Uniform Convergence of Function Sequences/Series

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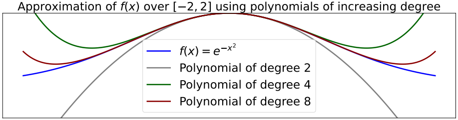
Motivation

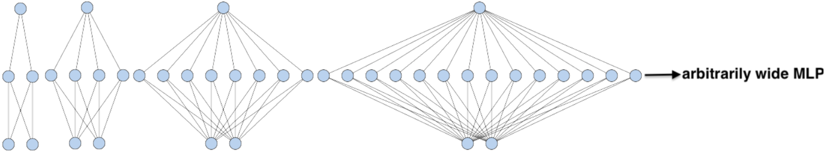
Why do we care about function sequences/series?

Motivation

- Stone-Weierstras theorem: can we "uniformly" well approximate an arbitrary continuous function f over an interval using polynomial sequence $\{P_n\}$?



- neural network approximation capabilities as width $\to \infty$



- sequential estimation/learning: $Y_1(\omega) = f_1(X_1(\omega))$, $Y_2(\omega) = f_2(X_1(\omega), X_2(\omega))$, ... How does the estimator $Y_n(\omega)$ behave? Can we get an arbitrarily good estimation of some unknown r.v. $Y(\omega)$ or parameter asymptotically?
- numerical computation: E.g. numerically approximate incomplete gamma function $\gamma(a, x) \equiv \int_0^x \mathrm{e}^{-t} t^{a-1} \, \mathrm{d}t \approx x^a \sum_{n=0}^{N} \frac{(-1)^n x^n}{(a+n)n!}$.
- network analysis (social, citation, collaboration): converge from increasing sized graphs to an asymptotic graphon (symmetric function $W : [0,1]^2 \mapsto [0,1]$)

Pointwise Convergence

We confine our attention to complex- or real-valued functions.

Definition (pointwise convergence)

Suppose $\{f_n\}, n = 1, 2, 3, \ldots$, is a sequence of functions defined on a set E, and suppose that the sequence of numbers $\{f_n(x)\}$ converges for every $x \in E$. We can then define a limit function f, by

$$f(x) = \lim_{n \to \infty} f_n(x) \quad (x \in E). \tag{1}$$

We say that "$\{f_n\}$ converges to f pointwise on E".

summary: $\{f_n\}$ converges to f pointwise on E iff $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in E$.

Remark (Series)

Similarly, if $\sum f_n(x)$ converges for $x \in E$, and if we define the limit function as $f(x) = \sum_{n=1}^{\infty} f_n(x) \quad (x \in E)$. We say that "$\sum f_n$ converges to f pointwise on E".

Uniform Convergence

Motivation: The rate of convergence is "uniform" across the entire domain.

Definition (uniform convergence)

We say that a sequence of functions $\{f_n\}, n = 1, 2, 3, \ldots$, converges uniformly on E to a function f if for every $\varepsilon > 0$ there is an integer N such that $n \geq N$ implies

$$|f_n(x) - f(x)| \leq \varepsilon \tag{2}$$

for all $x \in E$.

summary: $\{f_n\}$ converges to f uniformly on E iff $\forall \epsilon > 0$, $\exists N$ s.t. $\sup_{x \in E} |f_n(x) - f(x)| \leq \varepsilon$ for all $n > N$.

Remark (Series)

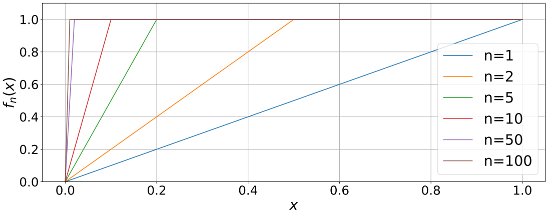
Similarly, the series $\sum f_n(x)$ converges uniformly on E if the sequence $\{s_n\}$ of partial sums defined by $s_n(x) := \sum_{i=1}^{n} f_i(x)$ converges uniformly on E.

Comparison

1. **pointwise convergence:** $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in E$.
   equivalent ε-δ ***description:*** *for every* $x > 0$, *and for every* $\varepsilon \in E$, *there is an integer* $N(\varepsilon, x)$, *depending on* ε *and on* x, *s.t.* $|f_n(x) - f(x)| \leq \varepsilon$ *holds if* $n \geq N$.
2. **uniform convergence:** *for every* $\varepsilon > 0$ *there is an integer* $N(\varepsilon)$, *depending only on* ε, *s.t.* $|f_n(x) - f(x)| \leq \varepsilon$ *if* $n \geq N$ *for all* $x \in E$.

- **Observation:** uniform convergence $\Rightarrow$ pointwise convergence.
- **Question:** uniform convergence $\Leftarrow$ pointwise convergence?

uniform convergence $\not\Leftarrow$ pointwise convergence



Consider $\{f_n(x)\}$ defined on the interval $[0, 1]$ by:

$$f_n(x) = \begin{cases} nx & \text{for } 0 \le x \le \frac{1}{n} \\ 1 & \text{for } \frac{1}{n} < x \le 1 \end{cases}$$

pointwisely convergent to

$$f(x) = \begin{cases} 0 & \text{for } x = 0 \\ 1 & \text{for } 0 < x \le 1 \end{cases}$$

but not uniformly:

$$\left| f_n\left(\frac{1}{2n^2}\right) - f\left(\frac{1}{2n^2}\right) \right| = \left| n \cdot \frac{1}{2n^2} - 1 \right| = 1 - \frac{1}{2n} \ge \frac{1}{2}$$

Uniform Convergence and Continuity

Theorem (Interchanging Two Limits)

Suppose $f_n \to f$ uniformly on a set E in a metric space. Let x be a limit point of E, and assume the existence of $\lim_{t\to x} f_n(t)$ for all n, we have

$$\lim_{t\to x}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\lim_{t\to x} f_n(t).$$

basic theorem $\Rightarrow$ theorems about continuous, differentiable.

Proof Hint: Let a ε be given.

Step 1: Define $A_n \coloneqq \lim_{t\to x} f_n(t)$. Uniform convergence $\Rightarrow \exists N$ s.t. $|f_n(t) - f_m(t)| \le \varepsilon \ \forall n, m \ge N, \forall t \in E$. It follows that $|A_n - A_m| \le \varepsilon$, a Cauchy sequence. So, it converges to a limit, denoted by A.

Step 2: $|f(t) - A| \le |f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A|$. Prove each item $< \varepsilon/3$ for large enough n and for t sufficiently close to x. We therefore have $|f(t) - A| < \varepsilon$ when t is sufficiently close to x.

Corollary (Continuity Preserved under Uniform Convergence)

If $\{f_n\}$ is a sequence of continuous functions on E, and if $f_n \to f$ uniformly on E, then f is continuous on E.

Example: pointwise convergence may lead to discontinuous

Let

$$f_n(x) = \frac{x^2}{(1+x^2)^n} \quad (x \text{ real } ; n = 0, 1, 2, \dots),$$

and consider

$$f(x) = \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n}.$$

Since $f_n(0) = 0$, we have $f(0) = 0$. For $x \neq 0$, the last series is a convergent geometric series with sum $1 + x^2$. Hence

$$f(x) = \begin{cases} 0 & (x = 0), \\ 1 + x^2 & (x \neq 0). \end{cases}$$

So, a pointwise convergent series of continuous functions may have a discontinuous sum.

Uniform Convergence and Differentiation

- $f_n \to f$ uniformly $\Rightarrow$ the convergence of $f_n' \to f'$? No!
  Example:
  $$f_n(x) = \frac{\sin nx}{\sqrt{n}} \quad (x \textit{ real}, n = 1, 2, 3, \ldots),$$
  and
  $$f(x) = \lim_{n\to\infty} f_n(x) = 0.$$
  Then $f'(x) = 0$, *and* $f_n'(x) = \sqrt{n}\cos nx$, *so that* $\{f_n'\}$ *does not converge to* f'. *For instance,* $f_n'(0) = \sqrt{n} \to +\infty$ *as* $n \to \infty$, *whereas* $f'(0) = 0$.
  So we need stronger hypotheses to get $f_n' \to f'$!

Theorem (Interchanging Limit and Differentiation)

Suppose $\{f_n\}$ *is a sequence of functions, differentiable on* $[a, b]$ *and such that* $\{f_n(x_0)\}$ *converges for some point* x_0 *on* $[a, b]$. *If* $\{f_n'\}$ *converges uniformly on* $[a, b]$, *then* $\{f_n\}$ *converges uniformly on* $[a, b]$, *to a function* f, *and*

$$f'(x) = \lim_{n\to\infty} f_n'(x) \quad (a \leq x \leq b).$$

Uniform Convergence and Differentiation

Proof Hint: Let $\varepsilon > 0$ be given.

Step 1: Prove $\{f_n\} \to f$ uniformly:
Uniform convergence of $\{f_n'\}$ leads to $|f_n'(t) - f_m'(t)| < \frac{\varepsilon}{2(b-a)} \quad (a \leq t \leq b)$ if $n, m \geq N_0$. Apply the mean value theorem to $f_n - f_m$ and get $|f_n(x) - f_m(x) - f_n(t) + f_m(t)| \leq \frac{|x-t|\varepsilon}{2(b-a)} \leq \frac{\varepsilon}{2}$. Finally use
$|f_n(x) - f_m(x)| \leq |f_n(x) - f_m(x) - f_n(x_0) + f_m(x_0)| + |f_n(x_0) - f_m(x_0)| < \varepsilon$
when $n, m > N_1$ for all x.

Step 2: Prove $\{\phi_n\} \to \phi(t)$ uniformly: Fix a x and define

$$\phi_n(t) = \frac{f_n(t) - f_n(x)}{t - x}, \quad \phi(t) = \frac{f(t) - f(x)}{t - x}$$

for $a \leq t \leq b, t \neq x$.

Step 3: Interchanging limits: Since $\{f_n\}$ converges to f, it follows that

$$\lim_{n\to\infty} \phi_n(t) = \phi(t)$$

uniformly for $a \leq t \leq b, t \neq x$. Theorem (Interchanging Two Limits) show that

$$\lim_{t\to x} \phi(t) = \lim_{n\to\infty} f_n'(x);$$

Analytic Functions/Power Series

Lemma (Uniform Convergence Test for Series)

Suppose $\{f_n\}$ is a sequence of functions defined on E, and suppose

$$|f_n(x)| \leq M_n \quad (x \in E, n = 1, 2, 3, \ldots).$$

Then $\sum f_n$ converges uniformly on E if $\sum M_n$ converges.

Theorem

Suppose the real-valued series $\sum_{n=0}^{\infty} c_n x^n$ converges for $|x| < R$, and define

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \quad (|x| < R). \tag{3}$$

Then the series converges uniformly on $[-R+\varepsilon, R-\varepsilon]$, no matter which $\varepsilon > 0$ is chosen. The function f is continuous and differentiable in $(-R, R)$, and

$$f'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1} \quad (|x| < R). \tag{4}$$

Proof Hint: Use the above Lemma and "Interchanging Limit and Differentiation".

Post-Course Tasks and Recommended Readings

- Self-write the complete proofs of discussed theorems.
- Complete all assigned homework.
- Delve into Convergence Types: L_1 convergence, almost surely convergence, pointwise convergence, and uniform convergence.
- Investigate the Approximation Capabilities of Neural Networks [1].
- Explore the intricacies of Sequential Estimation [2].
- Explore Neural Ordinary Differential Equations [3] and the associated existence and uniqueness of solutions: Picard–Lindelöf theorem [4] where the function sequence and uniform convergence are used in the proof.

References

1 Hornik, K., 1991. Approximation capabilities of multilayer feedforward networks. Neural networks, 4(2), pp.251-257.

2 Chapter 3, Stochastic processes, detection, and estimation, MIT 6.432 Course Notes.

3 Chen, Ricky TQ, et al. "Neural ordinary differential equations." Advances in neural information processing systems 31 (2018).

4 Chapter II, Theorem 1.1., Hartman, P., 2002. Ordinary differential equations. Society for Industrial and Applied Mathematics.