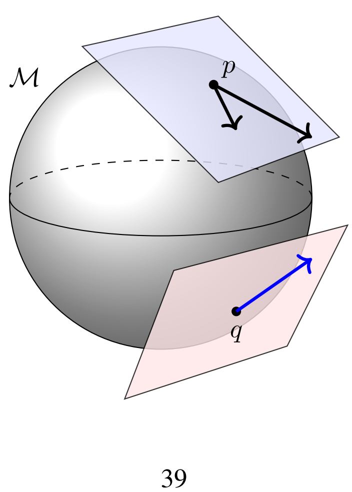
$$TM := \coprod_{p \in M} T_pM,$$

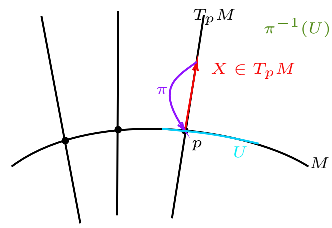
equipped with the **canonical projection map**

$$\begin{aligned} \pi :\ TM &\to M \\ X &\mapsto p, \end{aligned}$$

where p is the **unique** $p \in M$ such that $X \in T_pM$.

- T_pM and T_qM are totally different vector spaces if $p \neq q$ (**disjoint union** is therefore guaranteed). To define a smooth change between vectors from those tangent spaces at each point, we introduce the tangent bundle and then the smooth change is defined as the **smooth section** between base manifold to total manifold.





- Currently the TM is only a **set bundle**. We need assign it a manifold structure and show it is indeed have the locally trivial property to make it a fibre bundle.
- Later we will define **connection** to numerically compare the vectors from different tangent space. Currently, we can only talk about the smoothness.

13. **equip TM with the structure of a smooth manifold:** Let $\mathscr{A}_M$ be a smooth atlas on M and let $(U, x) \in \mathscr{A}_M$. If $X \in \operatorname{preim}_\pi(U) \subseteq TM$, then $X \in T_{\pi(X)}M$, by definition of π. Moreover, since $\pi(X) \in U$, we can expand X in terms of the basis induced by the chart (U, x):

$$X = X^a \left(\frac{\partial}{\partial x^a}\right)_{\pi(X)},$$

where $X^1, \dots, X^{\dim M} \in \mathbb{R}$. We can then define the **map**

$$\begin{aligned} \xi : \operatorname{preim}_\pi(U) &\to x(U) \times \mathbb{R}^{\dim M} \cong_{\text{set}} \mathbb{R}^{2\dim M} \\ X &\mapsto (x(\pi(X)), X^1, \dots, X^{\dim M}). \end{aligned}$$

We then equip TM with the **initial topology**, we claim that **the pair (preim$_\pi(U), \xi)$ is a chart** on TM and

$$\mathscr{A}_{TM} := \{(\operatorname{preim}_\pi(U), \xi) \mid (U, x) \in \mathscr{A}_M\}$$

is a **smooth atlas** on TM.

- Note that, from its definition, it is clear that ξ is a bijection. We can show that $(\operatorname{preim}_\pi(U), \xi)$ is a **chart** but we omit the details.
- We also can show that $\mathscr{A}_{TM}$ is a **smooth atlas**: any two charts $(\operatorname{preim}_\pi(U), \xi), (\operatorname{preim}_\pi(\widetilde{U}), \widetilde{\xi}) \in \mathscr{A}_{TM}$ are $\mathcal{C}^\infty$-compatible.
- The **locally trivial** property then follows immediately: the required homeomorphism map is $x^{-1} \circ \operatorname{proj}_1 \circ \xi : \operatorname{preim}_\pi(U) \to U \times \mathbb{R}^{\dim M}$:

$$\begin{tikzcd} \operatorname{preim}_\pi(U) \arrow[d, "\pi"'] & & U \times \mathbb{R}^{\dim M} \arrow[ll, "x^{-1} \circ \operatorname{proj}_1 \circ \xi"'] \arrow[dll, "\operatorname{proj}_1"] \\ U & & \end{tikzcd}$$

14. **cotangent bundle:** Similarly, one can construct the **cotangent bundle** T^*M to M by defining

$$T^*M := \coprod_{p \in M} T_p^*M$$

and going through the above again, using the dual basis $\{(\mathrm{d}x^a)_p\}$ instead of $\{\left(\frac{\partial}{\partial x^a}\right)_p\}$.

15. (r, s) **tensor bundle:** Similarly, (r, s) tensor bundle of M is defined as

$$T_s^rM := \coprod_{p \in M} (T_s^r)_pM.$$

12 Tensor Field and Tensor Space Theory II: Over A Ring

A summary:

- Following from the bundles from last Section 11, we construct **vector, covector, and tensor fields** using **smooth sections**.
- The **global** push-forward for **vector fields** needs additional condition **diffeomorphism**. However **global** pull-back for **covector fields** is still always possible.
- We show vector fields have **module properties** with the ring $\mathcal{C}^\infty(M)$ and then formally introduce **rings and modules over a ring**.
- We show **bases for modules** and when it is possible. **Module R-linear map** and **module isomorphism** are also defined.
- From modules, we show applications like constructing **covector fields** and **tensor fields** using $\mathcal{C}^\infty(M)$ multilinear map.

1. **vector field (equivalent definition I):** Let M be a smooth manifold, and let $TM \xrightarrow{\pi} M$ be its tangent bundle. A **vector field** on M is a smooth[16] **section** of the tangent bundle, i.e. a smooth map $\sigma:\ M \to TM$ such that $\pi \circ \sigma = \mathrm{id}_M$.

$$\begin{tikzcd} TM \arrow[d, shift left, "\pi"] \\ M \arrow[u, shift left, "\sigma"] \end{tikzcd}$$

- **notation:** We denote the **set of all vector fields on** M by $\Gamma(TM)$, i.e.

$$\Gamma(TM) := \{\sigma:\ M \to TM \mid \sigma \text{ is smooth and } \pi \circ \sigma = \mathrm{id}_M\}.$$

- **vector field equivalent definition II:** A vector field σ on M is a **derivation** on the algebra $\mathcal{C}^\infty(M)$, i.e. an **$\mathbb{R}$-linear map**

$$\sigma:\ \mathcal{C}^\infty(M) \xrightarrow{\sim} \mathcal{C}^\infty(M)$$

satisfying the **Leibniz rule** (w.r.t. dot multiplication (i.e. pointwise multiplication) on algebra $\mathcal{C}^\infty(M)$)

$$\sigma(fg) = g\,\sigma(f) + f\,\sigma(g).$$

  - ∗ This definition is similar to **tangent space equivalent definition III** in Section 10, where we use bimodule there. Here, we define the derivation globally from $\mathcal{C}^\infty(M) \xrightarrow{\sim} \mathcal{C}^\infty(M)$.
  - ∗ Later we will see we can also view vector field as $\mathcal{C}^\infty(M)$-linear map from $\Gamma(T^*M)$ to $\mathcal{C}^\infty(M)$. Don't be confused. Even we can map function f to covector field $\mathrm{d}f$, $\mathcal{C}^\infty(M)$-linear map means $g(p)\mathrm{d}_p f$ **not** $g(p)f(p)$ pointwisely.

- Example: **vector fields locally from chart:** Let (U, x) be a chart on M. For each $1 \leq a \leq \dim M$, the map

$$\begin{aligned} \frac{\partial}{\partial x^a}:\ U &\to TU \\ p &\mapsto \left(\frac{\partial}{\partial x^a}\right)_p \end{aligned}$$

is a vector field on the **submanifold** U. We can also think of this as a **derivation**, i.e., **$\mathbb{R}$-linear map**, over the algebra $\mathcal{C}^\infty(U)$:

$$\begin{aligned} \frac{\partial}{\partial x^a}:\ \mathcal{C}^\infty(U) &\xrightarrow{\sim} \mathcal{C}^\infty(U) \\ f &\mapsto \frac{\partial}{\partial x^a}(f) = \partial_a(f \circ x^{-1}). \end{aligned}$$

By abuse of notation, one usually denotes the right hand side as $\partial_a f$.

[16] In this note, we only consider smooth sections as vector fields.

2. **covector field (equivalent definition I):** Analogously, we could define the **covector fields** as smooth sections on T^*M.

$$\begin{tikzcd} T^*M \arrow[d, shift left, "\pi"] \\ M \arrow[u, shift left, "\omega"] \end{tikzcd}$$

   - **notation:** We denote the **set of all covector fields on** M by $\Gamma(TM)$, i.e.

$$\Gamma(T^*M) := \{\omega : \; M \to T^*M \mid \omega \text{ is smooth and } \pi \circ \omega = \mathrm{id}_M\}.$$

   - Example I: **vector fields locally from chart:** Let (U, x) be a chart on M. For each $1 \leq a \leq \dim M$, the map, the $\mathrm{d}x^{a_i}$ appearing above are the covector fields (i.e. called differentiable 1-forms in Section 13) on the **submanifold** U

$$\begin{aligned} \mathrm{d}x^a : \; U &\to T^*U \\ p &\mapsto \mathrm{d}_p x^a \end{aligned}$$

   - Example II: **vector fields from smooth function:** More generally, we have that for $f \in \mathcal{C}^\infty(M)$, $\mathrm{d}f$ is a covector field.

3. **tensor field (equivalent definition I):** Analogously, we define the (r, s) **tensor fields** as smooth sections on (r, s) **tensor bundle** $T^r_s M$. We denote the set of all (r, s) tensor fields on M by $\Gamma(T^r_s M)$.

4. **push-forward pointwisely:** Let $\phi : \; M \to N$ be smooth. The **push-forward** ϕ_* is defined as

$$\begin{aligned} \phi_* : \; TM &\to TN \\ X &\mapsto (\phi_*)_{\pi(X)}(X). \end{aligned}$$

   - Recall that, given a smooth map $\phi : \; M \to N$, the push-forward $(\phi_*)_p$ is a linear map that takes in a tangent vector in T_pM and outputs a tangent vector in $T_{\phi(p)}N$.
   - Any vector $X \in TM$ must belong to T_pM for some $p = \pi(X) \in M$. The map ϕ_* takes one vector $X \in TM$ and applies the push-forward at the "right" point, producing a vector in TN.
   - **compositions of push-forward:** We have $(g \circ f)_* = g_* \circ f_*$.
   Roughly speaking, we need the following **chain rule** to prove it. More strictly, we need a local representation. But here I just show the R^n case with d being the traditional derivative.

$$(g \circ f)_*(X|_a) = (d(g \circ f)(a)(X|_a))_{(g \circ f)(a)} = \underbrace{(dg(f(a))_{g(f(a))})}_{g_*}\underbrace{(df(a))_{f(a)})}_{f_*}(X|_a)$$

5. **pull-back pointwisely:** One can similarly define **pull-back** ϕ^* where **at each point** $p \in M$, where $(\phi^*)_p$ is defined as in Section 11.
   - Better **not** to think ϕ^* as a map from $T^*N \to T^*M$. It is not a map pointwise at each point in N. We first select one point $p \in M$, and look at covector space $T^*_{\phi(p)}N$ at $\phi(p) \in N$. A covector space T^*_pM at $p \in M$ is then constructed.
   - **compositions of pull-back:** We have $(g \circ f)^* = g^* \circ f^*$.

6. **global push-forward for vector fields:** We now construct a map $\Phi_* : \; \Gamma(TM) \to \Gamma(TN)$ that allows us to push vector fields on M forward to vector fields on N. Assume $\phi : \; M \to N$ is a **diffeomorphism**.

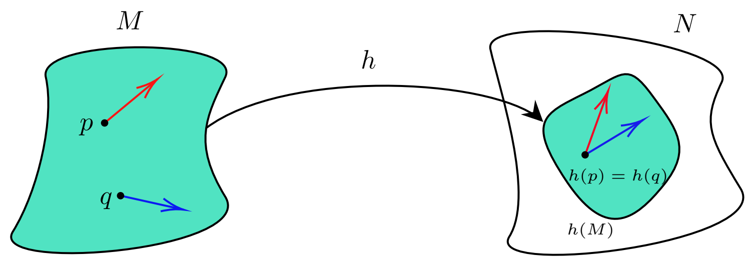
$$\begin{tikzcd} TM \arrow[r, "\phi_*"] & TN \\ M \arrow[u, "\sigma"] \arrow[r, "\phi"] & N \arrow[u, "\Phi_*(\sigma)"'] \end{tikzcd}$$

If $\sigma \in \Gamma(TM)$, we can define the **push-forward** $\Phi_*(\sigma) \in \Gamma(TN)$ as

$$\Phi_*(\sigma) := \phi_* \circ \sigma \circ \phi^{-1}.$$

- Do not be confused here. We still push-forward tangent space at p on manifold M to tangent space at $\phi(p)$ on manifold N. The above diagram is just for beauty.
- Why we need **diffeomorphism**? See the figure.
  - ∗ The map ϕ may fail to be injective.
  - ∗ The map ϕ may fail to be surjective.

  Both issues will make the push-forwarded vector field on N not well-defined.



7. **global pull-back for covector fields:** Let $\phi: M \to N$ be smooth and let $\omega \in \Gamma(T^*N)$. We define the **pull-back** $\Phi^*(\omega) \in \Gamma(T^*M)$ of ω as

$$\begin{aligned} \Phi^*(\omega): \ & M \to T^*M \\ & p \mapsto \Phi^*(\omega)(p), \end{aligned}$$

where

$$\begin{aligned} \Phi^*(\omega)(p): \ & T_pM \xrightarrow{\sim} \mathbb{R} \\ & X \mapsto \Phi^*(\omega)(p)(X) := \omega(\phi(p))(\phi_*(X)), \end{aligned}$$

as in the following diagram

$$\begin{tikzcd} T^*M & T^*N \arrow[l, "\phi^*"'] \\ M \arrow[u, "\Phi^*(\omega)"] \arrow[r, "\phi"] & N \arrow[u, "\omega"'] \end{tikzcd}$$

- Covector fields can be pull-back globally **without diffeomorphism** assumption.
- **pull-back of tensor fields:**
  - ∗ If $\phi: M \to N$ is smooth, we can define the **pull-back of contravariant field**, i.e. $(0, q)$) tensor fields, **without diffeomorphism** assumption.
  - ∗ If $\phi: M \to N$ is a **diffeomorphism**, then we can define the **pull-back of any smooth (p, q) tensor field** $\tau \in \Gamma(T^r_sN)$ as

    $$\begin{aligned} &\Phi^*(\tau)(p)(\omega_1, \ldots, \omega_r, X_1, \ldots, X_s) \\ &\qquad := \tau(\phi(p))\big((\phi^{-1})^*(\omega_1), \ldots, (\phi^{-1})^*(\omega_r), \phi_*(X_1), \ldots, \phi_*(X_s)\big), \end{aligned}$$

    with $\omega_i \in T^*_pM$ and $X_i \in T_pM$.

8. **preliminary $\mathcal{C}^\infty(M)$-module property for $\Gamma(TM)$:** We can equip the set $\Gamma(TM)$ with the following operations.
   - **pointwise addition:**

$$\begin{aligned} \oplus: \ & \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM) \\ & (\sigma, \tau) \mapsto \sigma \oplus \tau, \end{aligned}$$

   where

$$\begin{aligned} \sigma \oplus \tau: \ & M \to \Gamma(TM) \\ & p \mapsto (\sigma \oplus \tau)(p) := \sigma(p) + \tau(p). \end{aligned}$$

   Note that the $+$ on the right hand side above is the addition in T_pM.

- **pointwise multiplication operation:**

$$\begin{aligned}\odot:\ \mathcal{C}^\infty(M)\times\Gamma(TM) &\to \Gamma(TM)\\ (f,\sigma) &\mapsto f\odot\sigma,\end{aligned}$$

  where

$$\begin{aligned}f\odot\sigma:\ M &\to \Gamma(TM)\\ p &\mapsto (f\odot\sigma)(p) := f(p)\sigma(p).\end{aligned}$$

- So $\Gamma(TM)$ is similar to a vector space over a ring $\mathcal{C}^\infty(M)$, we call it a $\mathcal{C}^\infty(M)$**-module**.
- $\Gamma(TM)$ can also be viewed as a $\mathbb{R}$-vector space with a *global* scaling. However it is not that interesting because if we take this view a basis for this vector space is necessarily **uncountably infinite**.
- $\mathcal{C}^\infty(M)$-module property for $\Gamma(T^*M)$ follows analogously.

9. **ring:** See my algebra notes for the definition and details. In this course, the ring defined in that note is called a unital ring and we only care about unital ring in this course.
   - Example: $(\mathcal{C}^\infty(M),+,\bullet)$, where $\bullet$ is pointwise multiplication of maps, is a commutative, unital ring, but not a division ring and hence, not a field.
10. **general R-module:** Let $(R,+,\cdot)$ be a unital ring. A triple $(M,\oplus,\odot)$ is called an R**-module** if the maps

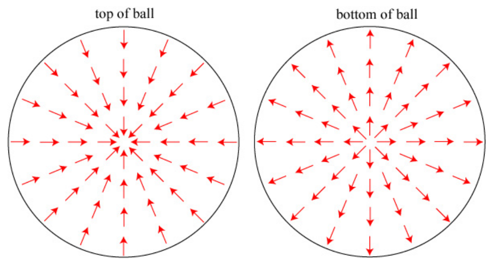
$$\begin{aligned}\oplus:\ M\times M &\to M\\ \odot:\ R\times M &\to M\end{aligned}$$

    satisfy the vector space axioms, i.e. $(M,\oplus)$ is an **abelian group** and for all $r,s\in R$ and all $m,n\in M$, we have
    (a) $r\odot(m\oplus n) = (r\odot m)\oplus(r\odot n)$;
    (b) $(r+s)\odot m = (r\odot m)\oplus(s\odot m)$;
    (c) $(r\cdot s)\odot m = r\odot(s\odot m)$;
    (d) $1\odot m = m$.
    - Most definitions we had for vector spaces carry over unaltered to modules, including that of a basis, i.e. a linearly independent spanning set.
    - Above is a **left** R-module. The definition of a **right** R-module is completely analogous with multiplication on the right. Moreover, if R and S are two unital rings, then we can define M to be an R-S**-bimodule** if it is a left R-module and a right S-module.
      * Note, R-R**-bimodule** is called the R**-bimodule** and we have used it in Section 10 for derivation.
      * If R is **commutative**, then left R-modules are often define to be the same as right R-modules and are simply called R-modules.
    - Examples:
      † Any ring R is trivially a module over itself. For example, let $R=\mathbb{Z}/6\mathbb{Z}$. R is a module over itself.
      † The triple $(\Gamma(TM),\oplus,\odot)$ is a $\mathcal{C}^\infty(M)$-module.
    - **notation:** We will usually denote $\oplus$ by $+$ and suppress the $\odot$, as we did with vector spaces.
11. **bases for modules:** Unlike a vector space, an R-module need not have a basis. However if D is a division ring, then any D-module V admits a basis.
    - The proof needs the axiom of choice, more specifically, the Zorn's lemma. See [3] for more details. Roughly speaking, axiom of choice is quite useful in topology like product compactness proof and hilbert space basis proof.
    - Example I: Let $M=\mathbb{R}^2$ and consider $v\in\Gamma(T\mathbb{R}^2)$. It is a fact from standard vector analysis that any such v can be written uniquely as

$$v = v^1e_1 + v^2e_2$$

for some $v^1, v^2 \in \mathcal{C}^\infty(\mathbb{R}^2)$ and $e_1, e_2 \in \Gamma(T\mathbb{R}^2)$. Hence, even though $\Gamma(T\mathbb{R}^2)$ is a $\mathcal{C}^\infty(\mathbb{R}^2)$-module and $\mathcal{C}^\infty(\mathbb{R}^2)$ is **not** a division ring, **it still has a basis**. Note that the coefficients in the linear expansion of v are functions. This example shows that the converse to the above theorem is not true: if D is not a division ring, then a D-module may or may not have a basis.

- **hairy ball theorem:** Let $M = S^2$. There is **no non-vanishing** smooth tangent vector field on **even**-dimensional n-spheres.



Hence, we can multiply any smooth vector field $v \in \Gamma(TS^2)$ by a function $f \in \mathcal{C}^\infty(S^2)$ which is zero everywhere except where v is, obtaining $fv = 0$ despite $f \neq 0$ and $v \neq 0$. Therefore, **there is no set of linearly independent vector fields on** S^2, much less a basis.

- Locally, for vector, covector and general tensor fields on a manifold, we always have a basis using local chart. See above **(co-)vector fields locally from chart** and **local representation of n-form** in next Section 13. This is because locally it behaves like $\mathbb{R}^d$ as the above Example I.

- **frame of tangent bundle:** We now give an example of the bases, called **frame**, for $\mathcal{C}^\infty(M)$-module property for $\Gamma(TM)$. An ordered k tuple $(X_1, \ldots, X_k)$ of vector fields defined on some subset $A \subseteq M$
  - ∗ is **linearly independent** if $\left(X_1|_p, \ldots, X_k|_p\right)$ is a **linearly independent** k-tuple in T_pM for each $p \in A$,
  - ∗ **spans the tangent bundle** if the k-tuple $\left(X_1|_p, \ldots, X_k|_p\right)$ spans T_pM at each $p \in A$.

  (a) **local frame:** A **local frame** for M is an ordered n-tuple of vector fields $(E_1, \ldots, E_n)$ defined on an open subset $U \subseteq M$ that is **linearly independent and spans the tangent bundle**. Thus the vectors $\left(E_1|_p, \ldots, E_n|_p\right)$ form a basis for T_pM at each $p \in U$.

  (b) **global frame and parallelizable:** It is called a **global frame** if $U = M$. Global frames do not generally exist. If a manifold admits a **global frame**, we call it **parallelizable**.

  - ◦ **notation:** We often use the shorthand notation (E_i) to denote a frame $(E_1, \ldots, E_n)$. So if M has dimension d, then to check that an ordered n-tuple of vector fields $(E_1, \ldots, E_n)$ is a local frame, it suffices to check either that it is linearly independent or that it spans the tangent bundle.
  - ◦ **coordinate frame:** Recall the **vector fields locally from chart** U defined in the beginning of this section.

$$\left\{\left(\frac{\partial}{\partial x^1}\right), \ldots, \left(\frac{\partial}{\partial x^d}\right)\right\}$$

    constitutes a basis for the tangent space T_pM at each point p, and it's known as the **coordinate frame basis**.
  - ◦ **equivalent condition of parallelizable:** "tangent bundle over a manifold is trivial" $\Leftrightarrow$ "the manifold is paralelizable". Because if it's parallelizable, you can find a global frame field of the manifold. This frame field induces a bundle isomorphism to $M \times \mathbb{R}^n$. Conversely, if the tangent bundle is trivial, then a trivialization is a global frame field.

12. **direct sum of module:** The **direct sum** of two R-modules M and N is the R-module $M \oplus N$, which has $M \times N$ as its underlying set and operations (inherited from M and N) defined componentwise.
    - **notation:** While we have been using $\oplus$ to temporarily distinguish two "plus-like" operations in different spaces, the symbol $\oplus$ is the standard notation for the direct sum.
13. **module terminology:** An R-module M is said to be
    - **finitely generated** if it has a finite generating set;
    - **free** is it has a basis;
    - **projective** if it is a direct **summand** of a free R-module F, i.e.

    $$M \oplus Q = F$$

    for some R-module Q.
    - If a **finitely generated** module R-module F is **free**, and $d \in \mathbb{N}$ is the cardinality of a finite basis, then

    $$F \cong_{\text{mod}} = \underbrace{R \oplus \cdots \oplus R}_{d \text{ copies}} =: R^d.$$

    One can show that if $R^d \cong_{\text{mod}} R^{d'}$, then $d = d'$ and hence, the concept of dimension is well-defined for finitely generated, free modules.
    - Examples:
      † $R := \mathbb{Z}/6\mathbb{Z}$ a **free** module over itself. Because $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \cong R$, we have that $\mathbb{Z}/2\mathbb{Z}$, considered as an R-module, is **projective**, but it is **not free** since any non-trivial direct sum of R (must take R as a direct sum summand, otherwise the coefficient is not unique) would have at least 6 elements.
      † The Baer-Specker group $\mathbb{Z}^{\mathbb{N}}$, which is **not finitely generated**, is an example of a **torsion-free** $\mathbb{Z}$-module which is **not free**.
      † $\Gamma(T\mathbb{R}^2)$ is free while $\Gamma(TS^2)$ is not.
      † Every free module is also projective.
14. **R-linear map** Let M and N be two (left) R-modules. A map $f: M \to N$ is said to be an **R-linear map**, or an **R-module homomorphism**, if

    $$\forall\, r \in R : \forall\, m_1, m_2 \in M : \; f(rm_1 + m_2) = rf(m_1) + f(m_2),$$

    where it should be clear which operations are in M and which in N.
    - **module isomorphism**: A **bijective module homomorphism**[17] is said to be a **module isomorphism**, and we write $M \cong_{\text{mod}} N$ if there exists a module isomorphism between them.
    - **right R-modules homomorphism:** If M and N are right R-modules, then the linearity condition is written as

    $$\forall\, r \in R : \forall\, m_1, m_2 \in M : \; f(m_1 r + m_2) = f(m_1)r + f(m_2).$$

15. **vector fibre bundle:** A **vector fibre bundle** is a fibre bundle in which the fibre is a **vector space**.
    - Example: tangent bundle to a manifold.
    - **Serre-Swan Theorem:** Let E be a vector fibre bundle over a smooth manifold M. Then, the set $\Gamma(E)$ of all smooth section of E over M is a **finitely generated, projective** $\mathcal{C}^\infty(M)$-module.
      * An immediate consequence of the theorem is that, for any vector fibre bundle E over M, there exists a $\mathcal{C}^\infty(M)$-module Q such that the direct sum $\Gamma(E) \oplus Q$ is free. If Q can be chosen to be the trivial module $\{0\}$, then $\Gamma(E)$ is itself free, as it is the case with $\Gamma(T\mathbb{R}^2)$. In a sense, the module Q **quantifies the failure** of $\Gamma(E)$ to have a basis.

[17] Recall that the inverse of a bijective linear map is automatically linear.

16. **hom-set:** Let P, Q be finitely generated (projective) modules over a commutative ring R. Then
$$\mathrm{Hom}_R(P, Q) := \{\phi : P \xrightarrow{\sim} Q \mid \phi \text{ is } R\text{-linear}\}$$
is again **a finitely generated (projective)** R**-module**, with operations defined pointwisely.
    - **covector field equivalent definition II:** We have the **dual of a module = covector field**, where the dual of a module is defined as
    $$\mathrm{Hom}_{\mathcal{C}^\infty(M)}(\Gamma(TM), \mathcal{C}^\infty(M)) =: \Gamma(TM)^*.$$
    One can show that $\Gamma(TM)^*$ coincides with $\Gamma(T^*M)$, i.e. the set of covector fields.
17. **tensor field equivalent definition II:** Let M be a smooth manifold. A smooth (r, s) **tensor field** τ on M is a $\mathcal{C}^\infty(M)$**-multilinear map**
$$\tau : \underbrace{\Gamma(T^*M) \times \cdots \times \Gamma(T^*M)}_{r \text{ copies}} \times \underbrace{\Gamma(TM) \times \cdots \times \Gamma(TM)}_{s \text{ copies}} \to \mathcal{C}^\infty(M).$$
    - The equivalence of this to the bundle definition is due to the **pointwise nature of tensors**. For instance, a covector field $\omega \in \Gamma(T^*M)$ can act on a vector field $X \in \Gamma(TM)$ to yield a smooth function $\omega(X) \in \mathcal{C}^\infty(M)$ by
    $$(\omega(X))(p) := \omega(p)(X(p)).$$
    Then, we see that for any $f \in \mathcal{C}^\infty(M)$, we have
    $$(\omega(fX))(p) = \omega(p)(f(p)X(p)) = f(p)\omega(p)(X(p)) =: (f\omega(X))(p)$$
    and hence, the map $\omega : \Gamma(TM) \xrightarrow{\sim} \mathcal{C}^\infty(M)$ is $\mathcal{C}^\infty(M)$-linear.
        * $fw(X)$ is inside the module $\Gamma(T^*M)$ is because of the above module property. Here it shows the $\mathcal{C}^\infty$-linear of the input variable $X \in \Gamma(TM)$. So we have $\Gamma(T^*M) \subseteq$ "the set of all the $\mathcal{C}^\infty$ maps from $\Gamma(TM) \to \mathcal{C}^\infty(M)$".
        * But note, one $\mathcal{C}^\infty$-linear map $\Gamma(TM) \to \mathcal{C}^\infty(M)$ is mentioned in "dual of a module". So it is in $\Gamma(TM)^*$ (which can be shown then to be $\Gamma(T^*M)$)
    - Similarly, the set $\Gamma(T^r_s M)$ of all (r, s) smooth tensor fields on M can be made into a $\mathcal{C}^\infty(M)$-module, with module operations defined **pointwise**.
    - Therefore, we can think of tensor fields on M either as sections of some tensor bundle on M or as a $\mathcal{C}^\infty(M)$-multilinear map as above.
18. **tensor product:** We define the tensor product of tensor fields as:
$$\begin{aligned} \otimes : \Gamma(T^p_q M) \times \Gamma(T^r_s M) &\to \Gamma(T^{p+r}_{q+s} M) \\ (\tau, \sigma) &\mapsto \tau \otimes \sigma \end{aligned}$$
analogously to what we had with tensors on a vector space, i.e.
$$\begin{aligned} &(\tau \otimes \sigma)(\omega_1, \ldots, \omega_p, \omega_{p+1}, \ldots, \omega_{p+r}, X_1, \ldots, X_q, X_{q+1}, \ldots, X_{q+s}) \\ &\qquad := \tau(\omega_1, \ldots, \omega_p, X_1, \ldots, X_q)\, \sigma(\omega_{p+1}, \ldots, \omega_{p+r}, X_{q+1}, \ldots, X_{q+s}), \end{aligned}$$
with $\omega_i \in \Gamma(T^*M)$ and $X_i \in \Gamma(TM)$.

13 Grassmann Algebra and De Rham Cohomology

In Section 9, we define n**-form** (a special $(0, n)$-tensor) and volume (top) form. Using the field concepts from last Section 12, we now state the **differential** n**-form**, which is a $(0, n)$**-tensor field**. All the concepts like Grassmann algebra, exterior derivative and de Rham cohomology are based on the differential forms.

A summary:

- ▶ We define **differential** n**-form**, and recall the pull-back again, which is always well-defined according to previous sections.
- ▶ We define the **Grassmann algebra** for a manifold with the algebra dot product being the **wedge (or called exterior) product**.
- ▶ The **exterior derivative** is then defined, and some properties are shown.

▶ We finally define **de Rham cohomology** using the concepts **closed** and **exact** from exterior derivative.

1. Let M be a smooth manifold. A **differential n-form** on M is a $(0,n)$ smooth **tensor field** ω which is **totally antisymmetric**, i.e.
$$\omega(X_1,\ldots,X_n)=\operatorname{sgn}(\pi)\,\omega(X_{\pi(1)},\ldots,X_{\pi(n)}),$$
for any $\pi\in S_n$, with $X_i\in\Gamma(TM)$.
   - Sometimes, we simply called **differential n-form** as **n-form**. But please don't mix up with the tensor n-form. Here **n-form** indicates a tensor field.
   - **orientable:** A manifold M is said to be **orientable** if it admits an oriented atlas, i.e. an atlas in which all chart transition maps, which are maps between open subsets of $\mathbb{R}^{\dim M}$, have a **positive determinant**.
     * If M is orientable, then there exists a nowhere vanishing top form ($n=\dim M$) on M providing the volume.
   - **degree:** If ω is an n-form, then n is said to be the **degree** of ω.
   - **notation:** We denote by $\Omega^n(M)$ the set of all differential n-forms on M, which then becomes a $\mathcal{C}^\infty(M)$-module by defining the addition and multiplication operations pointwise on M.
     * So we have $\Omega^0(M)\equiv\mathcal{C}^\infty(M)$ and $\Omega^1(M)\equiv\Gamma(T^0_1M)\equiv\Gamma(T^*M)$.
   - Similar to the **tensor n-form**, we have $\Omega^n(M)=\{0\}$ for $n>\dim M$.
2. **(global) pull-back of differential forms:** Note, we have defined the general pull-back of **contravariant field** in the last Section 12 which is always well-defined. Here we just specialise the pull-back to differential forms since differential n-form is special **contravariant n-tensor field**. Let $\phi:\ M\to N$ be a smooth map and let $\omega\in\Omega^n(N)$. Then we define the **pull-back** $\Phi^*(\omega)\in\Omega^n(M)$ of ω as
$$\begin{aligned}\Phi^*(\omega):\ &M\to T^*M\\ &p\mapsto\Phi^*(\omega)(p),\end{aligned}$$
where
$$\Phi^*(\omega)(p)(X_1,\ldots,X_n):=\omega(\phi(p))\big(\phi_*(X_1),\ldots,\phi_*(X_n)\big),$$
for $X_i\in T_pM$.
   - The map Φ^* on $\Omega^0(M)$ is simply
$$\begin{aligned}\Phi^*:\ &\Omega^0(M)\to\Omega^0(M)\\ &f\mapsto\Phi^*(f):=f\circ\phi.\end{aligned}$$
3. **wedge (exterior) product** Let M be a smooth manifold. We define the **wedge (exterior) product** of forms as the map
$$\begin{aligned}\wedge:\ &\Omega^n(M)\times\Omega^m(M)\to\Omega^{n+m}(M)\\ &(\omega,\sigma)\mapsto\omega\wedge\sigma,\end{aligned}$$
where
$$(\omega\wedge\sigma)(X_1,\ldots,X_{n+m}):=\frac{1}{n!\,m!}\sum_{\pi\in S_{n+m}}\operatorname{sgn}(\pi)(\omega\otimes\sigma)(X_{\pi(1)},\ldots,X_{\pi(n+m)})$$
and $X_1,\ldots,X_{n+m}\in\Gamma(TM)$.
   - **notation:** By convention, for any $f,g\in\Omega^0(M)$ and $\omega\in\Omega^n(M)$, we set
$$f\wedge g:=fg\qquad\text{and}\qquad f\wedge\omega=\omega\wedge f=f\omega.$$
   - **Why not just using the tensor product?** Because the tensor product of two forms is not necessarily still a form, i.e. **satisfying totally antisymmetric**.
   - **The wedge product is $\mathcal{C}^\infty(M)$-bilinear:**
$$(f\omega_1+\omega_2)\wedge\sigma=f\,\omega_1\wedge\sigma+\omega_2\wedge\sigma,$$
for all $f\in\mathcal{C}^\infty(M)$, $\omega_1,\omega_2\in\Omega^n(M)$ and $\sigma\in\Omega^m(M)$, and similarly for the second argument.

- * This will be used later to define the **Grassmann algebra dot product.**
- **graded commutative of $\wedge$ over single n-form:** Let $\omega \in \Omega^n(M)$ and $\sigma \in \Omega^m(M)$. Then

$$\omega \wedge \sigma = (-1)^{nm} \sigma \wedge \omega.$$

  We say that $\wedge$ is **graded commutative**, that is, it satisfies a version of anticommutativity which depends on the degrees of the forms.
- Example: Suppose that $\omega, \sigma \in \Omega^1(M)$. Then, for any $X, Y \in \Gamma(TM)$

$$\begin{aligned}(\omega \wedge \sigma)(X,Y) &= (\omega \otimes \sigma)(X,Y) - (\omega \otimes \sigma)(Y,X) \\ &= (\omega \otimes \sigma)(X,Y) - \omega(Y)\sigma(X) \\ &= (\omega \otimes \sigma)(X,Y) - (\sigma \otimes \omega)(X,Y) \\ &= (\omega \otimes \sigma - \sigma \otimes \omega)(X,Y).\end{aligned}$$

  Hence

$$\omega \wedge \sigma = \omega \otimes \sigma - \sigma \otimes \omega.$$

  Note is only true when ω and σ are pure degree forms, rather than linear combinations of forms of different degrees.

4. **local representation of n-form:** If (U, x) is a chart on M, then every n-form $\omega \in \Omega^n(U)$ can be expressed **locally** on U as

$$\omega = \omega_{a_1 \cdots a_n} \mathrm{d}x^{a_1} \wedge \cdots \wedge \mathrm{d}x^{a_n},$$

where $\omega_{a_1 \cdots a_n} \in \mathcal{C}^\infty(U)$, $1 \le a_1 < \cdots < a_n \le \dim M$ are increasing sequence and $\mathrm{d}x^{a_1} \wedge \cdots \wedge \mathrm{d}x^{a_n}$ has been defined as above.

- The $\mathrm{d}x^{a_i}$ appearing above are the covector fields (1-forms) appeared in last Section 12:

$$\mathrm{d}x^{a_i} : p \mapsto \mathrm{d}_p x^{a_i}.$$

- **wedge product in local representation:** If $\omega = \omega_{a_1 \cdots a_n} \mathrm{d}x^{a_1} \wedge \cdots \wedge \mathrm{d}x^{a_n}$, and $\lambda = \lambda_{b_1 \cdots a_m} \mathrm{d}x^{b_1} \wedge \cdots \wedge \mathrm{d}x^{b_m}$, we have

$$\omega \wedge \lambda = \omega_{a_1 \cdots a_n} \lambda_{b_1 \cdots a_m} (\mathrm{d}x^{a_1} \wedge \cdots \wedge \mathrm{d}x^{a_n}) \wedge (\mathrm{d}x^{b_1} \wedge \cdots \wedge \mathrm{d}x^{b_m}).$$

- **pull-back in local representation:** Assume that $x^1, \ldots, x^{\dim M}$ are coordinates on M, that $y^1, \ldots, y^{\dim N}$ are coordinates on N, and that these coordinate systems are related by the formulas $y^i = \phi_i\left(x^1, \ldots, x^{\dim M}\right)$ for all i. Locally on N, ω can be written as

$$\omega = \omega_{i_1 \cdots i_k} \mathrm{d}y^{i_1} \wedge \cdots \wedge \mathrm{d}y^{i_k},$$

  where, $i_1 < \cdots < i_k$ and for each choice of $i_1, \ldots, i_k$, $\omega_{i_1 \cdots i_k}$ is a real-valued function of $y^1, \ldots, y^{\dim N}$. Using the **linearity** of pullback and its **compatibility with exterior product** (See **a summary of pull-back**), the pullback of ω has the formula

$$\Phi^*\omega = \sum_{i_1 < \cdots < i_k} (\omega_{i_1 \cdots i_k} \circ \phi) \, \mathrm{d}\phi_{i_1} \wedge \cdots \wedge \mathrm{d}\phi_{i_k}.$$

  Each exterior derivative $\mathrm{d}\phi_i$ can be expanded in terms of $\mathrm{d}x^1, \ldots, \mathrm{d}x^{\dim M}$. The resulting k-form can be written using **Jacobian matrices**:

$$\Phi^*\omega = \sum_{i_1 < \cdots < i_k} \sum_{j_1 < \cdots < j_k} (\omega_{i_1 \cdots i_k} \circ \phi) \frac{\partial\left(\phi_{i_1}, \ldots, \phi_{i_k}\right)}{\partial\left(x^{j_1}, \ldots, x^{j_k}\right)} \mathrm{d}x^{j_1} \wedge \cdots \wedge \mathrm{d}x^{j_k}.$$

  Here, $\frac{\partial\left(\phi_{i_1}, \ldots, \phi_{i_k}\right)}{\partial\left(x^{j_1}, \ldots, x^{j_k}\right)}$ denotes the **determinant** of the Jacobian matrix. So rudin [4][Definition 10.11] can be viewed from the manifold perspective! In rudin, we take map from $\mathbb{R}^m$ to a surface in $\mathbb{R}^n$ with $m \le n$.
- Compare with ruding:
  - * In rudin, we directly define forms using local representation. The real meaning of the forms are reflected in integration [4][section 10, eq. 35]. The **totally antisymmetric** comes from the **determinant** of the Jacobian matrix.

* In this course, the differential form are defined using tensor fields with **totally antisymmetric** requirement. The integrand after pull-back will then have the **determinant**.

So **Jacobian determinant = totally antisymmetric**

5. **Grassmann algebra:** We first introduce a space which is closed under the action of wedge product $\wedge$: Let M be a smooth manifold. Define the $\mathcal{C}^\infty(M)$-module
$$\mathrm{Gr}(M) \equiv \Omega(M) := \bigoplus_{n=0}^{\dim M} \Omega^n(M).$$
The **Grassmann algebra** on M is the algebra $(\Omega(M), +, \cdot, \wedge)$, where
$$\wedge : \ \Omega(M) \times \Omega(M) \to \Omega(M)$$
is the extension of the previously defined $\wedge : \ \Omega^n(M) \times \Omega^m(M) \to \Omega^{n+m}(M)$.
   - Direct sum of modules has the Cartesian product of the modules as underlying set and module operations defined **componentwise**.
   - "Algebra" here we really mean **"algebra over a module"**.
   - Example: Let $\psi = \omega + \sigma$, where $\omega \in \Omega^1(M)$ and $\sigma \in \Omega^3(M)$. Of course, this "+" is neither the addition on $\Omega^1(M)$ nor the one on $\Omega^3(M)$, but rather that on $\Omega(M)$.
6. **commutator (Lie bracket):** Let M be a smooth manifold and let $X, Y \in \Gamma(TM)$. The **commutator (or Lie bracket)** of X and Y is defined as
$$\begin{aligned}[X, Y] : \ \mathcal{C}^\infty(M) &\xrightarrow{\sim} \mathcal{C}^\infty(M) \\ f &\mapsto [X, Y](f) := X(Y(f)) - Y(X(f)),\end{aligned}$$
where we are using the definition of vector fields as $\mathbb{R}$-linear maps $\mathcal{C}^\infty(M) \xrightarrow{\sim} \mathcal{C}^\infty(M)$.
   - This commutator is used to defined the **exterior derivative** below.
   - **Lie algebra from commutator:** We use this commutator to define a **Lie algebra** $(\Gamma(TM), +, \cdot, [-, -])$ over $\mathbb{R}$. According to the definition of **Lie algebra** in Section 10, we need to check $\mathbb{R}$-bilinear (as a algebra product), alternativity and the Jacobi identity (additional condition of the $[-, -]$ operator).
   - The brackets appear everywhere when you want to obtain natura tensors on your manifold. Maybe the simplest example of this is the torsion operator of a connection. $T(X, Y) = \nabla_X Y - \nabla_Y X$ is **not tensorial** in X and Y. Indeed, $T(fX, Y) = f(\nabla_X Y - \nabla_Y X) - (Y.f)X$. The last term involves the derivative of f and you want to get rid of it. Now observe that
$$[fX, Y] = f[X, Y] - (Y(f)) \cdot X$$
so the same term appears in the non-tensoriality of the bracket. This leads to the good definition of the torsion: $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$ and this is **tensorial** in X and Y.
   - Due to Schwarz's theorem on symmetric second derivatives, we have
$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0.$$
This is very useful and can be used to get **local representations**.
7. **exterior derivative:** The **exterior derivative** on M is the $\mathbb{R}$-linear operator
$$\begin{aligned}\mathrm{d} : \ \Omega^n(M) &\xrightarrow{\sim} \Omega^{n+1}(M) \\ \omega &\mapsto \mathrm{d}\omega\end{aligned}$$
with $\mathrm{d}\omega$ being defined as
$$\begin{aligned}\mathrm{d}\omega(X_1, \ldots, X_{n+1}) := &\sum_{i=1}^{n+1} (-1)^{i+1} X_i\big(\omega(X_1, \ldots, \widehat{X_i}, \ldots, X_{n+1})\big) \\ &+ \sum_{i<j} (-1)^{i+j} \omega\big([X_i, X_j], X_1, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_{n+1}\big),\end{aligned}$$
where $X_i \in \Gamma(TM)$ and the hat denotes omissions.

- This map is antisymmetric: this follows quite immediately from the fact that ω itself is antisymmetric and that the bracket is ; 2) that this map is tensorial: this is really the crucial point. As we have mentioned above, Lie brackets appear here to ensure tensorial. Denote the right hand side as $\widetilde{d\omega}$. We have

$$\begin{aligned}
\widetilde{d\omega}\left(fX_1, X_2, \cdots, X_{k+1}\right) =& fX_1\left(\omega\left(X_2, \cdots, X_{k+1}\right)\right) \\
&+ \sum_{i>1}(-1)^{i-1}X_i\left(\omega\left(fX_1, \cdots, \widehat{X_i}, \cdots, X_{k+1}\right)\right) \\
&+ \sum_{i>1}(-1)^{i+1}\omega\left(\left[fX_1, X_i\right], X_2, \cdots, \widehat{X_i}, \cdots, X_{k+1}\right) \\
&+ \sum_{1<i<j}(-1)^{i+j}\omega\left(\left[X_i, X_j\right], fX_1, \cdots, \widehat{X_i}, \cdots, \widehat{X_j}, \cdots, X_{k+1}\right) \\
=& f\widetilde{d\omega}\left(X_1, \cdots, X_{k+1}\right) \\
&+ \sum_{i>1}(-1)^{i-1}\left(X_i f\right)\omega\left(X_1, \cdots, \widehat{X}_i, \cdots, X_{k+1}\right) \\
&- \sum_{i>1}(-1)^{i+1}\left(X_i f\right)\omega\left(X_1, \cdots, \widehat{X}_i, \cdots, X_{k+1}\right) \\
=& f\widetilde{d\omega}\left(X_1, \cdots, X_{k+1}\right).
\end{aligned}$$

- Don't be confused, $\mathbb{R}$-linear is w.r.t. d(input), while the $\mathcal{C}^\infty$-linear is w.r.t. $\mathrm{d}\omega$(input). d(input) is not $\mathcal{C}^\infty$-linear, instead it satisfies the following **Leibniz rule**.
- **compare to covariant derivative:** That the operator d is only well-defined when it acts on forms, i.e., **contravariant tensor field**. In order to define a derivative operator on **covariant and general tensor fields** we will need to add extra structure, called **connection**, to our differentiable manifold.
- **relation to differential operator d_p in Section 11:** We can extend that definition in Section 11 to define the following ($\mathbb{R}$-linear) operator:

$$\begin{aligned}
\mathrm{d}:\ \mathcal{C}^\infty(M) &\xrightarrow{\sim} \Gamma(T^*M) \\
f &\mapsto \mathrm{d}f
\end{aligned}$$

where, of course, $\mathrm{d}f:\ p \mapsto \mathrm{d}_p f$.

  * So, we can also understand this as an example of **exterior derivative** that takes in 0-forms and outputs 1-forms

  $$\mathrm{d}:\ \Omega^0(M) \xrightarrow{\sim} \Omega^1(M).$$

  Alternatively, we can view $\mathrm{d}f$ as the **$\mathbb{R}$-linear map**

  $$\begin{aligned}
  \mathrm{d}f:\ \Gamma(TM) &\xrightarrow{\sim} \mathcal{C}^\infty(M) \\
  X &\mapsto \mathrm{d}f(X) = X(f).
  \end{aligned}$$

  * **local representation:** As we mentioned above, locally on some chart (U, x) on M, the covector field (or 1-form) $\mathrm{d}f$ can be expressed as

  $$\mathrm{d}f = \lambda_a\,\mathrm{d}x^a$$

  for some smooth functions $\lambda_i \in \mathcal{C}^\infty(U)$. To determine what they are, we simply apply *both sides* to the vector fields induced by the chart. We have

  $$\mathrm{d}f\left(\frac{\partial}{\partial x^b}\right) = \frac{\partial}{\partial x^b}(f) = \partial_b f$$

  and

  $$\lambda_a\,\mathrm{d}x^a\left(\frac{\partial}{\partial x^b}\right) = \lambda_a\frac{\partial}{\partial x^b}(x^a) = \lambda_a\,\delta^a_b = \lambda_b.$$

  Hence, the local expression of $\mathrm{d}f$ on (U, x) is

  $$\mathrm{d}f = \partial_a f\,\mathrm{d}x^a.$$

  * **Leibniz rule:** Note that the operator d satisfies the Leibniz rule

  $$\mathrm{d}(fg) = g\,\mathrm{d}f + f\,\mathrm{d}g.$$

- **general local representation:** From linearity, so without loss of generality, we may first assume

$$\omega = f\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^k$$

in a local chart U. Note that $[\partial_i, \partial_j] = 0$. It follows that for any increasing indices $j_1 < \cdots < j_{k+1}$, the right hand side of

$$\widetilde{\mathrm{d}\omega}\left(\partial_{j_1}, \cdots, \partial_{j_{k+1}}\right) = \sum_i (-1)^{i-1}\partial_{j_i}\left(\omega\left(\partial_{j_1}, \cdots, \hat{\partial}_{j_i}, \cdots, \partial_{j_{k+1}}\right)\right)$$

vanishes except for the case $j_1 = 1, \cdots, j_k = k$ and $i = k+1$ (and thus $j_i \geq k+1$). In other words, the only non-zero terms in all possible expressions $\widetilde{\mathrm{d}\omega}\left(\partial_{j_1}, \cdots, \partial_{j_{k+1}}\right)$ are

$$\widetilde{\mathrm{d}\omega}\left(\partial_1, \cdots, \partial_k, \partial_r\right) = (-1)^k \partial_r(f).$$

It follows that

$$\widetilde{\mathrm{d}\omega} = \sum_{r>k} (-1)^k \partial_r(f) \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^k \wedge \mathrm{d}x^r = \sum \partial_r(f) \mathrm{d}x^r \wedge \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^k.$$

From linearity, now suppose ω is a k-form on M, so that locally

$$\omega = \omega_{a_1 \cdots a_n} \mathrm{d}x^{a_1} \wedge \cdots \wedge \mathrm{d}x^{a_n}.$$

Then, we have

$$\begin{aligned} \mathrm{d}\omega &= \mathrm{d}\omega_{a_1 \cdots a_n} \wedge \mathrm{d}x^{a_1} \wedge \cdots \wedge \mathrm{d}x^{a_n} \\ &= \partial_b \omega_{a_1 \cdots a_n} \mathrm{d}x^b \wedge \mathrm{d}x^{a_1} \wedge \cdots \wedge \mathrm{d}x^{a_n}, \end{aligned}$$

- **graded Leibniz rule w.r.t. the wedge product:** Let $\omega \in \Omega^n(M)$ and $\sigma \in \Omega^m(M)$. Then
$$\mathrm{d}(\omega \wedge \sigma) = \mathrm{d}\omega \wedge \sigma + (-1)^n \omega \wedge \mathrm{d}\sigma.$$

8. **symplectic manifold:** Let M be a smooth manifold. A **2-form** $\omega \in \Omega^2(M)$ is said to be a **symplectic form** on M if
   (a) $\mathrm{d}\omega = 0$ and
   (b) it is **non-degenerate**, i.e.
   $$(\forall\ Y \in \Gamma(TM) : \omega(X, Y) = 0) \Rightarrow X = 0.$$

   A manifold equipped with a symplectic form is called a **symplectic manifold**.
   - **difference with Riemannian manifolds:** The metric tensor for Riemannian manifolds is **symmetric**. So the key is nondegenerate bilinear forms that are **symmetric vs. alternating.**
9. **a summary of pull-back of n-form:**
   - **distribute over the wedge product:**
     Let $\phi: M \to N$ be smooth, $\omega \in \Omega^n(N)$ and $\sigma \in \Omega^m(N)$. Then, we have
     $$\Phi^*(\omega \wedge \sigma) = \Phi^*(\omega) \wedge \Phi^*(\sigma).$$
   - **commute with wedge product:** Let $\phi: M \to N$ be smooth. For any $\omega \in \Omega^n(N)$, we have
     $$\Phi^*(\mathrm{d}\omega) = \mathrm{d}(\Phi^*(\omega)).$$
   - **$\mathbb{R}$-linear:**
     $$\begin{aligned} \Phi^*(c\sigma) &= c\Phi^*(\sigma), \\ \Phi^*(\omega + \sigma) &= \Phi^*(\omega) + \Phi^*(\sigma). \end{aligned}$$
10. **closed and exact:** Let M be a smooth manifold and let $\omega \in \Omega^n(M)$. We say that ω is
    - **closed** if $\mathrm{d}\omega = 0$;
    - **exact** if $\exists\ \sigma \in \Omega^{n-1}(M) : \omega = \mathrm{d}\sigma$.
11. **exact implies closed:** Let M be a smooth manifold. The operator
    $$\mathrm{d}^2 \equiv \mathrm{d} \circ \mathrm{d} : \Omega^n(M) \to \Omega^{n+2}(M)$$
    is identically zero, i.e. $\mathrm{d}^2 = 0$.
    - **antisymmetrization and symmetrization**: Given an object which carries some indices, say $T_{a_1,\ldots,a_n}$, we define the **antisymmetrization** of $T_{a_1,\ldots,a_n}$ as
      $$T_{[a_1 \cdots a_n]} := \frac{1}{n!} \sum_{\pi \in S_n} \operatorname{sgn}(\pi)\, T_{\pi(a_1) \cdots \pi(a_n)}.$$
      Similarly, the **symmetrization** of $T_{a_1,\ldots,a_n}$ is defined as
      $$T_{(a_1 \cdots a_n)} := \frac{1}{n!} \sum_{\pi \in S_n} T_{\pi(a_1) \cdots \pi(a_n)}.$$

* Examples:

$$T_{[ab]} = \frac{1}{2}(T_{ab} - T_{ba}), \qquad T_{(ab)} = \frac{1}{2}(T_{ab} + T_{ba})$$
$$T_{[abc]} = \frac{1}{6}(T_{abc} + T_{bca} + T_{cab} - T_{bac} - T_{cba} - T_{acb})$$
$$T_{(abc)} = \frac{1}{6}(T_{abc} + T_{bca} + T_{cab} + T_{bac} + T_{cba} + T_{acb})$$

* Of course, we can (anti)symmetrize only some of the indices

$$T^{ab}{}_{[cd]e} = \frac{1}{2}(T^{ab}{}_{cde} - T^{ab}{}_{dce}).$$

* It is easy to check that in a **contraction** (i.e. a sum), we have

$$T_{a_1\cdots a_n} S^{a_1\cdots[a_i\cdots a_j]\cdots a_n} = T_{a_1\cdots[a_i\cdots a_j]\cdots a_n} S^{a_1\cdots a_n}$$

and

$$T_{a_1\cdots(a_i\cdots a_j)\cdots a_n} S^{a_1\cdots[a_i\cdots a_j]\cdots a_n} = 0.$$

- proof sketch:

$$\begin{aligned} \mathrm{d}^2\omega &= \partial_c\partial_b\omega_{a_1\cdots a_n}\mathrm{d}x^c \wedge \mathrm{d}x^b \wedge \mathrm{d}x^{a_1} \wedge \cdots \wedge \mathrm{d}x^{a_n} \\ &= \partial_{(c}\partial_{b)}\omega_{a_1\cdots a_n}\mathrm{d}x^{[c} \wedge \mathrm{d}x^{b]} \wedge \mathrm{d}x^{a_1} \wedge \cdots \wedge \mathrm{d}x^{a_n} \\ &= 0. \end{aligned}$$

Since this holds for any ω, we have $\mathrm{d}^2 = 0$.

12. **de Rham cohomology:** We can extend the action of d to the zero vector space $0 := \{0\}$ by mapping the zero in 0 to the zero function in $\Omega^0(M)$. In this way, we obtain the chain of $\mathbb{R}$-linear maps

$$0 \xrightarrow{\mathrm{d}} \Omega^0(M) \xrightarrow{\mathrm{d}} \Omega^1(M) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega^n(M) \xrightarrow{\mathrm{d}} \Omega^{n+1}(M) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega^{\dim M}(M) \xrightarrow{\mathrm{d}} 0,$$

where we now think of the spaces $\Omega^n(M)$ as $\mathbb{R}$-vector spaces. The equation $\mathrm{d}^2 = 0$ is equivalent to

$$\mathrm{im}(\mathrm{d}:\ \Omega^n(M) \to \Omega^{n+1}(M)) \subseteq \ker(\mathrm{d}:\ \Omega^{n+1}(M) \to \Omega^{n+2}(M))$$

for all $0 \le n \le \dim M - 2$. Moreover, we have

$$\begin{aligned} \omega \in \Omega^n(M) \text{ is closed } &\Leftrightarrow\ \omega \in \ker(\mathrm{d}:\ \Omega^n(M) \to \Omega^{n+1}(M)) \\ \omega \in \Omega^n(M) \text{ is exact } &\Leftrightarrow\ \omega \in \mathrm{im}(\mathrm{d}:\ \Omega^{n-1}(M) \to \Omega^n(M)). \end{aligned}$$

- **notation:**

$$\begin{aligned} Z^n &:= \ker(\mathrm{d}:\ \Omega^n(M) \to \Omega^{n+1}(M)), \\ B^n &:= \mathrm{im}(\mathrm{d}:\ \Omega^{n-1}(M) \to \Omega^n(M)), \end{aligned}$$

so that Z^n is the space of closed n-forms and B^n is the space of exact n-forms.
  * $\mathrm{d}^2 = 0$ implies that $B^n \subseteq Z^n$ for all n. So B^n is a vector subspace of Z^n.
- question: We have the question of whether every closed form is exact and vice versa, i.e. whether the implications

$$Z^n = B^n$$

hold in general. The answer is **no** in general.
- **special simply connected domain (Poincaré lemma):** Let $M \subseteq \mathbb{R}^d$ be a simply connected domain. Then

$$Z^n = B^n\,, \qquad \forall\ n > 0.$$

- **de Rham cohomology group:** Let M be a smooth manifold. The n-th **de Rham cohomology group** on M is the quotient $\mathbb{R}$-vector space

$$H^n(M) := Z^n/B^n.$$

  * It is a quotient group as $Z^n/\sim$, where $\sim$ is the equivalence relation $\omega \sim \sigma\ :\Leftrightarrow \omega - \sigma \in B^n$.

* **question reformulation:** Does the following is true

$$H^n(M) \cong_{\text{vec}} 0.$$

* de Rham states that $H^n(M)$ only depends on the global topology of M. In other words, the cohomology groups are topological invariants. This is remarkable because $H^n(M)$ is defined in terms of exterior derivatives, which have everything to do with the local differentiable structure of M, and a given topological space can be equipped with several inequivalent differentiable structures.
* Example I: Let M be any smooth manifold. We have

$$H^0(M) \cong_{\text{vec}} \mathbb{R}^{(\text{\# of connected components of } M)}$$

since the closed 0-forms are just the locally constant smooth functions on M. As an immediate consequence, we have

$$H^0(\mathbb{R}) \cong_{\text{vec}} H^0(S^1) \cong_{\text{vec}} \mathbb{R}.$$

* Example II: By Poincaré lemma, we have

$$H^n(M) \cong_{\text{vec}} 0$$

for any simply connected $M \subseteq \mathbb{R}^d$. In [4][Theorem 10.39,] the convex and open set is a speical case of the simply connected.

14 Lie Groups and Their Associated Lie Algebras

$$\begin{tikzcd}
\text{set} \arrow[r] & \text{topological space} \arrow[r] & \text{topological manifold} \arrow[r] & \text{differentiable manifold} \arrow[r] \arrow[d] & \text{principal fibre bundle} \arrow[r] & \text{associated fibre bundles} \\
 & & & \text{Lie group} \arrow[ur] \arrow[d, shift left] & & \\
 & & & \text{Lie algebra} \arrow[u, shift left] & &
\end{tikzcd}$$

"The miracle of Lie theory is that a curved object, a Lie group G, can be almost completely captured by a a flat one, the tangent space T_eG of G at the identity or any other points."

A summary:

- We give the definition of **Lie group** G, which is a also **manifold**.
- The **left translation map**is then introduced for this group. It is also a diffeomorphism from the manifold perspective.
- We then show the **Lie algebra** of a Lie group, which is a subalgebra $\Gamma(TG)$ and is Lie algebra homomorphic to T_eG.
- Finally, we mentioned some general facts of left translation for general fields.

1. **Lie group:** A **Lie group** is a **group** $(G, \bullet)$, where G is a **smooth manifold** and the maps

$$\begin{aligned} \mu: \ & G \times G \to G \\ & (g_1, g_2) \mapsto g_1 \bullet g_2 \end{aligned}$$

and

$$\begin{aligned} i: \ & G \to G \\ & g \mapsto g^{-1} \end{aligned}$$

are both **smooth**.

   - Note that the product manifold $G \times G$ inherits a smooth atlas from the smooth atlas of G.
   - The **dimension** of a Lie group $(G, \bullet)$ is the dimension of G as a manifold.

- Examples:
  † n**-dimensional translation group:** $(\mathbb{R}^n, +)$. This is a commutative (or abelian) Lie group.
  † $\mathrm{U}(1)$**:** Let $S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$ and let $\cdot$ be the usual multiplication of complex numbers. Then $(S^1, \cdot)$ is a commutative Lie group usually denoted $\mathrm{U}(1)$.
  † **general linear group:** Let $\mathrm{GL}(n, \mathbb{R}) = \{\phi : \mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^n \mid \det \phi \neq 0\}$. The condition $\det \phi \neq 0$ is a so-called *open condition*, meaning that $\mathrm{GL}(n, \mathbb{R})$ can be identified with an open subset of $\mathbb{R}^{n^2}$, from which it then inherits a smooth structure from manifold $\mathbb{R}^{n^2}$.
  † **orthogonal group:** Let V be an n-dimensional $\mathbb{R}$-vector space equipped with a pseudo inner product with **signature** (p, q) (See Section 9). We can define the set
  $$\mathrm{O}(p, q) := \{\phi : V \xrightarrow{\sim} V \mid \forall\, v, w \in V : (\phi(v), \phi(w)) = (v, w)\}.$$
  The pair $(\mathrm{O}(p, q), \circ)$ is a Lie group called the **orthogonal group** w.r.t. the pseudo inner product $(-, -)$. This is, in fact, a Lie subgroup of $\mathrm{GL}(p + q, \mathbb{R})$. Some notable examples are $\mathrm{O}(3, 1)$, which is known as the *Lorentz group* in relativity, and $\mathrm{O}(3, 0)$, which is the 3-dimensional rotation group.

2. **Lie group homomorphism:** Let $(G, \bullet)$ and $(H, \circ)$ be Lie groups. A map $\phi : G \to H$ is called **Lie group homomorphism** if it is a **group homomorphism** and a **smooth map**.
   - A **Lie group isomorphism** is a **group homomorphism** which is also a **diffeomorphism** (not only smooth).
3. **left translation:** To every element of a Lie group there is associated a special map. Let $(G, \bullet)$ be a Lie group and let $g \in G$. The map
   $$\begin{aligned} \ell_g :\ & G \to G \\ & h \mapsto \ell_g(h) := g \bullet h \equiv gh \end{aligned}$$
   is called the **left translation** by g.
   - **notation:** If there is no danger of confusion, we usually suppress the $\bullet$ notation.
   - In group theory, this is called **group action**, from which we get the equivalent class called **orbit**. Orbit-Stabiliser theorem is quite important for this topic. See my abstract algebra notes.
   - Let G be a Lie group. For any $g \in G$, the left translation map $\ell_g : G \to G$ is a **diffeomorphism** but **not** group homomorphism (so **not** Lie group homomorphism).
     ∗ From group action, we have
     $$\ell_g \circ \ell_h = \ell_{gh}$$
     ∗ The inverse map is $(\ell_g)^{-1} = \ell_{g^{-1}}$, since
     $$\ell_{g^{-1}} \circ \ell_g = \ell_g \circ \ell_{g^{-1}} = \mathrm{id}_G.$$
     ∗ Note that, in general, ℓ_g is **not** an isomorphism (or homomorphism) of groups since $ghk \neq ghgk$. Similarly, a right translation that defined analogously as the above left translation is **not** group isomorphism. However, the group **conjugation** action is **Lie group isomorphism (automorphism)**. Recall conjugation:
     $$\begin{aligned} \Psi : G \times G &\to G \\ (g, h) &\mapsto ghg^{-1} \end{aligned}$$
     For each g, we define $\Psi_g(h) = ghg^{-1}$. We have
     $$\Psi_g(hk) = ghkg^{-1} = ghg^{-1}gkg^{-1} = \Psi_g(h)\Psi_g(k)$$
     Later we will use to define the Adjoint map on Lie group.
4. **push-forward map from left translation:** Since $\ell_g : G \to G$ is a **diffeomorphism**, we have a well-defined push-forward map
   $$\begin{aligned} (L_g)_* :\ & \Gamma(TG) \to \Gamma(TG) \\ & X \mapsto (L_g)_*(X) \end{aligned}$$
   where
   $$\begin{aligned} (L_g)_*(X) :\ & G \to TG \\ & h \mapsto (L_g)_*(X)(h) := (\ell_g)_*(X(g^{-1}h)). \end{aligned}$$

- **diagram:** We can draw the diagram

$$
\begin{tikzcd}
TG \arrow[r, "(\ell_g)_*"] & TG \\
G \arrow[u, "X"] \arrow[r, "\ell_g"] & G \arrow[u, "(L_g)_*(X)"']
\end{tikzcd}
$$

  Note that this is exactly the same as our previous

$$\Phi_*(\sigma) := \phi_* \circ \sigma \circ \phi^{-1}.$$

- **equivalent representations I:** By introducing the notation $X|_h := X(h)$, so that $X|_h \in T_hG$, we can write

$$(L_g)_*(X)|_h := (\ell_g)_*(X|_{g^{-1}h}).$$

- **equivalent representations II:** Alternatively, recalling that the map ℓ_g is a diffeomorphism and relabelling the elements of G, we can write this as

$$(L_g)_*(X)|_{gh} := (\ell_g)_*(X|_h).$$

- **equivalent representations III:** A further reformulation comes from considering the vector field $X \in \Gamma(TG)$ as an $\mathbb{R}$-linear map $X: \mathcal{C}^\infty(G) \xrightarrow{\sim} \mathcal{C}^\infty(G)$. Then, for any $f \in \mathcal{C}^\infty(G)$

$$(L_g)_*(X)(f) := X(f \circ \ell_g).$$

- **group action on push forward:** Let G be a Lie group. Recall the compositions of push-forward in Section 12, for any $g, h \in G$, we have

$$(L_g)_* \circ (L_h)_* = (L_{gh})_*.$$

  This appies to pointwise push-forward as well, i.e.

$$\big((\ell_{g_1})_* \circ (\ell_{g_2})_*\big)(X|_h) = (\ell_{g_1 g_2})_*(X|_h)$$

  for any $g_1, g_2, h \in G$ and $X|_h \in T_hG$.

5. **left-invariant vector field:** Let G be a Lie group. A vector field $X \in \Gamma(TG)$ is said to be **left-invariant** if

$$\forall\ g \in G:\ (L_g)_*(X) = X.$$

- **equivalent condition from above II**: We can require this to hold pointwise

$$\forall\ g, h \in G:\ (\ell_g)_*(X|_h) = X|_{gh}.$$

  equivalent condition from above III: By recalling the last reformulation of the push-forward, we have that $X \in \Gamma(TG)$ is left-invariant if, and only if

$$\forall\ f \in \mathcal{C}^\infty(G):\ X(f \circ \ell_g) = X(f) \circ \ell_g.$$

  Note this is because $(\ell_g)_*(X|_h) = X(f \circ \ell_g)(h)$ and $X|_{gh} f = Xf(gh) = [(Xf) \circ \ell_g](h)$.

- **notation:** We use $\mathcal{L}(G) \subseteq \Gamma(TG)$ to denote the set of all left-invariant vector fields on G is .
- $\mathcal{L}(G)$ **is** $\mathbb{R}$**-vector subspace of** $\Gamma(TG)$**:** We have $\mathcal{L}(G)$ is closed under

$$
\begin{aligned}
+:&\ \mathcal{L}(G) \times \mathcal{L}(G) \to \mathcal{L}(G) \\
\cdot:&\ \mathcal{C}^\infty(G) \times \mathcal{L}(G) \to \mathcal{L}(G),
\end{aligned}
$$

  only for the **constant functions** in $\mathcal{C}^\infty(G)$. Therefore, $\mathcal{L}(G)$ is **not** a $\mathcal{C}^\infty(G)$-submodule of $\Gamma(TG)$, but it is an $\mathbb{R}$-vector subspace of $\Gamma(TG)$.

6. **associated Lie algebra:** Recall in Section 13, we defined **Lie algebra** $(\Gamma(TM), +, \cdot, [-,-])$ over $\mathbb{R}$. Here we can check

   Let G be a Lie group. Then $\mathcal{L}(G)$ is a Lie subalgebra of $\Gamma(TG)$.

- We only need to show **left-invariant** property of $[X, Y]$

$$\begin{aligned}[X,Y](f\circ \ell_g) &:= X(Y(f\circ \ell_g)) - Y(X(f\circ \ell_g))\\ &= X(Y(f)\circ \ell_g) - Y(X(f)\circ \ell_g)\\ &= X(Y(f))\circ \ell_g - Y(X(f))\circ \ell_g\\ &= (X(Y(f)) - Y(X(f)))\circ \ell_g\\ &= [X,Y](f)\circ \ell_g.\end{aligned}$$

Hence, $[X, Y]$ is **left-invariant**.
- Let G be a Lie group. $\mathcal{L}(G)$ is called **associated Lie algebra** of G.

7. **key theorem:** Let G be a Lie group with identity element $e \in G$. Then $\mathcal{L}(G) \cong_{\text{vec}} T_eG$.
   - We construct a **linear isomorphism** $j:\ T_eG \xrightarrow{\sim} \mathcal{L}(G)$. Define
   $$\begin{aligned} j:\ T_eG &\to \Gamma(TG)\\ A &\mapsto j(A),\end{aligned}$$
   where
   $$\begin{aligned} j(A):\ G &\to TG\\ g &\mapsto j(A)|_g := (\ell_g)_*(A) \in T_gG.\end{aligned}$$
   linear isomorphism is from the linearity of push-forward.
   - Here the key is to imagine the isomorphism push every vector from T_eG to $\mathcal{L}(G)$ over the manifold.
   - **inverse of j:** We have the inverse of j is $j^{-1}:\ \mathcal{L}(G) \xrightarrow{\sim} T_eG$:
   $$\begin{aligned} j^{-1}:\ \Gamma(TG) &\to T_eG\\ X &\mapsto j^{-1}(X) = X|_e.\end{aligned}$$
   - $(\Gamma(TM), +, \cdot, [-,-])$ over $\mathbb{R}$ which is infinite-dimensional. By contrast, here we have that the space $\mathcal{L}(G)$ is finite-dimensional and $\dim \mathcal{L}(G) = \dim G$ from $\cong_{\text{vec}} T_eG$.
8. **Lie algebra homomorphism:** Recall the **algebra homomorphism** from Section 10. For the special Lie algebra, let $(L_1, [-,-]_{L_1})$ and $(L_2, [-,-]_{L_2})$ be Lie algebras over the same field. A linear map $\phi:\ L_1 \xrightarrow{\sim} L_2$ is a **Lie algebra homomorphism** if
   $$\forall\ x, y \in L_1:\ \phi([x,y]_{L_1}) = [\phi(x), \phi(y)]_{L_2}.$$
   - **Lie algebra isomorphism:** If ϕ is **bijective**, then it is a **Lie algebra isomorphism**.
     * **notation:** We write $L_1 \cong_{\text{Lie alg}} L_2$ for **Lie algebra isomorphism**.
9. **equip T_eG with Lie algebra structure:** By using the bracket $[-,-]_{\mathcal{L}(G)}$ on $\mathcal{L}(G)$ we can define, for any $A, B \in T_eG$
   $$[A,B]_{T_eG} := j^{-1}\big([j(A), j(B)]_{\mathcal{L}(G)}\big),$$
   where $j^{-1}(X) = X|_e$. Equipped with these brackets, we have
   $$\mathcal{L}(G) \cong_{\text{Lie alg}} T_eG.$$
   - **notation:** We often write $\mathfrak{g} := T_eG$ with the Lie bracket in $\mathfrak{g}$ defined as above.
10. **some general facts of left-invariant:**
    (a) **left-invariant tensor field:** Since ℓ_g is (for each $g \in G$) a **diffeomorphism** $G \to G$, its **pull-back** L_g^* may be applied to an arbitrary tensor field on a group G, the result being again a tensor field on G. A tensor field T of type $\begin{pmatrix} p \\ q \end{pmatrix}$ on G is said to be **left-invariant** if it satisfies
    $$L_g^* T = T \quad \text{for all } g \in G$$
    - Left-invariant tensor fields are smooth and uniquely specified by its value at the unit element $e \in G$ of the group (or at any other point $h \in G$ as well). Hint: from definition, recall that we need $T(gh) = \ell_{g*}T(h)$. For any fixed h, by varying g we get the full vector field. If $h = e$, value at the point e gives $T(g) = L_{g*}T(e)$. Smoothness is a consequence of the multiplication.

- So the above left-invariant vector fields is just a special case of left-invariant tensor fields.
- The special $(0,0)$ tensor field is a constant function on the manifold.

(b) Let E_a be a basis of the tangent space in the unit element and let e_a denote the **left-invariant vector fields** on G generated by E_a; there thus holds

$$e_a(g) = L_{g*}E_a \quad E_a = e_a(e)$$

We have

(i). the fields e_a constitute a global **frame** on G.

(ii). any Lie group is a **parallelizable** as well as **orientable** manifold.

(iii). the vector field $V = V^a e_a$ is left-invariant if and only if it has constant components V^a with respect to the left-invariant frame field e_a

(iv). if $\hat{e}_a = A_a^b e_b$ is any other left-invariant frame field, then the transition matrix A_a^b is necessarily constant.

Hint: (i) a linear dependence of the vectors $e_a(g)$ at a point g would need the linear dependence of E_a (contradiction); (ii) the definition of being parallelizable; a volume form is given by $e^1 \wedge \cdots \wedge e^n$; (iii) $L_g^*(V^a e_a) = \left(L_g^* V^a\right) e_a \overset{!}{=} V^a e_a$, so that the components V^a are to be left-invariant, so must be constant; (iv) consider the consequence of item (iii).

11. From the existence of global frame, we know that given any $X \in T_g G$, we can the find a $A \in T_e G$ such that $j(A)|_g = X$.

15 Classification of Complex Lie algebras and Dynkin diagrams

While it is possible to classify Lie algebras more generally, we will only consider the classification of **finite-dimensional complex Lie algebras**, i.e. Lie algebras $(L, [-,-])$. For example, any complex Lie group G, which is a complex manifold, gives rise to a associated complex Lie algebra as shown in Section 14. We need complex to ensure it is a space over an **algebraically closed field**[18].

A summary:

- ▶ We introduce the **Levi's decomposition theorem** for Lie algebras **up to Lie algebra isomorphism**.
- ▶ The **adjoint map** and the **Killing form** are then defined.
- ▶ We then introduce **Cartan subalgebra, fundamental roots, the Weyl group and the Cartan matrix**. **Dynkin diagram** associated to a Cartan matrix is defined. The classification hinges on the existence of this Cartan subalgebra.
- ▶ The **classification** of Lie simple algebra is finaly stated using Dynkin diagram .

1. **span of two Lie subalgebras:** If A, B are Lie subalgebras of a Lie algebra $(L, [-,-])$ over K, then
$$[A, B] := \operatorname{span}_K\left(\{[x,y] \in L \mid x \in A \text{ and } y \in B\}\right)$$
is again a Lie subalgebra of L.
2. **abelian Lie algebra:** A Lie algebra L is said to be **abelian** if
$$\forall\, x, y \in L:\ [x,y] = 0.$$
which means $[L,L] = 0$, where 0 denotes the trivial Lie algebra $\{0\}$.
   - **commute:** The $(\operatorname{End}(V), +, \cdot, \circ)$ defined in Section 10 is just matrix for finite dimensional vector space V. Recall that in matrix analysis [5], we have the concept of **commute** of a family L of matrix. **Abelian** is the restatement of **matrix commute** in Lie algebra.
   - **abelian is not interesting:**
     - ∗ All zero means no information for the bracket operator.
     - ∗ Given any two abelian Lie algebras, every linear isomorphism between their underlying vector spaces is automatically a Lie algebra isomorphism. Therefore, for each $n \in \mathbb{N}$, there is (up to isomorphism) **only one** abelian n-dimensional Lie algebra.

[18] This is to ensure the existence of eigenvalue-eigenvectors.

* Note, however, for general group classification, abelian is still interesting. See my abstract algebra notes.

3. **ideal:** An **ideal** I of a Lie algebra L is a Lie **subalgebra** such that $[I, L] \subseteq I$, i.e.

$$\forall\, x \in I : \forall\, y \in L : \; [x, y] \in I.$$

   - **trivial ideals:** The ideals 0 and L are called the **trivial ideals** of L.

4. **(semi-)simple:** A Lie algebra L is said to be
   - **simple:** if it is **non-abelian** and it contains **no non-trivial ideals**;
   - **semi-simple:** if it is a **direct sum** of simple Lie algebras.

5. **a summary of equivalent conditions for semi-simple:** Some of the concepts in the following will be defined later:
   - L is **semi-simple**;
   - the **Killing form**, $k(x, y) = \mathrm{tr}(\mathrm{ad}(x)\,\mathrm{ad}(y))$, is **non-degenerate**;
   - L has **no non-zero abelian ideals**;
   - L has **no non-zero solvable ideals**;
   - the **radical** (maximal solvable ideal) of L is zero.
     * No non-zero solvable ideals and zero radical will be understood by the following Levi's decomposition theorem.
     * No non-zero abelian ideals does not indicate no ideal since the subspace $L_1 \oplus 0$ in $L_1 \oplus L_2$ will be a ideal, but it is not an abelian ideal. Note abelian ideals are solvable.

6. **derived subalgebra:** Let L be a Lie algebra. The Lie subalgebra

$$L' := [L, L]$$

   is called the **derived subalgebra** of L.
   - **derived series:** We can form a sequence of Lie subalgebras

$$L \supseteq L' \supseteq L'' \supseteq \cdots \supseteq L^{(n)} \supseteq \cdots$$

     called the **derived series** of L.
   - **solvable:** A Lie algebra L is **solvable** if there exists $k \in \mathbb{N}$ such that $L^{(k)} = 0$.
     * All **abelian ideals** of any Lie Algebra are **solvable.**
     * In group theory, **solvable** definition is different. See my abstract algebra notes.

7. **direct sum of Lie algebra:** Let L_1 and L_2 be Lie algebras. It has two conditions:
   (a) The **direct sum** $L_1 \oplus_{\mathrm{Lie}} L_2$ has $L_1 \oplus L_2$ as its underlying vector space and Lie bracket defined as

$$[x_1 + x_2, y_1 + y_2]_{L_1 \oplus_{\mathrm{Lie}} L_2} := [x_1, y_1]_{L_1} + [x_2, y_2]_{L_2}$$

   for all $x_1, y_1 \in L_1$ and $x_2, y_2 \in L_2$.
   (b) By identifying L_1 and L_2 with the subspaces $L_1 \oplus 0$ and $0 \oplus L_2$ of $L_1 \oplus L_2$ respectively, we require

$$[L_1, L_2]_{L_1 \oplus_{\mathrm{Lie}} L_2} = 0.$$

   - Recall the vanilla direct sum is defined in Section 12 **without** the $[L_1, L_2] = 0$ condition.
   - **notation:** In the following, we will drop the "Lie" subscript and understand $\oplus$ to mean $\oplus_{\mathrm{Lie}}$ whenever the summands are Lie algebras.

8. **semi-direct sum:** Let R and L be Lie algebras. It has two conditions:
   (a) The **semi-direct sum** $R \oplus_s L$ has $R \oplus L$ as its underlying vector space
   (b) Lie bracket satisfying

$$[R, L]_{R \oplus L} \subseteq R,$$

   i.e. R is an ideal of $R \oplus_s L$.

9. **Levi's decomposition theorem:** Any finite-dimensional complex Lie algebra L can be decomposed as
$$L = R \oplus_s (L_1 \oplus \cdots \oplus L_n)$$
where R is a **solvable Lie algebra** and $L_1, \ldots, L_n$ are **simple Lie algebras**.
   - No general classification of solvable Lie algebras is known. In contrast, the **finite dimensional, simple, complex Lie algebras** have been classified completely.
   - R is called the **radical (maximal solvable ideal)**.
   - If Lie Algebra L is semi-simple, **radical** of L has to be **zero**. Hence, all **abelian ideals** are zero. We now can interpret the above 3 equivalent conditions for semi-simple.
   - The simple Lie algebras are the basic building blocks to build any semi-simple Lie algebra. We will later introduce **Dynkin diagram** to classify **simple Lie algebras.**

10. **adjoint map:** Let L be a Lie algebra over k and let $x \in L$. The **adjoint map** w.r.t. x is the K-linear map
$$\begin{aligned} \mathrm{ad}_x :\ & L \xrightarrow{\sim} L \\ & y \mapsto \mathrm{ad}_x(y) := [x, y]. \end{aligned}$$
   - The linearity of ad_x follows from the linearity of the bracket in the **second argument**.
   - **adjoint endomorphism:** The **linearity in the first argument** of the bracket implies that the map
$$\begin{aligned} \mathrm{ad} :\ & L \xrightarrow{\sim} \mathrm{End}(L) \\ & x \mapsto \mathrm{ad}(x) := \mathrm{ad}_x\,. \end{aligned}$$
is also linear.
     * **homomorphism:** Recall that $\mathrm{End}(L)$ is a Lie algebra with bracket $[\phi, \psi] := \phi \circ \psi - \psi \circ \phi$. The map $\mathrm{ad} :\ L \xrightarrow{\sim} \mathrm{End}(L)$ is a **Lie algebra homomorphism**:
$$\mathrm{ad}([x, y]) = [\mathrm{ad}(x), \mathrm{ad}(y)].$$
     * We may call it **adjoint endomorphism** or **adjoint action**. It will be used for adjoint representation of Lie algebra.

11. **Killing form:** Let L be a Lie algebra over K. The **Killing form** on L is the K-bilinear map
$$\begin{aligned} \kappa :\ & L \times L \to K \\ & (x, y) \mapsto \kappa(x, y) := \mathrm{tr}(\mathrm{ad}_x \circ \mathrm{ad}_y), \end{aligned}$$
where tr is the usual **trace** on the vector space $\mathrm{End}(L)$.
   - **symmetric:** Note that the Killing form is **not** a tensor "form". In fact, since L is finite-dimensional, the trace is cyclic and thus κ is **symmetric** (note tensor form instead requires antisymmetric), i.e.
$$\forall\ x, y \in L :\ \kappa(x, y) = \kappa(y, x).$$
   - **associativity:** Let L be a Lie algebra. For any $x, y, z \in L$, we have
$$\kappa([x, y], z) = \kappa(x, [y, z]).$$
     * **antisymmetric w.r.t.** k**:** The associativity property of κ w.r.t. the bracket can be restated by saying that, for any $y \in L$, the linear map ad_y is **antisymmetric** w.r.t. κ, i.e.
$$\forall\ x, y \in L :\ \kappa(\mathrm{ad}_y(x), z) = -\kappa(x, \mathrm{ad}_y(z)).$$
   - **Cartan's criterion:** A Lie algebra L is **semi-simple** if, and only if, the Killing form κ is **non-degenerate**, i.e.
$$(\forall\ y \in L : \kappa(x, y) = 0) \Rightarrow x = 0.$$
Hence, if L is semi-simple, then κ is a **pseudo inner product** on L.
   - Killing form on the Lie algebra of a compact Lie group is always **negative semi-definite**. In other words, we have $\kappa(X, X)$ is always negative or zero.

12. **basis representation:** Let L be a Lie algebra over K and let $\{E_i\}$ be a **basis**. Then, we have

$$[E_i, E_j] = C^k{}_{ij} E_k$$

for some $C^k{}_{ij} \in K$. The numbers $C^k{}_{ij}$ are called the **structure constants** of L w.r.t. the basis $\{E_i\}$.

- **antisymmetry:** Using structure constants, the **antisymmetry of the Lie bracket** is

$$C^k{}_{ij} = -C^k{}_{ji}$$

- **Jacobi identity:** Lie Jacobi identity becomes

$$C^n{}_{im} C^m{}_{jk} + C^n{}_{jm} C^m{}_{ki} + C^n{}_{km} C^m{}_{ij} = 0.$$

- **adjoint map:** $(\mathrm{ad}_{E_i})^k{}_j = C^k{}_{ij}$
- **Killing form:** $\kappa_{ij} = C^m{}_{ik} C^k{}_{jm}$:

$$\begin{aligned} \kappa_{ij} &:= \kappa(E_i, E_j) \\ &= \mathrm{tr}(\mathrm{ad}_{E_i} \circ \mathrm{ad}_{E_j}) \\ &= (\mathrm{ad}_{E_i} \circ \mathrm{ad}_{E_j})^k{}_k \\ &= (\mathrm{ad}_{E_i})^m{}_k (\mathrm{ad}_{E_j})^k{}_m \\ &= C^m{}_{ik} C^k{}_{jm}, \end{aligned}$$

13. **Cartan subalgebra:** Let L be a d-dimensional semi-simple Lie algebra. A **Cartan subalgebra** H of L is a **maximal Lie subalgebra** of L with the following property: there exists a basis $\{h_1, \ldots, h_r\}$ of H which can be extended to a basis $\{h_1, \ldots, h_r, e_1, \ldots, e_{d-r}\}$ of L such that $e_1, \ldots, e_{d-r}$ are **eigenvectors** of $\mathrm{ad}(h)$ for any $h \in H$, i.e.

$$\forall\ h \in H : \exists\ \lambda_\alpha(h) \in \mathbb{C} :\ \mathrm{ad}(h) e_\alpha = \lambda_\alpha(h) e_\alpha,$$

for each $1 \leq \alpha \leq d - r$. (This is called **maximal toral subalgebra**, i.e. diagonalizable over an algebraically closed field.)

- **Cartan-Weyl basis:** The basis $\{h_1, \ldots, h_r, e_1, \ldots, e_{d-r}\}$ is known as a **Cartan-Weyl basis** of L.
- **eigenvalue function is $\mathbb{C}$-linear functional:** From linearity of $\mathrm{ad}(h)$, we know $\lambda_\alpha(h)$ is a **$\mathbb{C}$-linear functional** over H: $\lambda_\alpha :\ H \xrightarrow{\sim} \mathbb{C}$, and thus $\lambda_\alpha \in H^*$.
  - **root set:** The maps $\lambda_1, \ldots, \lambda_{d-r} \in H^*$ are called the **roots** of L. The collection

  $$\Phi := \{\lambda_\alpha \mid 1 \leq \alpha \leq d - r\} \subseteq H^*$$

  is called the **root set** of L.
- Let L be a finite-dimensional semi-simple complex Lie algebra. Then
  - **existence:** L possesses a Cartan subalgebra;
  - **rank:** All Cartan subalgebras of L have the same dimension, called the **rank** of L;
  - **abelian:** Any of Cartan subalgebra of L is **abelian** [6][sec.8]. See also below maximality property.
- **maximality property:** We have

$$\big(\forall\ h \in H : [h, x] = 0\big) \Leftrightarrow x \in H.$$

  - From the **maximality property**, if λ_α were the zero map, then we would have $e_\alpha \in H$. Thus, we must have $0 \notin \Phi$.
- Recall the **simultaneously diagonalizable** in [5]. $\forall\ h \in H :\ \mathrm{ad}(h) e_\alpha = \lambda_\alpha(h) e_\alpha$ for each $1 \leq \alpha \leq d - r$ just roughly means the set $\{e_1, \ldots, e_{d-r}\}$ simultaneously make all $\mathrm{ad}(h)$ diagonal (of course we still need h_i with 0 eigenvalues to be the diagonal elements).
- **Antisymmetry** of each $\mathrm{ad}(h)$ w.r.t. the **Killing form** κ leads to

$$\lambda \in \Phi\ \Rightarrow\ -\lambda \in \Phi.$$

  Hence Φ is **not** a linearly independent subset of H^*.
- $\dim L = \dim H + |\Phi| = \dim H^* + |\Phi|$

14. **fundamental roots:** A set of **fundamental roots** $\Pi := \{\pi_1, \dots, \pi_f\}$ is a subset $\Pi \subseteq \Phi$ such that
    (a) Π is a linearly independent subset of H^*;
    (b) for each $\lambda \in \Phi$, there exist $n_1, \dots, n_f \in \mathbb{N}$ and $\varepsilon \in \{+1, -1\}$ such that
    $$\lambda = \varepsilon \sum_{i=1}^{f} n_i \pi_i.$$
    - **notation:** We can write the last equation more concisely as $\lambda \in \operatorname{span}_{\varepsilon,\mathbb{N}}(\Pi)$.
        * Note $\operatorname{span}_{\varepsilon,\mathbb{N}}(\Pi) \neq \operatorname{span}_{\mathbb{Z}}(\Pi)$. For any $\lambda \in \Phi$, the coefficients of $\pi_1, \dots, \pi_f$ in the expansion above always have the same sign.
    - **existence:** Let L be a finite-dimensional semi-simple complex Lie algebra. Then a set $\Pi \subseteq \Phi$ of fundamental roots always **exists**.
    - **basis:** we have $\operatorname{span}_{\mathbb{C}}(\Pi) = H^*$, that is, Π is a basis of H^*.
        * So we have $|\Pi|$ equals the **rank** of L.
15. **pseudo inner product on dual space:** A **pseudo inner product** $B(-,-)$ on a finite-dimensional vector space V over K induces a **linear isomorphism**
    $$\begin{aligned} i:\ V &\xrightarrow{\sim} V^* \\ v &\mapsto i(v) := B(v,-) \end{aligned}$$
    which can be used to define a pseudo inner product $B^*(-,-)$ on V^* as
    $$\begin{aligned} B^*:\ V^* \times V^* &\to K \\ (\phi,\psi) &\mapsto B^*(\phi,\psi) := B(i^{-1}(\phi), i^{-1}(\psi)). \end{aligned}$$
    - The **non-degeneracy** condition in pseudo inner product is need to ensure the inverse i^{-1}. See the [3][Theorem 5.25] (even in that book it only considers inner product, the proof still applies to pseudo inner product).
    - If we restrict κ to the **Cartan subalgebra**, it remains as a **pseudo inner product** on **Cartan subalgebra**. Note in general the non-degeneracy condition may fail when considered on a subspace.
    $$\begin{aligned} \lambda_\alpha(h_j)\kappa(h_i, e_\alpha) &= \kappa(h_i, \lambda_\alpha(h_j)e_\alpha) \\ &= \kappa(h_i, [h_j, e_\alpha]) \\ &= \kappa([h_i, h_j], e_\alpha) \\ &= \kappa(0, e_\alpha) \\ &= 0. \end{aligned}$$
    Since $\lambda_\alpha \neq 0$, there is some h_j such that $\lambda_\alpha(h_j) \neq 0$ and hence $\kappa(h_i, e_\alpha) = 0$. By linearity, we have $\kappa(h, e_\alpha) = 0$ for any $h \in H$ and any e_α. Let $h \in H \subseteq L$. Since κ is non-degenerate on L, we have $\big(\forall\ x \in L : \kappa(h,x) = 0\big) \Rightarrow h = 0$. Expand $x \in L$ in the Cartan-Weyl basis as$x = h' + e$, where $h' := x^i h_i$ and $e := x^\alpha e_\alpha$. Then, we have
    $$\kappa(h,x) = \kappa(h,h') + x^\alpha \kappa(h, e_\alpha) = \kappa(h,h').$$
    Thus, the non-degeneracy condition reads
    $$\big(\forall\ h' \in H : \kappa(h,h') = 0\big) \Rightarrow h = 0.$$
    We can now define pseudo inner product on H^*:
    $$\begin{aligned} \kappa^*:\ H^* \times H^* &\to \mathbb{C} \\ (\mu,\nu) &\mapsto \kappa^*(\mu,\nu) := \kappa(i^{-1}(\mu), i^{-1}(\nu)), \end{aligned}$$
    where $i:\ H \xrightarrow{\sim} H^*$ is the linear isomorphism induced by κ.
16. **inner product on real subalgebra:** Define the real subalgebra $H^*_{\mathbb{R}} := \operatorname{span}_{\mathbb{R}}(\Pi)$. Note that we have the following chain of inclusions
    $$\Pi \subseteq \Phi \subseteq \operatorname{span}_{\varepsilon,\mathbb{N}}(\Pi) \subseteq \underbrace{\operatorname{span}_{\mathbb{R}}(\Pi)}_{H^*_{\mathbb{R}}} \subseteq \underbrace{\operatorname{span}_{\mathbb{C}}(\Pi)}_{H^*}.$$
    The restriction of κ^* to $H^*_{\mathbb{R}}$ is surprising! Instead of being weakened, the non-degeneracy of κ^* gets strengthened to **positive definiteness**.
    - For any $\alpha, \beta \in H^*_{\mathbb{R}}$, we have $\kappa^*(\alpha,\beta) \in \mathbb{R}$.
    - $\kappa^*:\ H^*_{\mathbb{R}} \times H^*_{\mathbb{R}} \to \mathbb{R}$ is an **inner product** on $H^*_{\mathbb{R}}$. Let $\alpha, \beta \in H^*_{\mathbb{R}}$. Then, we can define:
        * **length:** the **length** of α as $|\alpha| := \sqrt{\kappa^*(\alpha,\alpha)}$;

* **angle:** the **angle** between α and β as $\varphi := \cos^{-1}\left(\frac{\kappa^*(\alpha,\beta)}{|\alpha||\beta|}\right)$.

17. **Weyl transformation and Weyl group:** For any $\lambda \in \Phi \subseteq H^*_{\mathbb{R}}$, define the linear map

$$\begin{aligned} s_\lambda :\ H^*_{\mathbb{R}} &\xrightarrow{\sim} H^*_{\mathbb{R}} \\ \mu &\mapsto s_\lambda(\mu), \end{aligned}$$

where

$$s_\lambda(\mu) := \mu - 2\frac{\kappa^*(\lambda,\mu)}{\kappa^*(\lambda,\lambda)}\,\lambda.$$

The map s_λ is called a **Weyl transformation** and the set

$$W := \{s_\lambda \mid \lambda \in \Phi\}$$

is a group under composition of maps, called the **Weyl group**.

(a) The Weyl group W is generated by the fundamental roots in Π, in the sense that for some $1 \leq n \leq r$, with $r = |\Pi|$,

$$\forall\ w \in W : \exists\ \pi_1, \dots, \pi_n \in \Pi :\ w = s_{\pi_1} \circ s_{\pi_2} \circ \cdots \circ s_{\pi_n};$$

(b) Every root can be produced from a fundamental root by the action of W, i.e.

$$\forall\ \lambda \in \Phi : \exists\ \pi \in \Pi : \exists\ w \in W :\ \lambda = w(\pi);$$

(c) The Weyl group permutes the roots, that is,

$$\forall\ \lambda \in \Phi : \forall\ w \in W :\ w(\lambda) \in \Phi.$$

* Consider, for any $\pi_i, \pi_j \in \Pi$, we have

$$s_{\pi_i}(\pi_j) := \pi_j - 2\frac{\kappa^*(\pi_i,\pi_j)}{\kappa^*(\pi_i,\pi_i)}\,\pi_i.$$

Since $s_{\pi_i}(\pi_j) \in \Phi$ and $\Phi \subseteq \mathrm{span}_{\varepsilon,\mathbb{N}}(\Pi)$, for all $1 \leq i \neq j \leq r$ we must have

$$-2\frac{\kappa^*(\pi_i,\pi_j)}{\kappa^*(\pi_i,\pi_i)} \in \mathbb{N}.$$

18. **Dynkin matrix:** The **Cartan matrix** of a Lie algebra is the $r \times r$ matrix C with entries

$$C_{ij} := 2\frac{\kappa^*(\pi_i,\pi_j)}{\kappa^*(\pi_i,\pi_i)},$$

where the C_{ij} should not be confused with the structure constants ${C^k}_{ij}$.

* **existence, uniqueness and inverse:** To every simple finite-dimensional complex Lie algebra there corresponds a **unique** Cartan matrix and **vice versa** (up to relabelling of the basis elements).
* **diagonal entry:** $C_{ii} = 2$ (no summation implied), the diagonal entries of C are all equal to 2.
* **off-diagonal entry:** The off-diagonal entries are either **zero or negative**. In general, $C_{ij} \neq C_{ji}$, so the Cartan matrix is **not** symmetric, but if $C_{ij} = 0$, then necessarily $C_{ji} = 0$.

19. **bond number:** Given a Cartan matrix C, the ij-th **bond number** is

$$n_{ij} := C_{ij}C_{ji} \qquad \text{(no summation implied)}.$$

Note that we have

$$\begin{aligned} n_{ij} &= 4\,\frac{\kappa^*(\pi_i,\pi_j)}{\kappa^*(\pi_i,\pi_i)}\,\frac{\kappa^*(\pi_j,\pi_i)}{\kappa^*(\pi_j,\pi_j)} \\ &= 4\left(\frac{\kappa^*(\pi_i,\pi_j)}{|\pi_i||\pi_j|}\right)^2 \\ &= 4\cos^2\varphi, \end{aligned}$$

where φ is the angle between π_i and π_j. For $i \neq j$, the angle φ is neither zero nor $180°$, hence $0 \leq \cos^2\varphi < 1$, and therefore

$$n_{ij} \in \{0, 1, 2, 3\}.$$

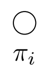
Since $C_{ij} \leq 0$ for $i \neq j$, the only possibilities are

C_{ij}	C_{ji}	n_{ij}
0	0	0
-1	-1	1
-1	-2	2
-1	-3	3

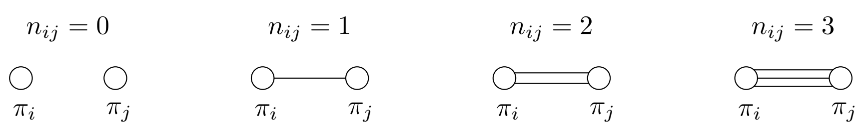
- **different length:** If $n_{ij} = 2$ or 3, then the corresponding fundamental roots have **different lengths**, i.e. either $|\pi_i| < |\pi_j|$ or $|\pi_i| > |\pi_j|$. We also have the following result.
  - The roots of a simple Lie algebra have, at most, two distinct lengths.
- Swapping any pair of C_{ij} and C_{ji} gives a Cartan matrix which represents the same Lie algebra as the original matrix, with two elements from the Cartan-Weyl basis swapped. This is why we have not included $(-2, -1)$ and $(-3, -1)$ in the table above.

20. **Dynkin diagram:** A **Dynkin diagram** associated to a Cartan matrix is constructed as follows.

    (a) Draw a circle for every fundamental root in $\pi_i \in \Pi$;

    (b) Draw n_{ij} lines between the circles representing the roots π_i and π_j;

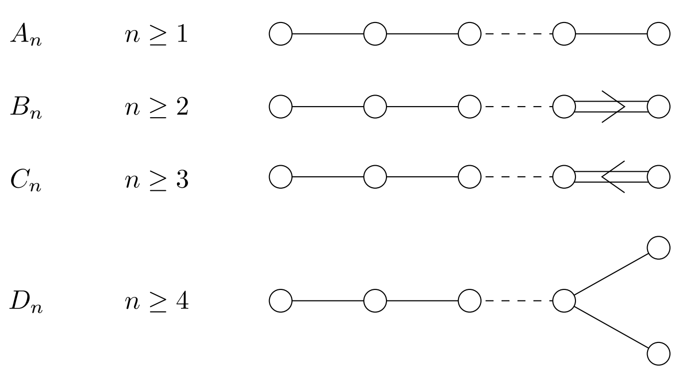
    (c) If $n_{ij} = 2$ or 3, draw an arrow on the lines from the longer root to the shorter root.

    - Dynkin diagrams completely characterise any set of **fundamental roots**, from which we can reconstruct the **entire root set** by using the **Weyl transformations**. The root set can then be used to produce a **Cartan-Weyl basis**. From the basis, of course the full simple Lie algebra is determined.

21. **Cartan classification theorem:** Any **simple finite-dimensional complex Lie algebra** can be reconstructed from its set of fundamental roots Π, which only come in the following **connected** Dynkin diagrams.

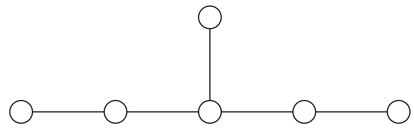
    (a) There are 4 **infinite families**

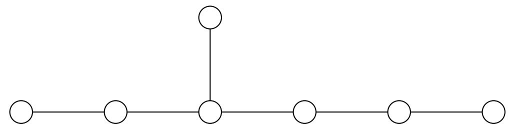
    where the restrictions on n ensure that we don't get repeated diagrams (the diagram D_2 is excluded since it is disconnected and does not correspond to a simple Lie algebra)

    (b) **five exceptional cases**

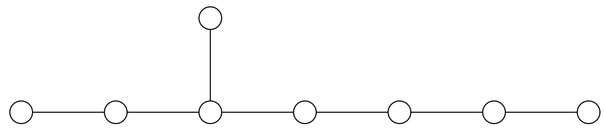
E_6



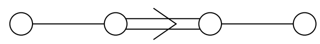
E_7



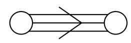
E_8



F_4



G_2



- Note the finite-dimensional **semi-simple** complex Lie algebras are direct sums of simple Lie algebras, and correspond to **disconnected** Dynkin diagrams whose **connected components** are the ones listed above.

22. Here we summarize the main flow for getting the classification of **simple** Lie algebra.

    (a) We first define the **adjoint map** $\mathrm{ad}_x \in \mathrm{End}(L)$ and the **Killing form** κ:

    $$\begin{aligned} \mathrm{ad}_x \colon L &\xrightarrow{\sim} L \\ y &\mapsto \mathrm{ad}_x(y) := [x, y]. \end{aligned}$$

    $$\begin{aligned} \kappa : L \times L &\to K \\ (x, y) &\mapsto \kappa(x, y) := \mathrm{tr}\,(\mathrm{ad}_x \circ \mathrm{ad}_y) \end{aligned}$$

    - Note κ is **nondegenerate** for simple (and semi-simple) Lie algebra.

    (b) We then define the **Cartan subalgebra** H of L which is a maximal Lie subalgebra that has the following property:

    - there exists a basis $\{h_1, \ldots, h_r\}$ of H which can be extended to a basis $\{h_1, \ldots, h_r, e_1, \ldots, e_{d-r}\}$ of L such that $e_1, \ldots, e_{d-r}$ are eigenvectors of $\mathrm{ad}(h)$ for any $h \in H$, i.e.

      $$\forall\, h \in H : \exists\, \lambda_\alpha(h) \in \mathbb{C} : \mathrm{ad}(h) e_\alpha = \lambda_\alpha(h) e_\alpha,$$

      for each $1 \leq \alpha \leq d - r$.

    The basis $\{h_1, \ldots, h_r, e_1, \ldots, e_{d-r}\}$ is known as a **Cartan-Weyl basis** of L.

    - Note, κ is still **nondegenerate** when restrict to H. We can then define κ^* for H^*.

    (c) Note for each α we then have $\lambda_\alpha \in H^*$. The maps $\lambda_1, \ldots, \lambda_{d-r} \in H^*$ are called the **roots** of L. The collection

    $$\Phi := \{\lambda_\alpha \mid 1 \leq \alpha \leq d - r\} \subseteq H^*$$

    is called the **root set** of L.

    - One can show that if λ_α were the zero map, then we would have $e_\alpha \in H$. Thus, we must have $0 \notin \Phi$:

      $$(\forall\, h \in H : [h, x] = 0) \Rightarrow x \in H$$

    - Note that a consequence of the antisymmetry of each $\mathrm{ad}(h)$ w.r.t. the Killing form κ is that

      $$\lambda \in \Phi \Rightarrow -\lambda \in \Phi.$$

      Hence Φ is **not a linearly independent** subset of H^*.

(d) We then define the **fundamental roots** $\Pi := \{\pi_1, \dots, \pi_f\}$ is a subset $\Pi \subseteq \Phi$ such that
 i). Π is a **linearly independent subset** of H^*;
 ii). for each $\lambda \in \Phi$, there exist $n_1, \dots, n_f \in \mathbb{N}$ and $\varepsilon \in \{+1, -1\}$ such that

$$\lambda = \varepsilon \sum_{i=1}^{f} n_i \pi_i$$

 - We have $\operatorname{span}_{\mathbb{C}}(\Pi) = H^*$, that is, the fundamental roots set Π is a basis of H^*.

(e) We now turn our attention to the **real subalgebra** $H^*_{\mathbb{R}} := \operatorname{span}_{\mathbb{R}}(\Pi)$. Note that we have the following chain of inclusions

$$\Pi \subseteq \Phi \subseteq \operatorname{span}_{\varepsilon,\mathbb{N}}(\Pi) \subseteq \underbrace{\operatorname{span}_{\mathbb{R}}(\Pi)}_{H^*_{\mathbb{R}}} \subseteq \underbrace{\operatorname{span}_{\mathbb{C}}(\Pi)}_{H^*}.$$

 - For any $\alpha, \beta \in H^*_{\mathbb{R}}$, we have $\kappa^*(\alpha, \beta) \in \mathbb{R}$.
 - $\kappa^* : H^*_{\mathbb{R}} \times H^*_{\mathbb{R}} \to \mathbb{R}$ is an **inner product** on $H^*_{\mathbb{R}}$.

(f) We then define the **Weyl transformation** and **Weyl group:** For any $\lambda \in \Phi \subseteq H^*_{\mathbb{R}}$, define the linear map

$$\begin{aligned} s_\lambda : H^*_{\mathbb{R}} &\xrightarrow{\sim} H^*_{\mathbb{R}} \\ \mu &\mapsto s_\lambda(\mu), \end{aligned}$$

where

$$s_\lambda(\mu) := \mu - 2\frac{\kappa^*(\lambda, \mu)}{\kappa^*(\lambda, \lambda)}\lambda.$$

The map s_λ is called a Weyl transformation and the set

$$W := \{s_\lambda \mid \lambda \in \Phi\}$$

is a group under composition of maps, called the Weyl group.

 - Weyl group has good properties:
 Every root can be produced from a fundamental root by the action of W, i.e.

$$\forall\, \lambda \in \Phi : \exists\, \pi \in \Pi : \exists\, w \in W : \lambda = w(\pi)$$

 The Weyl group permutes the roots, that is,

$$\forall\, \lambda \in \Phi : \forall\, w \in W : w(\lambda) \in \Phi.$$

(g) We now can define **Cartan matrix:**

$$s_{\pi_i}(\pi_j) := \pi_j - 2\frac{\kappa^*(\pi_i, \pi_j)}{\kappa^*(\pi_i, \pi_i)}\pi_i$$

Since $s_{\pi_i}(\pi_j) \in \Phi$ and $\Phi \subseteq \operatorname{span}_{\varepsilon,\mathbb{N}}(\Pi)$, for all $1 \leq i \neq j \leq r$ we must have

$$-2\frac{\kappa^*(\pi_i, \pi_j)}{\kappa^*(\pi_i, \pi_i)} \in \mathbb{N}$$

Then, the Cartan matrix of a Lie algebra is the $r \times r$ matrix C with entries

$$C_{ij} := 2\frac{\kappa^*(\pi_i, \pi_j)}{\kappa^*(\pi_i, \pi_i)}$$

 - $C_{ii} = 2$.
 - In general $C_{ij} \leq 0$ if $i \neq j$ and C is not symmetric, but if $C_{ij} = 0$, then necessarily $C_{ji} = 0$.

(h) Given a Cartan matrix C, the ij-th **bond number** is

$$n_{ij} := C_{ij}C_{ji} \qquad \text{(no summation implied).}$$

 - We have $n_{ij} \in \{0, 1, 2, 3\}$. So the only possibilities for $C_{ij} \leq 0$ for $i \neq j$ are listed in the above table.
 - Now, we can state the Dynkin diagram which represent the set of **fundamental roots**, and from which we can reconstruct the **entire root set** by using the **Weyl transformations**. The root set can then be used to produce a **Cartan-Weyl basis**. From the basis, of course the full simple Lie algebra is determined.

16 One Example That Utilize All Previous Section

Recall in the previous section, we go through the following diagram. We now show an example called **relativistic spin group** (or **special linear group of degree 2 over** $\mathbb{C}$) that utilize all the following starting from set and ending with Lie algebra.

$$\begin{array}{ccccccc} \text{set} & \longrightarrow & \text{topological space} & \longrightarrow & \text{topological manifold} & \longrightarrow & \text{differentiable manifold} \\ &&&&&& \downarrow \\ &&&&&& \text{Lie group} \\ &&&&&& \uparrow\downarrow \\ &&&&&& \text{Lie algebra} \end{array}$$

1. $\mathrm{SL}(2,\mathbb{C})$ **as a set:** We define the following subset of $\mathbb{C}^4 := \mathbb{C}\times\mathbb{C}\times\mathbb{C}\times\mathbb{C}$
$$\mathrm{SL}(2,\mathbb{C}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{C}^4 \,\middle|\, ad - bc = 1 \right\},$$
where the array is just an alternative notation for a quadruple.
2. $\mathrm{SL}(2,\mathbb{C})$ **as a topological space:** First establish a topology $(\mathbb{C}, \mathcal{O}_{\mathbb{C}})$ using the open balls: $B_r(z) := \{y \in \mathbb{C} \mid |z-y| < r\}$ and define $\mathcal{O}_{\mathbb{C}}$ implicitly by
$$U \in \mathcal{O}_{\mathbb{C}} \;:\Leftrightarrow\; \forall\, z \in U : \exists\, r > 0 : B_r(z) \subseteq U.$$
We can then equip $\mathbb{C}^4$ with the **product topology** and then we can finally define the **subset topology**
$$\mathcal{O} := (\mathcal{O}_{\mathbb{C}})|_{\mathrm{SL}(2,\mathbb{C})},$$
The pair $(\mathrm{SL}(2,\mathbb{C}), \mathcal{O})$ is a topological space.
   - We have $(\mathbb{C}, \mathcal{O}_{\mathbb{C}}) \cong_{\mathrm{top}} (\mathbb{R}^2, \mathcal{O}_{\mathrm{std}})$.
   - It is a **connected topological space**, and we will need this property later on.
3. $\mathrm{SL}(2,\mathbb{C})$ **as a topological manifold:** Recall that a topological space $(M, \mathcal{O})$ is a complex topological manifold if each point $p \in M$ has an open neighbourhood $U(p)$ which is **homeomorphic** to an open subset of $\mathbb{C}^d$. Equivalently, there must exist a $\mathcal{C}^0$**-atlas** $\mathscr{A}$ **of charts** (U_α, x_α), where the U_α are open and cover M and each x is a **homeomorphism** onto a subset of $\mathbb{C}^d$.
   (a) **first chart:** Let U be the set
   $$U := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}) \,\middle|\, a \neq 0 \right\}$$
   and define the map
   $$\begin{aligned} x : \quad & U \to x(U) \subseteq \mathbb{C}^* \times \mathbb{C} \times \mathbb{C} \\ & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a, b, c), \end{aligned}$$
   where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. One can show that U is an open subset of $(\mathrm{SL}(2,\mathbb{C}), \mathcal{O})$ and x is a homeomorphism with inverse
   $$\begin{aligned} x^{-1} : \quad & x(U) \to U \\ & (a, b, c) \mapsto \begin{pmatrix} a & b \\ c & \frac{1+bc}{a} \end{pmatrix}. \end{aligned}$$
   (b) **second chart:** However, since $\mathrm{SL}(2,\mathbb{C})$ contains elements with $a = 0$, the chart (U, x) does not cover the whole space, and hence we need at least one more chart. We thus define the set
   $$V := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}) \,\middle|\, b \neq 0 \right\}$$

and the map

$$\begin{aligned} y: \quad & V \to x(V) \subseteq \mathbb{C} \times \mathbb{C}^* \times \mathbb{C} \\ & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a, b, d). \end{aligned}$$

Similarly to the above, V is open and y is a homeomorphism with inverse

$$\begin{aligned} y^{-1}: \quad & x(V) \to V \\ & (a, b, d) \mapsto \begin{pmatrix} a & b \\ \frac{ad-1}{b} & d \end{pmatrix}. \end{aligned}$$

- An element of $\mathrm{SL}(2, \mathbb{C})$ cannot have both a and b equal to zero, for otherwise $ad - bc = 0 \neq 1$.

(c) **atlas:** So now we have $\mathscr{A}_{\text{top}} := \{(U, x), (V, y)\}$ is a $\mathcal{C}^0$-atlas, the triple $(\mathrm{SL}(2, \mathbb{C}), \mathcal{O}, \mathscr{A}_{\text{top}})$ is a **3-dimensional, complex, topological manifold.**

- **maximal atlas:** If needed, we can equip it with the **maximal atlas** that is $\mathcal{C}^0$-compatible with $\mathscr{A}_{\text{top}}$.

4. $\mathrm{SL}(2, \mathbb{C})$ **as a complex differentiable manifold:** We have to check that every **transition map** between charts in $\mathscr{A}_{\text{top}} := \{(U, x), (V, y)\}$ is differentiable:

$$\begin{aligned} y \circ x^{-1}: \ & x(U \cap V) \to y(U \cap V) \\ & (a, b, c) \mapsto y(\begin{pmatrix} a & b \\ c & \frac{1+bc}{a} \end{pmatrix}) = (a, b, \tfrac{1+bc}{a}). \end{aligned}$$

Similarly, we have

$$\begin{aligned} x \circ y^{-1}: \ & y(U \cap V) \to x(U \cap V) \\ & (a, b, c) \mapsto y(\begin{pmatrix} a & b \\ \frac{ad-1}{b} & d \end{pmatrix}) = (a, b, \tfrac{ad-1}{b}). \end{aligned}$$

The transition maps are complex **differentiable**. Therefore, the atlas $\mathscr{A}_{\text{top}}$ is a **differentiable atlas.**

- By defining $\mathscr{A}$ to be the **maximal differentiable atlas** containing $\mathscr{A}_{\text{top}}$, we have that $(\mathrm{SL}(2, \mathbb{C}), \mathcal{O}, \mathscr{A})$ is a **3-dimensional, complex differentiable manifold.**

5. $\mathrm{SL}(2, \mathbb{C})$ **as a (non-commutative) group:** We define an operation

$$\begin{aligned} \bullet: \ & \mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C}) \to \mathrm{SL}(2, \mathbb{C}) \\ & (\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} e & f \\ g & h \end{pmatrix}) \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \bullet \begin{pmatrix} e & f \\ g & h \end{pmatrix}, \end{aligned}$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \bullet \begin{pmatrix} e & f \\ g & h \end{pmatrix} := \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}.$$

- This operation $\bullet$ is the same as matrix multiplication. The closeness of the $\bullet$ follows from the determinant of a product is the product of the determinants.
- Other conditions for a group like the associative, identity element and inverse can also be checked.
  * **identity element:** $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C})$ is the **identity element**.
  * **inverse:** We have $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

6. $\mathrm{SL}(2, \mathbb{C})$ **as a Lie group:** We equipped $\mathrm{SL}(2, \mathbb{C})$ with **both a group and a manifold** structure. In order to obtain a Lie group structure, we have to check that these following two maps

$$\begin{aligned} \mu: \ & \mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C}) \to \mathrm{SL}(2, \mathbb{C}) \\ & (\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} e & f \\ g & h \end{pmatrix}) \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \bullet \begin{pmatrix} e & f \\ g & h \end{pmatrix} \end{aligned}$$

and

$$
\begin{aligned}
i:\ \mathrm{SL}(2,\mathbb{C}) &\to \mathrm{SL}(2,\mathbb{C})\\
\begin{pmatrix} a & b\\ c & d\end{pmatrix} &\mapsto \begin{pmatrix} a & b\\ c & d\end{pmatrix}^{-1}
\end{aligned}
$$

are differentiable w.r.t. the differentiable structure on $\mathrm{SL}(2,\mathbb{C})$.

- For the **inverse map** i, we have to show that the map $y \circ i \circ x^{-1}$ is differentiable in the usual for any pair of charts $(U,x),(V,y) \in \mathscr{A}$.

$$
\begin{tikzcd}
U \subseteq \mathrm{SL}(2,\mathbb{C}) \arrow[r, "i"] \arrow[d, "x"] & V \subseteq \mathrm{SL}(2,\mathbb{C}) \arrow[d, "y"] \\
x(U) \subseteq \mathbb{C}^3 \arrow[r, "y\circ i\circ x^{-1}"] & y(V) \subseteq \mathbb{C}^3
\end{tikzcd}
$$

Since $\mathrm{SL}(2,\mathbb{C})$ is **connected**, the differentiability of the transition maps in $\mathscr{A}$ implies that if $y \circ i \circ x^{-1}$ is differentiable for any two given charts, then it is differentiable for all charts in $\mathscr{A}$. Hence, we can simply let (U,x) and (V,y) be the two charts on $\mathrm{SL}(2,\mathbb{C})$ defined above. Then, we have

$$
(y \circ i \circ x^{-1})(a,b,c) = (y \circ i)\Big(\begin{pmatrix} a & b\\ c & \frac{1+bc}{a}\end{pmatrix}\Big) = y\Big(\begin{pmatrix} \frac{1+bc}{a} & -b\\ -c & a\end{pmatrix}\Big) = \big(\tfrac{1+bc}{a}, -b, a\big)
$$

which is certainly complex differentiable as a map between open subsets of $\mathbb{C}^3$ (recall that $a \neq 0$ on $x(U)$).

- Checking that μ is complex differentiable is slightly more involved, since we first have to equip $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$ with a suitable "product differentiable structure" and then proceed as above. We do not include the details here.

We can finally conclude that $((\mathrm{SL}(2,\mathbb{C}),\mathcal{O},\mathscr{A}),\bullet)$ is a **3-dimensional complex Lie group.**

7. **The Lie algebra of** $\mathrm{SL}(2,\mathbb{C})$**:** Recall that to every Lie group G, there is an associated **left-invariant Lie algebra** $\mathcal{L}(G)$, where

$$
\mathcal{L}(G) := \{X \in \Gamma(TG) \mid \forall\, g,h \in G : (\ell_g)_*(X|_h) = X_{gh}\},
$$

which we then proved to be isomorphic to the Lie algebra T_eG with Lie bracket

$$
[A,B]_{T_eG} := j^{-1}([j(A),j(B)]_{\mathcal{L}(G)})
$$

induced by the Lie bracket on $\mathcal{L}(G)$ via the isomorphism j

$$
j(A)|_g := (\ell_g)_*(A).
$$

In the case of $\mathrm{SL}(2,\mathbb{C})$, the left translation map by $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$ is

$$
\begin{aligned}
\ell_{\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)}:\ \mathrm{SL}(2,\mathbb{C}) &\to \mathrm{SL}(2,\mathbb{C})\\
\begin{pmatrix} e & f\\ g & h\end{pmatrix} &\mapsto \begin{pmatrix} a & b\\ c & d\end{pmatrix} \bullet \begin{pmatrix} e & f\\ g & h\end{pmatrix}
\end{aligned}
$$

- **notation:** We use the standard notation $\mathfrak{sl}(2,\mathbb{C}) \coloneqq \mathcal{L}(\mathrm{SL}(2,\mathbb{C}))$, and we have

$$
\mathfrak{sl}(2,\mathbb{C}) \cong_{\text{Lie alg}} T_{\left(\begin{smallmatrix} 1 & 0\\ 0 & 1\end{smallmatrix}\right)} \mathrm{SL}(2,\mathbb{C}).
$$

8. **local chart representation of** $\mathfrak{sl}(2,\mathbb{C})$**:**

   (a) **select a chart:** Recall that if (U,x) is a chart on a manifold M and $p \in U$, then the chart (U,x) induces a basis of the tangent space T_pM. We shall use our previously defined chart (U,x) on $\mathrm{SL}(2,\mathbb{C})$ as a neighbour of the identity element, where $U := \{\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{C}) \mid a \neq 0\}$ and

$$
\begin{aligned}
x:\quad U &\to x(U) \subseteq \mathbb{C}^3\\
\begin{pmatrix} a & b\\ c & d\end{pmatrix} &\mapsto (a,b,c).
\end{aligned}
$$

We will keep writing the d to avoid having a fraction in a matrix in a subscript. We get an induced coordinate basis for the tangent space:

$$\left\{ \left(\frac{\partial}{\partial x^i}\right)_{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)} \in T_{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)} \mathrm{SL}(2,\mathbb{C}) \,\middle|\, 1 \le i \le 3 \right\}$$

so that any $A \in T_{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)} \mathrm{SL}(2,\mathbb{C})$ can be written as

$$A = \alpha \left(\frac{\partial}{\partial x^1}\right)_{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)} + \beta \left(\frac{\partial}{\partial x^2}\right)_{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)} + \gamma \left(\frac{\partial}{\partial x^3}\right)_{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)},$$

for some $\alpha, \beta, \gamma \in \mathbb{C}$.

(b) **objective:** Since the Lie bracket is **bilinear**, its action on these basis vectors uniquely extends to the whole of $T_{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)} \mathrm{SL}(2,\mathbb{C})$ by linear continuation. Hence, we simply have to determine the action of the Lie bracket of $\mathfrak{sl}(2,\mathbb{C})$ on the images under the isomorphism j of **these basis vectors**.

(c) After tedious computation, we have an expansion of the images of the basis of $T_{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)} \mathrm{SL}(2,\mathbb{C})$ under j:

$$\begin{aligned}
j\left(\left(\frac{\partial}{\partial x^1}\right)_{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)}\right) &= x^1 \frac{\partial}{\partial x^1} - x^2 \frac{\partial}{\partial x^2} + x^3 \frac{\partial}{\partial x^3} \\
j\left(\left(\frac{\partial}{\partial x^2}\right)_{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)}\right) &= x^1 \frac{\partial}{\partial x^2} \\
j\left(\left(\frac{\partial}{\partial x^3}\right)_{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)}\right) &= x^2 \frac{\partial}{\partial x^1} + \tfrac{1+x^2x^3}{x^1} \frac{\partial}{\partial x^3}.
\end{aligned}$$

(d) **calculate the bracket in $\mathfrak{sl}(2,\mathbb{C})$ of every pair:** We use the following

$$\left[j\left(\left(\frac{\partial}{\partial x^i}\right)_{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)}\right), j\left(\left(\frac{\partial}{\partial x^k}\right)_{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)}\right) \right] = \left[D^m{}_i \frac{\partial}{\partial x^m}, D^n{}_k \frac{\partial}{\partial x^n} \right].$$

Expanding the above and then **applying** j^{-1} (recall this is the just evaluation at the identity), to these vector fields, we finally see that the induced Lie bracket on $T_{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)} \mathrm{SL}(2,\mathbb{C})$ satisfies

$$\begin{aligned}
\left[\left(\frac{\partial}{\partial x^1}\right)_{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)}, \left(\frac{\partial}{\partial x^2}\right)_{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)} \right] &= 2 \left(\frac{\partial}{\partial x^2}\right)_{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)} \\
\left[\left(\frac{\partial}{\partial x^1}\right)_{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)}, \left(\frac{\partial}{\partial x^3}\right)_{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)} \right] &= -2 \left(\frac{\partial}{\partial x^3}\right)_{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)} \\
\left[\left(\frac{\partial}{\partial x^2}\right)_{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)}, \left(\frac{\partial}{\partial x^3}\right)_{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)} \right] &= \left(\frac{\partial}{\partial x^1}\right)_{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)}.
\end{aligned}$$

Hence, the structure constants of $T_{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)} \mathrm{SL}(2,\mathbb{C})$ w.r.t. the coordinate basis are

$$C^2{}_{12} = 2, \qquad C^3{}_{13} = -2, \qquad C^1{}_{23} = 1,$$

with all other being either zero or related to these by antisymmetry.

17 Dynkin Diagrams from Lie algebras, and Vice Versa

We first use the example $\mathfrak{sl}(2,\mathbb{C})$ from Section 16 to construct its associated Dynkin diagram. Then from a given Dynkin diagram, we restore a Lie algebra.

1. $\mathfrak{sl}(2,\mathbb{C})$ **is a simple Lie algebra:** Denote the basis from the end of Section 16 as $\{X_1, X_2, X_2\}$ of $\mathfrak{sl}(2,\mathbb{C})$, w.r.t. which we have

$$
\begin{aligned}
[X_1, X_2] &= 2X_2, \\
[X_1, X_3] &= -2X_3, \\
[X_2, X_3] &= X_1.
\end{aligned}
$$

(a) **Killing form of** $\mathfrak{sl}(2,\mathbb{C})$**:** Recall that it has components

$$\kappa_{ij} = C^m{}_{in} C^n{}_{jm},$$

with all indices ranging from 1 to 3. Since κ is symmetric, we only need to determine κ_{ij} for $i \leq j$. By writing the components in a 3×3 array, we find

$$[\kappa_{ij}] = \begin{pmatrix} 8 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 8 \end{pmatrix},$$

- The Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ is semi-simple. Since the diagonal entries of κ are all non-zero, the Killing form is non-degenerate. By Cartan's criterion, this implies that $\mathfrak{sl}(2,\mathbb{C})$ is semi-simple.
- Killing form on the Lie algebra of a compact Lie group is always negative semi-definite. Hence, we can conclude that $\mathrm{SL}(2,\mathbb{C})$ is **not** a compact Lie group.

(b) **simple:** The Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ is **simple** (not only semi-simple). Recall that a Lie algebra is said to be simple if it contains no non-trivial ideals. Assume we have the following ideal:

$$I := \{\alpha X_1 + \beta X_2 + \gamma X_3 \mid \alpha, \beta, \gamma \text{ restricted so that } I \text{ is an ideal}\}.$$

Since the bracket is bilinear, it suffices to check the result of bracketing an arbitrary element of I with each of the basis vectors of $\mathfrak{sl}(2,\mathbb{C})$. We find

$$
\begin{aligned}
[\alpha X_1 + \beta X_2 + \gamma X_3, X_1] &= -2\beta X_1 + 2\gamma X_3, \\
[\alpha X_1 + \beta X_2 + \gamma X_3, X_2] &= 2\alpha X_2 - \gamma X_1, \\
[\alpha X_1 + \beta X_2 + \gamma X_3, X_3] &= -2\alpha X_3 + \beta X_1.
\end{aligned}
$$

We need to choose α, β, γ so that the results always land back in I. Of course, we can choose $\alpha, \beta, \gamma \in \mathbb{C}$ and $\alpha = \beta = \gamma = 0$, which correspond respectively to the trivial ideals $\mathfrak{sl}(2,\mathbb{C})$ and 0. If none of α, β, γ is zero, then you can check that the right hand sides above are linearly independent, so that I contains three linearly independent vectors. Since the only n-dimensional subspace of an n-dimensional vector space is the vector space itself, we have $I = L$. Thus, we are left with the following cases:

i) if $\alpha = 0$, then $I \subseteq \mathrm{span}_{\mathbb{C}}(\{X_2, X_3\})$ and hence we must have $\beta = \gamma = 0$ as well;
ii) if $\beta = 0$, then $I \subseteq \mathrm{span}_{\mathbb{C}}(\{X_1, X_3\})$, hence we must have $\alpha = 0$, so that in fact $I \subseteq \mathrm{span}_{\mathbb{C}}(\{X_3\})$, and hence $\gamma = 0$ as well;
iii) if $\gamma = 0$, then $I \subseteq \mathrm{span}_{\mathbb{C}}(\{X_1, X_2\})$, hence we must have $\alpha = 0$, so that in fact $I \subseteq \mathrm{span}_{\mathbb{C}}(\{X_2\})$, and hence $\beta = 0$ as well.

In all cases, we have $I = 0$. Therefore, there are no non-trivial ideals of $\mathfrak{sl}(2,\mathbb{C})$.

(c) **roots:** For any $h \in H$, there exists a $\xi \in \mathbb{C}$ such that $h = \xi X_1$, and hence we have

$$
\begin{aligned}
\mathrm{ad}(h)X_2 &= \xi[X_1, X_2] = 2\xi X_2, \\
\mathrm{ad}(h)X_3 &= \xi[X_1, X_3] = -2\xi X_3.
\end{aligned}
$$

Recall that in the section on Lie algebras, we re-interpreted these eigenvalue equations in terms of functionals $\lambda_2, \lambda_3 \in H^*$

$$
\begin{aligned}
\lambda_2 : \quad H &\xrightarrow{\sim} \mathbb{C} & \qquad \lambda_3 : \quad H &\xrightarrow{\sim} \mathbb{C} \\
\xi X_1 &\mapsto 2\xi, & \xi X_1 &\mapsto -2\xi
\end{aligned}
$$

Then, the **root set** is $\Phi = \{\lambda_2, \lambda_3\}$.

(d) **fundamental roots:** Recall that $\Pi \subset \Phi$ as the fundamental root set satisfies

i) Π is a linearly independent subset of H^*;
ii) for any $\lambda \in \Phi$, we have $\lambda \in \mathrm{span}_{\epsilon,\mathbb{N}}(\Pi)$.

We can choose $\Pi := \{\lambda_2\}$.

(e) **Weyl group:** Since $|\Pi| = 1$, the **Weyl group** is generated by the single **Weyl transformation**

$$
\begin{aligned}
s_{\lambda_2} : \ H^*_{\mathbb{R}} &\to H^*_{\mathbb{R}} \\
\mu &\mapsto \mu - 2\frac{\kappa^*(\lambda_2, \mu)}{\kappa^*(\lambda_2, \lambda_2)}\lambda_2.
\end{aligned}
$$

- Recall that we can **recover the entire root** set Φ by acting on the fundamental roots with Weyl transformations. Indeed, we have

$$s_{\lambda_2}(\lambda_2) = \lambda_2 - 2\frac{\kappa^*(\lambda_2, \lambda_2)}{\kappa^*(\lambda_2, \lambda_2)}\lambda_2 = \lambda_2 - 2\lambda_2 = -\lambda_2 = \lambda_3,$$

as expected.

(f) **Cartan matrix:** Since there is only one fundamental root, the **Cartan matrix** is actually just a 1×1 matrix.

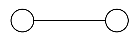
$$C = (2).$$

(g) **Dynkin diagram:** The **Dynkin diagram** of $\mathfrak{sl}(2, \mathbb{C})$ is simply

- Hence, with reference to the Cartan classification, we have $A_1 = \mathfrak{sl}(2, \mathbb{C})$.

2. **reconstruction of A_2 from its Dynkin diagram:**

(a) **Dynkin diagram:** We will start from the Dynkin diagram



(b) **two fundamental roots:** We immediately see that we have two fundamental roots, i.e. $\Pi = \{\pi_1, \pi_2\}$, since there are two circles in the diagram.

(c) **Cartan matrix:** The **bond number** is $n_{12} = C_{12}C_{21} = 1$, so the two fundamental roots have the same length. Recall Cartan matrix elements are **non-positive integers**, the only possibility is $C_{12} = C_{21} = -1$, so that we have

$$C = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

(d) **fundamental roots from Cartan matrix:** To determine the angle φ between π_1 and π_2, recall that

$$1 = n_{12} = 4\cos^2\varphi,$$

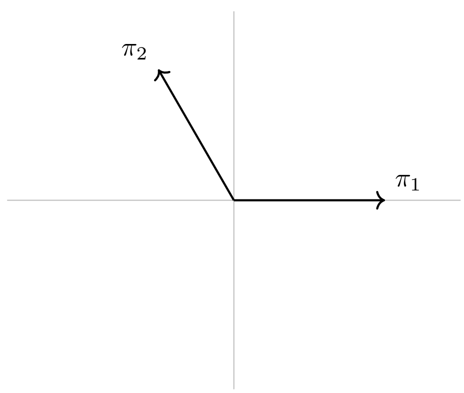
and hence $|\cos\varphi| = \frac{1}{2}$. By definition, we have

$$\cos\varphi = \frac{\kappa^*(\pi_1, \pi_2)}{|\pi_1|\,|\pi_2|},$$

and therefore

$$0 > C_{12} = 2\frac{\kappa^*(\pi_1, \pi_2)}{\kappa^*(\pi_1, \pi_1)} = 2\frac{|\pi_1|\,|\pi_2|\cos\varphi}{\kappa^*(\pi_1, \pi_1)} = 2\frac{|\pi_2|}{|\pi_1|}\cos\varphi.$$

It follows that $\cos\varphi < 0$, and hence $\varphi = 120°$. We can thus plot the two fundamental roots in a 2-dim **real** plane $H^*_{\mathbb{R}}$ as follows.



- Since $H^* = \mathrm{span}_{\mathbb{C}}(\Pi)$, we have $\dim H^* = 2$, thus the **dimension of the Cartan subalgebra** H is also 2.
- Since we are deal with a real vector space $H^*_{\mathbb{R}}$, from vector isomorphism, we can therefore using the 2-dim real plane (**not** the complex plane).

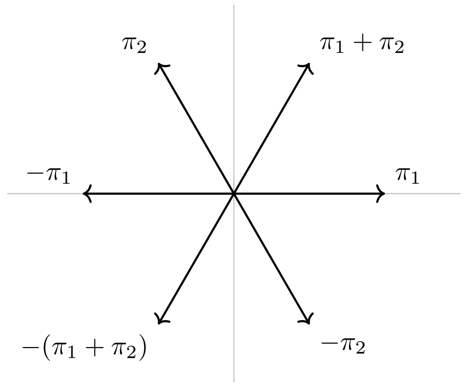
- We can select any axis to build the plane coordinate system. Note, finally, we can only get a Lie algebra **up to Lie algebra isomorphism**.
- Please note, each vector in the plane represent a **linear functional**!

(e) **entire root set using Weyl group:** We can determine all the other roots in Φ by repeated action of the **Weyl group**. For instance, we easily find that $s_{\pi_1}(\pi_1) = -\pi_1$ and $s_{\pi_2}(\pi_2) = -\pi_2$. We also have

$$s_{\pi_1}(\pi_2) = \pi_2 - 2\frac{\kappa^*(\pi_1, \pi_2)}{\kappa^*(\pi_1, \pi_1)}\pi_1 = \pi_2 - 2(-\tfrac{1}{2})\pi_1 = \pi_1 + \pi_2.$$

Finally, we have $s_{\pi_1+\pi_2}(\pi_1 + \pi_2) = -(\pi_1 + \pi_2)$. Any further action by **Weyl transformations** simply permutes these roots. Hence, we have the **entire root set**

$$\Phi = \{\pi_1, -\pi_1, \pi_2, -\pi_2, \pi_1 + \pi_2, -(\pi_1 + \pi_2)\}$$



- **Cartan-Weyl basis:** Since $|\Phi| = 6$, we know that any **Cartan-Weyl basis** of the Lie algebra A_2 must have $2 + 6 = 8$ elements. Hence, the dimension of A_2 is 8.

(f) **Lie bracket behaviour–basis representation with structure constants:** To complete our reconstruction of A_2, we would now like to understand how its bracket behaves, i.e. we need to compute the **structure constants**:

$$[E_i, E_j] = C^k{}_{ij} E_k$$

Note that since $\dim A_2 = 8$, the structure constants $C^k{}_{ij}$ consist of $8^3 = 512$ complex numbers. But a lot of them are redundant.

Denote by $\{h_1, h_2, e_3, \ldots, e_8\}$ a Cartan-Weyl basis of A_2, so that $H = \mathrm{span}_{\mathbb{C}}(\{h_1, h_2\})$ and the e_α are eigenvectors for every $h \in H$.

- Since A_2 is simple, H is **abelian** and hence

$$[h_1, h_2] = 0 \quad \Rightarrow \quad C^k{}_{12} = C^k{}_{21} = 0, \quad \forall\, 1 \leq k \leq 8.$$

- To each e_α, for $3 \leq \alpha \leq 8$, there is an associated $\lambda_\alpha \in \Phi$ such that

$$\forall\, h \in H : \ \mathrm{ad}(h)e_\alpha = \lambda_\alpha(h)e_\alpha.$$

In particular, for the basis elements h_1, h_2,

$$[h_1, e_\alpha] = \mathrm{ad}(h_1)e_\alpha = \lambda_\alpha(h_1)e_\alpha,$$
$$[h_2, e_\alpha] = \mathrm{ad}(h_2)e_\alpha = \lambda_\alpha(h_2)e_\alpha,$$

so that we have

$$C^1{}_{1\alpha} = C^2{}_{1\alpha} = 0, \quad C^\alpha{}_{1\alpha} = \lambda_\alpha(h_1), \quad \forall\, 3 \leq \alpha \leq 8,$$
$$C^1{}_{2\alpha} = C^2{}_{2\alpha} = 0, \quad C^\alpha{}_{2\alpha} = \lambda_\alpha(h_2), \quad \forall\, 3 \leq \alpha \leq 8.$$

- Finally, we need to determine $[e_\alpha, e_\beta]$. By using the Jacobi identity, we have

$$\begin{aligned}
[h, [e_\alpha, e_\beta]] &= -[e_\alpha, [e_\beta, h]] - [e_\beta, [h, e_\alpha]] \\
&= -[e_\alpha, -\lambda_\beta(h)e_\beta] - [e_\beta, \lambda_\alpha(h)e_\alpha] \\
&= \lambda_\beta(h)[e_\alpha, e_\beta] + \lambda_\alpha(h)[e_\alpha, e_\beta] \\
&= (\lambda_\alpha(h) + \lambda_\beta(h))[e_\alpha, e_\beta],
\end{aligned}$$

that is,

$$\mathrm{ad}(h)[e_\alpha, e_\beta] = (\lambda_\alpha(h) + \lambda_\beta(h))[e_\alpha, e_\beta] \text{ for all } h.$$

* If $\lambda_\alpha + \lambda_\beta = \lambda_\gamma \in \Phi$ for some $3 \le \gamma \le 8$, we have $[e_\alpha, e_\beta] = \xi e_\gamma$ for some $\xi \in \mathbb{C}$. Let us label the roots in our previous plot as

λ_3	λ_4	λ_5	λ_6	λ_7	λ_8
π_1	π_2	$\pi_1 + \pi_2$	$-\pi_1$	$-\pi_2$	$-(\pi_1 + \pi_2)$

Then, for example

$$\mathrm{ad}(h)[e_3, e_4] = (\pi_1 + \pi_2)(h)[e_3, e_4],$$

and hence $[e_3, e_4]$ is an eigenvector of $\mathrm{ad}(h)$ with eigenvalues $(\pi_1 + \pi_2)(h)$. But so is e_5! Hence, we must have $[e_3, e_4] = \xi e_5$ for some $\xi \in \mathbb{C}$. Similarly, $[e_5, e_7] = \xi e_3$, and so on.

* If $\lambda_\alpha + \lambda_\beta \notin \Phi$ and $\neq 0$, then in order for the equation above to hold, we must have $[e_\alpha, e_\beta] = 0$. This is because the root set has covered all the eigenvalues and eigenvector pairs. If for example $\mathrm{ad}(h)(ae_1 + be_2) = a\,\mathrm{ad}(h)e_1 + b\,\mathrm{ad}(h)e_2 = a\pi_1(h)e_1 + b\pi_2(h)e_2 \neq (\pi_1(h) + \pi_2(h))(ae_1 + be_2)$.
* If $\lambda_\alpha(h) + \lambda_\beta(h) = 0$ for all h, i.e. $\lambda_\alpha + \lambda_\beta = 0$ as a functional. Then we must have $[e_\alpha, e_\beta] \in H$. This follows from the **maximality property** of the Cartan subalgebra H.

Summarising, we have

$$[e_\alpha, e_\beta] = \begin{cases} \xi e_\gamma & \text{if } \lambda_\alpha + \lambda_\beta = \lambda_\gamma \in \Phi \\ \in H & \text{if } \lambda_\alpha + \lambda_\beta = 0 \\ 0 & \text{otherwise} \end{cases}$$

and these relations con be used to determine the remaining structure constants.

18 Representation theory of Lie groups and Lie algebras

In previous section, we took a more abstract approach to define a Lie group its associated Lie algebra. Representation theory is a branch of mathematics that studies abstract algebraic structures by representing their elements as **linear transformations of vector spaces**. In essence, a representation makes an abstract algebraic object more concrete by describing its elements by matrices and their algebraic operations (for example, matrix addition, matrix multiplication). The **theory of matrices and linear operators** is well-understood, so representations of more abstract objects in terms of familiar linear algebra objects helps glean properties and sometimes simplify calculations on more abstract theories.

A summary:

- ► We give the definition of **representation of Lie algebra** as $\mathrm{End}(V)$ for a vector space V and the **adjoint representation** of Lie algebra.
- ► We give the definition of **Casimir operator associated with a representation of Lie algebra**.
- ► We give the definition of **representation of Lie group** as $\mathrm{GL}(V)$ and the **Adjoint map**. The representation of adjoint representation in the general linear group is shown in the matrix form. The relation between adjoint representation of Lie group and its algebra is stated.

1. **Ado's theorem:** Every finite-dimensional Lie algebra L over a field K of characteristic zero can be viewed as a **Lie algebra of square matrices under the commutator bracket**. More precisely, L has a linear representation ρ over K, on a finite-dimensional vector space V, that is a **faithful** representation, making L **Lie algebra isomorphic to a subalgebra of the endomorphisms of V, i.e.** $\mathrm{End}(V)$.
2. **representation of Lie algebra:** Let L be a Lie algebra. A **representation** of L is a **Lie algebra homomorphism**

$$\rho : L \xrightarrow{\sim} \mathrm{End}(V),$$

where V is some finite-dimensional vector space over the same field as L. In other words,

$$\forall\, x, y \in L : \rho([x, y]) = [\rho(x), \rho(y)] := \rho(x) \circ \rho(y) - \rho(y) \circ \rho(x).$$

- Note, on $\mathrm{End}(V)$, the natural Lie bracket is $[a, b] = a \circ b - b \circ a$ where the $\circ$ denotes the composition of endomorphisms.
- **representation space:** The vector space V is called the **representation space** of ρ.

- **dimension of representation:** The **dimension** of the representation ρ is $\dim V$.
- Example: Consider the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$. We constructed a basis $\{X_1, X_2, X_3\}$ satisfying the relations

$$\begin{aligned}
[X_1, X_2] &= 2X_2, \\
[X_1, X_3] &= -2X_3, \\
[X_2, X_3] &= X_1.
\end{aligned}$$

Let $\rho : \mathfrak{sl}(2,\mathbb{C}) \xrightarrow{\sim} \mathrm{End}(\mathbb{C}^2)$ be the linear map defined by

$$\rho(X_1) := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \rho(X_2) := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \rho(X_3) := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

From linearity, we only to check that ρ is a representation of $\mathfrak{sl}(2,\mathbb{C})$, we calculate

$$[\rho(X_1), \rho(X_2)] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \rho([X_1, X_2]).$$

Similarly, we find

$$\begin{aligned}
[\rho(X_1), \rho(X_3)] &= \rho([X_1, X_3]), \\
[\rho(X_2), \rho(X_3)] &= \rho([X_2, X_3]).
\end{aligned}$$

By linear continuation, $\rho([x,y]) = [\rho(x), \rho(y)]$ for any $x, y \in \mathfrak{sl}(2,\mathbb{C})$ and hence, ρ is a 2-**dimensional representation** of $\mathfrak{sl}(2,\mathbb{C})$ with representation space $\mathbb{C}^2$. Note that we have

$$\begin{aligned}
\mathrm{im}_\rho(\mathfrak{sl}(2,\mathbb{C})) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{End}(\mathbb{C}^2) \;\middle|\; a + d = 0 \right\} \\
&= \{\phi \in \mathrm{End}(\mathbb{C}^2) \mid \mathrm{tr}\,\phi = 0\}.
\end{aligned}$$

This is how $\mathfrak{sl}(2,\mathbb{C})$ is often defined in physics courses, i.e. as the algebra of 2×2 complex traceless matrices.

3. **homomorphism of representations:** Let L be a Lie algebra and let

$$\rho_1 : L \xrightarrow{\sim} \mathrm{End}(V_1), \qquad \rho_2 : L \xrightarrow{\sim} \mathrm{End}(V_2)$$

be representations of L. A linear map $f : V_1 \xrightarrow{\sim} V_2$ is a **homomorphism of representations** if

$$\forall\, x \in L : \; f \circ \rho_1(x) = \rho_2(x) \circ f.$$

- **equivalent statement:** Equivalently, if the following diagram commutes for all $x \in L$.

$$\begin{tikzcd}
V_1 \arrow[r, "f"] \arrow[d, "\rho_1(x)"'] & V_2 \arrow[d, "\rho_2(x)"] \\
V_1 \arrow[r, "f"] & V_2
\end{tikzcd}$$

- **inverse is also homomorphism of representations:** If in addition $f : V_1 \xrightarrow{\sim} V_2$ is a linear isomorphism, then $f^{-1} : V_2 \xrightarrow{\sim} V_1$ is automatically a homomorphism of representations:

$$\begin{aligned}
f \circ \rho_1(x) = \rho_2(x) \circ f \;&\Leftrightarrow\; f^{-1} \circ (f \circ \rho_1(x)) \circ f^{-1} = f^{-1} \circ (\rho_2(x) \circ f) \circ f^{-1} \\
&\Leftrightarrow\; \rho_1(x) \circ f^{-1} = f^{-1} \circ \rho_2(x).
\end{aligned}$$

- **isomorphism of representations:** An **isomorphism of representations** of Lie algebras is a **bijective homomorphism of representations**.
  - $*$ Isomorphic representations necessarily have the **same dimension**.
- Example: $\mathfrak{so}(3,\mathbb{R})$ is the Lie algebra of the rotation group $\mathrm{SO}(3,\mathbb{R})$. It has a basis $\{J_1, J_2, J_3\}$ satisfying

$$\begin{aligned}
[J_1, J_2] &= J_3, \\
[J_2, J_3] &= J_1, \\
[J_3, J_1] &= J_2.
\end{aligned}$$

(a) **representation I:** Define a linear map $\rho_{\mathrm{vec}} : \mathfrak{so}(3,\mathbb{R}) \xrightarrow{\sim} \mathrm{End}(\mathbb{R}^3)$ by

$$\rho_{\mathrm{vec}}(J_1) := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \rho_{\mathrm{vec}}(J_2) := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad \rho_{\mathrm{vec}}(J_3) := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

You can easily check that this is a representation of $\mathfrak{so}(3,\mathbb{R})$.

(b) **representation II:** Define the linear map

$$\rho_{\text{spin}} : \mathfrak{so}(3, \mathbb{R}) \xrightarrow{\sim} \text{End}(\mathbb{C}^2),$$

with

$$\rho_{\text{spin}}(J_1) := -\frac{\mathrm{i}}{2}\sigma_1, \qquad \rho_{\text{spin}}(J_2) := -\frac{\mathrm{i}}{2}\sigma_2, \qquad \rho_{\text{spin}}(J_3) := -\frac{\mathrm{i}}{2}\sigma_3,$$

where $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

You can again check that this is a representation of $\mathfrak{so}(3, \mathbb{R})$. Note here we need to take $\mathbb{C}^2$ as a 4-dimensional $\mathbb{R}$-vector space.

Since

$$\dim \mathbb{R}^3 = 3 \neq 4 = \dim \mathbb{C}^2,$$

the representations ρ_{vec} and ρ_{spin} are **not** isomorphic.

4. **two special representations:** Any (non-abelian) Lie algebra always has at least two special representations.
   (a) **trivial representation:** Let L be a Lie algebra. A **trivial representation** of L is defined by

$$\begin{aligned} \rho_{\text{trv}} : \; & L \xrightarrow{\sim} \text{End}(V) \\ & x \mapsto \rho_{\text{trv}}(x) := 0, \end{aligned}$$

   where 0 denotes the trivial endomorphism on V. $\forall\, x, y \in L : \; \rho_{\text{trv}}([x, y]) = 0 = [\rho_{\text{trv}}(x), \rho_{\text{trv}}(y)]$.
   (b) **adjoint representation of Lie algebra:** The **adjoint representation** of L is

$$\begin{aligned} \rho_{\text{adj}} : \; & L \xrightarrow{\sim} \text{End}(L) \\ & x \mapsto \rho_{\text{adj}}(x) := \text{ad}(x). \end{aligned}$$

   This is because ad is a **Lie algebra homomorphism**.
5. **faithful:** A representation $\rho : \; L \xrightarrow{\sim} \text{End}(V)$ is called **faithful** if ρ is **injective**, i.e.

$$\dim(\text{im}_\rho(L)) = \dim L.$$

   (a) If L is trivial, then any representation is faithful, since now dimension is 0.
   (b) **adjoint representation is faithful:** We have

$$\begin{aligned} \text{ad}(x) = \text{ad}(y) &\Leftrightarrow \forall\, z \in L : \text{ad}(x)z = \text{ad}(y)z \\ &\Leftrightarrow \forall\, z \in L : [x, z] = [y, z] \\ &\Leftrightarrow \forall\, z \in L : [x - y, z] = 0. \end{aligned}$$

   If L is trivial, then any representation is faithful. Otherwise, there is some non-zero $z \in L$, hence we must have $x - y = 0$, so $x = y$, and thus ad is **injective**.
   (c) All representations considered so far are faithful, except for the trivial representations.
   (d) **direct sum representation:** Given two representations $\rho_1 : \; L \xrightarrow{\sim} \text{End}(V_1)$, $\rho_2 : L \xrightarrow{\sim} \text{End}(V_2)$, the **direct sum representation** is

$$\begin{aligned} \rho_1 \oplus \rho_2 : \; & L \xrightarrow{\sim} \text{End}(V_1 \oplus V_2) \\ & x \mapsto (\rho_1 \oplus \rho_2)(x) := \rho_1(x) \oplus \rho_2(x) \end{aligned}$$

   (e) **tensor product representation:**

$$\begin{aligned} \rho_1 \otimes \rho_2 : \; & L \xrightarrow{\sim} \text{End}(V_1 \times V_2) \\ & x \mapsto (\rho_1 \otimes \rho_2)(x) := \rho_1(x) \otimes \text{id}_{V_2} + \text{id}_{V_1} \otimes \rho_2(x). \end{aligned}$$

   * Example: The **direct sum representation** $\rho_{\text{vec}} \oplus \rho_{\text{spin}} : \; \mathfrak{so}(3, \mathbb{R}) \xrightarrow{\sim} \text{End}(\mathbb{R}^3 \oplus \mathbb{C}^2)$ given in block-matrix form by

$$(\rho_{\text{vec}} \oplus \rho_{\text{spin}})(x) = \left(\begin{array}{c|c} \rho_{\text{vec}}(x) & 0 \\ \hline 0 & \rho_{\text{spin}}(x) \end{array}\right)$$

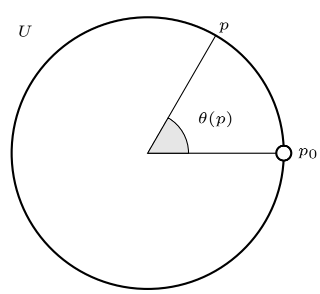
   is a 7-dimensional representation of $\mathfrak{so}(3, \mathbb{R})$.

6. **reducible:** A representation $\rho: L \xrightarrow{\sim} \mathrm{End}(V)$ is called **reducible** if there exists a non-trivial vector **subspace** $U \subseteq V$ which is **invariant under the action of** ρ, i.e.
$$\forall\, x \in L : \forall\, u \in U : \ \rho(x)u \in U.$$
In other words, ρ restricts to a representation $\rho|_U : L \xrightarrow{\sim} \mathrm{End}(U)$.
A representation is **irreducible** if it is not reducible.
   - Just like the simple Lie algebras are the building blocks of all semi-simple Lie algebras, the irreducible representations of a semi-simple Lie algebra are the building blocks of all finite-dimensional representations of the Lie algebra.
   - Examples: The representation $\rho_{\text{vec}} \oplus \rho_{\text{spin}} : \mathfrak{so}(3,\mathbb{R}) \xrightarrow{\sim} \mathrm{End}(\mathbb{R}^3 \oplus \mathbb{C}^2)$ is **reducible** since, for example, we have a subspace $\mathbb{R}^3 \oplus 0$ such that
   $$\forall\, x \in \mathfrak{so}(3,\mathbb{R}) : \forall\, u \in \mathbb{R}^3 \oplus 0 : \ (\rho_{\text{vec}} \oplus \rho_{\text{spin}})(x)u \in \mathbb{R}^3 \oplus 0.$$
   While the representations ρ_{vec} and ρ_{spin} are both **irreducible**.
7. **Casimir operator (associated to a representation):** To every representation ρ of a **compact Lie algebra** (i.e. the Lie algebra of a compact Lie group) there is associated an operator Ω_ρ, called the **Casimir operator**. We will need some preparation in order to define it.
   - **ρ-Killing form:** Let $\rho: L \xrightarrow{\sim} \mathrm{End}(V)$ be a representation of a complex Lie algebra L. We define the **ρ-Killing form** on L as
   $$\begin{aligned} \kappa_\rho : \ & L \times L \xrightarrow{\sim} \mathbb{C} \\ & (x,y) \mapsto \kappa_\rho(x,y) := \mathrm{tr}(\rho(x) \circ \rho(y)). \end{aligned}$$
     * The Killing form in Section 15 is just κ_{ad}.
     * Similarly to κ_{ad}, every κ_ρ is **symmetric** and **associative** w.r.t. the Lie bracket of L.
     * **semi-simple $\Rightarrow$ non-degenerate**: Let $\rho: L \xrightarrow{\sim} \mathrm{End}(V)$ be a **faithful** representation of a complex **semi-simple** Lie algebra L. Then, κ_ρ is **non-degenerate**.
     * Similar to Section 15, **non-degenerate** κ_ρ induces an **isomorphism** $L \xrightarrow{\sim} L^*$ via
     $$L \ni x \mapsto \kappa_\rho(x, -) \in L^*.$$
     Recall that if $\{X_1, \dots, X_{\dim L}\}$ is a basis of L, then the dual basis $\{\widetilde{X}^1, \dots, \widetilde{X}^{\dim L}\}$ of L^* is defined by$\widetilde{X}^i(X_j) = \delta^i_j$. By using the isomorphism induced by κ_ρ, we can find some $\xi_1, \dots, \xi_{\dim L} \in L$ such that we have $\kappa(\xi_i, -) = \widetilde{X}^i$ or, equivalently,
     $$\forall\, x \in L : \ \kappa_\rho(x, \xi_i) = \widetilde{X}^i(x).$$
     We thus have
     $$\kappa_\rho(X_i, \xi_j) = \delta_{ij} := \begin{cases} 1 & \text{if } i \neq j \\ 0 & \text{otherwise.} \end{cases}$$
     * Let $\{X_i\}$ and $\{\xi_j\}$ be defined as above. Then
     $$[X_j, \xi_k] = \sum_{m=1}^{\dim L} C^k{}_{mj} \xi_m,$$
     where $C^k{}_{mj}$ are the **structure constants** with respect to $\{X_i\}$.
   - **Casimir operator associated to a representation:** Let $\rho: L \xrightarrow{\sim} \mathrm{End}(V)$ be a faithful representation of a complex (compact) Lie algebra L and let $\{X_1, \dots, X_{\dim L}\}$ be a basis of L. The **Casimir operator** associated to the representation ρ is the **endomorphism** $\Omega_\rho : V \xrightarrow{\sim} V$
   $$\Omega_\rho := \sum_{i=1}^{\dim L} \rho(X_i) \circ \rho(\xi_i).$$
     * **commutes with every endomorphism in $\mathrm{im}_\rho(L)$:** Let Ω_ρ the Casimir operator of a representation $\rho: L \xrightarrow{\sim} \mathrm{End}(V)$. Then
     $$\forall\, x \in L : \ [\Omega_\rho, \rho(x)] = 0,$$
     that is, Ω_ρ **commutes with every endomorphism in** $\mathrm{im}_\rho(L)$.

* **Schur lemma:** If $\rho: L \xrightarrow{\sim} \mathrm{End}(V)$ is **irreducible**, then any operator $S \in \mathrm{End}(V)$ which commutes with every endomorphism in $\mathrm{im}_\rho(L)$ has the form
$$S = c_\rho \mathrm{id}_V$$
for some constant $c_\rho \in \mathbb{C}$ (or $\mathbb{R}$, if L is a real Lie algebra).
* So for **irreducible** ρ we have $\Omega_\rho = c_\rho \mathrm{id}_V$ for some c_ρ.
* Furthermore, the Casimir operator of **irreducible** $\rho: L \xrightarrow{\sim} End(V)$ is $\Omega_\rho = c_\rho \mathrm{id}_V$, where
$$c_\rho = \frac{\dim L}{\dim V}.$$

- Example I: Consider again the Lie algebra $\mathfrak{so}(3,\mathbb{R})$. With representation I, We have
$$[(\kappa_{\rho_{\mathrm{vec}}})_{ij}] = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$
Thus, $\kappa_{\rho_{\mathrm{vec}}}(J_i, \xi_j) = \delta_{ij}$ requires that we define $\xi_i := -\frac{1}{2} J_i$. Then, we have
$$\Omega_{\rho_{\mathrm{vec}}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$
Hence $\Omega_{\rho_{\mathrm{vec}}} = c_{\rho_{\mathrm{vec}}} \mathrm{id}_{\mathbb{R}^3}$ with $c_{\rho_{\mathrm{vec}}} = 1$, which agrees with our previous obeservation since $\frac{\dim \mathfrak{so}(3,\mathbb{R})}{\dim \mathbb{R}^3} = \frac{3}{3} = 1$.
- Example II: Similary, with representation II ρ_{spin}, we have
$$[(\kappa_{\rho_{\mathrm{spin}}})_{ij}] = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$
So $\xi_i := -J_i$, and we have $\Omega_{\rho_{\mathrm{spin}}} = \frac{3}{4} \mathrm{id}_{\mathbb{C}^2}$, in accordance with the fact that $\frac{\dim \mathfrak{so}(3,\mathbb{R})}{\dim \mathbb{C}^2} = \frac{3}{4}$.

8. **representations of Lie groups:**
   - **automorphism group (or general linear group):** Given a vector space V, recall that the **automorphism group (or general linear group)** consisting of the **invertible endomorphisms** with the composition as group operation.
     * **notation:**
     $$\mathrm{GL}(V) \equiv \mathrm{Aut}(V) := \{\phi \in \mathrm{End}(V) \mid \det \phi \neq 0\},$$
     * $\mathrm{GL}(V)$ is a **manifold** that is established with a topology and a differentiable structure.
     * Moreover, if V is a **finite-dimensional** K-vector space, then $V \cong_{\mathrm{vec}} K^{\dim V}$ and hence the group $\mathrm{GL}(V)$ can be given the structure of a Lie group via
     $$\mathrm{GL}(V) \cong_{\mathrm{Lie\,grp}} \mathrm{GL}(K^{\dim V}) := \mathrm{GL}(\dim V, K).$$
   - **representation:** A **representation** of a Lie group $(G, \bullet)$ is a **Lie group homomorphism**
   $$R: G \to \mathrm{GL}(V)$$
   for some **finite-dimensional** vector space V.
     * Recall that $R: G \to \mathrm{GL}(V)$ is a **Lie group homomorphism** if it is **smooth** and has **group homomorphism**
     $$\forall\, g_1, g_2 \in G : R(g_1 \bullet g_2) = R(g_1) \circ R(g_2).$$
     Note that, as is the case with any group homomorphism, we have
     $$R(e) = \mathrm{id}_V \qquad \text{and} \qquad R(g^{-1}) = R(g)^{-1}.$$
     * Example: Consider the Lie group $\mathrm{SO}(2,\mathbb{R})$. As a smooth manifold, $\mathrm{SO}(2,\mathbb{R})$ is isomorphic to the circle S^1. Let $U = S^1 \setminus \{p_0\}$, where p is any point of S^1, so that we can define a chart $\theta: U \to [0, 2\pi) \subseteq \mathbb{R}$ on S^1 by mapping each point in U to an "angle" in $[0, 2\pi)$.



The operation

$$p_1 \bullet p_2 := (\theta(p_1) + \theta(p_2)) \mod 2\pi$$

endows S^1 with the structure of a Lie group. Then, a representation of $\mathrm{SO}(2, \mathbb{R})$ is given by

$$\begin{aligned} R: \ \mathrm{SO}(2, \mathbb{R}) &\to \mathrm{GL}(\mathbb{R}^2) \\ p &\mapsto \begin{pmatrix} \cos\theta(p) & \sin\theta(p) \\ -\sin\theta(p) & \cos\theta(p) \end{pmatrix}. \end{aligned}$$

We can check that

$$R(p_1 \bullet p_2) = R(p_1) \circ R(p_2).$$

9. **adjoint representation of Lie group:** Let G be a Lie group, and let

$$\Psi : G \to \mathrm{Aut}(G)$$

be the mapping $g \mapsto \Psi_g$, where recall that $\Psi_g : G \to G$ given by the **inner automorphism (conjugation)** as we introduced in Section 14:

$$\Psi_g(h) = ghg^{-1}.$$

For each g in G, define Ad_g to be the **differential or push-forward** of Ψ_g at the **identity**

$$\mathrm{Ad}_g = (\mathrm{d}\Psi_g)_e \equiv (\Psi_{g*})_e : T_eG \xrightarrow{\sim} T_eG$$

where T_eG is the tangent space at the identity e. We have the adjoint representation of **adjoint representation of Lie group:**

$$\begin{aligned} \mathrm{Ad}: \ G &\to \mathrm{GL}(T_eG) \\ g &\mapsto \mathrm{Ad}_g \end{aligned}$$

- So given a Lie group G, we can always find a representation of Lie group with V being the Lie algebra T_eG of G!
- This Ψ, the mapping $g \mapsto \Psi_g$, is a **Lie group homomorphism**[19] from G to $\mathrm{Aut}(G)$. This is from the group action property.

$$\Psi_{gf}(h) = gfh(gf)^{-1} = g(fhf^{-1})g^{-1} = \Psi_g \circ \Psi_f(h).$$

- For each g, Ψ_g is a **Lie group automorphism** from G to G as we have shown in Section 14: We have

$$\Psi_g(mn) = gmng^{-1} = gmg^{-1}gng^{-1} = \Psi_g(m)\Psi_g(n)$$

- Since Ψ, the mapping $g \mapsto \Psi_g$, is a **Lie group homomorphism**, $g \mapsto \mathrm{Ad}_g$ too is a **Lie group homomorphism**: This is from the **compositions of push-forward:**

$$\mathrm{Ad}_{gf} = \mathrm{Ad}_g \circ \mathrm{Ad}_f$$

  * So Ad_g is again a **group action**! We have the properties like $\mathrm{Ad}_{g^{-1}} \circ \mathrm{Ad}_g = \mathrm{Ad}_g \circ \mathrm{Ad}_{g^{-1}} = \mathrm{id}_{T_eG}$.
- If G is an immersed Lie subgroup of the general linear group $\mathrm{GL}_n(\mathbb{C})$ (called immersely linear Lie group), then
  * The Lie algebra $\mathfrak{g}$ consists of matrices and the exponential map is the matrix exponential

$$\exp(X) = e^X$$

    for matrices X with small operator norms. (See Section 19.)
  * Thus, for g in G and X in $\mathfrak{g}$, taking the derivative of $\Psi_g(\exp(tX)) = ge^{tX}g^{-1}$ at $t = 0$, one gets:

$$\mathrm{Ad}_g(X) = gXg^{-1}$$

    where on the right we have the products of matrices. Note here we have used the curve $exp(tX)$ has derivative X at $t = 0$. (See Section 19.)

[19] Note, roughly speaking, here and below we only consider group homomorphism and the smoothness not checked.

10. **representation of Lie algebra from representation of Lie group:** Since a representation R of a Lie group G is required to be **smooth**, we can always consider its **differential or push-forward** at the **identity**

$$(R_*)_e : \ T_eG \xrightarrow{\sim} T_{\mathrm{id}_V}\mathrm{GL}(V) \equiv \mathrm{GL}(V).$$

Since for any $A, B \in T_eG$ we have

$$(R_*)_e[A, B] = [(R_*)_e A, (R_*)_e B],$$

the map $(R_*)_e$ is a representation of the Lie algebra of G on the vector space $\mathrm{GL}(V)$.

- **adjoint representation of Lie algebra from adjoint representation of Lie group:** We have

$$(\mathrm{Ad}_*)_e = \mathrm{ad},$$

where ad is the **adjoint representation of $\mathfrak{g} \equiv T_eG$**. The proof need exponential map defined in Section 19.

  * Recall in Section 15, we define the following **adjoint representation for a Lie algebra**

$$\mathrm{ad}_x(y) = [x, y]$$

  * Taking the derivative of the adjoint map $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ at the identity element gives the **equivalent definition of adjoint representation** of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ of G :

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$$
$$x \mapsto \mathrm{ad}_x = \mathrm{d}(\mathrm{Ad})_e(x)$$

where $\mathrm{Der}(\mathfrak{g}) = \mathrm{Lie}(\mathrm{Aut}(\mathfrak{g}))$ is the Lie algebra of $\mathrm{Aut}(\mathfrak{g})$ which may be identified with the derivation algebra of $\mathfrak{g}$. The proof needs [7][Prop 1.9.]. See `https://en.wikipedia.org/wiki/Adjoint_representation`.

  * If G is an immersely linear Lie group (a Lie subgroup), $\mathrm{Ad}_g(Y) = gYg^{-1}$ and thus with $g = e^{tX}$

$$\mathrm{Ad}_{e^{tX}}(Y) = e^{tX}Ye^{-tX}$$

Taking the derivative of this at $t = 0$, we have:

$$\mathrm{ad}_X Y = XY - YX$$

11. A summary of the Adjoint and adjoint mappings here:

$\Psi\colon G \to \mathrm{Aut}(G)$	$\Psi_g\colon G \to G$
Lie group homomorphism: • $\Psi_{gh} = \Psi_g\Psi_h$	Lie group automorphism: • $\Psi_g(ab) = \Psi_g(a)\Psi_g(b)$ • $(\Psi_g)^{-1} = \Psi_{g^{-1}}$
$\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g})$	$\mathrm{Ad}_g\colon \mathfrak{g} \to \mathfrak{g}$
Lie group homomorphism: • $\mathrm{Ad}_{gh} = \mathrm{Ad}_g\mathrm{Ad}_h$	Lie algebra automorphism: • Ad_g is linear • $(\mathrm{Ad}_g)^{-1} = \mathrm{Ad}_{g^{-1}}$ • $\mathrm{Ad}_g[x, y] = [\mathrm{Ad}_g x, \mathrm{Ad}_g y]$
$\mathrm{ad}\colon \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$	$\mathrm{ad}_x\colon \mathfrak{g} \to \mathfrak{g}$
Lie algebra homomorphism: • ad is linear • $\mathrm{ad}_{[x,y]} = [\mathrm{ad}_x, \mathrm{ad}_y]$	Lie algebra derivation: • ad_x is linear • $\mathrm{ad}_x[y, z] = [\mathrm{ad}_x y, z] + [y, \mathrm{ad}_x z]$

19 Reconstruction of A Lie Group from Its Algebra: The Exponential Map

$$\begin{array}{ccccccccccc} \text{set} & \longrightarrow & \text{topological space} & \longrightarrow & \text{topological manifold} & \longrightarrow & \text{differentiable manifold} & \longrightarrow & \text{principal fibre bundle} & \longrightarrow & \text{associated fibre bundles} \\ & & & & & & \downarrow & \nearrow & & & \\ & & & & & & \text{Lie group} & & & & \\ & & & & & & \uparrow\downarrow & & & & \\ & & & & & & \text{Lie algebra} & & & & \end{array}$$

A summary:

- For any Lie group G, we could get Lie algebra $\mathcal{L}(G) \cong_{\text{Lie alg}} T_eG \equiv \mathfrak{g}$. For the Lie algebra $\mathcal{L}(G)$, at least we could recover the G of a **neighbour around its identity** e. How large is the neighbour, the full G? It depends. Note, in this neighbour we have that the correspondence between the points on G and the vector in $\mathcal{L}(G)$ is **bijective** through the so called **exponential map**.
- We then state the **one-to-one correspondence** between one-parameter subgroup, elements of the Lie algebra of the group and the left-invariant vector fields.

1. **integral curve:** Let M be a smooth manifold and let $Y \in \Gamma(TM)$. An **integral curve** of Y is smooth curve $\gamma : \ (-\varepsilon, \varepsilon) \to M$, with $\varepsilon > 0$, such that
$$\forall\ \lambda \in (-\varepsilon, \varepsilon) : \ X_{\gamma,\gamma(\lambda)} = Y|_{\gamma(\lambda)}.$$
   - **local existence and uniqueness of integral curve that pass a point:** It follows from the **local existence and uniqueness** of solutions to ordinary differential equations [8] that,
     * **existence:** Given any $Y \in \Gamma(TM)$ and any $p \in M$, there **exist** $\varepsilon > 0$ and a **smooth curve** $\gamma : \ (-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = p$ which is an **integral curve** of Y.
     * **uniqueness:** If γ_1 and γ_2 are both integral curves of Y through p, i.e. $\gamma_1(0) = \gamma_2(0) = p$, then $\gamma_1 = \gamma_2$ on the intersection of their domains of definition.
     * **reparametrization?:** Given $\gamma(t)$, an integral curve of a vector field Y on M, let $\hat{\gamma}(t) := \gamma(\sigma(t))$ be a reparametrized curve. The only possible $\sigma(t) = t +$ constant that makes $\hat{\gamma}$ is still an integral curve of the vector field Y is $\sigma(t) = t +$ constant. hint: $(d/dt)f(\gamma(\sigma(t))) = \sigma'(t)(d/dt)f(\gamma(t))$, so that $\dot{\hat{\gamma}} = \sigma'\dot{\gamma}$; [$\sigma(t) = t+$ constant]. $\hat{\gamma}$ is another trajectory, such that we traverse the same set of points on M at different moments of time. At one point the new speed is $\sigma'(t)$ times the old one at any point $\gamma(t)$. Since the velocity vector of an integral curve may not be changed, $\sigma'(t) = 1$ is the only possible result. This means that the only possibility to change the trajectory is to traverse the same path either sooner or later.
     * Let γ be an integral curve of a vector field Y on M, which starts from $P \equiv \gamma(0) \in M$. The integral curve (of the same field Y) $\hat{\gamma}$, which starts from $Q \equiv \gamma(a\ )$, is $\hat{\gamma}(t) := \gamma(t + a)$
   - **maximal integral curve:** The **maximal integral curve** of $Y \in \Gamma(TM)$ through $p \in M$ is the **unique integral curve** $\gamma : \ I^p_{\max} \to M$ of Y through p, where
$$I^p_{\max} := \bigcup \{I \subseteq \mathbb{R} \mid \text{there exists an integral curve } \gamma : \ I \to M \text{ of } Y \text{ through } p\}.$$
     * **complete vector field:** A vector field is **complete** if $I^p_{\max} = \mathbb{R}$ for all $p \in M$.
     * **compact $\Rightarrow$ complete:** On a **compact** manifold, every vector field is **complete**.
     * **left-invariant $\Rightarrow$ complete:** Every **left-invariant** vector field on a Lie group is **complete**. So we have a maximal integral curves of a left-invariant vector field. This is crucial in the construction of the map that allows us to go from a Lie algebra to a Lie group.
     * Example: See the ball in hairy ball theorem in Section 12. Note the image of one curve may be not compact since we may never reach the point where the vector field vanishes. However, please note the theorem is talking about the domain manifold is compact. We still can get complete curves with the vector size become smaller and smaller if we move with $\lambda \to \infty$. With a **non-compact** manifold, the field may suddenly vanishes and we cannot move any more.
   - **diffeomorphism push-forward integral curve to integral curve:** Recall in Section 11 we mention the **curve push-forward** under diffeomorphism. Now, if let $f : M \to M$

be a **diffeomorphism** and let $\gamma(t)$ be the **integral curve** of a field Y which starts in $x \in M$. We have that the curve $f(\gamma(t))$ is then the **integral curve** of the field f_*Y which starts in $f(x)$.

2. **exponential map:** Let G be a Lie group. Recall that given any $A \in T_eG$, we can define the uniquely determined **left-invariant vector field** $K_A := j(A)$ via the isomorphism $j : T_eG \xrightarrow{\sim} \mathcal{L}(G)$ as

$$K_A|_g := (\ell_g)_*(A).$$

Then let $\gamma^A : \mathbb{R} \to G$ be the **maximal integral curve** of K_A through $e \in G$. The **exponential map** is defined as

$$\begin{aligned} \exp : \; & T_eG \to G \\ & A \mapsto \exp(A) := \gamma^A(1) \end{aligned}$$

- **comparison between $\exp$ and $j(A)$:** The map $\exp$ is different from $j(A)$. It is we first select the $j(A)$ and then get the "location" at time 1. However, there is **one to one correspondence** between them. See below one-parameter subgroup.
  (a) They have **range difference:**
    * The output of $j(\cdot)$ is a vector field while $\exp(\cdot)$ is a point on the manifold.
    * In some sense, $j(\cdot)$ will cover the full G. This is because left translation ℓ_g is a **diffeomorphism** on G.
    * However, for $\exp(\cdot)$, we will always travel from e, so it can only reach the **connected component** of G containing the identity. See below for more details.
  (b) Furthermore, as shown in Section 14, from the existence of global frame, we know that given any $X \in T_gG$, we can the find a unique $A \in T_eG$ such that $K_A|_g \equiv j(A)|_g = X$. But for $\exp$ we only have the following **local diffeomorphism** in general.
- **local diffeomorphism:** The map $\exp$ is smooth and a **local diffeomorphism** around $0 \in T_eG$, i.e. there exists an open set $U \subseteq T_eG$ containing 0 such that the restriction

$$\exp|_U : \; U \to \exp(U) \subseteq G$$

is **bijective** and both $\exp|_U$ and $(\exp|_U)^{-1}$ are **smooth**.
  * Note that the maximal integral curve of X^0 is the constant curve $\gamma^0(\lambda) \equiv e$, and hence we have $\exp(0) = e$. It means that we can recover a **neighbourhood** of the identity of G from a neighbourhood of the identity of T_eG.
- **How large is $\exp(T_eG)$?:** Let G be a Lie group. We can only say that the image of $\exp : \; T_eG \to G$ is a subset of the **connected component** of G containing the identity. In some cases, $\exp$ is **surjective** so that the neighbor is the full G.
  (a) G is connected and compact. Since T_eG is a vector space, it is non-compact. Hence, if G is compact, $\exp$ cannot be injective. This is because diffeomorphism will keep the compactness.
  (b) G is connected and nilpotent (for example, G connected and abelian).
  (c) $G = \mathrm{GL}_n(\mathbb{C})$.
  * Example: Let $B : \; V \times V$ be a pseudo inner product on V. Then recall that

$$\mathrm{O}(V) := \{\phi \in \mathrm{GL}(V) \mid \forall \; v, w \in V : B(\phi(v), \phi(w)) = B(v, w)\}$$

is called the **orthogonal group** of V with respect to B. Every $\phi \in \mathrm{O}(V)$ has determinant 1 or -1. Since $\det$ is multiplicative, we have a **subgroup**

$$\mathrm{SO}(V) := \{\phi \in \mathrm{O}(V) \mid \det \phi = 1\}.$$

These are, in fact, Lie subgroups of $\mathrm{GL}(V)$. The Lie group $\mathrm{SO}(V)$ is connected while

$$\mathrm{O}(V) = \mathrm{SO}(V) \cup \{\phi \in \mathrm{O}(V) \mid \det \phi = -1\}$$

is disconnected. Since $\mathrm{SO}(V)$ contains id_V, we have

$$\mathfrak{so}(V) := T_{\mathrm{id}_V} \mathrm{SO}(V) = T_{\mathrm{id}_V} \mathrm{O}(V) =: \mathfrak{o}(V)$$

and

$$\exp(\mathfrak{so}(V)) = \exp(\mathfrak{o}(V)) = \mathrm{SO}(V).$$

where $\mathfrak{so}(V)$ is the Lie algebra associated with Lie group $\mathrm{SO}(V)$.
- **parameterization of G near e:** Because of the **diffeomorphism** (especially bijective), choosing a basis $A_1, \ldots, A_{\dim G}$ of T_eG provides a convenient parameterization in the form of $\exp(\lambda^i A_i)$ for the points in the neighbor U mentioned above in G near e.

* Example: Consider, for example, the **Lorentz group**

$$\mathrm{O}(3,1) \equiv \mathrm{O}(\mathbb{R}^4) = \{\Lambda \in \mathrm{GL}(\mathbb{R}^4) \mid \forall\, x, y \in \mathbb{R}^4 : B(\Lambda(x), \Lambda(y)) = B(x, y)\},$$

where $B(x, y) := \varepsilon_{\mu\nu} x^\mu y^\nu$, with $0 \le \mu, \nu \le 3$ (x^μ is the components of x) and

$$[\varepsilon^{\mu\nu}] = [\varepsilon_{\mu\nu}] := \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The Lorentz group $\mathrm{O}(3,1)$ is 6-dimensional, hence so is the **Lorentz algebra** $\mathfrak{o}(3,1)$. For convenience, instead of denoting a basis of $\mathfrak{o}(3,1)$ as $\{M^i \mid i = 1, \ldots, 6\}$, we will denote it as $\{M^{\mu\nu} \mid 0 \le \mu, \nu \le 3\}$ and require that the indices μ, ν be anti-symmetric, i.e.

$$M^{\mu\nu} = -M^{\nu\mu}.$$

Then $M^{\mu\nu} = 0$ when $\rho = \sigma$, and the set $\{M^{\mu\nu} \mid 0 \le \mu, \nu \le 3\}$, while technically not linearly independent, contains the 6 independent elements that we want to consider as a basis. These basis elements satisfy the following bracket relation

$$[M^{\mu\nu}, M^{\rho\sigma}] = \eta^{\nu\sigma} M^{\mu\rho} + \eta^{\mu\rho} M^{\nu\sigma} - \eta^{\nu\rho} M^{\mu\sigma} - \eta^{\mu\sigma} M^{\nu\rho}.$$

Any element $\lambda \in \mathfrak{o}(3,1)$ can be expressed as linear combination of the $M^{\mu\nu}$,

$$\lambda = \tfrac{1}{2}\omega_{\mu\nu} M^{\mu\nu}$$

where the indices on the coefficients $\omega_{\mu\nu}$ are also anti-symmetric, and the factor of $\frac{1}{2}$ ensures that the sum over all μ, ν counts each anti-symmetric pair only once. Then, we have

$$\Lambda = \exp(\lambda) = \exp(\tfrac{1}{2}\omega_{\mu\nu} M^{\mu\nu}) \in \mathrm{O}(3,1).$$

The subgroup of $\mathrm{O}(3,1)$ consisting of the the space-orientation preserving Lorentz transformations, or *proper* Lorentz transformations, is denoted by $\mathrm{SO}(3,1)$. The subgroup consisting of the time-orientation preserving, or *orthochronous*, Lorentz transformations is denoted by $\mathrm{O}^+(3,1)$. The Lie group $\mathrm{O}(3,1)$ is disconnected: its **four connected components** from which we care about the connected component $\mathrm{SO}^+(3,1) := \mathrm{SO}(3,1) \cap \mathrm{O}^+(3,1)$, also called the *restricted Lorentz group*, consisting of the proper orthochronous Lorentz transformations; Since $\mathrm{id}_{\mathbb{R}^4} \in \mathrm{SO}^+(3,1)$, we have $\exp(\mathfrak{o}(3,1)) = \mathrm{SO}^+(3,1)$. Then $\{M^{\mu\nu}\}$ provides a nice **parameterization** of $\mathrm{SO}^+(3,1)$ since, if we choose

$$[\omega_{\mu\nu}] = \begin{pmatrix} 0 & \psi_1 & \psi_2 & \psi_3 \\ -\psi_1 & 0 & \varphi_3 & -\varphi_2 \\ -\psi_2 & -\varphi_3 & 0 & \varphi_1 \\ -\psi_3 & \varphi_2 & -\varphi_1 & 0 \end{pmatrix}$$

then the Lorentz transformation $\exp(\frac{1}{2}\omega_{\mu\nu} M^{\mu\nu}) \in \mathrm{SO}^+(3,1)$ corresponds to a boost in the (ψ_1, ψ_2, ψ_3) direction and a space rotation by $(\varphi_1, \varphi_2, \varphi_3)$. Indeed, in physics one often thinks of the Lie group $\mathrm{SO}^+(3,1)$ as being generated by $\{M^{\mu\nu}\}$.
A **representation of the Lie algebra is** $\rho : \; T_{\mathrm{id}_{\mathbb{R}^4}} \mathrm{SO}^+(3,1) \xrightarrow{\sim} \mathrm{End}(\mathbb{R}^4)$ is given by

$$\rho(M^{\mu\nu})^a{}_b := \eta^{\nu a}\delta^\mu_b - \eta^{\mu a}\delta^\nu_b$$

which is probably how you have seen the $M^{\mu\nu}$ themselves defined in some previous course on relativity theory. Using this representation, we get a corresponding representation

$$R : \; \mathrm{SO}^+(3,1) \to \mathrm{GL}(\mathbb{R}^4)$$

via the exponential map by defining

$$R(\Lambda) = \exp(\tfrac{1}{2}\omega_{\mu\nu}\rho(M^{\mu\nu})).$$

Then, the map $\exp$ becomes the usual exponential (series) of matrices.

3. **one-parameter subgroup:** A **one-parameter subgroup** of a Lie group G is a **Lie group homomorphism**

$$\xi : \; \mathbb{R} \to G,$$

with $\mathbb{R}$ understood as a Lie group under ordinary addition "+". In other words,

$$\gamma(t+s) = \gamma(t)\gamma(s) \quad \gamma(0) = e \quad t, s \in \mathbb{R}.$$

We next state some facts:

- Denote $K_A \in \mathcal{L}(G)$ as the **left-invariant vector field** on G which is generated by a vector $A \in \mathfrak{g} \equiv T_e G$.
  (a) Its **integral curve** $\gamma^A(t)$ starting from e is a **oneparameter subgroup** since we have

$$\gamma^A(t+s) = \gamma^A(t)\gamma^A(s) \quad \gamma^A(0) = e.$$

  hint: the curve $\Gamma(t) := \gamma^A(t+s)$ is the integral curve of K_A starting at $\gamma^A(s)$. Since $L_{g*}K_A = K_A$ for any g as K_A is left-invariant, the curve $\Gamma(t)$ is also the integral curve of the field $L_{\gamma^A(s)*}K_A$. Note, by the push-forward, the integral curve of $L_{\gamma^A(s)*}K_A$ is $\gamma^A(s)\gamma^A(t)$. Put together $\gamma^A(t+s) = \gamma^A(t)\gamma^A(s)$.

  (b) If, in turn, $\gamma(t)$ is an arbitrary **one-parameter subgroup**, then it is necessarily the **integral curve of the left- invariant field** K_A with $A \equiv K_A(e) = \dot\gamma(0)$. The complete trajectory $\gamma(t)$ then turns out to be **determined by its initial velocity**, i.e. by the tangent vector $\dot\gamma(0) = A$ at the starting point e.
  hint: we have

$$\begin{aligned}\dot\gamma(t) &= \frac{d}{ds}\Big|_{s=0}\gamma(t+s) = \frac{d}{ds}\Big|_0 \gamma(t)\gamma(s) = \frac{d}{ds}\Big|_0 L_{\gamma(t)}\gamma(s) = L_{\gamma(t)*}\frac{d}{ds}\Big|_0 \gamma(s) = L_{\gamma(t)*}X \\ &= K_A(\gamma(t))\end{aligned}$$

- Since we have the correspondence $A \Leftrightarrow \gamma^A(t) \Leftrightarrow K_A$, there is a **one-to-one correspondence:**

elements of the Lie algebra of the group $\Longleftrightarrow$ one-parameter subgroups $\Longleftrightarrow$ left-invariant vector fields

- The one-parameter subgroup satisfies

$$\gamma^X(kt) = \gamma^{kX}(t) \quad k \in \mathbb{R}$$

  since they have the same initial velocity. This enables one to express the one-parameter subgroup in terms of an exponential map in the form

$$\gamma^X(t) = \exp tX \equiv e^{tX}$$

- **explicit form on** $\mathrm{GL}(n, \mathbb{R})$**:** Using [9][11.1.10], the local representation of left-invariant vector fields of $\mathrm{GL}(n, \mathbb{R})$, we have the equation

$$\dot{x}(t) = x(t)C, x(0) = \mathbb{I}_n.$$

  The solution of this (matrix) equation is

$$x(t) = \exp tC \equiv e^{tC} := \mathbb{I}_n + tC + \frac{t^2}{2!}C^2 + \cdots$$

4. **exponential map under Lie group homomorphism:** Let G and H be Lie groups and let $\phi : G \to H$ be a **Lie group homomorphism**. Then, for all $A \in T_{e_G}G$, we have

$$\phi(\exp(A)) = \exp((\phi_*)_{e_G}A).$$

- **equivalent statement:** the following diagram commutes.

$$\begin{tikzcd}
T_{e_G}G \arrow[r, "(\phi_*)_{e_G}"] \arrow[d, "\exp"'] & T_{e_H}H \arrow[d, "\exp"] \\
G \arrow[r, "\phi"] & H
\end{tikzcd}$$

- **Adjoint map and exponential map:** In particular, for $\phi \equiv \mathrm{Ad}_g : G \to G$, we have

$$\mathrm{Ad}_g(\exp(A)) = \exp((\mathrm{Ad}_{g*})_e A).$$

20 Principal Fibre Bundles

$$\begin{tikzcd}
\text{set} \arrow[r] & \text{topological space} \arrow[r] & \text{topological manifold} \arrow[r] & \text{differentiable manifold} \arrow[r] \arrow[d] & \text{principal fibre bundle} \arrow[r] & \text{associated fibre bundles} \\
& & & \text{Lie group} \arrow[ur] \arrow[d, shift right] & & \\
& & & \text{Lie algebra} \arrow[u, shift right] & &
\end{tikzcd}$$

Very roughly speaking, a principal fibre bundle is a bundle whose typical fibre is a Lie group. Principal fibre bundles are so immensely important because they allow us to understand any fibre bundle with fibre F on which a Lie group G acts. These are then called associated fibre bundles, and will be discussed later on.

A summary:

- ▶ We first introduce the **left and right Lie group action** G **on manifold** M. The concept of **orbit** as equivalence relation on M and **stabiliser** as subgroup G are then defined.

- ► We then introduce **principal fibre bundles** from the above equivalence classes from a **free action**.
- ► The **frame bundle** is then introduced in detail.
- ► We state the **principal bundle morphisms** which requires the extra ρ-**equivariant** condition besides the bundle morphism condition.
- ► Finally, we state the **restriction and extension** of principal fibre bundles in terms of Lie subgroup.

1. **left group action:** Let $(G, \bullet)$ be a Lie group and let M be a smooth manifold. A smooth map

$$\begin{aligned} \triangleright\colon\ G \times M &\to M \\ (g, p) &\mapsto g \triangleright p \end{aligned}$$

   satisfying

   (a)
   (b) $\forall\ p \in M :\ e \triangleright p = p$;
   (c) $\forall\ g_1, g_2 \in G : \forall\ p \in M :\ (g_1 \bullet g_2) \triangleright p = g_1 \triangleright (g_2 \triangleright p)$,

   is called a **left Lie group action**, or **left G-action**, on M.

   - **left G-manifold:** A manifold equipped with a left G-action is called a **left G-manifold**.
   - The smooth structures on G and M were only used in the requirement that $\triangleright$ be smooth. By dropping this condition, we obtain the usual definition of a group action on a set. So concepts like **orbits** and **stabilisers** can then be defined.
   - Example: Let G be a Lie group and let $R :\ G \to \mathrm{GL}(V)$ be a **representation** of G on a vector space V. Define a map

$$\begin{aligned} \triangleright\colon\ G \times V &\to V \\ (g, v) &\mapsto g \triangleright v := R(g)v. \end{aligned}$$

     We easily check that $e \triangleright v := R(e)v = \mathrm{id}_V v = v$ and

$$\begin{aligned} (g_1 \bullet g_2) \triangleright v &:= R(g_1 \bullet g_2)v \\ &= (R(g_1) \circ R(g_2))v \\ &= R(g_1)(R(g_2)v) \\ &= g_1 \triangleright (g_2 \triangleright v), \end{aligned}$$

     for any $v \in V$ and any $g_1, g_2 \in G$.

     $*$ In some sense, we can therefore think of left G-actions as "generalised" representations of G on some manifold.

2. **right G-action:** Similarly, a **right G-action** on M is a smooth map

$$\begin{aligned} \triangleleft\colon\ M \times G &\to M \\ (p, g) &\mapsto p \triangleleft g \end{aligned}$$

   satisfying

   (a) $\forall\ p \in M :\ p \triangleleft g = p$;
   (b) $\forall\ g_1, g_2 \in G : \forall\ p \in M :\ p \triangleleft (g_1 \bullet g_2) = (p \triangleleft g_1) \triangleleft g_2$.

   - **from left to right action:** Let $\triangleright$ be a left G-action on M. Then

$$\begin{aligned} \triangleleft\colon\ M \times G &\to M \\ (p, g) &\mapsto p \triangleleft g := g^{-1} \triangleright p \end{aligned}$$

     is a right G-action on M.
   - In some sense, left and right action are dual. But later, within the context of **principal and associated fibre bundles**, we will attach **separate** "meanings" to left and right actions. See Section 21.

- **change of basis viewed from Lie group actions:** Recall that if we have a basis $e_1, \ldots, e_{\dim M}$ of T_pM and $X^1, \ldots, X^{\dim M}$ are the components of some $X \in T_pM$ in this basis, then under a change of basis

$$\widetilde{e}_a = A^b{}_a e_b,$$

we have $X = \widetilde{X}^a \widetilde{e}_a$, where

$$\widetilde{X}^a = (A^{-1})^a{}_b X^b.$$

Once expressed in terms of principal and associated fibre bundles, we will see that the "recipe" of labelling the basis by lower indices and the vector components by upper indices, as well as their transformation law, can be understood as a **right action** of $\mathrm{GL}(\dim M, \mathbb{R})$ on the basis and a **left action** of the same $\mathrm{GL}(\dim M, \mathbb{R})$ on the components. See Section 21 for more details.

3. **ρ-equivariant:** Let G, H be Lie groups, let $\rho : G \to H$ be a Lie group homomorphism and let

$$\triangleright\colon\ G \times M \to M,$$
$$\blacktriangleright\colon\ H \times N \to N$$

be left actions of G and H on some smooth manifolds M and N, respectively. Then, a smooth map $f : M \to N$ is said to be **ρ-equivariant** if the diagram

$$\begin{tikzcd} G \times M \arrow[r, "\rho\times f"] \arrow[d, "\triangleright"'] & H \times N \arrow[d, "\blacktriangleright"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

where $(\rho \times f)(g, p) := (\rho(g), f(p)) \in H \times N$, commutes. Equivalently,

$$\forall\, g \in G : \forall\, p \in M :\ f(g \triangleright p) = \rho(g) \blacktriangleright f(p).$$

- ρ-equivariant maps are the "action-preserving" maps between the G-manifold M and the H-manifold N.
- **equivalent diagram representation:** We sometime draw it in the following form

$$\begin{tikzcd} M \arrow[r, "u"] & N \\ G \times M \arrow[r, "\rho\times f"] \arrow[u, "\triangleright"] & H \times N \arrow[u, "\blacktriangleright"'] \\ M \arrow[r, "f"] \arrow[u, "i_1"] & N \arrow[u, "i_1"'] \end{tikzcd}$$

- If $\rho = \mathrm{id}_G$ or $f = \mathrm{id}_M$, the notion of f being ρ-equivariant reduces to
  - $\rho = \mathrm{id}_G$: homomorphism of G-manifolds as $f(g \triangleright p) = g \blacktriangleright f(p)$. In this case, we may suppress the above diagram to the following

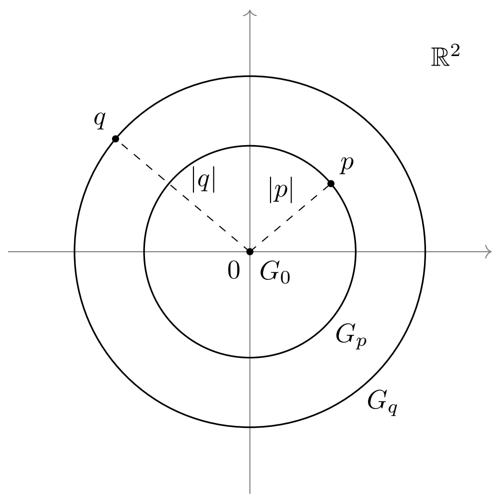
$$\begin{tikzcd} M \arrow[r, "u"] & N \\ M \arrow[r, "u"] \arrow[u, "G\triangleright"] & N \arrow[u, "G\blacktriangleright"'] \end{tikzcd}$$

  Note that $M \xrightarrow{G\triangleright} P$ is a shorthand for the inclusion of M into the product $G \times M$ followed by the left action $\triangleright$, i.e.

$$M \xrightarrow{G\triangleright} M \quad = \quad M \xrightarrow{i_1} G \times M \xrightarrow{\triangleright} M$$

* $f = \mathrm{id}_M$: homomorphism of left actions on M as $g \triangleright p = \rho(g) \blacktriangleright p$.

4. **orbit:** Let $\triangleright\colon\ G \times M \to M$ be a left G-action. For each $p \in M$, we define the **orbit** of p as the set
$$G_p := \{q \in M \mid \exists\ g \in G : q = g \triangleright p\}.$$
   - Alternatively, the orbit of p is the image of G under the map $(- \triangleright p)$.
   - **orbit as equivalence class:** Let $\triangleright\colon\ G \times M \to M$ be an action on M. Define a relation on M
$$p \sim q\ :\Leftrightarrow\ \exists\ g \in G : q = g \triangleright p.$$
   Then $\sim$ is an **equivalence relation** on M.
   - **orbit space as quotient space:** Let $\triangleright\colon\ G \times M \to M$ be an action on M. The **orbit space** of M is
$$M/G := M/\!\sim = \{G_p \mid p \in M\}.$$
   - **transitive:** A left G-action $\triangleright\colon\ G \times M \to M$ is said to be **transitive** if for all $p, q \in M$, there exists $g \in G$ such that $p = g \triangleright p$.
     * Note, G always acts **transitively** on G_p.
   - Example: Consider the action induced by representation of $\mathrm{SO}(2, \mathbb{R})$ as rotation matrices in $\mathrm{End}(\mathbb{R}^2)$. The orbit of any $p \in \mathbb{R}^2$ is the circle of radius $|p|$ centred at the origin.



   The orbit space the partition of $\mathbb{R}^2$ into concentric circles centred at the origin, plus the origin itself. So $\mathrm{SO}(2, \mathbb{R})$ is **not** transitive on $\mathbb{R}^2$. However, we can say it is **transitive** on the circles!

5. **stabiliser:** Let $\triangleright\colon\ G \times M \to M$ be a G-action on M. The **stabiliser** of $p \in M$ is
$$S_p := \{g \in G \mid g \triangleright p = p\}.$$
   - Note that for each $p \in M$, the stabiliser S_p is a **subgroup** of G.
   - A left G-action $\triangleright\colon\ G \times M \to M$ is said to be **free** if for all $p \in M$, we have $S_p = \{e\}$.
   - **stabiliser and orbit relation:** We have the following **bijection** ϕ
$$\begin{aligned} \phi : G/S_p &\to G_p \\ gS_p &\mapsto g \triangleright p \end{aligned}$$
   where $gS_p := \{g \bullet h | h \in S_p\}$ is a equivalent class called left coset induced from the subgroup (See my algebra notes).
   - Let $\triangleright\colon\ G \times M \to M$ be a **free action**. Then
$$g_1 \triangleright p = g_2 \triangleright p \quad \Leftrightarrow \quad g_1 = g_2.$$
   That means under free action, we have a one to one map between G and the orbit S_p for any point p on the manifold. Furthermore, $\triangleright\colon\ G \times M \to M$ is a **free** action, then
$$\forall\ p \in G :\ G_p \cong_{\text{diff}} G.$$

- For **right** action, the above definitions and conclusions also follow analogously.
- Examples:
  - † The action $\triangleright\colon\ G \times V \to V$ induced by a representation $R:\ G \to \mathrm{GL}(V)$ is never free since we always have $S_0 = G$.
  - † Consider the action $\triangleright\colon\ \mathrm{T}(n) \times \mathbb{R}^n \to \mathbb{R}^n$ of the n-dimensional translation group $\mathrm{T}(n)$ on $\mathbb{R}^n$. We have $\mathrm{T}(n)_p = \mathbb{R}^n$ for every $p \in \mathbb{R}^n$, a subjective. It is also easy to show that this action is free and transitive.
  - † Define $\triangleright\colon\ \mathrm{SO}(2,\mathbb{R}) \times \mathbb{R}^2 \backslash\{0\} \to \mathbb{R}^2 \backslash\{0\}$ to coincide with the action induced by the representation of $\mathrm{SO}(2,\mathbb{R}^2)$ on $\mathbb{R}^2$ for each non-zero point of $\mathbb{R}^2$. Then this action is **free**, since we have $S_p = \{\mathrm{id}_{\mathbb{R}^2}\}$ for $p \neq 0$, and the previous proposition implies
$$\forall\ p \in \mathbb{R}^2 \setminus \{0\}:\ \mathrm{SO}(2,\mathbb{R})_p \cong_{\text{diff}} \mathrm{SO}(2,\mathbb{R}) \cong_{\text{diff}} S^1.$$

6. **principal G-bundle:** Let G be a Lie group. A smooth bundle (E, π, M) is called a **principal G-bundle** if E is equipped with a **free right G-action** and
$$\begin{tikzcd} E \arrow[d, "\pi"'] & E \arrow[d, "\rho"] \\ M \arrow[r, phantom, "\cong_{\text{bdl}}"] & E/G \end{tikzcd}$$
where ρ is the **quotient map**, defined by sending each $p \in E$ to its equivalence class (i.e. orbit) in the orbit space E/G.
   - Since the right action of G on E is **free**, for each $p \in E$ we have
$$\mathrm{preim}_\rho(G_p) \cong_{\text{diff}} G_p \cong_{\text{diff}} G.$$
   So roughly speaking, a principal bundle is a bundle whose fibre at each point is a Lie group.
   - A principal G-bundle is a bundle which is **isomorphic** to a bundle whose **fibres are the orbits** under the right action of G, which are themselves **isomorphic to G** since the action is free.
   - The isomorphism in our definition enforces the **fibre-wise transitivity** since G acts transitively on each G_p by the definition of orbit.
7. **frame bundle:** We now define the frame bundle using **principal G-bundle**.
   (a) Let M be a smooth manifold. Consider the space
$$L_pM := \{(e_1, \dots, e_{\dim M}) \mid e_1, \dots, e_{\dim M} \text{ is a basis of } T_pM\} \cong_{\text{vec}} \mathrm{GL}(\dim M, \mathbb{R}).$$
   We define the frame bundle of M as
$$LM := \coprod_{p \in M} L_pM$$
   with the obvious projection map $\pi:\ LM \to M$ sending each basis $(e_1, \dots, e_{\dim M})$ to the unique point $p \in M$ such that $(e_1, \dots, e_{\dim M})$ is a basis of T_pM.
      - By proceeding similarly to the case of the tangent bundle, we can equip LM with a smooth structure inherited from that of M. We then find
$$\dim LM = \dim M + \dim T_pM = \dim M + (\dim M)^2.$$
   (b) We would now like to make $LM \xrightarrow{\pi} M$ into a principal $\mathrm{GL}(\dim M, \mathbb{R})$-bundle. We define a right $\mathrm{GL}(\dim M, \mathbb{R})$-action on LM by
$$(e_1, \dots, e_{\dim M}) \triangleleft g := (g^a{}_1 e_a, \dots, g^a{}_{\dim M} e_a),$$
   where $g^a{}_b$ are the components of the endomorphism $g \in \mathrm{GL}(\dim M, \mathbb{R})$ with respect to the standard basis on $\mathbb{R}^n$.
      - Note that if $(e_1, \dots, e_{\dim M}) \in L_pM$, we must also have $(e_1, \dots, e_{\dim M}) \triangleleft g \in L_pM$.
      - This action is **free** since
$$(e_1, \dots, e_{\dim M}) \triangleleft g = (e_1, \dots, e_{\dim M}) \Leftrightarrow (g^a{}_1 e_a, \dots, g^a{}_{\dim M} e_a) = (e_1, \dots, e_{\dim M})$$
      and hence, by linear independence, $g^a{}_b = \delta^a_b$, so $g = \mathrm{id}_{\mathbb{R}^n}$.

- Note that since all bases of each T_pM are related by some $g \in \mathrm{GL}(\dim M, \mathbb{R})$, $\triangleleft$ is also **fibre-wise transitive**.

(c) We now have to show that

$$\begin{tikzcd} LM \arrow[d, "\pi"'] \\ M \end{tikzcd} \cong_{\mathrm{bdl}} \begin{tikzcd} LM \arrow[d, "\rho"] \\ LM/\,\mathrm{GL}(\dim M, \mathbb{R}) \end{tikzcd}$$

i.e. that there exist smooth maps u and f such that the diagram

$$\begin{tikzcd} LM \arrow[r, shift left, "u"] \arrow[d, "\pi"'] & LM \arrow[l, shift left, "u^{-1}"] \arrow[d, "\rho"] \\ M \arrow[r, shift left, "f"] & LM/\,\mathrm{GL}(\dim M, \mathbb{R}) \arrow[l, shift left, "f^{-1}"] \end{tikzcd}$$

commutes. We can simply choose $u = u^{-1} = \mathrm{id}_{LM}$, while we define f as

$$\begin{aligned} f: \ M &\to LM/\,\mathrm{GL}(\dim M, \mathbb{R}) \\ p &\mapsto \mathrm{GL}(\dim M, \mathbb{R})_{(e_1, \dots, e_{\dim M})}, \end{aligned}$$

where $(e_1, \dots, e_{\dim M})$ is some basis of T_pM, i.e. $(e_1, \dots, e_{\dim M}) \in \mathrm{preim}_\pi(\{p\})$. $LM \xrightarrow{\pi} M$ is a principal G-bundle, called the **frame bundle** of M.

- Note that f is **well-defined** since every basis of T_pM gives rise to the same orbit in the orbit space $LM/\,\mathrm{GL}(\dim M, \mathbb{R})$.
  * Moreover, it is **injective** since
  $$f(p) = f(p') \ \Leftrightarrow \ \mathrm{GL}(\dim M, \mathbb{R})_{(e_1, \dots, e_{\dim M})} = \mathrm{GL}(\dim M, \mathbb{R})_{(e'_1, \dots, e'_{\dim M})},$$
  which is true only if $(e_1, \dots, e_{\dim M})$ and $(e'_1, \dots, e'_{\dim M})$ are basis of the same tangent space, so $p = p'$.
  * It is clearly **surjective** since every orbit in $LM/\,\mathrm{GL}(\dim M, \mathbb{R})$ is the orbit of some basis of some tangent space T_pM at some point $p \in M$.
- The **inverse** map is given explicitly by
$$\begin{aligned} f^{-1}: \qquad LM/\,\mathrm{GL}(\dim M, \mathbb{R}) &\to M \\ \mathrm{GL}(\dim M, \mathbb{R})_{(e_1, \dots, e_{\dim M})} &\mapsto \pi((e_1, \dots, e_{\dim M})). \end{aligned}$$
- Finally, we have
$$(\rho \circ \mathrm{id}_{LM})(e_1, \dots, e_{\dim M}) = \mathrm{GL}(\dim M, \mathbb{R})_{(e_1, \dots, e_{\dim M})} = (f \circ \pi)(e_1, \dots, e_{\dim M})$$
and thus $LM \xrightarrow{\pi} M$ is a principal G-bundle, called the **frame bundle** of M.

8. **principal bundle morphisms:** Recall that a bundle morphism (also called simply a bundle map) between two bundles (E, π, M) and (E', π', M') is a pair of maps (u, f) such that the diagram

$$\begin{tikzcd} E \arrow[r, "u"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\ M \arrow[r, "f"] & M' \end{tikzcd}$$

commutes, that is, $f \circ \pi = \pi' \circ u$. Let (P, π, M) and (Q, π', N) both be principal G-bundles. A **principal bundle morphism** from (P, π, M) to (Q, π', N) is a pair of smooth maps (u, f) such that the diagram

$$\begin{tikzcd} P \arrow[r, "u"] & Q \\ P \arrow[u, "\triangleleft G"] \arrow[r, "u"] \arrow[d, "\pi"'] & Q \arrow[u, "\blacktriangleleft G"'] \arrow[d, "\pi'"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

commutes, that is for all $p \in P$ and $g \in G$, we have

$$\begin{aligned} (f \circ \pi)(p) &= (\pi' \circ u)(p) \\ u(p \lhd g) &= u(p) \blacktriangleleft g. \end{aligned}$$

- **compared with bundle morphisms:** one extra condition of homomorphism of G-manifolds: $u(p \lhd g) = u(p) \blacktriangleleft g$. Below it will go back to require the ρ-equivariant map.
- **principal bundle isomorphism:** A principal bundle morphism between two principal G-bundles is an **isomorphism or diffeomorphism of principal bundles** if it is also a **bundle isomorphism**.
- **principal bundle morphism definition with different group:** Let (P, π, M) be a principal G-bundle, let (Q, π', N) be a principal H-bundle, and let $\rho : G \to H$ be a Lie group homomorphism. A **principal bundle morphism** from (P, π, M) to (Q, π', N) is a pair of smooth maps (u, f) such that the diagram

$$\begin{tikzcd} P \times G \arrow[r, "u \times \rho"] & Q \times H \\ P \arrow[u, "\lhd"] \arrow[r, "u"] \arrow[d, "\pi"'] & Q \arrow[u, "\blacktriangleleft"'] \arrow[d, "\pi'"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

commutes, $\forall\, p \in P : \forall\, g \in G$ and u is a **ρ-equivariant map**:

$$\begin{aligned} (f \circ \pi)(p) &= (\pi' \circ u)(p) \\ u(p \lhd g) &= u(p) \blacktriangleleft \rho(g). \end{aligned}$$

- **principal bundle isomorphism definition with different group:** A principal bundle morphism between principal G-bundle and a principal H-bundle is an **isomorphism (or diffeomorphism) of principal bundles** if it is also a bundle **isomorphism** and ρ is a **Lie group isomorphism**.

9. **principal G-bundles over the same base manifold $\Rightarrow$ morphism must be diffeomorphism:** Let (P, π, M) and (Q, π', M) be principal G-bundles over the same base manifold M. Then, any $u : P \to Q$ such that (u, id_M) is a principal bundle morphism is necessarily a diffeomorphism.

$$\begin{tikzcd} P \arrow[rr, "u"] & & Q \\ P \arrow[u, "\lhd G"] \arrow[rr, "u"] \arrow[dr, "\pi"'] & & Q \arrow[u, "\blacktriangleleft G"'] \arrow[dl, "\pi'"] \\ & M & \end{tikzcd}$$

we check bijective of u. Let $p_1, p_2 \in P$ be such that $u(p_1) = u(p_2)$. Then

$$\pi(p_1) = \pi'(u(p_1)) = \pi'(u(p_2)) = \pi(p_2),$$

that is, p_1 and p_2 belong to the same fibre. As the action of G on P is fibre-wise transitive, there is a unique $g \in G$ such that $p_1 = p_2 \lhd g$. Then

$$u(p_1) = u(p_2 \lhd g) = u(p_2) \blacktriangleleft g = u(p_1) \blacktriangleleft g,$$

so $g \in S_{u(p_1)}$, but since $\blacktriangleleft$ is free, we have $g = e$ and thus

$$p_1 = p_2 \lhd e = p_2.$$

Therefore u is injective. Surjective of u is trivial.

10. **trivial principal G-bundle:** A principal G-bundle (P, π, M) if it is called **trivial** if it is isomorphic as a principal G-bundle to the principal G-bundle $(M \times G, \pi_1, M)$ where π_1 is the projection onto the first component and the **action** is defined as

$$\begin{aligned} \blacktriangleleft \colon\ & (M \times G) \times G \to M \times G \\ & ((p, g), g') \mapsto (p, g) \blacktriangleleft g' := (p, g \bullet g'). \end{aligned}$$

$$\begin{tikzcd} P \arrow[r, "u"] & M \times G \\ P \arrow[u, "\triangleleft G"] \arrow[r, "u"] \arrow[dr, "\pi"'] & M \times G \arrow[u, "\blacktriangleleft G"'] \arrow[d, "\pi_1"] \\ & M \end{tikzcd}$$

- **existence of global section $\Rightarrow$ whether a principal bundle is trivial:** A principal G-bundle (P, π, M) is trivial if, and only if, there exists a smooth section $\sigma \in \Gamma(P)$, that is, a smooth $\sigma : M \to P$ such that $\pi \circ \sigma = \mathrm{id}_M$.
  - $*$ This is quite interesting. A smooth global section is enough to determine the principle bundle is trivial or not!
  - $*$ So if we have a principal frame bundle that has a global section then it must be trivial.
  - $*$ The existence of a section on the frame bundle LM can be reduced to the existence of $(\dim M)$ non-everywhere vanishing linearly independent vector fields on M. Since no such vector field exists on even-dimensional spheres, LS^{2n} is always non-trivial (**hairy ball theorem**).

($\Rightarrow$) Suppose is trivial. We can define

$$\begin{aligned} \sigma :\ & M \to P \\ & m \mapsto u^{-1}(m, e), \end{aligned}$$

where e is the identity of G.

($\Leftarrow$) Suppose that there exists a smooth section $\sigma : M \to P$. Let $p \in P$ and consider the point $\sigma(\pi(p)) \in P$. We have

$$\pi(\sigma(\pi(p))) = \mathrm{id}_M(\pi(p)) = \pi(p),$$

hence $\sigma(\pi(p))$ and p belong to the same fibre, and thus there exists a unique group element in G which links the two points via $\triangleleft$. Since this element depends on both σ and p, let us denote it by $\chi_\sigma(p)$. Then, χ_σ defines a function

$$\begin{aligned} \chi_\sigma :\ & P \to G \\ & p \mapsto \chi_\sigma(p) \end{aligned}$$

and we can write

$$\forall\ p \in P :\ p = \sigma(\pi(p)) \triangleleft \chi_\sigma(p).$$

In particular, for any other $g \in G$ we have $p \triangleleft g \in P$ and thus

$$p \triangleleft g = \sigma(\pi(p \triangleleft g)) \triangleleft \chi_\sigma(p \triangleleft g) = \sigma(\pi(p)) \triangleleft \chi_\sigma(p \triangleleft g),$$

where the second equality follows from the fact that the fibres of P are precisely the orbits under the action of G. On the other hand, we can act on the right with an arbitrary $g \in G$ directly to obtain

$$p \triangleleft g = (\sigma(\pi(p)) \triangleleft \chi_\sigma(p)) \triangleleft g = \sigma(\pi(p)) \triangleleft (\chi_\sigma(p) \bullet g).$$

Combining the last two equations yields

$$\sigma(\pi(p)) \triangleleft \chi_\sigma(p \triangleleft g) = \sigma(\pi(p)) \triangleleft (\chi_\sigma(p) \bullet g)$$

and hence

$$\chi_\sigma(p \triangleleft g) = (\chi_\sigma(p) \bullet g).$$

We can now define the map

$$\begin{aligned} u_\sigma :\ & P \to M \times G \\ & p \mapsto (\pi(p), \chi_\sigma(p)). \end{aligned}$$

By our previous conclusion, it suffices to show that u_σ is a principal bundle morphism.

$$\begin{tikzcd} P \arrow[r, "u_\sigma"] & M \times G \\ P \arrow[u, "\triangleleft G"] \arrow[r, "u_\sigma"] \arrow[dr, "\pi"'] & M \times G \arrow[u, "\blacktriangleleft G"'] \arrow[d, "\pi_1"] \\ & M \end{tikzcd}$$

By definition, we have

$$(\pi_1 \circ u_\sigma)(p) = \pi_1(\pi(p), \chi_\sigma(p)) = \pi(p)$$

for all $p \in P$, so the lower triangle commutes. Moreover, we have

$$\begin{aligned} u_\sigma(p \lhd g) &= (\pi(p \lhd g), \chi_\sigma(p \lhd g)) \\ &= (\pi(p), \chi_\sigma(p) \bullet g)) \\ &= (\pi(p), \chi_\sigma(p)) \blacktriangleleft g \\ &= u_\sigma(p) \blacktriangleleft g \end{aligned}$$

for all $p \in P$ and $g \in G$, so the upper square also commutes and hence (P, π, M) is a trivial bundle.

- Note even we do not need to use u_σ^{-1} above since we use the morphism must be diffeomorphism, we still can write down it as
$$u_\sigma^{-1}(m, g) = \sigma(m) \lhd g$$
. This will be used locally in Section 23.

11. **restriction and extension:** Assumptions
    (a) Let H be a **closed Lie subgroup** of G.
    (b) Let (P, π, M) be a principal H-bundle and (Q, π', M) a principal G-bundle with the same base space.

    If there exists a **principal bundle morphism** (u, f) from (P, π, M) to (Q, π', M), i.e. a smooth bundle morphism which is equivariant with respect to the **inclusion** of H into G,

$$\begin{tikzcd} P \times G & & Q \times H \arrow[ll, "u \times i"'] \\ P \arrow[u, "\lhd"] & & Q \arrow[ll, "u"'] \arrow[u, "\blacktriangleleft"'] \arrow[d, "\pi'"'] \\ M \arrow[from=u, "\pi"] \arrow[rr, "f"] & & M \end{tikzcd}$$

    where i is the inclusion map, then (P, π, M) is called an H**-restriction** of (Q, π', M), while (Q, π', M) is called a G**-extension** of (P, π, M).

    - Let H be a closed Lie subgroup of G. We have that
      (a) Any principal H-bundle can be extended to a principal G-bundle.
      (b) A principal G-bundle (P, π, M) can be restricted to a principal H-bundle if, and only if, the bundle $(P/H, \pi', M)$ has a **section**.
    - Examples:
      † The bundle $(LM/\,\mathrm{SO}(d), \pi, M)$ always has a section, and since $\mathrm{SO}(d)$ is a closed Lie subgroup of $\mathrm{GL}(d, \mathbb{R})$, the frame bundle can be restricted to a principal $\mathrm{SO}(d)$-bundle. This is related to the fact that any manifold can be equipped with a Riemannian metric.
      † The bundle $(LM/\,\mathrm{SO}(1, d-1), \pi, M)$ may or may not have a section. For example, the bundle $(LS^2/\,\mathrm{SO}(1,1), \pi, S^2)$ does not admit any section, and hence we cannot restrict $(LS^2/\,\mathrm{SO}(1,1), \pi, S^2)$ to a principal $\mathrm{SO}(1,1)$-bundle, even though $\mathrm{SO}(1,1)$ is a closed Lie subgroup of $\mathrm{GL}(2, \mathbb{R})$. This is related to the fact that the 2-sphere cannot be equipped with a Lorentzian metric.

21 Associated Fibre Bundles

$$\begin{tikzcd} \text{set} \arrow[r] & \text{topological space} \arrow[r] & \text{topological manifold} \arrow[r] & \text{differentiable manifold} \arrow[r] \arrow[d] & \text{principal fibre bundle} \arrow[r, red] & \text{associated fibre bundles} \\ & & & \text{Lie group} \arrow[ur] \arrow[d, shift left] & & \\ & & & \text{Lie algebra} \arrow[u, shift left] & & \end{tikzcd}$$

An associated fibre bundle is a fibre bundle which is associated (in a precise sense) to a principal G-bundle. Associated bundles are related to their underlying principal bundles in a way that models the transformation law for components under a change of basis.

A summary:

- We first give the definition of **associated fibre bundle.**
- We then introduce three important examples: **tangent bundle**, (p, q)**-tensor bundle** and (p, q)**-tensor** ω**-density bundle** that are all associated to the **frame bundle**.
- The associated bundle morphism is the defined.

1. **associated fibre bundle:** Let (P, π, M) be a principal G-bundle (recall it is a **right** G-action) and let F be a smooth manifold equipped with a **left** G-action $\triangleright$. We define
   (a) $P_F := (P \times F)/\sim_G$, where $\sim_G$ is the **equivalence relation**
   $$(p, f) \sim_G (p', f') \quad :\Leftrightarrow \quad \exists\, g \in G : \begin{cases} p' &= p \triangleleft g \\ f' &= g^{-1} \triangleright f \end{cases}$$
   Note, P_F is the quotient set, and $[p, f]$ is a point in P_F, the equivalent class that contains (p, f).
   (b) The **projection map**
   $$\begin{aligned} \pi_F : \quad & P_F \to M \\ & [p, f] \mapsto \pi(p), \end{aligned}$$
   which is well-defined since, if $[p', f'] = [p, f]$, then for some $g \in G$
   $$\pi_F([p', f']) = \pi_F([p \triangleleft g, g^{-1} \triangleright f]) := \pi(p \triangleleft g) = \pi(p) =: \pi_F([p, f]).$$

   The **associated bundle** (to (P, π, M), F and $\triangleright$) is the bundle (P_F, π_F, M).
   - **notation:** we have simplified the notation of $[(p, f)]$ to $[p, f]$.
   - Roughly speaking, the associated fibre bundle makes that the base manifold keeps $P/G \cong_{\text{diff}} M$, and now the fibre becomes F.
   - The set $[p, f]$ includes **all the pairs** $(p \triangleleft g, g^{-1} \triangleright f)$, i.e. $\{(p \triangleleft g, g^{-1} \triangleright f) \mid g \in G\}$.
2. **tangent bundle is associated to frame bundle:** Recall that the frame bundle (LM, π, M) is a principal $\mathrm{GL}(d, \mathbb{R})$-bundle, where $d = \dim M$, with **right** G-action $\triangleleft$: $LM \times G \to LM$ given by
   $$(e_1, \dots, e_d) \triangleleft g := (g^a{}_1 e_a, \dots, g^a{}_d e_a).$$
   Let $F := \mathbb{R}^d$ (as a smooth manifold) and define a **left** action
   $$\begin{aligned} \triangleright\colon\ & \mathrm{GL}(d, \mathbb{R}) \times \mathbb{R}^d \to \mathbb{R}^d \\ & (g, x) \mapsto g \triangleright x, \end{aligned}$$
   where
   $$(g \triangleright x)^a := g^a{}_b x^b.$$
   Then $(LM_{\mathbb{R}^d}, \pi_{\mathbb{R}^d}, \mathbb{R}^d)$ is the **associated bundle**. In fact, we have a **bundle isomorphism**
   $$\begin{tikzcd} LM_{\mathbb{R}^d} \arrow[r, "u"] \arrow[d, "\pi_{\mathbb{R}^d}"'] & TM \arrow[d, "\pi"] \\ M \arrow[r, "\mathrm{id}_M"] & M \end{tikzcd}$$
   where (TM, π, M) is the tangent bundle of M, and u is defined as
   $$\begin{aligned} u : \quad & LM_{\mathbb{R}^d} \to TM \\ & [(e_1, \dots, e_d), x] \mapsto x^a e_a. \end{aligned}$$

- The inverse map $u^{-1} : TM \to LM_{\mathbb{R}^d}$ works as follows. Given any $X \in TM$, pick any basis $(e_1, \dots, e_d)$ of the tangent space at the point $\pi(X) \in M$, i.e. any element of $L_{\pi(X)}M$. Decompose X as $x^a e_a$, with each $x^a \in \mathbb{R}$, and define

$$u^{-1}(X) := [(e_1, \dots, e_d), x].$$

  * The map u^{-1} is well-defined since, while the pair $((e_1, \dots, e_d), x) \in LM \times \mathbb{R}^d$ clearly depends on the choice of basis, the equivalence class

$$[(e_1, \dots, e_d), x] \in LM_{\mathbb{R}^d} := (LM \times \mathbb{R}^d)/\sim_G$$

    does not. It includes all pairs $((e_1, \dots, e_d) \triangleleft g, g^{-1} \triangleright x)$ for every $g \in \mathrm{GL}(d, \mathbb{R})$, i.e. every choice of basis together with the "right" components $x \in \mathbb{R}^d$.

3. (p,q)**-tensor bundle is associated to frame bundle:** Consider the principal $\mathrm{GL}(d,\mathbb{R})$-bundle (LM, π, M) again, with the same right action as before. This time we define

$$F := (\mathbb{R}^d)^{\times p} \times (\mathbb{R}^{d^*})^{\times q} := \underbrace{\mathbb{R}^d \times \cdots \times \mathbb{R}^d}_{p \text{ times}} \times \underbrace{\mathbb{R}^{d^*} \times \cdots \times \mathbb{R}^{d^*}}_{q \text{ times}}$$

with **left** $\mathrm{GL}(d,\mathbb{R})$-action $\triangleright$: $\mathrm{GL}(d,\mathbb{R}) \times F \to F$ given by

$$(g \triangleright f)^{a_1 \cdots a_p}{}_{b_1 \cdots b_q} := g^{a_1}{}_{\widetilde{a}_1} \cdots g^{a_p}{}_{\widetilde{a}_p} (g^{-1})^{\widetilde{b}_1}{}_{b_1} \cdots (g^{-1})^{\widetilde{b}_q}{}_{b_q} f^{\widetilde{a}_1 \cdots \widetilde{a}_p}{}_{\widetilde{b}_1 \cdots \widetilde{b}_q}.$$

Then, the **associated bundle** (LM_F, π_F, M) thus constructed is **isomorphic** to $(T^p_q M, \pi, M)$, the (p,q)-tensor bundle on M.

4. (p,q)**-tensor** ω**-density bundle is associated to frame bundle:** Let M be a smooth manifold and let (LM, π, M) be its frame bundle, with right $\mathrm{GL}(d,\mathbb{R})$-action as above. Let $F := (\mathbb{R}^d)^{\times p} \times (\mathbb{R}^{d^*})^{\times q}$ and define a left $\mathrm{GL}(d,\mathbb{R})$-action on F by

$$(g \triangleright f)^{a_1 \cdots a_p}{}_{b_1 \cdots b_q} := (\det g^{-1})^{\omega}\, g^{a_1}{}_{\widetilde{a}_1} \cdots g^{a_p}{}_{\widetilde{a}_p} (g^{-1})^{\widetilde{b}_1}{}_{b_1} \cdots (g^{-1})^{\widetilde{b}_q}{}_{b_q} f^{\widetilde{a}_1 \cdots \widetilde{a}_p}{}_{\widetilde{b}_1 \cdots \widetilde{b}_q},$$

where $\omega \in \mathbb{Z}$. Then the associated bundle (LM_F, π_F, M) is called the (p,q)**-tensor** ω**-density bundle** on M.

   - Its sections are called (p,q)**-tensor densities of weight** ω.
   - If $\omega = 0$, we recover the (p,q)**-tensor bundle** on M.
   - **scalar density:** If $F = \mathbb{R}$ (i.e. $p = q = 0$), the left action reduces to

$$(g \triangleright f) = (\det g^{-1})^{\omega} f,$$

     which is the transformation law for a **scalar density of weight** ω.
   - **compare to tensor fields:** If $\mathrm{GL}(d,\mathbb{R})$ is restricted in such a way that we always have $(\det g^{-1}) = 1$, then tensor densities are indistinguishable from ordinary tensor fields. So in special relativity with the **Lorentz group** $\mathrm{O}(3,1)$ (which is a subgroup of the orthogonal group, see Section 19), they are the same.
   - **determinant:** Recall that if B is a **bilinear form** on a K-vector space V, the determinant of B is **not** independent from the choice of basis. If $\{e_a\}$ and $\{e'_b := g^a{}_b e_a\}$ are both basis of V, where $g \in \mathrm{GL}(\dim V, K)$, then with $f = \det B$, we have

$$(g \triangleright \det B) = (\det g^{-1})^2 \det B.$$

     Here we now know that determinant of a bilinear form is a **scalar density of weight** 2**.**
   - **integration:** We can only integrate the scalar density (a form) but not a general function! Recall in integration if we change a basis, an additional factor of det will appear. The scalar density will then vanish the effect of basis transformation.

5. **important clarification:**

   (a) If $G = \mathrm{GL}(d,\mathbb{R})$ the general linear group, then we get that tangent bundle "equals to" the frame bundle in the sense of bundle isomorphism! However, now, in some sense we have more structure, the group action. In this case, we care about tensors (field) that can be transformed under the $\mathrm{GL}(d,\mathbb{R})$ group.

(b) However if we restrict G to other **Lie subgroup** of $\mathrm{GL}(d, \mathbb{R})$ like the $\mathrm{SO}(d)$ group or **Lorentz group** $\mathrm{O}(3,1)$ (which is also a subgroup of the orthogonal group), the associated fibre bundle now of course does **not** "equal to" the tangent bundle. Now, the transformation between tensors are restricted to the G group. For example if G is the lorentz group, then we only care about the tensors (field) like the tangent vectors that can only be transformed using lorentz transformation.

(c) For the tangent bundle and the general (p,q)-tensor bundle on M, they are associated to the **same frame principle** $\mathrm{GL}(d,\mathbb{R})$**-bundle.** So if we transform using a action g from $\mathrm{GL}(d,\mathbb{R})$, all the tensors in the tangent bundle or the (p,q)-tensor bundle are **transformed simultaneous** with the same transform g! They are tied together now.

6. **associated bundle morphism:** Let (P_F, π_F, M) to (Q_F, π'_F, N) be the associated bundles (with the same fibre F) of two principal G-bundles (P, π, M) and (Q, π', N). An **associated bundle morphism** between the associated bundles is a bundle morphism $(\widetilde{u}, v)$ between them such that

   (a) for some u, the pair (u, v) is a principal bundle morphism between the underlying principal G-bundles, and

   (b) we need
$$\widetilde{u}([p,f]) := [u(p), f].$$

   - **diagram representation:** The following two diagrams both commute.

$$\begin{tikzcd} P_F \arrow[r, "\widetilde{u}"] \arrow[d, "\pi_F"'] & Q_F \arrow[d, "\pi'_F"] \\ M \arrow[r, "v"] & N \end{tikzcd} \qquad \begin{tikzcd} P \arrow[r, "u"] & Q \\ P \arrow[u, "\lhd G"] \arrow[r, "u"] \arrow[d, "\pi"'] & Q \arrow[u, "\blacktriangleleft G"'] \arrow[d, "\pi'"] \\ M \arrow[r, "v"] & N \end{tikzcd}$$

   - **associated bundle isomorphism:** An associated bundle morphism $(\widetilde{u}, v)$ is an **associated bundle isomorphism** if $\widetilde{u}$ and v are invertible and $(\widetilde{u}^{-1}, v^{-1})$ is also an associated bundle morphism.
   - Note two associated F-fibre bundles may be **isomorphic as bundles** but **not as associated bundles.**

7. **trivial associated bundle:**

   (a) Recall that an F-fibre bundle (E, π, M) is called **trivial** if there exists a bundle isomorphism

$$\begin{tikzcd} F \arrow[r] & E \arrow[r, "u"] \arrow[dr, "\pi"'] & M \times F \arrow[d, "\pi_1"] \\ & & M \end{tikzcd}$$

   (b) Recall a principal G-bundle is called **trivial** if there exists a principal bundle isomorphism

$$\begin{tikzcd} P \arrow[r, "u"] & M \times G \\ P \arrow[u, "\lhd G"] \arrow[r, "u"] \arrow[dr, "\pi"'] & M \times G \arrow[u, "\blacktriangleleft G"'] \arrow[d, "\pi_1"] \\ & M \end{tikzcd}$$

An **associated bundle** (P_F, π_F, M) is called **trivial** if the underlying principal G-bundle (P, π, M) is **trivial**.

   - A trivial associated bundle is a trivial fibre bundle. But the converse does not hold.

8. **important theorem:** (See Section 25 for more details!) The sections

$$\sigma : M \to P_F$$

of an associated bundle $P_F \xrightarrow{\pi_F} M$ (associated to (P, π, M)) are in the one-to-one correspondence to G**-equivariant** F-valued functions

$$\phi : P \to P_F$$

on the underline principle bundle, where ϕ is G**-equivariant means that** $\phi(p \lhd g) = g^{-1} \rhd \phi(p)$.

- **compare with vector field:** For tangent bundle and vector field, we cannot simply say a vector field is a vector valued function on each point on the base manifold M. As we have explained in Section 7, locally, we may think it as a function because of **local trivialization**, but globally, it is not. However, if we now use associated bundle TM to LM, we can view the vector field as a vector valued function on each point on the principle GL-bundle LM.
- hint: give $\phi : P \to P_F$ we construct the section $\sigma_\phi : M \to P_F$ as

$$\sigma_\phi := [p, \phi(p)]$$

where $p \in \pi^{-1}(x)$ for $x \in M$. Conversely, given a section we can then construct a F valued function on P.

22 Connections and The Connection 1-forms

$$\begin{tikzcd}
\text{set} \arrow[r] & \text{topological space} \arrow[r] & \text{topological manifold} \arrow[r] & \text{differentiable manifold} \arrow[r] \arrow[d] & \text{principal fibre bundle} \arrow[r] \arrow[d, Rightarrow] & \text{associated fibre bundles} \arrow[d, Rightarrow] \\
 & & & \text{Lie group} \arrow[ur] \arrow[d, shift left] & \text{connection on it} \arrow[r, bend right] & \text{connection on it} \\
 & & & \text{Lie algebra} \arrow[u, shift left] & &
\end{tikzcd}$$

The idea of a connection is to make a choice of how to "connect" the individual points in "neighbouring" fibres in a principal fibre bundle. What a connection really is, is just additional structure on a principal bundle consisting of **an assignment of an horizontal subspace H_pP to each** $p \in P$ with the compatibility with the right action of the Lie group. Such an assignment is, in fact, equivalent to a certain **Lie-algebra-valued one-form on the principal bundle (not the base manifold).**

Later, we will see that a connection on a principal bundle induces a **parallel transport map on the principal bundle**, which in turn induces a **parallel transport map on any of its associated bundles** (e.g. the associated **vector bundles** where each fibre carry a vector space structure, see Section 24).

$$\begin{array}{c}
\text{connection (and parallel transport) on } P \\
\Downarrow \\
\text{connection (and parallel transport) on any associated fibre bundle} \\
\Downarrow \\
\text{covariant derivative (and parallel transport) on any associated vector bundle}
\end{array}$$

A summary:

- We define the **vertical subspace** as the kernel of the linear mapping namely the push-forward of the projection π. And then the **horizontal subspace** is introduced as the complementary space of the **vertical subspace**.
- We give the connection definition as an **an assignment of an horizontal subspace H_pP to each** $p \in P$.
- We give the connection definition as **Lie-algebra-valued one-form** on the **principal bundle** which is equivalent to the above definition.

▶ Finally, as a side mark, we explain what is a **Lie-algebra-valued one-form** and how the pull-back works on it.

$$\begin{tikzcd} P \\ P \arrow[u, "\triangleleft G"] \arrow[d, "\pi"] \\ M \end{tikzcd}$$

1. **vector field induced from element in T_eG:** Let (P, π, M) be a principal G-bundle. Given $A \in T_eG$, we define $X^A \in \Gamma(TP)$ by

$$\begin{aligned} X_p^A :\ \mathcal{C}^\infty(P) &\xrightarrow{\sim} \mathbb{R} \\ f &\mapsto [f(p \triangleleft \exp(tA))]'(0), \end{aligned}$$

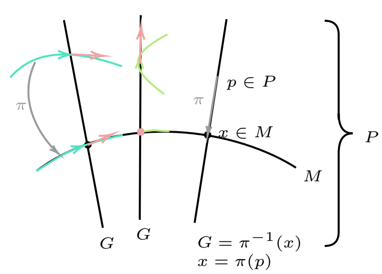
where the derivative is to be taken with respect to t.
We also define the maps
$$i : T_eG \to \Gamma(TP)$$
with
$$\begin{aligned} i_p :\ T_eG &\to T_pP \\ A &\mapsto X_p^A, \end{aligned}$$

   - i is a **Lie algebra homomorphism:**
$$i[A, B] = [i(A), i(B)]$$
   - $p \triangleleft \exp(tA)$ is just a curve go through p.

2. **vertical subspace:** Let (P, π, M) be a principal bundle and let $p \in P$. The **vertical subspace** at p is the **vector subspace of** T_pP given by

$$\begin{aligned} T_pP \supseteq V_pP &:= \ker((\pi_*)_p) \\ &= \{X_p \in T_pP \mid (\pi_*)_p(X_p) = 0\}. \end{aligned}$$

   - Visualization of the push-forward of π_* and the **vertical subspace**:



   - Recall the **curve push-forward**, in order to make the projection of the curve under π_* to be zeros. We need the curve around p is fully inside the fibre $G_p \cong_{\text{diff}} G$. Roughly speaking, in some sense the kernel "equals" to the full fibre. Note, here the explanation is not that rigorous since G is only a manifold, not a vector space. please do not think velocity of the curve in G here, think in the curve way that the full curve is in G.
   - **important conclusion:** For all $A \in T_eG$ and $p \in P$, we have $X_p^A \in V_pP$.
     * This is because the entire curve $\gamma(t) = p \triangleleft \exp(tA)$ is fully inside G_p! The action of G simply permutes the elements within each fibre.
     * So $i_p :\ T_eG \xrightarrow{\sim} V_pP$.
   - Furthermore, the map $i_p :\ T_eG \xrightarrow{\sim} V_pP$ is now a **bijection.**

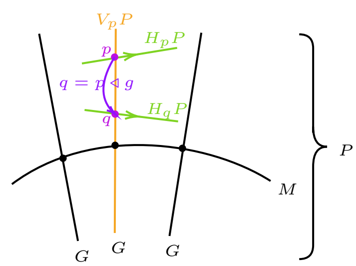
    * So it is a **linear isomorphism**.
    * Roughly speaking, this is because the dimension of T_eG equals the dimension of the Lie group (as a manifold).

3. **horizontal subspace:** Let (P, π, M) be a principal bundle and let $p \in P$. A **horizontal subspace** at p a vector subspace H_pP of T_pP which is **complementary to** V_pP, i.e.

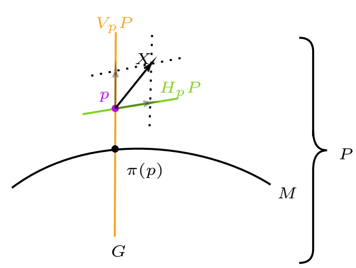
$$T_pP = H_pP \oplus V_pP.$$

   - Note the **vertical subspace is fixed**. But the choice of horizontal space at $p \in P$ is **not unique**. However, once a choice is made, there is a **unique decomposition** of each $X_p \in T_pP$ as
$$X_p = \mathrm{hor}(X_p) + \mathrm{ver}(X_p),$$
with $\mathrm{hor}(X_p) \in H_pP$ and $\mathrm{ver}(X_p) \in V_pP$.

4. **connection (as an assignment of horizontal subspace):** A **connection** on a principal G-bundle (P, π, M) is a choice of horizontal space at each $p \in P$ such that
   (a) For all $g \in G$, $p \in P$ and $X_p \in H_pP$, we have
$$(\triangleleft\, g)_* X_p \in H_{p \triangleleft g}P,$$
where $(\triangleleft\, g)_*$ is the **push-forward** of the map $(- \triangleleft g) : P \to P$ and it is a **bijection**. We can also write this condition more concisely as
$$(\triangleleft\, g)_*(H_pP) = H_{p \triangleleft g}P.$$



   (b) For every smooth $X \in \Gamma(TP)$, the two summands in the unique decomposition
$$X|_p = \mathrm{hor}(X|_p) + \mathrm{ver}(X|_p)$$
at each $p \in P$, extend to **smooth vector field** $\mathrm{hor}(X), \mathrm{ver}(X) \in \Gamma(TM)$.



   - The definition formalises the idea that the assignment of an H_pP to each $p \in P$ should be "**smooth**"z:
     * (a) indicates smooth within each fibre
     * (b) indicates smooth between different fibres.
   - Note, for each $X_p \in T_pP$, both $\mathrm{hor}(X_p)$ and $\mathrm{ver}(X_p)$ depend on the choice of H_pP.
   - **horizontal subspace connection $\Rightarrow$ Lie-algebra-valued one-form**:
$$\omega : \Gamma(T_P) \xrightarrow{\sim} T_eG$$

with

$$\begin{aligned} \omega_p:\ T_pP &\xrightarrow{\sim} T_eG \\ X_p &\mapsto \omega_p(X_p) := i_p^{-1}(\mathrm{ver}(X_p)) \end{aligned}$$

The map $\omega:\ p \to \omega_p$ sending each $p \in P$ to the T_eG-valued one-form ω_p is called the **connection one-form** with respect to the connection.

 * This is a one form because of the linearity of both $i_p^{-1}(\cdot)$ and $\mathrm{ver}(\cdot)$.
 * It is a one form on P **not** on the base manifold M.

- **Lie-algebra-valued one-form ⇒ horizontal subspace (i.e. the connection)**: The choice of horizontal spaces can be recovered from ω by

$$H_pP = \ker(\omega_p).$$

5. **connection (as a Lie-algebra-valued one-form):** Of course, not every (Lie-algebra-valued) one-form on P is such that $\ker(\omega_p)$ gives a connection on the principal bundle. What we would now like to do is to study **3 crucial properties of** ω. We may **re-define** the notion of connection in terms of a connection one-form using the properties without first defining a horizontal subspace connection.

A connection one-form ω with respect to a connection satisfies

(a) For all $p \in P$, we have $\omega_p(X_p^A) = A$, that is $\omega_p \circ i_p = \mathrm{id}_{T_eG}$. In short $\omega(X^A) = A$.

$$\begin{tikzcd} T_eG \arrow[r, "i_p"] \arrow[dr, "\mathrm{id}_{T_eG}"'] & V_pP \arrow[d, "\omega_p|_{V_pP}"] \\ & T_eG \end{tikzcd}$$

(b) $((\lhd\, g)^*\omega)|_p(X_p) = (\mathrm{Ad}_{g^{-1}})(\omega_p(X_p))$

$$\begin{tikzcd} T_pP \arrow[r, "\omega_p"] \arrow[dr, "((\lhd g)^*\omega)|_p"'] & T_eG \arrow[d, "(\mathrm{Ad}_{g^{-1}})"] \\ & T_eG \end{tikzcd}$$

(c) ω is a **smooth** one-form.

We first prove that for all $p \in P$, $g \in G$ and $A \in T_eG$, we have

$$(\lhd\, g)_* X_p^A = X_{p \lhd g}^{(\mathrm{Ad}_{g^{-1}})A}.$$

Let $f \in \mathcal{C}^\infty(P)$ be arbitrary. We have

$$\begin{aligned} (\lhd\, g)_* X_p^A(f) &= X_p^A(f \circ (- \lhd g)) \\ &= [f(p \lhd \exp(tA) \lhd g)]'(0) \\ &= [f(p \lhd g \lhd g^{-1} \lhd \exp(tA) \lhd g)]'(0) \\ &= [f(p \lhd g \lhd (g^{-1} \bullet \exp(tA) \bullet g)]'(0) \\ &= [f(p \lhd g \lhd \mathrm{Ad}_{g^{-1}}(\exp(tA))]'(0) \\ &= [f(p \lhd g \lhd \exp(t(\mathrm{Ad}_{g^{-1}})A)]'(0) \\ &= X_{p \lhd g}^{(\mathrm{Ad}_{g^{-1}})A}(f), \end{aligned}$$

which is what we wanted.

(a) Since $X_p^A \in V_pP$, by definition of ω we have

$$\omega_p(X_p^A) := i_p^{-1}(\mathrm{ver}(X_p^A)) = i_p^{-1}(X_p^A) = A.$$

(b) First observe that the left hand side is linear in X_p. Consider the two cases

b.1) Suppose that $X_p \in V_pP$. Then $X_p = X_p^A$ for some $A \in T_eG$. Hence

$$\begin{aligned} ((\lhd\, g)^*\omega)|_p(X_p^A) &= \omega_{p\lhd g}((\lhd\, g)_* X_p^A) \\ &= \omega_{p \lhd g}\left(X_{p\lhd g}^{(\mathrm{Ad}_{g^{-1}})A}\right) \\ &= (\mathrm{Ad}_{g^{-1}})A \\ &= (\mathrm{Ad}_{g^{-1}})(\omega_p(X_p^A)) \end{aligned}$$

b.2) Suppose now that $X_p \in H_pP = \ker(\omega_p)$. Then

$$((\lhd\, g)^*\omega)|_p(X_p) = \omega_{p\lhd g}((\lhd\, g)_* X_p) = 0$$

since $(\lhd\, g)_* X_p \in H_{p\lhd g}P = \ker(\omega_{p\lhd g})$. But also $(\mathrm{Ad}_{g^{-1}})(\omega_p(\mathrm{hor}(X_p))) = 0$ since $\omega_p(\mathrm{hor}(X_p)) = 0$.

(c) We have $\omega = i^{-1} \circ \mathrm{ver}$ and both i^{-1} and ver are smooth.

6. **more explanation of connection one-form:** For example, if the pair (u, f) is a **principal bundle automorphism** of (P, π, M), i.e. if the diagram

$$\begin{tikzcd} P \arrow[r, "u"] & P \\ P \arrow[u, "\lhd G"] \arrow[r, "u"] \arrow[d, "\pi"'] & P \arrow[u, "\lhd G"'] \arrow[d, "\pi"] \\ M \arrow[r, "f"] & M \end{tikzcd}$$

commutes, we should be able to pull a connection one-form ω on P back to another connection one-form $u^*\omega$ on P as

$$u^*\omega : \; X \mapsto (u^*\omega)(X) := \omega(u_*(X))$$

$$\begin{tikzcd} \Gamma(TP) \arrow[r, "\omega"] & T_eG \\ \Gamma(TP) \arrow[u, "u"] \arrow[ur, "u^*\omega"'] & \end{tikzcd}$$

- Recall that for a one-form $\omega : \Gamma(TN) \xrightarrow{\sim} \mathcal{C}^\infty(N)$, we defined

$$\begin{aligned} \Phi^*(\omega) : \Gamma(TM) &\xrightarrow{\sim} \mathcal{C}^\infty(M) \\ X &\mapsto \omega(\Phi_*(X)) \end{aligned}$$

  for any diffeomorphism $\phi : M \to N$. One might be worried about whether this and similar definitions apply to Lie-algebra-valued one-forms but, in fact, they do.
- Now, if $u : P \to P$ is a diffeomorphism of P, then $u_* X \in \Gamma(TP)$ and so $u^*\omega$ is again a Lie-algebra-valued one-form.
- For pull-back of the Lie-algebra-valued one-forms under other smooth maps, the situation is the same.

23 Local Representations of A Connection on The Base Nanifold: YangMills Fields

In the last section, we consider the connection as a Lie-algebra-valued one-form on the principal bundle. By **first selecting a local section**, we now study how we can express this connection one-form **locally on the base manifold or the local trivialization product space using pull-back** of the principal bundle since we may care about the practical calculation.

A summary:

► We first introduce the **Yang-Mills field** which is a Lie-algebra-valued one-forms **locally** defined on the subset $U \subseteq M$ using the **pull-back** from the Lie-algebra-valued one-form defined on the principal bundle.

$$\begin{array}{c} \text{Lie-algebra-valued one-form on the principal bundle} \\ \Big\Downarrow \text{pull-back} \\ \text{Lie-algebra-valued one-form locally on the base manifold or the local local trivialisation} \end{array}$$

- ► **Maurer-Cartan form** is used to give the transformation between the **Yang-Mills field** and another **local representation** of connection using local trivialization. **Maurer-Cartan form** is Lie-algebra-valued one-form on the Lie group G. By using examples, we illustrate the local calculation of **Maurer-Cartan form**.
- ► We define the **gauge map** which is used to glue the Yang-Mills fields on several open subsets of our manifold. Examples are used for illustration.

1. **Yang-Mills fields and local representations:** Let $U \subseteq M$ be some open subset on which the local representations are studied. Let $\sigma : U \to P$ be a local section of P, i.e. $\pi \circ \sigma = \mathrm{id}_U$.

$$\begin{tikzcd}
 & U \times G \arrow[rr, "h"] & & P \\
 & U \times G \arrow[u, "\triangleleft' G"] \arrow[rr, "h"] \arrow[dd, "\pi_1"] & & P \arrow[u, "\triangleleft G"'] \arrow[dd, "\pi"'] \\
h^*\omega \arrow[ur, dashed, hook] & & \omega \arrow[ur, dashed, hook] & \\
 & U \arrow[rr, "\mathrm{id}"] & & U \arrow[uu, bend right, "\sigma"'] \\
 & & \omega^U = \sigma^*\omega \arrow[ur, dashed, hook] &
\end{tikzcd}$$

Given a connection one-form ω on P, such a local section σ induces

   (a) **Yang-Mills field (on the base manifold):** a **Yang-Mills field** $\omega^U : \Gamma(TU) \xrightarrow{\sim} T_e G$ is given by
$$\omega^U := \sigma^* \omega;$$

   (b) **local representation (under local trivialisation):** a **local trivialisation** of the principal bundle P, is the map introduced in Section 20:
$$\begin{aligned} h : \ & U \times G \to P \\ & (m, g) \mapsto \sigma(m) \triangleleft g; \end{aligned}$$
   A **local representation under local trivialisation** of ω on U is then given by
$$h^*\omega : \Gamma(T(U \times G)) \xrightarrow{\sim} T_e G.$$

   - So Yang-Mills field is just a Lie-algebra-valued one form.
   - Note that, at each point $(m, g) \in U \times G$, we have
$$T_{(m,g)}(U \times G) \cong_{\text{Lie alg}} T_m U \oplus T_g G.$$
   - **equivalence:** Both the Yang-Mills field ω^U and the local representation $h^*\omega$ encode the **same** information carried by ω locally on U. Since $h^*\omega$ involves $U \times G$ while ω^U doesn't, one might guess that $h^*\omega$ gives a more "accurate" picture of ω on U than the Yang-Mills field. But in fact, this is not the case. They both contain the **same amount of local information** about the connection one-form ω.
   - The above diagram is **not** principle bundle isomorphism, we are only care the local performances. But we still can define the pull-backs.

2. **Yang-Mills field for the frame bundle from a chart induced local section:**

   (a) Assume we have a (Lie-algebra-valued one-form) connection one-form
$$\omega : \Gamma(LM) \xrightarrow{\sim} T_e \, \mathrm{GL}(\dim M, \mathbb{R})$$
   Since $\mathrm{GL}(\dim M, \mathbb{R})$ can be identified with an open subset of $\mathbb{R}^{(\dim M)^2}$, we have
$$T_e \, \mathrm{GL}(\dim M, \mathbb{R}) \cong_{\text{Lie alg}} \mathbb{R}^{(\dim M)^2},$$

where $\mathbb{R}^{(\dim M)^2}$ is understood as the algebra of $\dim M \times \dim M$ square **matrices**, with bracket induced by matrix multiplication. So the connection one-form is given in terms of $(\dim M)^2$ functions,

$$\omega^i{}_j : \Gamma(LM) \xrightarrow{\sim} \mathbb{R}, \qquad 1 \leq i, j \leq \dim M.$$

(b) **select a local section**: any chart (U, x) of a smooth manifold M induces a local section $\sigma : U \to LM$ of the frame bundle of M by

$$\sigma(m) := \left(\left(\frac{\partial}{\partial x^1} \right)_m, \ldots, \left(\frac{\partial}{\partial x^{\dim M}} \right)_m \right) \in L_m M.$$

(c) **calculate Yang-Mills field:** The associated Yang-Mills field $\omega^U := \sigma^* \omega$ is, at each point $m \in U$, a Lie-algebra-valued one-form on the vector space $T_m U$. By using the coordinate induced basis and its dual basis, we can **express the one-form** $(\omega^U)_m$ **in terms of components** as

$$(\omega^U)_m = \omega^U_\mu(m) \, (\mathrm{d}x^\mu)_m,$$

where $1 \leq \mu \leq \dim M$ and

$$\omega^U_\mu(m) := (\omega^U)_m \left(\left(\frac{\partial}{\partial x^\mu} \right)_m \right) \in T_e G.$$

- **Christoffel symbol:** Again, we can identify each $\omega^U_\mu(m)$ with a square $\dim M \times \dim M$ **matrix** and define the symbol

$$\Gamma^i{}_{j\mu}(m) := (\omega^U(m))^i{}_{j\mu} := (\omega^U_\mu(m))^i{}_j,$$

usually referred to as the **Christoffel symbol**.

- Note that, even though all three indices i, j, μ run from 1 to $\dim M$, the numbers $\Gamma^i{}_{j\mu}(m)$ do **not** constitute the components of a $(1, 2)$-tensor on U. Only the μ index transforms as a one-form component index, while the i, j indices simply label different one-forms in terms of the $(\dim M)^2$ functions.

3. **relation between the Yang-Mills field and the local representation:** For all $v \in T_m U$ and $\gamma \in T_g G$, we have

$$(h^* \omega)_{(m,g)}(v, \gamma) = (\mathrm{Ad}_{g^{-1}})(\omega^U(v)) + \Xi_g(\gamma),$$

where Ξ_g is the **Maurer-Cartan form**

$$\begin{aligned} \Xi_g : \quad T_g G &\xrightarrow{\sim} T_e G \\ K_A|g &\mapsto A. \end{aligned}$$

- As shown in Section 14, from the existence of global frame, we know that given any $X \in T_g G$, we can the find a unique $A \in T_e G$ such that $K_A|_g \equiv j(A)|_g = X$.
- Maurer-Cartan form only depends on the Lie group (and its Lie algebra), **not** on the principal bundle P or the restriction $U \subseteq M$.

4. **calculation of the Maurer-Cartan form of the Lie group** $\mathrm{GL}(d, \mathbb{R})$**:**

(a) **the natural chart:** Let $(\mathrm{GL}^+(d, \mathbb{R}), x)$ be a chart on $\mathrm{GL}(d, \mathbb{R})$, where $\mathrm{GL}^+(d, \mathbb{R})$ denotes an open subset of $\mathrm{GL}(d, \mathbb{R})$ containing the identity $\mathrm{id}_{\mathbb{R}^d}$, and let $x^i{}_j$: $\mathrm{GL}^+(d, \mathbb{R}) \to \mathbb{R}$ denote the corresponding coordinate functions

$$\begin{tikzcd} \mathrm{GL}^+(d, \mathbb{R}) \arrow[r, "x"] \arrow[dr, "x^i{}_j"'] & x(\mathrm{GL}^+(d, \mathbb{R})) \subseteq \mathbb{R}^{d^2} \arrow[d, "\mathrm{proj}^i{}_j"] \\ & \mathbb{R} \end{tikzcd}$$

so that $x^i{}_j(g) := g^i{}_j$. Recall that the coordinate functions are smooth maps on the chart domain, i.e. we have $x^i{}_j \in \mathcal{C}^\infty(\mathrm{GL}^+(d, \mathbb{R}))$.

(b) **left-invariant vector field:** recall that to each $A \in T_{\mathrm{id}_{\mathbb{R}^d}} \mathrm{GL}(d, \mathbb{R})$ there is associated a left-invariant vector field

$$K_A : \mathcal{C}^\infty(\mathrm{GL}^+(d, \mathbb{R})) \xrightarrow{\sim} \mathcal{C}^\infty(\mathrm{GL}^+(d, \mathbb{R}))$$

which, at each point $g \in \mathrm{GL}(d, \mathbb{R})$, is the tangent vector to the curve

$$\gamma^A(t) := g \bullet \exp(tA).$$

- Recall **tangent space equivalent definition II** in Section 10, we can select any curve as long as the tangent vectors have the same directional derivative. So the above $\gamma^A(t)$ works. At point 0, it is a curve $\exp(tA)$, while at any g, the **curve push-forward** in Section 11 results in $\gamma^A(t)$.
- Action on a function f on the manifold $\mathrm{GL}(d, \mathbb{R})$, we have

$$\begin{aligned}(K_A f)(g) &= [f(g \bullet \exp(tA))]'(0) \\ &= [f(g \bullet \mathrm{e}^{tA})]'(0) \\ &= \left(f(g^i{}_k (\mathrm{e}^{tA})^k{}_j)\right)'(0).\end{aligned}$$

(c) Consider the action of K_A on the coordinate functions, we have:

$$\begin{aligned}(K_A x^i{}_j)(g) &= [x^i{}_j(g \bullet \exp(tA))]'(0) \\ &= \left(x^i{}_j(g^i{}_k (\mathrm{e}^{tA})^k{}_j)\right)'(0) \\ &= g^i{}_k A^k{}_j.\end{aligned}$$

- Note, recall in Section 19, we have shown for a matrix Lie group, the exponential map is just the ordinary exponential

$$\exp(A) = \mathrm{e}^A := \sum_{n=0}^{\infty} \frac{1}{n!} A^n.$$

(d) **get Maurer-Cartan form:** Hence, we can write

$$K_A|_g = g^i{}_k A^k{}_j \left(\frac{\partial}{\partial x^i{}_j}\right)_g$$

from which we can read-off the **Maurer-Cartan form** of $\mathrm{GL}(d, \mathbb{R})$

$$(\Xi_g)^i{}_j := (g^{-1})^i{}_k (\mathrm{d}x^k{}_j)_g.$$

Indeed, we can quickly check that

$$\begin{aligned}(\Xi_g)^i{}_j(K_A) &= (g^{-1})^i{}_k (\mathrm{d}x^k{}_j)_g \left(g^p{}_r A^r{}_q \left(\frac{\partial}{\partial x^p{}_q}\right)_g\right) \\ &= (g^{-1})^i{}_k g^p{}_r A^r{}_q \delta^k_p \delta^q_j \\ &= (g^{-1})^i{}_p g^p{}_r A^r{}_j \\ &= \delta^i_r A^r{}_j \\ &= A^i{}_j.\end{aligned}$$

5. **The gauge map:**

(a) **set-up:** We can then try to reconstruct the global connection by glueing the Yang-Mills fields on several open subsets of our manifold.

$$\begin{tikzcd} & P \arrow[dl, "\pi"'] \arrow[dr, "\pi"] & \\ U_1 \arrow[ur, bend left, "\sigma_1"] & & U_2 \arrow[ul, bend right, "\sigma_2"'] \end{tikzcd}$$

Suppose, for instance, that we have two open subsets $U_1, U_2 \subseteq M$ and consider the Yang-Mills fields associated to two local connections σ_1, σ_2. If ω^{U_1} and ω^{U_2} are both local versions of a unique connection one-form, then is $U_1 \cap U_2 \neq \varnothing$, the Yang-Mills fields ω^{U_1} and ω^{U_2} should satisfy some **compatibility condition** on $U_1 \cap U_2$.

(b) **gauge map:** Within the above set-up, the **gauge map** is the map

$$\Omega :\ U_1 \cap U_2 \to G$$

where, for each $m \in U_1 \cap U_2$, the Lie group element $\Omega(m) \in G$ satisfies

$$\sigma_2(m) = \sigma_1(m) \lhd \Omega(m).$$

- Note that since the G-action $\lhd$ on P is **free**, for each m there exists a unique $\Omega(m)$ satisfying the above condition, and hence the gauge map Ω is well-defined.

(c) **transformation between local representations:** Under the above assumptions, we have

$$(\omega^{U_2})_m = (\mathrm{Ad}_{\Omega^{-1}(m)})(\omega^{U_1}) + (\Omega^* \Xi_g)_m.$$

6. **transformation between local representations on frame bundle:** Consider again the frame bundle LM of some manifold M. Assume we have two overlapped chart (U_1, x) and $(U_2, \widetilde{x})$ with both the above natural chart repesentation.

(a) **evaluate $\Omega^* \Xi_g$:** This is the **pull-back** along Ω of the Maurer-Cartan form. Since $\Xi_g :\ T_g G \to T_e G$ and $\Omega :\ U_1 \cap U_2 \to G$, we have $\Omega^* \Xi_g :\ T_{\Omega^{-1}(g)}(U_1 \cap U_2) \to T_e G$. Let x be a chart map near the point $m \in U_1 \cap U_2$. We have

$$\begin{aligned}
((\Omega^* \Xi_g)_m)^i{}_j \left(\left(\frac{\partial}{\partial x^\mu} \right)_m \right) &= (\Xi_{\Omega(m)})^i{}_j \left(\Omega_* \left(\frac{\partial}{\partial x^\mu} \right)_m \right) \\
&= (\Omega(m)^{-1})^i{}_k (\mathrm{d}\widetilde{x}^k{}_j)_{\Omega(m)} \left(\Omega_* \left(\frac{\partial}{\partial x^\mu} \right)_m \right) \\
&= (\Omega(m)^{-1})^i{}_k \left(\Omega_* \left(\frac{\partial}{\partial x^\mu} \right)_m \right) (\widetilde{x}^k{}_j) \\
&= (\Omega(m)^{-1})^i{}_k \left(\frac{\partial}{\partial x^\mu} \right)_m (\widetilde{x}^k{}_j \circ \Omega) \\
&= (\Omega(m)^{-1})^i{}_k \left(\frac{\partial}{\partial x^\mu} \right)_m (\Omega(m))^k{}_j.
\end{aligned}$$

hence, we can write

$$\begin{aligned}
((\Omega^* \Xi_g)_m)^i{}_j &= (\Omega(m)^{-1})^i{}_k \left(\frac{\partial}{\partial x^\mu} \right)_m (\Omega(m))^k{}_j \mathrm{d}x^\mu \\
&=: (\boldsymbol{\Omega}^{-1} \mathrm{d}\boldsymbol{\Omega})^i{}_j.
\end{aligned}$$

(b) **evaluate $(\mathrm{Ad}_{\Omega^{-1}(m)})(\omega^{U_1})$:** Recall that Ad_g is the push-forward of Ψ_g as a linear endomorphism of $T_e G$. Moreover, from Section 18, we know that since here $G = \mathrm{GL}(d, \mathbb{R})$ is a matrix Lie group, we have

$$((\mathrm{Ad}_g)A)^i{}_j = g^i{}_k A^k{}_l (g^{-1})^l{}_j =: (\boldsymbol{g} \boldsymbol{A} \boldsymbol{g}^{-1})^i{}_j.$$

Hence, we have

$$(\mathrm{Ad}_{\Omega^{-1}(m)})(\omega^{U_1}) = (\Omega(m)^{-1})^i{}_k ((\omega^{U_1})_m)^k{}_l (\Omega(m))^l{}_j$$

(c) **full transformation:** Altogether, we find that the transition rule for the Yang-Mills fields on the intersection of U_1 and U_2 is given by

$$(\omega^{U_2})^i{}_{j\mu} = (\Omega^{-1})^i{}_k (\omega^{U_1})^k{}_{l\mu} \Omega^l{}_j + (\Omega^{-1})^i{}_k \partial_\mu (\Omega^{-1})^k{}_j.$$

(d) **an example:** As an application, consider the spacial case in which the sections σ_1 and σ_2 are from the local sections induced by coordinate charts (U_1, x) and $(U_2, \widetilde{x})$. Then we have

$$\begin{aligned}
\Omega^i{}_j &= \frac{\partial \widetilde{x}^i}{\partial x^j} := \partial_j (\widetilde{x}^i \circ x^{-1}) \circ x \\
(\Omega^{-1})^i{}_j &= \frac{\partial x^i}{\partial \widetilde{x}^j} := \partial_j (x^i \circ \widetilde{x}^{-1}) \circ y
\end{aligned}$$

and hence

$$(\omega^{U_2})^i{}_{j\nu} = \frac{\partial \widetilde{x}^\mu}{\partial x^\nu} \left(\frac{\partial x^i}{\partial \widetilde{x}^k} (\omega^{U_1})^k{}_{l\mu} \frac{\partial \widetilde{x}^l}{\partial x^j} + \frac{\partial x^i}{\partial \widetilde{x}^k} \frac{\partial^2 \widetilde{x}^k}{\partial x^\mu \partial x^j} \right).$$

This is the **transformation law for the Christoffel symbols**.

24 Parallel Transport and Connection on Associated Fibre Bundles

$$
\begin{tikzcd}
\text{set} \arrow[r] & \text{topological space} \arrow[r] & \text{topological manifold} \arrow[r] & \text{differentiable manifold} \arrow[r] \arrow[d] & \text{principal fibre bundle} \arrow[r] \arrow[d, Rightarrow] & \text{associated fibre bundles} \arrow[d, Rightarrow] \\
 & & & \text{Lie group} \arrow[ur] \arrow[d, shift left] & \text{connection on it} \arrow[r, bend right] & \text{connection on it} \\
 & & & \text{Lie algebra} \arrow[u, shift left] & &
\end{tikzcd}
$$

Now, we will see that a connection on a principal bundle induces a **parallel transport map on the principal bundle**, which in turn induces a **parallel transport map on any of its associated bundles** (e.g. the associated **vector bundles** where each fibre carry a vector space structure). The idea of parallel transport on a principal bundle hinges on that of horizontal lift of a curve on the base manifold to a curve on the principal bundle in the sense that the projection to the base manifold of this curve gives the curve we started with. In particular, if the principal bundle is equipped with a **connection**, we would like to impose some extra conditions on this lifting, so that it "connects" nearby fibres in a nice way. We will then consider the same idea on an associated bundle and see how we can induce a **derivative operator** if the associated bundle is a vector bundle.

$$
\begin{gathered}
\text{connection (and parallel transport) on } P \\
\Downarrow \\
\text{connection (and parallel transport) on any associated fibre bundle} \\
\Downarrow \\
\text{covariant derivative (and parallel transport) on any associated vector bundle}
\end{gathered}
$$

A summary:

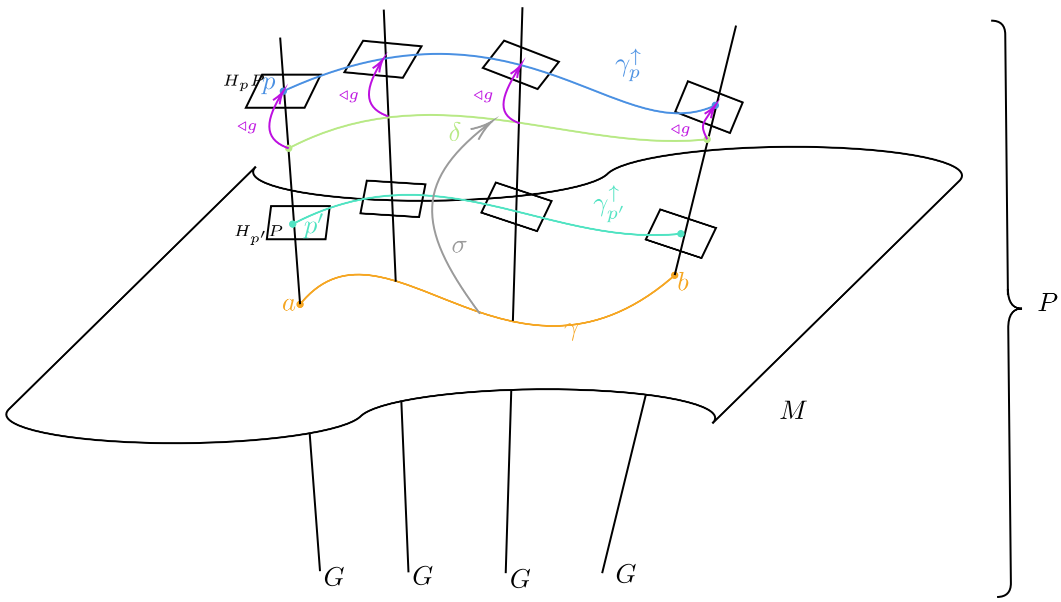
- Given a **connection on** the principal bundle, we define the **horizontal lifts of curve on the base manifold to the principal bundle.**

- The solution of the horizontal lift by using an ODE is shown to given by a **path-ordered exponential**.

- The **parallel transport** is then defined from the **horizontal lifts**, from which holonomy groups are defined if the curve on the base manifold are loops.

- Finally, **horizontal lifts to the associated bundle** are introduced. If the associated bundle is a **vector** bundle, we can then define **covariant derivative geometrically** (which is however quite difficult to implement, and we will instead take another point view in Section 25).

1. **horizontal lifts of curve on the base manifold to the principal bundle:**



Let (P, π, M) be a principal G-bundle equipped with a **connection** ω and let $\gamma : [0,1] \to M$ be a curve on M. The **horizontal lift of** γ **through** $p_0 \in P$ is the **unique** curve

$$\gamma^\uparrow : [0,1] \to P$$

with $\gamma^\uparrow(0) = p_0 \in \mathrm{preim}_\pi(\{\gamma(0)\})$ satisfying

(a) $\pi \circ \gamma^\uparrow = \gamma$;
(b) $\forall\ \lambda \in [0,1] :\ \mathrm{ver}(X_{\gamma^\uparrow, \gamma^\uparrow(\lambda)}) = 0$;
(c) $\forall\ \lambda \in [0,1] :\ \pi_*(X_{\gamma^\uparrow, \gamma^\uparrow(\lambda)}) = X_{\gamma, \gamma(\lambda)}$.

- **explanation:** Intuitively, a horizontal lift of a curve γ on M is a curve $\gamma^\uparrow$ on P such that
  - each point $\gamma^\uparrow(\lambda) \in P$ belongs to the fibre of $\gamma(\lambda)$ (condition (a)),
  - the tangent vectors to the curve $\gamma^\uparrow$ have **no vertical component** (condition (b)), i.e. they lie entirely in the horizontal spaces at each point, and
  - finally the **push-forward** of the tangent vector to $\gamma^\uparrow$ at $\gamma^\uparrow(\lambda)$ coincides with the tangent vector to the curve γ at $\pi(\gamma^\uparrow(\lambda)) = \gamma(\lambda)$.
- **uniqueness:** Note that the **uniqueness** in the above definition only stems from the choice of $p_0 \in \mathrm{preim}_\pi(\{\gamma(0)\})$. A curve on M has several horizontal lifts to a curve on P, but there is only one such curve going through each point $p_0 \in \mathrm{preim}_\pi(\{\gamma(0)\})$. Clearly, different horizontal lifts **cannot intersect** each other.
  - "cannot intersect" comes from the fact

$$(\triangleleft\ g)_*(H_pP) = H_{p \triangleleft g}P.$$

    and G acts **freely**.

2. **solution to get the horizontal lift:** How to find the horizontal lift? **Use ODE!** To get an explicit expression for the horizontal lift through $p_0 \in P$ of a curve $\gamma : [0,1] \to M$, we use **two steps**:

(a) "Generate" the horizontal lift by starting from some arbitrary curve **using a local section** σ as

$$\delta : [0,1] \to P$$

such that $\pi \circ \delta = \gamma$.

(b) Find **an action curve** $g : (0,1) \to G$ so that

$$\gamma^\uparrow(\lambda) = \delta(\lambda) \triangleleft g(\lambda).$$

The suitable curve g will be the solution to an ordinary differential equation with initial condition $g(0) = g_0$, where g_0 is the unique element in G such that

$$\delta(0) \triangleleft g_0 = p_0 \in P.$$

We will explicitly solve (locally) this differential equation for $g : [0,1] \to P$ by a path-ordered integral over the **local Yang-Mills field**.

- **condition for the action curve:** We state without proofs that the (first order) ODE satisfied by the curve $g : [0,1] \to G$ is

$$(\mathrm{Ad}_{g(\lambda)^{-1}})(\omega_{\delta(\lambda)}(X_{\delta,\delta(\lambda)})) + \Xi_{g(\lambda)}(X_{g,g(\lambda)}) = 0$$

  with the initial condition $g(0) = g_0$.

  $*$ If G is a matrix group, then the above ODE takes the form

$$g(\lambda)^{-1}(\omega_{\delta(\lambda)}(X_{\delta,\delta(\lambda)}))g(\lambda) + g(\lambda)^{-1}\dot{g}(\lambda) = 0$$

  where $\dot{g}(\lambda)$ denotes the derivative with respect to λ of the matrix entries of g. Equivalently, by multiplying both sides on the left by $g(\lambda)$,

$$\dot{g}(\lambda) = -(\omega_{\delta(\lambda)}(X_{\delta,\delta(\lambda)}))g(\lambda).$$

- **use local Yang-Mills field:** Consider a chart (U, x) on the base manifold M, such that the image of γ is entirely contained in U. The local section $\sigma : U \to P$ induces a **Yang-Mills field** ω^U**.** Note that since we have

$$\sigma_*(X_{\gamma,\gamma(\lambda)}) = X_{\delta,\delta(\lambda)},$$

  and hence

$$\begin{aligned}
\omega_{\delta(\lambda)}(X_{\delta,\delta(\lambda)}) &= \omega_{\delta(\lambda)}(\sigma_*(X_{\gamma,\gamma(\lambda)})) \\
&= (\sigma^*\omega)_{\gamma(\lambda)}(X_{\gamma,\gamma(\lambda)}) \\
&= (\omega^U)_{\gamma(\lambda)}(X_{\gamma,\gamma(\lambda)}) \\
&= \omega^U_\mu(\gamma(\lambda))(\mathrm{d}x^\mu)_{\gamma(\lambda)}\left(X^\nu_\gamma(\gamma(\lambda))\left(\frac{\partial}{\partial x^\nu}\right)_{\gamma(\lambda)}\right) \\
&= \omega^U_\mu(\gamma(\lambda))X^\nu_\gamma(\gamma(\lambda))(\mathrm{d}x^\mu)_{\gamma(\lambda)}\left(\left(\frac{\partial}{\partial x^\nu}\right)_{\gamma(\lambda)}\right) \\
&= \omega^U_\mu(\gamma(\lambda))X^\nu_\gamma(\gamma(\lambda))\,\delta^\mu_\nu \\
&= \omega^U_\mu(\gamma(\lambda))X^\mu_\gamma(\gamma(\lambda)).
\end{aligned}$$

  Thus, in the special case of a matrix Lie group, we have the so called **horizontal lift ODE:**

$$\dot{g}(\lambda) = -\Gamma_\mu(\gamma(\lambda))\,\dot{\gamma}^\mu(\lambda),$$

  where $\Gamma_\mu := \omega^U_\mu$ and $\dot{\gamma}^\mu(\lambda) := X^\mu_\gamma(\gamma(\lambda))$, together with the initial condition $g(0) = g_0$.

3. **solution of the horizontal lift ODE by a path-ordered exponential:** As a first step towards the solution of our ODE, consider

$$g(t) := g_0 - \int_0^t \mathrm{d}\lambda\,\Gamma_\mu(\gamma(\lambda))\,\dot{\gamma}^\mu(\lambda)g(\lambda).$$

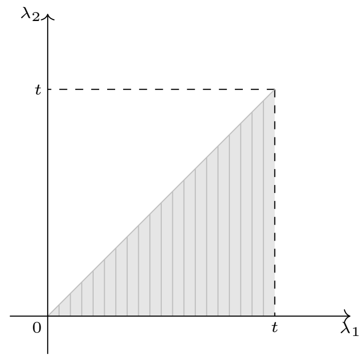
   This doesn't seem to have brought us far since the function g that we would like to determine appears again on the right hand side. However, we can now **iterate this definition** to obtain

$$\begin{aligned}
g(t) &= g_0 - \int_0^t \mathrm{d}\lambda_1\,\Gamma_\mu(\gamma(\lambda_1))\dot{\gamma}^\mu(\lambda_1)\left(g_0 - \int_0^{\lambda_1}\mathrm{d}\lambda_2\,\Gamma_\nu(\gamma(\lambda_2))\dot{\gamma}^\nu(\lambda_2)\right) \\
&= g_0 - \int_0^t \mathrm{d}\lambda_1\,\Gamma_\mu(\gamma(\lambda_1))\dot{\gamma}^\mu(\lambda_1)g_0 + \int_0^t \mathrm{d}\lambda_1 \int_0^{\lambda_1}\mathrm{d}\lambda_2\,\Gamma_\mu(\gamma(\lambda_1))\dot{\gamma}^\mu(\lambda_1)\Gamma_\nu(\gamma(\lambda_2))\dot{\gamma}^\nu(\lambda_2)g(\lambda_2).
\end{aligned}$$

Matters seem to only get worse, until one realises that the first integral no longer contains the unknown function g. Hence, the above expression provides a "first-order" approximation to g. It is clear that we can get higher-order approximations by iterating this process

$$\begin{aligned} g(t) = g_0 &- \int_0^t \mathrm{d}\lambda_1\, \Gamma_\mu(\gamma(\lambda_1))\dot{\gamma}^\mu(\lambda_1) g_0 \\ &+ \int_0^t \mathrm{d}\lambda_1 \int_0^{\lambda_1} \mathrm{d}\lambda_2\, \Gamma_\mu(\gamma(\lambda_1))\dot{\gamma}^\mu(\lambda_1)\Gamma_\nu(\gamma(\lambda_2))\dot{\gamma}^\nu(\lambda_2) g_0 \\ &\vdots \\ &+ (-1)^n \int_0^t \mathrm{d}\lambda_1 \int_0^{\lambda_1} \mathrm{d}\lambda_2 \cdots \int_0^{\lambda_{n-1}} \mathrm{d}\lambda_n\, \Gamma_\mu(\gamma(\lambda_1))\dot{\gamma}^\mu(\lambda_1)\cdots\Gamma_\nu(\gamma(\lambda_n))\dot{\gamma}^\nu(\lambda_n)\, g(\lambda_n). \end{aligned}$$

Note how the range of each integral depends on the integration variable of the previous integral. It would much nicer if we could have the same range in each integral. In fact, there is a standard trick to achieve this. The region of integration in the double integral is



and if the integrand $f(\lambda_1, \lambda_2)$ is invariant under the exchange $\lambda_1 \leftrightarrow \lambda_2$, we have

$$\int_0^t \mathrm{d}\lambda_1 \int_0^{\lambda_1} \mathrm{d}\lambda_2\, f(\lambda_1, \lambda_2) = \frac{1}{2} \int_0^t \mathrm{d}\lambda_1 \int_0^t \mathrm{d}\lambda_2\, f(\lambda_1, \lambda_2).$$

Generalising to n dimensions, we have

$$\int_0^t \mathrm{d}\lambda_1 \cdots \int_0^{\lambda_{n-1}} \mathrm{d}\lambda_n\, f(\lambda_1, \ldots, \lambda_n) = \frac{1}{n!} \int_0^t \mathrm{d}\lambda_1 \cdots \int_0^t \mathrm{d}\lambda_n\, f(\lambda_1, \ldots, \lambda_n)$$

if f is invariant under any permutation of its arguments. Moreover, since each term in our integrands only depends on one integration variable at a time, we can use

$$\int_0^t \mathrm{d}\lambda_1 \cdots \int_0^t \mathrm{d}\lambda_n\, f_1(\lambda_1)\cdots f_n(\lambda_n) = \left(\int_0^t \mathrm{d}\lambda_1\, f(\lambda_1)\right)\cdots\left(\int_0^t \mathrm{d}\lambda_n\, f(\lambda_n)\right)$$

so that, in our case, we would have

$$\begin{aligned} g(t) &= \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\int_0^t \mathrm{d}\lambda\, \Gamma_\mu(\gamma(\lambda))\dot{\gamma}^\mu(\lambda)\right)^n\right) g_0 \\ &= \exp\left(-\int_0^t \mathrm{d}\lambda\, \Gamma_\mu(\gamma(\lambda))\dot{\gamma}^\mu(\lambda)\right) g_0. \end{aligned}$$

However, our integrands are Lie-algebra-valued (that is, matrix valued), and since the factors therein need **not commute**. So the traditional $\exp$ does not work. Instead, we need to define

$$g(t) = \mathrm{P}\exp\left(-\int_0^t \mathrm{d}\lambda\, \Gamma_\mu(\gamma(\lambda))\dot{\gamma}^\mu(\lambda)\right) g_0,$$

where the **path-ordered exponential** $\mathrm{P}\exp$ is defined to yield the correct expression for $g(t)$.

- Note, the **path-ordered exponential** is just a definition. In some sense it is useless since if we what to know it we need to still write down the infinite long integration.

4. **a summary of the horizontal lifts of curve:** For a principal G-bundle (P, π, M), where G is a matrix Lie group, the horizontal lift of a curve $\gamma: [0,1] \to U$ through $p_p \in \text{preim}_\pi(\{U\})$, where (U, x) is a chart on M, is given in terms of a local section $\sigma: U \to P$ by the explicit expression

$$\gamma^\uparrow(\lambda) = (\sigma \circ \gamma)(\lambda) \triangleleft \left(\mathrm{P}\exp\left(-\int_0^\lambda \mathrm{d}\widetilde{\lambda}\, \Gamma_\mu(\gamma(\widetilde{\lambda}))\dot{\gamma}^\mu(\widetilde{\lambda})\right) g_0\right).$$

5. **parallel transport map on principal fibre bundle:** Let $\gamma_p^{\uparrow} : [0,1] \to P$ be the horizontal lift through $p \in \text{preim}_\pi(\{\gamma(0)\})$ of the curve $\gamma : [0,1] \to M$. The **parallel transport map along** γ is the map

$$T_\gamma : \text{preim}_\pi(\{\gamma(0)\}) \to \text{preim}_\pi(\{\gamma(1)\})$$
$$p \mapsto \gamma_p^{\uparrow}(1).$$

   - The parallel transport is bijection between the fibres $\text{preim}_\pi(\{\gamma(0)\})$ and $\text{preim}_\pi(\{\gamma(1)\})$. It is injective since there is a unique horizontal lift of γ through each point $p \in \text{preim}_\pi(\{\gamma(0)\})$, and horizontal lifts through different points do not intersect. It is surjective since for each $q \in \text{preim}_\pi(\{\gamma(1)\})$ we can find a p such that $q = \gamma_p^{\uparrow}(1)$ as follows. Let $\widetilde{p} \in \text{preim}_\pi(\{\gamma(0)\})$. Then $\gamma_{\widetilde{p}}^{\uparrow}(1)$ belongs to the same fibre as q and hence there exists a unique $g \in G$ such that $q = \gamma_{\widetilde{p}}^{\uparrow}(1) \lhd g$. Recall that

$$\gamma_{\widetilde{p}}^{\uparrow}(\lambda) = (\sigma \circ \gamma)(\lambda) \lhd (\mathrm{P}\exp(\cdots)g_0)$$

where g_0 is the unique $g_0 \in G$ such that $\widetilde{p} = (\sigma \circ \gamma)(0) \lhd g_0$. Define $p \in \text{preim}_\pi(\{\gamma(0)\})$ by

$$p := \widetilde{p} \lhd g = (\sigma \circ \gamma)(0) \lhd (g_0 \bullet g).$$

Then we have

$$\begin{aligned}\gamma_p^{\uparrow}(1) &= (\sigma \circ \gamma)(1) \lhd (\mathrm{P}\exp(\cdots)g_0 \bullet g)\\ &= (\sigma \circ \gamma)(1) \lhd (\mathrm{P}\exp(\cdots)g_0) \lhd g\\ &= \gamma_{\widetilde{p}}^{\uparrow}(1) \lhd g\\ &= q.\end{aligned}$$

6. **loops and holonomy groups:** From loops we get the holonomy groups
   - **loop:** Consider the **loops**, i.e. curves $\gamma : [0,1] \to M$ for which $\gamma(0) = \gamma(1)$. Fix some $p \in \text{preim}_\pi(\{\gamma(0)\})$. The condition that $\pi \circ \gamma_p^{\uparrow} = \gamma$ then implies that $\gamma_p^{\uparrow}(0)$ and $\gamma_p^{\uparrow}(1)$ belong to the same fibre. Hence, there exists a **unique** $g_\gamma \in G$ such that

$$\gamma_p^{\uparrow}(1) = \gamma_p^{\uparrow}(0) \lhd g_\gamma = p \lhd g_\gamma.$$

   - **holonomy group:** Let ω be a connection one-form on the principal G-bundle (P, π, M). Let $\gamma : [0,1] \to M$ be a loop with base-point $a \in M$, i.e. $\gamma(0) = \gamma(1) = a$. The subgroup of G

$$\text{Hol}_a(\omega) := \{g_\gamma \mid \gamma_p^{\uparrow}(1) = p \lhd g_\gamma \text{ for some loop } \gamma\}$$

   is called the **holonomy group** of ω on P at the base-point a.

7. **horizontal lifts to the associated bundle:** Let (P, π, M) be a principal G-bundle and ω a connection one-form on P. Let (P_F, π_F, M) be an associated fibre bundle of P on whose typical fibre F the Lie group G acts on the left by $\rhd$.
   - Let $\gamma : [0,1] \to M$ be a curve on M.
   - Let $\gamma_p^{\uparrow}$ be its horizontal lift to P through $p \in \text{preim}_\pi(\{\gamma(0)\})$.
   - Then the **horizontal lift** of γ to the associated bundle P_F through the point $[p, f] \in P_F$ is the curve

$$\overset{P_F}{\gamma}{}^{\uparrow}_{[p,f]} : [0,1] \to P_F$$
$$\lambda \mapsto [\gamma_p^{\uparrow}(\lambda), f]$$

     * Recall $P_F := (P \times F)/\sim_G$, where $\sim_G$ is the **equivalence relation**

$$(p, f) \sim_G (p', f') \quad :\Leftrightarrow \quad \exists\, g \in G : \begin{cases} p' &= p \lhd g \\ f' &= g^{-1} \rhd f \end{cases}$$

     Hence, even in some sense f is kept fixed in $[\gamma_p^{\uparrow}(\lambda), f]$, the **equivalence class** actually has changes when $\gamma_p^{\uparrow}(\lambda)$ changes.

8. **parallel transport map on any associated bundle:** The **parallel transport map** on the associated bundle is given by

$$T_\gamma^{P_F} : \text{preim}_{\pi_F}(\{\gamma(0)\}) \to \text{preim}_{\pi_F}(\{\gamma(1)\})$$
$$[p, f] \mapsto \overset{P_F}{\gamma}{}^{\uparrow}_{[p,f]}(1).$$

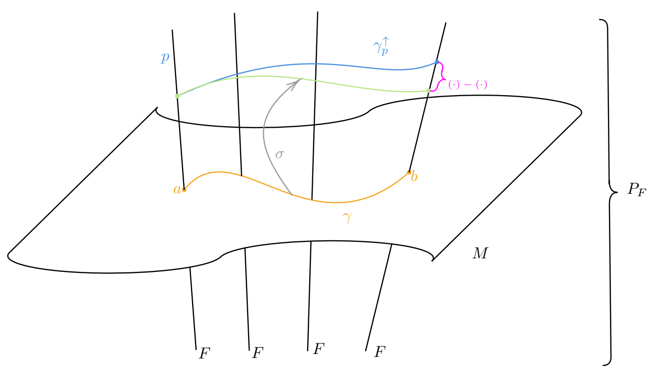
9. **"covariant derivative" on associated vector bundle:** If F is a vector space with the vector operation $+$ and $\triangleright$: $G \times F \to F$ is fibre-wise **linear** (i.e. for each fixed $g \in G$, the map $(g \triangleright -): F \to F$ is linear), then (P_F, π_F, M) is called a **vector bundle**. The basic idea of a covariant derivative is:
   (a) **section σ:** Let $\sigma: U \to P_F$ be a **local section** of the associated bundle.
   (b) **derivative along tangent vector $X \in T_m M$:** We would like to define the **derivative of** σ at the point $m \in U \subseteq M$ in the direction $X \in T_m M$.
   (c) **solution:** By definition, there exists a curve $\gamma: (-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = m$ such that $X = X_{\gamma,m}$. Then for any $0 \le t < \varepsilon$, the points $\gamma^{\overset{P_F}{\uparrow}}_{[\sigma(m)]}(t)$ and $\sigma(\gamma(t))$ lie in the **same fibre** of P_F. We define the derivative of σ at the point m in the direction X, or the derivative of σ along γ at $\gamma(0) = m$, by

$$\lim_{t \to 0} \frac{\sigma(\gamma(t)) - \gamma^{\overset{P_F}{\uparrow}}_{[\sigma(m)]}(t)}{t}$$

   - Note $\sigma(\gamma(t)) - \gamma^{\overset{P_F}{\uparrow}}_{[\sigma(m)]}(t)$ is in P_F, how we can do the subtraction? This is because they are **bundle isomorphic** to the bundle with fibre being F at each point m. Here we can think in the following way because of **local trivialization**: The projection under π_F of them are at the same point, say $m' \in M$. We first fix a $g \in G$, then for the **equivalence class**, we select a value from F, say $[p_1, f_1]$ for $\sigma(\gamma(t))$ and $[p_2, f_2]$ for $\gamma^{\overset{P_F}{\uparrow}}_{[\sigma(m)]}(t)$, where $\pi(p_1) = \pi(p_2) = m'$. But since the fibres are vector spaces, we can write the difference

$$\sigma(\gamma(t)) - \gamma^{\overset{P_F}{\uparrow}}_{[\sigma(m)]}(t) := f_1 - f_2,$$

     where the minus sign denotes the additive inverse in the vector space $\mathrm{preim}_{\pi_F}(\{\gamma(t)\})$.
   - Note, we have defined a **derivative of a section in the direction of tangent vector $X \in T_m M$.**
   - This approach to the concept of covariant derivative is very intuitive and geometric, but it is a disaster from a technical point of view as it is quite difficult to implement. We will soon present a more practical approach in Section 25.



25 Covariant Derivatives

26 A Summary of Connection, Parallel transport and Covariant Derivative

1. We have two points of view for connection on principal bundle,
   (a) Connection is an assignment of an **horizontal subspace** $H_p P$ to each $p \in P$:
   Let (P, π, M) be a **principal bundle** and let $p \in P$. The **vertical subspace** (this space is fixed) at p is the vector subspace of $T_p P$ given by

$$\begin{aligned} V_p P &:= \ker\left((\pi_*)_p\right) \\ &= \left\{X_p \in T_p P \mid (\pi_*)_p (X_p) = 0\right\}. \end{aligned}$$

After that we can choose **horizontal subspace** at p a vector subspace H_pP of T_pP which is complementary to V_pP, i.e.

$$T_pP = H_pP \oplus V_pP.$$

Note here different choice of H_pP will give different decomposition:

$$X|_p = \operatorname{hor}\left(X|_p\right) + \operatorname{ver}\left(X|_p\right)$$

The assignment of an H_pP to each $p \in P$ should be "smooth" within each fibre as well as between different fibres.

(b) **Lie-algebra-valued one-form:**
The above definition is equivalent to a certain **Lie-algebra-valued one-form** on the principal bundle P.

$$\begin{aligned} \omega_p : T_pP &\xrightarrow{\sim} T_eG \\ X_p &\mapsto \omega_p\left(X_p\right) := i_p^{-1}\left(\operatorname{ver}\left(X_p\right)\right) \end{aligned}$$

The map ω is a T_eG-valued one-form (taking the tangent vector space at each point p as input) is called the **connection one-form** w.r.t. the connection. The above ω is a **one-form** from a choice of horizontal spaces (i.e. a connection) because $\operatorname{ver}\left(X_p\right)$ is determined by the connection.
From the other direction, given a Lie-algebra-valued one-form ω , the choice of horizontal spaces can be recovered from ω by

$$H_pP = \ker\left(\omega_p\right).$$

- Of course we cannot say every Lie-algebra-valued one-form is a connection, additional properties need to be satisfied. See Theorem 21.5.

We then have two local representations (using pull back to different space) for the one form:

(a) Yang-Mills field (pull back to the (local subset of) base space U)
(b) local trivialisation of the principal bundle P (pull back to the product space $U \times G$, locally)

Their relation is shown in Theorem 22.2. where the **Maurer-Cartan form** is involved.

For an important example of local representation using Yang-Mills field, we show the **Christoffel symbol** under the setting of **frame bundle**. Note, **Christoffel symbol** $\Gamma_\mu(m)$ is Lie-algebra-valued and of course can use some $\mathrm{GL}(V)$ representation like the matrix.

- $\Gamma^i_{j\mu}(m)$ therefore **do not** constitute the components of a $(1, 2)$-tensor on U. μ is the index for the μ-th basis of $\mathrm{d}x^\mu$ (since w is one form and take a tangent vector as the input).

2. A connection on a principal bundle induces a **parallel transport map on the principal bundle: horizontal lifts to the principal bundle**
Let $\gamma : [0, 1] \to M$ be a curve on M. We need the concept of **horizontal lift** of γ through $p_0 \in P$, which is the unique curve

$$\gamma^\uparrow : [0, 1] \to P$$

with $\gamma^\uparrow(0) = p_0 \in \operatorname{preim}_\pi(\{\gamma(0)\})$ satisfying

(a) $\pi \circ \gamma^\uparrow = \gamma$;
(b) $\forall\, \lambda \in [0, 1] : \operatorname{ver}\left(X_{\gamma^\uparrow, \gamma^\uparrow(\lambda)}\right) = 0$;
(c) $\forall\, \lambda \in [0, 1] : \pi_*\left(X_{\gamma^\uparrow, \gamma^\uparrow(\lambda)}\right) = X_{\gamma, \gamma(\lambda)}$.

Intuitively, a horizontal lift of a curve γ on M is a curve $\gamma^\uparrow$ on P such that

(a) each point $\gamma^\uparrow(\lambda) \in P$ belongs to the fibre of $\gamma(\lambda)$
(b) the tangent vectors to the curve $\gamma^\uparrow$ have no vertical component; i.e. they lie entirely in the **horizontal spaces** at each point.
(c) the projection of the tangent vector to $\gamma^\uparrow$ at $\gamma^\uparrow(\lambda)$ coincides with the tangent vector to the curve γ at $\pi\left(\gamma^\uparrow(\lambda)\right) = \gamma(\lambda)$.

Note that the uniqueness in the above definition only stems from the choice of $p_0 \in \mathrm{preim}_\pi(\{\gamma(0)\})$. A curve on M has several horizontal lifts to a curve on P, but there is only one such curve going through each point $p_0 \in \mathrm{preim}_\pi(\{\gamma(0)\})$. Clearly, different horizontal lifts cannot intersect each other.

- How to find the horizontal lift? Use ODE!
  Our strategy to write down an **explicit expression for the horizontal lift** through $p_0 \in P$ of a curve $\gamma : [0,1] \to M$ is to proceed in two steps:
  (a) "Generate" the horizontal lift by starting from **some arbitrary curve** $\delta : [0,1] \to P$ such that $\pi \circ \delta = \gamma$ by action of a suitable curve $g : (0,1) \to G$ so that
  $$\gamma^{\uparrow}(\lambda) = \delta(\lambda) \triangleleft g(\lambda).$$
  The suitable curve g will be the solution to an ordinary differential equation with initial condition $g(0) = g_0$, where g_0 is the unique element in G such that
  $$\delta(0) \triangleleft g_0 = p_0 \in P.$$
  (b) We will explicitly solve (locally) this differential equation for $g : [0,1] \to P$ by a path ordered integral over the local **Yang-Mills field**.
  In Theorem 23.2. and Corollary 23.3., we are trying to get the ODE whose solution is the curve $g(t)$. The solution is
  $$g(t) = \mathrm{P}\exp\left(-\int_0^t \mathrm{d}\lambda \Gamma_\mu(\gamma(\lambda))\dot{\gamma}^\mu(\lambda)\right) g_0,$$
  where the **path-ordered exponential** $\mathrm{P}\exp$. Furthermore, we then have the **horizontal lifted curve:**
  $$\gamma^{\uparrow}(\lambda) = (\sigma \circ \gamma)(\lambda) \triangleleft \left(\mathrm{P}\exp\left(-\int_0^\lambda \mathrm{d}\tilde{\lambda}\Gamma_\mu(\gamma(\tilde{\lambda}))\dot{\gamma}^\mu(\tilde{\lambda})\right) g_0\right).$$

Finally, let $\gamma_p^{\uparrow}: [0,1] \to P$ be the **horizontal lift** through $p \in \mathrm{preim}_\pi(\{\gamma(0)\})$ of the curve $\gamma : [0,1] \to M$. The **parallel transport map** along γ is the map
$$\begin{aligned} T_\gamma \colon \mathrm{preim}_\pi(\{\gamma(0)\}) &\to \mathrm{preim}_\pi(\{\gamma(1)\}) \\ p &\mapsto \gamma_p^{\uparrow}(1). \end{aligned}$$
which is a **bijection** between the fibres $\mathrm{preim}_\pi(\{\gamma(0)\})$ and $\mathrm{preim}_\pi(\{\gamma(1)\})$.

3. From parallel transport map on the principal bundle, we have a **horizontal lift of the curve** and **parallel transport map** on **any of its associated bundles**. See P195.
4. If the fibres of the associated bundle carry a **vector space structure** together with a **linear group action**, then the parallel transport can be used to define a **covariant derivative of a section** on the associated bundle. See P196. Note now, we can define the different because **the lifted curve and the section in the same fibre which is a vector space.**

   - The above idea of very intuitive and geometric, but it is a disaster from a technical point of view as it is quite difficult to implement. However, the local sections $\sigma : U \to P_F$ are in **bijective correspondence** with **G-equivariant** (see Section 21) functions $\phi : \mathrm{preim}_\pi(U) \subseteq P \to F$, where the G-equivariance condition is
     $$\forall\, g \in G : \forall\, p \in \mathrm{preim}_\pi(U) : \phi(p \triangleleft g) = g^{-1} \triangleright \phi(p)$$
     (a) We first define $\mathrm{D}\phi(X)$, which we can also write as $\mathrm{D}_X\phi$, is $\mathcal{C}^\infty(P)$-linear in the X, additive in the ϕ and satisfies the Leinz property the form of covariant derivative. However, it also clearly **not a covariant derivative on** P_F since $X \in TP$ rather than $X \in TM$ and ϕ is a G-equivariant function $P \to F$ rather than a local section o (P_F, π_F, M).
     (b) We can obtain a covariant derivative from D by introducing a **local trivialisation** on the bundle (P, π, M).

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