Introduction to Mathematical Analysis with Machine Learning

Academic Year	AY2024/25
Semester	1
Course Type	Mathematics
Pre-requisites	None
Credits (AU)	5
Grading	Score Grading
Contact Hours	6-hour lecture per week. In total, 58 lecture hours per semester.
Proposal Date	20th Jan. 2024
Proposed By	Qiyu Kang

Course Aims

This course is meticulously structured to provide students with an in-depth introduction to the core principles of mathematical analysis, laying a strong foundation for their further academic and professional growth in a variety of fields, with a special emphasis on machine learning. By engaging deeply with the detailed study and verification processes inherent in mathematical analysis, the course aspires to:

- **Nurture Analytical Reasoning:** Through systematic exploration of core concepts, students will develop a heightened capacity for analytical thought, enabling them to dissect complex mathematical problems and arrive at logical solutions.

- **Demystify Advanced Mathematical Language:** With an emphasis on the precise language of advanced mathematics, students will become fluent in mathematical expressions, symbols, and structures, making them adept at both understanding and formulating complex mathematical arguments.

- **Lay Groundwork for Advanced Studies:** This course serves as a precursor to more specialized studies in Real Analysis, Complex Analysis, Topology, and Differential Equations, among other topics. It establishes the requisite knowledge base for students to confidently navigate and excel in more advanced mathematical terrain.

- **Bridge Theory with Modern Application:** Recognizing the growing importance of mathematical analysis in today's technological landscape, the course provides insights into its theoretical applications within the machine learning community. ***Through case studies, discussions, and problem-solving sessions, students will gain a perspective on how foundational mathematics underpins and shapes cutting-edge machine learning algorithms and models.***

By the end of this course, students will not only have acquired a robust understanding of mathematical analysis but will also be well-prepared to apply these concepts in both theoretical and practical settings, reflecting the evolving demands of the scientific and technological sectors.

Learning Outcomes (LO)

By the end of this course, students will have honed the following competencies:

1. **The Real and Complex Number Systems:** Understand and describe the properties of ordered sets. State and prove the properties of fields. Recognize the axiom of completeness and its significance. Explore the extended real number system and its properties.

2. **Basic Topology:** Define and distinguish between open and closed sets. Understand the concept of limit points. Analyze sequences and describe their convergence behavior in metric spaces. Identify and prove properties of compact sets. Understand and explain the Bolzano-Weierstrass Theorem. Define connected sets and describe their properties.

   Application in Machine Learning: Apply topological concepts in clustering algorithms and neural network topologies, ensuring efficient data representation and processing.

3. **Numerical Sequences and Series:** Determine the convergence or divergence of sequences. Utilize the Cauchy criterion to assess sequence convergence. Analyze infinite series and determine their convergence or divergence. Apply various tests (e.g., comparison, root, and ratio tests) to assess series convergence. Understand the properties of power series and their radius of convergence.

   Application in Machine Learning: See topic LO. 7 for more details.

4. **Continuity:** Understand and apply the definition of continuity. Prove properties of continuous functions. Determine the boundedness and compactness of continuous functions on compact sets. Describe types of discontinuities and understand their significance.

   Application in Machine Learning: Leverage the notion of continuity in understanding and designing activation functions in neural networks, and in ensuring smooth loss landscapes for optimization.

5. **Differentiation:** Understand the concept of differentiation and its geometric significance. Prove and utilize properties of differentiable functions. Understand and apply the mean value theorem. Determine the behavior of functions using differentiation (e.g., monotonicity, extrema).

   Application in Machine Learning: Utilize differentiation in backpropagation, gradient descent optimization, and understanding model sensitivity/robustness.

6. **The Riemann-Stieltjes Integral:** Understand the construction of the Riemann-Stieltjes integral. Determine the integrability of functions. Relate integrability with continuity and boundedness. Prove and apply properties of the Riemann-Stieltjes integral. Explore applications of the Riemann-Stieltjes integral in various contexts.

   Application in Machine Learning: Integral calculus principles underpin many kernel methods and probabilistic model formulations in machine learning. Recent advances in neural networks explore the layers as discretizations of continuous transformations governed by ordinary differential equations (ODEs). Solving these ODEs necessitates integral calculus techniques.

7. **Sequences and Series of Functions:** Understand the pointwise and uniform convergence of sequences of functions. Analyze the relationship between uniform convergence and continuity, integration, and differentiation. Prove and apply the Weierstrass M-test. Explore the properties and implications of power series. Understand and apply the concepts of the Taylor series and its convergence.

   Application in Machine Learning: Understand the convergence behaviors crucial for iterative algorithms and in modeling time series data. Delve into its significance in neural network approximation capabilities, gain insights into its role in sequential estimation and learning, and appreciate its relevance in numerical computations.

8. **Some Special Functions:** Define and understand properties of the exponential, logarithm, and trigonometric functions in a rigorous manner. Analyze the behavior and properties of functions through their power series representations. Understand and explore the Gamma function. Prove and apply functional equations and other identities of these special functions.

   Application in Machine Learning: Understand special functions in the design and functionality of neural network activations. Elucidate the significance of the Gamma function in the theoretical foundations of machine learning, highlighting its role in distribution modeling and Bayesian methodologies.

9. **Functions of Several Variables:** Understand the linear transformation. Apply the notion of partial derivatives and the gradient. Explore the inverse function theorem and the implicit function theorem. Understand the concept of change of variables in multiple integrals.

   Application in Machine Learning: Use multi-variable calculus in multi-input models, understand high-dimensional data spaces, and optimize complex systems.

10. **Integration of Differential Forms:** Understand the construction and properties of differential forms. Analyze the integration of differential forms over regions in R^n. Apply Stoke's theorem and understand its significance. Explore the divergence theorem and its applications in various contexts. Analyze and apply the concept of exterior differentiation.

    Application in Machine Learning: Grasp how integration shapes vector field interpretations in deep learning and structured data. Recognize its influence in differential geometry and geometric learning.

Course Content

No	Topic	Hours
1	The Real and Complex Number Systems	6
2	Basic Topology	6
3	Numerical Sequences and Series	6
4	Continuity	6

5	Differentiation	6
6	The Riemann-Stieltjes Integral	6
7	Sequences and Series of Functions	6
8	Some Special Functions	4
9	Functions of Several Variables	6
10	Integration of Differential Forms	6
	Total	**58**

Assessment (Includes both continuous and summative assessment)

Component	Course Learning Out Tested	Weighting	Team/ Individual	Assessment rubrics
1. Continuous Assessment 1 (CA1): Quiz 1	1, 2, 3	15%	Individual	
2. Continuous Assessment 2 (CA 2): Quiz 2	4, 5, 6	10%	Individual	
3. Continuous Assessment 3 (CA 3): Homework (after each topic)	1, 2, 3, 4, 5, 6, 7, 8, 9, 10	10%	Individual	Appendix 1
4. Continuous Assessment 4 (CA 4) Readiness assessment	1, 2, 3, 4, 5, 6, 7, 9	5%	Individual	Appendix 2
5. Final Examination	1, 2, 3, 4, 5, 6, 7, 8, 9, 10	60%	Individual	
Total		**100%**		

Formative Feedback

- Examination outcomes
- Marker's summary of exam performance
- Quiz results and solutions
- Homework and readiness assessment evaluations
- Discussions during office hours.

Learning & Teaching Approach

Approach	How does this approach support students in achieving the learning outcomes?
LECTURE	Course materials covering all topics
TUTORIAL	-
LABORATORY	-

Readings & References

Textbook
1. Rudin, Walter. Principles of mathematical analysis. McGraw-Hill, 3rd Edition, 1976.

References
1. Stein, Elias M., and Rami Shakarchi. Real analysis: measure theory, integration, and Hilbert spaces. Princeton University Press, 2009.
2. Folland, Gerald B. Real analysis: modern techniques and their applications. Vol. 40. John Wiley & Sons, 1999.

Course Policy & Student Responsibility

General: Students are expected to take all scheduled projects and tests by due dates. Students are expected to take responsibility to follow up with course notes, projects and course related announcements. Students are expected to participate in all tutorial discussions and activities.

Absenteeism: A major component of your final grade is based on continuous assessments. Missing these without an official leave of absence will result in zero marks, negatively impacting your overall grade.

Academic Integrity

Good academic work depends on honesty and ethical behaviour. Good academic work relies on honesty and ethical behavior. It is crucial that as a student, you understand and apply the principles of academic integrity in all your work at SUTD. Not knowing these principles does not excuse academic dishonesty. *You should make sure to learn and avoid any forms of dishonesty, including plagiarism, academic fraud, collusion, and cheating.*

Planned Weekly Schedule

Week	Topic	Course LO	Readings/ Activities
1	The Real and Complex Number Systems	1	
2	Basic Topology	2	RA
3	Numerical Sequences and Series	3	Quiz 1, covering lecture materials in Weeks 1 to 3
4	Continuity	4	RA
5	Differentiation	5	RA
6	The Riemann-Stieltjes Integral	6	Quiz 2, covering lecture materials in Weeks 4 to 6
7	Sequences and Series of Functions	7	RA
8	Some Special Functions	8, 9	RA
9	Functions of Several Variables	9	RA

10	Integration of Differential Forms	10	RA

Appendix 1: Assessment Criteria for Homework (CA3)

Learning outcomes measured:

Problem solving.

In accomplishing these learning outcomes of the homework, you would also have shown your achievement of Learning Outcomes 1 to 10 of the course.

Marks	Marking Scheme
0%	Absent
30%	Poor – Not able to solve the problem. Almost all the concepts are wrong.
50%	Marginal – Not able to fully solve the problem; however, some concepts are correct and a small partial solution is found.
60%	Average – Not able to fully solve the problem; however, about half of the concepts are correct and about half of the solution is found.
70%	Good – Not able to fully solve the problem; however, more than half of the concepts are correct and more than half of the solution is found.
80%	Excellent – The problem is fully solved, with very small errors.
90%	Outstanding – The problem is fully solved without any errors.

Appendix 2: Assessment Criteria for Readiness Assessment (CA4)

There will be a total of 3 Readiness Assessments (RAs). Each RA will be given points using the criteria in the table below, with maximum 3 points for each RA. The total points for a student will be converted to the marks for this CA, with a maximum 5% of the final marks.

Standards	**Criteria**
Good (3 points)	Demonstrates good understanding of Learning Outcomes 1, 2, 3, 4, 5, 6, 7, and 9 with thoughtful completion of all questions.
Average (2 points)	Demonstrates an understanding of Learning Outcomes 1, 2, 3, 4, 5, 6, 7, and 9 with some minor gaps.
Below Average (1 points)	Puts in effort to complete the questions but has weak understanding of Learning Outcomes 1, 2, 3, 4, 5, 6, 7, and 9 with glaring mistakes.
Unsatisfactory (0 point)	No significant effort to complete the questions or did not submit any solutions.